THE UNIVERSITY OF MICHIGAN
COLLEGE OF ENGINEERING
Department of Mechanical Engineering
Heat Transfer Laboratory

Technical Report No. 4

BOUNDARY LAYER ANALYSIS OF TWO-PHASE (LIQUID-GAS) FLOW OVER A CIRCULAR CYLINDER AND OSCILLATING FLAT PLATE

M. E. Goldstein
Wen-Jei Yang
J. A. Clark

ORA Project 05065

under contract with:

AERONAUTICAL RESEARCH LABORATORY, OAR
AERONAUTICAL SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
CONTRACT NO. AF 33(657)-8368
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

August 1965

This report was also a dissertation submitted by the first author in partial fulfillment of the re-requirements for the degree of Doctor of Philosophy in The University of Michigan, 1965.

ACKNOWLEDGMENTS

The authors wish to express their appreciation to Professors Vedat S. Arpaci and Stanley Jacobs, for their interest and cooperation in serving as members of the Doctoral Thesis Advisory Committee.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Concluded)

LIST OF ILLUSTRATIONS

NOMENCLATURE

Notation for Cylinder Problem

Latin Letters

a_o	=	constant appearing in the expansion for δ^*
A_n	=	function of the f_{2n-1}'s and their derivatives and the b_{2n}'s appearing in the ordinary differential equations for the momentum problem, $n = 1,3,5,\ldots$
b_{2n}	=	constants appearing in the expansion for δ^*; $n = 0,1,2\ldots$
B_n	=	function of F_{2n}'s and b_{2n}'s appearing in the ordinary differential equation for the energy problem; $n = 0,2,4$
C_D	=	drag coefficient for liquid drops
C_p	=	specific heat at constant pressure of liquid
C_{p_g}	=	specific heat at constant pressure of gas
D^2	=	partial differential operator in momentum equations
D^2_{xy}	=	D^2, rewritten in terms of x,y
$\bar{D}^2$	=	dimensionless version of D^2
$\tilde{D}^2$	=	dimensionless version of D^2
E^2	=	$2X_e^2 \quad R_o U_\infty/\nu$
f_{2n-1}	=	$n = 1,2,\ldots$, function of η in expansion for ψ^*
F_{2n}	=	$n = 0,1,2\ldots$, function of η in expansion for Θ^*
G_n	=	$n = 0,2,4,\ldots$, function of f_{2n-1}'s and F_{2n}'s appearing in the differential equations for F_{2n}
H_n	=	$n = 1,3,5,\ldots$, function of b_{2n}'s f_{2n-1}'s and their derivatives appearing in differential equations for f_{2n-1}'s

$\hat{i}$	=	unit vector in x_o direction
$\hat{j}$	=	unit vector in y_o direction
J	=	$- \frac{x_e}{X_{e_\infty}} V_r^*$
k	=	thermal conductivity of the liquid
k_g	=	thermal conductivity of the gas
K_n	=	linear operator for energy equations
L_n	=	linear operator for momentum equations
$\dot{m}$	=	$- \int_0^{\tilde{x}} \frac{x_e}{X_{e_\infty}} V_r^* \, d\tilde{x}$
$\hat{n}$	=	unit outward drawn normal to the cylinder
N'	=	number of liquid drops lying along a fluid line whose length is equal to the radius of the cylinder
N_o	=	number of drops per unit volume
N_u	=	Nusselt number of the liquid
N_{u_g}	=	Nusselt number of the gas
P	=	pressure in the liquid
P_g	=	pressure in the gas
$\bar{P}$	=	$P/1/2 \; \rho \; U_\infty^2$
$\tilde{P}$	=	$P/1/2 \; \rho \; U_\infty^2$
P_g^*	=	$P_g/ \; 1/2 \; \rho_g U_\infty^2$
P_r		Prandtl Number of liquid
P_{r_g}		Prandtl Number of gas
q_w	=	heat flux at cylinder wall

q_n	=	terms of expansion for $N_u/\sqrt{\frac{2U_\infty R_o}{\nu}}$
$\dot{q}_g$	=	heat flux into the gas at boundary of liquid film
r	=	radial coordinate
r_d	=	radius of drop
$\vec{r}_d$	=	position vector of drop
R_o	=	radius of cylinder
R_e	=	$2R_oU_\infty/\nu$
s	=	surface of liquid film
t	=	time
T	=	temperature of liquid
T_w	=	temperature of cylinder wall
T_∞	=	temperature of flow system at infinity
t^*	=	$t\ U_\infty/R_o$
u	=	v_φ; velocity of liquid in film in asimuthal direction
u_n^*	=	terms in expansion for u/U_∞
u_g	=	x_o component of gas velocity
u_g^*	=	u_g/U_∞
U_∞	=	velocity of liquid and gas at infinity
v_φ	=	velocity of liquid in film in asimuthal direction
v_r	=	velocity of liquid in film in radial direction
v_g	=	y_o component of gas velocity
v_g^*	=	V_g/U_∞
V_φ	=	asimuthal component of drop velocity at edge of liquid film

V_r	=	radial component of drop velocity at edge of liquid film
V_φ^*	=	V_φ/U_∞
V_r^*	=	V_r/U_∞
$\vec{V}$	=	$\hat{i}\, V_{xo} + \hat{j}\, V_{yo}$
V_{xo}	=	drop velocity in x_o direction
V_{yo}	=	drop velocity in y_o direction
V_{xo}^*	=	V_{xo}/U_∞
V_{yo}^*	=	V_{yo}/U_∞
$\vec{V}_g$	=	$\hat{i}\, u_g + \hat{j}\, v_g$
$\vec{V}_g^*$	=	$\vec{v}_g/U_\infty$
$\vec{V}^*$	=	$\vec{V}/U_\infty$
V_{yof}	=	drop velocity in y_o direction at edge of liquid film
V_{xof}	=	drop velocity in x_o direction at edge of liquid film
V_{yof}^*	=	V_{yof}/U_∞
V_{xof}^*	=	V_{xof}/U_∞
ΔV^*	=	$\sqrt{(V_{xo}^* - u_g^*)^2 + (V_{yo}^* - v_g^*)^2}$
x	=	φ: distance along surface of cylinder divided by R_o
$\overline{x}$	=	x
$\tilde{x}$	=	x
x_o	=	coordinate in direction of flow at infinity
x_d	=	x_o position of drop
x_o^*	=	x_o/R_o

x_d^*	=	x_d/R_o
x_{d_f}	=	x_o position of drop at surface of liquid film
x_{df}^*	=	x_{df}/R_o
X_e	=	volume fraction of liquid outside of film
X_{e_∞}	=	volume fraction of liquid at infinity
y	=	coordinate in radial direction: $\frac{r}{R_o} - 1$
$\bar{y}$	=	$y/2X_{e_\infty}$
$\tilde{y}$	=	$y/2\ \sqrt{\frac{2R_oU_\infty}{\nu}}$
y_o	=	coordinate perpendicular to flow at infinity
y_d	=	y_o coordinate of drop
$y_{d\infty}$	=	y_o coordinate of drop at infinity
y_{df}	=	y_o coordinate of drop at surface of film
y_o^*	=	y_o/R_o
y_d^*	=	y_d/R_o
$y_{d_\infty}^*$	=	$y_{d\infty}/R_o$
y_{df}^*	=	y_{df}/R_o
$Z(\tilde{x})$	=	dimensionless gas shear stress term

Greek Letters

α_{2n}	=	terms in expansion for $\tilde{\delta}J$
β_{2n}	=	terms in expansion for $\tilde{\delta}\ \dot{\mathcal{m}}$
γ	=	$\cos^{-1}.\ \left\{ - \left[1 - \frac{24(\rho_g/\rho)}{E^2}\ \frac{(\sin\cdot 2x)^2 \dot{\mathcal{m}}}{\tau_{eff}^3} \right] \right\}$
γ_n	=	$n = 1,3,5,\ldots,$ terms in series expansion of $Z(\tilde{x})$

δ	=	thickness of liquid film
δ^*	=	$\delta(\xi)/2R_o X_{e\infty}$
$\bar{\delta}$	=	$\delta(\bar{x})/2R_o X_{e\infty}$
$\tilde{\delta}$	=	$\frac{\hat{\delta}}{2}\sqrt{\frac{2R_oU_\infty}{\nu}}$
$\hat{\delta}$	=	δ/R_o
ζ	=	function arising from integration of stream function
η	=	$y/\hat{\delta}$
θ	=	angle between tangent to cylinder and tangent to surface of liquid film
θ^*	=	$\frac{T - T_\infty}{T_w - T_\infty}$
μ	=	viscosity of liqued
μ_g	=	viscosity of gas
ν	=	μ/ρ; kinematic viscosity of liquid
ν_g	=	μ_g/ρ_g; kinematic viscosity of gas
ξ	=	x/R_o
ρ	=	density of liquid
ρ_g	=	density of gas
τ_{rr}	=	radial component of shear stress in liquid film
$\tau_{r\varphi}$	=	azimuthal and radial components of shear stress in liquid film
$\tau_{\varphi\varphi}$	=	azimuthal component of shear stress in liquid film
τ_g	=	shear stress exerted by gas on surface of film
τ_w	=	shear stress exerted by liquid on surface of cylinder

τ_n = n = 1,3,5,..., terms in expansion for $\tau_w \sqrt{\frac{2U_\infty R_o}{\nu}} \Big/ 1/2\rho U_\infty^2$

τ_{eff} = $\left\{ \left[\frac{4}{E} \; \frac{\rho_g}{\rho} \sqrt{\frac{\nu_g}{\nu}} \; Z(\tilde{x}) - \frac{x_e}{x_{e_\infty}} \; V_r^* \; V_\varphi^* \right] \right\}$

φ = azimuthal coordinate

φ_d = azimuthal coordinate of drop at surface of film

ψ = stream function in liquid film

ψ^* = $\psi/U_\infty R_o X_{e\infty}$

$\tilde{\psi}$ = $\psi/U_\infty R_o X_{e\infty}$

$\bar{\psi}$ = $\psi/U_\infty R_o X_{e\infty}$

Special notation

∇_o = $\hat{i} \frac{\partial}{\partial x_o} + \hat{j} \frac{\partial}{\partial y_o}$

∇_d = $\hat{i} \frac{\partial}{\partial x_d} + \hat{j} \frac{\partial}{\partial y_d}$

∇_o^* = $R_o \; \nabla_o$

∇_d^* = $R_o \; \nabla_d$

Subscripts

d = drop

g = gas

i = integer

j = integer

∞ = upstream conditions

m = integer

n = integer

NOMENCLATURE (Continued)

Notation for Oscillating Flat Plate

Latin Letters

$a, a_{1n}, a_{2n}, \tilde{a}_{2n}$	=	coefficients in expansion for δ^* (constants)
A	=	area of control surface
$A_{c.s.}$	=	area of control surface
$B_n, \tilde{B}_n$	=	terms appearing in momentum boundary conditions function of $f_{m,n}$'s, $a_{m,n}$'s
$C_n, \tilde{C}_n$	=	terms appearing in energy boundary conditions function of $f_{m,n}$'s $F_{m,n}$'s
c_p	=	specific heat at constant pressure of liquid
c_{p_g}	=	specific heat at constant pressure of gas
$e_{1n}, e_{2n}, \tilde{e}_{2n}$	=	terms appearing in expansion for τ_n^* ; $n = 0,1,2,\ldots$
$E_n, \tilde{E}_n$	=	nonhomogeneous terms appearing in differential equations for $F_{2n}, \tilde{F}_{2n}$'s
$f_o, f_{1n}, f_{2n}, \tilde{f}_{2n}$	=	coefficients in expansions for ψ^* (functions of η)
$F_o, F_{1n}, F_{2n}, \tilde{F}_{2n}$	=	coefficients in expansions for Θ^* (functions of η)
$g_{1n}, g_{2n}, \tilde{g}_{2n}$	=	terms of expansion of u_n^*
$G_{1n}, G_{2n}, \tilde{G}_{2n}$	=	terms of expansion for g_n^*
$h_n, \tilde{h}_n$	=	terms appearing in expansion for u_n^*
$H_n, \tilde{H}_n$	=	nonhomogeneous terms appearing in the differential equations for $f_{2n}, \tilde{f}_{2n}$'s
i	=	$\sqrt{-1}$
k	=	thermal conductivity of liquid
k_g	=	thermal conductivity of gas

NOMENCLATURE (Continued)

K_n	=	differential operator for energy equations
L_n	=	differential operator for momentum equations
N_{u_x}	=	Nusselt number
N_{u_g}	=	Nusselt number of gas
P_r	=	Prandtl number of liquid
P_{r_g}	=	Prandtl number of gas
$P_n, \tilde{P}_n$	=	nonhomogeneous terms appearing in momentum equations
q_w	=	heat flux at plate
q_g	=	heat flux into gas at edge of film
q_n^*	=	coefficient of ϵ in expansion for local-Nusselt number
$r_n, \tilde{r}_n$	=	terms appearing in expansion for τ_n^*
$R_n, \tilde{R}_n$	=	terms appearing in expansion for q_n^*
s'	=	upper surface of liquid film
s''	=	vertical boundary of control surface
t	=	time
T	=	temperature
T_w	=	temperature of plate
T_∞	=	temperature at infinity
u	=	x-component of velocity in film
u_n^*	=	coefficient of ϵ in expansion for u/U_∞
U	=	$U_\infty(1 + \frac{\epsilon}{2} \sin \cdot \omega t)$
U_∞	=	velocity of liquid drops and gas at infinity

v	=	y-component of velocity in film
$V_{c.s.}^{(x)}$	=	x-component of velocity of edge of liquid film
$V_{c.s}^{(y)}$	=	y-component of velocity of edge of liquid film
$V_{c.s.}^{\perp}$	=	normal component of velocity of edge of liquid film
x_1	=	x-distance in x-y coordinates
x	=	coordinate direction along plate in coordinate system fixed to plate
X_e	=	volume fraction of liquid in free stream
y	=	coordinate direction perpendicular to plate in co-ordinate system fixed to plate
Z_1, Z_2	=	arbitrary complex quantities

reek Letters

α_n	=	arbitrary complex quantities with α_n imaginary for $n = 0,4,8,\ldots$, α_n real for $n = 2,6,\ldots$ $\alpha_n = 0$ for $n = 1,3,5,\ldots$
δ	=	thickness of liquid film
δ^*	=	$\delta \sqrt{\frac{\omega}{\nu X_e}}$
δ_n^*	=	$n = 0,1,2,\ldots$, coefficient of ϵ^n in expansion for δ^*
ϵ	=	dimensionless amplitude of oscillation
η	=	y/δ
Θ^*	=	$(T - T_\infty)/(T_w - T_\infty)$
Θ_n^*	=	coefficient of ϵ^n in expansion for Θ^*
λ	=	equal to ± 1
μ	=	viscosity of liquid

μ_g = viscosity of gas

ν = kinematic viscosity of liquid

ν_g = kinematic viscosity of gas

ξ = $\sqrt{\frac{\omega x}{X_e U_\infty}}$

ρ = density of liquid

ρ_g = density of gas

τ = ωt

τ_{xy} = component of liquid stress tensor

τ_g = shear stress exerted by gas at edge of film

τ_w = shear stress at plate

τ_n^* = coefficient of ϵ^n in expansion for τ_w

φ = slope of liquid film and phase angle

ψ = stream function

ψ^* = $\frac{\psi}{U_\infty X_e} \sqrt{\frac{\omega}{X_e \nu}}$

ψ_n^* = coefficient of ϵ^n in expansion for ψ^*

ω = frequency of oscillation of plate

Subscripts

avg. = time average value

g = gas

w = surface of plate

n = integer

steady = steady state value

∞ = upstream conditions

ABSTRACT

An analysis is carried out of a two phase (gas/liquid) flow over a circular cylinder and over an oscillating flat plate. The flow is assumed two dimensional and gravity, vaporization of the liquid phase, and compressibility are ignored. The liquid is assumed to be in the form of small drops far upstream from the body. The liquid film which forms on the surface of the body due to drop impingement is analyzed extensively. In the case of the cylinder the analysis is started from the full incompressible Navier-Stokes and Energy equations in the film which are simplified by using dimensional arguments. The order of magnitude of the physical properties of the fluids is taken to be those of air/water mixtures. The analysis is carried out for two ranges of a parameter appearing in the problem. For low values of the parameter the liquid drop trajectories deviate appreciably from straight lines and are obtained numerically. For the flat plate the boundary layer approximations are assumed to hold *a priori* in the film. The plate is taken to be oscillating sinusoidally in its plane with small amplitude and small frequency; and the drops are assumed to move in straight lines.

After appropriate changes of variable the solutions to the governing equations and boundary conditions are carried out by a series expansion technique which results in a series of ordinary differential equations which have been solved numerically on a 7090 computer.

The solutions are used to calculate velocity and temperature profiles in the film and also such physical quantities as local film thickness, local Nusselt Number, and local skin friction.

The analysis shows that in general there is a significant increase in heat transfer as well as skin friction over what would be obtained from a single component gas flow. In the case of the flat plate only a very small permanent change in the heat transfer was found due to the oscillations. In the case of the cylinder a peaking in the heat transfer, film thickness and skin friction was found with respect to the parameter E^2 (the product of the volume fraction of the liquid in the free stream, squared, and the diameter Reynolds number based on liquid properties). The peaking occurs at a value of this parameter of about unity. The theoretical prediction compares favorably with the experimental results of Acrivos *et al*. (ARL-64-116).

CHAPTER I

INTRODUCTION

For some years now there has been a growing interest in the artificial creation of two phase flows for the purpose of significantly increasing the attainable level of surface heat transfer rates from solid bodies. Up until recently much of the interest has come from the expanding nuclear power field with much emphasis on systems in which solid particles, such as graphite dust, have been added to a coolent flow.

A large number of papers and reports have also appeared dealing with two-phase (gas-liquid) hydrodynamics and heat transfer phenomena. Among them are a large number on forced convective flows of steam/water mixtures. Data have been presented for channels of circular[5,6,15,27] rectangular[8] and annular[3,5] cross-section with the flow vertically upwards, downwards[26] and horizontal[8,15] over a wide range of values of steam quality, mass velocity and heat flux. In addition details are also given of investigations with refrigerents[25] and with two-component systems[11] both with[22] and without[14] appreciable vaporization of the liquid phase.

A number of impirical correlations have been suggested[3,6,26,27] and some analysis has been carried out.[7]

An analysis has also been carried out for two-phase solid/liquid flows.

However due to the fact that relatively recent Russian liquid injection tests in flow across a tube bank have yielded increases in the measured heat transfer rates of an order of magnitude.[9] Research programs have currently been undertaken to investigate the heat transfer characteristics of external gas/liquid flow systems. Tests have been conducted at the Marquandt Corporation on a system which consisted of an air stream containing water droplets flowing over a heated vertical cylinder.[1] An intensification of the heat transfer coefficient from 2.5 to nine times was noted. An exploratory analytic investigation was carried out by Tifford on the boundary layer characteristics of gas/liquid spray systems, with particular emphasis on heat transfer characteristics.[29]

Experiments on single component systems have shown that oscillations of the system may result in significantly increased heat transfer performance. Analytical two dimensional studies[20,21,24,28] however have not confirmed these results and it is believed that this disagreement between analysis and experiment may be due to the fact that the increased heat transfer is due to three dimensional effects. Nevertheless it may be possible that the two dimensional effects of oscillations may be augmented considerably by the increased mass which results in a two component system.

In any event the use of two-phase heat transfer systems with oscillations is strongly encouraged by the following two important factors:

1. For a given wall temperature, which is often limited by the physical properties of material, a higher heat-transfer rate can be obtained.

2. For a given heat-transfer surface area, which is often limited by the space and other considerations, a large increase in heat-transfer rate can be obtained.

It is anticipated that a detailed analytical study would provide significant information on the dependence of the augmentative effects upon the choice of operating conditions for such a gas/liquid heat transfer system and would aid in the understanding of the physical mechanisms involved.

The present study has therefore been undertaken in order to make some progress toward developing such detailed analytical investigations. Since however the physical situation is quite complex and changes considerably with the geometry and operating conditions the study was limited to two specific systems each in a certain range of operating conditions.

Thus the present work is to study the momentum and heat transfer in a two-component (gas/liquid) steady flow over a circular cylinder and over a flat plate oscillating sinusoidally in its plane.

Analytical results are obtained through the use of the perturbation technique. These results include velocity and temperature profiles, local friction factors, and heat transfer coefficients.

Numerical computations are performed by means of the Runge-Cutta method for a gas/liquid system which corresponds to air-water mixture.

For low values of Reynolds Numbers and small drop sizes it has been found that the liquid drop trajectories deviate appreciably from straight lines.[30] Analytical and numerical studies have been carried out to determine drop trajectories around bodies in connection with icing problems by several investigators. Tribus[30] has calculated trajectories about cylinders and airfoils on a differential analyzer. Langmuir[19] has considered the cylinder analytically, and Bergran[4] and Guibert *et al*.[17] have studied airfoils numerically.

It was disclosed that the presence of the liquid in the gas stream caused a significant increase in both the heat transfer and friction. The oscillations (in the case of the flat plate) caused only very small time average changes. Theoretical predictions on heat transfer and friction for the steady two phase flow over a circular cylinder agree qualitatively with the qualitative results of Tifford.[29]

CHAPTER II

ANALYSIS

A. General Considerations

The physical system to be studied consists of a two-phase (gas/liquid) flow over a heated surface. Two cases are to be treated: A steady flow over a circular cylinder and a steady flow over a flat plate oscillating in its own plane.

The experiments conducted at Marquandt Corporation indicate that when a streaming gas containing liquid drops impinges on a solid body a liquid film is formed on the surface of the body and is caused to flow in the down stream direction.

The following assumptions are imposed on the solution:

(1) The liquid drops contained in the streaming gas are uniformly distributed in the gas and move with the velocity of the gas far upstream.

(2) The flow is taken to be two dimensional and laminar.

(3) Due to inertial effects some of the drops in the free stream splash on the surface of the liquid film. The incoming drops contribute their mass, momentum and heat content to the film. However, the unsteady effects due to random drop impingement are negligible.

(4) There are a sufficient number of drops in the free stream so that only the time average effect of drop impingement need be considered.

(5) It is disclosed, from Marquadt's experiments that waves formed on the surface of the liquid layer and that under certain conditions splashing occured at this surface. These latter two effects are also neglected.

(6) On top of the liquid film a gas boundary layer forms which joins the flow in the region of the body to the external flow field. The usual boundary layer assumptions apply in this region.

(7) Surface tension effects on the surface of the liquid layer are neglected.

(8) The effects of compressibility and heat generated by dissipation can be ignored. The effects of gravity are also neglected.

(9) The ratio of the gas/liquid properties will be taken to be of the order of those of air and water.

(10) All fluid properties will be taken as constant. (e.g., the liquid and the gas are separately assumed incompressible).

(11) No appreciable vaporization occurs.

B. Gas/Liquid Flow Over an Infinite Circular Cylinder

1. FORMULATION

With these assumptions, the following governing differential equations and boundary conditions are obtained for the two component (gas/liquid) flow about the infinite circular cylinder for three regions of the flow field: (1) The liquid film; (2) the gas/liquid free stream flow; (3) the gas/liquid boundary layer.

a. Equations and Boundary Conditions for the Liquid Film

Consider the element of volume shown in Figure 1 bounded by $d\varphi$, $d\delta$ and ds. The law of conservation of momentum about 0 yields:

$$- \tau_{r\varphi}(R_o+\delta,\varphi)[R_o+\delta]^2 d\varphi+\tau_{\varphi\varphi}(R_o+\delta,\varphi)[R_o+\delta]d\delta$$

$$+ \tau_g(0,\varphi)[R_o+\delta]\cos.\theta\ ds+P_g(0,\varphi)[R_o+\delta]\sin.\theta\ ds$$

$$= \rho[v_\varphi(R_o+\delta,\varphi)]^2[R_o+\delta]\ d\delta$$

$$- \rho v_r(R_o+\delta,\varphi)v_\varphi(R_o+\delta,\varphi)[R_o+\delta]^2\ d\varphi$$

$$- X_e(R_o+\delta,\varphi)\ \rho V_\varphi(R_o+\delta,\varphi)[R_o+\delta][V_\varphi(R_o+\delta,\varphi)\sin.\theta-$$

$$V_r(R_o+\delta,\varphi)\cos.\theta]\ ds$$

where terms in the square of small quantities have been neglected.

A force balance in the radial direction yields upon neglecting the squares of small quantities:

$$- \tau_{rr}(R_o+\delta,\varphi)[R_o+\delta]d\varphi + \tau_{r\varphi}(R_o+\delta,\varphi)ds-P_g(0,\varphi)\cos.\theta\ ds$$

$$+ \tau_g(0,\varphi)\sin.\theta ds = \rho v_\varphi(R_o+\delta,\varphi)V_r(R_o+\delta,\varphi)\ d\delta$$

$$- \rho[(v_r(R_o+\delta,\varphi)]^2[R_o+\delta]d\varphi$$

$$- X_e(R_o+\delta,\varphi)\rho V_r(R_o+\delta,\varphi)[V_\varphi(R_o+\delta,\varphi)\sin\theta\ -$$

$$- V_r(R_o+\delta,\varphi)\cos.\theta]ds$$

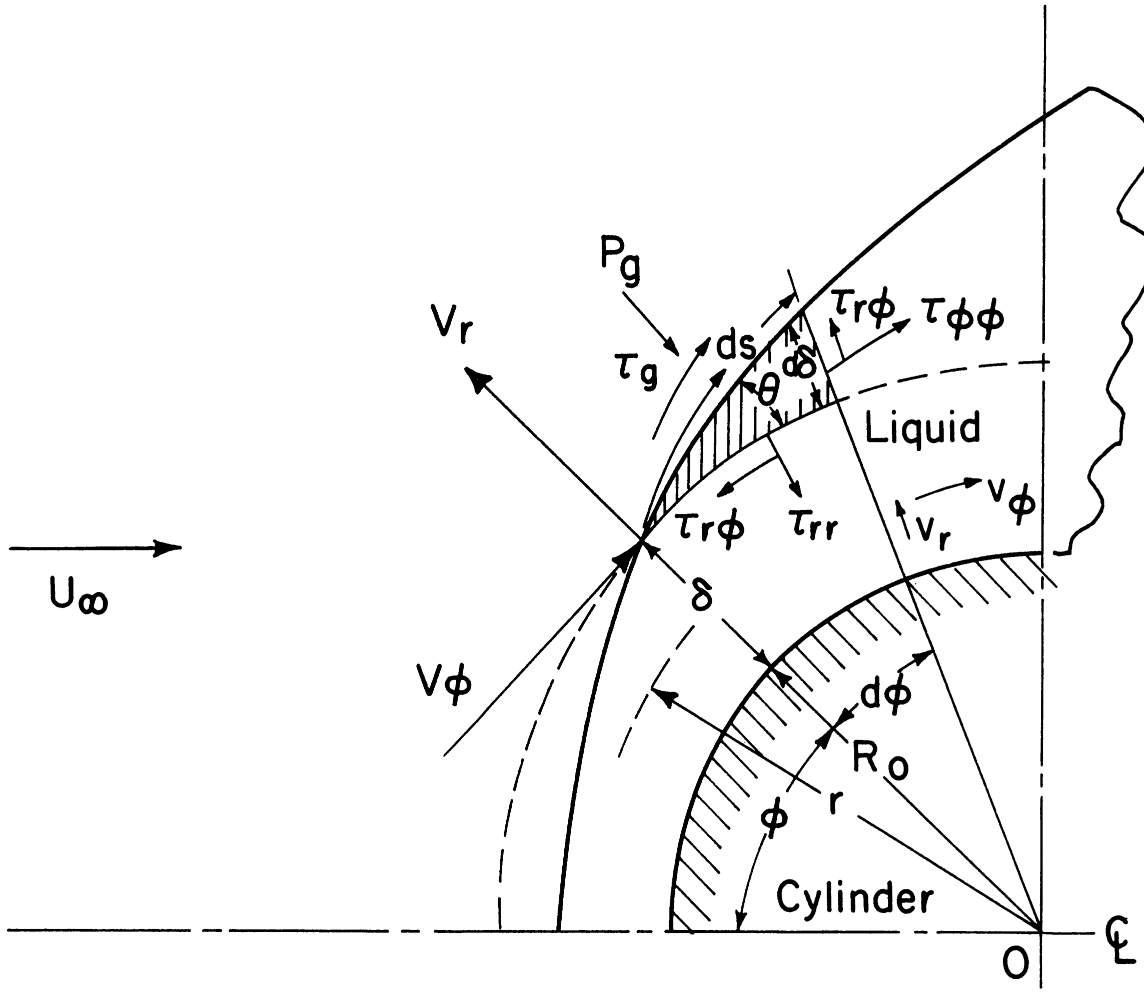

Figure 1. Coordinates and control volume for liquid film on cylinder.

And the law of conservation of mass is:

$$\rho v_\varphi(R_o+\delta,\varphi)d\delta - \rho v_r(R_o+\delta,\varphi)[R_o+\delta]d\varphi =$$

$$X_e(R_o+\delta,\varphi)\rho[V_\varphi(R_o+\delta,\varphi)\sin.\theta - V_r(R_o+\delta,\varphi)\cos.\theta]ds$$

From elementary calculus we have the relations:

$$(ds)^2 = \left[(R_o+\delta)^2 + \left(\frac{d\delta}{d\varphi}\right)^2\right](d\varphi)^2$$

$$\mathrm{Tan}.\varphi = \frac{1}{R_o+\delta}\ \frac{d\delta}{d\varphi}$$

$$ds\ \cos.\theta = [R_o+\delta]d\varphi$$

$$ds\ \sin.\theta = d\delta$$

After dividing the conservation laws by $d\varphi$ and using the above relations between the differentials one obtains:

$$\left.\begin{aligned}
&-\tau_{r\varphi}(R_o+\delta)+\tau_{\varphi\varphi}\frac{d\delta}{d\varphi}+(R_o+\delta)\tau_g+P_g\frac{d\delta}{d\varphi}\\
&=\rho v_\varphi^2\frac{d\delta}{d\varphi} - \rho v_r v_\varphi(R_o+\delta) - \rho X_e V_\varphi\left[V_\varphi\frac{d\delta}{d\varphi}\right.\\
&\left. - (R_o+\delta)V_r\right]\\
&\tau_{r\varphi}\frac{d\delta}{d\varphi} - (R_o+\delta)\tau_{rr} - (R_o+\delta)P_g + \tau_g\frac{d\delta}{d\varphi}\\
&= \rho v_r v_\varphi\frac{d\delta}{d\varphi} - \rho v_r^2(R_o+\delta) - \rho X_e V_r\left[V_\varphi\frac{d\delta}{d\varphi}\right.\\
&\left. - (R_o+\delta)V_r\right]\\
&\rho v_\varphi\frac{d\delta}{d\varphi} - \rho(R_o+\delta)v_r = \rho X_e\left[V_\varphi\frac{d\delta}{d\varphi} - (R_o\right.\\
&\left. +\delta)V_r\right]
\end{aligned}\right\}\ \begin{aligned}&\text{at}\\ &r = R_o+\delta\\ &0 \le \varphi\end{aligned}$$

Using the last of these two relations to simplify the first two, one arrives at:

$$\left.\begin{aligned}
&- \tau_{r\varphi}(R_o+\delta) + \tau_{\varphi\varphi}\frac{d\delta}{d\varphi} + (R_o+\delta)\tau_g + P_g\frac{d\delta}{d\varphi} = \\
&\rho X_e(v_\varphi - V_\varphi)\left[V_\varphi \frac{d\delta}{d\varphi} - (R_o+\delta)V_r\right] \\
&- \tau_{rr}(R_o+\delta) + \tau_{r\varphi}\frac{d\delta}{d\varphi} - P_g(R_o+\delta) + \tau_g\frac{d\delta}{d\varphi} = \\
&= \rho X_e(v_r - V_r)\left[V_\varphi \frac{d\delta}{d\varphi} - (R_o+\delta)V_r\right] \\
&\rho v_\varphi \frac{d\delta}{d\varphi} - \rho v_r(R_o+\delta) = \rho X_e\left[V_\varphi \frac{d\delta}{d\varphi} - V_r(R_o+\delta)\right]
\end{aligned}\right\} \quad \begin{aligned} r &= R_o+\delta \\ 0 &\leq \varphi \end{aligned} \qquad 1)$$

Within the film the liquid is assumed to satisfy the incompressible Navier-Stokes equations and the continuity equation. These equations and the relations of the components of the stress tensor to the velocity field are:[18]

$$\left.\begin{aligned}
&v_r\frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \\
&\nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \varphi^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi} - \frac{v_r}{r^2}\right) \\
&v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r v_\varphi}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \\
&\nu\left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r^2}\right)
\end{aligned}\right\} \qquad 2)$$

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} = 0 \qquad 3)$$

$$\left.\begin{aligned}
\tau_{rr} &= -P + 2\mu \frac{\partial v_r}{\partial r} \\
\tau_{\varphi\varphi} &= -P + 2\mu\left(\frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r}\right) \\
\tau_{r\varphi} &= \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \varphi} + \frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r}\right)
\end{aligned}\right\} \qquad 4)$$

Since the equation of continuity Eq. (3) may be rewritten as:

$$\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_\varphi}{\partial \varphi} = 0 \qquad 3^*)$$

it will be satisfied identically if one introduces the usual form of the stream function ψ for cylindrical coordinates. That is:

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad ; \quad v_\varphi = \frac{\partial \psi}{\partial r} \qquad 5)$$

Now since:

$$d\psi = \frac{\partial \psi}{\partial r}dr + \frac{\partial \psi}{\partial \varphi}d\varphi$$

we have:

$$\begin{aligned}
\frac{d\psi}{ds} &= \left.\frac{\partial \psi}{\partial r}\right|_{r=R_o+\delta} \cdot \frac{dr}{ds} + \left.\frac{\partial \psi}{\partial \varphi}\right|_{r=R_o+\delta} \cdot \frac{d\varphi}{ds} \\
&= \left.\frac{\partial \psi}{\partial r}\right|_{r=R_o+\delta} \cdot \frac{d\delta}{ds} + \left.\frac{\partial \psi}{\partial \varphi}\right|_{r=R_o+\delta} \cdot \frac{d\varphi}{ds}
\end{aligned}$$

$$\frac{d\psi}{ds}\frac{ds}{d\varphi} = \left.\frac{\partial \psi}{\partial r}\right|_{r=R_o+\delta} \cdot \frac{d\delta}{ds}\frac{ds}{d\varphi} + \left.\frac{\partial \psi}{\partial \varphi}\right|_{r=R_o+\delta}$$

$$\left.\frac{d\psi}{d\varphi}\right|_{\text{along } s} = \left.\frac{\partial\psi}{\partial r}\right|_{r=R_o+\delta} \cdot \left.\frac{d\delta}{d\varphi}\right|_{\text{along } s} + \left.\frac{\partial\psi}{\partial\varphi}\right|_{r=R_o+\delta}$$

$$\left.\frac{\partial\psi}{\partial\varphi}\right|_{\text{along } s} = v_\varphi \frac{d\delta}{d\varphi} - (R_o+\delta)v_r \quad \text{at} \quad r = R_o+\delta \tag{6)}$$

Now the vanishing of the velocity at the wall of the cylinder requires that:

$$v_r = v_\varphi = 0 \quad \text{at} \quad r = R_o \;;\; 0 \leq \varphi \tag{7)}$$

in order to satisfy this condition it is sufficient to take:

$$\psi = \frac{\partial\psi}{\partial r} = 0 \quad \text{at} \quad r = R_o \;;\; 0 \leq \varphi \tag{7*)}$$

Introducing (5) into (2), yields:

$$\left.\begin{aligned}
&\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial r\partial\varphi} - \frac{\partial\psi}{\partial\varphi}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial\varphi}\frac{\partial\psi}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial\varphi}\\
&+ \nu\left\{r\frac{\partial^3\psi}{\partial r^3} + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial^3\psi}{\partial r\partial\varphi^2} - \frac{2}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right\}\\
&\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial\varphi^2} - \frac{\partial\psi}{\partial\varphi}\frac{\partial^2\psi}{\partial r\partial\varphi} + r\left(\frac{\partial\psi}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial\psi}{\partial\varphi}\right)^2 = \frac{1}{\rho r^2}\frac{\partial P}{\partial r}\\
&+ \nu\left\{\frac{1}{r}\frac{\partial^3\psi}{\partial\varphi^3} + r\frac{\partial^3\psi}{\partial r^2\partial\varphi} + \frac{\partial^2\psi}{\partial r\partial\varphi} + \frac{2}{r}\frac{\partial\psi}{\partial\varphi}\right\}
\end{aligned}\right\} \tag{2*)}$$

Using (6) in the last of Eqs. (1), one obtains:

$$\frac{\partial\psi}{\partial\varphi} = \rho X_e \left[V_\varphi \frac{d\delta}{d\varphi} - V_r(R_o+\delta)\right] \quad \text{for } r = R_o+\delta$$

or integrating both sides with respect to ψ along s, gives:

$$\psi - \psi\Big|_{\varphi=o} = \rho \int_{o}^{\varphi} X_e \left[V_\varphi \frac{d\delta}{d\varphi} - V_r(R_o+\delta) \right] d\varphi \quad ; \quad r=R_o+\delta$$

but since ψ must be an odd function of φ, $\psi\Big|_{\varphi=o} = 0$ and one has:

$$\psi = \rho \int_{o}^{\varphi} X_e \left[V_\varphi \frac{d\delta}{d\varphi} - V_r(R_o+\delta) \right] d\varphi \; : \text{ at } r = R_o+\delta \qquad 1^*)$$

Now using Eqs. (4) in the first two of Eqs. (1) and substituting Eqs. (5) into the resulting equations, rewriting Eqs. (2*) and collecting Eqs. (1*) and (7*) yields:

$$r=R_o+\delta,\; 0 \le \varphi \quad \left\{ \begin{aligned} & \frac{\partial\psi}{\partial r} \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\varphi}\right) - \frac{1}{r}\frac{\partial\psi}{\partial\varphi}\frac{\partial^2\psi}{\partial r^2} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial P}{\partial\varphi} + \nu\frac{\partial}{\partial r} D^2\psi \\ & \frac{1}{r}\frac{\partial\psi}{\partial\varphi}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\varphi}\right) - \frac{\partial\psi}{\partial r}\left(\frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \nu\frac{1}{r}\frac{\partial}{\partial\varphi} D^2\psi \\ & D^2 \equiv \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial\varphi^2} \\ & v_r = -\frac{1}{r}\frac{\partial\psi}{\partial\varphi} \quad ; \quad v_\varphi = \frac{\partial\psi}{\partial r} \end{aligned} \right\} \qquad 8)$$

$$
r=R_o+\delta,\quad 0 \le \varphi \quad
\left\{
\begin{aligned}
&\mu \left[\frac{\partial^2\psi}{\partial\varphi^2} (R_o+\delta)^{-1} - (R_o+\delta)^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\psi}{\partial r} \right) + 2 \frac{d\delta}{d\varphi} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\psi}{\partial\varphi} \right) \right] - P \frac{d\delta}{d\varphi} \\
&+ (R_o+\delta)\tau_g + P_g \frac{d\delta}{d\varphi} = \rho X_e \left(\frac{\partial\psi}{\partial r} - V_\varphi \right) \left[V_\varphi \frac{d\delta}{d\varphi} - (R_o+\delta)V_r \right] \\
&\qquad P(R_o+\delta) - \mu \left[\frac{\partial^2\psi}{\partial\varphi^2} (R_o+\delta)^{-2} \frac{d\delta}{d\varphi} - (R_o+\delta) \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\psi}{\partial r} \right) \frac{d\delta}{d\varphi} + \right. \\
&\qquad \left. +2(R_o+\delta) \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\psi}{\partial\varphi} \right) \right] - P_g(R_o+\delta) + \tau_g \frac{d\delta}{d\varphi} = \\
&\qquad = -\rho X_e \left[\frac{\partial\psi}{\partial\varphi} (R_o+\delta)^{-1} + V_r \right] \left[V_\varphi \frac{d\delta}{d\varphi} - (R_o+\delta)V_r \right] \\
&\qquad \psi = \rho \int_0^\varphi X_e \left[V_\varphi \frac{\partial\delta}{d\varphi} - (R_o+\delta)V_r \right] d\varphi
\end{aligned}
\right\} \qquad 9)
$$

$$
r=R_o,\quad 0 \le \varphi \qquad \psi = \frac{\partial\psi}{\partial r} = 0 \qquad 10)
$$

These equations i.e. Eqs. (8), (9), and (10) would completely determine the flow in the liquid film if the gas conditions (P_g and τ_g) and the liquid drop conditions (X_e,V_φ and V_r) were known at the edge of the liquid film. Thus the hydrodynamic problem in the film is completely determined by these relations.*

Since the hydrodynamic equations are decoupled from the energy equations by the assumption of constant density and viscosity, the flow problem can be solved without consideration of the energy and the temperature field within the film can be solved for once the velocity field is known.

*Note that Eqs. (8) constitute a fourth-order system in r and that (9) and (10) specify five boundary conditions. But the thickness of the layer is unknown and the additional condition is necessary to determine this.

With the assumptions that (1) the effects due to kinetic energy changes across this volume are negligible, and (2) the temperature of the drops is not significantly reduced from that of the free stream when they pass through the gas boundary layer, the energy balance produces:

$$k \frac{\partial T}{\partial r}(R_o+\delta,\varphi)\left[R_o+\delta\right] d\varphi - k \frac{\partial T}{\partial \varphi}(R_o+\delta,\varphi)\left[R_o+\delta\right]^{-1} d\delta$$

$$+ \dot{q}_g\, ds = - \rho C_p T(R_o+\delta,\varphi) v_\varphi(R_o+\delta,\varphi)\, d\delta$$

$$+ \rho C_p T(R_o+\delta,\varphi) v_r(R_o+\delta,\varphi)\left[R_o+\delta\right] d\varphi$$

$$+ X_e(R_o+\delta,\varphi)\; C_p \rho T_\infty \Big[V_\varphi(R_o+\delta,\varphi)\sin\Theta\, -$$

$$- V_r(R_o+\delta,\varphi)\cos\Theta\Big]\, ds$$

Dividing through by $d\varphi$, using the relations given above between ds $d\varphi$ and $\sin\Theta$ and the last of Eqs.(1) one gets:

$$\text{at } r=R_o+\delta,\; 0 \le \varphi \quad \begin{cases} k \dfrac{\partial T}{\partial r}(R_o+\delta) - k(R_o+\delta)^{-1} \dfrac{\partial T}{\partial \varphi}\dfrac{d\delta}{d\varphi} + \dot{q}_g \left[(\delta+R_o)^2 + \left(\dfrac{d\delta}{d\varphi}\right)^2\right]^{1/2} \\ = X_e C_p \rho (T_\infty - T)\left[V_\varphi \dfrac{d\delta}{d\varphi} - (R_o+\delta)V_r\right] \end{cases} \qquad 11)$$

Under the above restrictions the energy equation is:

$$\frac{1}{r}\left(\frac{\partial \psi}{\partial r}\frac{\partial T}{\partial \varphi} - \frac{\partial \psi}{\partial \varphi}\frac{\partial T}{\partial r}\right) = \frac{k}{\rho C_p}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \qquad 12)$$

with the boundary condition at the wall:

$$T = T_w \quad \text{at} \quad r = R_o \quad ; \quad 0 \le \varphi \qquad 13)$$

Thus Eqs. (11), (12) and (13) determine the temperature field inside the liquid film completely once $\dot{q}_g$ and the velocity field are known. Now the order of magnitude of the various terms in the governing equations is to be investigated. The following new variables are introduced:

$$y \equiv \frac{r}{R_o} - 1 \quad ; \quad x \equiv \varphi \quad ; \quad \hat{\delta} \equiv \delta/R_o$$

and Eqs. (8) through (13) become*:

$$0 \le y \le \hat{\delta}, \quad 0 \le x \quad \left\{ \begin{aligned} &\psi_y \frac{\partial}{\partial y}\left[(y+1)^{-1}\psi_x\right] - (y+1)^{-1}\psi_x\psi_{yy} = -\frac{R_o^2}{\rho}(y+1)^{-1}P_x + \nu\frac{\partial}{\partial y}D^2_{xy}\psi \\ &(y+1)^{-1}\psi_x\frac{\partial}{\partial y}\left[(y+1)^{-1}\psi_x\right] - \psi_y\left[(y+1)^{-2}\psi_{xx} + (y+1)^{-1}\psi_y\right] = \\ &-\frac{R_o^2}{\rho}P_y - \nu(y+1)^{-1}\frac{\partial}{\partial x}D^2_{xy}\psi \\ &D^2_{xy} \equiv \frac{1}{1+y}\frac{\partial}{\partial y}(1+y)\frac{\partial}{\partial y} + \frac{1}{(1+y)^2}\frac{\partial^2}{\partial x^2} \\ &v_\varphi = \frac{1}{R_o}\frac{\partial\psi}{\partial y} \\ &(1+y)^{-1}\left[\psi_y T_x - \psi_x T_y\right] = \frac{k}{\rho C_p}\left[T_{yy} + (1+y)^{-2}T_{xx} + (1+y)^{-1}T_y\right] \end{aligned} \right\} \quad 14)$$

$$\text{at } y = 0, \; 0 \le x \qquad \psi = \psi_y = 0 \; ; \; T = T_w \qquad 15)$$

*The subscript notation for partial derivatives is being used. Thus

$$\psi_y \equiv \frac{\partial\psi}{\partial y}, \quad \psi_x \equiv \frac{\partial\psi}{\partial x}, \quad \psi_{yy} \equiv \frac{\partial^2\psi}{\partial y^2}, \text{ etc.}$$

$$
y=\delta,\ 0\le x \quad
\left\{
\begin{aligned}
&\nu\ \{\psi_{xx}(1+\hat{\delta})^{-1} - (1+\delta)^2 \frac{\partial}{\partial y}\left[(y+1)^{-1}\psi_y\right] + 2\hat{\delta}_x \frac{\partial}{\partial y}\left[(y+1)^{-1}\psi_x\right]\} \\
&+ \frac{R_o^2}{\rho}\tau_g(1+\hat{\delta}) + \frac{R_o^2}{\rho}(P_g-P)\frac{d\hat{\delta}}{dx} = x_e R_o^2\left[\frac{1}{R_o}\psi_y - V_\varphi\right]\left[V_\varphi \hat{\delta}_x - \right. \\
&\left. - (1+\hat{\delta})V_r\right] \\
&\frac{R_o^2}{\rho}(P-P_g)(1+\hat{\delta}) - \nu\ \{\psi_{xx}(1+\delta)^{-2}\hat{\delta}_x - (1+\hat{\delta})\frac{\partial}{\partial y}\left[(1+y)^{-1}\frac{\partial\psi}{\partial y}\right]\hat{\delta}_x \\
&+ 2(1+\hat{\delta})\frac{\partial}{\partial y}\left[(1+y)^{-1}\frac{\partial\psi}{\partial x}\right]\} + \frac{R_o^2}{\rho}\tau_g\hat{\delta}_x = \\
&= R_o^2 X_e\left[\frac{1}{R_o}\psi_x(1+\hat{\delta})^{-1} + V_r\right]\left[V_\varphi\hat{\delta}_x - (1+\hat{\delta})V_r\right] \\
&\frac{\psi}{R_o} = \int_o^x X_e\left[V_\varphi\hat{\delta}_x - V_r(1+\hat{\delta})\right]dx \\
&k\frac{\partial T}{\partial y}(1+\hat{\delta}) - k(1+\hat{\delta})^{-1}\frac{\partial T}{\partial x}\hat{\delta}_x + R_o\dot{q}_g\left[(1+\hat{\delta})^2 + \hat{\delta}_x^2\right]^{1/2} \\
&= X_e R_o C_p \rho(T_\infty - T)\left[V_\varphi\hat{\delta}_x - (1+\hat{\delta})V_r\right]
\end{aligned}
\right\} \quad 16)
$$

Now interest is focused on the asymptotic behavior of these equations as $X_{e_\infty} \to 0$ and $\frac{2R_oU_\infty}{\nu} \to \infty$ that is when $X_{e_\infty} \ll 1$; $2R_oU_\infty/_\nu \gg 1$. For large Reynolds numbers and small volume fractions of the liquid in the free stream, these equations will behave differently depending on whether the parameter $E^2 \equiv X_{e_\infty}^2\ 2R_oU_\infty/_\nu$ is less than or greater than 1.

We want to make these equations dimensionless in such a way that all the coefficients of the parameters appearing in the resulting equations are of order 1 in order to determine which terms can be considered negligably small. But this requires that the terms be made dimensionless in different ways depending upon whether E^2 is less or greater than 1.* Therefore these two cases have to be treated separately.

Case a) $E^2 \geq 1$

Under these circumstances the numerical results show that:

$$\psi = O(U_\infty R_0 X_{e_\infty}) \quad \text{and} \quad y, \hat{\delta} = O(2X_{e_\infty})$$

Therefore the dimensionless quantities:

$$\bar{\psi} = \psi / U_\infty R_0 X_{e_\infty} \; ; \; \bar{x} = x \; ; \; \bar{y} = y/2X_{e_\infty} \; ;$$

$$\bar{\delta} = \delta/2X_{e_\infty} = \delta/2R_0X_{e_\infty} \; ; \; \overset{*}{V}_\varphi = V_\varphi/U_\infty$$

$$\overset{*}{V}_r = V_r/U_\infty \; ; \; \bar{P} = P/1/2\, \rho U_\infty^2$$

are introduced and Eqs. (14), (15) and (16) become

$$0 \leq \bar{y} \leq \bar{\delta}, \quad 0 \leq \bar{x}$$

$$\bar{\psi}_{\bar{y}} \frac{\partial}{\partial \bar{y}} \left\{ (1+2X_{e_\infty}\bar{y})^{-1} \bar{\psi}_{\bar{x}} \right\} - (1+2X_{e_\infty}\bar{y})^{-1} \bar{\psi}_{\bar{x}} \bar{\psi}_{\bar{y}\bar{y}} = - 2(1+2X_{e_\infty}\bar{y})^{-1} \bar{P}_{\bar{x}} + \frac{1}{E^2} \frac{\partial}{\partial \bar{y}} \bar{D}^2 \bar{\psi}$$

$$\left[(1+2X_{e_\infty}\bar{y})^{-1} \bar{\psi}_{\bar{x}} \frac{\partial}{\partial \bar{y}} \{ (1+2X_{e_\infty}\bar{y})^{-1} \bar{\psi}_{\bar{x}} \} - \bar{\psi}_{\bar{y}} \{ (1+2X_{e_\infty}\bar{y})^{2} \bar{\psi}_{\bar{x}\bar{x}} + \frac{(1+2X_e\bar{y})^{-1}}{2X_e} \bar{\psi}_{\bar{y}} \} \right] 4X_{e_\infty}^2 = - 2\bar{P}_{\bar{y}} - \frac{4X_{e_\infty}^2}{E^2} (1+2X_{e_\infty}\bar{y}) \frac{\partial}{\partial \bar{x}} D^{-2}\bar{\psi}$$

*It can be verified from the numerical results that all the dimensionless terms are of the order 1.

$$\bar{D}^2 = (1+2X_{e_\infty}\bar{y})^{-1} \frac{\partial}{\partial\bar{y}} (1+2X_{e_\infty}\bar{y}) \frac{\partial}{\partial\bar{y}} +$$

$$+ (1+2X_{e_\infty}\bar{y})^{-2} 4X_{e_\infty}^2 \frac{\partial}{\partial\bar{x}^2}$$

$$(1+2X_{e_\infty}\bar{y})^{-1} \left[\bar{\psi}_{\bar{y}}T_{\bar{x}} - \bar{\psi}_{\bar{x}}T_{\bar{y}}\right] = \frac{1}{P_rE^2} \left[T_{\bar{y}\bar{y}} +\right.$$

$$\left.+ 2X_{e_\infty}(1+2X_{e_\infty}\bar{y})^{-1} T_{\bar{y}} + 4X_{e_\infty}^2 (1+2X_{e_\infty}\bar{y})^{-2} T_{\bar{x}\bar{x}}\right]$$

$\bar{y}=\bar{\delta}$

$0\leq\bar{x}$

$$-(1+2X_{e_\infty}\bar{\delta})^2 \frac{\partial}{\partial\bar{y}}\left[(1+2X_{e_\infty}\bar{y})\bar{\psi}_{\bar{y}}\right] + 4X_{e_\infty} \{(1+2X_{e_\infty}\bar{\delta})\bar{\psi}_{\bar{x}\bar{x}}$$

$$+ 2\bar{\delta} \frac{\partial}{\partial\bar{y}}\left[(1+2X_{e_\infty}\bar{y})\bar{\psi}_{\bar{x}}\right]\} + \frac{4X_{e_\infty}R_o}{\mu U_\infty} \tau_g(1+2X_{e_\infty}\bar{\delta})$$

$$+ 2E^2 \frac{(P_g-P)}{1/2\rho U_\infty^2} \bar{\delta}_x = E^2\left[\bar{\psi}_{\bar{y}} - 2V_\varphi^*\right]\left[2X_{e_\infty}\bar{\delta}_{\bar{x}}V_\varphi^* -\right.$$

$$\left. - (1+2X_{e_\infty}\bar{\delta}) V_r^*\right] \frac{X_e}{X_{e_\infty}}$$

$$X_{e_\infty}^2 \{\bar{\psi}_{\bar{x}\bar{x}}(1+2X_{e_\infty}\bar{\delta})^{-2}\bar{\delta}_{\bar{x}}-(1+2X_{e_\infty}\bar{\delta}) \frac{\partial}{\partial\bar{y}}\left[(1+2X_{e_\infty})^{-1}\bar{\psi}_{\bar{x}}\right]\bar{\delta}_{\bar{x}}$$

$$+ 2(1+2X_{e_\infty}\bar{\delta}) \frac{\partial}{\partial\bar{y}}\left[(1+2X_{e_\infty}\bar{y})^{-1} \frac{\partial\bar{\psi}}{\partial\bar{x}}\right]\} + \frac{2X_{e_\infty}^3R_o}{\nu\rho U_\infty} \tau_g\bar{\delta}_{\bar{x}}$$

$$+ \frac{E^2}{2} \frac{(P-P_g)}{1/2\rho U_\infty^2} (1+2X_{e_\infty}\bar{\delta}) = X_{e_\infty}E^2\left[X_{e_\infty}\bar{\psi}_{\bar{x}}(1+2X_{e_\infty}\bar{\delta})\right.$$

$$\left.+ V_r^*\right]\left[2X_{e_\infty}\bar{\delta}_{\bar{x}}V_\varphi^* - (1+2X_{e_\infty}\bar{\delta})V_r^*\right] \frac{X_e}{X_{e_\infty}}$$

$$\bar{\psi} = \int_0^{\bar{x}} \frac{X_e}{X_{e_\infty}} \left[V_\varphi^* 2X_{e_\infty} \frac{d\bar{\delta}}{d\bar{x}} - V_r^*(1+2X_{e_\infty}\bar{\delta})\right] d\bar{x}$$

$$T_{\bar{y}} (1+2X_{e_\infty}\bar{\delta}) - 4X_{e_\infty}^2 (1+2X_{e_\infty}\bar{\delta})^{-1} T_{\bar{x}}\bar{\delta}_{\bar{x}} +$$

$$+ \frac{2X_{e_\infty}R_o}{k} \dot{q}_g \left[(1+2X_{e_\infty}\bar{\delta})^2 + 4X_{e_\infty}\bar{\delta}_{\bar{x}}^2\right]^{1/2} =$$

$$\left. = \frac{X_e}{X_{e_\infty}} E^2 P_r (T_\infty - T) \left[V_\varphi^* \, 2X_{e_\infty} \bar{\delta}_{\bar{x}} - (1 + 2X_{e_\infty} \bar{\delta}) V_r^* \right] \right]$$

$$\bar{y} = 0 \; ; \; 0 \leq \bar{x} \qquad \bar{\psi} = \bar{\psi}_y = 0 \; ; \; T = T_w$$

Now taking the limit as $X_{e_\infty} \to 0$ and retaining all terms relating to the two-phase region one gets:

$$\text{at } 0 \leq \bar{y} \leq \bar{\delta},\; 0 \leq \bar{x} \quad \left\{ \begin{array}{l} \bar{\psi}_{\bar{y}} \bar{\psi}_{\overline{xy}} - \bar{\psi}_{\bar{x}} \bar{\psi}_{\overline{yy}} = -2\bar{P}_{\bar{x}} + \frac{1}{E^2} \bar{\psi}_{\overline{yyy}} \\ \frac{\partial \bar{P}}{\partial \bar{y}} = 0 \\ \bar{\psi}_{\bar{y}} T_{\bar{x}} - \bar{\psi}_{\bar{x}} T_{\bar{y}} = \frac{1}{P_r E^2} T_{\overline{yy}} \end{array} \right\} \qquad 14a)$$

$$\text{at } \bar{y} = \bar{\delta},\; 0 \leq \bar{x} \quad \left\{ \begin{array}{l} -\bar{\psi}_{\overline{yy}} = E^2 \left\{ \frac{X_e}{X_{e_\infty}} \left[2V_\varphi^* - \bar{\psi}_{\bar{y}} \right] V_r^* + \frac{-2\tau_g}{X_{e_\infty} \rho U_\infty^2} \right\} \\ P - P_g + 2X_{e_\infty} \tau_g \bar{\delta}_{\bar{x}} = 0 \\ \bar{\psi} = -\int_0^x \frac{X_e}{X_{e_\infty}} V_r^* \, d\bar{x} \\ T_{\bar{y}} + \frac{2X_{e_\infty} R_o}{k} \dot{q}_g = -\frac{X_e}{X_{e_\infty}} P_r E^2 (T_\infty - T) V_r^* \end{array} \right\} \qquad 15a)$$

$$\text{at } \bar{y} = 0 \; ; \; 0 \leq \bar{x} \quad \{ \bar{\psi} = \bar{\psi}_y = 0 \; ; \; T = T_w \} \qquad 16a)$$

Where the pressure term in the first of Eqs. (15a) has been eliminated by substituting the second equation into it.

Case b) $\qquad E^2 \leq 1$

Under these circumstances the numerical results show that:

$$\psi = O(U_\infty R_o X_{e_\infty}) \quad \text{and} \quad \hat{\delta},\; y = O\left(\frac{2}{\sqrt{\frac{2R_o U_\infty}{\nu}}} \right)$$

Therefore the dimensionless quantities:

$$\tilde{\psi} = \psi / U_\infty R_o X_{e_\infty} \; ; \; \tilde{x} = x \; ; \; \tilde{y} = \frac{y}{2}\sqrt{\frac{2R_oU_\infty}{\nu}} \; ;$$

$$\tilde{\delta} = \frac{\hat{\delta}}{2}\sqrt{\frac{2R_oU_\infty}{\nu}} \; ; \; V_\varphi^* = V_\varphi / U_\infty \; ; \; V_r^* = V_r / U_\infty$$

$$P = \tilde{P} / {}_{1/2\rho U_\infty^2} \; ; \quad R_e \equiv \frac{2R_oU_\infty}{\nu}$$

are introduced and Eqs. (14), (15) and (16) now become:

$$0 \le \tilde{y} \le \tilde{\delta}, \quad 0 \le \tilde{x}$$

$$E^2 \left\{ \tilde{\psi}_{\tilde{y}} \frac{\partial}{\partial \tilde{y}} \left[\left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-1} \tilde{\psi}_{\tilde{x}} \right] - \left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-1} \tilde{\psi}_{\tilde{x}} \tilde{\psi}_{\tilde{y}\tilde{y}} \right\}$$

$$= -2\left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-1} \tilde{P}_{\tilde{x}} + E \frac{\partial}{\partial \tilde{y}} \tilde{D}^2 \tilde{\psi}$$

$$4X_{e_\infty}^2 E \left\{ \left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-1} \tilde{\psi}_{\tilde{x}} \frac{\partial}{\partial \tilde{y}} \left[\left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-1} \tilde{\psi}_{\tilde{x}} \right] - \right.$$

$$\left. \tilde{\psi}_{\tilde{y}} \left[\left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-2} \tilde{\psi}_{\tilde{x}\tilde{x}} + \frac{\sqrt{R_e}}{2}\left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-1} \tilde{\psi}_{\tilde{y}} \right] \right\} =$$

$$-2\tilde{P}_{\tilde{y}} - 4X_{e_\infty}^2 \left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-1} \frac{\partial}{\partial \tilde{x}} \tilde{D}^2 \tilde{\psi}$$

$$\tilde{D}^2 = \left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-1} \frac{\partial}{\partial \tilde{y}} \left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right) \frac{\partial}{\partial \tilde{y}} +$$

$$\left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-2} \frac{4}{R_e} \frac{\partial^2}{\partial \tilde{x}^2}$$

$$\left(1+\frac{2\tilde{y}}{\sqrt{R_e}}\right)^{-1} \left[\tilde{\psi}_{\tilde{y}} T_{\tilde{x}} - \tilde{\psi}_{\tilde{x}} T_{\tilde{y}} \right] = \frac{1}{P_r E} \left[T_{\tilde{y}\tilde{y}} + \right.$$

$$\left. + \left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-1} \frac{2}{\sqrt{R_e}} T_{\tilde{y}} + \left(1+\frac{2}{\sqrt{R_e}}\tilde{y}\right)^{-2} \frac{4}{R_e} T_{\tilde{x}\tilde{x}} \right]$$

at $\tilde y = \tilde\delta$, $0 \le \tilde x$:

$$-\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)^2 \frac{\partial}{\partial\tilde y}\left[\left(1+\frac{2}{\sqrt{R_e}}\tilde y\right)^{-1}\tilde\psi_{\tilde y}\right] \quad \frac{4}{R_e}\left\{\tilde\psi_{\tilde x\tilde x}\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)^{-1}\right.$$

$$\left. + 2\tilde\delta\frac{\partial}{\partial\tilde y}\left[\left(1+\frac{2}{\sqrt{R_e}}\tilde y\right)\tilde\psi_{\tilde x}\right]\right\} + \frac{2}{X_{e_\infty}1/2\rho U_\infty^2}\left[\tau_g\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)\right.$$

$$\left. + \frac{2}{\sqrt{R_e}}(P_g-P)\tilde\delta_{\tilde x}\right] = \frac{X_e}{X_{e_\infty}}\left[E\tilde\psi_{\tilde y} - 2V_\varphi^*\right]\left[\frac{2}{\sqrt{R_e}}\tilde\delta_{\tilde x}V_\varphi^* -\right.$$

$$\left.\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)V_r^*\right]$$

$$E\left[\frac{(P-P_g)}{1/2\rho U_\infty^2}\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right) + \frac{2}{\sqrt{R_e}}\tilde\delta_{\tilde x}\frac{\tau_g}{1/2\rho U_\infty^2}\right]$$

$$-X_{e_\infty}^2\left\{\tilde\psi_{\tilde x\tilde x}\frac{2}{\sqrt{R_e}}\tilde\delta_{\tilde x}\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)^{-2} - \left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)\frac{\partial}{\partial\tilde y}\left[\left(1+\frac{2}{\sqrt{R_e}}\tilde y\right)^{-1}\tilde\psi_{\tilde y}\right]\tilde\delta_{\tilde x}\right.$$

$$\left. + 2\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)\frac{\partial}{\partial\tilde y}\left[\left(1+\frac{2}{\sqrt{R_e}}\tilde y\right)^{-1}\tilde\psi_{\tilde x}\right]\right\}$$

$$= X_{e_\infty}E\frac{X_e}{X_{e_\infty}}\left[X_{e_\infty}\tilde\psi_{\tilde x}\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)^{-1} + V_r^*\right]\left[\frac{2}{\sqrt{R_e}}\tilde\delta_{\tilde x}V_\varphi^* \quad -\right.$$

$$\left. - \left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)V_r^*\right]$$

$$\tilde\psi = \int_0^{\tilde x}\frac{X_e}{X_{e_\infty}}\left[V_\varphi^*\frac{2}{\sqrt{R_e}}\tilde\delta_{\tilde x} - V_r^*\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)\right]d\tilde x$$

$$\frac{\partial T}{\partial\tilde y} - \frac{4}{R_e}\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)T_{\tilde x}\tilde\delta_{\tilde x} + \frac{2R_o\dot q_g}{k\sqrt{R_e}}\left[\left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)^2 + \frac{4}{R_e}\tilde\delta_{\tilde x}^2\right]^{1/2}$$

$$= EP_r\frac{X_e}{X_{e_\infty}}(T_\infty - T)\left[V_\varphi^*\frac{2}{\sqrt{R_e}}\tilde\delta_{\tilde x} - \left(1+\frac{2}{\sqrt{R_e}}\tilde\delta\right)V_r^*\right]$$

at $\tilde y = 0$; $0 \le \tilde x$; $\tilde\psi = \tilde\psi_{\tilde y} = 0$; $T = T_w$

Now taking the limit as $R_e \to \infty$ and $X_{e_\infty} \to 0$ and retaining all those terms relating to the two-phase region yields:

$$
\begin{array}{l} 0 \le \tilde{y} \le \tilde{\delta} \\ 0 \le \tilde{x} \end{array}
\left\{
\begin{aligned}
& E^2 \left[\tilde{\psi}_{\tilde{y}} \tilde{\psi}_{\tilde{x}\tilde{y}} - \tilde{\psi}_{\tilde{x}} \tilde{\psi}_{\tilde{y}\tilde{y}} \right] = - 2\tilde{P}_{\tilde{x}} + E\, \tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}} \\
& \frac{\partial \tilde{P}}{\partial \tilde{y}} = 0 \\
& \tilde{\psi}_{\tilde{y}} T_{\tilde{x}} - \tilde{\psi}_{\tilde{x}} T_{\tilde{y}} = \frac{1}{P_r E}\, T_{\tilde{y}\tilde{y}}
\end{aligned}
\right. \qquad \text{14b)}
$$

$$
\begin{array}{l} \text{at} \\ \tilde{y} = \tilde{\delta} \\ 0 \le \tilde{x} \end{array}
\left\{
\begin{aligned}
& - \tilde{\psi}_{\tilde{y}\tilde{y}} + \frac{2\tau_g}{X_{e_\infty} 1/2\rho U^2} \\
& = - \frac{X_e}{X_{e_\infty}} \left[E\tilde{\psi}_{\tilde{y}} - 2V_\varphi^* \right] V_r^* \\
& P - P_g + \frac{2}{R_e}\, \tilde{\delta}_{\tilde{x}}\, \tau_g = 0 \\
& \tilde{\psi} = - \int_0^{\tilde{x}} \frac{X_e}{X_{e_\infty}} V_r^* \, d\tilde{x} \\
& \frac{\partial T}{\partial \tilde{y}} + \frac{2R_o \dot{q}_g}{k\sqrt{R_e}} = E\, P_r \frac{X_e}{X_{e_\infty}} (T - T_\infty)\, V_r^*
\end{aligned}
\right. \qquad \text{15b)}
$$

$$
\text{at } \tilde{y} = 0 \; ; \; 0 \le \tilde{x} \; \{ \tilde{\psi} = \tilde{\psi}_{\tilde{y}} = 0 \; ; \; T = T_w \} \qquad \text{16b)}
$$

Where the pressure term has been eliminated in the first of Eqs. (15b) by use of the second Eq. (15b).

In order to proceed further it is necessary to bring in the phenomenon which occurs in the two-phase region.

b. Consideration of the Two-Phase Region

The two-phase region may be thought of as consisting of a boundary layer (on top of the liquid film) and a free stream flow outside of the boundary layer.* However due to the presence of the liquid drops there is the possibility of shear stresses being exerted on the gas outside of the boundary layer even though the mean velocity gradients are not steep.

In any case we can conclude that the gas and liquid interact with one another only in the following ways:

BI 1) The gas and liquid can interact at the surface of the liquid film by exerting a mutual shearing stress on one another. This is characterized by the term τ_g appearing above.

BI 2) There is a pressure coupling between the gas and liquid at the surface of the liquid film. This is characterized by the term P_g.

BI 3) There is a heat transfer by conduction at the surface of the film from the liquid to the gas (or the opposite direction). This is characterized by the term $\dot{q}_g$.

DI 1) The gas and the liquid drops can exert mutual frictional effects on one another. The drops due to their inertia tend to move in straight lines. The gas on the other hand tends to flow around the cylinder. A difference in velocity therefore develops between the gas and the liquid drops in the neighborhood of the cylinder, and as a result there is a mutual shearing stress between the two of them.

*And, of course, the region of the wake and separated flow which will not be considered here.

DI 2) The gas can exert pressure forces on the liquid drops.*

Thus BI 1) through BI 3) are interactions which occur at the surface of the liquid film. We may anticipate that under certain conditions these interactions will have only a small effect on hydrodynamic and heat transfer behavior of the film. Because of the large difference in density between the gas and the liquid we will expect that the interaction BI 1) will have only a small effect on the film. We further expect that the interaction BI 2) will be dominated by the momentum carried into the film by the liquid drops provided that the velocity of the drops and the volume fraction of the liquid in the free stream is sufficiently high. And finally we might anticipate that the interaction BI 3) will be dominated by the enthalpy flux into the film due to the drops. Thus for two given fluids we anticipate, for sufficiently high velocities and amount of liquid, that the presence of the gas will have a negligible effect on the film at least as far as the interactions BI 1) through BI 3) are concerned. At the other extreme the liquid film will be moving at a very low velocity, will have a relatively uniform pressure, and there will be only a small temperature drop across it relative to the gas (since the thermal conductivity of the liquid is much higher than for the gas). Under these conditions we will expect that the presence of the film has only a small effect on the gas boundary layer. It is further possible that these two domains will over lap. Thus there is a

---

*Note that we have already neglected the heat transfer effects between the drops and the gas.

domain where BI 1) through BI 3) are negligible for the behavior of the film, a domain where BI 1) through BI 3) are negligible for the gas boundary layer, and a region where the coupling between the two is mutually weak (i.e., the above two domains overlap).* But of course the lower boundary of the domain where the gas has only a small effect on the film (BI 1) through BI 3)) will depend also upon the dynamics of the drops (i.e., DI 1) and DI 2) since if the drops are strongly deflected by the gas stream little momentum, mass, and enthalpy can be carried into the film by the drops. Therefore before proceeding further with the interactions which occur at the surface of the film we will consider the interactions which effect the drop trajectories (i.e., DI 1) and DI 2)). In order to do this we should divide DI 1) and DI 2) up a little differently. We distinguish now between the total drag force exerted on the drops by the gas due to the difference in velocity between them and the force exerted on the drops by the gas due to the pressure gradients in the gas. The drops will be taken to be spherical and relatively small. Then the equations of motion for a single drop can be written as**:

$$4/3\pi r_d^3\ \rho\ \frac{d\vec{V}}{dt} = -\ 4/3\pi r_d^3\ \ \nabla_o P_g\ -\ \frac{1}{2}\ C_D \pi r_d^3 \rho_g |\vec{V}-\vec{V}_g|(\vec{V}-\vec{V}_g)$$

*Note that even if at the lower end of the domain where the gas influences the film there is a fairly large influence exerted on the gas by the liquid only a small error will be introduced if we replace the interaction on the film by that which would result if only the gas was present.

**| | denotes the absolute value.

$$\nabla_o \equiv \hat{i}\frac{\partial}{\partial x_o} + \hat{j}\frac{\partial}{\partial y_o} \quad ; \quad \vec{V} = \hat{i}\, V_{x_o} + \hat{j}\, V_{y_o} = \frac{d\vec{r}_d}{dt}$$

$$\vec{r}_d = \hat{i}\, x_d + \hat{j}\, y_d \quad ; \quad \vec{V}_g = \hat{i}\, u_g + \hat{j}\, v_g$$

$$\vec{V} \rightarrow \hat{i}\, U_\infty \quad \text{for} \quad \vec{r}_d \rightarrow \infty$$

where C_D is of course a function of $|\vec{V} - \vec{V}_g|$

If we introduce the notation:

$$\nabla_d \equiv \hat{i}\frac{\partial}{\partial x_d} + \hat{j}\frac{\partial}{\partial y_d}$$

then it follows from elementary mechanics that:

$$\frac{dV}{dt} = \vec{V}\cdot \nabla_d \vec{V}$$

We introduce the dimensionless quantities:

$$x_o^* = x_o/R_o \quad ; \quad y_o^* = y_o/R_o \quad ; \quad x_d^* = x_d/R_o \quad ; \quad \nabla_o^* = R_o \nabla_o$$

$$\nabla_d^* = R_o \nabla_d \quad ; \quad \vec{V}^* = \vec{V}/U_\infty \quad ; \quad V_{x_o}^* = V_{x_o}/U_\infty \quad ; \quad \vec{V}_g^* = \vec{V}_g/U_\infty$$

$$u_g^* = u_g/U_\infty \quad ; \quad P_g^* = P_g/\tfrac{1}{2}\rho_g U_\infty^2 \quad ; \quad \text{etc.}$$

and we get:

$$\vec{V}^*\cdot\nabla_d^* \vec{V}^* = -2\frac{\rho_g}{\rho}\nabla_o^* P_g^* - 3/8\; C_D \frac{1}{(r_d/R_o)} \frac{\rho_g}{\rho} |\vec{V}^* - \vec{V}_g^*|(\vec{V}^* - \vec{V}_g^*) \qquad 17)$$

Now the drag forces exerted by the gas on the drops must equal minus the drag force exerted by the drops on the gas. Therefore if N_o is the number of drops per unit volume then on the average (i.e., we assume that there are a sufficient number of drops per unit volume so that only the average effect influences the overall gas motion) the force exerted by the drops on the gas per unit volume is:

$$+ 1/2 \; N_o C_D \pi \; r_d^2 \; \rho_g |\vec{V}^* - \vec{V}_g^*| (\vec{V}^* - \vec{V}_g^*)$$

in the region outside the gas boundary layer where the average velocity gradients are not steep the frictional effects which are due to these gradients can be neglected, and the equation of motion for the gas can be written as

$$\rho_g \vec{V}_g \cdot \nabla_o \vec{V}_g = - \nabla_o P_g + 1/2 \; N_o C_D \pi \; r_d^2 \rho_g |\vec{V} - \vec{V}_g| (\vec{V} - \vec{V}_g)$$

Now it is obvious that N_o is related to the volume fraction of the liquid X_e by the following formula:

$$N_o (4/3) \pi r_d^3 = X_e$$

Using this and the dimensionless quantities already defined the equation of motion for the gas becomes:

$$\vec{V}_g^* \cdot \nabla_o^* \vec{V}_g^* = - 2 \nabla_o^* P_g^* + 3/8 \frac{C_D}{(r_d/R_o)} X_e |\vec{V}^* - \vec{V}_g^*| (\vec{V}^* - \vec{V}_g^*) \qquad 18)$$

From Eq. (17) it might be anticipated that under certain conditions the drop trajectories may be straight lines (i.e., their motion is not appreciably influenced by the motion of the gas).

We are now is a position to determine the lower limit of the domain where the presence of the gas has a negligible effect on the liquid film.

In order to do this it is necessary to consider the ratio of properties of the gas and the liquid specifically. We will limit ourselves in what follows to air and water since these are the substances of principal interest.

From the first of Eqs. (15a) and (15b) it can be concluded that the shear stress on the surface of the liquid film will be negligibly small if:

$$2\,\frac{X_e}{X_{e_\infty}}\,V_\varphi^* V_r^* \gg \frac{2\tau_g}{X_e \rho U_\infty^2} = \frac{1}{E}\left[\sqrt{\frac{\nu_g}{\nu}}\cdot\frac{\rho_g}{\rho}\right]\left[\tau_g\sqrt{\frac{2U_\infty R_o}{\nu_g}}\Big/\frac{1}{2}\,\rho_g U_\infty^2\right] \qquad 19)$$

By substituting the second of Eqs. (15a) and (15b) into the first of Eqs. (14a) and(14b), respectively, it can be concluded that the pressure forces exerted on the liquid film by the gas are negligibly small whenever:

$$E^2 \geq 1 \quad ; \quad 2\bar{\bar{P}} = 2(P_g - 2X_{e\infty}\,\tau_g\bar{\delta}_{\bar{x}})\Big/1/2\rho U_\infty^2 =$$

$$2\,\frac{\rho_g}{\rho}\left[\frac{P_g}{1/2\rho_g U_\infty^2} - \frac{2X_{e\infty}^2}{E}\,\bar{\bar{\delta}}_{\bar{x}}\sqrt{\frac{\nu_g}{\nu}}\left\{\tau_g\sqrt{\frac{2U_\infty R_o}{\nu_g}}\Big/1/2\rho_g U_\infty^2\right\}\right] \ll 1 \qquad 20a)$$

or whenever:

$$E^2 \geq 1 \quad ; \quad 2\tilde{P} = 2(P_g - \frac{2}{\sqrt{R_e}}\,\tau_g\tilde{\delta}_{\tilde{x}})\Big/1/2\rho U_\infty^2 =$$

$$2\,\frac{\rho_g}{\rho}\left[\frac{P_g}{1/2\rho_g U_\infty^2} - \frac{2}{\sqrt{R_e}}\,\tilde{\delta}_{\tilde{x}}\sqrt{\frac{\nu_g}{\nu}}\left\{\tau_g\sqrt{\frac{2U_\infty R_o}{\nu_g}}\Big/1/2\rho_g U_\infty^2\right\}\right] \ll E \qquad 20b)$$

And finally from the last of Eqs. (15a) and (15b) it can be concluded that the gas conduction effects are negligible whenever:

$$\frac{X_e}{X_{e_\infty}} (T - T_\infty) V_r^* \gg \dot{q}_g \frac{2X_{e_\infty} R_o}{k\ P_r\ E} =$$

$$\left[\frac{N_{ug}}{\sqrt{\frac{2R_o U_\infty}{\nu_g}}\ P_{rg}}\right] \left[\frac{C_{pg}}{C_p} \frac{\rho_g}{\rho} \sqrt{\frac{\nu_g}{\nu}}\right] \frac{(T - T_\infty)}{E}$$

that is whenever:

$$\left[\frac{N_{ug}}{\sqrt{\frac{2R_o U_\infty}{\nu_g}}\ P_{rg}}\right] \left[\frac{C_{pg}}{C_p} \frac{\rho_g}{\rho} \sqrt{\frac{\nu_g}{\nu}}\right] \ll E V_r^* \frac{X_e}{X_{e_\infty}} \qquad 21)$$

These inequalities* are seen to depend upon the quantities $\frac{X_e}{X_{e_\infty}} V_r^*$ and V_φ^* which in turn depend upon the liquid drop trajectories. If the drop trajectories were straight lines we would have from Figures 1 and 2:

$$\left.\begin{aligned} X_e &= X_{e_\infty} \\ V_r^* &= - \cos \bar{x} = - \cos . \tilde{x} \\ V_\varphi^* &= \sin \bar{x} = \sin x \end{aligned}\right\} \qquad 22)$$

We now proceed to investigate under what conditions Eq.(22) hold with sufficient accuracy. Now if the drops are uniformly distributed then the number of drops per unit length of line in the plain of the flow and perpendicular to the velocity at infinity is $N_o^{1/3}$. And the number N' lying along the line segment $\overline{ab}$ in Figure 3 is $N_o^{1/3} R_o$ or since:

$$N_o = X_e / \left(\frac{4}{3}\right) \pi r_d^3$$

*(14) and (21).

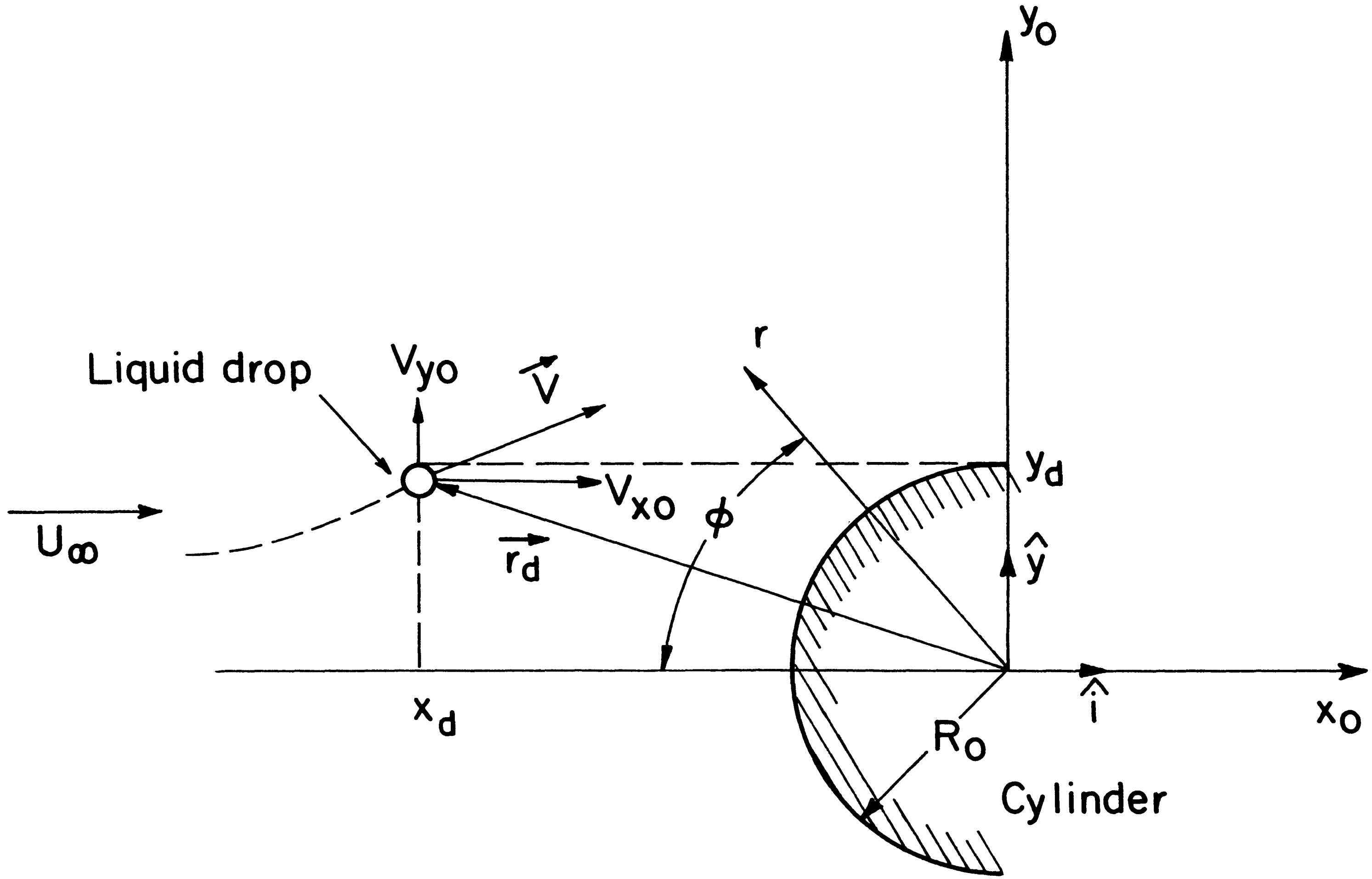

Figure 2. Coordinates for dynamical equations of liquid droplet.

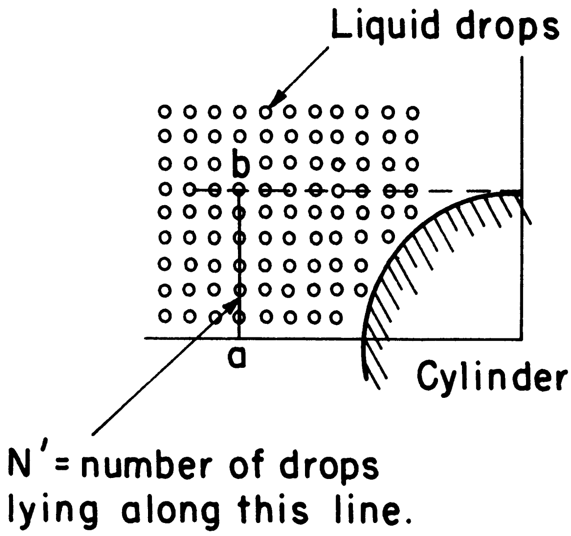

Figure 3. Sketch for linear drop density.

we have:

$$N' = \frac{X_e^{1/3}}{\left(\frac{4}{3}\pi\right)^{1/3} \frac{rd}{R_o}} \qquad \hat{23})$$

Now the number N' must be large enough so that the assumption that the unsteady effects due to drop impingement on the liquid film can be neglected, is valid. This fixes the minimum value of N'. From Eq. (17) one can see that if the dimensionless quantities are of the order unity then the particle trajectories will be straight if $^{\rho}g/\rho$ and $3/8C_D \frac{1}{(r_d/R_o)} \frac{\rho g}{\rho}$ are both much less than 1. But if the drops did not influence the gas appreciably then $\nabla_o^* P_g^*$ would certainly be of order 1 and even if the presence of the drops could influence the gas pressure field by a factor of 100 the ratio $^{\rho}g/\rho$ for air/water mixtures is of the order of 10^{-3} so that this term is almost certainly negligible.

On the other hand the coefficient of $3/8\ C_D \frac{1}{(r_d/R_o)} \frac{\rho_g}{\rho}$ might conceivably be small almost everywhere if the liquid drops actually caused the gas to follow their trajectories. This is what we might expect if there were a large mass of liquid present. But with a large amount of liquid present (i.e., large X_e) we could take (r_d/R_o) large and still satisfy ($\hat{23}$)and with large (r_d/R_o), $3/8C_D \frac{1}{(r_d/R_o)} \frac{\rho g}{\rho}$ will certainly be small. We are interested rather in the other extreme when X_e is small. It is in this case that 23) requires that (r_d/R_o) be small and therefore $3/8C_D \frac{1}{(r_d/R_o)}$ will be large not only from the (r_d/R_o) in the denominator but also from the fact that C_D increases considerably with a decrease in the drop Reynolds number and therefore (for a fixed Reynolds number based on the cylinder diameter) with decreasing r_d/R_o. Now a comparison of Eqs. (17) and (18) show that the ratio of the drag force per unit volume acting on the gas to the drag force per unit volume acting on the drops is given by:

$$X_e \Big/ \left(\frac{\rho_g}{\rho}\right)$$

We anticipate (and the numerical results below verify) that this minimum value of X_e^* is somewhat less than (ρ_g/ρ) so that the gas velocity field at the minimum value of X_e is less affected by the liquid than the liquid velocity by the gas. Hence we conclude that at the minimum value of X_e for which the particle trajectories are straight lines the gas can not be affected much by the liquid if the liquid is not effected much by the gas. Therefore we anticipate the gas to behave as potential flow and the coefficient of $3/8C_D \dfrac{1}{(r_d/R_o)} \dfrac{\rho_g}{\rho}$ must be small. Of course the minimum value of X_e depends also on the gas Reynolds number. That is for lower gas Reynolds numbers the minimum value of X_e must be larger in order to satisfy $(\hat{23})$ and still make $3/8C_D \dfrac{1}{(r_d/R_o)} \dfrac{\rho_g}{\rho}$ small. But the numerical results obtained below for air/water mixtures show that for reasonable values of (r_d/R_o) the minimum value of X_e is always such for all gas Reynolds numbers that we can not expect to strong an effect on the gas

*I.e., minimum value of X_e for which the particle trajectories are straight and there is a sufficient number of liquid drops.

by the liquid. In order to obtain numerical values for the minimum value of X_e the minimum value of E^2 and the required r_d/R_o for which the particle trajectories are straight lines for a given value of the gas Reynolds number we first choose a reasonable minimum value of N' of ten. And we have the following conditions:

$$\text{Drop Reynolds No.} = \left(\frac{r_d}{R_o}\right) \text{x Gas Reynolds No.}$$

$$C_D = \text{Function of Drop Reynolds No.*}$$

$$3/8 \frac{C_D}{(r_d/R_o)} \frac{\rho_g}{\rho} \leq .1$$

$$N' = \frac{X_e^{1/3}}{\left(\frac{4}{3}\pi\right)^{1/3} \left(\frac{r_d}{R_o}\right)} \geq 10$$

These relations serve to determine the desired minimum values and the results** are shown in Figure 7 for air/water mixtures. We have therefore shown that under reasonable restrictions we may expect the particle trajectories to be approximately straight. We will therefore impose this restriction temporarily on the analysis. And therefore require that Eqs.(22) hold. We now return to the consideration of the interaction which occurs at the surface of the liquid film and to what one might expect the minimum conditions to be for air/water mixtures for these interactions to have only a small effect on the liquid film.

*See Schlichting, p. 16.

**These were obtained from the computer program in Appendix I.

From Eq. (22) one gets:

$$-2\,\frac{X_e}{X_{e_\infty}}\,V^*_\varphi\,V^*_r \;=\; 2\cos\bar{X}\,\sin\bar{x} \;=\; \sin 2\bar{X} \;=\; \sin 2\breve{x}$$

hence

$$-2\,\frac{X_e}{X_{e_\infty}}\,V^*_\varphi\,V^*_r \;=\; O(1)$$

and the inequality (19) becomes

$$E \gg \left[\sqrt{\frac{\nu_g}{\nu}}\;\frac{\rho_g}{\rho}\right]\left[\tau_g\,\sqrt{\frac{2U_\infty R_o}{\nu_g}}\;\Big/\;1/2\,\rho_g U_\infty^{\;2}\right] \qquad 19^*)$$

Now using the inequality (19*) in the inequalities (20a) and (20b) and remembering that the limit as $X_e \to 0$ and $R_e \to \infty$ is being taken the inequalities (20a) and (20b) become:

$$E^2 \geq 1 \quad 2\bar{P} \;=\; 2\,\frac{\rho_g}{\rho}\left[\frac{P_g}{1/2\,\rho_g U_\infty^{\;2}}\right] \ll 1 \qquad 20a^*)$$

$$E^2 \leq 1 \quad 2\breve{P} \;=\; 2\,\frac{\rho_g}{\rho}\left[\frac{P_g}{1/2\,\rho_g U_\infty^{\;2}}\right] \ll E \qquad 20b^*)$$

and finally the inequality (21) becomes:

$$\left[\frac{N_{ug}}{\sqrt{\frac{2R_o U_\infty}{\nu_g}}\;P_{rg}}\right]\left[\frac{C_{pg}}{C_p}\;\frac{\rho_g}{\rho}\;\sqrt{\frac{\nu_g}{\nu}}\right] \ll E \qquad 21^*)$$

one can now estimate the minimum value of E at which the effects of the gas on the liquid film are small for air/water mixtures by evaluating the terms in the inequalities (19*) through (21*) for a single component gas boundary layer (i.e., evaluating them from the relations for single component flows over a cylinder). It should be noted that a low values of E^2 the presence of the liquid does not modify the gas flow very much since the liquid in the film (which is thin) is moving at a low velocity so it is relatively stationary as far as the gas boundary layer is concerned. Low values of E^2 also correspond to the range of parameters where the presence of the drops doesn't modify the potential flow very much.

Hence the inequalities (19*) through (21*) become using the numerical values obtained in Appendix II:

$$E \gg 2.2 \times 10^{-2} \qquad 19*A/W)$$

$$E^2 \geq 1 \quad 2\bar{P} \approx 5.2 \times 10^{-3} \qquad 20a*A/W)$$

$$E^2 \leq 1 \quad 2\tilde{P} \approx 5.2 \times 10^{-3} \ll E \qquad 20b*A/W)$$

$$E \gg 1.2 \times 10^{-3} \qquad 21*A/W)$$

Hence it can be concluded that for a value of $E \geq .1$ (or $E^2 \geq .01$) the terms in Eqs. (14a) through (16a) and Eqs. (14b) through (16b) which result from the presence of the gas can be neglected. Therefore using the relations (22) and the inequalities (19*A/W) through (21*A/W) we arrive at the following formulation:

$E^2 \geq 1$ (straight drop trajectories $\sim$ conditions of Appendix I)

$$0 \le \bar{y} \le \bar{\delta} \qquad 0 \le \bar{x} \qquad \left\{ \begin{aligned} &\bar{\psi}_{\bar{y}}\, \bar{\psi}_{\overline{xy}} - \bar{\psi}_{\bar{x}}\, \bar{\psi}_{\overline{yy}} = \frac{1}{E^2}\, \bar{\psi}_{\overline{yyy}} \\ &\bar{\psi}_{\bar{y}}\, T_{\bar{x}} - \bar{\psi}_{\bar{x}}\, T_{\bar{y}} = \frac{1}{P_r E^2}\, T_{\overline{yy}} \end{aligned} \right\} \tag{14a*}$$

$$\bar{y} = \bar{\delta} \qquad 0 \le \bar{x} \qquad \left\{ \begin{aligned} &\bar{\psi}_{\overline{yy}} = E^2 \,[2\sin \bar{x} - \bar{\psi}_{\bar{y}}]\, \cos \bar{x} \\ &\bar{\psi} = \sin \bar{x} \\ &T_{\bar{y}} = P_r E^2\, (T_\infty - T)\, \cos \bar{x} \end{aligned} \right\} \tag{15a*}$$

$$\bar{y} = 0 \qquad 0 \le \bar{x} \qquad \left\{ \bar{\psi} = \bar{\psi}_{\bar{y}} = 0 \quad ; \quad T = T_w \right\} \tag{16a*}$$

$.01 \le E^2 \le 1$ (straight drop trajectories ~ conditions of Appendix I apply)

$$0 \le \tilde{y} \le \tilde{\delta} \qquad 0 \le \tilde{x} \qquad \left\{ \begin{aligned} &\tilde{\psi}_{\tilde{y}}\, \tilde{\psi}_{\widetilde{xy}} - \tilde{\psi}_{\tilde{x}}\, \tilde{\psi}_{\widetilde{yy}} = \frac{1}{E}\, \tilde{\psi}_{\widetilde{yyy}} \\ &\tilde{\psi}_{\tilde{y}}\, T_{\tilde{x}} - \tilde{\psi}_{\tilde{x}}\, T_{\tilde{y}} = \frac{1}{P_r E}\, T_{\widetilde{yy}} \end{aligned} \right\} \tag{14b*}$$

$$\tilde{y} = \tilde{\delta} \qquad 0 \le \tilde{x} \qquad \left\{ \begin{aligned} &\tilde{\psi}_{\widetilde{yy}} = \left[2 \sin \tilde{x} - E\, \tilde{\psi}_{\tilde{y}} \right] \cos \tilde{x} \\ &\tilde{\psi} = \sin \tilde{x} \\ &T_{\tilde{y}} = E P_r\, (T_\infty - T)\, \cos \tilde{x} \end{aligned} \right\} \tag{15b*}$$

$$\tilde{y} = 0 \qquad 0 \le \tilde{x} \qquad \left\{ \tilde{\psi} = \tilde{\psi}_{\tilde{y}} = 0 \quad ; \quad T = T_w \right\} \tag{16b*}$$

We now notice that Eqs. (14a*) through (16a*) and Eqs. (14b*) through (16b*) are essentially the same. It is only necessary to use the relations $\tilde{y} = \bar{y}E$; $\tilde{\delta} = \bar{\delta}E$ to go from one to the other and thus there is a single formulation of the problem for $E^2 \ge .01$ and the condition of straight particle trajectories. And therefore only the Eqs. (14a*) through (16a*) will be needed to solve the problem.

Let us now consider the case when $E^2 \leq .01$. For reasonable values of the Reynolds number this implies that X_e is small.* Hence there is only a small range of parameters for which one might expect the particle trajectories to be straight and still have a sufficient number of drops to have a contuum. On the other hand, as argued above, one might expect that the presence of the liquid film does not influence the gas boundary layer very much and one may therefore evaluate the gas/liquid interaction terms in Eqs. (14a) through (14b) by assuming that the gas boundary layer and the pressure field in the region of the cylinder are the same as for single component gas flow about a cylinder. Also since we are considering the conditions when $E \leq .1$ we may take the limit as $E \to 0$. With one exception! In the energy relations the product EP_r appears, and since the Prnadtl number is of order ten we must retain these terms. It seems rather unlikely that a film will form for values of $E^2 \leq 10^{-4}$ hence only the range $.01 \leq E \leq .1$ will be considered. From the inequality (21*A/W) one sees that one may also neglect the gas conduction term for this range of parameters. From Schlichting, p. 153 (28) we have for single component flow around a cylinder:

$$\frac{\tau_g}{1/2\rho_g U_\infty^2}\sqrt{\frac{2U_\infty R_o}{\nu}} = 8\left\{\gamma_1 \tilde{x} - \sum_{n=2}^{\infty} \frac{(-1)^n(2n)}{(2n-1)!}\,\gamma_{(2n-1)}\,\tilde{x}^{2n-1}\right\}$$

*For air/water mixtures if the gas Reynolds number based on the cylinder diameter is 1000 than for $E^2 = .01$, $X_e = 10^{-3}$ larger Reynolds numbers and smaller values of E^2 correspond to lower values of X_e.

Where the γ_i's are the $f_i''(0)$'s there, and are listed in Table 9.1. It also follows from the general theory of potential flow that:

$$\frac{-\partial}{\partial \tilde{x}}\left[\frac{P_g}{1/2\rho_g U_\infty^2}\right] = 2 \sin. 2\tilde{x}$$

And finally after noting as in the derivation of the inequalities (20a*) and (20b*) from (20a) and (20b) that $\tilde{P} \approx \frac{\rho_g}{\rho} \quad \frac{P_g}{1/2\rho_g U_\infty^2}$ Eqs. (14b) through (16b) become:

$$.01 \leq E \leq .1$$

$$\begin{array}{l} 0 \leq \tilde{y} \leq \tilde{\delta} \\ 0 \leq \tilde{x} \end{array} \left\{ \begin{array}{l} \tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}} = - \dfrac{4\rho_g}{E\rho} \sin 2\tilde{x} \\ T_{\tilde{y}\tilde{y}} = P_r E \left[\tilde{\psi}_{\tilde{y}} T_{\tilde{x}} - \tilde{\psi}_{\tilde{x}} T_{\tilde{y}} \right] \end{array} \right\} \qquad 14c)$$

$$\begin{array}{l} \tilde{y} = \tilde{\delta} \\ 0 \leq \tilde{x} \end{array} \left\{ \begin{array}{l} \tilde{\psi}_{\tilde{y}\tilde{y}} = \dfrac{8\rho_g}{E\rho} \sqrt{\dfrac{\nu_g}{\nu}}\, Z(\tilde{x}) - 2 \dfrac{X_e}{X_{e_\infty}} V_\varphi^* V_r^* \\ \tilde{\psi} = - \displaystyle\int_0^{\tilde{x}} \dfrac{X_e}{X_{e_\infty}} V_r^* \, d\tilde{x} \\ T_{\tilde{y}} = E P_r \dfrac{X_e}{X_{e_\infty}} (T - T_\infty) V_r^* \end{array} \right\} \qquad 15c)$$

$$\begin{array}{l} \tilde{y} = 0 \\ 0 \leq \tilde{x} \end{array} \left\{ \tilde{\psi} = \tilde{\psi}_{\tilde{y}} = 0 \quad ; \quad T = T_w \right\} \qquad 16c)$$

$$Z(\tilde{x}) \equiv \gamma_1 \tilde{x} + \sum_{n=2}^{\infty} \frac{(-1)^n (2n)}{(2n-1)!} \gamma_{(2n-1)} \tilde{x}^{2n-1}$$

Of course V_φ^*, V_r^* and X_e/X_{e_∞} are still unknown. These must be obtained by integrating Eqs. (17) and (18) and using suitable boundary conditions.

i.e.,

$$\vec{V}^* \to \hat{i} \qquad x_d^2 + y_d^2 \to \infty$$

$$\vec{V}_g^* \to \hat{i} \qquad x_o^2 + y_o^2 \to \infty$$

$$V_g^* \cdot \hat{n} = 0 \quad x_o^2 + y_o^2 = R_o^2$$

where $\hat{n}$ is the normal to the cylinder. This in general constitutes a difficult task. However for sufficiently small X_e the presence of the liquid drops don't influence the gas motion very much. By combining Eqs. (17) and (18) we get:

$$\vec{V}_g^* \cdot \nabla_o^* \vec{V}_g^* = -2(1-X_e) \nabla_o^* P_g^* - \frac{X_e}{\rho_{g/\rho}} \vec{V}^* \cdot \nabla_d^* \vec{V}^*$$

Hence for $X_e \leq .1 \frac{\rho g}{\rho}$ one would expect the gas flow to be essentially a potential flow. Under these circumstances Eq. (17) can be solved for the components of the liquid velocity at the surface of the cylinder by replacing u_g^* and v_g^* by the corresponding values for potential* flow. Several numerical procedures have been developed for integrating this equation in connection with iceing problems on airfoils. See for example References.[19,30]

We now formulate the problem of drop trajectories for the case when the gas flow can be considered a potential flow to a sufficient approximation i.e., when* $X_e \leq .1\ \rho_{g/\rho}$.

We first recall from potential flow theory that the velocity components outside of the boundary layer and away from the region of the wake are given to a fair approximation by:

*This approximation is probably good for larger values of X_e if the drops are large enough so that their trajectories are fairly straight.

$$u_g^* = 1 - \frac{x_d^{*2} - y_d^{*2}}{(x_d^{*2} + y_d^{*2})^2}$$

$$v_g^* = - \frac{2x_d^* y_d^*}{(x_d^{*2} + y_d^{*2})^2}$$

The situation is as shown in Figures 2 and 4. Next $\nabla_o^* P_g^*$ must be of the order of unity and since $\rho_{g/\rho}$ is very small we are entirely justified in neglecting the pressure force term in Eq. (17). Now if one introduces the dimensionless time defined by:

$$t^* = \frac{t}{R_o} U_\infty$$

one may write

$$\vec{V}^* . \nabla_d^* \vec{V}^* = \dot{\vec{V}}^*$$

where the dot denotes differentiation with respect to $t.^*$ We also have:

$$V_{yo}^* = \dot{y}_d^* \quad ; \quad V_{xo}^* = \dot{x}_d^*$$

Equation (17) may now be rewritten in component form as:

$$\dot{y}_d^* = V_{yo}^*$$

$$\dot{x}_d^* = V_{xo}^*$$

$$\dot{V}_{xo}^* = 3/8 \frac{C_D}{(r_d/R_o)} \frac{\rho_g}{\rho} (V_{xo}^* - u_g^*) \; \Delta V^*$$

$$\dot{V}_{y_o}^* = 3/8 \frac{C_D}{(rd/R_o)} \frac{\rho_g}{\rho} (V_{y_o}^* - v_g^*) \; \Delta V^*$$

$$\Delta V^* \equiv (V^*_{x_o} - u^*_g)^2 + (V^*_{y_o} - v^*_g)^2$$

$$u^*_g = 1 - \frac{x^{*2}_d - y^{*2}_d}{(x^{*2}_d + y^{*2}_d)^2}$$

$$v^*_g = - \frac{2x^*_d y^*_d}{(x^{*2}_d + y^{*2}_d)^2}$$

This constitutes a system of four first order ordinary differential equations in y_d, x^*_d, $V^*_{x_o}$, and $V^*_{y_o}$. We must therefore specify four boundary conditions. For our purposes we take these as:

$$\text{at } t^* = 0 \qquad \begin{aligned} x^*_d &\to -\infty \\ y^*_d &\to y^*_{d_\infty} \\ \dot{x}^*_d &\to 1 \\ \dot{y}^*_d &\to 0 \end{aligned}$$

These are the equations for an individual drop. We must now relate the solutions of these equations to the quantities at the edge of the liquid film which are of interest to us. In order to do this we consider the control surface in Figure 4 bounded on two sides by drop trajectories, on the third side by the liquid film and on the fourth side by a line perpendicular to the particle trajectories at a distance far enough from the cylinder so that they are parallel. Then the particles only cross the control surface at the latter two sides (i.e., they enter at the upstream side and leave at the downstream side). Now from the previous developments we have that to the order of approximation of the quantities appearing in the liquid film equations the total flux of

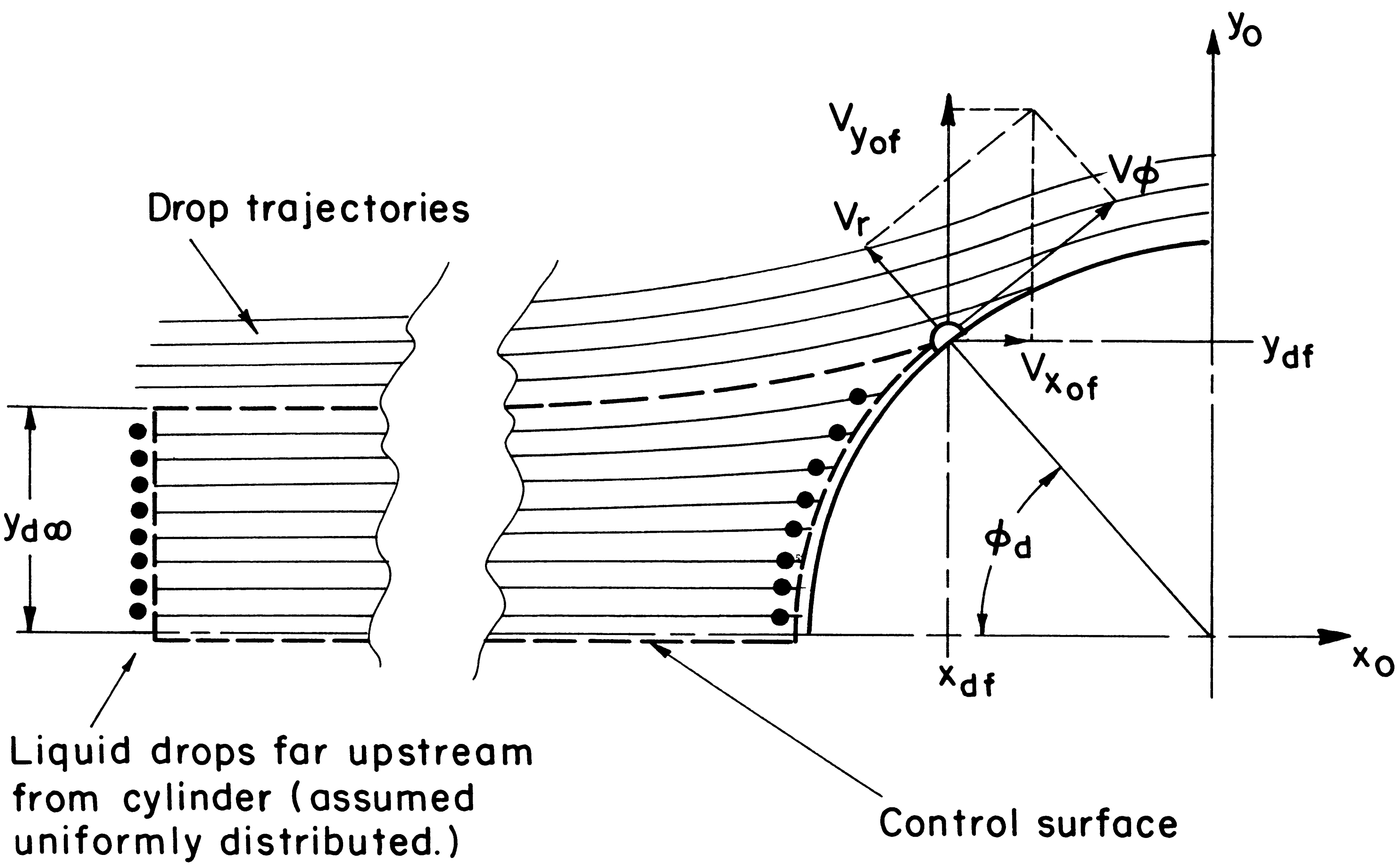

Figure 4. Control volume for the determination of physical quantities at the surface of liquid film from the trajectory data.

liquid across the downstream surface (which is the flux of liquid into the film between $\varphi = 0$ and $\varphi = \varphi_d$) is given by:*

$$-\int_0^{\varphi_d} X_e V_r \, R_o \, d\varphi$$

The flux through the downstream side is given by:

$$X_{e_\infty} U_\infty \, y_{d_\infty}$$

and since these must be equal to one another we have:

$$-\int_0^{\varphi_d} X_e \, V_r \, R_o \, d\varphi = X_{e_\infty} U_\infty y_{d_\infty}$$

$$-\int_0^{\varphi_d} (X_e/X_{e_\infty}) \, V_r^* \, d\varphi = y^*_{d_\infty}$$

differentiating both sides with respect to $y^*_{d_\infty}$ we get:

$$\frac{d}{dy^*_{d_\infty}} \left[-\int_0^{\varphi_d} (X_e/X_{e_\infty}) V_r^* \, d\varphi \right] = 1$$

$$= \frac{d}{d\varphi_d} \left[-\int_0^{\varphi_d} (X_e/X_{e_\infty}) V_r^* \, d\varphi \right] \frac{d\varphi_d}{dy^*_{d_\infty}}$$

$$= -(X_e/X_{e_\infty}) \, V_r^* \, \frac{d\varphi_d}{dy^*_{d_\infty}}$$

*This is essentially equivalent to the approximation that the film is thin.

Next we note that:

$$\varphi_d = \tan.^{-1} \frac{y^*_{d_f}}{x^*_{d_f}}$$

$$\cos.\varphi_d = -x^*_{d_f}$$

$$\sin.\varphi_d = y^*_{d_f}$$

$$\cos.\varphi_d d\varphi_d = dy^*_{d_f}$$

$$d\varphi_d = -\frac{dy^*_{d_f}}{x^*_{d_f}}$$

Hence:

$$-(X_e/X_{e_\infty})V^*_r = x^*_{d_f} \Big/ \frac{dy^*_{d_f}}{dy^*_{d_\infty}}$$

and we may write from elementary analytic geometry at the edge of the liquid film

$$V^*_\varphi = V^*_{x_{of}} \sin\varphi_d + V^*_{y_{of}} \cos\varphi_d$$

and remembering that we have set $\tilde{x} = \varphi$, the results may be summarized as follows:

$$\left[\begin{array}{l} \tilde{x} = \tan.^{-1} y^*_{d_f}/x^*_{d_f} \\ -\int_0^x (X_e/X_{e_\infty})V^*_r \, d\tilde{x} = y^*_{d_\infty} \end{array}\right\} \qquad 17\text{*A.C.})$$

$$\left\langle \begin{aligned} & -(X_e/X_{e_\infty})V_r^* = -x_{d_f}^* \Big/ \frac{dy_{d_f}^*}{dy_{d_\infty}^*} \\ & V_\varphi^* = -V_{x_{of}}^* x_{d_f}^* + V_{y_{of}}^* y_{d_f}^* \end{aligned} \right\rangle$$

We now notice that all the quantities on the right of the above relations can be obtained by integrating Eq. (17*) for different values of $y_{d_\infty}^*$. The quantities on the left are precisely those which are needed for the Eqs. (14c) through (16c). The calculations have been carried out numerically. These are discussed in Appendix XXV.

We will consider here only the cases when (1) $E^2 \geq .01$ and the drop trajectories are straight as formulated in (14a*) through (16a*) or equivalently in (14b*) through (16b*) (since these are the same equations), and when (2) $.01 \leq E \leq .1$ and the external gas flow field is not influenced by the motion of the drops as formulated in (14c) through (16c) and (17*) and (17*A.C.). In the next section we therefore turn to the solution of Eqs. (14a*) through (16a*), and (14c) through (16c).

2. SOLUTION

An examination of Eqs. (14a*) through (16a*) (or equivalently Eqs. (14b*) through (16b*)) shows that the position of a boundary of the domain of the problem is unknown and must be determined by the problem. That is the thickness $\bar{\delta}(\bar{x})$ is an unknown function. Since it is inconvenient to solve the problem in this form we will map the domain of the problem into one whose boundaries are known and in the process reintroduce

$\bar{\delta}(\bar{x})$ as a dependent variable (on the same footing as $\bar{\psi}$). In order to accomplish this let us introduce the following variables:

$$\eta \equiv \bar{y}/\bar{\delta} = \tilde{y}/\tilde{\delta} = y/\delta \; R_o$$

$$\xi \equiv \bar{x} = \tilde{x} = x/R_o$$

$$\psi^* \equiv \bar{\psi} = \tilde{\psi} = \psi/U_\infty R_o X_e$$

$$\delta^* \equiv \bar{\delta} = \tilde{\delta}/E = \delta/2R_o X_e$$

$$\Theta^* \equiv \frac{T - T_\infty}{T_w - T_\infty} \; .$$

Then we have:

$$\frac{\partial}{\partial \bar{y}} = \left(\frac{\partial \eta}{\partial \bar{y}}\right)_{\bar{x}} \frac{\partial}{\partial \eta} + \left(\frac{\partial \xi}{\partial \bar{y}}\right)_{\bar{x}} \frac{\partial}{\partial \xi} = \frac{1}{\bar{\delta}} \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial \bar{x}} = \left(\frac{\partial \eta}{\partial \bar{x}}\right)_{\bar{y}} \frac{\partial}{\partial \eta} + \left(\frac{\partial \xi}{\partial \bar{x}}\right)_{\bar{y}} \frac{\partial}{\partial \xi} = \frac{1}{\bar{\delta}} \frac{d\bar{\delta}}{d\bar{x}} \eta \frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial \bar{x} \partial \bar{y}} = \frac{1}{\bar{\delta}} \frac{\partial^2}{\partial \eta \partial \xi} - \frac{1}{\bar{\delta}^2} \frac{d\bar{\delta}}{d\bar{x}} \frac{\partial}{\partial \eta} \eta \frac{\partial}{\partial \eta}$$

Now using these in the first of Eqs. (14a*) we get:

$$\frac{1}{\delta^{*2}} \frac{\partial \psi^*}{\partial \eta} \frac{\partial^2 \psi^*}{\partial \eta \partial \xi} - \frac{1}{\delta^{*3}} \frac{d\delta^*}{d\xi} \frac{\partial \psi^*}{\partial \eta} \left[\frac{\partial}{\partial \eta} \left(\eta \frac{\partial \psi^*}{\partial \eta} \right) \right]$$

$$- \frac{1}{\delta^{*2}} \frac{\partial \psi^*}{\partial \xi} \frac{\partial^2 \psi^*}{\partial \eta^2 \xi} + \frac{1}{\delta^{*3}} \frac{d\delta^*}{d\xi} \eta \frac{\partial \psi^*}{\partial \eta} \frac{\partial^2 \psi^*}{\partial \eta^2} = \frac{1}{E^2 \delta^{*3}} \frac{\partial^3 \psi^*}{\partial \eta^3}$$

which becomes after some rearrangement:

$$\delta^* \left[\frac{\partial\psi^*}{\partial\eta}\frac{\partial^2\psi^*}{\partial\eta\partial\xi} - \frac{\partial\psi^*}{\partial\xi}\frac{\partial^2\psi^*}{\partial\eta^2}\right] - \frac{d\delta^*}{d\xi}\left(\frac{\partial\psi^*}{\partial\eta}\right)^2 = \frac{1}{E^2}\frac{\partial^3\psi^*}{\partial\eta^3}$$

and wer get from the second of Eqs. (14a*):

$$\delta^* \left[\frac{\partial\psi}{\partial\eta}\frac{\partial\theta^*}{\partial\xi} - \frac{\partial\theta^*}{\partial\eta}\frac{\partial\psi^*}{\partial\xi}\right] = \frac{1}{P_r E^2}\frac{\partial^2\theta^*}{\partial\eta^2}$$

and finally since $\eta = 1$ when $\bar{y} = \bar{\delta}$ and $\eta = 0$ when $\bar{y} = 0$ Eqs. (14a*) through (16a*) become:

$$\begin{array}{l} 0 \leq \eta \leq 1 \\ 0 \leq \xi \end{array} \left\{ \begin{array}{l} \delta^* \left[\frac{\partial\psi^*}{\partial\eta}\frac{\partial^2\psi^*}{\partial\eta\partial\xi} - \frac{\partial\psi^*}{\partial\xi}\frac{\partial^2\psi^*}{\partial\eta^2}\right] - \frac{d\delta^*}{d\xi}\left(\frac{\partial\psi^*}{\partial\eta}\right)^2 = \frac{1}{E^2}\frac{\partial^3\psi^*}{\partial\eta^3} \\ \delta^* \left[\frac{\partial\psi^*}{\partial\eta}\frac{\partial\theta^*}{\partial\xi} - \frac{\partial\theta^*}{\partial\eta}\frac{\partial\psi^*}{\partial\xi}\right] = \frac{1}{P_r E^2}\frac{\partial^2\theta^*}{\partial\eta^2} \end{array} \right\} \qquad 23)$$

$$\begin{array}{l} \eta = 1 \\ 0 \leq \xi \end{array} \left\{ \begin{array}{l} \frac{\partial^2\psi^*}{\partial\eta^2} = E^2\delta^{*2}\cos\xi\left[2\sin\xi - \frac{1}{\delta^*}\frac{\partial\psi^*}{\partial\eta}\right] \\ \psi^* = \sin\xi \\ \frac{\partial\theta^*}{\partial\eta} = -P_r E^2\,\theta^*\delta^*\cos\xi \end{array} \right\} \qquad 24)$$

$$\begin{array}{l} \eta = 0 \\ 0 \leq \xi \end{array} \left\{ \psi^* = \frac{\partial\psi^*}{\partial\eta} = 0 \;;\; \theta^* = 1 \right\} \qquad 25)$$

The same results could also have been obtained from Eqs. (14b*) through (16b*). The problem is now completely formulated except that we have not specified the boundary conditions in the ξ direction but these follow from the symmetry of the problem. That is δ^* and Θ^* must have even symmetry while ψ^* must have odd symmetry.

Because of these symmetries we seek a solution to the problem in the following form:

$$\psi^* = f_1(\eta)\xi - \sum_{n=2}^{\infty} \frac{(-1)^n(2n)}{(2n-1)!} f_{(2n-1)}(\eta)\xi^{2n-1} \qquad 26)$$

$$\delta^* = a_o \left[1 + \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!} \xi^{2n-1} \right] \qquad 27)$$

$$\Theta^* = F_o(\eta) + \sum_{n=1}^{\infty} \frac{F_{2n}(\eta)}{(2n)!} \xi^{2n} \qquad 28)$$

If we substitute Eqs. (26) and (27) into the first Eq. (23) and equate coefficients of like powers of ξ after a lengthly calculation which is carried out in Appendix III we arrive at:

$$\frac{1}{a_o E^2} f_1''' + f_1 f_1'' - f_1'^2 = 0$$

$$L_n(f_n) = (-1)^{\frac{n+1}{2}} \frac{n}{n+1} \left[f_1 f_1'' + (n-2) f_1'^2 \right] b_{n-1} + H_n \quad n = 3,5,7\ldots \qquad 29)$$

Where:

$$L_n \equiv \frac{1}{a_o E^2} \frac{d^3}{d\eta^3} + f_1 \frac{d^2}{d\eta^2} - (n+1) f_1' \frac{d}{d\eta} + n f_1''$$

And:

$$H_3 = 0$$

$$H_5 = \frac{20}{3} \left[4(f_3'^2 - f_3 f_3'') + b_2(f_3'' f_1 + 3 f_3 f_1'')\right]$$

$$H_7 = \frac{7}{4} \left[12(8 f_5' f_3' - 3 f_5'' f_3 - 5 f_3'' f_3) - b_2(18 f_5' f_1' + 80\left[f_3'^2 - 3 f_3 f_3''\right]\right.$$

$$\left. -9 f_5'' f_1 - 45 f_1'' f_5) - 10\, b_4(f_3'' f_1 + 4 f_1' f_3' + 3 f_3 f_1'')\right]$$

....

....

Upon substituting the expansion (26), (27), and (28) into the second of Eqs. (23) and equating coefficients of like powers of ξ we get after some calculation which is performed in Appendix IV:

$$K_n(F_n) = G_n \qquad n = 0,2,4,6 \ldots. \tag{30}$$

Where:

$$K_n \equiv \frac{1}{a_o E^2 P_r} \frac{d^2}{d\eta^2} + f_1 \frac{d}{d\eta} - n f_1'$$

$$G_o = 0$$

$$G_2 = (4 f_3 - b_2 f_1) F_o'$$

$$G_4 = 8(3 f_3 F_2' - 2 f_3' F_2) + 6 b_2(2 f_1' F_2 - f_1 F_2')$$

$$-(6 f_5 - 24 b_2 f_3 + b_4 f_1) F_o'$$

$$G_6 = 20(3 f_3 F_4' - 4 f_3' F_4) - 18(5 f_5 F_2' - 2 f_5' F_2)$$

$$+ 15b_2(4f_1'F_4 - f_1F_4' - 16f_3'F_2 + 24f_3F_2') +$$

$$+ 15b_4(2f_1'F_2 - f_1F_2') + (8f_7 - 90b_2f_5 + 60b_4f_3$$

$$-b_6f_1)F_o'$$

....

....

....

Next we get after substituting the expansions (26) and (27) into the first of Eqs. (24), noting that:

$$\sin\xi = \xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} - \frac{\xi^7}{7!} + \ldots..$$

$$\cos\xi = 1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots..$$

and equating coefficients of like powers of ξ (see Appendix V for the calculations):

$$\left.\begin{array}{l} \dfrac{2}{a_oE^2} f_1''(1) + 2f_1'(1) - 4a_o = 0 \\ \dfrac{2}{a_oE^2} f_n''(1) + 2f_n'(1) + \dfrac{2n}{n+1}\left\{(-1)^{\frac{n+1}{2}} b_{n-1}\left[4a_o - f_1'(1)\right] - \right. \\ \left. -2a_o + f_1'(1)\right\} + A_n \qquad n = 3,5,7 \end{array}\right\} \quad 31)$$

where:

$$A_3 = -a_o$$

$$A_5 = \frac{2}{3}\left\{15b_2\left[2a_o - f_1'(1)\right] + a_o(10b_2 - 1) + 10(b_2-1)\left[3b_2a_o + 2f_3'(1) - a_o\right]\right\}$$

$$A_7 = \frac{1}{4}\left\{105(b_2 - b_4)\left[2a_o - f_1'(1)\right] + 2a_o(21\,b_2 - 35b_4 - 1) + 42(b_2-1)\left[a_o(5b_4 - 10b_2 + 1) - 3f_5'(1)\right] + - 70(b_4 - 6b_2 + 1)\left[a_o(3b_2-1) + 2f_3'(1)\right]\right\}$$

....

....

Next we turn to the third of Eqs. (24). We substitute in the expansion (27) and (28) and equate coefficients of ξ. After the calculation performed in Appendix VI we arrive at:

$$\frac{1}{a_oE^2P_r}\,F_o'(1) + F_o(1) = 0$$

$$\frac{1}{a_oE^2P_r}\,F_n'(1) + F_n(1) + F_o(1)\left[b_n + (-1)^{\frac{n}{2}}\right] + B_n \;;\; n = 2,4,6,... \qquad 32)$$

Where:

$$B_2 = 0$$

$$B_4 = 6\left[b_2F_2(1) - F_2(1) - b_2F_o(1)\right]$$

$$B_6 = 15\left[b_2F_4(1) + b_4F_2(1) - F_4(1) - 6b_2F_2(1) - b_4\,F_o(1) + F_2(1) + b_2\,F_o(1)\right]$$

....

....

....

The second of Eqs. (24) yields upon comparison of:

$$\sin\xi = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}\,\xi^{2n-1}$$

with the expansion (26):

$$\left.\begin{aligned} f_1(1) &= 1 \\ f_n(1) &= \frac{1}{n+1} \quad n = 3,5,7\ \ldots. \end{aligned}\right\} \qquad 33)$$

And finally to satisfy the conditions (25) we must have

$$\left.\begin{aligned} f_n(0) &= \frac{df_n(0)}{d\eta} = 0 \ ; \quad n = 1,3,5,7,\ldots. \\ F_o(0) &= 1 \\ F_n(0) &= 0 \qquad n = 2,4,6\ldots. \end{aligned}\right\} \qquad 34)$$

We have now completely formulated the problem in terms of a set of simultaneous ordinary differential equations with their appropriate boundary conditions. These equations can be solved successively, i.e., the second equation involves the solution to the first but not of the third, fourth, etc. We also note that the equations in f_n are of the third order but that there are four boundary conditions. The fourth boundary condition is necessary to determine the a_o, b_2, b_4, etc. which are to be determined by the problem.

Before summarizing the results of this section we will derive explicit formulas for the velocity, shear stress, and heat transfer coefficients in terms of the solutions to the equations derived above.

We first recall that the velocity in the x direction (i.e., in the φ direction) v_φ is given terms of the stream function in Eqs. (14) by:

$$v_\varphi = \frac{1}{R_o} \frac{\partial \psi}{\partial y}$$

If we now change the symbol from v_φ to u (i.e., $v_\varphi = u$) and introduce the new variables of this section (i.e., ψ^*, η, δ^*) we get:

$$\frac{u}{U_\infty} = \frac{1}{2\delta^*} \frac{\partial \psi^*}{\partial \eta} \qquad 35)$$

Also the wall shear stress in the φ direction is given by*:

$$\tau_w = \frac{\mu}{R_o^2} \frac{\partial^2 \psi}{\partial y^2} \bigg|_{y=0}$$

Introducing the variables ψ^*, η, and δ^* we get:

$$\tau_w = \frac{\mu U_\infty R_o X_e}{4 R_o^2 X_e^2} \frac{1}{\delta^{*2}} \frac{\partial^2 \psi^*}{\partial \eta^2} \bigg|_{\eta=0}$$

or:

$$\frac{\tau_w \sqrt{\dfrac{2U_\infty R_o}{\nu}}}{1/2 \rho U_\infty^2} = \frac{1}{E\,\delta^{*2}} \frac{\partial^2 \psi^*}{\partial \eta^2} \bigg|_{\eta=0} \qquad 36)$$

---

*Note that from the last of Eq. (4) $\tau_{r\varphi} = \mu \left(\frac{1}{r}\frac{\partial v_r}{\partial \varphi} + \frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r}\right)$ and from (7) $v_r = v_\varphi = 0$ at $r = R_o$ and therefore $\frac{\partial v_r}{\partial \varphi} = 0$ at $r = R_o$ hence $\tau_{r\varphi}(R_o, \varphi) = \mu \frac{\partial v_\varphi}{\partial r}\bigg|_{r=R_o} = \frac{\mu}{R_o}\frac{\partial v_\varphi}{\partial y}\bigg|_{y=0} = \frac{\mu}{R_o^2}\frac{\partial^2 \psi}{\partial y^2}\bigg|_{y=0}$

Finally the heat flux at the wall q_w is given by:

$$q_w = -k \frac{\partial T}{\partial y}\bigg|_{r=R_o} = -\frac{k}{R_o}\frac{\partial T}{\partial y}\bigg|_{y=0}$$

and substitution of θ^*, η, and δ^* yields:

$$q_w = -\frac{k(T_w - T_\infty)}{2R_oX_e}\frac{1}{\delta*}\frac{\partial \theta^*}{\partial \eta}\bigg|_{\eta=0}$$

or introducing the Nusselt number $N_u \equiv q_w\, 2R_o/(T_\infty - T_w)k$ we get:

$$\frac{N_u}{\sqrt{\frac{2U_\infty R_o}{\nu}}} = \frac{1}{E\delta^*}\frac{\partial \theta^*}{\partial \eta}\bigg|_{\eta=0}$$

Now upon substitution of the expansions (26) and (27) we get from Eq. (35) after a calculation performed in Appendix VII

$$\frac{u}{U_\infty} = u_1\xi - \frac{u_3}{3!}\xi^3 + \frac{u_5}{5!}\xi^5 - \frac{u_7}{7!}\xi^7 + \ldots\ldots \qquad 38)$$

Where:

$$u_1 = \frac{1}{2a_o} f_1'$$

$$u_3 = \frac{1}{2a_o}(4f_3' + 3b_2f_1')$$

$$u_5 = \frac{1}{2a_o}\left\{6f_5' + 40b_2f_3' + 5(6b_2^2 - b_4)f_1'\right\}$$

$$u_7 = \frac{1}{2a_o}\left\{8f_7' + 126b_2f_5' + 140(6b_2^2 - b_4)f_3' + \right.$$

$$+ 7(90b_2^3 - 15b_2b_4 + b_6)f_1' \Big\}$$

....

....

....

After substituting the expansions (26) and (27) into Eq. (36) the calculation of Appendix VIII yields:

$$\frac{\tau_w\sqrt{\frac{2U_\infty R_o}{\nu}}}{1/2\rho U_\infty^2} = \tau_1 \xi - \frac{\tau_3}{3!}\xi^3 + \frac{\tau_5}{5!}\xi^5 - \frac{\tau_7}{7!}\xi^7 + \ldots\ldots \qquad 39)$$

where:

$$\tau_1 = \frac{1}{a_o^2E} f_1''(0)$$

$$\tau_3 = \frac{2}{a_o^2E}\Big[2f_3''(0) + 3b_2f_1''(0)\Big]$$

$$\tau_5 = \frac{2}{a_o^2E}\Big[3f_5''(0) + 40b_2f_3''(0) + 5(9b_2^2 - b_4)f_1''(0)\Big]$$

$$\tau_7 = \frac{2}{a_o^2E}\Big[4f_7''(0) + 126b_2f_5''(0) + 140(9b_2^2 - b_4)f_3''(0) +$$

$$+ 7(180b_2^3 - 45b_2b_4 + b_6)f_1''(0)\Big]$$

....

....

....

And finally the calculation of Appendix IX yields after substituting the expansions (27) and (28) into Eq. (37):

$$\frac{N_u}{\sqrt{\frac{2U_\infty R_o}{\nu}}} = q_o + \frac{q_2}{2!}\xi^2 + \frac{q_4}{4!}\xi^4 + \frac{q_6}{6!}\xi^6 + \ldots\ldots \qquad 40)$$

where:

$$q_o = \frac{1}{a_o E} F_o'(0)$$

$$q_2 = \frac{1}{a_o E}\left[F_2'(0) - b_2 F_o'(0)\right]$$

$$q_4 = \frac{1}{a_o E}\left[F_4'(0) - 6b_2 F_2'(0) + (6b_2^2 - b_4)F_o'(0)\right]$$

$$q_6 = \frac{1}{a_o E}\left[F_6'(0) - 15b_2 F_4'(0) + 15(6b_2^2 - b_4)F_2'(0) - \right.$$

$$\left. - (90b_2^3 - 15b_2 b_4 + b_6)F_o'(0)\right]$$

....

....

We now summarize the results of this section:

$$\left\{\begin{array}{l} \frac{1}{a_o E^2} f_1''' + f_1 f_1'' - f_1'^2 = 0 \quad ; \quad 0 \le \eta \le 1 \\ \frac{2}{a_o E^2} f_1''(1) + 2f_1'(1) - 4a_o = 0 \\ f_1(1) = 1 \\ f_1'(0) = 0 \\ f_1(0) = 0 \end{array}\right\} \qquad 41)$$

for $n=3,5,7\ldots$

$$\left[\begin{array}{l}
L_n(f_n) = (-1)^{\frac{n+1}{2}} \frac{n}{n+1}\left[f_1 f_1'' + (n-2) f_1'^2\right] b_{n-1} + \\
\quad + H_n(f_1, f_2, \ldots f_{n-2}, f_1', \ldots f_{n-2}', f_1'', \ldots f_{n-2}'', b_2, \ldots b_{n-3}) \\
L_n \equiv \frac{1}{a_o E^2} \frac{d^3}{d\eta^3} + f_1 \frac{d^2}{d\eta^2} - (n+1) f_1' \frac{d}{d\eta} + n f_1'' \\
\frac{2}{a_o E^2} f_n''(1) + 2 f_n'(1) + \frac{2n}{n+1}\left\{(-1)^{\frac{n+1}{2}} b_{n-1}\left[4a_o - f_1'(1)\right] - \right. \\
\quad \left. -2a_o + f_1'(1)\right\} + A_n\left\{f_1'(1), \ldots f_{n-2}'(1), a_o, b_2, \ldots b_{n-3}\right\} \\
f_n(1) = \frac{1}{n+1} \\
f_n(0) = 0 \\
f_n'(0) = 0
\end{array}\right] \qquad 42)$$

for $n=0,2,4,6\ldots$

$$\left[\begin{array}{l}
K_n(F_n) = G_n(f_1, \ldots f_{n+1}, F_o, \ldots F_{n-2}, a_o, b_2, \ldots b_n) \\
K_n = \frac{1}{a_o E^2 P_r} \frac{d^2}{d\eta^2} + f_1 \frac{d}{d\eta} - n f_1' \quad ; \quad 0 \leq \eta \leq 1 \\
\frac{1}{a_o E^2 P_r} F_n'(1) + F_n(1) = \epsilon(n) \left\{F_o(1)\left[b_n + (-1)^{\frac{n}{2}}\right] + \right. \\
\quad \left. + B_n\left[F_o(1), \ldots, F_{n-2}(1), a_o, b_2, \ldots b_{n-2}\right]\right\} \\
\quad F_n(0) = 1 - \epsilon(n) \\
\quad \epsilon(n) \equiv \begin{cases} 0 & n = 0 \\ 1 & n \neq 0 \end{cases}
\end{array}\right] \qquad 43)$$

We now notice that the differential Eq. in (41) is of the third order but it contains the unknown parameter a_o. There are thus three boundary conditions plus a fourth condition to determine a_o. Thus the problem for f_1 and a_o is completely specified and can in principle be solved. The situation is exactly the same for each of the Eqs. in (42) and they thus determine the pair f_n, b_{n-1} uniquely. It should be noted however that the solution to (41) and the solutions to the preceding Eqs. in (42) must first be found before any of the Eqs. (42) can be solved but this presents no problem since after the solution to (41) is determined the Eqs. of (42) can be solved successively. Finally the Eqs. of (43) are of second order and there are two boundary conditions. These depend of course on the solutions of Eqs. (41) and (42) and the solutions to the Eqs. (43) corresponding to lower value of n. Thus these equations can be successively solved once the solutions to (41) and (42) have been obtained. Having obtained the solutions to these equations the liquid film thickness (Eq. (27)) the liquid stream function (Eq. (26)) temperature (Eq. (28)) and velocity (Eq. (38)) can be computed. Also the local Nusselt number and shear stress can be obtained from Eq. (40) and (39), respectively. These equations have been solved numerically. The actual calculations are discussed below. Before considering these solutions we turn to the case when $.01 \leq E \leq .1$ and the presence of the liquid has no effect on the gas (i.e., there is potential flow upstream of the cylinder).

Integrating the first of Eqs. (14c) and using the first two boundary conditions (16c) we get:

$$\tilde{\psi} = -\frac{2}{3}\,\frac{1}{E}\,\frac{\rho_g}{\rho}\sin 2\tilde{x}\tilde{y}^3 + \zeta(\tilde{x})\,\tilde{y}^2 \qquad 44)$$

Using this in the first and second of Eqs. (15c) we get:

$$-\frac{4}{E}\,\frac{\rho_g}{\rho}\sin.2\tilde{x}\,\tilde{\delta} + 2\zeta(\tilde{x}) = \frac{8}{E}\,\frac{\rho_g}{\rho}\sqrt{\frac{\nu_g}{\nu}}\,Z(\tilde{x}) - $$
$$- 2\,\frac{X_e}{X_e}\,V_r^*\,V_\varphi^*$$

and:

$$-\frac{2}{3}\,\frac{1}{E}\,\frac{\rho_g}{\rho}\sin 2\tilde{x}\,\tilde{\delta}^3 + \zeta(\tilde{x})\tilde{\delta}^2 = -\int_0^{\tilde{x}} \frac{X_e}{X_{e\infty}}\,V_r^*\,d\tilde{x}$$

or:

$$\frac{4}{3}\,\frac{1}{E}\,\frac{\rho_g}{\rho}\sin.2\tilde{x}\,\tilde{\delta}^3 + \left\{\frac{4}{E}\,\frac{\rho_g}{\rho}\sqrt{\frac{\nu_g}{\nu}}\,Z(\tilde{x}) - \frac{X_e}{X_{e_\infty}}\,V_r^*\,V_\varphi^*\right\}\tilde{\delta}^2$$
$$+\int_0^{\tilde{x}} \frac{X_e}{X_{e_\infty}}\,V_r^*\,d\tilde{x} = 0 \qquad 45)$$

and:

$$\zeta(\tilde{x}) = \left\{\frac{4}{E}\,\frac{\rho_g}{\rho}\sqrt{\frac{\nu_g}{\nu}}\;Z(\tilde{x}) - \frac{X_e}{X_{e_\infty}}\;V_r^*\;V_\varphi^*\right\} + \frac{2}{E}\,\frac{\rho_g}{\rho}\sin 2\tilde{x}\,\tilde{\delta}$$

Next introducing the notation:

$$\tau_{eff.} \equiv \left\{\frac{4}{E}\,\frac{\rho_g}{\rho}\sqrt{\frac{\nu_g}{\nu}}\;Z(\tilde{x}) - \frac{X_e}{X_{e_\infty}}\,V_r^*\,V_\varphi^*\right\}$$
$$\dot{m} \equiv -\int_0^{\tilde{x}} \frac{X_e}{X_{e_\infty}}\,V_r^*\,d\tilde{x}$$

the solution to Eq. (45) can be written as[13]:

$$\tilde{\delta} = \frac{E}{4(\rho_g/\rho)}\;\frac{\tau_{eff.}}{\sin 2\tilde{x}}\;(2\cos.\frac{\gamma}{3} - 1) \qquad 46)$$

where:

$$\gamma = \cos^{-1}\left\{-\left[1 - \frac{24\left(\frac{\rho_g}{\rho}\right)^2}{E^2}\,\frac{\sin^2 2\tilde{x}}{\tau_{eff}^3}\right]\right\}$$

it should be noted that $\gamma + 4\pi$ and $\gamma + 2\pi$ would also be solutions to (45) but $\gamma + 2\pi$ corresponds to negative values of $\tilde{\delta}$ and $\gamma + 4\pi$ correspond to values of δ which increase with increasing τ_{eff} which can be ruled out on physical grounds.

Substitution of Eq. (44) into the second of Eqs. (14c) yields:

$$\theta^*_{\tilde{y}\tilde{y}} = -2P_r\frac{\rho_g}{\rho}\sin 2\tilde{x}\,\tilde{y}^2\,\tilde{\theta}^*_{\tilde{x}} + \frac{4}{3}P_r\frac{\rho_g}{\rho}\cos 2\tilde{x}\,\tilde{y}^3\,\theta^*_{\tilde{y}} + P_rE\left\{2\tilde{y}\,\zeta\,\theta^*_{\tilde{x}} - \tilde{y}^2\,\zeta'\,\theta^*_{\tilde{y}}\right\}$$

where we have introduced:

$$\theta^* \equiv \frac{T - T_\infty}{T_w - T_\infty}$$

We next note that for air/water mixtures

$$P_r \approx 10$$

$$\frac{\rho_g}{\rho} \approx 1.3 \times 10^{-3}$$

$$2\,P_r\frac{\rho_g}{\rho} \approx 2.6 \times 10^{-2}$$

Hence we will neglect the first two terms on the right of the energy equation and we have:

$$\theta^*_{\tilde{y}\tilde{y}} = P_rE\left\{2\tilde{y}\,\zeta\,\theta^*_{\tilde{x}} - \tilde{y}^2\,\zeta'\,\theta^*_{\tilde{y}}\right\} \qquad 47)$$

Next we introduce the variables:

$$\eta \equiv \tilde{y}/\tilde{\delta} = y/\delta$$

$$\xi \equiv \tilde{x}$$

and we have:

$$\frac{\partial}{\partial \tilde{y}} = \frac{1}{\tilde{\delta}} \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial \xi} - \frac{1}{\tilde{\delta}} \frac{d\tilde{\delta}}{d\xi} \eta \frac{\partial}{\partial \eta}$$

and Eq. (47) now becomes:

$$\Theta^*_{\eta\eta} = P_r E \tilde{\delta} \left\{ 2\eta \left[\tilde{\delta}^2 \zeta\right] \Theta^*_\xi - \eta^2 \Theta^*_\eta \frac{d}{d\xi} \left[\tilde{\delta}^2 \zeta\right] \right\} \qquad 48)$$

Now if we define $J(\tilde{x}) = J(\xi)$ by:

$$J \equiv - \frac{X_e}{X_{e_\infty}} V^*_r$$

It follows from the equation preceding (45):

$$\zeta \tilde{\delta}^2 = \int_0^\xi J d\xi + \frac{2}{3} \frac{1}{E} \frac{\rho_g}{\rho} (\sin 2\xi) \tilde{\delta}^3$$

Now using this in Eq. (48) and neglecting terms of the order of $2P_r \frac{\rho_g}{\rho}$ we get:

$$\Theta^*_{\eta\eta} = P_r E \tilde{\delta} \left\{ 2\eta \Theta^*_\xi \int_0^\xi J d\xi - \eta^2 \Theta^*_\eta J \right\} \qquad 49)$$

and from the last condition (15c) and (16c) we have:

$$\Theta^*_\eta = -P_r E \tilde{\delta} J \Theta^* \qquad \text{at } \eta = 1 \qquad 50)$$

$$\Theta^* = 1 \qquad \text{at } \eta = 0 \qquad 51)$$

We now notice that $\tilde{\delta}$ must be an even function of ξ, J must be an even function of ξ and $\int_0^\xi J d\xi$ must be an odd function of ξ. We therefore set:

$$\left.\begin{aligned} \tilde{\delta} J &= \sum_{n=0}^{\infty} \frac{\alpha_{2n}}{(2n)!} \xi^{2n} \\ \tilde{\delta} \int_0^\xi J d\xi &= \sum_{n=1}^{\infty} \frac{\beta_{2n}}{(2n)!} \xi^{2n-1} \end{aligned}\right\} \qquad 52)$$

and:

$$\Theta^* = \sum_{n=0}^{\infty} \frac{F_{2n}}{(2n)!} \xi^{2n} \qquad 53)$$

where as usual we take $0! = 1$

Next differentiating the second of Eqs. (52) with respect to ξ and setting $\xi = 0$ we get upon comparison with the first Eq. (52)

$$\beta_2 = \alpha_o \qquad 54)$$

Substitution of Eqs. (53) and (52) into Eq. (49) and equating the coefficients of like powers of ξ yields after the calculation performed in Appendix X:

$$0 \leq \eta \leq 1 \qquad K_k (F_{2k}) = G_k \frac{1}{\alpha_o} \quad k = 0,1,2, \ldots \qquad 55)$$

Where:

$$K_k \equiv \frac{1}{P_r E\ \alpha_o} \frac{d^2}{d\eta^2} + \eta^2 \frac{d}{d\eta} - 4k\eta$$

$$G_k \begin{cases} = 0 & k = 0 \\ = -\alpha_2\ \eta^2 F_o' & k = 1 \\ = \eta \sum_{n=1}^{k-1} \left[2\binom{2k}{2n-1} F_{2n}\ \beta_{2(k-n+1)} - \binom{2k}{2n} \eta F_{2n}'\ \alpha_{2(k-n)} \right] & \\ \qquad -\alpha_{2k}\ \eta^2\ F_o' & k > 1 \end{cases}$$

In obtaining these equations the binomial coefficients which are written as

$$\binom{m}{n}$$

have been introduced and the relation

$$\binom{m}{n} = \frac{m!}{(m-n)!\ n!}$$

has been used.

Substitution of the expansion (52) and (53) into the boundary condition (50) yields after equating the coefficients of like powers of ξ

$$\eta = 1 \quad ; \quad \frac{1}{P_r E} F_{2k}' = - \sum_{n=0}^{k} \binom{2k}{2n} \alpha_{2(k-n)} F_{2n} \qquad 56)$$

and the boundary condition (51) requires that

$$\eta = 0 \qquad F_{2n} \begin{cases} = 1 & n = 0 \\ = 0 & n > 0 \end{cases} \tag{57}$$

Now the heat flux at the wall q_w is given by:*

$$q_w = -\frac{k}{R_o} \frac{\partial T}{\partial y}\bigg|_{y=0}$$

and substitution of θ^*, η, and $\tilde{\delta}$ yields:

$$q_w = -\frac{k(T_w - T_\infty)}{2R_o} \sqrt{\frac{2R_oU_\infty}{\nu}} \frac{1}{\tilde{\delta}} \frac{\partial \theta^*}{\partial \eta}\bigg|_{\eta=0}$$

and the normalized Nusselt number is:*

$$\frac{N_u}{\sqrt{\frac{2U_\infty R_o}{\nu}}} = \frac{1}{\tilde{\delta}} \frac{\partial \theta^*}{\partial \eta}\bigg|_{\eta=0}$$

or substituting in the expansion (53) yields

$$\frac{N_u}{\sqrt{\frac{2U_\infty R_o}{\nu}}} = \frac{1}{\tilde{\delta}} \sum_{n=0}^{\infty} \frac{F'_{2n}(0)}{(2n)!} \xi^{2n}$$

Next the wall shear stress can be written as in the derivations of Eqs. (36):

$$\tau_w = \frac{\mu}{R_o^2} \frac{\partial^2 \psi}{\partial y^2}\bigg|_{y=0}$$

and introducing $\tilde{\psi}$ and $\tilde{y}$ we get:

*C.f. derivation of Eq. (37).

$$\frac{\tau_w \sqrt{\frac{2U_\infty R_o}{\nu}}}{1/2\rho U_\infty^2} = E \left.\frac{\partial^2 \tilde{\psi}}{\partial \tilde{y}^2}\right|_{\tilde{y}=0}$$

and using Eqs. (44) and (46) we have

$$\frac{\tau_w \sqrt{\frac{2U_\infty R_o}{\nu}}}{1/2\rho U_\infty^2} = 2E\,\tau_{eff.} + 4\,\frac{\rho_g}{\rho}\,\tilde{\delta}\sin 2\tilde{x}$$

Next we have:*

$$u = \frac{1}{R_o}\frac{\partial \psi}{\partial y}$$

and introducing $\tilde{\psi}$ and $\tilde{y}$ we get:

$$\frac{u}{U_\infty} = \frac{E}{2}\frac{\partial \tilde{\psi}}{\partial \tilde{y}}$$

using Eqs. (44) and (46) we arrive at:

$$\frac{u}{U_\infty} = E\,\tau_{eff}\,\tilde{y} + \frac{\rho_g}{\rho}(\sin.2\tilde{x})\,(2\tilde{\delta} - \tilde{y})\tilde{y}$$

and finally summarizing the results of this section we have:

$$\tilde{\delta} = \frac{E}{4(\rho_g/\rho)}\,\frac{\tau_{eff.}}{(\sin.2\tilde{x})}\left(2\cos.\frac{\gamma}{3} - 1\right)$$

$$\gamma = \cos^{-1}\left\{-\left[1 - \frac{24\left(\frac{\rho_g}{\rho}\right)^2}{E^2}\,\frac{(\sin.^2 2\tilde{x})\dot{m}}{\tau^3_{eff}}\right]\right\}$$

$$\frac{u}{U_\infty} = E\,\tau_{eff}\,\tilde{y} + \frac{\rho_g}{\rho}(\sin.2\tilde{x})\,(2\tilde{\delta} - \tilde{y})\tilde{y}$$

$$\frac{\tau_w \sqrt{\frac{2U_\infty R_o}{\nu}}}{1/2\rho U_\infty^2} = 2E\,\tau_{eff.} + 4\,\frac{\rho_g}{\rho}\,\tilde{\delta}\sin.2\tilde{x}$$

---

*C.f. derivation of Eq. (35).

$$\frac{N_u}{\sqrt{\frac{2U_\infty R_o}{\nu}}} = \frac{1}{\delta} \sum_{n=0}^{\infty} \frac{F'_{2n}(0)}{(2n)!} \xi^{2n}$$

Where the F_{2n}'s are determined by:

$$\left.\begin{array}{lll} 0 \leq \eta \leq 1 & ; & K_k(F_{2k}) = \frac{1}{\alpha_o} G_k \\ \eta = 1 & ; & \frac{1}{P_r E} F''_{2k} = \sum_{n=0}^{k} \binom{2k}{2n} \alpha_{2(k-n)} F_{2n} \\ \eta = 0 & ; & F_{2n} \begin{cases} = 1 & n = 0 \\ = 0 & n > 0 \end{cases} \end{array}\right] \quad k = 0,1,2,3, \ldots$$

$$J \equiv - \frac{X_e}{X_{e\infty}} V_r^*$$

$$\dot{m} \equiv - \int_0^{\tilde{x}} \frac{X_e}{X_{e_\infty}} V_r^* \, d\tilde{x}$$

$$\tilde{\delta} J = \sum_{n=0}^{\infty} \frac{\alpha_{2n}}{(2n)!} \xi^{2n}$$

$$\tilde{\delta} \dot{m} = \sum_{n=1}^{\infty} \frac{\beta_{2n}}{(2n)!} \xi^{2n-1}$$

$$\tau_{eff} \equiv \frac{4}{E} \frac{\rho_g}{\rho} \sqrt{\frac{\nu_g}{\nu}} Z(\tilde{x}) + JV_\varphi^*$$

$$G_k \begin{cases} = 0 \\ = - \alpha_2 \eta^2 F'_o \\ = \eta \sum_{n=1}^{k-1} \left[2 \binom{2k}{2n-1} F_{2n} \beta_{2(k-n+1)} - \binom{2k}{2n} \eta F'_{2n} \alpha_{2(k-n)} \right] \end{cases}$$

$$- \alpha_{2k} \eta^2 F'_o$$

$$K_k \equiv \frac{1}{P_r E \alpha_o} \frac{d^2}{d\eta^2} + \eta^2 \frac{d}{d\eta} - 4k\eta$$

$$Z(\tilde{x}) = \gamma_1 \tilde{x} - \sum_{n=2}^{\infty} \frac{(-1)^n (2n)}{(2n-1)!} \gamma_{(2n-1)} \tilde{x}^{2n-1}$$

γ_1	γ_3	γ_5	γ_7	γ_9	γ_{11}
1.2326	0.7244	1.0320	2.036833	.280140	67.637501

These formulas can now be used to calculate the desired physical quantities once J, $\dot{m}$, and V_φ^* are known as functions of $\tilde{x}$. These quantities have been obtained numerically (see Appendix XXV) and the results are tabulated for various values of the Reynolds number and drop size. The numerical calculations are described in Appendix XXVI.

Now the problem of the oscillating flat plate will be discussed.

C. Semi-Infinite Oscillating Flat Plate in a Streaming Gas/Liquid Flow with Oscillations Parallel to the Flow

1. FORMULATION

Now a semi-infinite oscillating flat plate which is maintained at a constant temperature T_w in a streaming gas which contains liquid drops will be considered. The plate is taken to be parallel to the velocity of the gas far upstream from it and to be oscillating sinusoidally in its plane with a small amplitude $U_\infty \frac{\epsilon}{2}$. Now we will use the flow model described above in the section entitled "Preliminary Consideration" and assume from the outset that the boundary layer assumptions hold with sufficient accuracy within the liquid film. The boundary layer approximations will hold with good accuracy in the film if it is sufficiently thin. But since the liquid capture area of the film in this case is only proportional to its thickness, whereas in the case of the cylinder it is equal to the cylinder cross sectional area one can be sure that the film will be thin. Next since the pressure field due to the presence of the plate is only a second order effect and the gas flow outside of the boundary layer is essentially parallel, it will further be assumed that the drop trajectories are straight lines and the volume fraction X_e of the liquid outside the film is everywhere the same.

The liquid film will now be considered. We set up an x, y coordinate system which is fixed to the plate. The velocity of the free stream U(t) relative to these coordinates is then U_∞ minus the velocity of oscillation of the plate which is taken as $-\frac{\epsilon}{2} U_\infty \sin. \omega t$

Thus:

$$U(t) = U_\infty \left(1 + \frac{\epsilon}{2} \sin. \omega t\right) \qquad 59)$$

Attention is now focused upon the control volume of Figure 5 which is bounded by the liquid free surface S', the surface perpendicular to the plate S'' and by the plate itself. Its cross sectional area is $A_{c.s.}$ If the x and y components of the velocity of the surface S' are $V_{c.s}^{(x)}$

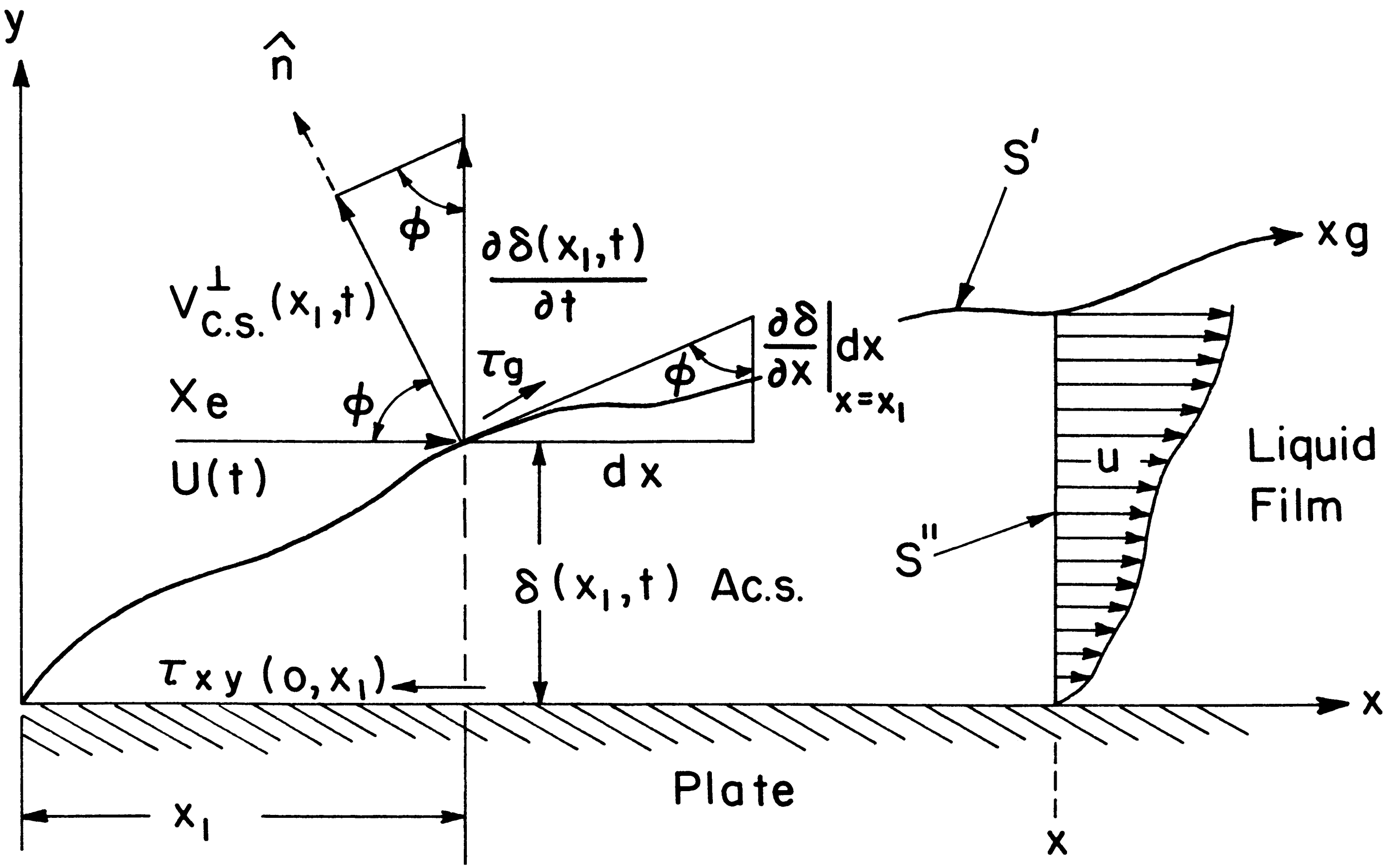

Figure 5. Coordinates and control volume for liquid layer on flat plate.

and $V^{(y)}_{c.s.}$, respectively, then they must be related to δ by:*

$$V^{(y)}_{c.s.} = \frac{\partial\delta}{\partial t} + V^{(x)}_{c.s.} \frac{\partial\delta}{\partial x}$$

and they must be related to the velocity of the control surface perpendicular to itself $V^{\perp}_{c.s.}$ by:

$$V^{\perp}_{c.s.} = - V^{(x)}_{c.s.} \cos.\varphi + V^{(y)}_{c.s.} \sin\varphi$$

$$= - V^{(x)}_{c.s.} \cos.\varphi + V^{(x)}_{c.s.} \frac{\partial\delta}{\partial x} \sin\varphi + \frac{\partial\delta}{\partial t} \sin\varphi$$

$$= \left[V^{(x)}_{c.s.} \left(\frac{\partial\delta}{\partial x} - \text{cotan}.\varphi \right) + \frac{\partial\delta}{\partial x} \right] \sin\varphi$$

but

$$\frac{\partial\delta}{\partial x} = \text{cotan}\varphi$$

and:

$$V^{\perp}_{c.s.} = \frac{\partial\delta}{\partial t} \sin\varphi \qquad 60)$$

Thus $V^{\perp}_{c.s.}$ is dependent only on $\frac{\partial\delta}{\partial t}$ and independent of what motion we may apply to the control surface along its length which is as it should be. The law of conservation of mass for the control volume may be written as:

---

*Note that:

$$\frac{d\delta}{dt} = V^{(y)}_{c.s.} = \frac{\partial\delta}{\partial t} + \frac{\partial\delta}{\partial S} \frac{\partial S}{\partial t} = \frac{\partial\delta}{\partial t} + \frac{\partial\delta}{\partial x} \frac{\partial x}{\partial S} \frac{\partial S}{\partial t}$$

$$= \frac{\partial\delta}{\partial t} + \frac{\partial\delta}{\partial x} \frac{dx}{dt}$$

$$\frac{d}{dt}\iint_{A_{c.s.}} \rho \, dA + \int_{s'} \rho X_e \left[-U \cos\varphi - V^{\perp}_{c.s.}\right] ds' + \int_{s''} \rho \, u \, ds'' = 0 \qquad 61)$$

And remembering that an accelerated coordinate system is being used the law of conservation of x-direction momentum may be written as:

$$\frac{d}{dt}\iint_{A_{c.s.}} \rho \, u \, dA + \int_{s'} \rho X_e U \left[-U \cos\varphi - V^{\perp}_{c.s.}\right] ds' + \int_{s''} \rho \, u^2 \, ds'' = -\int_0^x \tau_{x,y}(x_1,0,t)dx_1 + \iint_{A_{c.s.}} \rho \frac{dU}{dt} dy + \int_{s'} \tau_g \sin\varphi \, ds' \qquad 62)$$

where the pressure forces have been neglected since it is assumed that the boundary layer approximations apply. And finally an energy balance on the control volume is performed. In doing this, we will neglect a priori the effects due to kinetic energy changes, and assume that the temperature of the drops is not significantly reduced from that of the free stream when they pass through the gas boundary layer.

With these approximations one obtains:

$$\frac{d}{dt}\iint_{A_{c.s.}} \rho \, C_p T \, dA + \int_{s'} \rho X_e C_p T_\infty \left[-U \cos\varphi - V^{\perp}_{c.s.}\right] ds'$$

$$+\int_{s''} \rho C_p Tu \, ds'' = \int_0^x k \frac{\partial T}{\partial y}\bigg|_{y=0} dx_1 + $$

$$+\int_{s'} q_g \, ds' \qquad 63)$$

We next notice that:

$$\frac{\partial \delta}{\partial x} = \text{cotan.}\varphi$$

$$ds' = \sqrt{1 + \left(\frac{\partial \delta}{\partial x}\right)^2} \, dx = \sqrt{1 + \text{cotan}^2 \varphi} \, dx$$

$$= \frac{dx}{\sin\varphi}$$

$$dA = dy \, dx_1$$

$$ds'' = dy$$

and by using these and Eqs. (60) in Eqs. (61) through (63), one obtains:

$$\frac{d}{dt}\int_0^x \int_0^{\delta(x_1,t)} dy \, dx_1 - X_e \int_0^x \left[\frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x}\right] dx_1$$

$$+\int_0^{\delta} u(x,y,t) \, dy = 0 \qquad 61A)$$

$$\rho X_e \int_0^x U \left[\frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x}\right] dx_1 + \int_0^x \tau_g \, dx_1$$

$$= \rho \frac{d}{dt}\int_0^x \int_0^{\delta(x_1,t)} u \, dy \, dx_1 + \rho \int_0^{\delta} u^2(x,y,t) dy$$

$$- \rho \int_0^x \int_0^{\delta(x_1,t)} \frac{dU}{dt}\, dy + \int_0^x \tau_{xy}(x_1 0,t)\, dx_1 \qquad 62A)$$

$$\rho X_e C_p T \int_0^x \left[\frac{\partial\delta}{\partial t} + U\frac{\partial\delta}{\partial x}\right] dx_1 + \int_0^x \frac{q_g}{\sin\varphi}\, dx_1$$

$$= \rho C_p \int_0^{\delta(x,t)} T(x,y,t)\, u(x,y,t) dy + \rho C_p \frac{d}{dt}\int_0^x \int_0^{\delta(x_1,t)} T\, dy\, dx_1$$

$$- k \int_0^x \left.\frac{\partial T}{\partial y}\right|_{y=0} dx_1 \qquad 63A)$$

Now differentiating (61A) through (63A) with respect of x yields:

$$\frac{\partial\delta}{\partial t}\int_0^{\delta(x,t)} dy - X_e\left[\frac{\partial\delta}{\partial t} + U\frac{\partial\delta}{\partial x}\right] + \int_0^{\delta(x,t)} \frac{\partial u}{\partial x}\, dy$$

$$+ u(x,\delta,t)\frac{\partial\delta}{\partial x} = 0$$

$$\rho X_e U\left[\frac{\partial\delta}{\partial t} + U\frac{\partial\delta}{\partial x}\right] + \tau_g = \rho\frac{\partial}{\partial t}\int_0^{\delta(x,t)} u\, dy$$

$$+ \rho\int_0^{\delta(x,t)} 2u\frac{\partial u}{\partial x}\, dy + \rho u^2(x,\delta,t)\frac{\partial\delta}{\partial x} + \tau_{x,y}\;(x,0,t)$$

$$-\rho\int_0^{\delta(x,t)} \frac{dU}{dt}\, dy$$

$$\rho X_e C_p T_\infty\left[\frac{\partial\delta}{\partial t} + U\frac{\partial\delta}{\partial x}\right] + \frac{q_g}{\sin\varphi} = \rho C_p\int_0^{\delta(x,t)} \frac{\partial(ut)}{\partial x}\, dy$$

$$+ \rho C_p T(x,\delta,t)\, u(x,\delta,t)\frac{\partial\delta}{\partial x} + \rho C_p\frac{\partial}{\partial t}\int_0^{\delta(x,t)} T\, dy$$

$$- k\frac{\partial T}{\partial y}\;(x,0,t)$$

or performing the integrations with respect to t:

$$\left[\frac{\partial\delta}{\partial t} + u(x,\delta,t)\frac{\partial\delta}{\partial x}\right] = X_e\left[\frac{\partial\delta}{\partial t} + U\frac{\partial\delta}{\partial x}\right] - \int_0^{\delta(x,t)}\frac{\partial u}{\partial x}\,dy$$

$$\rho X_e U\left[\frac{\partial\delta}{\partial t} + U\frac{\partial\delta}{\partial x}\right] - \rho u(x,\delta,t)\left[\frac{\partial\delta}{\partial t} + u(x,\delta,t)\frac{\partial\delta}{\partial x}\right] + \tau_g =$$

$$\rho\int_0^{\delta(x,t)} 2u\frac{\partial u}{\partial x}\,dy + \rho\int_0^{\delta(x,t)}\frac{\partial u}{\partial t}\,dy + \tau_{x,y}(x,0,t) - \rho\int_0^{\delta(x,t)}\frac{dU}{dt}\,dy$$

$$\rho X_e C_p T_\infty\left[\frac{\partial\delta}{\partial t} + U\frac{\partial\delta}{\partial x}\right] - \rho C_p T(x,\delta,t)\left[\frac{\partial\delta}{\partial t} + u(x,\delta,t)\frac{\partial\delta}{\partial x}\right] +$$

$$+ \frac{q_g}{\sin\varphi} = \rho C_p\int_0^{\delta(x,t)}\left[u\frac{\partial T}{\partial x} + \frac{\partial T}{\partial t}\right]dy + \rho C_p\int_0^{\delta(x,t)} T\frac{\partial u}{\partial x}\,dy$$

$$- k\frac{\partial T}{\partial y}(x,0,t)$$

and finally using the first equation to eliminate $\left[\frac{\partial\delta}{\partial t} + u(x,\delta,t)\frac{\partial\delta}{\partial x}\right]$ from the last two we obtain:

$$\rho X_e\left[U - u(x,\delta,t)\right]\left[\frac{\partial\delta}{\partial t} + U\frac{\partial\delta}{\partial x}\right] + \tau_g = + \tau_{xy}(x,0,t)$$

$$\rho\int_0^{\delta(x,t)}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right]dy + \rho\int_0^{\delta(x,t)} u\frac{\partial u}{\partial x}\,dy - \rho u(x,\delta,t)\int^{\delta(x,t)}\frac{\partial u}{\partial x}\,dy$$

$$- \rho\int_0^{\delta(x,t)}\frac{dU}{dt}\,dy$$

$$\rho X_e C_p\left[T_\infty - T(x,\delta,t)\right]\left[\frac{\partial\delta}{\partial t} + U\frac{\partial\delta}{\partial x}\right] + \frac{q_g}{\sin\varphi} = - k\frac{\partial T}{\partial y}(x,0,t)$$

$$\rho C_p \int_0^{\delta(x,t)} \left[u \frac{\partial T}{\partial x} + \frac{\partial T}{\partial t} \right] dy + \rho C_p \int_0^{\delta(x,t)} T \frac{\partial u}{\partial x} dy - \rho C_p T(x,\delta,t) \int_0^{\delta(x,t)} \frac{\partial u}{\partial x} dy$$

In order to proceed further the continuity equation is now introduced, i.e.:

$$\frac{\partial u}{\partial x} = - \frac{\partial v}{\partial y}$$

By using this in the last two integrals on the right of the last two equations and performing an integration by parts one obtains:

$$\rho X_e \left[U - u(x,\delta,t) \right] \left[\frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \tau_g =$$

$$\int_0^{\delta(x,t)} \rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] dy + \tau_{xy}(x,0,t) - \rho \int_0^{\delta(x,t)} \frac{dU}{dt} dy \qquad 64)$$

$$\rho C_p X_e \left[T_\infty - T(x,\delta,t) \right] \left[\frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \frac{q_g}{\sin.\varphi} =$$

$$\int_0^{\delta(x,t)} \rho C_p \left[\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] dy - k \frac{\partial T}{\partial y} (x,0,t) \qquad 65)$$

Since the liquid in the film is assumed to satisfy the boundary layer approximations, (and since for flow over a flat plate it is consistent with these to neglect the pressure forces) since we are using an accelerated coordinate system, the governing differential equations become:

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = \frac{dU}{dt} + \mu \frac{\partial^2 u}{\partial y^2} \qquad 66)$$

$$\rho C_p \left[\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = k \frac{\partial^2 T}{\partial y^2} \qquad 67)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad 68)$$

and the shear stress $\tau_{xy}(x,0,t)$ is given by:

$$\tau_{xy}(x,0,t) = -\mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \qquad 69)$$

Using Eqs. (66) and (67), and (69) in Eqs. (64) and (65) and performing the integrations we obtain:

$$\left.\begin{array}{l} y = \delta \\ 0 < x \end{array}\right\{ \left[\begin{array}{l} \rho X_e \left[U - u\right]\left[\frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x}\right] + \tau_g = \mu \frac{\partial u}{\partial y} \\ \rho X_e C_p \left[T_\infty - T\right]\left[\frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x}\right] + \frac{q_g}{\sin\varphi} = k \frac{\partial T}{\partial y} \end{array}\right] \qquad 70)$$

Next we introduce the steam function ψ defined by:

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \qquad 71)$$

which satisfies equation (68) identically.

Now the boundary conditions at the plate are:

$$T = T_w \quad ; \quad u = v = 0 \quad ; \quad y = 0 \quad ; 0 < x$$

the last two of which will be satisfied by taking:

$$\psi = \frac{\partial \psi}{\partial y} = 0 \quad ; \quad y = 0, \quad 0 < x \qquad 72)$$

Using (71) in Eq. (61A) gives:

$$\frac{d}{dt} \int_0^x \int_0^{\delta(x_1,t)} dy\, dx_1 - X_e \int_0^x \left[\frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x_1}\right] dx_1$$

$$+\int_0^{\delta(x,t)} \frac{\partial\psi}{\partial y}\, dy = 0$$

and performing the integrations and using the boundary condition (72) gives

$$\frac{d}{dt}\int_0^x \delta(x_1,t)dx_1 - X_e \int_0^x \frac{\partial\delta}{\partial t}\, dx_1 - X_e\delta(x,t)U + \psi(x,\delta,t) = 0$$

or

$$\text{at } y = \delta \;;\; 0 < x \;;\; \psi = X_e U_\infty \delta + (X_e - 1)\int_0^x \frac{\partial\delta}{\partial t}\, dx_1 \qquad 73)$$

Now using Eqs. (71) and (59) in Eqs. (66), (67), (70) and (73) yields:

$$\begin{array}{l} 0\le y\le\delta \\ 0<x \end{array}\left\{\begin{array}{l} \dfrac{\partial^2\psi}{\partial t\partial y} + \dfrac{\partial\psi}{\partial y}\dfrac{\partial^2\psi}{\partial x\partial y} - \dfrac{\partial\psi}{\partial x}\dfrac{\partial^2\psi}{\partial y^2} = \dfrac{U_\infty\epsilon\omega}{2}\cos\omega t + \nu\dfrac{\partial^3\psi}{\partial y^3} \\[2ex] \dfrac{\partial T}{\partial t} + \dfrac{\partial\psi}{\partial y}\dfrac{\partial T}{\partial x} - \dfrac{\partial\psi}{\partial x}\dfrac{\partial T}{\partial y} = \dfrac{k}{\rho C_p}\dfrac{\partial^2 T}{\partial y^2} \end{array}\right\} \qquad 74)$$

$$\begin{array}{l} y = \delta \\ 0 < x \end{array}\left\{\begin{array}{l} \psi = X_e\delta\, U_\infty(1 + \frac{\epsilon}{2}\sin\omega t) + (X_e - 1)\displaystyle\int_0^x \frac{\partial\delta}{\partial t}\, dx_1 \\[2ex] X_e\left[\dfrac{\partial\psi}{\partial y} - U_\infty(1 + \frac{\epsilon}{2}\sin\omega t)\right]\left[\dfrac{\partial\delta}{\partial t} + U_\infty(1 + \frac{\epsilon}{2}\sin\omega t)\dfrac{\partial\delta}{\partial x}\right] \\[2ex] -\dfrac{\tau_g}{\rho} = -\nu\dfrac{\partial^2\psi}{\partial y^2} \\[2ex] X_e\left[T_\infty - T\right]\left[\dfrac{\partial\delta}{\partial t} + U_\infty(1 + \frac{\epsilon}{2}\sin\omega t)\dfrac{\partial\delta}{\partial x}\right] + \dfrac{q_g}{\rho C_p\sin\varphi} \\[2ex] = \dfrac{k}{\rho C_p}\dfrac{\partial T}{\partial y} \end{array}\right\} \qquad 75)$$

$$y = 0,\ 0 < x \quad \left\{ T = T_w \quad ; \quad \psi = \frac{\partial \psi}{\partial y} = 0 \right\} \tag{76}$$

These equations and boundary conditions would completely determine the problem of the liquid film if τ_g and q_g were known. Before discussing these, the following dimensionless variables are introduced:

$$\eta \equiv y/\delta$$

$$\xi \equiv \sqrt{\frac{\omega x}{X_e U_\infty}}$$

$$\psi^* \equiv \frac{\psi}{U_\infty X_e} \sqrt{\frac{\omega}{X_e \nu}}$$

$$\delta^* \equiv \delta \sqrt{\frac{\omega}{\nu X_e}} \tag{77}$$

$$\tau \equiv \omega t$$

$$\Theta^* \equiv \frac{T - T_\infty}{T_w - T_\infty}$$

It can be verified from the results of the calculations which this analysis leads to, that the terms of the Eqs. (73) through (76) expressed in terms of these new variables are of order unity. Now employing the chain rule, we get:

$$\frac{\partial}{\partial x} = \left(\frac{\partial \eta}{\partial x}\right)_{y,t} \frac{\partial}{\partial \eta} + \left(\frac{\partial \xi}{\partial x}\right)_{y,t} \frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial t} = \left(\frac{\partial \eta}{\partial t}\right)_{x,y} \frac{\partial}{\partial \eta} + \left(\frac{\partial \tau}{\partial t}\right)_{x,y} \frac{\partial}{\partial \tau}$$

$$\frac{\partial}{\partial y} = \left(\frac{\partial \eta}{\partial y}\right)_{x,t} \frac{\partial}{\partial \eta}$$

or

$$\left[\begin{array}{l} \dfrac{\partial}{\partial x} = \dfrac{\omega}{2X_e U_\infty}\left\{\left[\dfrac{1}{\xi}\dfrac{\partial}{\partial \xi} - \dfrac{1}{\delta^{*}\xi}\dfrac{\partial \delta^{*}}{\partial \xi}\,\eta\,\dfrac{\partial}{\partial \eta}\right]\right\} \\ \dfrac{\partial}{\partial t} = \omega\left\{\left[\dfrac{\partial}{\partial \tau} - \dfrac{1}{\delta^*}\dfrac{\partial \delta^*}{\partial \tau}\,\eta\,\dfrac{\partial}{\partial \eta}\right]\right\} \\ \dfrac{\partial}{\partial y} = \sqrt{\dfrac{\omega}{\nu X_e}}\,\dfrac{1}{\delta^*}\,\dfrac{\partial}{\partial \eta} \end{array}\right] \qquad 78)$$

and:

$$\left.\begin{array}{l} \dfrac{\partial^2}{\partial x \partial y} = \dfrac{1}{2}\,\dfrac{\omega}{X_e U_\infty}\sqrt{\dfrac{\omega}{\nu X_e}}\left\{\left[\dfrac{1}{\xi\delta^*}\,\dfrac{\partial^2}{\partial \xi \partial \eta} - \dfrac{1}{\delta^{*2}}\,\dfrac{\partial \delta^*}{\partial \xi}\,\dfrac{1}{\xi}\,\dfrac{\partial}{\partial \eta}\,\eta\,\dfrac{\partial}{\partial \eta}\right]\right\} \\ \dfrac{\partial^2}{\partial t \partial y} = \omega\sqrt{\dfrac{\omega}{\nu X_e}}\left\{\left[\dfrac{1}{\delta^*}\,\dfrac{\partial^2}{\partial \tau \partial \eta} - \dfrac{1}{\delta^*}\,\dfrac{\partial \delta^*}{\partial \tau}\,\dfrac{\partial}{\partial \eta}\,\eta\,\dfrac{\partial}{\partial \eta}\right]\right\} \\ \dfrac{\partial^3}{\partial y^3} = \dfrac{\omega}{\nu X_e}\sqrt{\dfrac{\omega}{\nu X_e}}\,\dfrac{1}{\delta^{*3}}\,\dfrac{\partial^3}{\partial \eta^3} \end{array}\right] \qquad 79)$$

Rearranging the second of Eqs. (75) one obtains:

$$\left(U_\infty - \frac{\partial \psi}{\partial y}\right)\left(\frac{\partial \delta}{\partial t} + U_\infty \frac{\partial \delta}{\partial x}\right) + \frac{\epsilon}{2}\,U_\infty \sin \omega t\left[\frac{\partial \delta}{\partial t} + 2U_\infty \frac{\partial \delta}{\partial x} - \frac{\partial \delta}{\partial x}\,\frac{\partial \psi}{\partial y}\right] + \frac{\epsilon^2}{4}\,U_\infty^2\,\frac{1}{2}(1 - \cos 2\omega t)\,\frac{\partial \delta}{\partial x} = \frac{\nu}{X_e}\,\frac{\partial^2 \psi}{\partial y^2} - \frac{\tau_g}{\rho X_e} \qquad \text{at } y = \delta$$

where the identity:

$$\sin^2 \omega t = 1/2\,(1 - \cos.\ 2\omega t)$$

has been used.

Using the relations (77) and (79) in this, one obtains:

$$\text{at } \eta = 1 \; ; \; \left(1 - \frac{X_e}{\delta^*}\frac{\partial\psi^*}{\partial\eta}\right)\left(X_e \frac{\partial\delta^*}{\partial\tau}\xi + \frac{1}{2}\frac{\partial\delta^*}{\partial\xi}\right) +$$

$$+ \frac{\epsilon}{2}\sin\tau \left(X_e\, \xi \frac{\partial\delta^*}{\partial\tau} + \frac{\partial\delta^*}{\partial\xi} - \frac{X_e}{2\delta^*}\frac{\partial\delta^*}{\partial\xi}\frac{\partial\psi^*}{\partial\eta}\right)$$

$$+ \frac{\epsilon^2}{16}(1 - \cos 2\tau)\frac{\partial\delta^*}{\partial\xi} = \frac{1}{\delta^{*2}}\frac{\partial^2\psi^*}{\partial\eta^2} -$$

$$- \frac{\tau_g}{X_e\rho_g U_\infty^2}\sqrt{\frac{U_\infty x}{\nu_g}}\left[\sqrt{\frac{\nu_g}{\nu}\frac{\rho_g}{\rho}}\right]$$

Now the order of magnitude of the last term on the right of this equation can be estimated by evaluating it for a single component gas boundary layer and evaluating the property ratios for a particular liquid/gas combination. Since the principle interest is in air/water mixtures, we will limit ourselves to these. Thus from Appendix II we have:

$$\frac{\tau_g}{\rho_g U_\infty^2}\sqrt{\frac{U_\infty x}{\nu}}\left[\sqrt{\frac{\nu_g}{\nu}\frac{\rho_g}{\rho}}\right] \cong 1.1 \times 10^{-3}$$

This term will be taken to be negligible if it is less than 0.1 since the term $\frac{1}{2}\frac{\partial\delta^*}{\partial\xi} \geq 1$ for all values of X_e as shown by the numerical calculations. And the above results shown for air/water mixtures will be less than 0.1 if:

$$X_e > 1.1 \times 10^{-2}$$

We therefore limit ourselves to values of X_e greater than this, and neglect the gas shear stress term, and we have:*

$$\delta^* (\delta^* - X_e \psi^*_\eta)(X_e \xi \delta^*_\tau + 1/2\, \delta^*_\xi) + \frac{\epsilon}{2} \delta^* \Big[\delta^*(X_e \xi \delta^*_\tau + \delta^*_\xi) - \frac{X_e}{2} \delta^*_\xi \psi^*_\eta\Big] \sin\tau + \frac{\epsilon^2}{16} \delta^{*2} \delta^*_\xi (1 - \cos 2\tau) = \xi \psi^*_{\eta\eta}$$

.... For $\eta = 1$; $0 < \xi^*$ 80)

Next using the relations (77) and (79) in the third of Eqs. (75), one obtains:

at $\eta = 1$

$$(T - T_\infty) \left[X_e \xi \frac{\partial \delta^*}{\partial \tau} + \frac{1}{2}\left(1 + \frac{\epsilon}{2} \sin \tau\right) \frac{\partial \delta^*}{\partial \xi}\right] = - \frac{1}{X_e P_r} \frac{\xi}{\delta^*} \frac{\partial (T-T_\infty)}{\partial \eta} + \frac{1}{X_e}\left[\left(\frac{q_g}{T-T_\infty}\right) \frac{x}{k_g} \Big/ \sqrt{\frac{U_\infty x}{\nu_g}} \left(\frac{\mu_g C_{pg}}{k_g}\right)\right]\left[\sqrt{\frac{\nu_g}{\nu}} \frac{\rho_g}{\rho} \frac{C_{pg}}{C_p}\right] \frac{(T-T_\infty)}{\sin\varphi}$$

We again estimate the order of magnitude of the last term in this equation for air/water mixture by calculating its value from the relations for a single component gas boundary layer. From Appendix II its value is then approximately:

$$\frac{1}{X_e} \quad 3.75 \times 10^{-4}\, (T - T_\infty)$$

---

*The subscript notation is used here to indicate partial derivatives i.e.,

$$\psi^*_\xi \equiv \frac{\partial \psi^*}{\partial \xi} \; ; \quad \delta^*_\xi \equiv \frac{\partial \delta^*}{\partial \xi} \; ; \quad \psi^*_\eta \equiv \frac{\partial \psi^*}{\partial \eta} \; ; \quad \psi^*_{\eta\eta} \equiv \frac{\partial^2 \psi^*}{\partial \eta^2} \; ; \quad \delta^*_\tau \equiv \frac{\partial \delta^*}{\partial \tau} \; ; \quad \text{etc.}$$

and thus it is negligible compared with the term:*

$$\frac{1}{2}\,\frac{\partial \delta^*}{\partial \xi}\,(T - T_\infty)$$

for $X_e > 10^{-2}$ which is the minimum value to be considered. Hence this term is neglected and we have

$$\text{at } \eta = 1 \;;\; 0 < \xi \;:\; \theta^* \delta^* \left[X_e\, \xi \frac{\partial \delta^*}{\partial \tau} + 1/2\,(1 + \frac{\epsilon}{2} \sin.\, \tau)\,\frac{\partial \delta^*}{\partial \tau} \right] = -\frac{1}{X_e P_r}\,\xi\,\frac{\partial \theta^*}{\partial \eta} \qquad 81)$$

Using the relations (77) and (78) in the first of Eqs. (75) yields:

$$\psi^* = \delta^*\,(1 + \frac{\epsilon}{2}\sin.\tau\,) + 2\,(X_e - 1)\int_0^\xi \frac{\partial \delta^*}{\partial\,\tau}\,\xi\, d\,\xi \;;\; \text{at } \eta = 1\;;$$

$$0 < \xi \qquad 82)$$

And using the relations (77), (78), and (79) in the first and second of Eqs. (74) gives:

$$X_e \left\{ \delta^{*2} \frac{\partial \psi^*}{\partial \tau \partial \eta} + \frac{1}{2}\,\frac{\delta^*}{\xi} \left[\frac{\partial^2 \psi^*}{\partial \xi \partial \eta}\,\frac{\partial \psi^*}{\partial \eta} - \frac{\partial \psi^*}{\partial \xi}\,\frac{\partial^2 \psi^*}{\partial \eta^2} \right] - \left[\frac{1}{2\xi}\,\frac{\partial \delta^*}{\partial \xi}\,\frac{\partial \psi^*}{\partial \eta} + \delta^* \frac{\partial \delta^*}{\partial \tau} \right] \frac{\partial \psi^*}{\partial \eta} - \eta \delta^* \frac{\partial \delta^*}{\partial \tau}\,\frac{\partial^2 \psi^*}{\partial \eta^2} \right\} = \qquad 82)$$

$$= \frac{\epsilon}{2}\,\delta^{*3} \cos\,\tau + \frac{\partial^3 \psi^*}{\partial \eta^3}$$

---

*Note that:

$$\sin.\varphi = \frac{1}{\sqrt{1 + \cot^2.\varphi}} = \frac{1}{\sqrt{1 + \left(\frac{\partial \delta}{\partial x}\right)^2}} \backsimeq 1$$

$$X_e P_r \left\{ \xi \delta^{*2} \frac{\partial \theta^*}{\partial \tau} + \frac{1}{2} \delta^* \left[\frac{\partial \theta^*}{\partial \xi} \frac{\partial \psi^*}{\partial \eta} - \frac{\partial \psi^*}{\partial \xi} \frac{\partial \theta^*}{\partial \eta} \right] - \right.$$

$$\left. - \xi \delta^* \frac{\partial \delta^*}{\partial \tau} \eta \frac{\partial \theta^*}{\partial \eta} \right\} = \xi \frac{\partial^2 \theta^*}{\partial \eta^2}$$

The boundary conditions (76) are in terms of (77):

$$\eta = 0 \quad ; \quad 0 < \xi \; : \quad \theta^* = 1, \quad \psi^* = \frac{\partial \psi^*}{\partial \eta} = 0 \qquad 83)$$

And finally collecting Eqs. (80) through (83) we have:

$$\begin{array}{l} 0 \le \eta \le 1 \\ 0 < \xi \end{array} \left\{ \begin{array}{l} X_e \left\{ \xi \delta^{*2} \dfrac{\partial^2 \psi^*}{\partial \tau \partial \eta} + \dfrac{1}{2} \delta^* \left[\dfrac{\partial^2 \psi^*}{\partial \xi \partial \eta} \dfrac{\partial \psi^*}{\partial \eta} - \dfrac{\partial \psi^*}{\partial \xi} \dfrac{\partial^2 \psi^*}{\partial \eta^2} \right] - \right. \\ \left. \dfrac{1}{2} \left[\dfrac{\partial \delta^*}{\partial \xi} \dfrac{\partial \psi^*}{\partial \eta} + 2 \xi \delta^* \dfrac{\partial \delta^*}{\partial \tau} \right] \dfrac{\partial \psi^*}{\partial \eta} - \eta \xi \delta^* \dfrac{\partial \delta^*}{\partial \tau} \dfrac{\partial^2 \psi^*}{\partial \eta^2} \right\} = \\ \dfrac{\epsilon}{2} \xi \delta^{*3} \cos \tau + \xi \dfrac{\partial^3 \psi^*}{\partial \eta^3} \\ X_e P_r \left\{ \xi \delta^{*2} \dfrac{\partial \theta^*}{\partial \tau} + \dfrac{1}{2} \delta^* \left[\dfrac{\partial \theta^*}{\partial \eta} \dfrac{\partial \psi^*}{\partial \eta} - \dfrac{\partial \psi^*}{\partial \xi} \dfrac{\partial \theta^*}{\partial \eta} \right] - \right. \\ \left. - \xi \delta^* \dfrac{\partial \delta^*}{\partial \tau} \eta \dfrac{\partial \theta^*}{\partial \eta} \right\} = \xi \dfrac{\partial^2 \theta^*}{\partial \eta^2} \end{array} \right\} \qquad 84)$$

$$\begin{array}{l} \eta = 1 \\ 0 < \xi \end{array} \left\{ \begin{array}{l} \psi^* = \delta^* \left(1 + \dfrac{\epsilon}{2} \sin \tau\right) + 2(X_e - 1) \displaystyle\int_0^{\xi} \dfrac{\partial \delta^*}{\partial \tau} \xi \, d\xi \\ \delta^* (\delta^* - X_e \psi^*_\eta)(X_e \xi \delta^*_\tau + \dfrac{1}{2} \delta^*_\xi) + \dfrac{\epsilon}{2} \delta^* \left[\delta^* (X_e \xi \delta^*_\tau + \right. \\ \left. + \delta^*_\xi) - \dfrac{X_e}{2} \delta^*_\xi \psi^*_\eta \right] \sin \tau + \dfrac{\epsilon^2}{16} \delta^{*2} \delta^*_\xi (1 - \cos 2\tau) = \\ \xi \psi^*_{\eta\eta} \end{array} \right\} \qquad 85)$$

$$\left[X_e P_r \theta^* \delta^* \left[X_e \xi \delta^*_\tau + \frac{1}{2}(1 + \frac{\epsilon}{2}\sin\tau)\,\delta^*_\xi \right] = -\ \xi \theta^*_\eta \right]$$

$$\begin{matrix} \eta = 0 \\ 0 < \xi \end{matrix} \quad \left\{ \quad \theta^* = 1 \quad ; \quad \psi^* = \frac{\partial \psi^*}{\partial \eta} = 0 \quad \right\} \qquad 86)$$

Equations (84) through (86) completely determine θ^*, δ^*, and ψ^* within the liquid film. The solution to this boundary value problem is now considered.

2. SOLUTION

First we remember that the amplitude of the oscillations is taken to be small and expand the solutions in powers of ϵ. Thus:

$$\left.\begin{aligned} \psi^* &= \psi^*_o + \epsilon\psi^*_1 + \epsilon^2\psi^*_2 + \ldots\ldots \\ \delta^* &= \delta^*_o + \epsilon\delta^*_1 + \epsilon^2\delta^*_2 + \ldots\ldots \\ \theta^* &= \theta^*_o + \epsilon\theta^*_1 + \epsilon^2\theta^*_2 + \ldots\ldots \end{aligned}\right\} \qquad 87)$$

We now substitute the first two of the assumed expansions (87) into the first of Eqs. (84), the coefficients of like powers of ϵ are equated, and after a rather lengthly calculation carried out in Appendix XI, we get:

$$\epsilon^o: \quad \xi\delta_o^{*2}\,\psi^*_{o\tau\eta} + \frac{1}{2}\delta^*_o\,(\psi^*_{o\xi\eta}\,\psi^*_{o\eta} - \psi^*_{o\eta\eta}\,\psi^*_{o\xi}) - \frac{1}{2}\,\delta^*_{o\xi}(\psi^*_{o\eta})^2$$

$$- \xi\delta^*_o\,\delta^*_{o\tau}\psi^*_{o\eta} - \xi\eta\delta^*_o\,\delta^*_{o\tau}\,\psi^*_{o\eta\eta} = \frac{1}{X_e}\,\xi\,\psi^*_{o\eta\eta\eta} \qquad 88\text{-}0)$$

$$\epsilon^1: \quad \xi\delta_o^{*2}\,\psi^*_{1\tau\eta} + 2\delta^*_o\,\delta^*_1\,\psi^*_{o\tau\eta} + \frac{1}{2}\,\delta^*_o\,(\psi^*_{1\xi\eta}\,\psi^*_{o\eta} + \psi^*_{o\xi\eta}\psi^*_{1\eta}$$

$$- \psi^*_{o\eta\eta}\,\psi^*_{1\xi} - \psi^*_{o\xi}\,\psi^*_{1\eta\eta}) + \frac{1}{2}\,\delta^*_1\,(\psi^*_{o\xi\eta}\,\psi^*_{o\eta} - \psi^*_{o\eta\eta}\,\psi^*_{o\xi})$$

$$- \frac{1}{2}\,\delta^*_{1\xi}\,(\psi^*_{o\eta})^2 - \xi(\delta^*_{1\tau}\,\delta^*_o + \delta^*_{o\tau}\,\delta^*_1)\,\psi^*_{o\eta} - \delta^*_{o\xi}\,\psi^*_{1\eta}\,\psi^*_{o\eta}$$

$$- \xi\,\delta^*_o\,\delta^*_{o\tau}\,\psi^*_{1\eta} - \xi\eta(\delta^*_1\,\delta^*_{o\tau} + \delta^*_{1\tau}\,\delta^*_o)\,\psi^*_{o\eta\eta} - \xi\eta\delta^*_o\,\delta^*_{o\tau}\,\psi^*_{1\eta\eta}$$

$$= \frac{\xi}{2X_e}\,\delta_o^{*3}\cos\tau + \frac{1}{X_e}\,\xi\,\psi^*_{1\eta\eta\eta} \tag{88-1}$$

$$\epsilon^2: \quad \xi\delta_o^{*2}\,\psi^*_{2\tau\eta} + 2\xi\delta^*_o\delta^*_1\,\psi^*_{1\tau\eta} + \xi(\delta_1^{*2} + 2\delta^*_2\delta^*_o)\,\psi^*_{o\tau\eta}$$

$$+ \frac{1}{2}\,\delta^*_o(\psi^*_{2\xi\eta}\,\psi^*_{o\eta} + \psi^*_{1\xi\eta}\,\psi^*_{1\eta} + \psi^*_{o\xi\eta}\,\psi^*_{2\eta} - \psi^*_{2\eta\eta}\,\psi^*_{o\xi} -$$

$$- \psi^*_{1\eta\eta}\,\psi^*_{1\xi} - \psi^*_{o\eta\eta}\,\psi^*_{2\xi}) + \frac{1}{2}\,\delta^*_1\,(\psi^*_{1\xi\eta}\,\psi^*_{o\eta} + \psi^*_{o\xi\eta}\,\psi^*_{1\eta}$$

$$- \psi^*_{o\eta\eta}\,\psi^*_{1\xi} - \psi^*_{o\xi}\,\psi^*_{1\eta\eta}) + \frac{1}{2}\,\delta^*_2\,(\psi^*_{o\xi\eta}\,\psi^*_{o\eta} - \psi^*_{o\eta\eta}\,\psi^*_{o\xi})$$

$$- \frac{1}{2}\,\delta^*_{2\xi}\,(\psi^*_{o\eta})^2 - \xi(\delta^*_{2\tau}\,\delta^*_o + \delta^*_{1\tau}\delta^*_1 + \delta^*_{o\tau}\delta^*_2)\,\psi^*_{o\eta} -$$

$$- \delta^*_{1\xi}\,\psi^*_{1\eta}\,\psi^*_{o\eta} - \xi(\delta^*_o\delta^*_{1\tau} + \delta^*_{o\tau}\delta^*_1)\,\psi^*_{1\eta} - \frac{1}{2}\,\delta^*_{o\xi}\,(\psi^{*2}_{1\eta}$$

$$+ 2\,\psi^*_{o\eta}\,\psi^*_{2\eta}) - \xi\delta^*_o\,\delta^*_{o\tau}\,\psi^*_{2\eta} - \xi\eta(\delta^*_2\,\delta^*_{o\tau} + \delta^*_1\delta^*_{1\tau} +$$

$$+ \delta^*_o\,\delta^*_{2\tau})\,\psi^*_{o\eta\eta} - \xi\eta(\delta^*_1\delta^*_{o\tau} + \delta^*_{1\tau}\delta^*_o)\,\psi^*_{1\eta\eta} - \xi\eta\delta^*_o\,\delta^*_{o\tau}\psi^*_{2\eta\eta}$$

$$= \frac{3\xi}{2X_e} \delta_o^{*2} \delta_1^* \cos\tau + \frac{1}{X_e} \xi \psi^*_{2\eta\eta\eta} \tag{88-2}$$

....

....

....

Next substitution of the first two of the expansions (87) into the first condition (85) gives upon equating the coefficients of like powers of ϵ:

at $\eta = 1$

$$\epsilon^o: \quad \psi_o^* = \delta_o^* + 2(X_e - 1)\int_o^\xi \frac{\partial\delta_o^*}{\partial\tau}\,\xi d\xi \tag{89-0}$$

$$\epsilon^1: \quad \psi_1^* = \delta_1^* + \frac{1}{2}\delta_o^* \sin\tau + 2(X_e - 1)\int_o^\xi \frac{\partial\delta_1^*}{\partial\tau}\,\xi d\xi \tag{89-1}$$

$$\epsilon^2: \quad \psi_2^* = \delta_2^* + \frac{1}{2}\delta_1^* \sin\tau + 2(X_e - 1)\int_o^\xi \frac{\partial\delta_2^*}{\partial\tau}\,\xi d\xi \tag{89-2}$$

....

....

....

Substitution of the first two of the expansions (87) into the second condition (85) and equating the coefficients of like powers of ϵ gives after the calculation performed in Appendix XII:

$$\epsilon^o: \quad \delta_o^*\left(X_e\,\xi\delta^*_{o\tau} + \frac{1}{2}\delta^*_{o\xi}\right)\left(\delta_o^* - X_e\psi^*_{o\eta}\right) = \xi\psi^*_{o\eta\eta} \tag{90-0}$$

$$\epsilon^1: \quad \left(\delta_o^* - X_e\psi^*_{o\eta}\right)\left[\delta_o^*\left(X_e\,\xi\delta^*_{1\tau} + \frac{1}{2}\delta^*_{1\xi}\right) + \delta_1^*\left(X_e\xi\delta^*_{o\tau}\right.\right.$$

$$
\text{at } \eta = 1 \quad \begin{cases}
+ \frac{1}{2}\,\delta^*_{o\xi}) + \frac{\delta^*_{o\xi}\delta^*_o}{4}\,\sin\tau \Big] + \delta^*_o(X_e\xi\delta^*_{o\tau} + \frac{1}{2}\,\delta^*_{o\xi})\Big[\delta^*_1 \\
- X_e\psi^*_{1\eta} + \frac{\delta^*_o}{2}\,\sin\tau\Big] = \xi\psi^*_{1\eta\eta} \qquad \text{90-1)} \\
\epsilon^2:\ (\delta^*_o - X_e\psi^*_{o\eta})\Big[\delta^*_o(X_e\xi\delta^*_{2\tau} + \frac{1}{2}\,\delta^*_{2\xi}) + \delta^*_2(X_e\xi\delta^*_{o\tau} \\
+ \frac{1}{2}\,\delta^*_{o\xi}) + \delta^*_1\,(X_e\xi\delta^*_{1\tau} + \frac{1}{2}\,\delta^*_{1\xi})\Big] + (\delta^*_1 - X_e\psi^*_{1\eta})\Big[\delta^*_o(X_e\xi\delta^*_{1\tau} \\
+ \frac{1}{2}\,\delta^*_{1\xi}) + \delta^*_1\,(X_e\xi\delta^*_{o\tau} + \frac{1}{2}\,\delta^*_{o\xi})\Big] + \delta^*_o(X_e\xi\delta^*_{o\tau} + \\
\frac{1}{2}\,\delta^*_{o\xi})(\delta^*_2 - X_e\psi^*_{2\eta}) + \frac{1}{2}\Big[2\,\delta^*_1\,\delta^*_o(X_e\xi\delta^*_{o\tau} + \delta^*_{o\xi}) + \\
\delta^{*2}_o(X_e\xi\delta^*_{1\tau} + \delta^*_{1\xi}) - \frac{X_e}{2}\,(\delta^*_o\,\delta^*_{1\xi} + \delta^*_1\delta^*_{o\xi})\,\psi^*_{o\eta} - \\
- \frac{X_e}{2}\,\delta^*_o\,\delta^*_{o\xi}\,\psi^*_{1\eta}\Big]\sin\tau + \frac{\delta^{*2}_o\delta^*_{o\xi}}{16}\,(1 - \cos 2\tau) = \\
= \xi\psi^*_{2\eta\eta} \qquad \text{90-2)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

and the boundary conditions (86) show that one must have:

$$\text{at } \eta = 0 \ :\ \psi^*_n = \frac{\partial\psi^*_n}{\partial\eta} = 0 \ ; \ n = 0, 1, 2 \ldots\ldots \qquad 91)$$

Now Eq. (88-0) with the boundary condition (89-0), (90-0), and (91) (with n = 0) will be satisfied by:

$$\psi^*_o = 2a\xi\ f_o(\eta) \ ; \ \delta^*_o = 2a\xi \qquad 92)$$

since (88-0) through (90-0) involve the time τ only implicitly. Substitution of (92) into (88-0) through (90-0) and (91) show that f_o and a must satisfy the following equations and boundary conditions:

$$0 < \eta \leq 1 \quad : \quad \frac{1}{2a^2X_e} f_o''' + f_o f_o'' = 0 \qquad 93)$$

$$\eta = 1 \quad : \quad \left\{ \begin{aligned} f_o &= 1 \\ \frac{1}{2a^2} f_o'' &= 1 - X_e f_o' \end{aligned} \right\} \qquad 94)$$

$$\eta = 0 \quad : \quad f_o = f_o' = 0 \qquad 95)$$

This as can easily be seen represents the steady state solution.

Now (92) is used to simplify the Eqs. (88-1) through (90-1) and (88-2) through (90-2). First substitutions of (92) into Eqs. (88-1) and (88-2) yields after a calculation performed in Appendix XIII:

$$\epsilon^1: \quad -\frac{1}{2a^2X_e} \psi^*_{1\eta\eta\eta} - f_o \psi^*_{1\eta\eta} + f_o' (\xi \psi_{1\eta\xi} - \psi^*_{1\eta}) - \xi f_o'' \psi^*_{1\xi}$$

$$+ 2\xi^2 \psi^*_{1\tau\eta} + (f_o'^2 - f_o f_o'') \delta^*_1 - \xi f_o'^2 \delta^*_{1\xi} - 2\xi^2 (f_o + \eta f_o'') \delta^*_{1\tau}$$

$$= \frac{2a}{X_e} \xi^3 \cos \tau \qquad 88\text{-}1a)$$

$$\epsilon^2: \quad 2a\xi \left[-\frac{1}{2a^2X_e} \psi^*_{2\eta\eta\eta} - f_o \psi^*_{2\eta\eta} + f_o' (\xi \psi^*_{2\xi\eta} - \psi^*_{2\eta}) - \xi f_o'' \psi^*_{2\xi} \right.$$

$$\left. + 2\xi^2 \psi^*_{2\tau\eta} + (f_o'^2 - f_o f_o'') \delta^*_2 - \xi f_o'^2 \delta^*_{2\xi} - 2\xi^2 (f_o' + \eta f_o'') \delta^*_{2\tau} \right]$$

$$= \frac{6a}{X_e} \xi^3 \delta_1^* \cos\tau - 4\xi^2\delta_1^* \psi_{1\tau\eta}^* - \xi\psi_{1\xi\eta}^* \psi_{1\eta}^* + \xi\psi_{1\eta\eta}^* \psi_{1\xi}^*$$

$$+ \psi_{1\eta}^{*2} - \delta_1^*(\xi f_o' \psi_{1\xi\eta}^* + f_o' \psi_{1\eta}^* - \xi f_o'' \psi_{1\xi}^* - f_o \psi_{1\eta\eta}^*)$$

$$+ 2\xi^2(f_o' + \eta f_o'') \delta_1^* \delta_{1\tau}^* + 2\xi f_o' \delta_{1\xi}^* \psi_{1\eta}^* + 2\xi^2 (\psi_{1\eta}^*$$

$$+ \eta \psi_{1\eta\eta}^*)\delta_{1\tau}^* \qquad 88\text{-}2a)$$

Substitution of Eqs. (92) into Eqs. (89-1) and (89-2) yields:

$$\eta = 1 \begin{cases} \epsilon^1: \quad \psi_1^* = \delta_1^* + a\,\xi \sin\tau + 2\,(X_e-1)\displaystyle\int_0^\xi \frac{\partial\delta_1^*}{\partial\tau}\,\xi d\xi & 89\text{-}1a) \\ \epsilon^2: \quad \psi_2^* = \delta_2^* + \frac{1}{2}\,\delta_1^* \sin\tau + 2\,(X_e-1)\displaystyle\int_0^\xi \frac{\partial\delta_2^*}{\partial\tau}\,\xi d\xi & 89\text{-}2a) \end{cases}$$

And finally using Eq. (92) in Eqs. (90-1) and (90-2) gives after some calculation performed in Appendix XIV:

$$\eta = 1 \begin{cases} \epsilon^1: \quad \frac{1}{2a^2}\psi_{1\eta\eta}^* = (1-X_e f_o')\left[2X_e\xi^2 \delta_{1\tau}^* + (\xi\delta_{1\xi}^* + \delta_1^*) + a\,\xi\sin\tau\right] + \delta_1^* - X_e\,\psi_{1\eta}^* + a\,\xi\sin\tau & 90\text{-}1a) \\ \epsilon^2: \quad \xi\psi_{2\eta\eta}^* = 2a\,\xi(1-X_e f_o')\left[a(\xi\delta_{2\xi}^* + \delta_2^*) + 2X_e a\xi^2\delta_{2\tau}^* + \delta_1^*(X_e\xi\delta_{1\tau}^* + \frac{1}{2}\delta_{1\xi}^*) + \frac{a}{2}(\xi\delta_{1\xi}^* + \delta_1^*)\sin\tau\right] + \\ \quad a(\delta_1^* - X_e\psi_{1\eta}^*)\left[(\xi\delta_{1\xi}^* + \delta_1^*) + 2\,X_e\xi^2\delta_{1\tau}^* + a\,\xi\sin\tau\right] \\ \quad + \frac{1}{2}(2a)^2\,\xi(\delta_2^* - X_e\psi_{2\eta}^*) + \frac{1}{4}(2a)^2\xi\left[(\xi\delta_{1\xi}^* + 2\delta_1^*) + 2\,X_e\xi^2\delta_{1\tau}^*\right]\sin\tau + \frac{1}{8}(2a)^2\,a\xi^2(1-\cos 2\tau) & 90\text{-}2a) \end{cases}$$

Next substitution of the expansions (87) into the second of Eqs. (84) and equating coefficients of like powers of ϵ yields after a calculation performed in Appendix XV:

$$\epsilon^{o}: \quad \Theta^*_{o\tau}\,\xi\delta^{*2}_{o} + \frac{1}{2}\,\delta^*_{o}\,(\Theta^*_{o\xi}\,\psi^*_{o\eta} - \psi^*_{o\xi}\,\Theta^*_{o\eta}) - \xi\eta\delta^*_{o}\,\delta^*_{o\tau}\,\Theta^*_{o\eta} = \frac{\xi}{X_e P_r}\,\Theta^*_{o\eta\eta} \qquad 96\text{-}0)$$

$$\begin{aligned}\epsilon^{1}: \quad & 2\xi\delta^*_{o}\,\delta^*_{1}\,\Theta^*_{o\tau} + \xi\delta^{*2}_{o}\,\Theta^*_{1\tau} + \frac{1}{2}\,\delta^*_{o}\,(\Theta^*_{1\xi}\,\psi^*_{o\eta} + \Theta^*_{o\xi}\,\psi^*_{1\eta} \\ & - \psi^*_{o\xi}\,\Theta^*_{1\eta} - \psi^*_{1\xi}\,\Theta^*_{o\eta}) + \frac{1}{2}\,\delta^*_{1}\,(\Theta^*_{o\xi}\,\psi^*_{o\eta} - \psi^*_{o\xi}\,\Theta^*_{o\eta}) \\ & - \xi\eta\delta^*_{o}\,\delta^*_{o\tau}\,\Theta^*_{1\eta} - \xi\eta\,(\delta^*_{1}\delta^*_{o\tau} + \delta^*_{o}\delta^*_{1\tau})\Theta^*_{o\eta} = \\ & = \frac{\xi}{P_r X_e}\,\Theta^*_{1\eta\eta} \qquad 96\text{-}1)\end{aligned}$$

$$\begin{aligned}\epsilon^{2}: \quad & \xi\delta^{*2}_{o}\,\Theta^*_{2\tau} + 2\xi\delta^*_{o}\delta^*_{1}\,\Theta^*_{1\tau} + \xi(\delta^{*2}_{1} + 2\delta^*_{o}\delta^*_{2})\,\Theta^*_{o\tau} \\ & + \frac{1}{2}\,\delta^*_{o}\,(\psi^*_{2\eta}\,\Theta^*_{o\xi} + \psi^*_{1\eta}\,\Theta^*_{1\xi} + \Theta^*_{2\xi}\,\psi^*_{o\eta} - \Theta^*_{2\eta}\,\psi^*_{o\xi} - \\ & \Theta^*_{1\eta}\,\psi^*_{1\xi} - \Theta^*_{o\eta}\,\psi^*_{2\xi}) + \frac{1}{2}\,\delta^*_{1}\,(\Theta^*_{1\xi}\,\psi^*_{o\eta} + \Theta^*_{o\xi}\,\psi^*_{1\eta} \\ & - \psi^*_{o\xi}\,\Theta^*_{1\eta} - \psi^*_{1\xi}\,\Theta^*_{o\eta}) - \frac{1}{2}\,\delta^*_{2}\,(\Theta^*_{o\xi}\,\psi^*_{o\eta} - \psi^*_{o\xi}\,\Theta^*_{o\eta}) \\ & - \xi\eta\,(\delta^*_{2}\,\delta^*_{o\tau} + \delta^*_{1}\delta^*_{1\tau} + \delta^*_{o}\,\delta^*_{2\tau})\,\Theta^*_{o\eta} - \xi\eta\,(\delta^*_{1}\,\delta^*_{o\tau} \\ & + \delta^*_{o}\delta^*_{1\tau})\,\Theta^*_{1\eta} - \xi\eta\,\delta^*_{o}\,\delta^*_{o\tau}\,\Theta^*_{2\eta} = \frac{\xi}{P_r X_e}\,\Theta^*_{2\eta\eta} \qquad 96\text{-}2)\end{aligned}$$

....

....

....

After substituting the expansions (87) into the third condition (85), equating coefficients of like powers of ϵ and performing the calculation of Appendix XVI, on obtains:

$$\eta = 1 \quad \begin{cases} \epsilon^0: & \Theta_0^* \delta_0^* (X_e \xi \delta_{0\tau}^* + \frac{1}{2} \delta_{0\xi}^*) = - \frac{1}{X_e P_r} \xi \Theta_{0\eta}^* \qquad 97\text{-}0) \\ \epsilon^1: & \Theta_1^* \delta_0^* (X_e \xi \delta_{0\tau} + \frac{1}{2} \delta_{0\xi}^*) + \Theta_0^* \left\{ \left[\delta_1^* (X_e \xi \delta_{0\tau}^* + \frac{1}{2} \delta_{0\xi}^*) + \right. \right. \\ & \left. \left. \delta_0^* (X_e \xi \delta_{1\tau}^* + \frac{1}{2} \delta_{1\xi}^*) \right] \right\} + \frac{1}{4} \Theta_0^* \delta_0^* \delta_{0\xi}^* \sin \tau = - \frac{1}{X_e P_r} \xi \Theta_{1\eta}^* \qquad 97\text{-}1) \\ \epsilon^2: & \Theta_2^* \delta_0^* (X_e \xi \delta_{0\tau}^* + \frac{1}{2} \delta_{0\xi}^*) + \Theta_1^* \left\{ \left[\delta_1^* (X_e \xi \delta_{0\tau}^* + \frac{1}{2} \delta_{0\xi}^*) \right. \right. \\ & \left. \left. + \delta_0^* (X_e \xi \delta_{1\tau}^* + \frac{1}{2} \delta_{1\xi}^*) \right] \right\} + \Theta_0^* \left\{ \left[\delta_2^* (X_e \xi \delta_{0\tau}^* + \frac{1}{2} \delta_{0\xi}^*) \right. \right. \\ & \left. \left. + \delta_1^* (X_e \xi \delta_{1\tau}^* + \frac{1}{2} \delta_{1\xi}^*) + \delta_0^* (X_e \xi \delta_{2\tau}^* + \frac{1}{2} \delta_{2\xi}^*) \right] \right\} + \\ & \frac{1}{4} \left\{ \left[\Theta_1^* \delta_0^* \delta_{0\xi}^* + \Theta_0^* \left[\delta_1^* \delta_{0\xi}^* + \delta_0^* \delta_{1\xi}^* \right] \right] \right\} \sin \tau = \\ & - \frac{1}{X_e P_r} \xi \Theta_{2\eta}^* \qquad 97\text{-}2) \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

And finally from the first boundary condition of (86), one must have:

$$\eta = 0 \quad \begin{cases} \epsilon^0: & \Theta_0^* = 1 \qquad 98\text{-}0) \\ \epsilon^1: & \Theta_1^* = 0 \qquad 98\text{-}1) \end{cases}$$

$$\epsilon^2: \quad \Theta^*_2 = 0 \qquad 98\text{-}2)$$

....

....

Using (92) to simplify (96-0) and (97-0), one obtains:

$$\frac{1}{2a^2 X_e P_r} \Theta^*_{o\eta\eta} = \xi \Theta^*_{o\xi} f'_o - f_o \Theta^*_{o\eta} + 2\,\xi^2 \Theta^*_{o\tau}$$

$$\eta = 1 \qquad \frac{1}{2a^2 X_e P_r} \Theta^*_{o\eta} + \Theta^*_o = 0$$

Since the time is involved only implicitly these equations will be satisfied along with (98-0) by taking:

$$\Theta^*_o = F_o(\eta)$$

And F_o is a solution of the following boundary value problem:

$$0 \leq \eta \leq 1 \qquad \frac{1}{2a^2 P_r X_e} F''_o + f_o F'_o = 0 \qquad 100)$$

$$\eta = 1 \quad : \quad \frac{1}{2a^2 P_r X_e} F'_o + F_o = 0 \qquad 101)$$

$$\eta = 0 \quad : \quad F_o = 1 \qquad 102)$$

Now Eqs. (99) and (92) are used to simplify Eqs. (96-1) and (96-2). This gives, after a calculation performed in Appendix XVII:

$$\epsilon^1: \qquad \frac{1}{2a^2 P_r X_e} \Theta^*_{1\eta\eta} - 2\xi^2 \Theta^*_{1\tau} + f_o \Theta^*_{1\eta} - \xi f'_o \Theta^*_{1\xi} =$$

$$- \frac{1}{2a\xi} (\xi \psi^*_{1\xi} + \delta^*_1 f_o + 2\eta\xi^2 \delta^*_{1\tau}) F'_o \qquad 96\text{-}1a)$$

$$\epsilon^2: \quad \frac{1}{2a^2 P_r X_e} \Theta^*_{2\eta\eta} - 2\xi^2 \Theta^*_{2\tau} + f_o \Theta^*_{2\eta} - \xi f'_o \Theta^*_{2\xi}$$

$$\frac{1}{2a^2\xi} \Big\{ \Big[- a(\xi\psi^*_{2\xi} + f_o \delta^*_2 + 2\xi^2\eta \delta^*_{2\tau}) F'_o -$$

$$- (\frac{1}{2} \delta^*_1 \psi^*_{1\xi} + \xi\eta \delta^*_1 \delta^*_{1\tau}) F'_o + a \Big[4 \xi^2 \delta^*_1 \Theta^*_{1\tau} +$$

$$(\psi^*_{1\eta} + f'_o \delta^*_1) \xi\Theta^*_{1\xi} - (\xi\psi^*_{1\xi} + f_o \delta^*_1 + 2\xi^2\eta\delta^*_{1\tau}) \Theta^*_{1\eta} \Big] \Big] \Big\} \qquad 96\text{-}2a)$$

And using Eqs. (99) and (92) to simplify Eqs. (97-1) and (97-2) yields, after a calculation performed in Appendix XVIII:

$$\eta = 1 \quad \epsilon^1: \quad \xi \left[\frac{1}{2a^2 X_e P_r} \Theta^*_{1\eta} + \Theta^*_1 \right] + \frac{1}{2a} F_o \left[(\delta^*_1 + \xi\delta^*_{1\xi}) + 2X_e \xi^2 \delta^*_{1\tau} \right]$$

$$+ \frac{1}{2} F_o \xi \sin\tau = 0 \qquad 97\text{-}1a)$$

$$\epsilon^2: \quad \xi \left[\frac{1}{2a^2 X_e P_r} \Theta^*_{2\eta} + \Theta^*_2 \right] + \frac{1}{2a} \Theta^*_1 \left[(\delta^*_1 + \xi\delta^*_{1\xi}) + 2X_e \xi^2 \delta^*_{1\tau} \right]$$

$$+ \frac{1}{2a} F_o \Big[(\delta^*_2 + \xi\delta^*_{2\xi}) + 2X_e \xi^2 \delta^*_{2\tau} + \frac{1}{a} \delta^*_1 (X_e \xi \delta^*_{1\tau} +$$

$$\frac{1}{2} \delta^*_{1\xi}) \Big] + \frac{1}{2} \xi \Theta^*_1 \sin\tau + \frac{1}{2} \cdot \frac{1}{2a} F_o (\delta^*_1 + \xi\delta^*_{1\xi}) \sin\tau$$

$$= 0 \qquad 97\text{-}2a)$$

The first order equations now may be solved, i.e., Eq. (88-1a) with the boundary conditions (89-1a) (90-1a) and (91) with (n = 1) for the momentum problem and Eq. (96-1a) with the boundary conditions (97-1a) and (98-1) for the energy problem:

Solutions are sought to these equations in the form:

$$\psi_1^* = \mathcal{R}e\, 2a \sum_{n=1}^{\infty} f_{1n}(\eta)\, \xi^n e^{i\tau} \qquad 103)$$

$$\delta_1^* = \mathcal{R}e\, 2a \sum_{n=1}^{\infty} a_{1n} \xi^n e^{i\tau} \qquad 104)$$

$$\theta_1^* = \mathcal{R}e \sum_{n=0}^{\infty} F_{1n}(\eta) \xi^n e^{i\tau} \qquad 105)$$

where $\mathcal{R}e$ denotes the real part and f_{1n}, a_{1n} and F_{1n} are complex quantities. A bar denotes the complex conjugate. Thus the complex conjugate of a_{1n} is $\bar{a}_{1n}$.

Upon substituting Eqs. (103) and (104) in Eq. (88-1a) through (90-1a) and (91)(with n = 1) and equating coefficients of like powers of ξ, it is found from the calculation of Appendix XIX that the f_{1n}'s and the a_{1n}'s ; n = 1,3,5,... must satisfy:

$$\left.\begin{array}{ll}
0 \leq \eta \leq 1 & \left\{ \begin{array}{l} L_n(f_{1n}) + \left[(n-1)f_o'^2 + f_o f_o''\right] a_{1n} = -\frac{1}{X_e}\delta_{3n} \\ -2i\,(1-\delta_{1n}) \left[(f_o' + \eta\, f_o'')\, a_{1(n-2)} - f'_{1(n-2)}\right] \end{array} \right. \\
\eta = 1 & \left\{ \begin{array}{l} f_{1n} = a_{1n} - \frac{i}{2}\delta_{1n} + 2i(1-\delta_{1n})\,(X_e - 1)\, a_{1(n-2)}/n \\ \frac{1}{2a^2} f''_{1n} = (1 - X_e f_o') \left[(n+1)a_{1n} - \frac{i}{2}\delta_{1n} + \right. \\ \left. 2i\,(1-\delta_{1n})\, X_e\, a_{1(n-2)}\right] + a_{1n} - X_e f'_{1n} - \frac{i}{2}\delta_{1n} \end{array} \right.
\end{array}\right\} \begin{array}{l} n = 1,3, \\ 5,7\ldots\ldots \end{array} \qquad 106)$$

$$\eta = 0 \qquad \left\{ f_{1n} = f'_{1n} = 0 \right.$$

Where:

$$L_n \equiv \frac{1}{2a^2X_e}\frac{d^3}{d\eta^3} + f_o\frac{d^2}{d\eta^2} - (n-1)\, f_o'\frac{d}{d\eta} + n\, f_o'' \qquad 107)$$

and δ_{mn} is the "Kroneka Delta" defined by:

$$\delta_{mn} = \begin{cases} 1\ ; & m = n \\ 0\ ; & m \neq n \end{cases}$$

and the f_{1n}'s and a_{1n}'s have the following restrictions:

$$\left.\begin{array}{l} f_{1n} = a_{1n} = 0 \quad \text{for} \quad n = 2,4,6,8,\ldots \\ f_{1n},\ a_{1n} = \text{pure imaginary quantity for } n = 1,5,9,13,\ldots \\ f_{1n},\ a_{1n} = \text{real quantity for } n = 3,7,11,\ldots \end{array}\right\} \qquad 108$$

by substituting Eqs. (103), (104) and (105) into Eqs. (96-1a), (97-1a) and (98-1), and equating coefficients of like powers of ξ, it is found from the calculation of Appendix XX that the F_{1n}'s; n = 0, 2, 4, ... must satisfy:

$$n=0,2,4,6,\ldots \left\{\begin{array}{l} 0\leq\eta\leq 1 \left\{\begin{array}{l} K_n(F_{1n}) = -F_o'\left[(n+1)f_{1(n+1)} + f_o a_{1(n+1)}\right. \\ \left. -2i(1-\delta_{on})\eta a_{1(n-1)}\right] + 2i(1-\delta_{on})F_{1(n-2)} \end{array}\right. \\ \eta=1 \left\{\begin{array}{l} \frac{1}{2a^2X_eP_r} F_{1n}' + F_{1n} + F_o\left[(2+n)a_{1(n+1)} + \right. \\ \left. 2i\,X_e(1-\delta_{on})\, a_{1(n-1)} - \frac{i}{2}\delta_{on}\right] = 0 \end{array}\right. \\ \eta=0 \left\{ F_{1n} = 0 \right. \end{array}\right\} \qquad 109)$$

Where:

$$K_n \equiv \frac{1}{2a^2 P_r X_e} \frac{d^2}{d\eta^2} + f_o \frac{d}{d\eta} - n f_o$$

and the F_{1n}'s have the following restrictions:

$$\left.\begin{array}{lll} F_{1n} = 0 & \text{for} & n = 1,3,5,7,\ldots \\ F_{1n} = & & \text{pure imaginary quantity for } n = 0,4,8,\ldots \\ F_{1n} = & & \text{real quantity for } n = 2,6,10,\ldots \end{array}\right\} \qquad 110)$$

Before proceeding to the second order equations it will be convenient to develop a certain relation between complex quantities. Let us consider two complex variables Z_1 and Z_2 and let us evaluate the product:

$$(\mathcal{Re}\, Z_1 e^{i\tau}) \quad (\mathcal{Re}\, Z_2 e^{i\tau})$$

we first note that:

$$\mathcal{Re}\, Z_1 e^{i\tau} = \frac{1}{2} (Z_1 e^{i\tau} + \overline{Z_1 e^{i\tau}})$$

$$= \frac{1}{2} (Z_1 e^{i\tau} + \overline{Z}_1 e^{-i\tau})$$

Hence:

$$(\mathcal{Re}\, Z_1 e^{i\tau}) (\mathcal{Re}\, Z_2 e^{i\tau}) = \frac{1}{4} (Z_1 e^{i\tau} + \overline{Z}_1 e^{i\tau}) (Z_2 e^{i\tau}$$

$$+ \overline{Z}_2 e^{-i\tau}) = \frac{1}{4} (Z_1 Z_2 e^{2i\tau} + \overline{Z}_1 \overline{Z}_2 e^{-2i\tau}) +$$

$$\frac{1}{4} (Z_1 \overline{Z}_2 + Z_2 \overline{Z}_1) = \frac{1}{2} \mathcal{Re}(Z_1 Z_2) e^{2i\tau} + \frac{1}{4} (Z_1 \overline{Z}_2 + \overline{Z_1 \overline{Z}_2})$$

and:

$$(\mathcal{R}e\, Z_1 e^{i\tau})\,(\mathcal{R}e\, Z_2 e^{i\tau}) = \frac{1}{2}\,\mathcal{R}e(Z_1 Z_2 e^{2i\tau} + Z_1 \bar{Z}_2) \qquad 111)$$

Next keeping this result in mind inspection of the right hand sides of Eqs. (88-2a), (89-2a), and (90-2a), and of Eqs. (96-2a) and (97-2a) shows that one might anticipate the solutions of the second order equations to be of the form:

$$\psi_2^* = \mathcal{R}e\, 2a \sum_{n=1}^{\infty} \left[f_{2n}(\eta) e^{2i\tau} + \tilde{f}_{2n}(\eta) \right] \xi^n \qquad 112)$$

$$\delta_2^* = \mathcal{R}e\, 2a \sum_{n=1}^{\infty} \left[a_{2n} e^{2i\tau} + \tilde{a}_{2n} \right] \xi^n \qquad 113)$$

$$\theta_2^* = \mathcal{R}e \sum_{n=0}^{\infty} \left[F_{2n}(\eta) e^{2i\tau} + \tilde{F}_{2n}(\eta) \right] \xi^n \qquad 114)$$

Upon substituting the expansions (112), (113), (103), and (104) into Eqs. (88-2a), (89-2a), (90-2a) and (91) (with $n = 2$) and using the relation (111) to simplify, it is found that two types of terms appear; those with a time dependence of the form $e^{2i\tau}$, and those which are independent of time. It is therefore necessary to equate the coefficients of $e^{2i\tau}$ with each other and the terms which are independent of time with each other. These calculations are performed in Appendix XXI. Then (Appendix XXI) the coefficients of like powers of ξ are equated in each of these expressions. It is then found from these calculations that the f_{2n}'s, and the a_{2n}'s for $n = 1,3,5\ldots$ must satisfy:

$$0 \leq \eta \leq 1 \quad \left\{ L_n(f_{2n}) = -\left[(n-1)f_o'^2 + f_o f_o''\right] a_{2n} + H_n \right.$$

$$\eta = 1 \quad \left\{ \begin{array}{l} f_{2n} = a_{2n} - \frac{i}{4} a_{1n} + 4i(1-\delta_{1n})(X_e - 1) \frac{a_{2(n-2)}}{n} \\ \frac{1}{2a^2} f''_{2n} = (1 - X_e f_o') \left[(n+1)(a_{2n} - \frac{i}{4} a_{1n}) + \right. \\ \left. 4i(1-\delta_{1n}) a_{2(n-2)}\right] - \frac{i}{2}(a_{1n} - X_e f'_{1n}) + a_{2n} \\ - X_e f'_{2n} - \frac{i}{4}\left[(n+2) a_{1n} + 2i(1-\delta_{1n}) X_e a_{1(n-2)}\right] \\ - \frac{\delta_{1n}}{8} + C_n \end{array} \right.$$

$$\eta = 0 \quad \left\{ f_{2n} = f'_{2n} = 0 \right.$$

$n=1,3,5,7\ldots$ 115)

Where:

$$n=1,3,5,\ldots \left\{ \begin{array}{l} H_n \equiv -\frac{3}{2X_e}(1-\delta_{1n})a_{1(n-2)} + 4i(1-\delta_{1n})\left[f'_{2(n-2)}\right. \\ \left. - (f_o' + \eta f_o'')a_{2(n-2)}\right] - \frac{1}{2}\left\{f_{11} f''_{1n} - (n-1) f'_{11} f'_{1n}\right. \\ + (1-\delta_{1n}) n f''_{11} f_{1n} + \left[2(n-1) f_o' f'_{11} + f_o'' f_{11} + f_o f''_{11}\right] a_{1n} \\ \left. + (1-\delta_{1n})\left[n f_o'' f_{1n} - (n-1) f_o' f'_{1n} + f_o f''_{1n}\right] a_{11}\right\} - \\ - i(1-\delta_{1n})\left\{(\eta f_o'' + f_o') a_{11} a_{1(n-2)} + (\eta f''_{11}\right. \\ \left. - f'_{11}) a_{1(n-2)}\right\} - P_n \end{array} \right.$$

116)

and

$$P_1 \equiv 0$$

$$P_3 \equiv 0$$

$$P_5 \equiv \frac{1}{2}\left\{3f_{13}\, f''_{13} - 2f'_{13}\, f'_{13} + \left[f_o f''_{13} + \right.\right.$$

$$\left.\left. + 3f''_o\, f_{13} + 2\, f'_o\, f'_{13}\right] a_{13}\right\} + i\left\{(\eta f''_o + f'_o)a_{11}\, a_{13}\right.$$

$$\left. + (\eta f''_{13} - f'_{13})a_{11}\right\}$$

....

....

....

For $\eta = 1$

$$C_1 \equiv (1 - X_e f'_o)\,\frac{a_{11} a_{11}}{2} + 2a_{11}\,(a_{11} - X_e f'_{11})$$

$$C_3 \equiv (1 - X_e f'_o)\left\{a_{11}\left[\frac{3}{2}\, a_{13} + iX_e\, a_{11}\right] + \frac{1}{2}\, a_{11}\, a_{13}\right\}$$

$$+ (a_{11} - X_e f'_{11})\left[4a_{13} + 2\, i\, X_e\, a_{11}\right] + 2a_{11}(a_{13} - X_e f'_{13})$$

$$C_5 \equiv (1 - X_e f'_o)\left\{a_{11}\left[\frac{5}{2}\, a_{15} + i\, X_e\, a_{13}\right] + a_{13}\left[\frac{3}{2}\, a_{13}\right.\right.$$

$$\left.\left. + i\, X_e\, a_{11}\right] + \frac{1}{2}\, a_{11}\, a_{15}\right\} + (a_{11} - X_e f'_{11})\left[6\, a_{15} + \right.$$

$$\left. + 2\, i\, X_e\, a_{13}\right] + (a_{13} - X_e f'_{13})\left[4a_{13} + 2\, i\, X_e a_{11}\right]$$

$$+ 2\, a_{11}\,(a_{15} - X_e f'_{15}) \qquad 117)$$

....

....

And that for $n = 2,4,6,\ldots$:

$$a_{2n} \equiv f_{2n} \equiv 0 \;;\; \text{for } n = 2,4,6,\ldots \qquad 118)$$

And comparison of Eqs. (108) with Eqs. (115) through (117) shows that

$$\left.\begin{aligned} a_{2n},\ f_{2n} &= \text{real quantities for } n = 1,5,\ldots \\ a_{2n},\ f_{2n} &= \text{pure imaginary quantities for } n = 3,7,\ldots \end{aligned}\right\} \quad 119)$$

The calculations also show that for n = 1,5,... the $\tilde{f}_{2n}$'s and the $\tilde{a}_{2n}$'s must satisfy:

$$0 \leq \eta \leq 1 \qquad \left\{ L_n(\tilde{f}_{2n}) = -\left[(n-1)f_0'^2 + f_0 f_0''\right]\tilde{a}_{2n} + \tilde{H}_n \right.$$

$$\eta = 1 \qquad \left\{ \begin{aligned} \tilde{f}_{2n} &= \tilde{a}_{2n} - \frac{i}{4}\bar{a}_{1n} \\ \frac{1}{2a^2}\tilde{f}''_{2n} &= (1 - X_e f_0')(n+1)\left(\tilde{a}_{2n} - \frac{i}{4}\bar{a}_{1n}\right) - \frac{i}{2}\left(\bar{a}_{1n} - \bar{f}'_{1n}\right) + \tilde{a}_{2n} - X_e\tilde{f}'_{2n} - \frac{i}{4}\left\{\left[(n+2)\bar{a}_{1n} - 2i(1 - \delta_{1n})X_e\,\bar{a}_{1(n-2)}\right]\right\} + \frac{\delta_{1n}}{8} + \tilde{C}_n \end{aligned} \right. \quad n=1,5,\ldots \quad 120)$$

$$\eta = 0 \qquad \left\{ \tilde{f}_{2n} = \tilde{f}'_{2n} = 0 \right.$$

Where:

$$n=1,5,\ldots \quad \left\{ \begin{aligned} \tilde{H}_n &= -\frac{3}{2X_e}(1-\delta_{1n})a_{1(n-2)} - \frac{1}{2}\Big\{ f_{11}\bar{f}''_{1n} - (n-1)f'_{11}\bar{f}'_{1n} + (1-\delta_{1n})nf''_{11}\bar{f}_{1n} + \left[2(n-1)f_0' f'_{11} + f_0'' f_{11} + f_0 f''_{11}\right]\bar{a}_{1n} \\ &\quad + (1-\delta_{1n})\left[nf_0'' f_{1n} - (n-1)f_0' f'_{1n} + f_0 f''_{1n}\right]\bar{a}_{11}\Big\} - \tilde{P}_n \\ \tilde{P}_1 &= 0 \end{aligned} \right\} \quad 121)$$

$$
\left[\begin{array}{l}
\tilde{P}_5 = \frac{1}{2}\left\{\left[3f_{13}\bar{f}''_{13} - 2\bar{f}'_{13}f_{13} + \left[f_o f''_{13}\right.\right.\right. \\
\left.\left. + 3f''_o f_{13} + 2\,f'_o\,f'_{13}\right]\ a_{13}\right] + i\ \left\{(\eta\bar{f}''_{11} + 3\,\bar{f}'_{11})\,a_{13}\right. \\
\left.\left. + (\eta\bar{f}''_{13} + 3\bar{f}'_{13})\ a_{11}\right]\right\} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}\right]
$$

And:

$$
\text{For } \eta=1 \left\{\left[\begin{array}{l}
\tilde{C}_1 = (1-X_e f'_o)\,\frac{1}{2}\,\bar{a}_{11}a_{11} + 2a_{11}(\bar{a}_{11} - X_e\bar{f}'_{11}) \\
\tilde{C}_5 = (1-X_e f'_o)\left\{a_{11}\left[\frac{5}{2}\,a_{15} + i\ X_e a_{13}\right] + \right. \\
\bar{a}_{13}\quad\left[\frac{3}{2}\,a_{13} + i\ X_e\ a_{11}\right] + \left.\frac{1}{2}\,a_{11}\,\bar{a}_{15}\right\} + (\bar{a}_{11} \\
- X_e\bar{f}'_{11})\left[6\ a_{15} + 2\ i\ X_e a_{13}\right] + (\bar{a}_{13} - X_e\bar{f}'_{13})\left[4a_{13}\right. \\
\left. + 2\ i\ X_e a_{11}\right]\ + (\bar{a}_{15} - X_e\ \bar{f}'_{15})\ 2a_{11} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}\right]\right. \qquad 122)
$$

And for $n \neq 1,5,\ldots$

$$
\tilde{a}_{2n} \equiv \tilde{f}_{2n}(\eta) \equiv 0 \quad ; \quad n \neq 1,5,9\ldots \qquad 123)
$$

And:

$$
\tilde{a}_{2n},\ \tilde{f}_{2n} \quad \text{real}
$$

Next substituting the expansions (103) through (105) and (112) through (114) into Eqs. (96-2a), (97-2), and using the relation (111) to simplify, it is again found that the only type of time dependent terms which appear are those whose time dependence is of the form $e^{2i\tau}$. Therefore the coefficients of $e^{2i\tau}$ are equated to one another and the time independent terms are equated to one another. After carrying out these operations in Appendix XXII, the coefficients of like powers of ξ are equated in each of the resulting expressions. From this is found that the F_{2n}'s for n = 0,2,4,...must satisfy:

$$\left.\begin{array}{ll} 0 \leq \eta \leq 1 & \left\{ \begin{array}{l} K_n(F_{2n}) = 4\,i\,(1-\delta_{on})F_{2(n-2)} - F_o' \Big[(n+1)\,f_{2(n+1)} \\ \quad + f_o\,a_{2(n+1)} + 4\,i\,(1-\delta_{on})\eta\,a_{2(n-1)}\Big] + E_n \end{array}\right. \\ \eta = 1 & \left\{ \begin{array}{l} \dfrac{1}{2a^2 P_r X_e} F_{2n}' + F_{2n} + F_o \Big\{(2+n)\,(a_{2(n+1)} - \frac{i}{4}\,a_{1(n+1)}) \\ \quad + 4\,i\,X_e(1-\delta_{on})a_{2(n-1)}\Big\} - \frac{i}{4}\,F_{1n} + B_n = 0 \end{array}\right. \\ \eta = 0 & \left\{ F_{2n} = 0 \right. \end{array}\right] \quad n=0,2,4,.. \qquad 124)$$

Where:

$$\begin{array}{l} E_o = -\frac{1}{2}\,a_{11}\,F_o'\,f_{11} - \frac{1}{2}\,F_{10}'\,(f_{11} + a_{11}\,f_o) \\ E_2 = -\frac{1}{2}\Big\{3\,a_{11}f_{13} + 2\,i\,a_{11}\,a_{11}\,\eta + a_{13}f_{11}\Big\}F_o' \\ \quad + \frac{1}{2}\Big\{2\,F_{12}\,(f_{11}' + f_o\,a_{11}) - F_{12}'\,(f_{11} + f_o\,a_{11}) \end{array}$$

$$0 \le \eta \le 1 \left\{ \begin{array}{l} - F'_{10}\,(3 f_{13} + f_o\, a_{13}) \Big\} + i\, a_{11}\,(2F_{10} - \eta\, F'_{10}) \\ E_4 = -\frac{1}{2} \Big\{ 5a_{11} f_{15} + 4\, i\, \eta\, a_{11}\, a_{13} + 3a_{13} f_{13} \\ + a_{15} f_{11} \Big\} F'_o + \frac{1}{2} \Big\{ 4F_{14}\,(f'_{11} + a_{11} f'_o) \\ - F'_{14}\,(f_{11} + f_o\, a_{11}) + 2F_{12}\,(f'_{13} + f'_o\, a_{13}) - \\ F'_{12}\,(3f_{13} + a_{13} f_o) - F'_{10}\,(5f_{15} + f_o a_{15}) \Big\} \\ + i\, a_{11}\,(2F_{12} - \eta\, F'_{12}) + i\, a_{13}\,(2F_{10} - \eta\, F'_{10}) \\ \dots\dots\dots \\ \dots\dots\dots \end{array} \right\} \qquad 125)$$

And:

$$\eta = 1 \left\{ \begin{array}{l} B_o = a_{11} F_{10} + \frac{1}{2} a_{11}\, a_{11}\, F_o \\ B_2 = \frac{1}{2} F_{10}(4a_{13} + 2\, i\, X_e a_{11}) + a_{11} F_{12} + a_{11}\,(\frac{3}{2} a_{13} \\ + i\, X_e\, a_{11})\, F_o + \frac{1}{2} a_{11}\, a_{13}\, F_o \\ B_4 = \frac{1}{2} F_{10}\,(6a_{15} + 2\, i\, X_e a_{13}) + \frac{1}{2} F_{12}\,(4a_{13} + 2\, i\, X_e a_{11}) \\ + F_{14}\, a_{11} + a_{11}\,(\frac{5}{2} a_{15} + i\, X_e a_{13})\, F_o + a_{13}\,(\frac{3}{2} a_{13} + \\ i\, X_e a_{11})\, F_o + \frac{1}{2} a_{11}\, a_{15}\, F_o \\ \dots\dots\dots \\ \dots\dots\dots \end{array} \right\} \qquad 126)$$

And for n = 1,3,5,... we find that:

$$F_{2n}(\eta) \equiv 0 \qquad ; \qquad \text{for} \quad n = 1,3,5,\ldots \qquad 127)$$

A comparison of Eqs. (108), (110) and (119) with Eqs. (124) through (126) shows that:

$$\left.\begin{array}{lll} F_{2n} & = & \text{real quantity for} \quad n = 0,4,8,\ldots \\ F_{2n} & = & \text{pure imaginary quantity for} \quad n = 2,6,\ldots \end{array}\right\} \qquad 128)$$

The calculation of Appendix XXII further shows that the $\tilde{F}_{2n}$ must satisfy for n = 0,4,...:

$$\left.\begin{array}{ll} 0 \le \eta \le 1 & \left\{ K_n(\tilde{F}_{2n}) = -F'_o \left\{ (n+1)\tilde{f}_{2(n+1)} + f_o \tilde{a}_{2(n+1)} \right\} + \tilde{E}_n \right. \\ \eta = 1 & \left\{ \begin{array}{l} \dfrac{1}{2a^2 P_r X_e} \tilde{F}'_{2n} + \tilde{F}_{2n} + F_o(2+n)\left(\tilde{a}_{2(n+1)} - \frac{i}{4}\bar{a}_{1(n+1)}\right) \\ -\frac{i}{4}\bar{F}_{1n} + \tilde{B}_n = 0 \end{array} \right. \\ \eta = 0 & \left\{ \tilde{F}_{2n} = 0 \right. \end{array}\right\} \text{for } n=0,4,\ldots \qquad 129)$$

Where:

$$\tilde{E}_o = -\frac{1}{2} F'_o a_{11} \bar{f}_{11} - \frac{1}{2} F'_{10} (\bar{f}_{11} + f_o \bar{a}_{11})$$

$$\tilde{E}_4 = -\frac{1}{2} \left\{ 5a_{11} \bar{f}_{15} + 3a_{13} \bar{f}_{13} + a_{15} \bar{f}_{11} \right\} F'_o +$$

$$\frac{1}{2} \left\{ 4F_{14} (\bar{f}'_{11} + \bar{a}_{11} f'_o) - F'_{14} (\bar{f}_{11} + f_o a_{11}) + 2F_{12}(\bar{f}'_{13} \right.$$

$$0 \leq \eta \leq 1 \left\{ \begin{array}{l} \left. + f_o' \bar{a}_{13}) - F_{12}' (3\bar{f}_{13} + f_o \bar{a}_{13}) - F_{10}' (5\bar{f}_{15} + f_o \bar{a}_{15}) \right\} \\ + i\, \bar{a}_{11} (2F_{12} + \eta\, F_{12}) + i\, a_{13} (2F_{10} - \eta\, \bar{F}_{10}') \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \right\} \qquad 130)$$

and:

$$\eta = 1 \left\{ \begin{array}{l} \tilde{B}_o = a_{11} \bar{F}_{10} + \frac{1}{2} a_{11} \bar{a}_{11} F_o \\ \tilde{B}_4 = \frac{1}{2} \bar{F}_{10} (6a_{15} + 2\, i\, X_e\, a_{13}) + \frac{1}{2} \bar{F}_{12}(4a_{13} + \\ 2\, i\, X_e\, a_{11}) + a_{11}\bar{F}_{14} + \bar{a}_{11} (\frac{5}{2} a_{15} + i\, X_e a_{13})\, F_o \\ + \bar{a}_{13} (\frac{3}{2} a_{13} + i\, X_e a_{11})\, F_o + \frac{1}{2} a_{11}\, \bar{a}_{15}\, F_o \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \right\} \qquad 131)$$

And for $n \neq 0,4,8,\ldots$ we find that:

$$\tilde{F}_{2n}(\eta) \equiv 0 \quad \text{for} \quad n \neq 0,4,8,\ldots \qquad 132)$$

And the $\tilde{F}_{2n}$'s are real otherwise.

The problem has now been completely formulated in terms of a set of simultaneous ordinary differential equations with their appropriate boundary conditions. These equations can be solved successively, i.e., once f_o and a are known one can solve for f_{11} and a_{11} etc. Then once these are known we can solve for f_{13} and a_{13}, etc. Once the solutions to the momentum equations are known (f_{mn}'s, a_{mn}'s), one can solve successively in the same way for the solution to the energy equation (i.e., the F_{mn}'s). The remarks at the end of Section II-B, , about the general nature of the equations and their solutions are also appropriate here. The discussion will not be repeated now.

It now remains to derive explicit formulae for the velocity, shear stress and heat transfer coefficients in terms of the solutions to the equations derived above.

From the first of Eqs. (71) one has:

$$u = \frac{\partial \psi}{\partial y}$$

And using the dimensionless quantities defined in (77) we obtain:

$$\frac{u}{U_\infty} = \frac{X_e}{\delta *} \frac{\partial \psi^*}{\partial \eta} \qquad 133)$$

Next the wall shear stress is given by:

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{\partial^2 \psi}{\partial y^2}\bigg|_{y=0}$$

And using the dimensionless quantities of (77) we have:

$$\frac{\tau_w \sqrt{\frac{U_\infty x}{\nu}}}{\frac{1}{2}\rho U_\infty^2} = X_e \frac{2\xi}{\delta *^2} \frac{\partial^2 \psi^*}{\partial \eta^2}\bigg|_{\eta=0} \qquad 134)$$

And finally the heat flux at the wall is given by:

$$q_w = -k \frac{\partial T}{\partial y}\bigg|_{y=0} \qquad 135)$$

and using (77) we get:

$$\frac{N_{u_x}}{\sqrt{\frac{U_\infty x}{\nu}}} \equiv \frac{q_w x}{k(T_\infty - T_w)} \Bigg/ \sqrt{\frac{U_\infty x}{\nu}} = \sqrt{X_e}\ \frac{\xi}{\delta *}\ \frac{\partial \theta^*}{\partial \eta}\Bigg|_{\eta=0} \qquad 136)$$

Substitution of the expressions (87) into the relations (133), (134), and (136) yields after the calculation of Appendix XXIII:

$$\frac{u}{U_\infty} = u_o^* + \epsilon u_1^* + \epsilon^2 u_2^* + \ldots \qquad 137)$$

$$\frac{\tau_w \sqrt{\frac{U_\infty x}{\nu}}}{\frac{1}{2}\rho\, U_\infty^2} = X_e(\tau_o^* + \epsilon\tau_1^* + \epsilon^2\tau_2^* + \ldots) \qquad 138)$$

$$\frac{N_{u_x}}{\sqrt{\frac{U_\infty x}{\nu}}} = \sqrt{X_e}\,(q_o^* + \epsilon q_1^* + \epsilon^2 q_2^* + \ldots) \qquad 139)$$

Where:

$$0 \leq \eta \leq 1 \quad \left\{\begin{array}{l} u_o^* = \dfrac{X_e \psi_{o\eta}^*}{\delta_o^*} \\ u_1^* = \dfrac{X_e}{\delta_o^*}\left[\psi_{1\eta}^* - \dfrac{\delta_1^*}{\delta_o^*}\ \psi_{o\eta}^*\right] \\ u_2^* = \dfrac{X_e}{\delta_o^*}\left[\psi_{2\eta}^* - \dfrac{\delta_2^*}{\delta_o^*}\ \psi_{o\eta}^* + \dfrac{\delta_1^*}{\delta_o^*}\left(\dfrac{\delta_1^*}{\delta_o^*}\psi_{o\eta}^* - \psi_{1\eta}^*\right)\right] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \qquad 140)$$

$$\eta = 0 \quad \left\{\begin{array}{l} \tau_o^* = \dfrac{2\xi\ \psi_{o\eta\eta}^*}{\delta_o^{*2}} \\ \tau_1^* = \dfrac{2\xi}{\delta_o^{*2}}\left[\psi_{1\eta\eta}^* - \dfrac{2\delta_1^*}{\delta_o^*}\ \psi_{o\eta\eta}^*\right] \end{array}\right\} \qquad 141)$$

$$\tau_2^* = \frac{2\xi}{\delta_o^{*2}}\left[\psi_{2\eta\eta} - \frac{2\delta_2^*}{\delta_o^*}\psi^*_{o\eta\eta} - \frac{\delta_1^*}{\delta_o^*}\left(2\psi^*_{1\eta\eta} - \frac{3\delta_1^*}{\delta_o^*}\psi^*_{o\eta\eta}\right)\right]$$

....

....

$$\eta = 0 \quad \left\{\begin{aligned} q_o^* &= \frac{\xi}{\delta_o^*}\,\theta^*_{o\eta} \\ q_1^* &= \frac{\xi}{\delta_o^*}\left[\theta^*_{1\eta} - \frac{\delta_1^*}{\delta_o^*}\,\theta^*_{o\eta}\right] \\ q_2 &= \frac{\xi}{\delta_o^*}\left[\theta^*_{2\eta} - \frac{\delta_2^*}{\delta_o^*}\,\theta^*_{o\eta} - \frac{\delta_1^*}{\delta_o^*}\left(\theta^*_{1\eta} - \frac{\delta_1^*}{\delta_o^*}\,\theta^*_{o\eta}\right)\right] \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \end{aligned}\right. \qquad 142)$$

Now using the relations (91), (99), (103), (104), (105), (112), (113), and (114) with the conditions (108), (110), (119), (123), (127) and (132) in the Eqs. (140) through (142) we get after performing the calculations of Appendix XXIV:

$$\left.\begin{aligned} u_o^* &= X_e f_o' \\ u_1^* &= \mathcal{R}e\sum_{n=0}^{\infty} g_{1n}\,\xi^n\, e^{i\tau} \\ u_2 &= \mathcal{R}e\sum_{n=0}^{\infty} (g_{2n}\, e^{2i\tau} + \tilde{g}_{2n})\,\xi^n \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \end{aligned}\right\} \qquad 143)$$

where for $n = 0,2,4,\ldots$:

$$
0 \leq \eta \leq 1 \quad
\begin{cases}
g_{1n} = X_e(f'_{1(n+1)} - f_o\, a_{1(n+1)}) \\
g_{2n} = X_e(f'_{2(n+1)} - f'_o\, a_{2(n+1)} - h_n) \\
\tilde{g}_{2n} = X_e(\tilde{f}'_{2(n+1)} - f'_o\, \tilde{a}_{2(n+1)} - \tilde{h}_n) \\
\text{and:} \\
h_o = \frac{1}{2} a_{11} (f'_{11} - f_o\, a_{11}) \\
h_2 = \frac{1}{2} \left\{ a_{13}\, f'_{11} + a_{11}\, f'_{13} - 2\, a_{11}\, a_{13}\, f'_o \right\} \\
h_4 = \frac{1}{2} \left\{ a_{15}\, f'_{11} + a_{13}\, f'_{13} + a_{11}\, f'_{15} - f'_o(2a_{11}\, a_{15} + a_{13}\, a_{13}) \right\} \\
\cdots\cdots\cdots \\
\cdots\cdots\cdots \\
\tilde{h}_o = \frac{1}{2} \bar{a}_{11} (f'_{11} - f'_o\, a_{11}) \\
\tilde{h}_4 = \frac{1}{2} \left\{ \bar{a}_{15} f'_{11} + \bar{a}_{13} f'_{13} + \bar{a}_{11} f'_{15} - f_o(2\bar{a}_{11} a_{15} + a_{13} \bar{a}_{13}) \right\} \\
\cdots\cdots\cdots \\
\cdots\cdots\cdots \\
\text{and:} \\
h_n = 0 \quad \text{for} \quad n = 1,3,5,\ldots \\
\tilde{h}_n = 0 \quad \text{for} \quad n \neq 0,4,5,\ldots
\end{cases}
\qquad 144)
$$

$$\left.\begin{aligned}
\tau_0^* &= \frac{1}{a} f_0'' \\
\tau_1^* &= \mathcal{R}e\, \frac{1}{a} \sum_{n=0}^{\infty} e_{1n}\, \xi^n\, e^{i\tau} \\
\tau_2^* &= \mathcal{R}e\, \frac{1}{a} \sum_{n=0}^{\infty} (e_{2n}\, e^{2i\tau} + \tilde{e}_{2n})\, \xi^n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \qquad 145)$$

Where:

$$\eta = 0 \quad \left\{\begin{aligned}
e_{1n} &= f''_{1(n+1)} - 2f_0''\, a_{1(n+1)} \\
e_{2n} &= f''_{2(n+1)} - 2f_0''\, a_{2(n+1)} - r_n \\
\tilde{e}_{2n} &= \tilde{f}''_{2(n+1)} - 2f_0''\, \tilde{a}_{2(n+1)} - \tilde{r}_n \\
&\text{and:} \\
r_0 &= \frac{1}{2} a_{11} (2f''_{11} - 3f_0''\, a_{11}) \\
r_2 &= \frac{1}{2} \left[2a_{13}\, f_{11}'' + 2a_{11}\, f_{13}'' - 6a_{11}\, a_{13}\, f_0''\right] \\
r_4 &= \frac{1}{2} \left[2a_{15}\, f_{11}'' + 2a_{13}\, f_{13}'' + 2a_{11}\, f_{15}'' - 3f_0'' \right. \\
&\qquad \left. (2a_{11}\, a_{15} + a_{13}\, a_{13})\right] \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\tilde{r}_0 &= \frac{1}{2} \bar{a}_{11} (2f_{11}'' - 3f_0''\, a_{11})
\end{aligned}\right. \qquad 146)$$

$$\tilde{r}_4 = \frac{1}{2}\Big[2\bar{a}_{15}\, f_{11}'' + 2\bar{a}_{13}\, f_{13}'' + 2\bar{a}_{11}\, f_{15}'' - 3f_0''\,(2\bar{a}_{11}\, a_{15} + \bar{a}_{13}\, a_{13})\Big]$$

....

....

and:

$$r_n = 0 \quad \text{for} \quad n = 1,3,5,\ldots$$

$$\tilde{r}_n = 0 \quad \text{for} \quad n \neq 0,4,8,\ldots$$

$$\left.\begin{aligned} q_0^* &= \frac{1}{2a} F_0' \\ q_1^* &= \mathcal{R}e\, \frac{1}{2a}\sum_{n=0}^{\infty} G_{1n}\, \xi^n\, e^{i\tau} \\ q_2^* &= \mathcal{R}e\, \frac{1}{2a}\sum_{n=0}^{\infty}\Big[G_{2n}\, e^{2i\tau} + \tilde{G}_{2n}\Big]\xi^n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad 147)$$

Where:

$$G_{1n} = F_{1n}' - a_{1(n+1)} F_0'$$

$$G_{2n} = F_{2n}' - a_{2(n+1)} F_0' - R_n$$

$$\tilde{G}_{2n} = \tilde{F}_{2n}' - \tilde{a}_{2(n+1)} F_0' - \tilde{R}_n$$

and:

$$R_0 = \frac{1}{2}\Big[a_{11}\, F_{10}' - a_{11}\, a_{11}\, F_0'\Big]$$

$$\eta = 0 \quad R_2 = \frac{1}{2}\Big[a_{13}\, F_{10}' + a_{11}\, F_{12}' - 2a_{11}\, a_{13}\, F_0'\Big]$$

$$R_4 = \frac{1}{2}\Big[a_{15} F'_{10} + a_{13} F'_{12} + a_{11} F'_{14} - (2a_{11} a_{15} + a_{13} a_{13})F'_0\Big]$$

....

....

$$\tilde{R}_0 = \frac{1}{2}\Big[\bar{a}_{11} F'_{10} - a_{11} \bar{a}_{11} F'_0\Big]$$

$$\tilde{R}_4 = \frac{1}{2}\Big[\bar{a}_{15} F_{10} + \bar{a}_{13} F'_{12} + \bar{a}_{11} F'_{14} - (2\bar{a}_{11} a_{15} + a_{13} \bar{a}_{13})F'_0\Big]$$

....

....

and:

$R_n = 0$ for $n = 1,3,5,\ldots$

$\tilde{R}_n = 0$ for $n \neq 0,4,8,\ldots$

It is convenient to have explicit formulas for the amplitude and phase of the first order harmonics (i.e., the coefficient of ϵ to the first power) of the various quantities defined above. To accomplish this we note that they all can be put into the form:

$$Z = \mathcal{R}e\ \xi^{\lambda} \sum_{n=0}^{\infty} \alpha_n \xi^n e^{i\tau}$$

Where:

α_n imaginary for $n = 0,4,8,\ldots$

α_n real for $n = 2,6,\ldots\ldots$

$\alpha_n = 0$ for $n = 1,3,5,7,\ldots$

and:

$$\lambda = 0 \text{ or } 1$$

Then:

$$Z = \mathcal{R}e\ \xi^{\lambda} \left| \sum_{n=0}^{\infty} \alpha_n \xi^n \right| e^{i(\tau + \varphi')}$$

$$= -\mathcal{R}i\ \xi^{\lambda} \left| \sum_{n=0}^{\infty} \alpha_n \xi^n \right| e^{i(\tau + \varphi' + \frac{\pi}{2})}$$

$$= \xi^{\lambda} \left| \sum_{n=0}^{\infty} \alpha_n \xi^n \right| \sin.\ (\tau + \varphi' + \tfrac{\pi}{2})$$

$$= \left| \xi^{\lambda} \sum_{n=0}^{\infty} \alpha_n \xi^n e^{i\tau} \right| \sin.\ (\tau + \varphi' + \tfrac{\pi}{2})$$

$$= |Z| \sin.\ (\tau - \varphi)$$

Where:

$$\varphi' \equiv \arg. \sum_{n=0}^{\infty} \alpha_n \xi^n = \tan.^{-1} \left[\frac{\mathcal{I}m \sum_{n=0}^{\infty} \alpha_n \xi^n}{\mathcal{R}e \sum_{n=0}^{\infty} \alpha_n \xi^n} \right]$$

$$|Z| = \xi^{\lambda} \left| \sum_{n=0}^{\infty} \alpha_n \xi^n \right|$$

and:

$$\varphi = -(\varphi' - \frac{\pi}{2})$$

Hence:

$$|Z| = \xi^{\lambda}\sqrt{\sum_{n=0}^{\infty}\alpha_n \xi^n \sum_{m=0}^{\infty}\bar{\alpha}_m \xi^m} = \xi^{\lambda}\sqrt{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\alpha_n\bar{\alpha}_m \xi^{n+m}}$$

$$= \xi^{\lambda}\left\{\alpha_0\bar{\alpha}_0 + (\alpha_2\bar{\alpha}_0 + \alpha_0\bar{\alpha}_2)\xi^2 + (\alpha_2\bar{\alpha}_2 + \alpha_4\bar{\alpha}_0 + \alpha_0\bar{\alpha}_4)\xi^4 + \ldots\right\}^{1/2}$$

$$= \xi^{\lambda}|\alpha_0|\left\{1 + \left[\frac{|\alpha_2|^2}{|\alpha_0|^2} + \frac{2\alpha_4\bar{\alpha}_0}{|\alpha_0|^2}\right]\xi^4 + \ldots\ldots\right\}^{1/2}$$

Hence the amplitude is given by:

$$|Z| = |\alpha_0|\xi^{\lambda}\left\{1 + 1/2\left[\frac{|\alpha_2|^2}{|\alpha_0|^2} + \frac{2\alpha_4\bar{\alpha}_0}{|\alpha_0|^2}\right]\xi^4 + \ldots\ldots\right\} \qquad 150)$$

and:

$$\varphi' = \tan^{-1}\left\{-i\,\frac{\alpha_0 + \alpha_4\xi^4 + \ldots}{\alpha_2\xi^2 + \ldots}\right\} = \tan^{-1}\left[\frac{-i}{\xi^2}\left\{\frac{\alpha_0}{\alpha_2} + \frac{\alpha_4}{\alpha_2}\xi^4 + \ldots\right\}\right]$$

and therefore:

$$\varphi' = -\frac{\pi}{2} - \tan^{-1}\left[\frac{-i}{\xi^2}\left\{\frac{\alpha_0}{\alpha_2} + \frac{\alpha_4}{\alpha_2}\xi^4 + \ldots\ldots\right\}\right] \qquad 151)$$

Now comparison of (149) with (104), (143), (145), and (147) shows that we may write:

$$u_1^* = |u_1^*| \sin.(\tau - \varphi_{u_1})$$

$$\tau_1^* = |\tau_1^*| \sin(\tau - \varphi_{\tau_1})$$

$$q_1^* = |q_1^*| \sin.(\tau - \varphi_{q_1})$$

$$\delta_1^* = |\delta_1^*| \sin.(\tau - \varphi_{\delta_1})$$

Where:

$$|u_1^*| = |g_{10}| \left\{ 1 + \frac{1}{2} \left[\frac{|g_{12}|^2}{|g_{10}|^2} + \frac{2g_{14}\,\bar{g}_{10}}{|g_{10}|^2} \right] \xi^4 + \ldots. \right\}$$

$$\varphi_{u_1} = -\frac{\pi}{2} - \tan^{-1} \left\{ \frac{1}{\xi^2} \left[\frac{g_{14}}{ig_{12}} \xi^4 + \frac{g_{10}}{ig_{12}} + \ldots. \right] \right\}$$

$$|\tau_1^*| = \frac{1}{a} |e_{10}| \left\{ 1 + \frac{1}{2} \left[\frac{|e_{12}|^2}{|e_{10}|^2} + \frac{2e_{14}\bar{e}_{10}}{|e_{10}|^2} \right] \xi^4 + \ldots \right\}$$

$$\varphi_{\tau_1} = -\frac{\pi}{2} - \tan^{-1} \left\{ \frac{1}{\xi^2} \left[\frac{e_{10}}{ie_{12}} + \frac{e_{14}}{ie_{12}} \xi^4 + \ldots \right] \right\}$$

$$|g_1^*| = \frac{|G_{10}|}{2a} \left\{ 1 + \frac{1}{2} \left[\frac{|G_{12}|^2}{|G_{10}|^2} + \frac{2G_{14}\bar{G}_{10}}{|G_{10}|^2} \right] \xi^4 + \ldots \right\}$$

$$\varphi_{q_1} = -\frac{\pi}{2} - \tan^{-1} \left\{ \frac{1}{\xi^2} \left[\frac{G_{10}}{iG_{12}} + \frac{G_{14}}{iG_{12}} \xi^4 + \ldots \right] \right\}$$

$$|\delta_1^*| = 2a\xi |a_{11}| \left\{ 1 + \frac{1}{2} \left[\frac{|a_{13}|^2}{|a_{11}|^2} + \frac{2a_{15}\bar{a}_{11}}{|a_{11}|^2} \right] \xi^4 + \ldots \right\}$$

$$\varphi_{\delta_1} = -\frac{\pi}{2} - \tan^{-1} \left\{ \frac{1}{\xi^2} \left[\frac{a_{11}}{ia_{13}} + \frac{a_{15}}{ia_{13}} \xi^4 + \ldots \right] \right\}$$

CHAPTER III

NUMERICAL PROCEDURES

For both the flat plate problem and the cylinder problem with $E^2 \geq .01$, there is the task of solving successively a system of ordinary differential equations. That is, the equations under consideration are such that the solution to the n'th equation depends on that of the n-1st equation, n-2nd equation, etc., but not on that of the n+1st, n+2nd, etc. The momentum equations (which of course must be solved before the energy equation) are of the third order. They each involve however an arbitrary parameter which must be determined by an additional boundary condition imposed on each of the equations. Thus there are two boundary conditions at $\eta=0$ (independent variable in the notation) and two at $\eta=1$.

In the case of the cylinder problem with $E^2 \leq .01$, only the energy equation is formulated in terms of a system of consectutive ordinary differential equations. In all cases however, once the solutions to the momentum equations are obtained the consecutive ordinary differential equations resulting from the energy equation can be solved. These are of second order with two boundaryconditions; one at $\eta=0$ and one at $\eta=1$.

The Runge-Cutta method which has been used to solve all these equations is a numerical procedure which performs a marching solution to a system of n simultaneous first order equations. That is, it calculates numerically the value of the solution at any desired number of points.

In order to apply this method to equations of the type considered here, some additional procedures must be used with it.

First the n'th order equations which are to be solved must be reduced to a system of n first order equations. This is easy to do and an illustrative example will further clarify the procedure.

Example:

Consider the third order differential equation:

$$x''' + ax'' + bx' = f(x)$$

and let us define:

$$u_1 \equiv x$$

$$u_2 \equiv x'$$

$$u_3 \equiv x''$$

It then follows that

$$u_1' = u_2$$

$$u_2' = u_3$$

$$u_3' = f(u_1) - au_3 - bu_2$$

which is the desired decomposition.

Next for a differential equation of order n the first n-1 derivitives must be specified at a single point (η=0 in this case) before the Runge-Cutta method can be used to calculate the solution to the equation.

In order to accomplish this the derivitives at $\eta=0$ which are unknown are first guessed at. The solution is carried out to $\eta=1$ by the Runge-Cutta method. Now the desired solution f_d must satisfy a condition of the form:

$$G(f_d(1), f_d'(1), \ldots f_d^{(n-1)}(1)) = 0$$

at this point. The calculated solution f_c is then substituted into these relations and the errors R:

$$R = G(f_c(1), f_c'(1), \ldots f_c^{(n-1)}(1)) - G(f_d(1), f_d'(1), \ldots, f_d^{(n-1)}(1))$$

are noted. On the basis of these errors the initial guesses are modified. Since many equations had to be solved it was desirable to have an efficient scheme for iterating on the initial conditions. This was accomplished by incorporating a procedure which could remember how much of a percentage change in R was caused by the previous change in the initial conditions and using this information to calculate what would be the most suitable change in the initial conditions. For example, suppose it is known that the previous change in the initial conditions decreased R by only a small percentage, then it would be desirable to make the next change in the initial conditions in the same direction as the previous one (since it decreased R and that is what is required) but it should be much larger (since the previous change in the initial conditions only caused a small percentage change in R and we want R to go to zero as rapidly as possible).

In order to make these ideas quantitative consider the equation:

$$\frac{\Delta I_{N+1}}{\Delta I_N} + 1 = - \frac{CSCH.(R_N/R_{N-1}-1)}{CSCH.(1)}$$

which is sketched in Figure 6. Where ΔI_N is the N'th increment in the initial conditions, R_N is the value of the error in the boundary condition after the initial conditions were incremented by ΔI_N. R_{N-1} is the error that existed before this, and ΔI_{N+1} is the new value of the increment. First it is noted that if R_N/R_{N-1} is negative then that means that ΔI_N was made too large and it should be decreased by a certain percentage which is less than one (otherwise the error would be increased over the previous one).

Hence:

$$- 1 < \frac{\Delta I_{N+1}}{\Delta I_N} < 0$$

is the proper change for ΔI to decrease R_N in this case. That this function accomplishes this, can be varified from Figure 6. Now also the more negative R_N/R_{N-1} is the closer ΔI_{N+1} should be to ΔI_N in order to bring the error back closer to zero.

Next it should be noted that $0 \leq R_N/R_{N-1} < 1$ means that the previous change in ΔI_N accomplished a reduction in R_N and therefore the next change in ΔI_N should be in the same direction as the previous one. But if R_N/R_{N-1} is close to 1 then the change in ΔI_N was not to effection in reducing R_N and hence ΔI_{N+1} should be much larger than ΔI_N and similarily if R_N/R_{N-1} is close to zero then it was very effective and ΔI_{N+1} need now only be a small change compared with ΔI_N

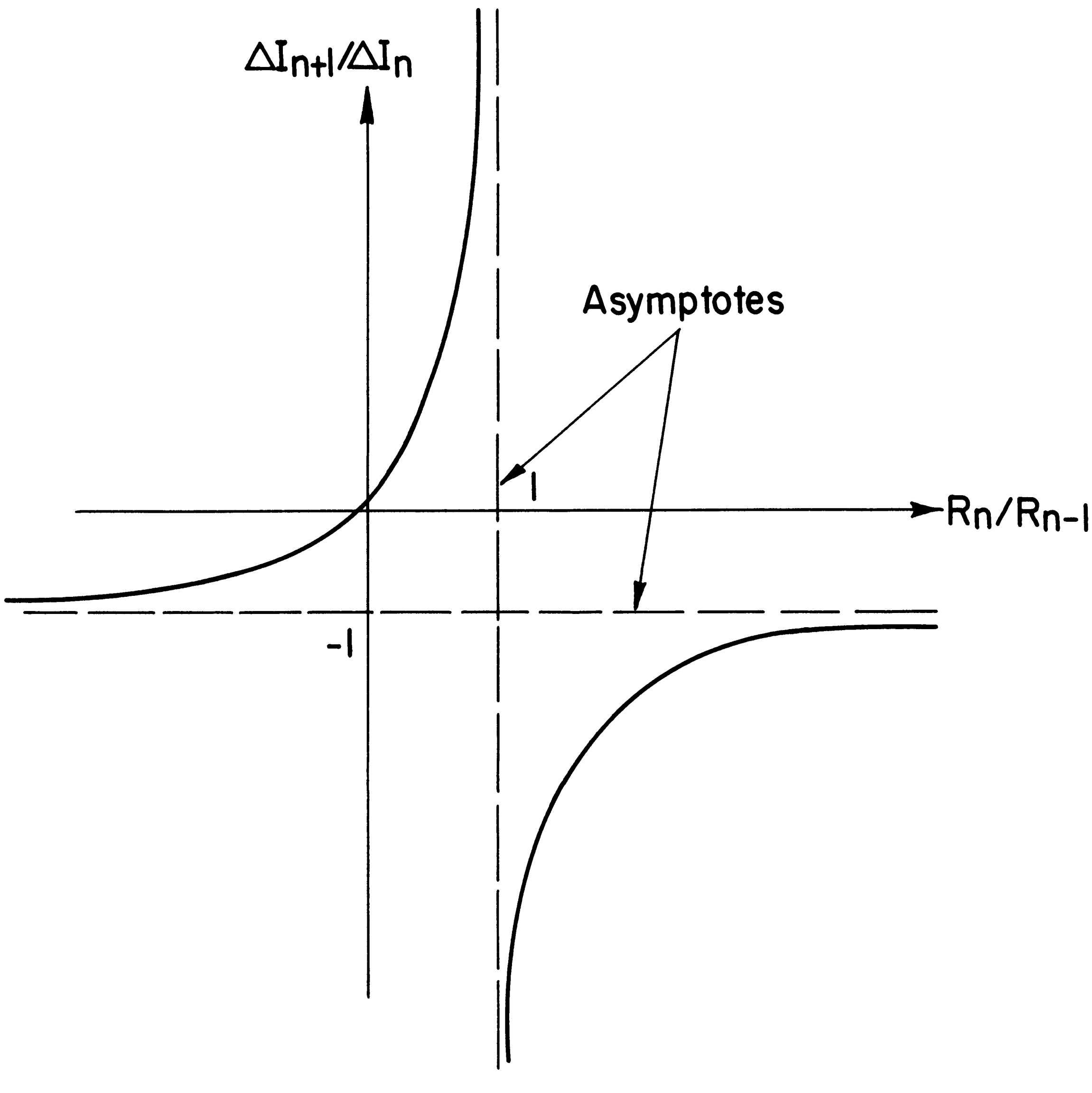

Figure 6. Plot of $\Delta I_{n+1}/\Delta I_n$ versus R_n/R_{n-1} for iterative procedure.

in order to bring the error close to 0. Finally notice that if R_N/R_{N+1} is greater than 1, then the change, ΔI_N, was in the wrong direction and ΔI_{N+1} should be in the opposite direction; i.e., $\Delta I_{N+1}/\Delta I_N$ should be negative. Note also that it is now necessary to change ΔI_N by such an amount that it not only compensates for the previous ΔI_N moving the error further away from 0 but that it also decreases R_N. It should further by noted that R_N/R_{N-1} close to one means that ΔI_N didn't change R_N very much, and a large change in ΔI_{N+1} will be necessary and conversely. That this will all be accomplished by the given function can easily be varified from Figure 6. It has been found that this function works very well in practice and that it usually requires less than ten iterations to get the desired solution when it is employed.

It is still necessary to discuss how the parameters appearing in the momentum equations are to be determined. It can easily be seen that the same method as is used for the initial conditions can be used here. That is, an initial guess can be made for the value of the parameters. Then, the error in the boundary conditions can be checked and a correction can be made using the same function as described above. In the case of these equations, however, iterations must be performed on both an initial condition and the values of a parameter in order to satisfy the two boundary conditions at $\eta=1$. This presents no problem since for a given assumed value for the initial condition, iterations can be performed on the parameter until one of the boundary conditions at $\eta=1$ is satisfied. When this is accomplished, the error in the other boundary condition can be noted, the function described above can be used to calculate a new value for the initial condition, and so on until both boundary conditions are satisfied.

In the case of the energy equations only a single iteration loop need be performed.

Thus the procedure for both the oscillating flat plate problem and the circular cylinder is to solve each of the equations successively using the iteration procedure described above and storing the solutions to be used in the higher order equations.

The Runge-Cutta method is also used for integrating the equations for the drop trajectories, but since this is an initial value problem no special difficulties are presented here.

There calculations are all performed on a 7090 computer using the "MAD" language. The individual programs are listed in Appendix XXV.

CHAPTER IV

DISCUSSION AND RESULTS

A. Convergence of Solutions

The computer programs listed in the appendix were run on the IBM 7090 digital computer located at The University of Michigan Computing Center. For those programs (Appendix XXV) which involved the integration of differential equations by the Runge-Kutta method it was necessary to determine the proper number of subdivisions into which the domain should be divided in order to get accurate answers (see Runge-Kutta Subroutine in Michigan Executive System). This was accomplished experimentally on the machine by calculating the solutions using various numbers of subdivisions. The results of these experiments for the differential equations describing the liquid film on the cylinder with $E^2 \geq .01$ are shown in Figures 7-a and 7-b. It can be seen from these that increasing the number of subdivisions past 100 results in no significant change in the solution. Of course, a point will be reached when round off errors become excessive and the solutions will begin to diverge again. But this could not be demonstrated, since it would involve excessive machine time to reach this point. It should be noted that the higher order terms are tabulated in these figures since the lower order terms were found to reach a stable value at a much smaller number of subdivisions.

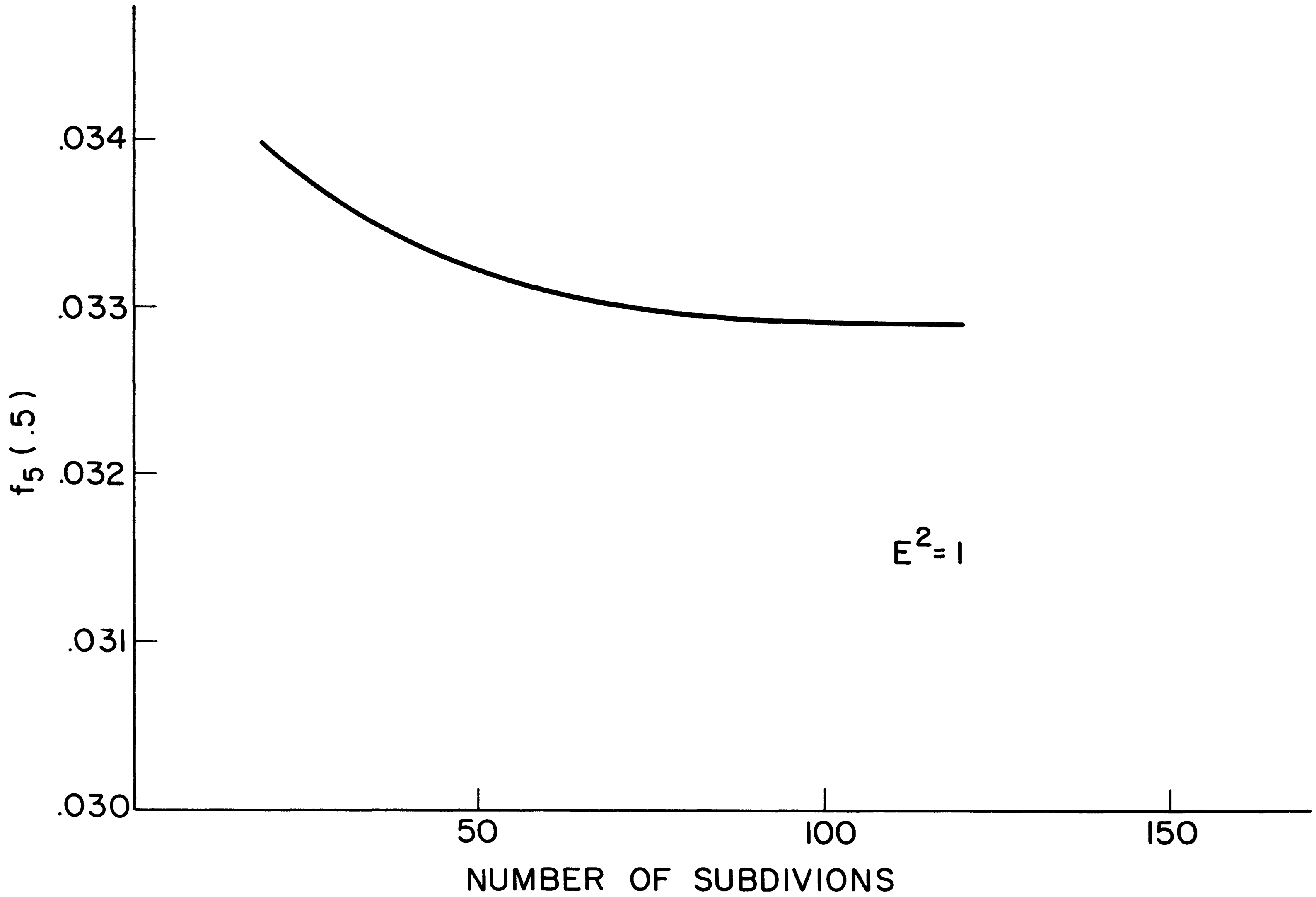

Figure 7-a. Convergence of function f_5 with respect to number of subdivisions used for numerical calculations in cylinder problem.

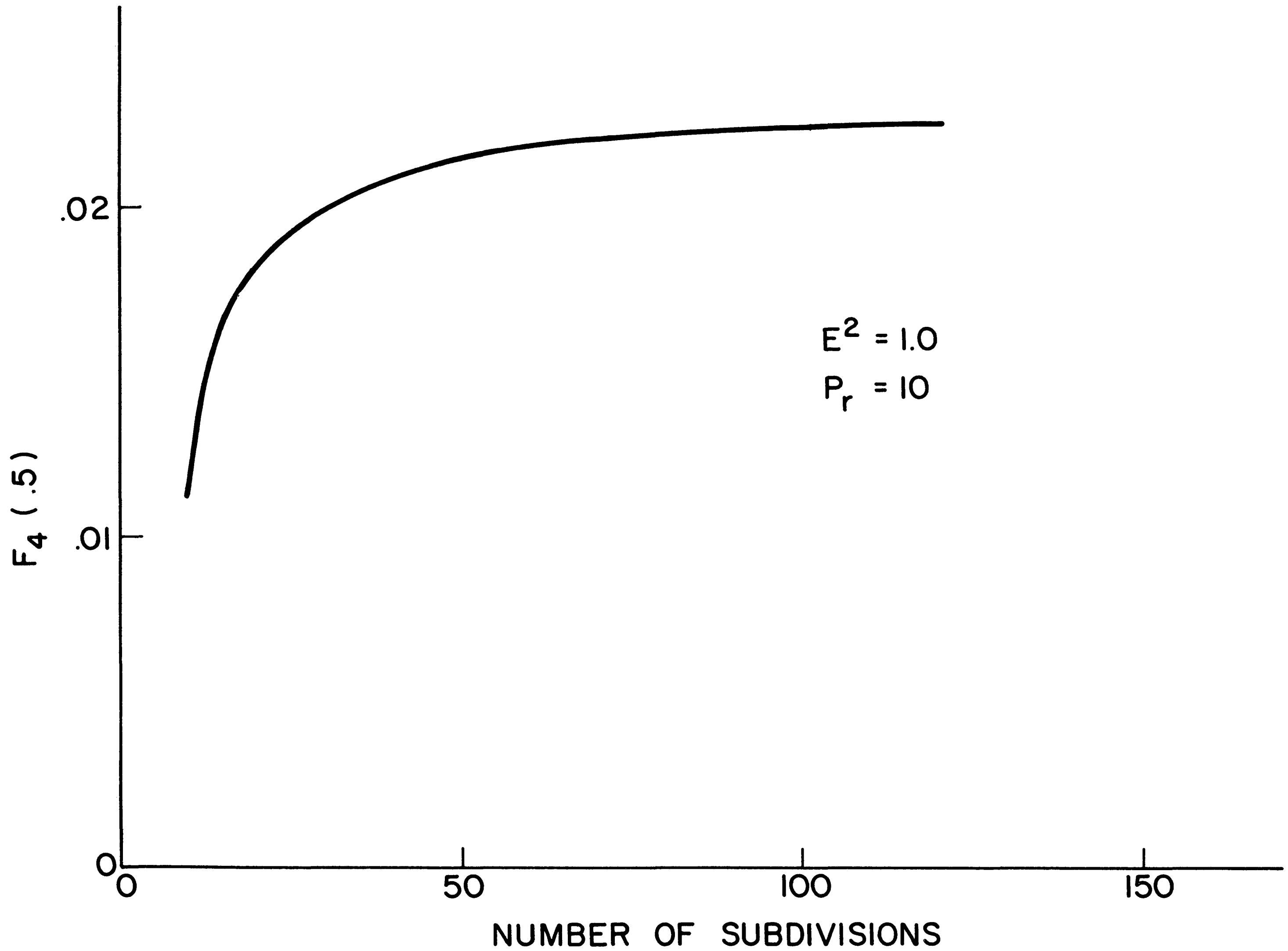

Figure 7-b. Convergence of function F_4 with respect to number of subdivisions used for numerical calculations in cylinder problem.

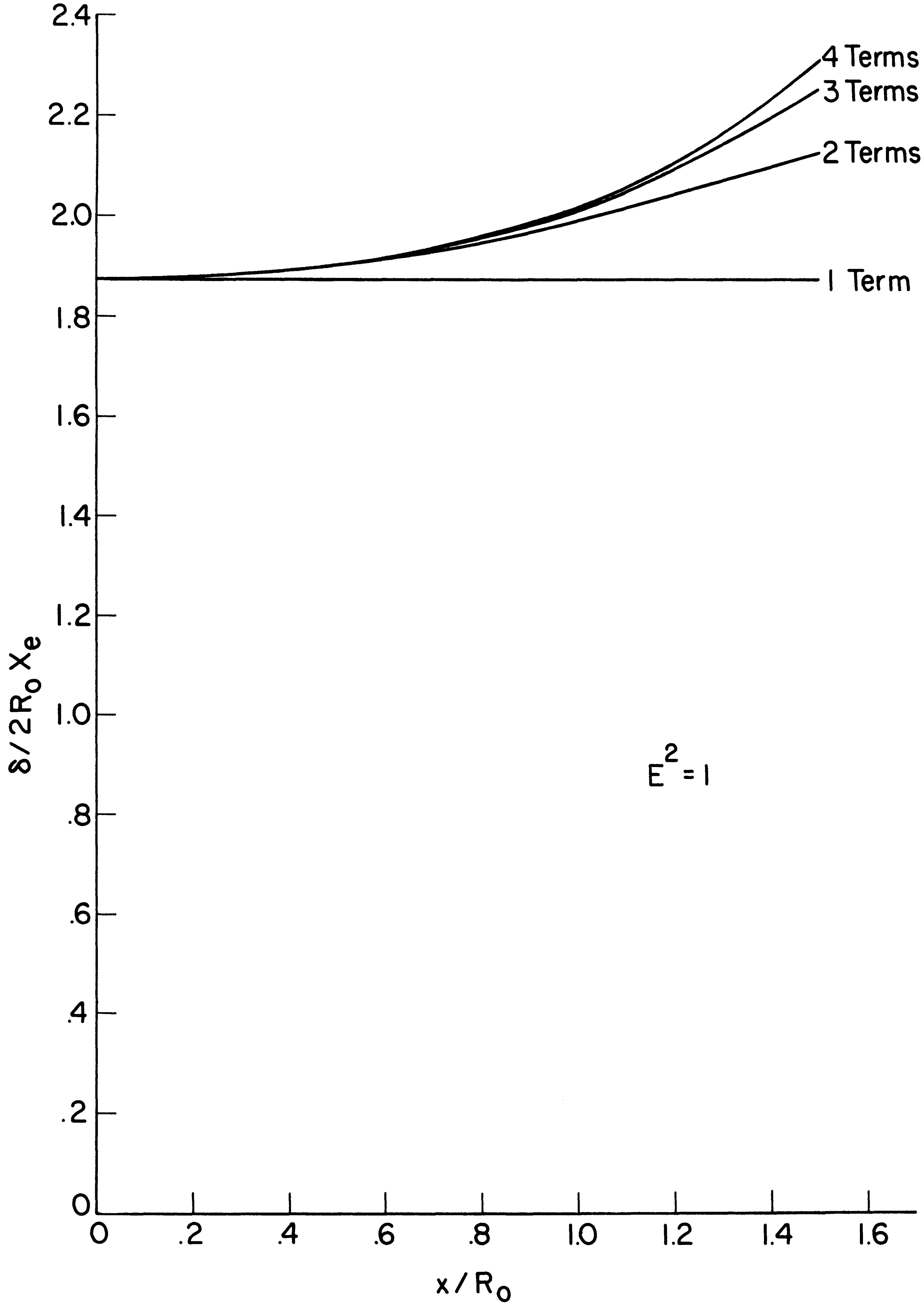

Figure 7-c. Convergence of series for liquid film thickness with respect to the number of terms retained in cylinder problem.

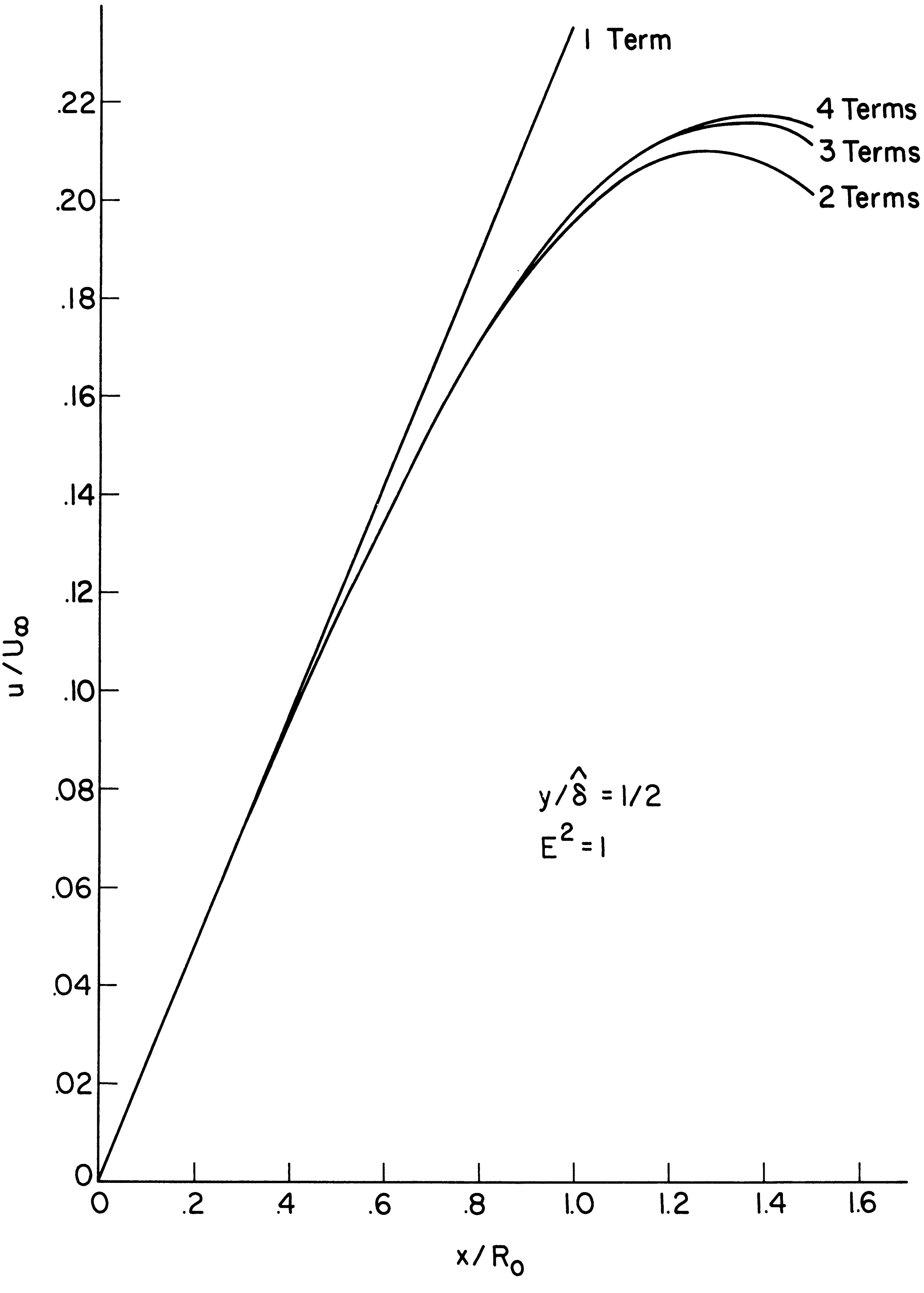

Figure 7-d. Convergence of series for liquid velocity with respect to the numer of terms retained in cylinder problem.

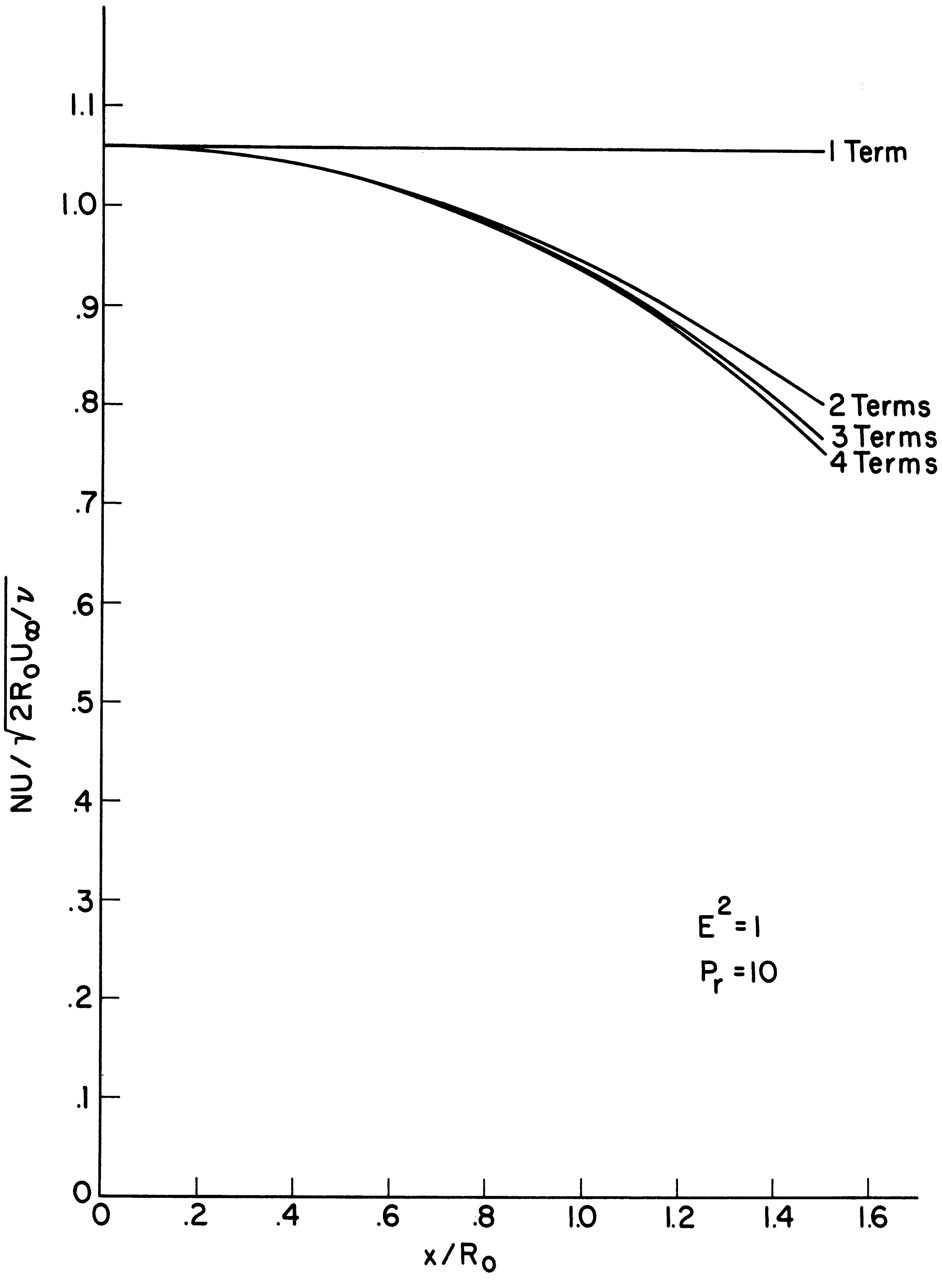

Figure 7-e. Convergence of series for Nusselt number with respect to the number of terms retained in cylinder problem.

Similarly the results of experiments on the equations for drop trajectories are shown in Figures 8-a and 8-b, and those for the energy equation in the liquid film with $E \leq .1$ are shown in Figure 9-a. The results for the equations describing the liquid film on the oscillating flat plate are shown in Figures 10-a through 10-d. In the last two cases, again only the higher order terms are shown since the lower order ones become stable at a lower number of subdivisions. In the case of the flat plate, it can be seen that a hundred subdivisions will give very satisfactory answers.

It is in general impossible to prove convergence of the expansions that have been imployed in obtaining solutions to the partial differential equations. In the case of the cylinder problem for both large and small E it is hoped that a convergent series was obtained. In the case of the flat plate only an asymptotic series was sought,[31] (a coordinate expansion plus a parameter expansion in Van Dyke's terminology). The best one can hope to do is to show that increasing the number of terms retained in the expansion has a small effect on the solution after a certain number of terms.

This has been carried out for the expansions used in the cylinder problem (for both large and small E) and the results are shown in Figures 7-c through 7-e for $E^2 \geq .01$ and in Figure 9-b and 9-c for $E \leq .1$. Because of the extreme complexity of the solutions for the oscillating flat plate problem this could not reasonably be done.

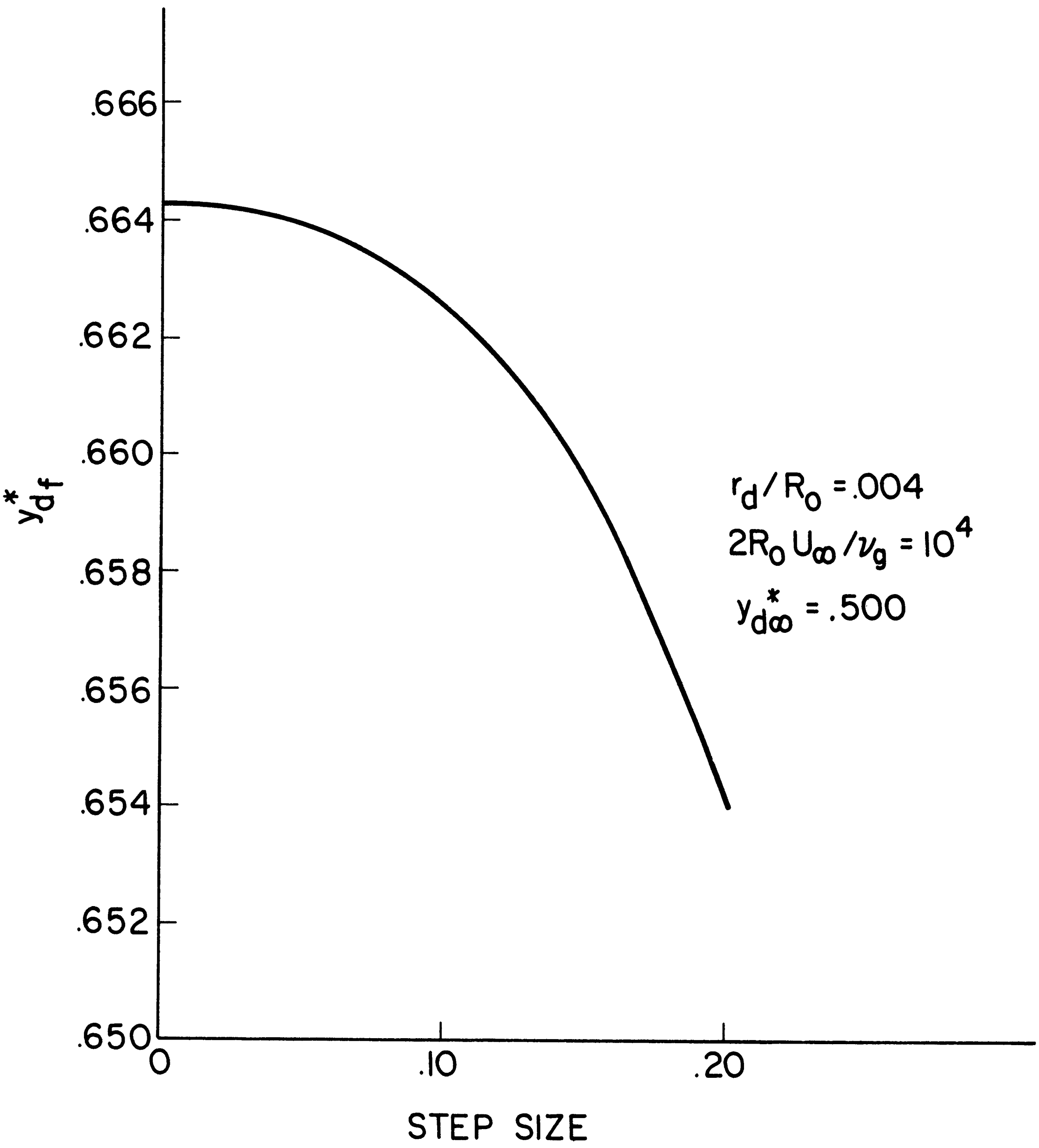

Figure 8-a. Convergence of solution for $y^*_{d_f}$ with respect to size of increment used in integrating drop trajectories.

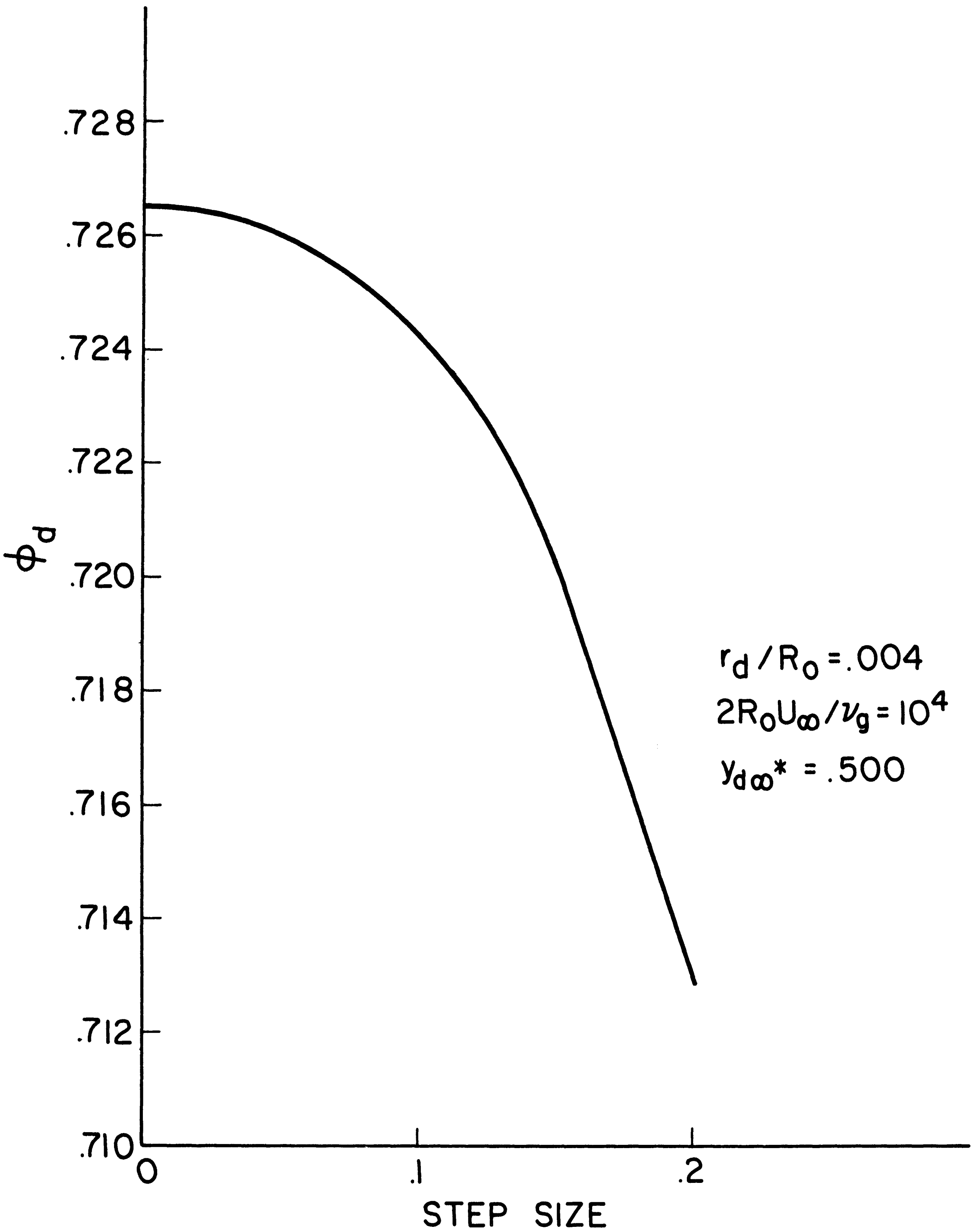

Figure 8-b. Convergence of solution for φ_d with respect to size of increment used in integrating drop trajectories.

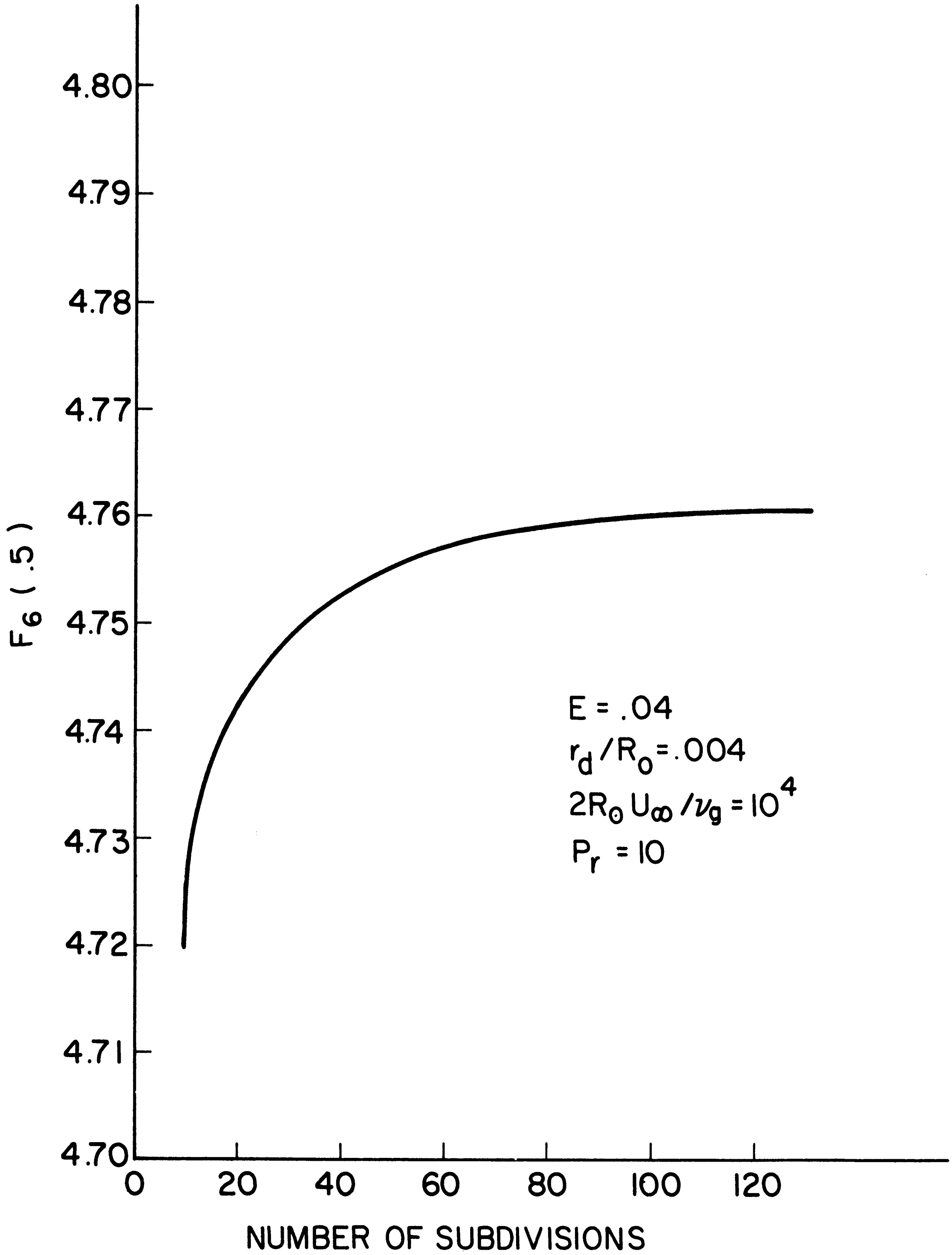

Figure 9-a. Convergence of function F_6 with respect to number of subdivisions for cylinder problem with $E \leq .1$.

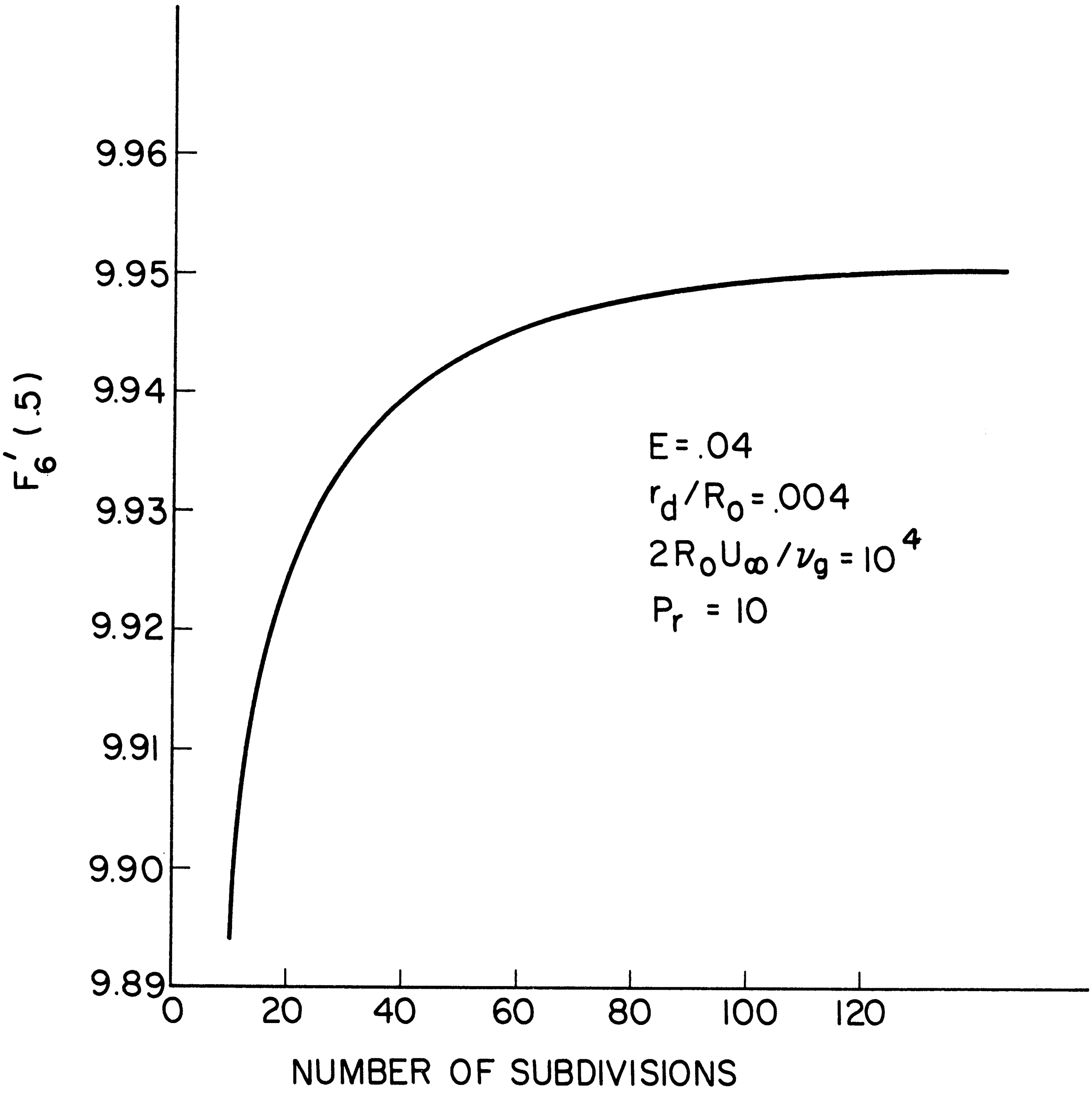

Figure 9-b. Convergence of function F_6' with respect to number of subdivisions for cylinder problem with $E \leq .1$.

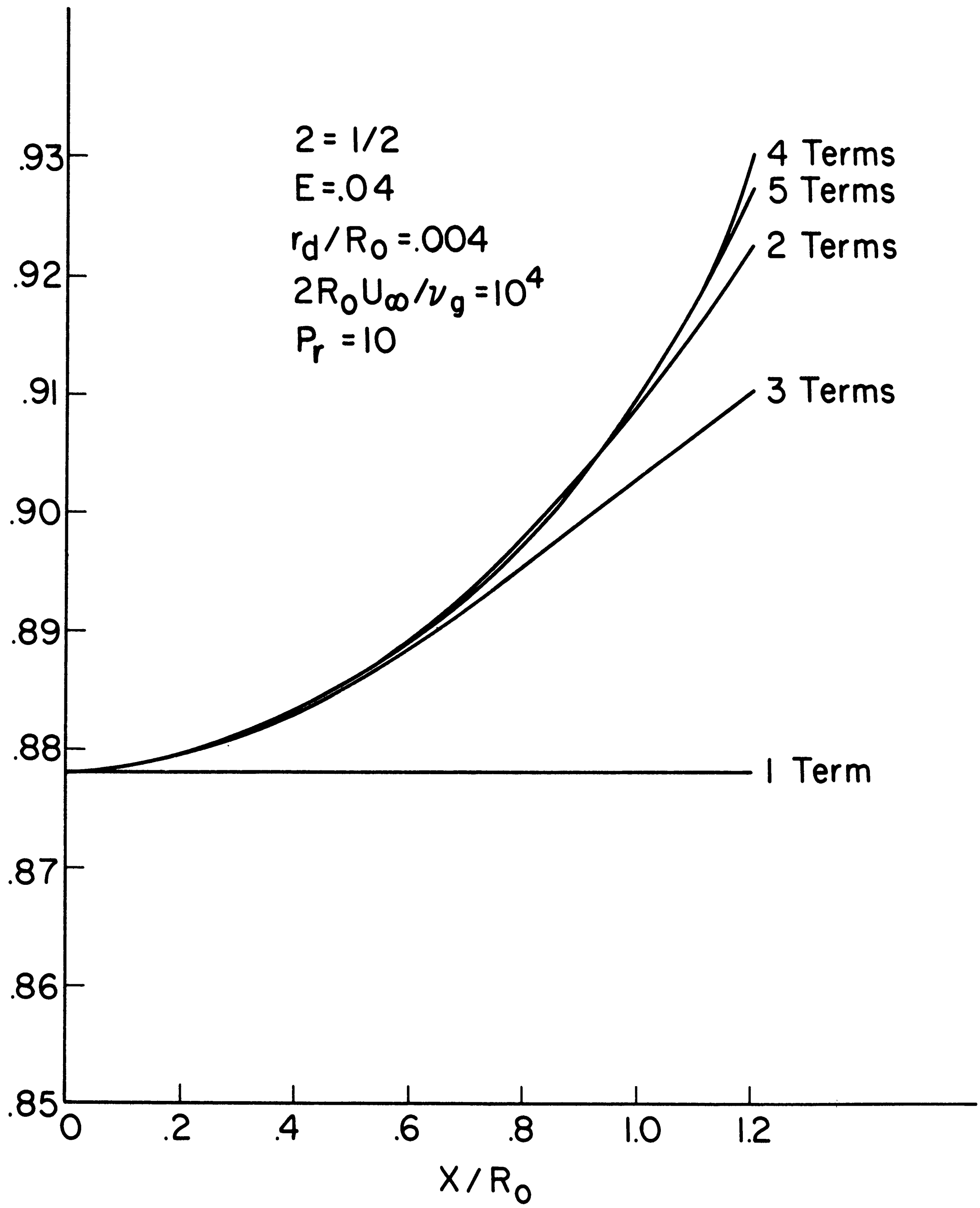

Figure 9-c. Convergence of solutions with respect to number of terms retained in expansion for cylinder problem with E $\leq$ .1.

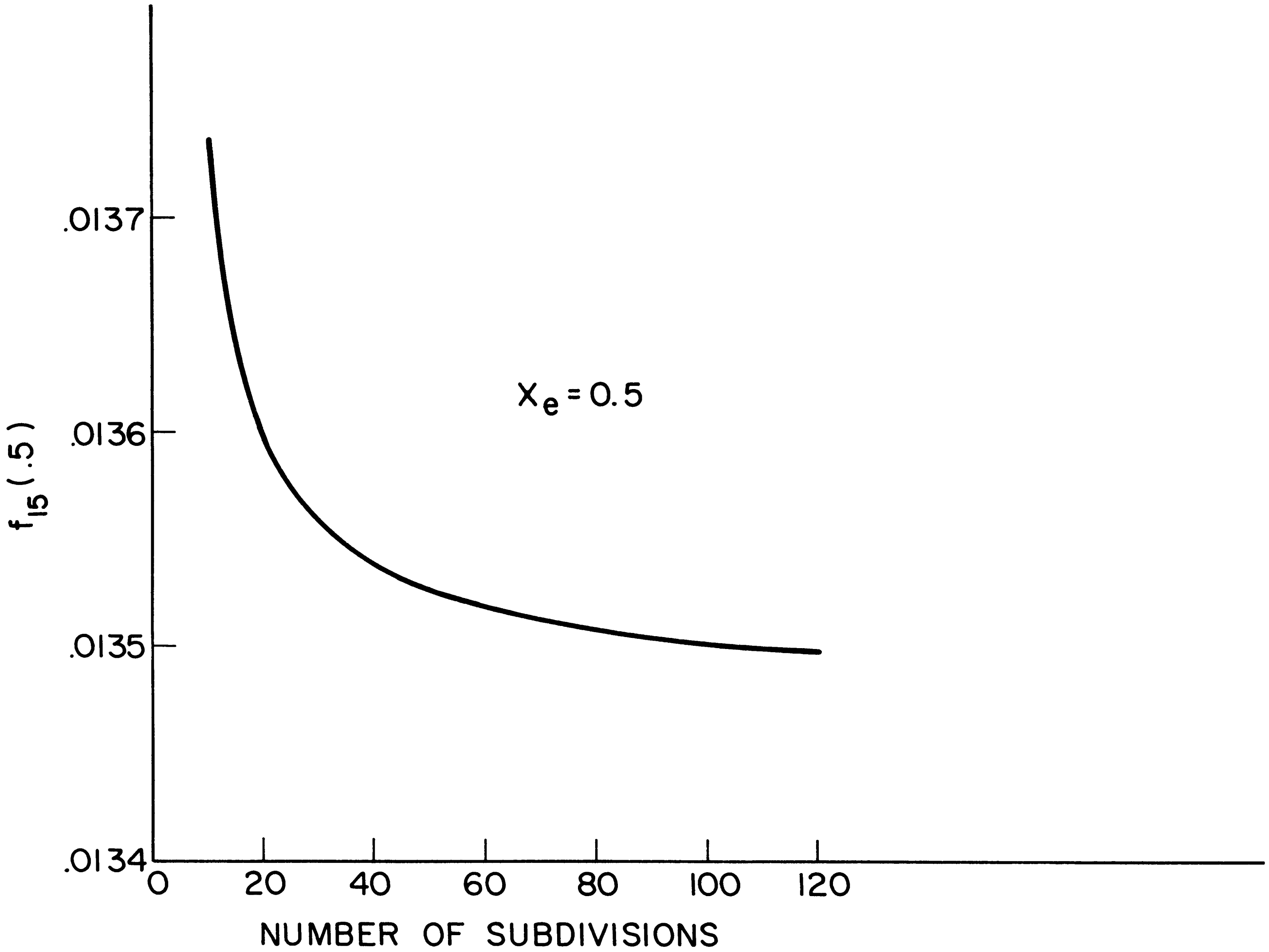

Figure 10-a. Convergence of function f_{15} with respect to number of subdivisions used for n8merical calculations in flat plate problem.

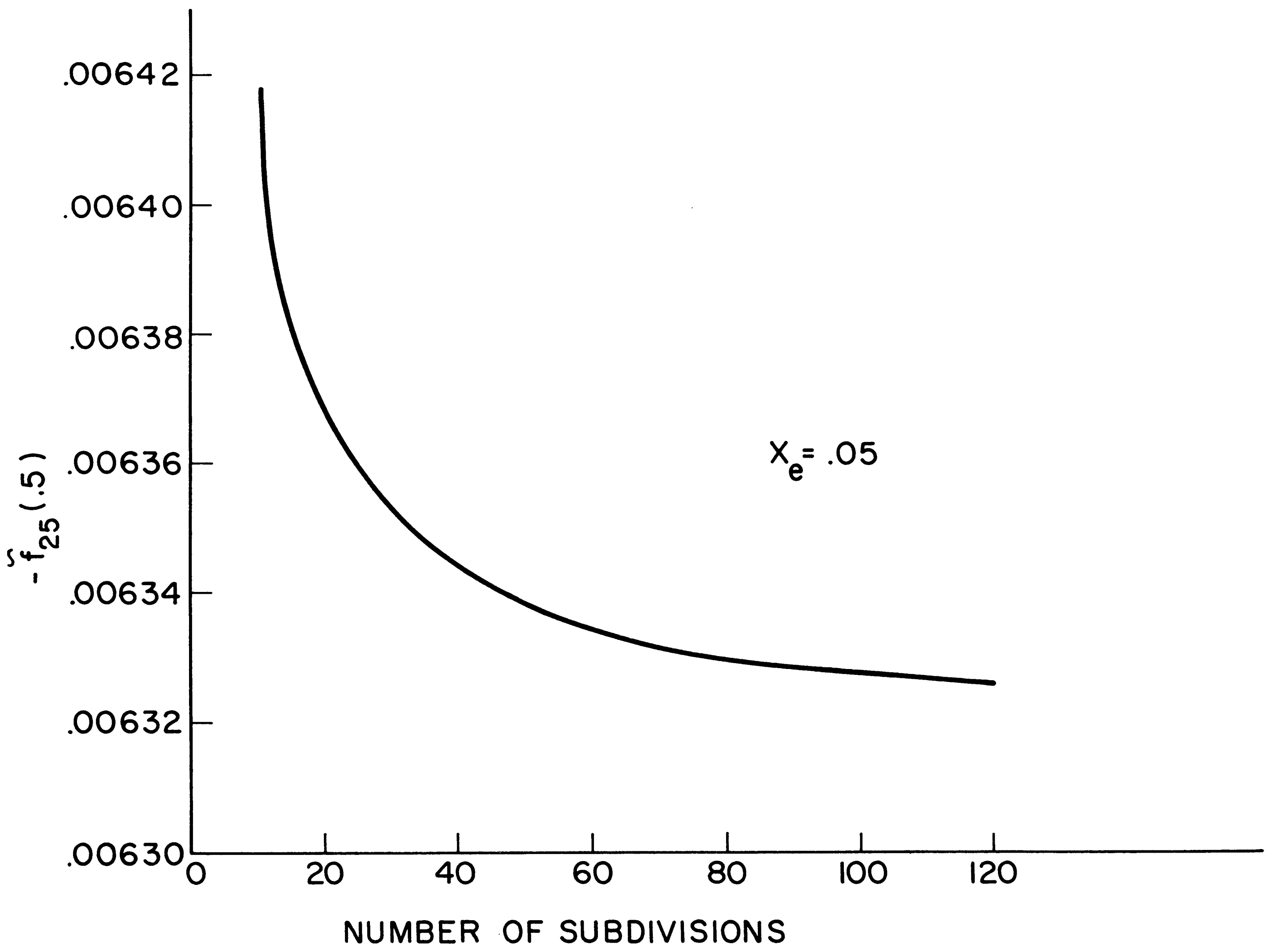

Figure 10-b. Convergence of function $\tilde{f}_{25}$ with respect to number of subdivisions used for numerical calculations in flat plate problem.

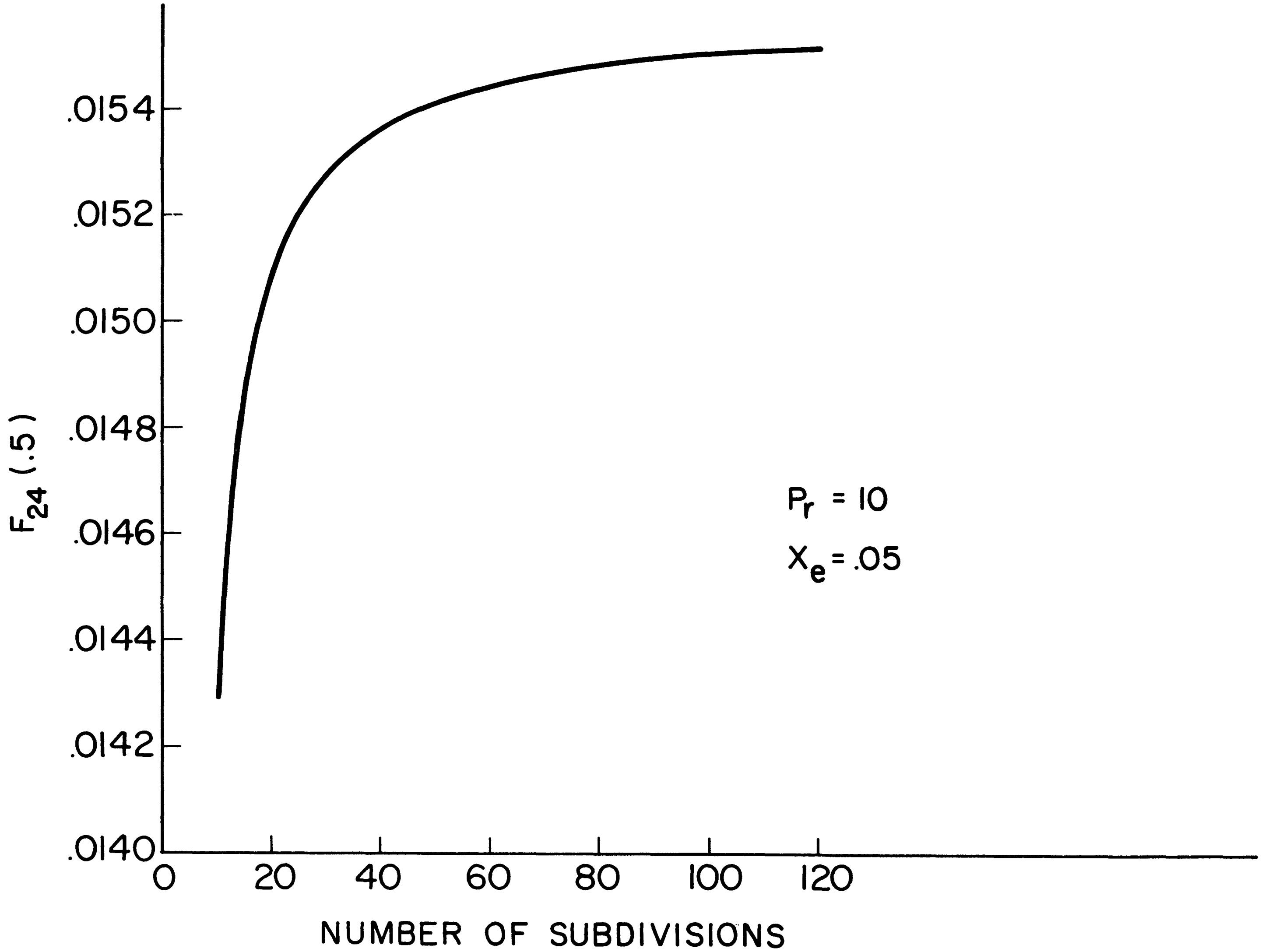

Figure 10-c. Convergence of function F_{24} with respect to number of subdivisions used for numerical calculations in flat plate problem.

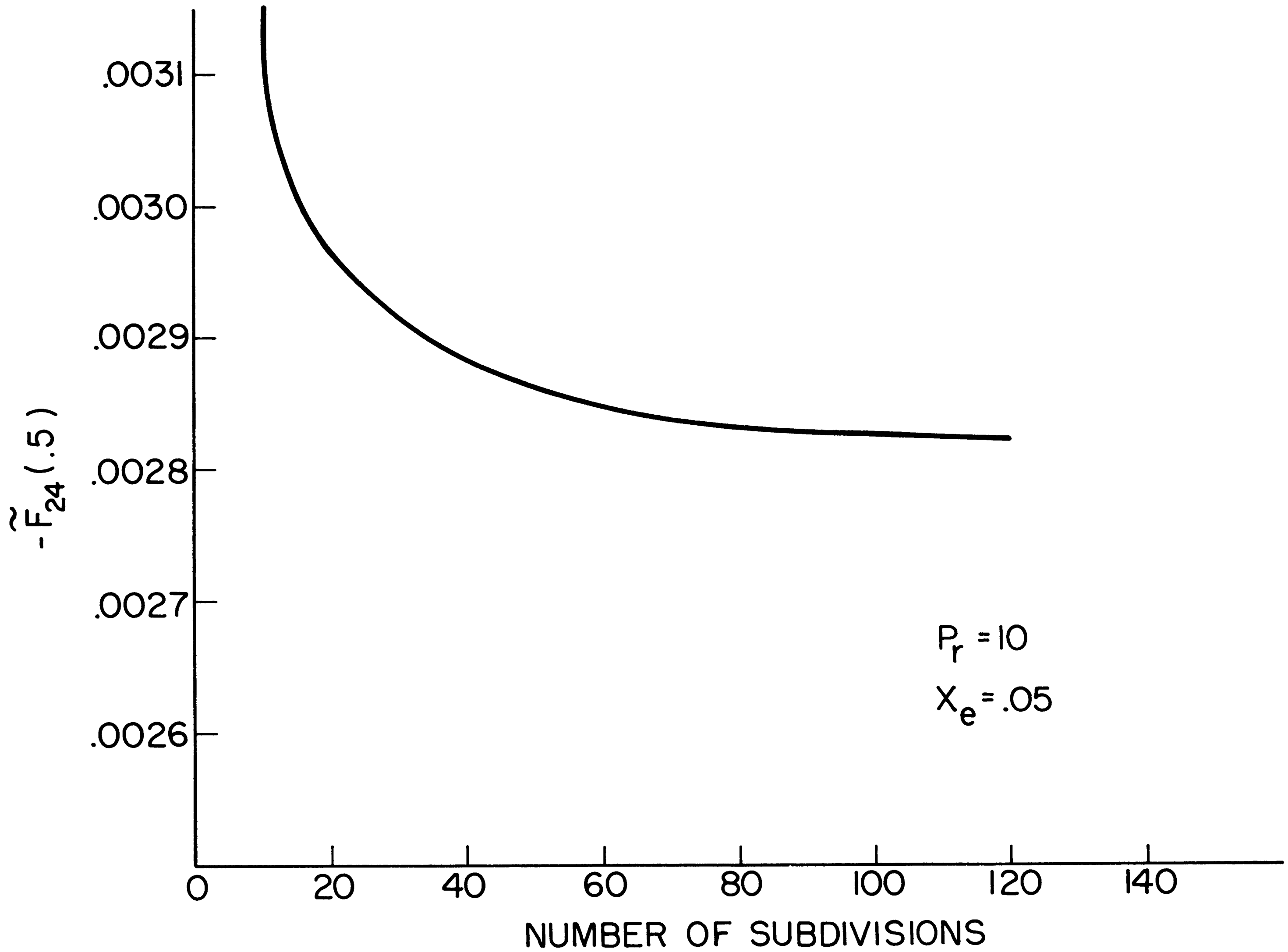

Figure 10-d. Convergence of function $\tilde{F}_{24}$ with respect to number of subdivisions used for numerical calculations in flat plate problem.

B. Numerical Results

1. THE CIRCULAR CYLINDER

The solutions to Eqs. (41) through (43) have been obtained numerically by means of the computer program listed in Appendix XVV. The results are available for values of E^2 between .01 and 100 and Prandtl numbers between 5 and 13.

These results can be used to calculate ψ^*, δ^*, and θ^* by using Eqs. (26) through (28). They have also been used to calculate the terms of Eqs. (38) through (40) for velocity, skin friction and Nusselt numbers respectively. The liquid film thickness has been plotted in Figures 11-a and 11-b for various values of E. The local wall shear stress is plotted in Figure 13-a and the local Nusselt number has been plotted in Figures 14-a, 14-b, and 15. Typical velocity profiles are plotted in Figure 12. For $E \leq 0.1$ it is first necessary to calculate the liquid drop trajectories before any of the physical quantities such as skin friction can be found in the film. The calculation of the drop trajectories has been carried out by using the computer program listed in Appendix XXV to integrate Eqs. (17*). This program also calculates the quantities of Eqs. (17*A.C.) which must be known to solve the governing equations for the film. Typical drop trajectories are shown in Figures 16-a through 16-i. The quantities of (17*A.C.) were tabulated for values of r_d/R_0 ranging between 0.0004 and 0.01 and values of $2R_0U_\infty/\nu_g$ between 10^3 and 10^5.

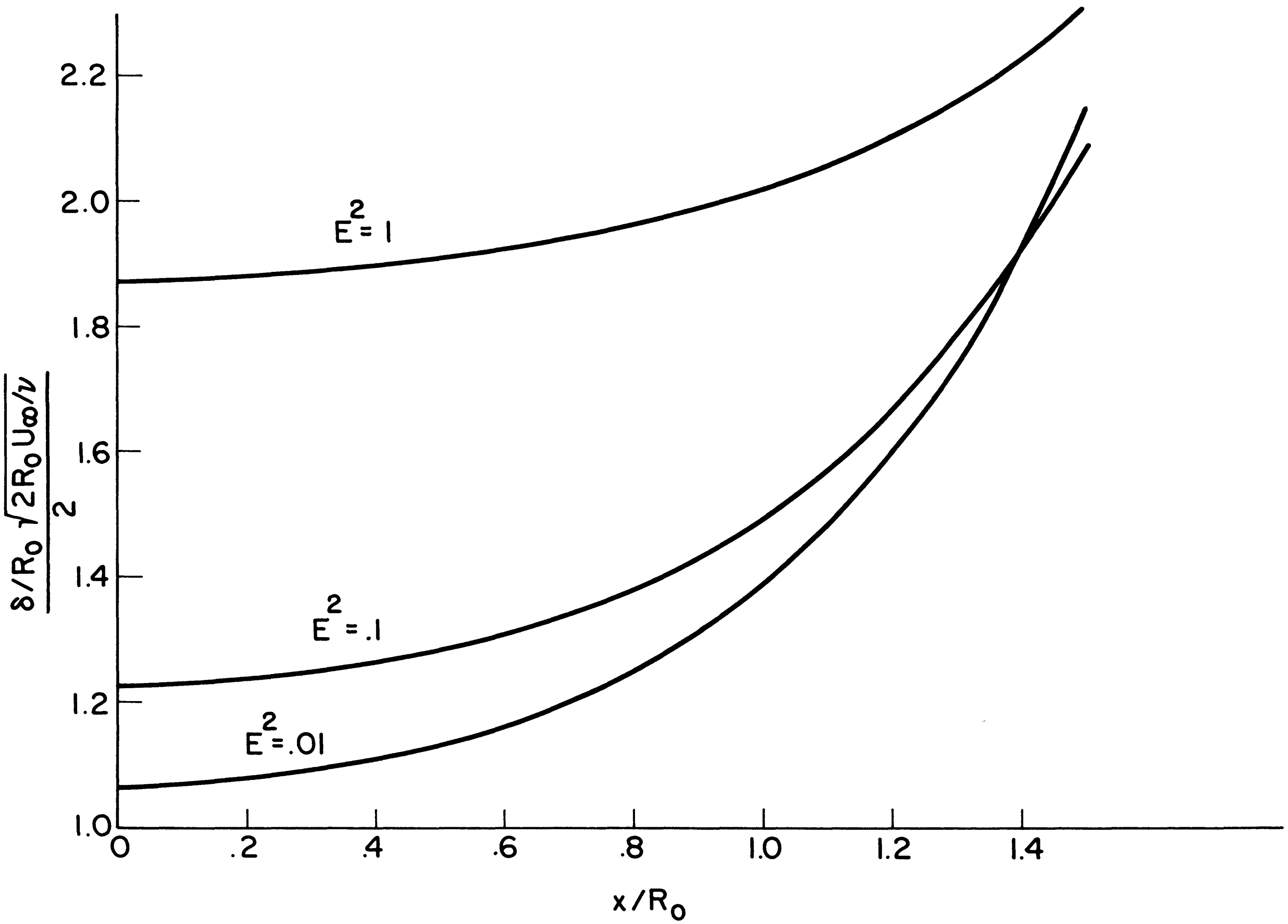

Figure 11-a. Local liquid film thickness for cylinder with $E^2 \geq .01$.

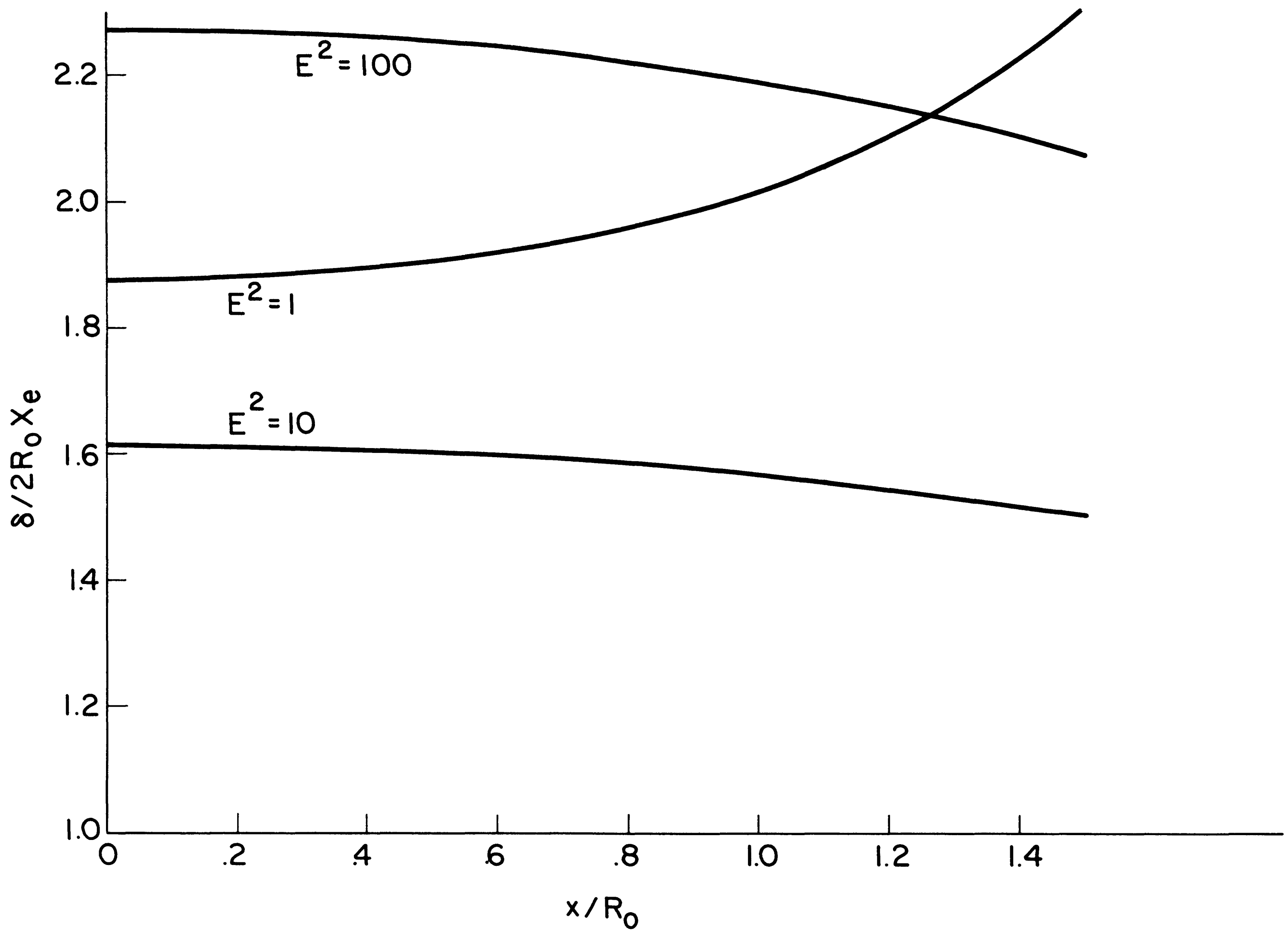

Figure 11-b. Local liquid film thickness for cylinder with $E^2 \geq .01$.

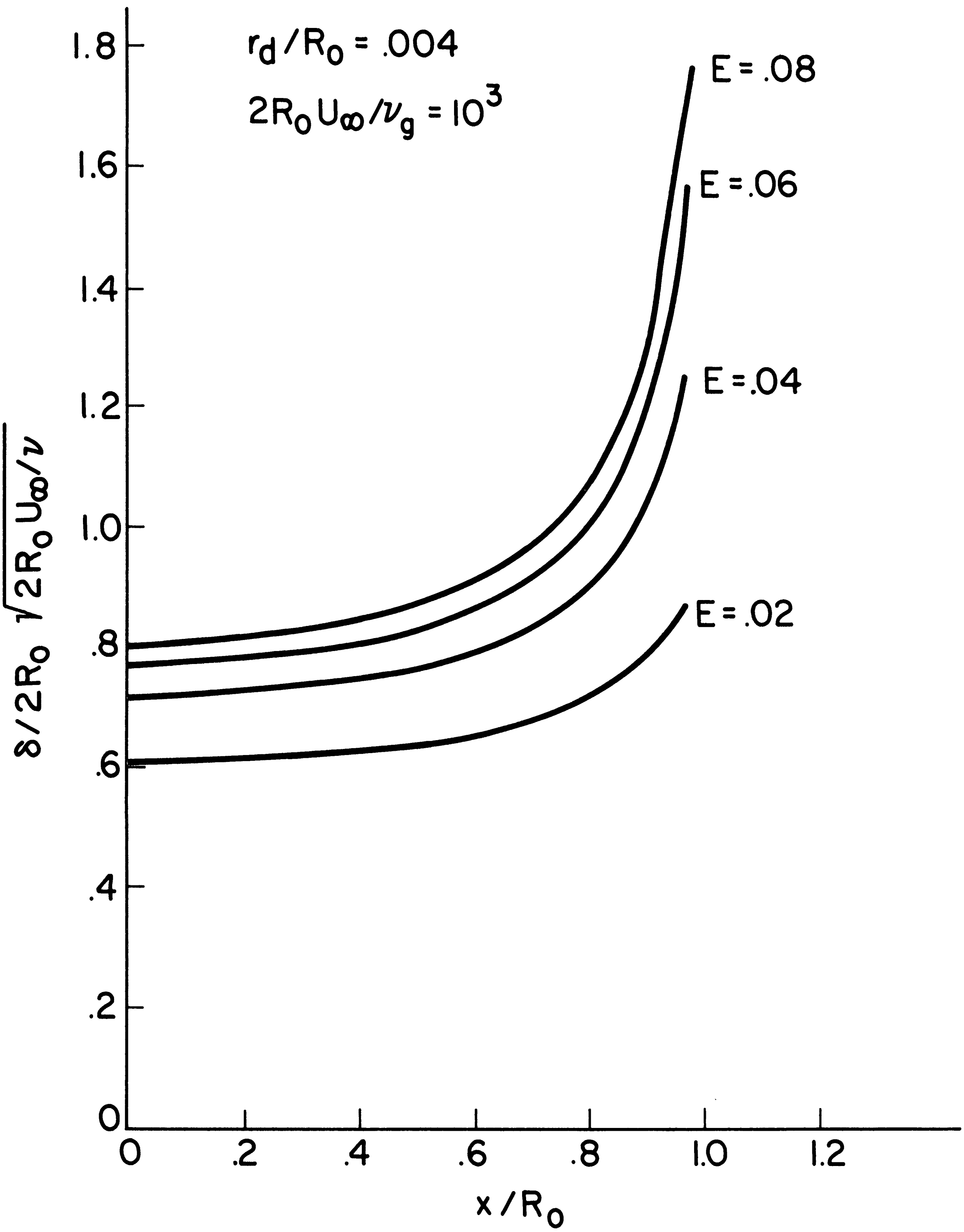

Figure 11-c. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_o = .004$ and $2R_oU_\infty/\nu_g = 10^3$.

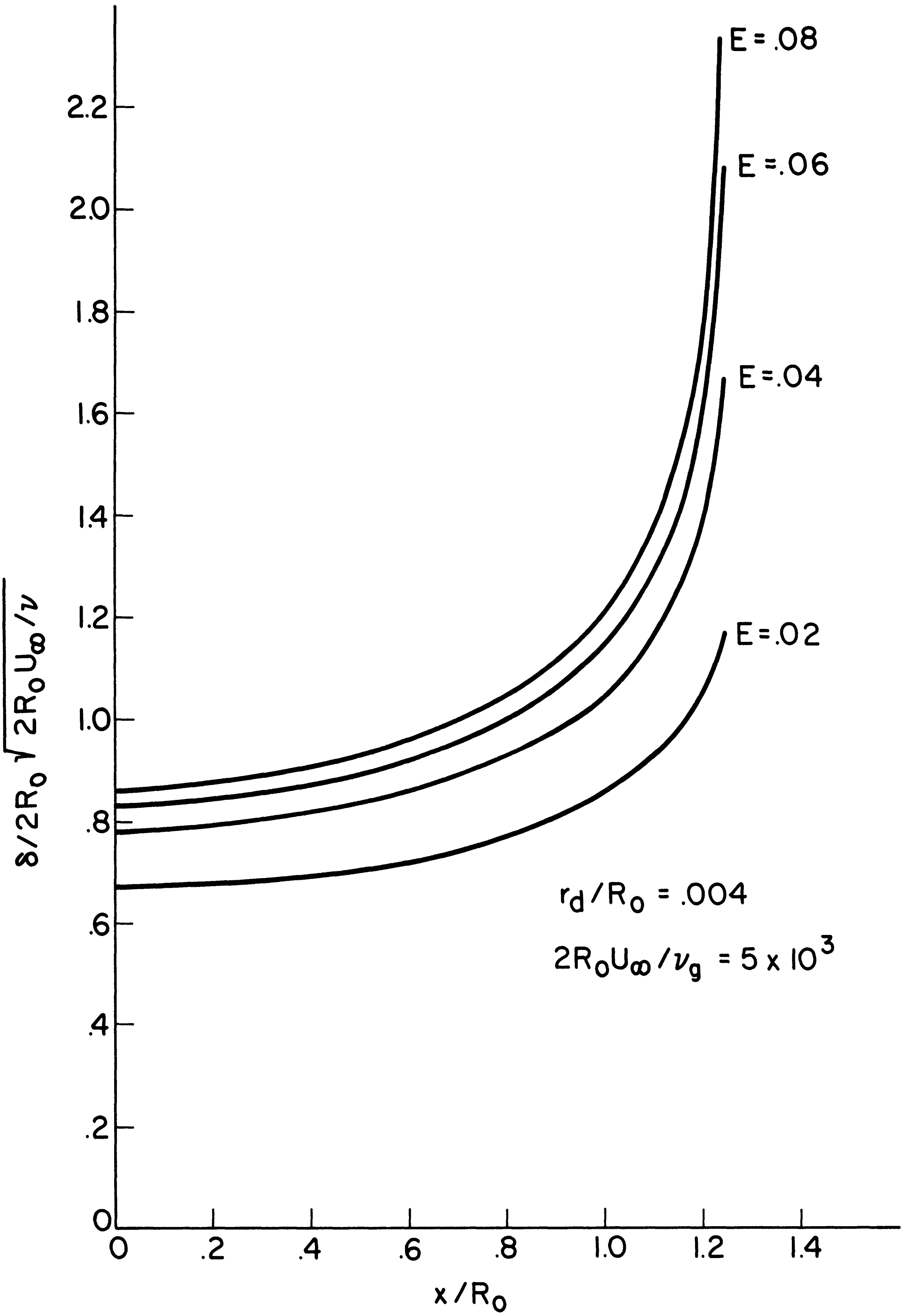

Figure 11-d. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_0 = .004$ and $2R_0U_\infty/\nu_g = 5\text{x}10^3$.

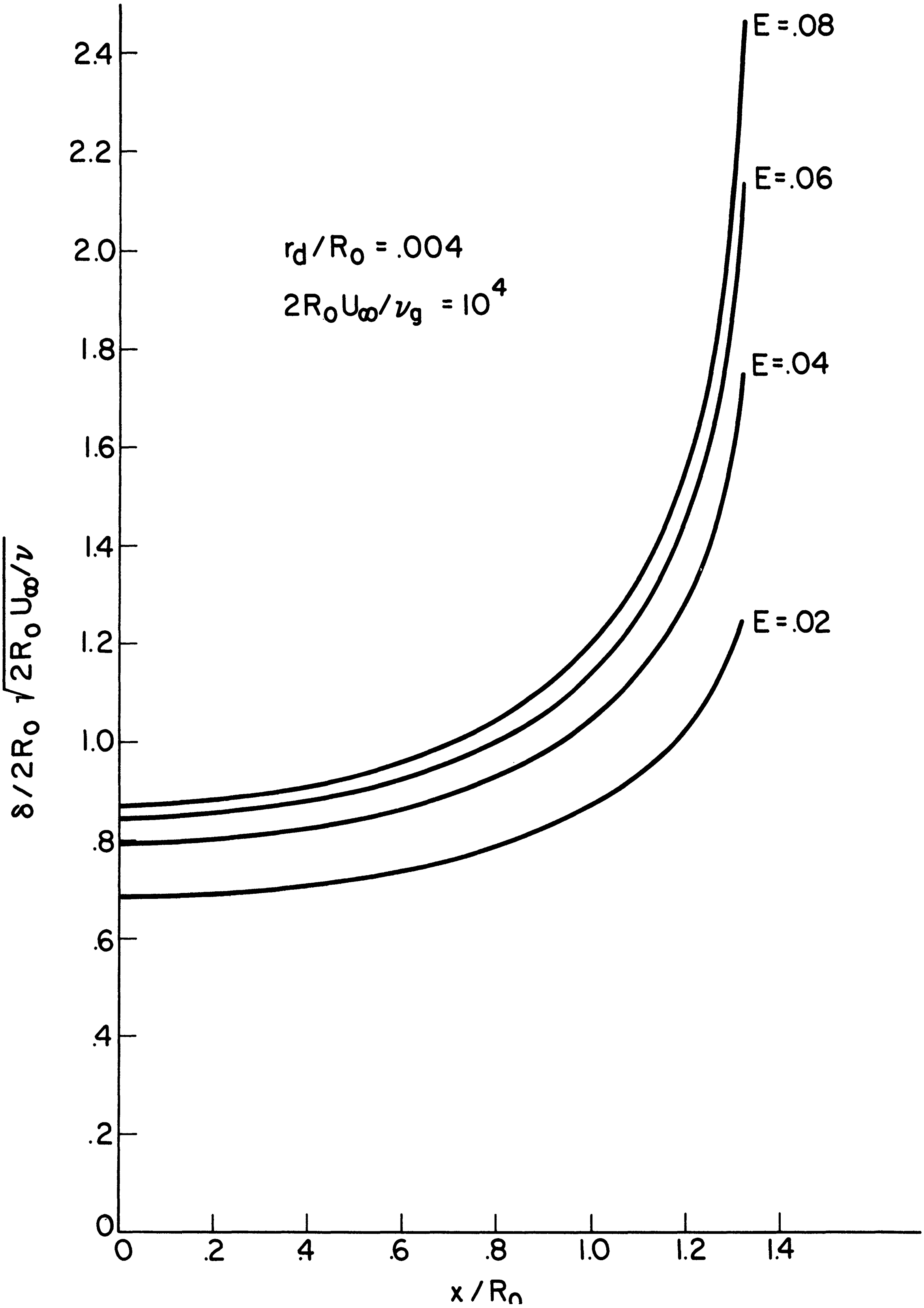

Figure 11-e. Local liquid film thickness for cylinder with $E \leq 0.1$, r_d/R_o = .004 and $2R_oU_\infty/\nu_g = 10^4$.

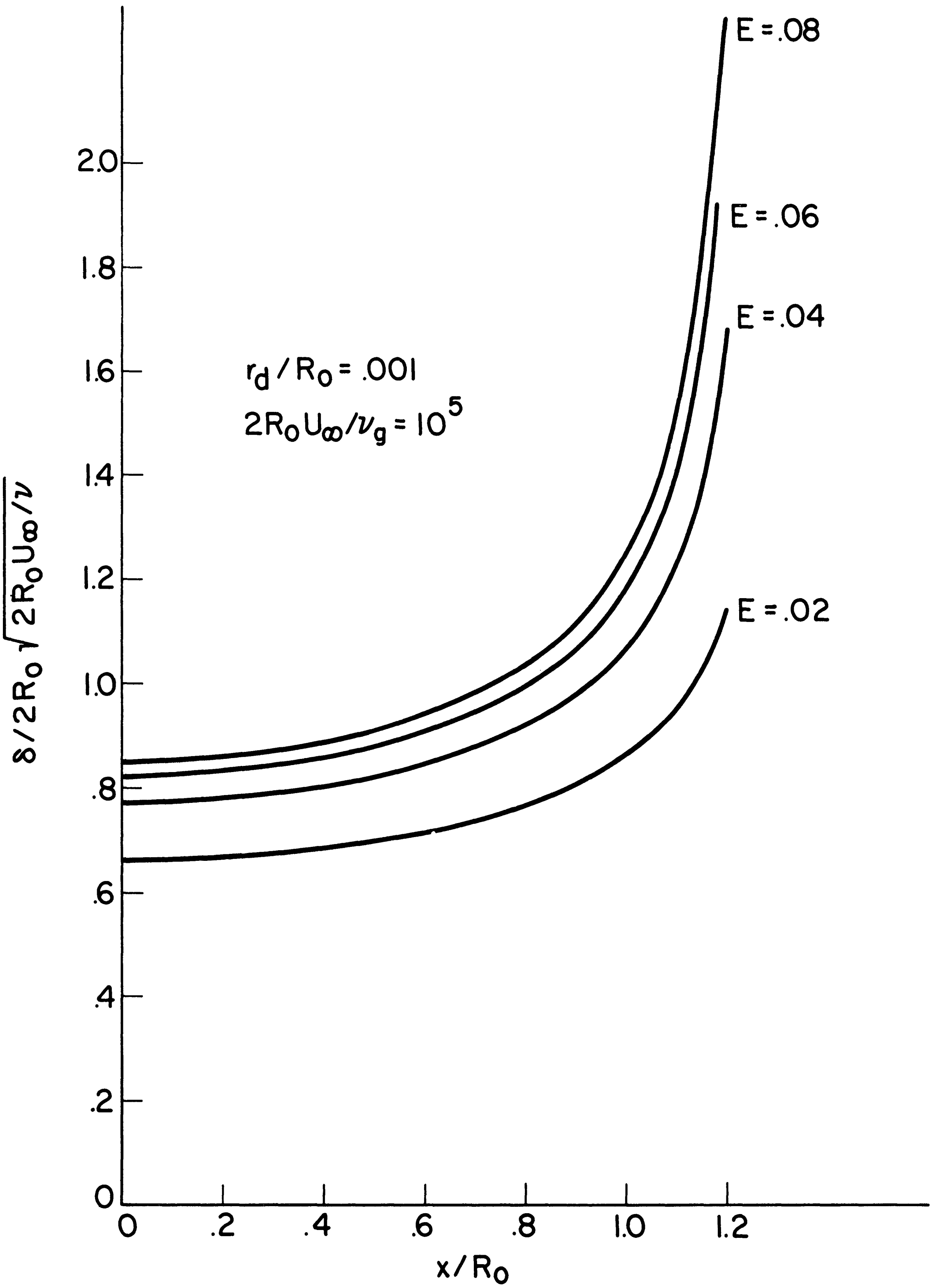

Figure 11-f. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_o = .001$ and $2R_oU_\infty/\nu_g = 10^5$.

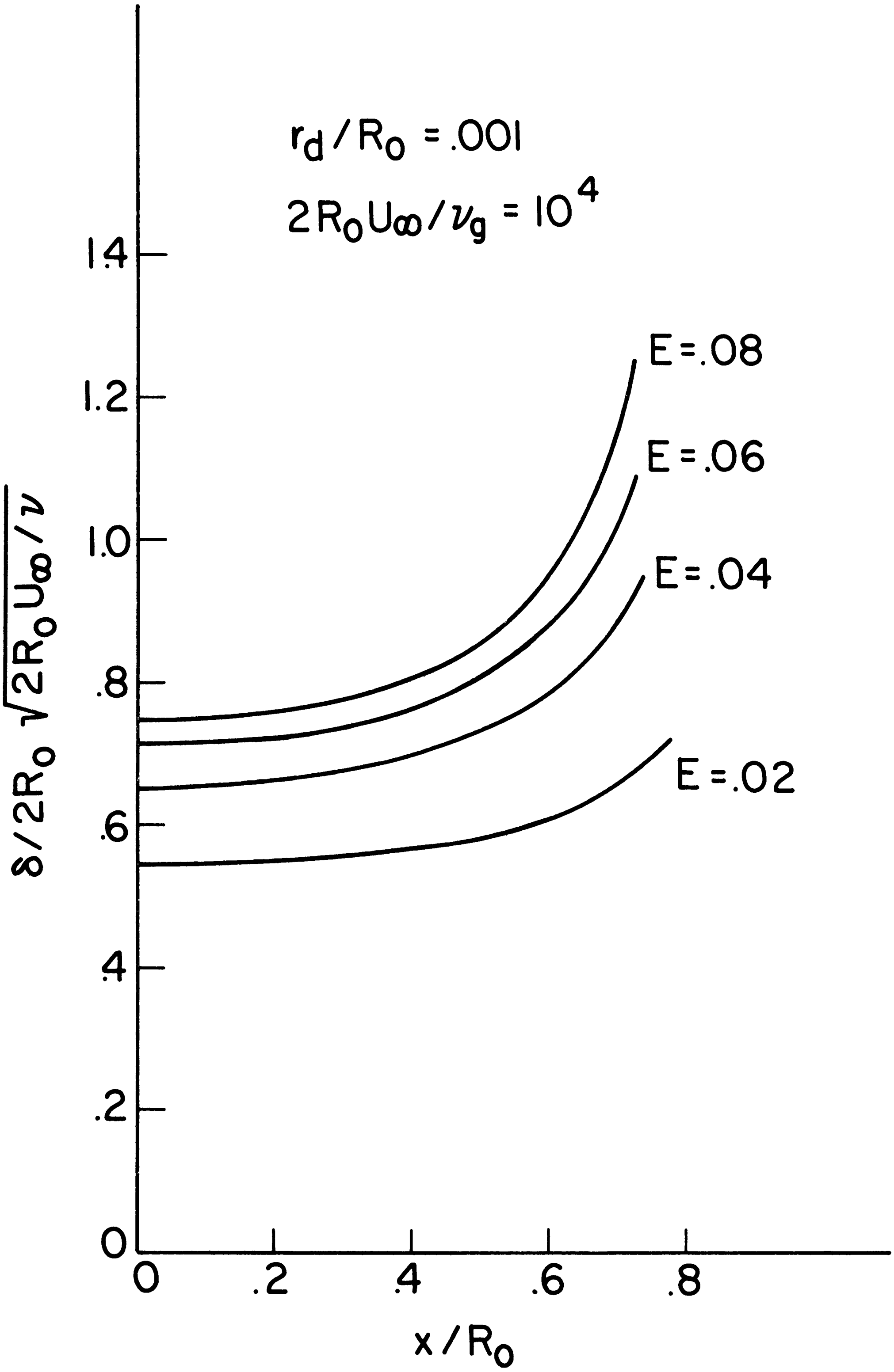

Figure 11-g. Local liquid film thickness for cylinder with $E \leq 0.1$, r_d/R_o = .001 and $2R_oU_\infty/\nu_g = 10^4$.

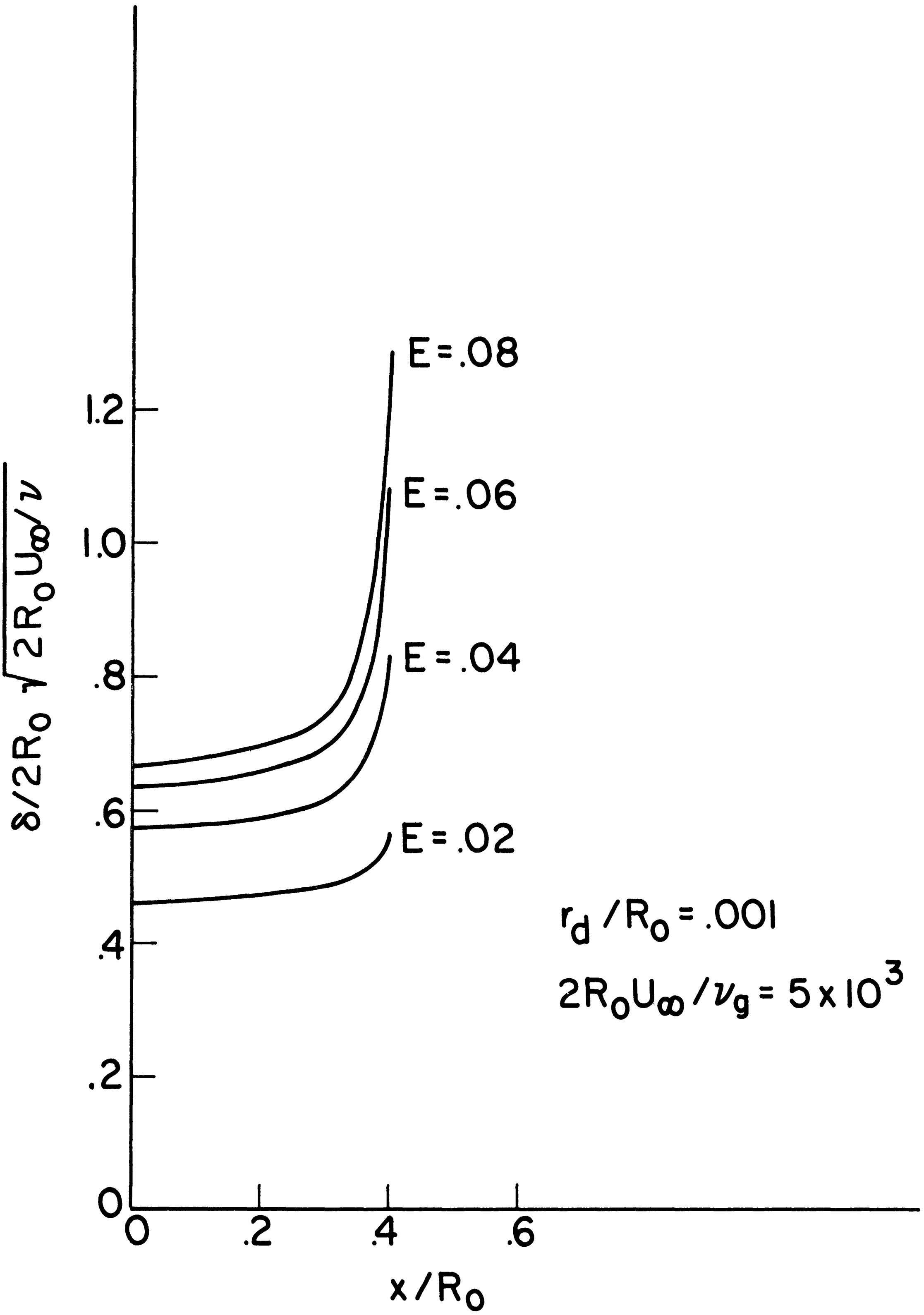

Figure 11-h. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_o = .001$ and $2R_oU_\infty/\nu_g = 5x10^3$.

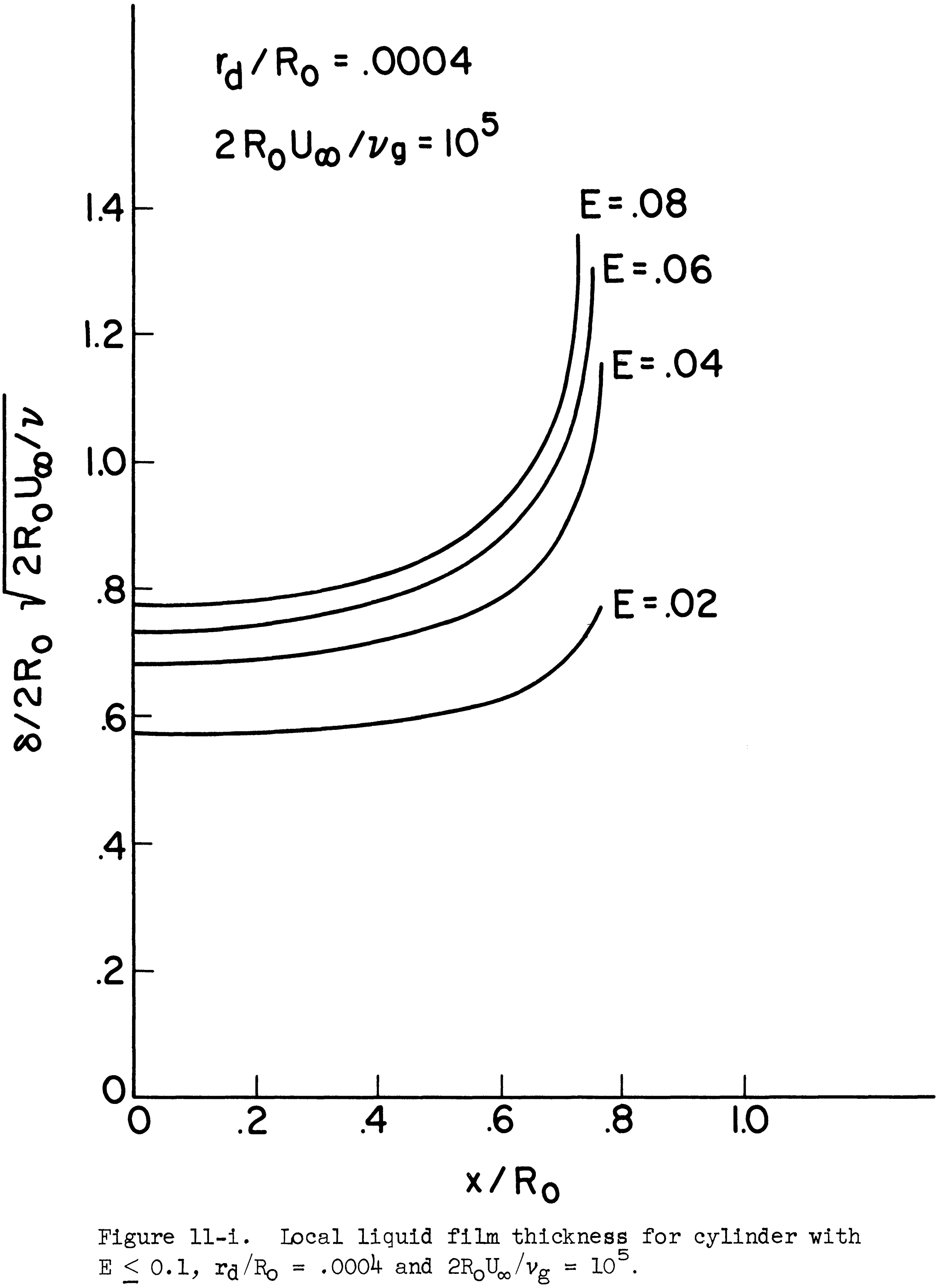

Figure 11-1. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_0 = .0004$ and $2R_0U_\infty/\nu_g = 10^5$.

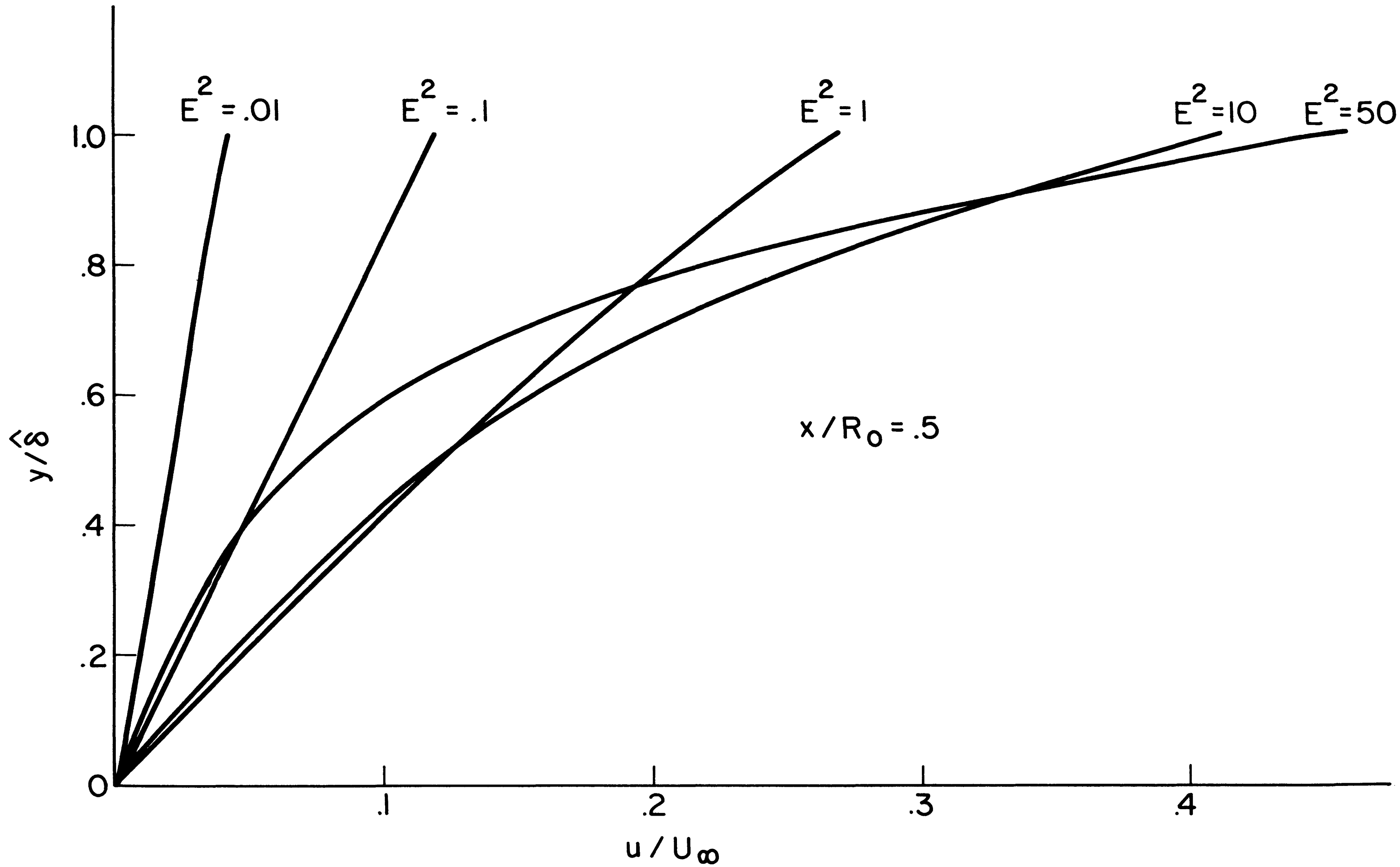

Figure 12. Velocity profile in liquid film at $x/R_o = .5$ for cylinder with $E^2 \geq .01$.

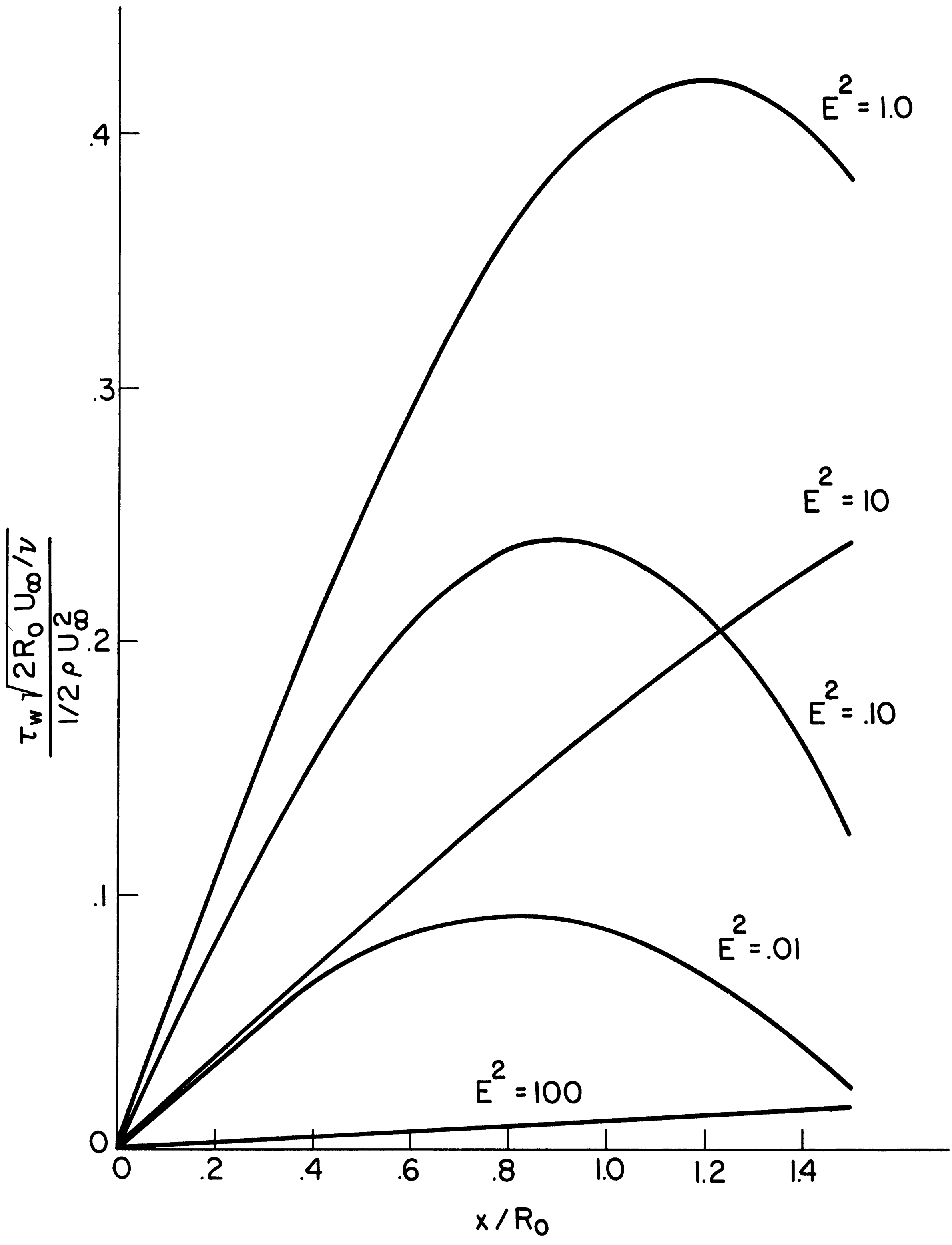

Figure 13-a. Local wall shear stress for cylinder with $E^2 \geq .01$.

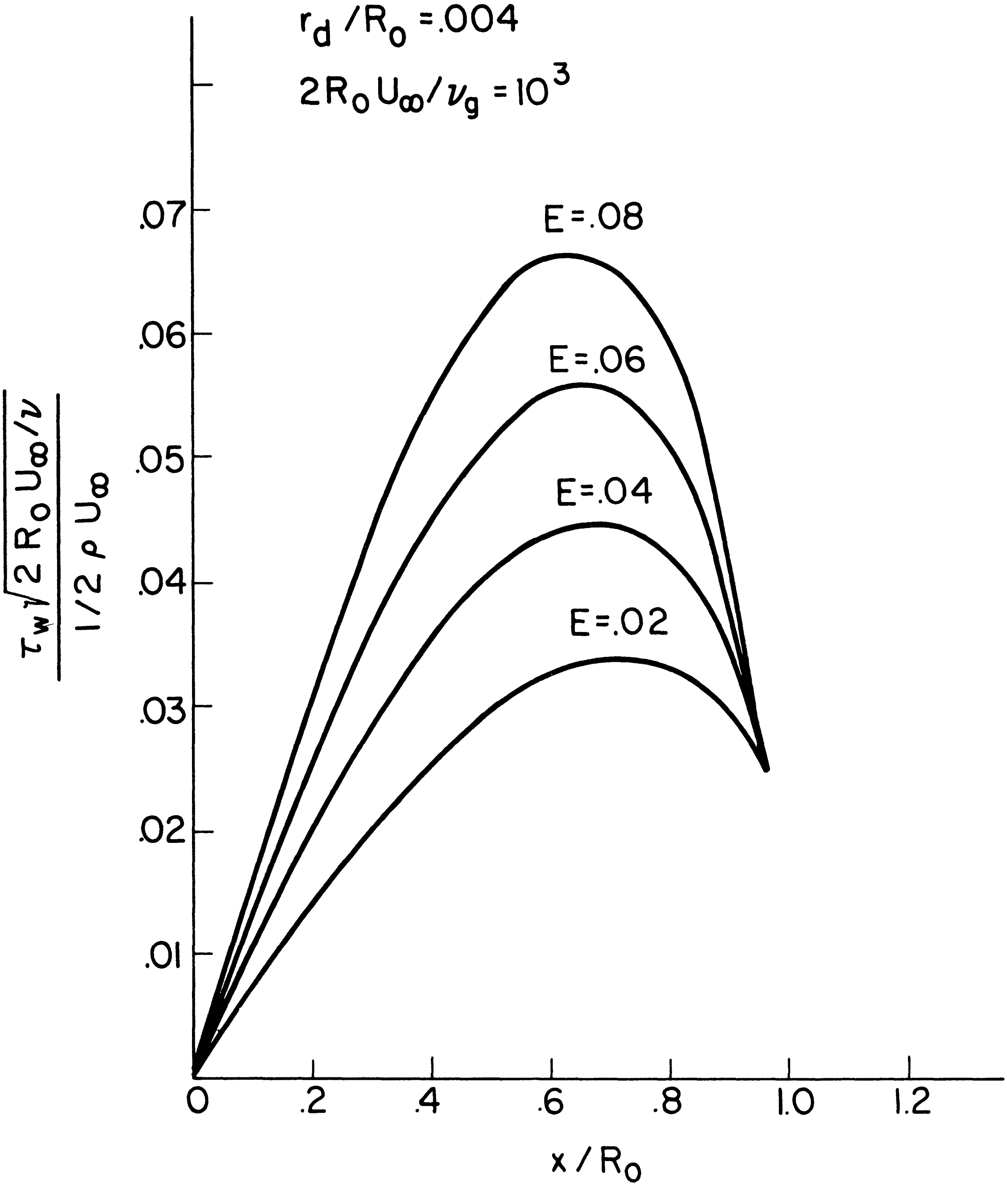

Figure 13-b. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_0 = .004$ and $2R_0U_\infty/\nu_g = 10^3$.

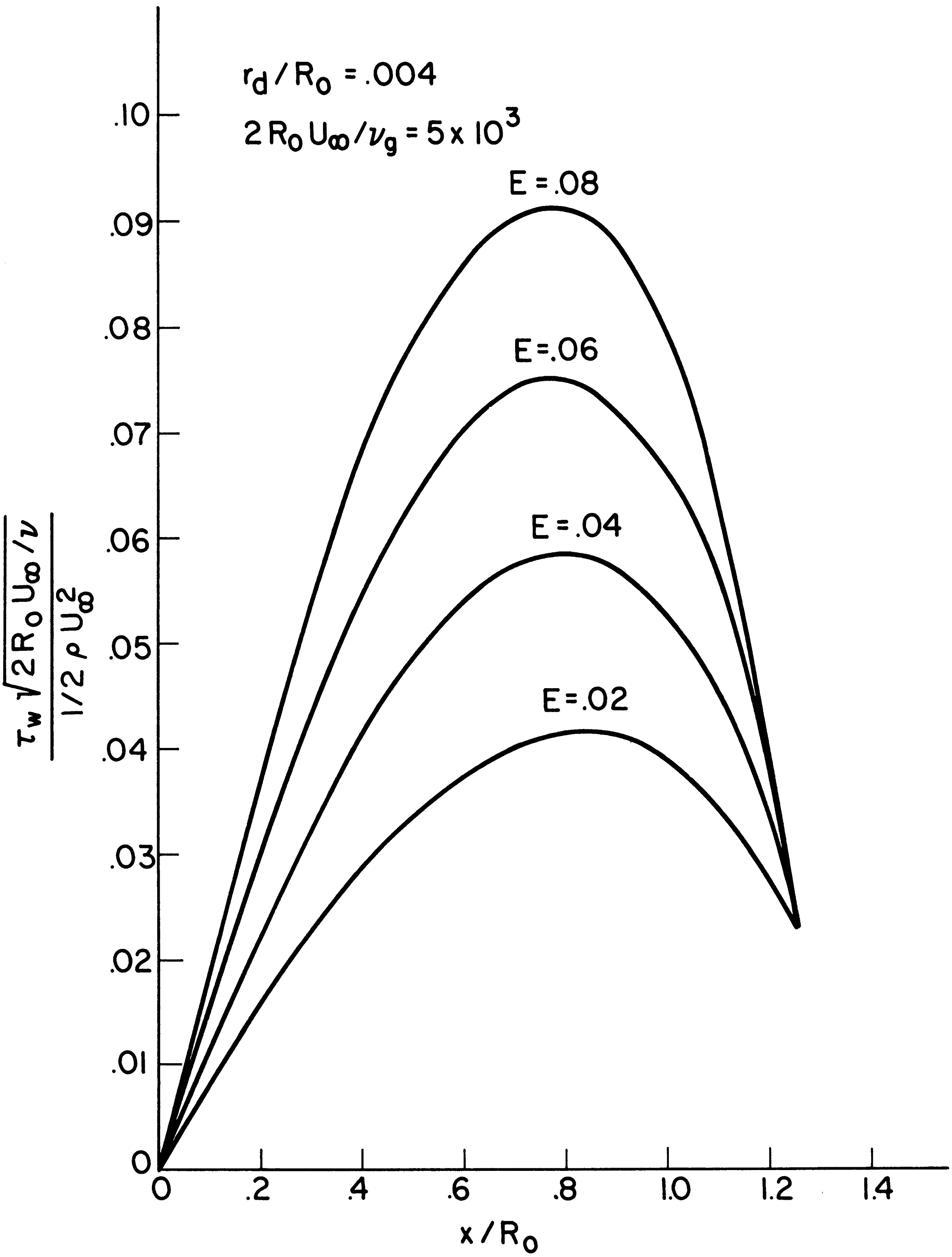

Figure 13-c. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_o = .004$ and $2R_oU_\infty/\nu_g = 5\times10^3$.

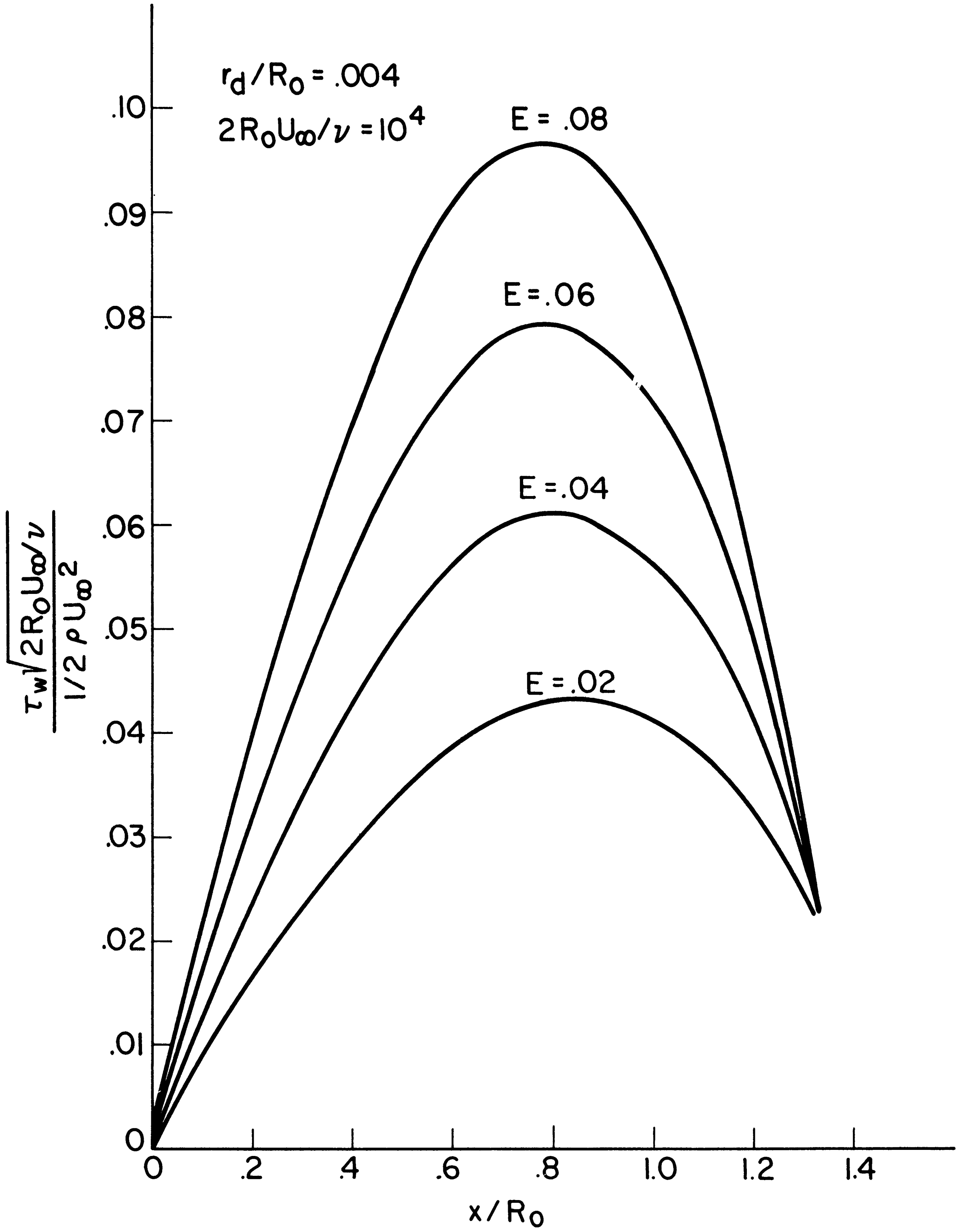

Figure 13-d. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_0 = .004$ and $2R_0U_\infty/\nu_g = 10^4$.

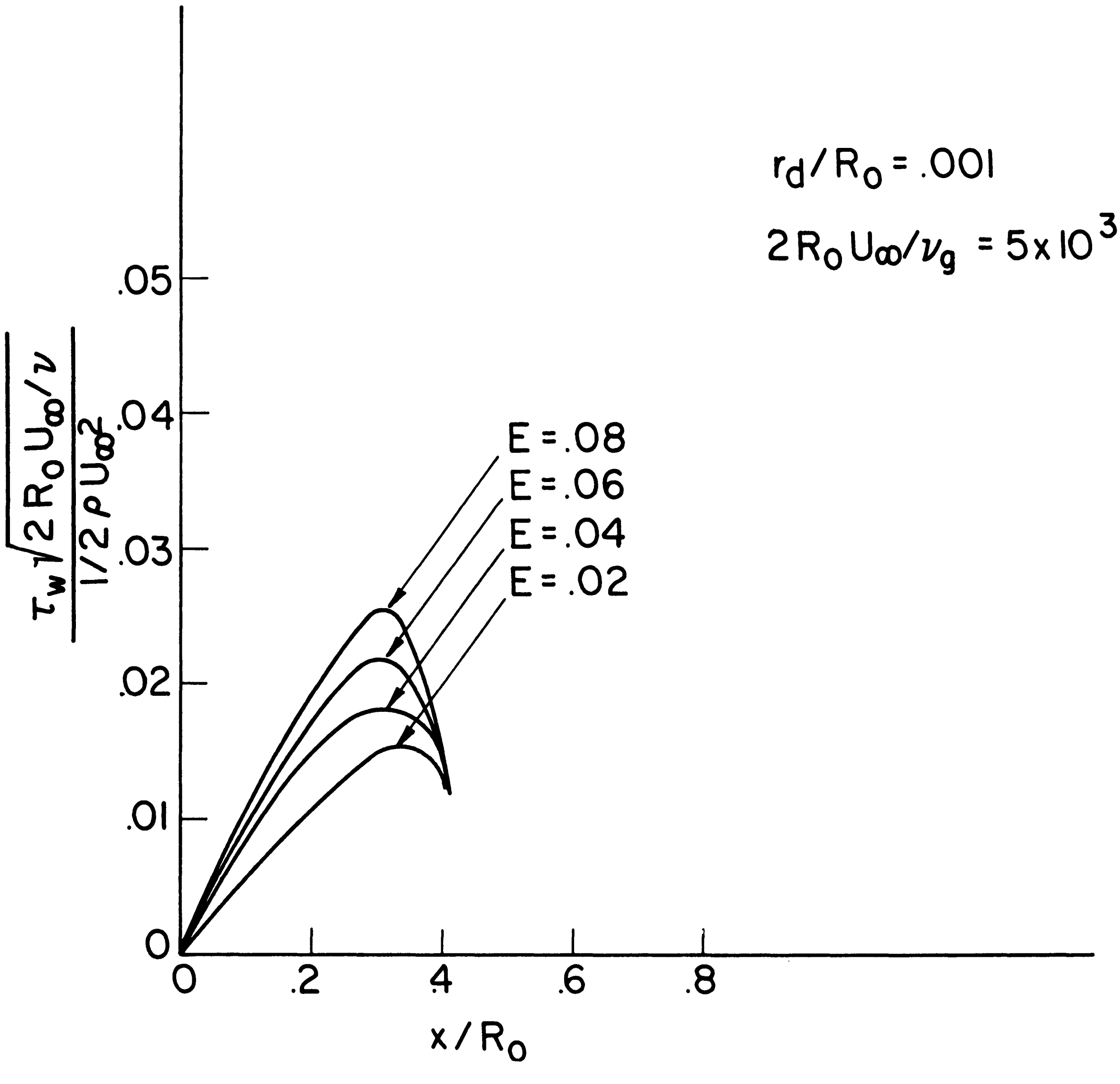

Figure 13-e. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_0 = .001$ and $2R_0U_\infty/\nu_g = 5\times10^2$.

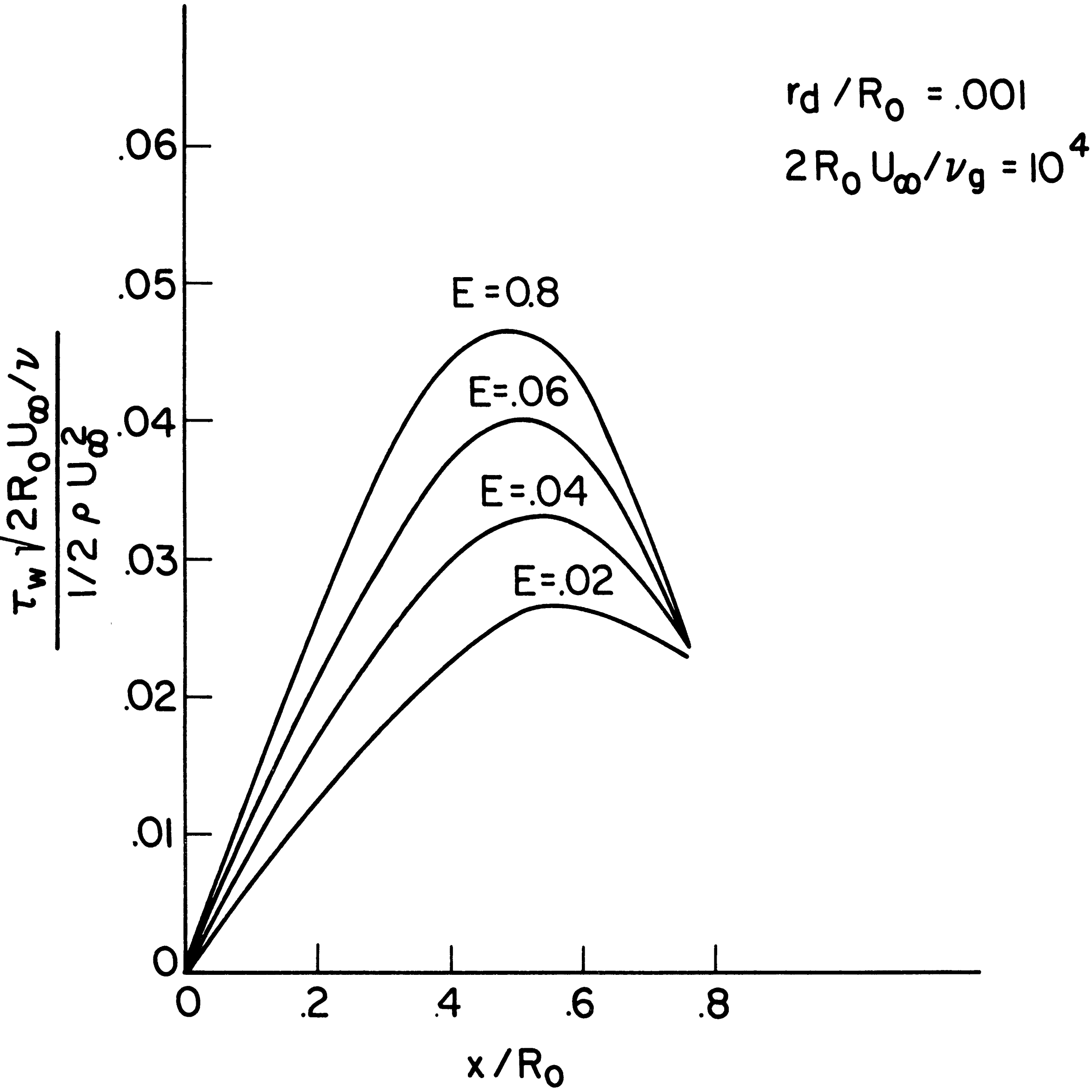

Figure 13-f. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_0 = .001$ and $2R_0U_\infty/\nu_g = 10^4$.

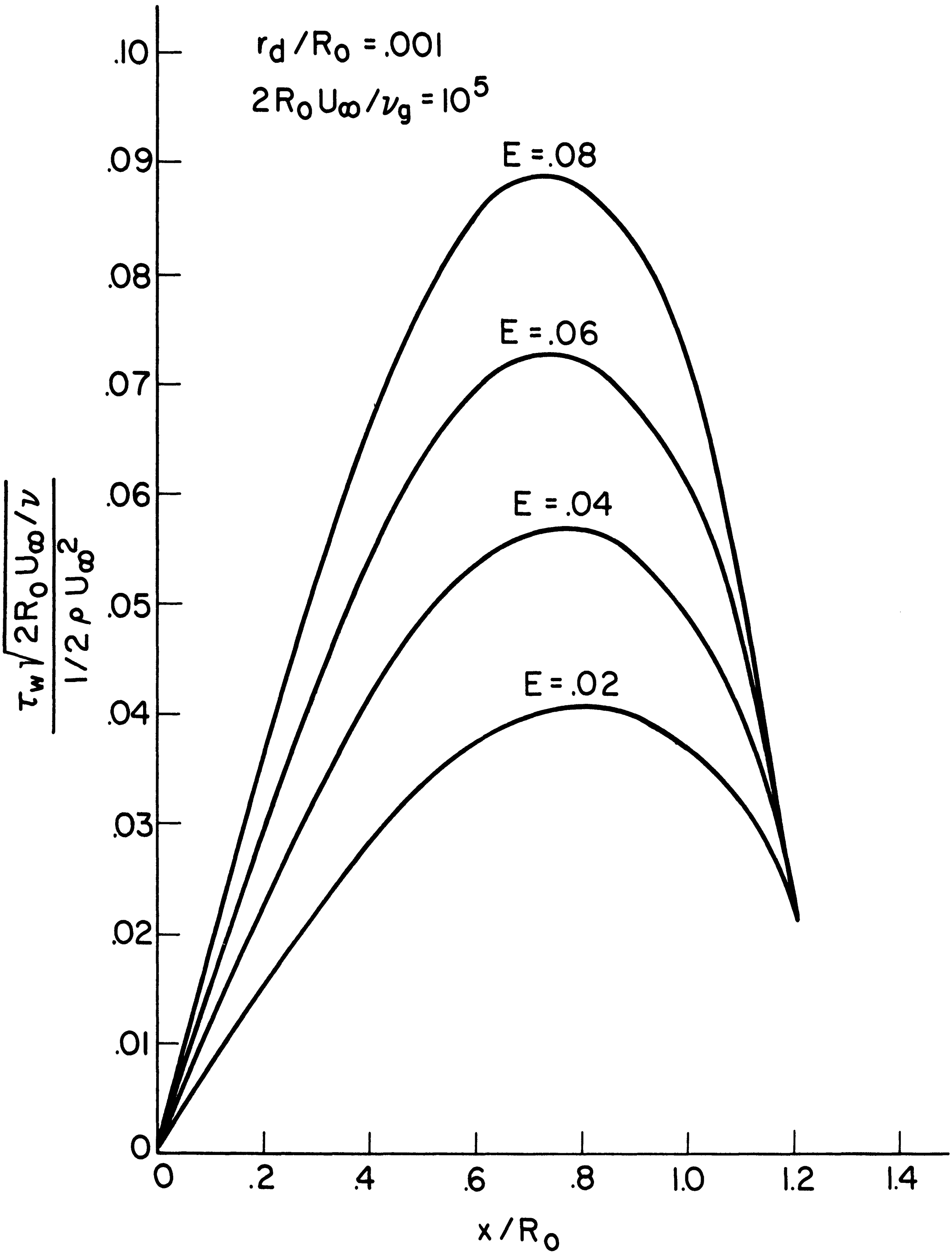

Figure 13-g. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_0 = .001$ and $2R_0U_\infty/\nu_g = 10^5$.

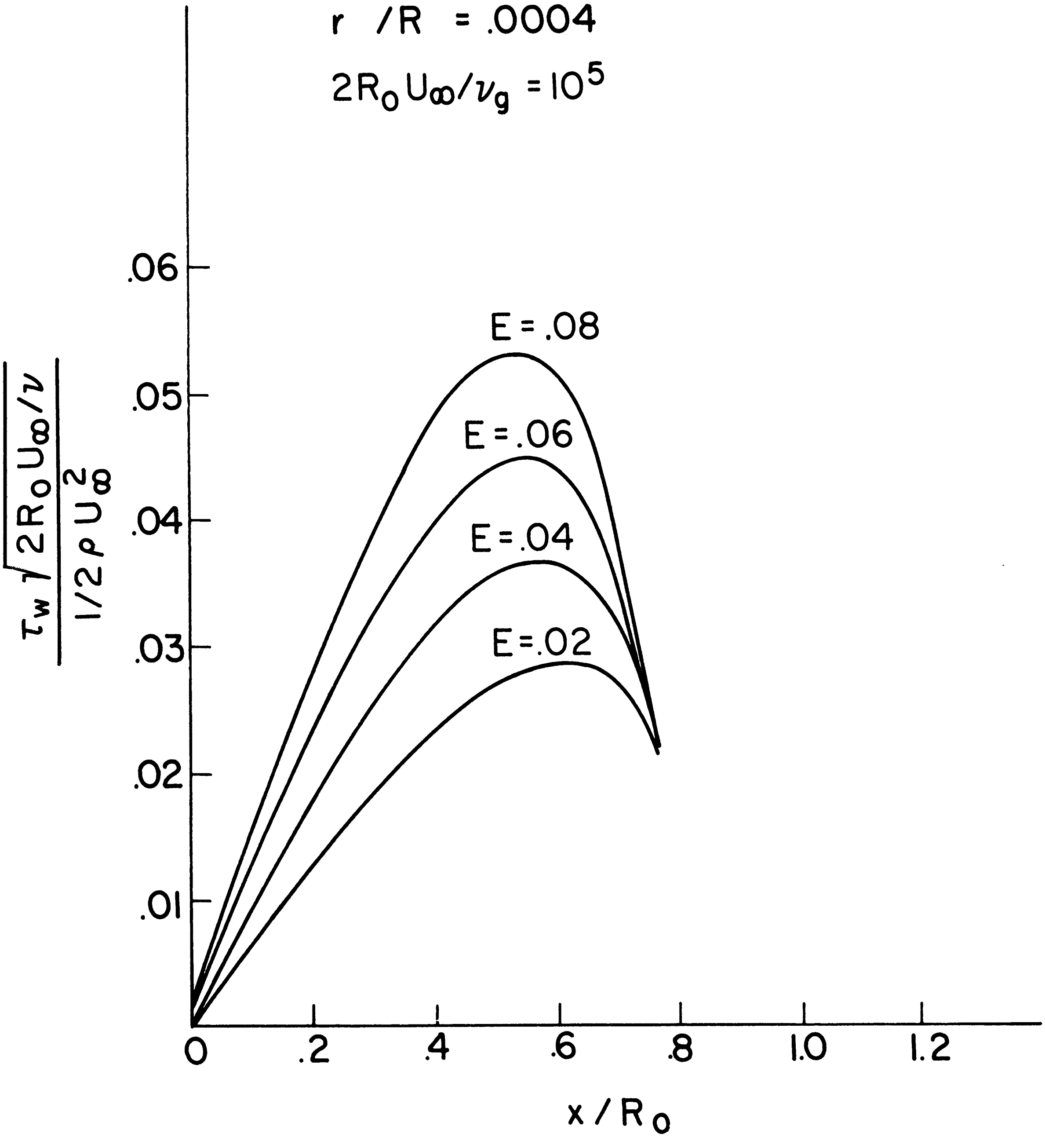

Figure 13-h. Local wall shear stress for cylinder with $E \leq 0.1$, r_d/R_o = .0004 and $2R_oU_\infty/\nu_g = 10^5$.

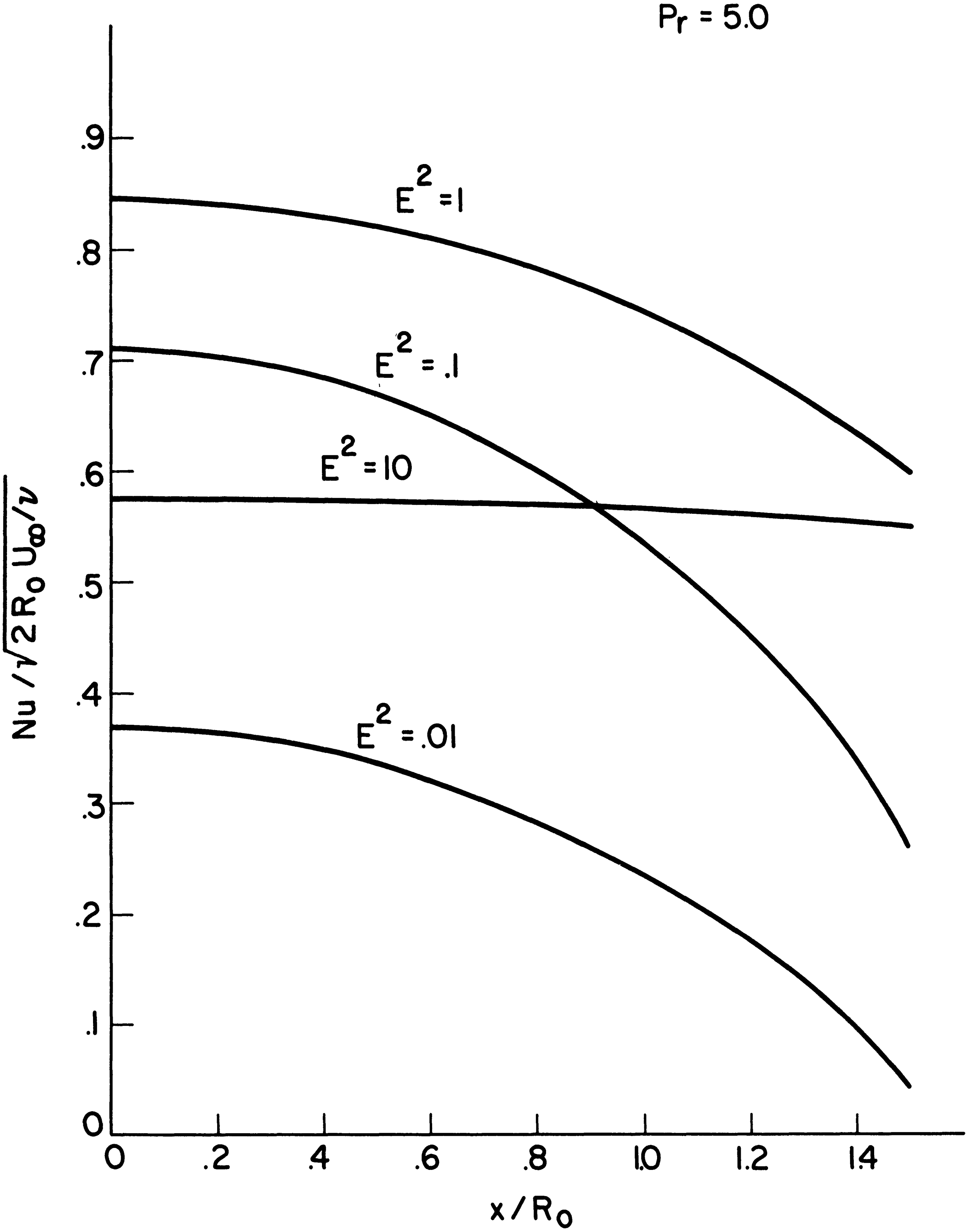

Figure 14-a. Local Nusselt number for cylinder with $P_r = 5.0$ and $E \geq 0.01$.

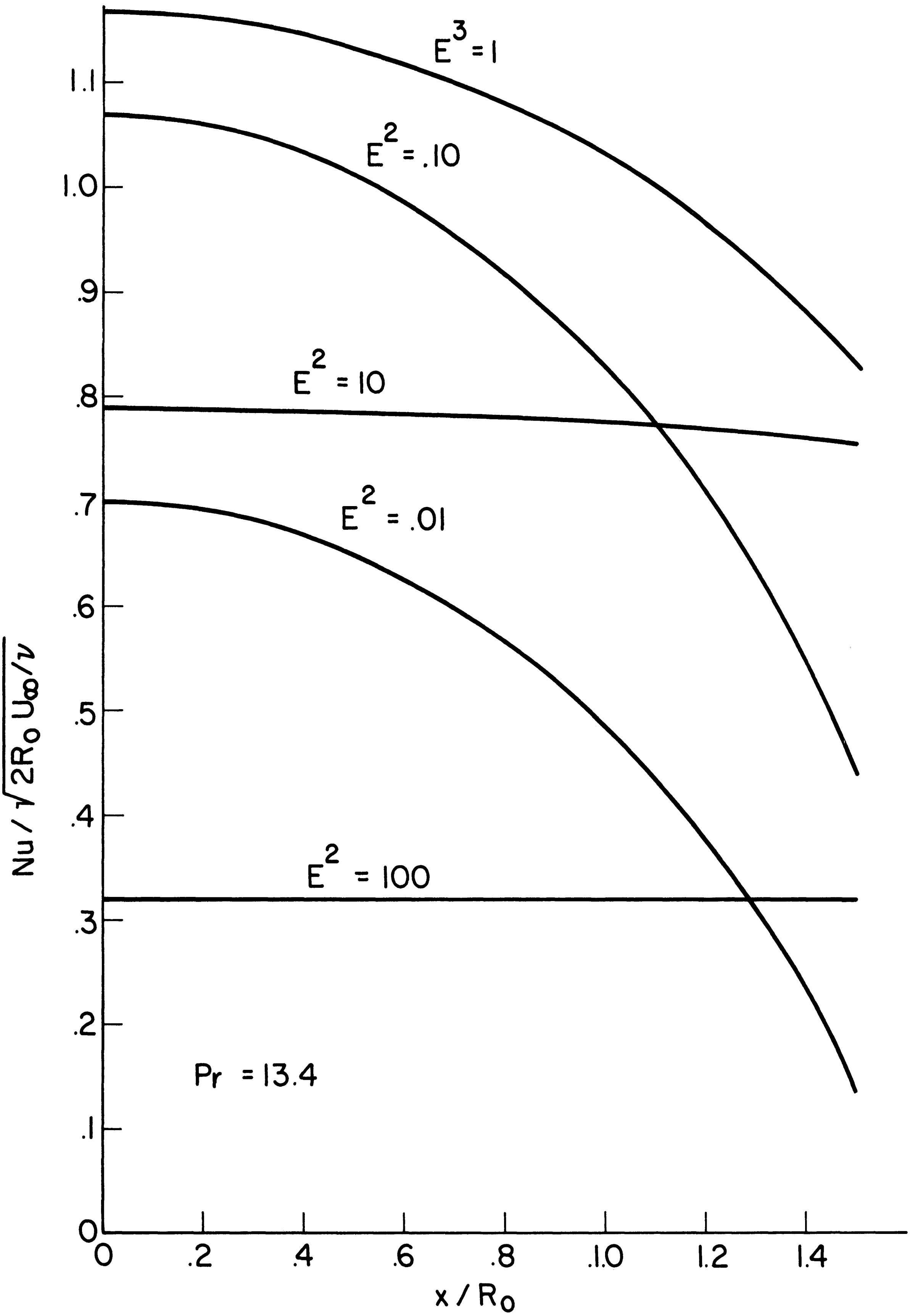

Figure 14-b. Local Nusselt number for cylinder with P_r = 13.4 and E $\geq$ 0.01.

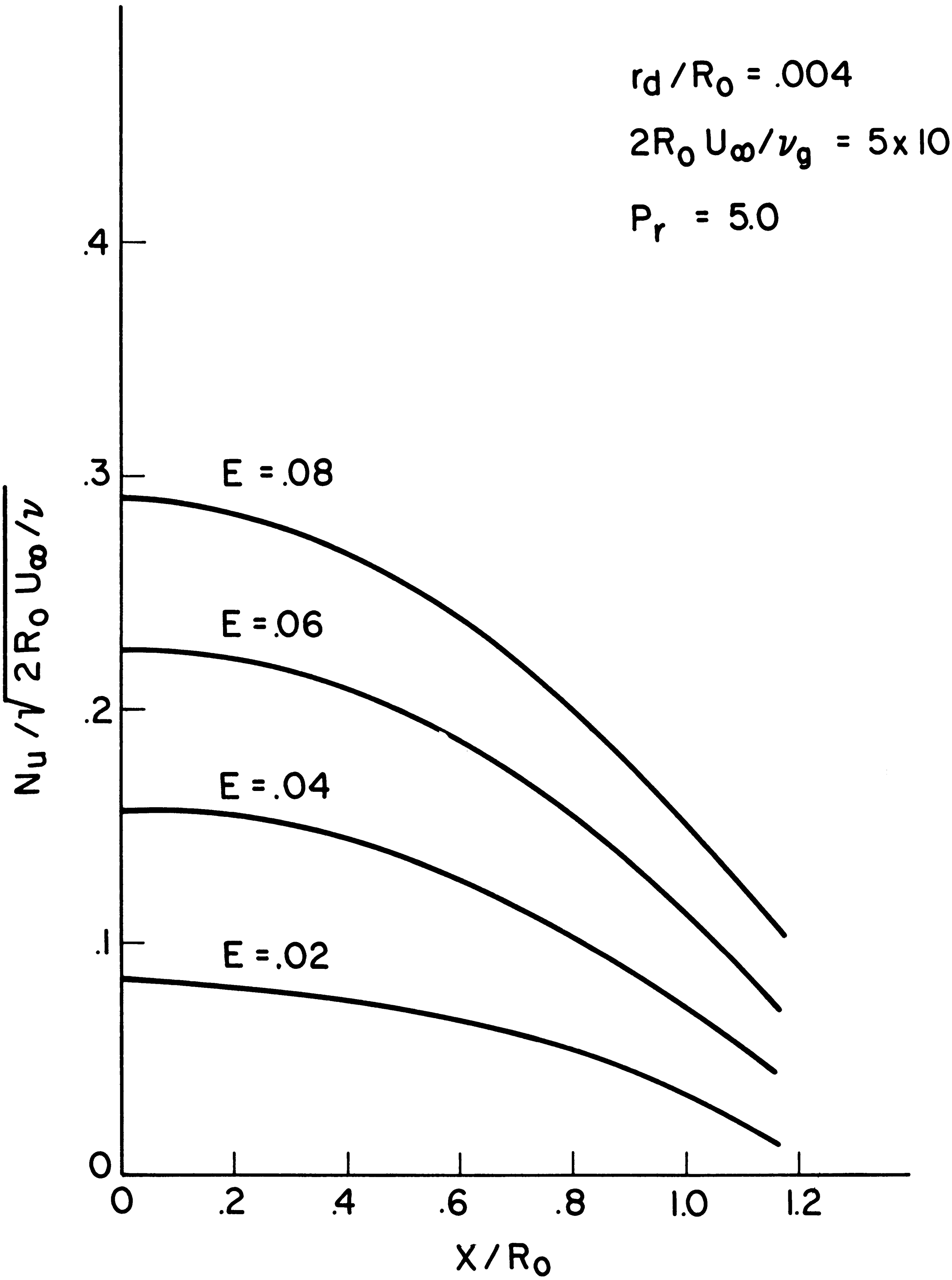

Figure 14-c. Local Nusselt number for cylinder with $P_r = 5.0$, $E \leq 0.1$, $r_d/R_o = .004$ and $2R_oU_\infty/\nu_g = 5 \times 10^3$.

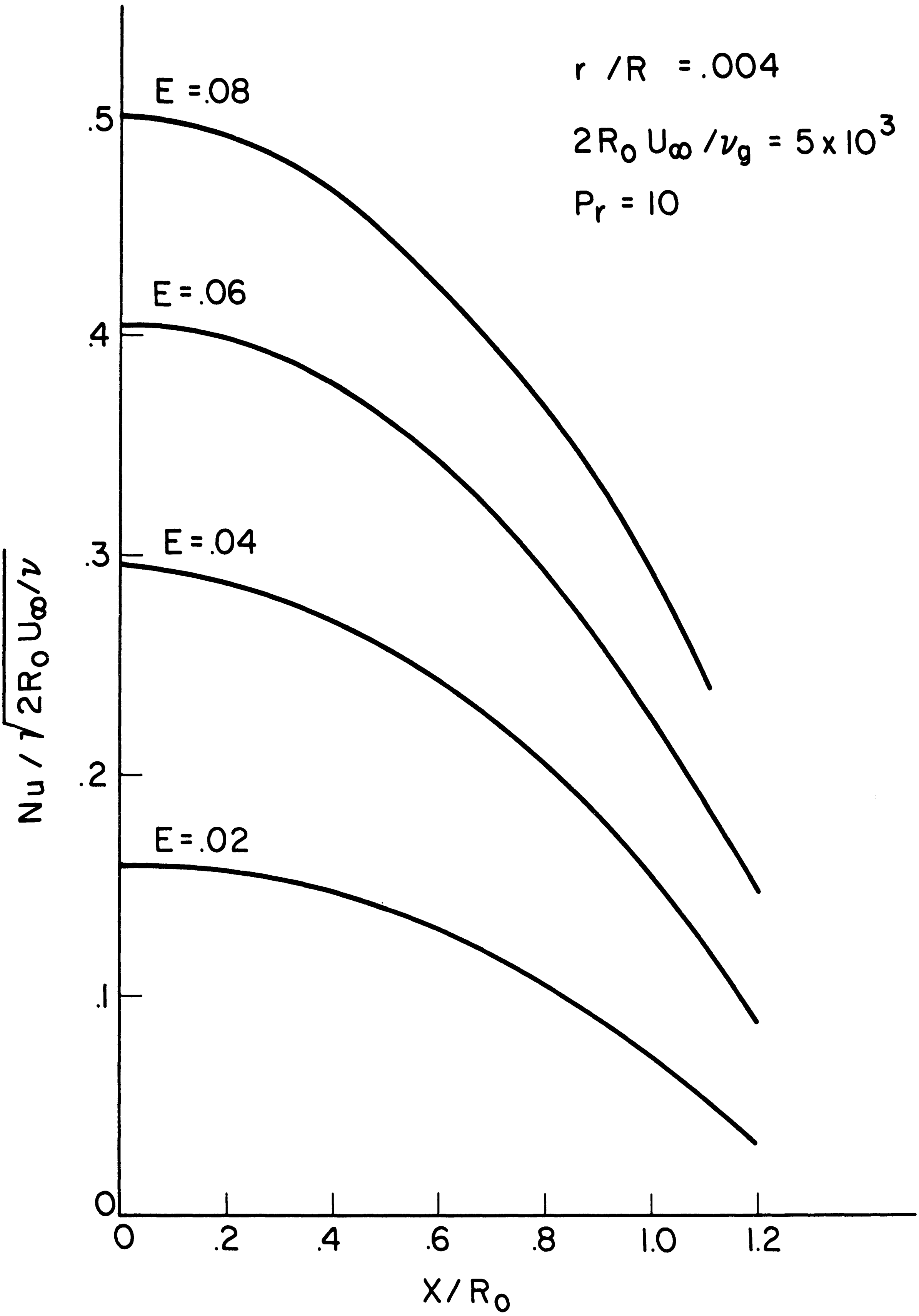

Figure 14-d. Local Nusselt number for cylinder with P_r = 10, $E \leq 0.1$, r_d/R_o = .004 and $2R_oU_\infty/\nu_g = 5\text{x}10^3$.

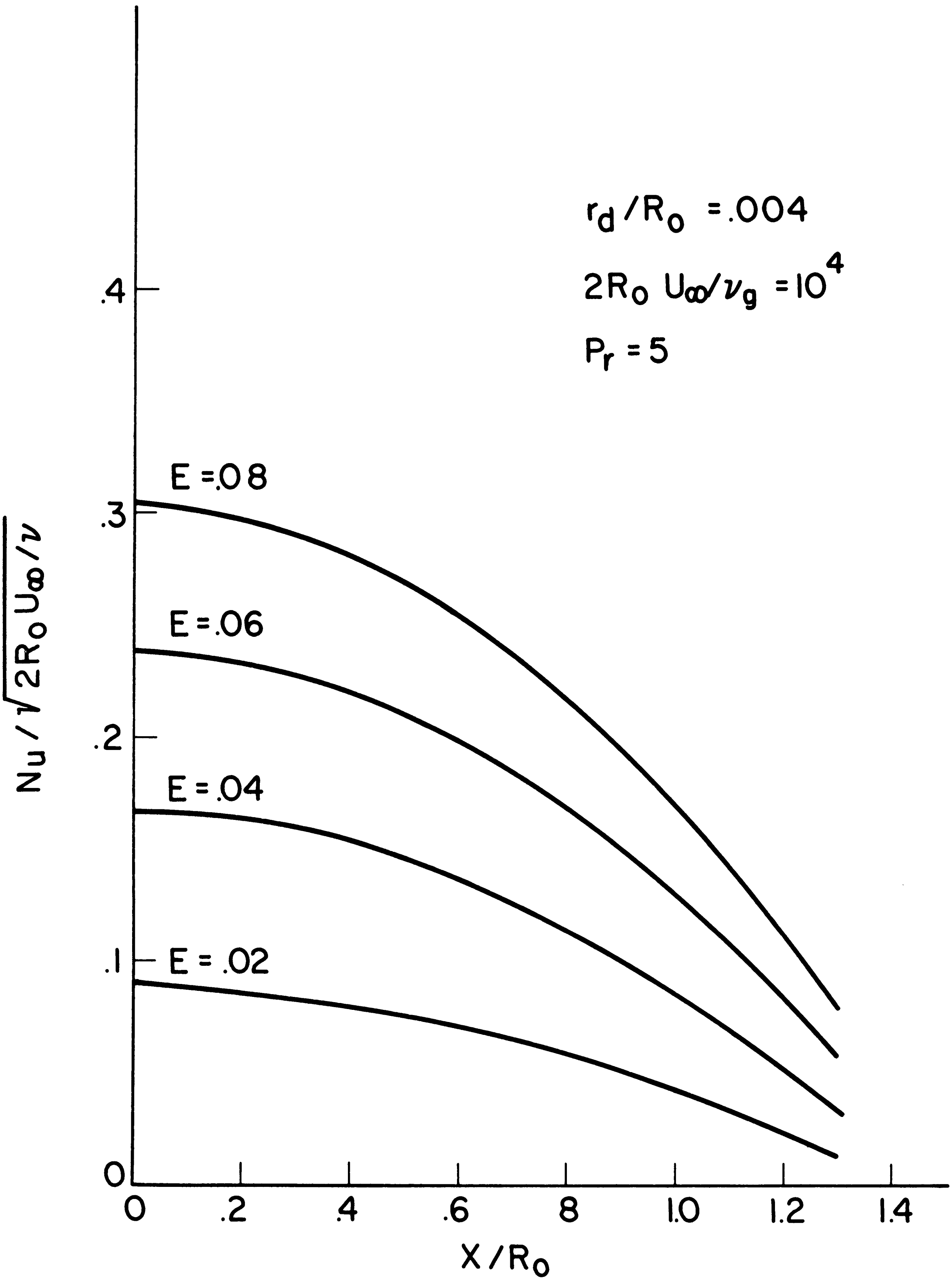

Figure 14-e. Local Nusselt number for cylinder with $P_r = 5$, $E \leq 0.1$, $r_d/R_o = .004$ and $2R_oU_\infty/\nu_g = 10^4$.

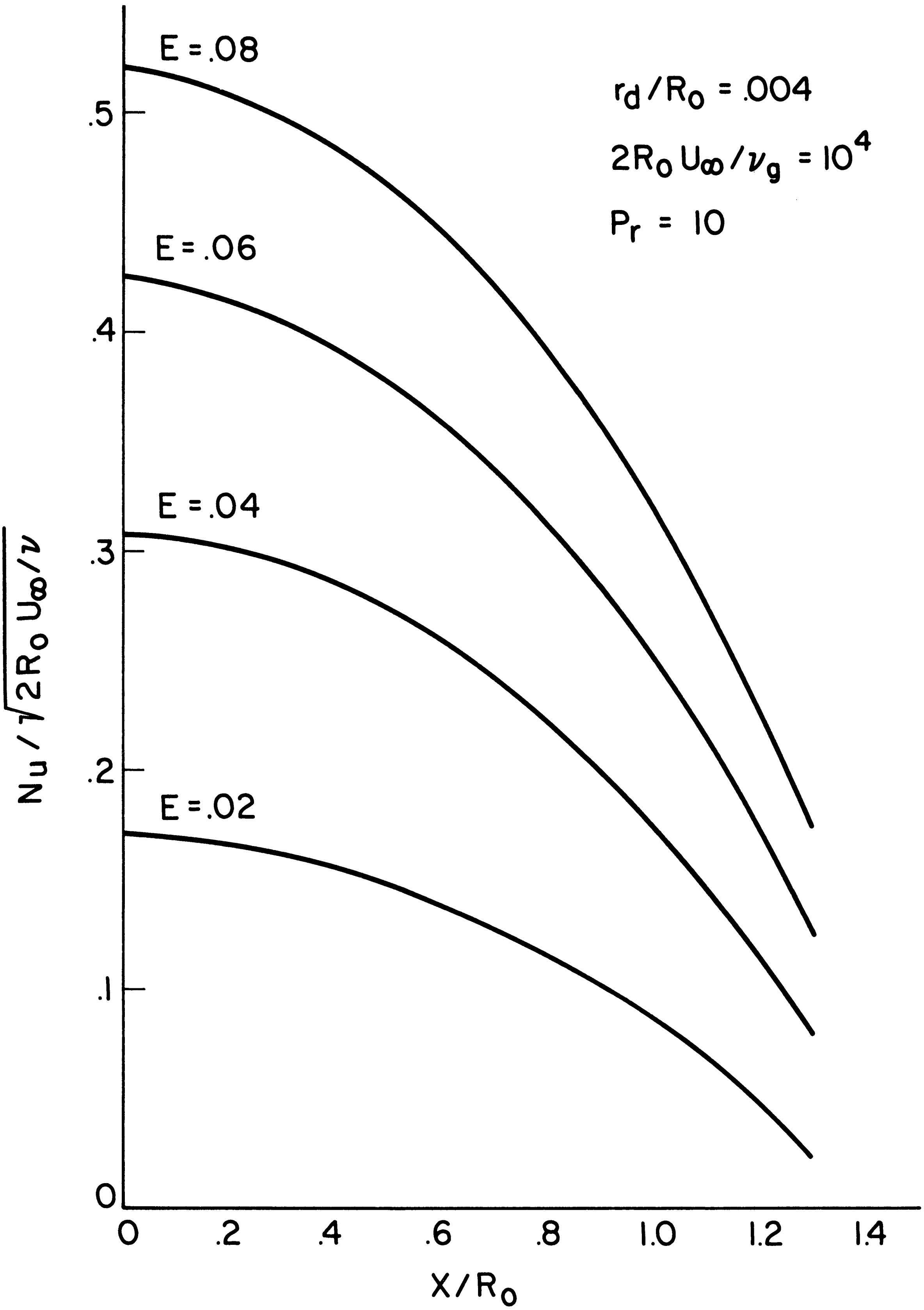

Figure 14-f. Local Nusselt number for cylinder with P_r = 10, $E \leq 0.1$, r_d/R_o = .004 and $2R_oU_\infty/\nu_g = 10^4$.

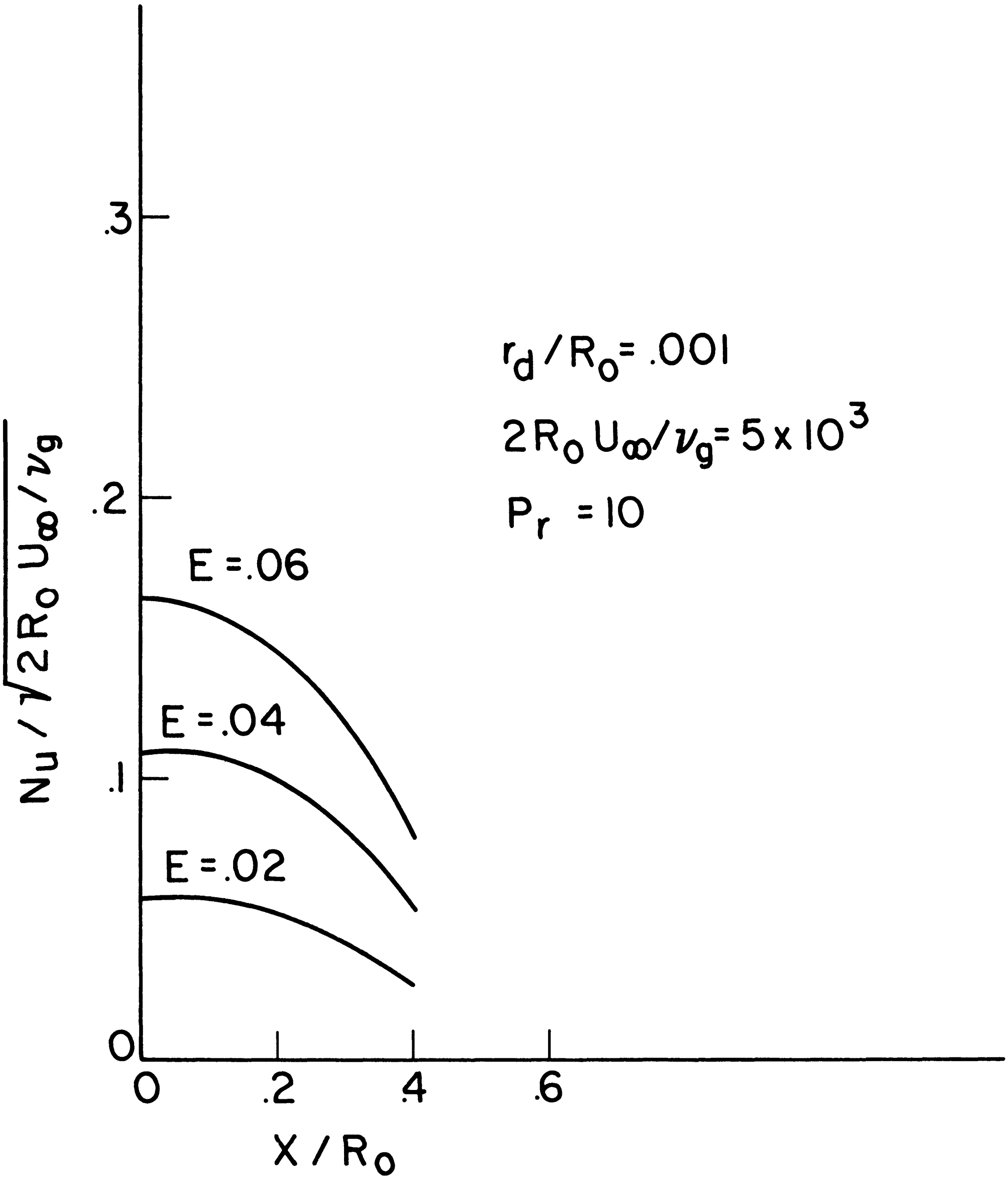

Figure 14-g. Local Nusselt number for cylinder with $P_r = 10$, $E \leq .1$, $r_d/R_o = .001$ and $2R_oU_\infty/\nu_g = 5\text{x}10^3$.

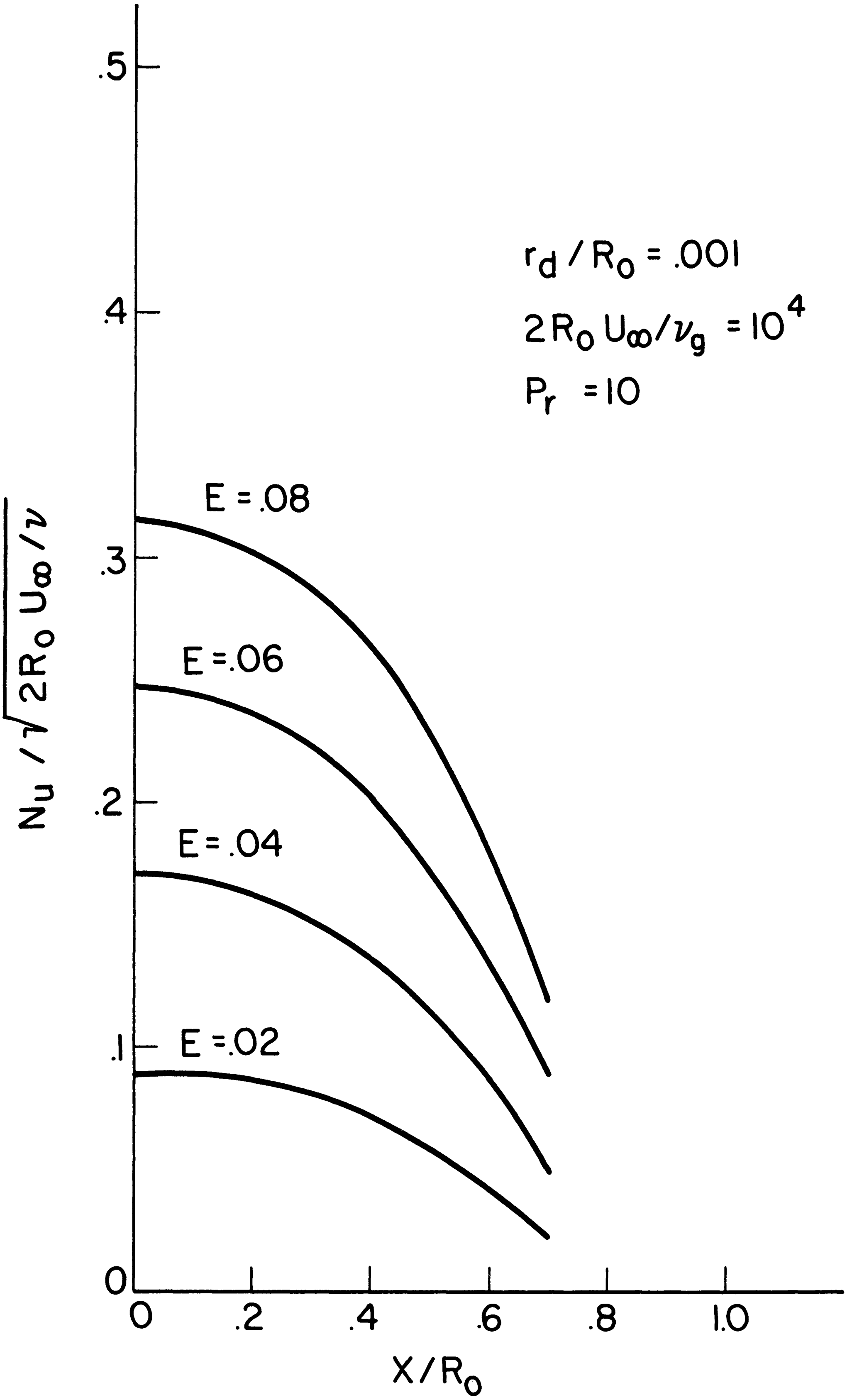

Figure 14-h. Local Nusselt number for cylinder with $P_r = 10$, $E \leq .1$, $r_d/R_o = .001$ and $2R_oU_\infty/\nu_g = 10^4$.

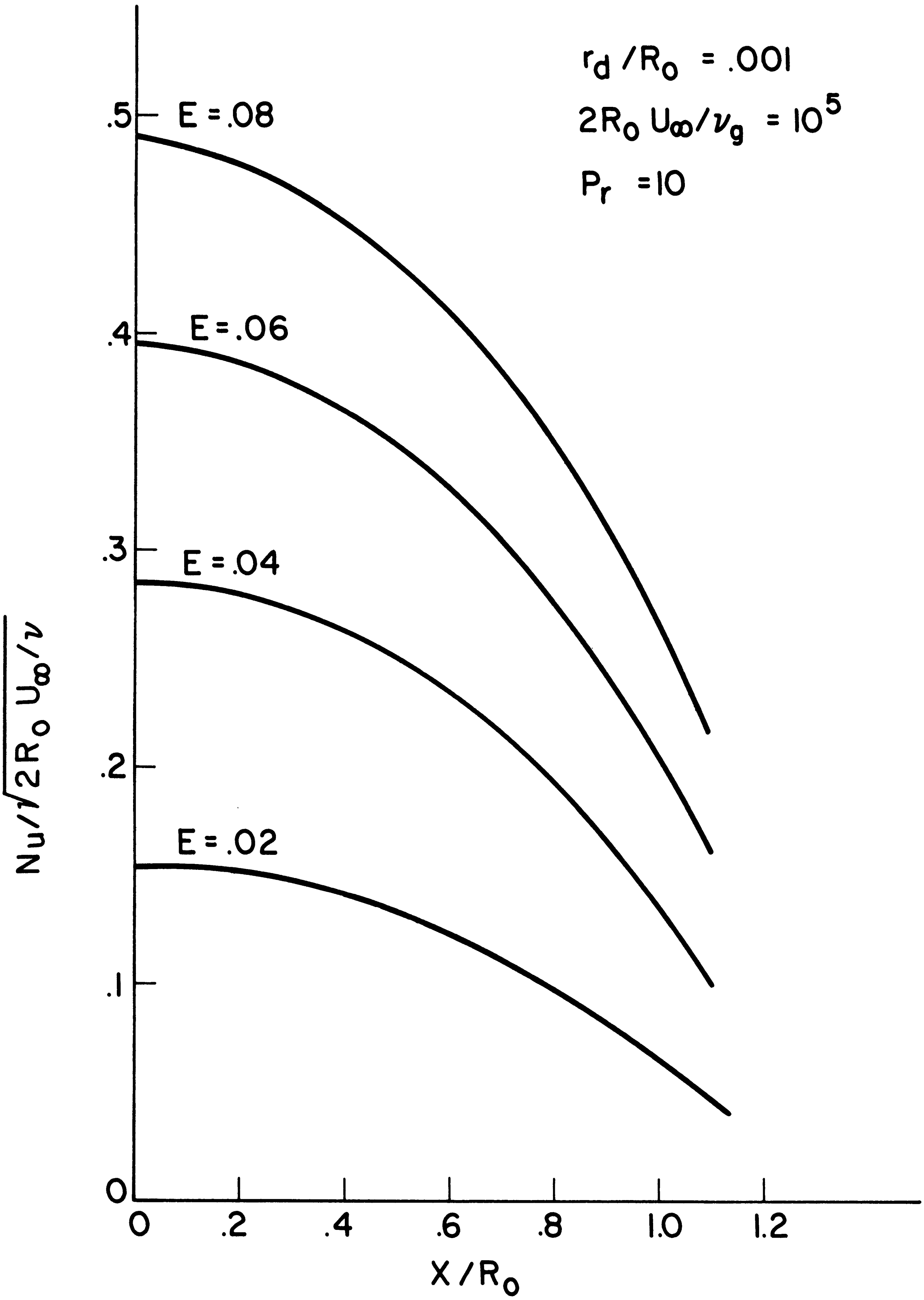

Figure 14-1. Local Nusselt number for cylinder with P_r = 10, $E \leq .1$, r_d/R_o = .001 and $2R_oU_\infty/\nu_g = 10^5$.

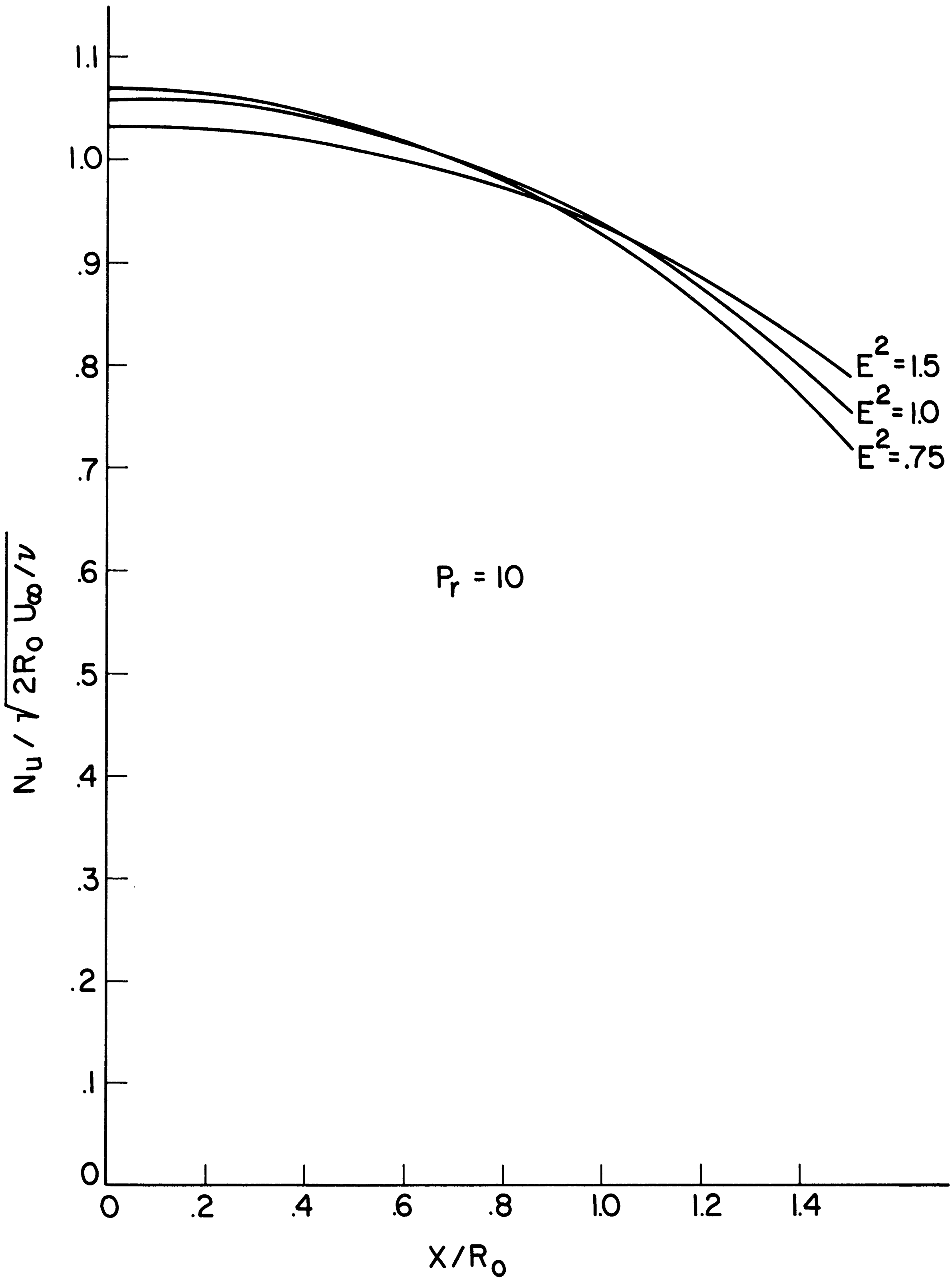

Figure 15. Peak values of local Nusselt number for cylinder with P_r = 10.

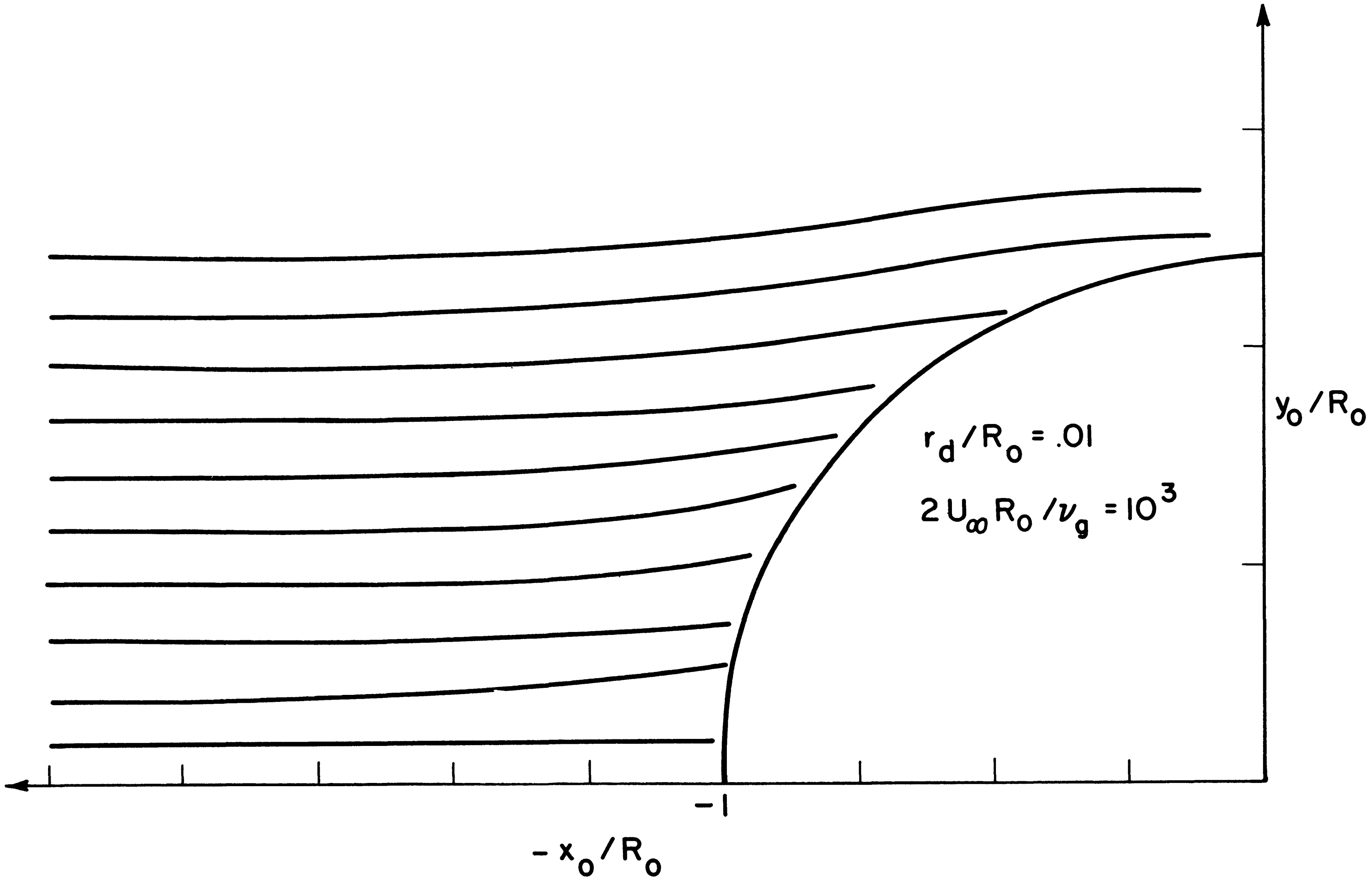

Figure 16-a. Drop trajectories around cylinder with $r_d/R_o = .01$ and $2R_oU_\infty/\nu_g = 10^3$.

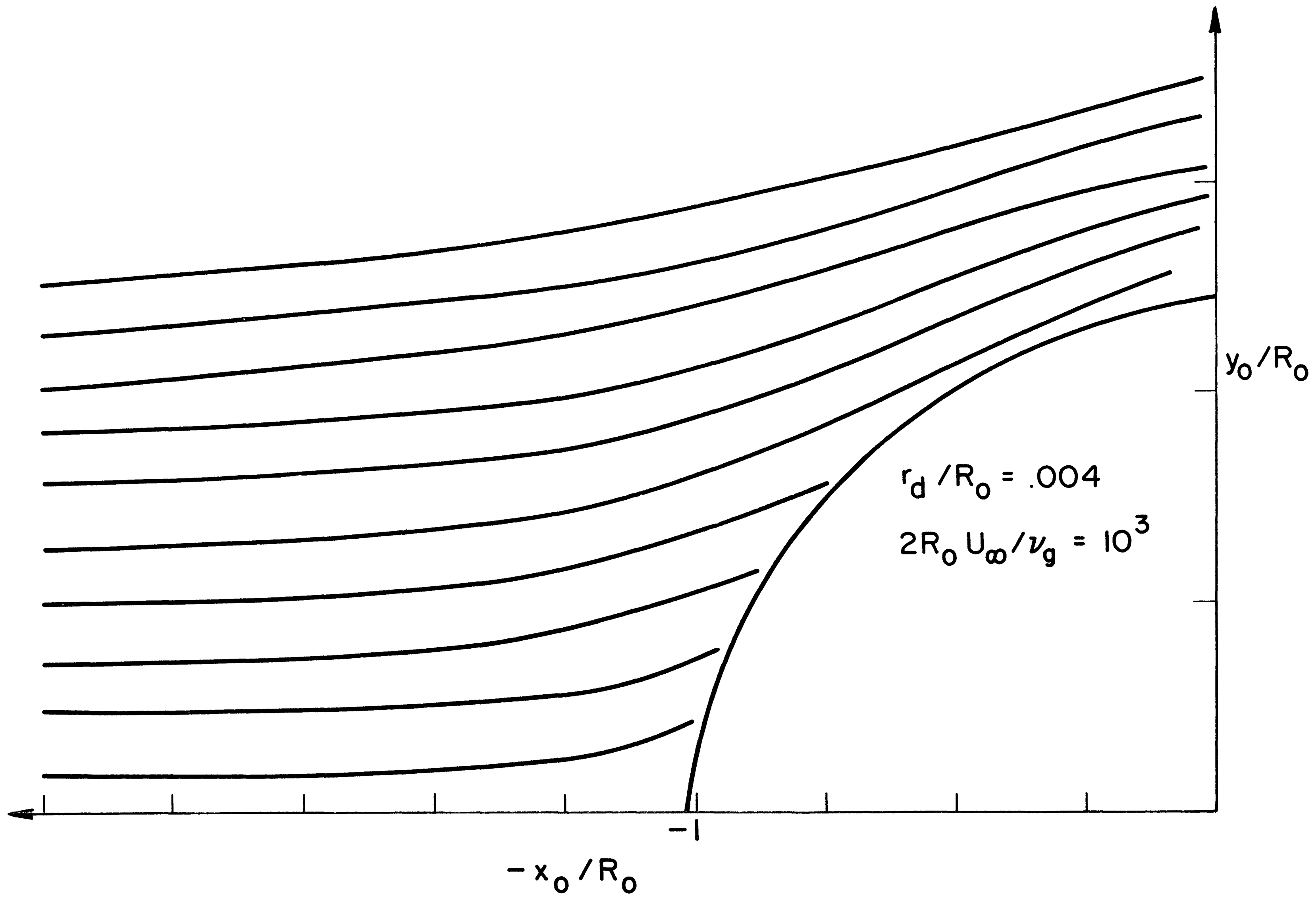

Figure 16-b. Drop trajectories around cylinder with $r_d/R_o = .004$ and $2R_oU_\infty/\nu_g = 10^3$.

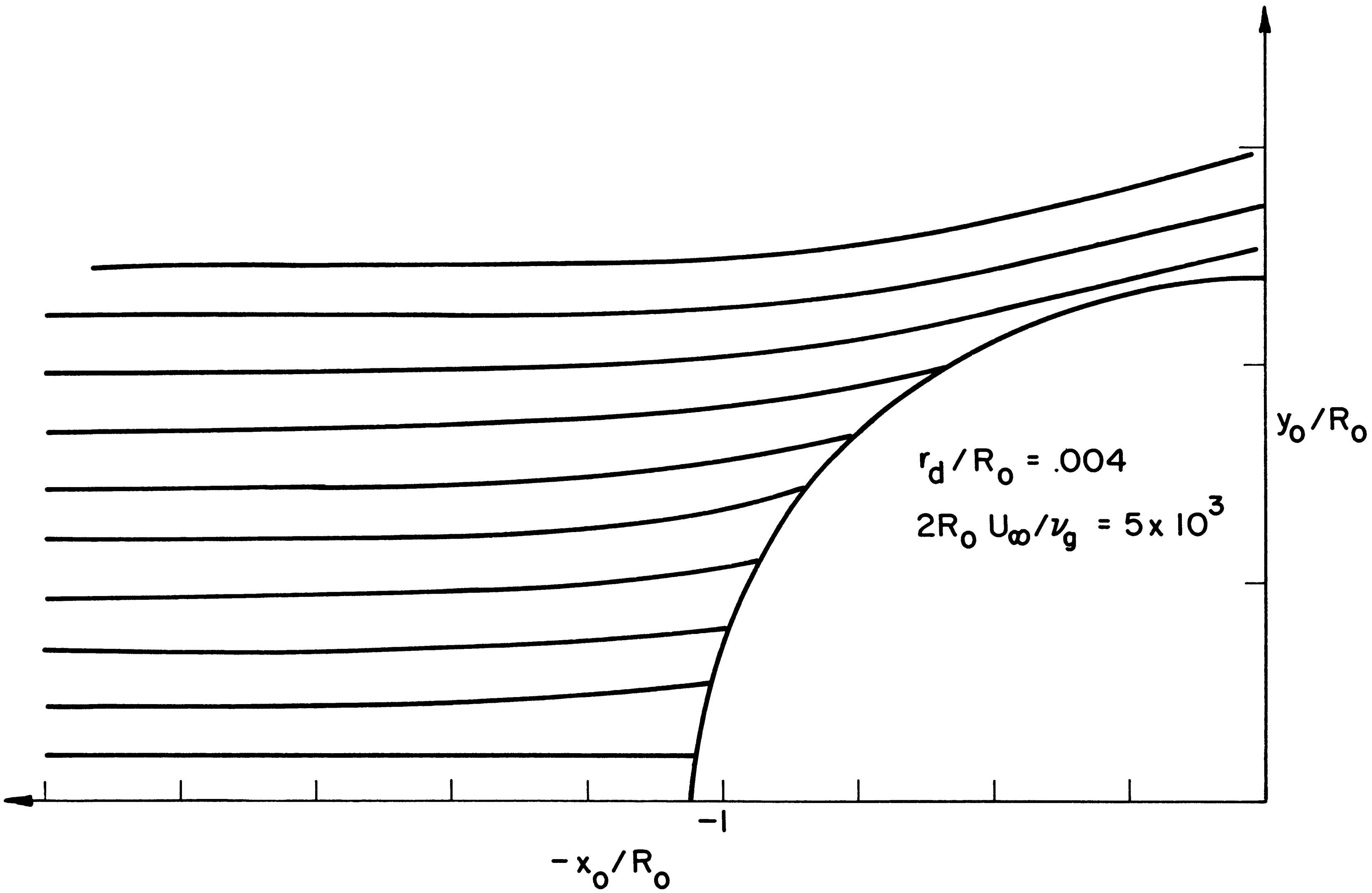

Figure 16-c. Drop trajectories around cylinder with $r_d/R_o = .004$ and $2R_oU_\infty/\nu_g = 5x10^3$.

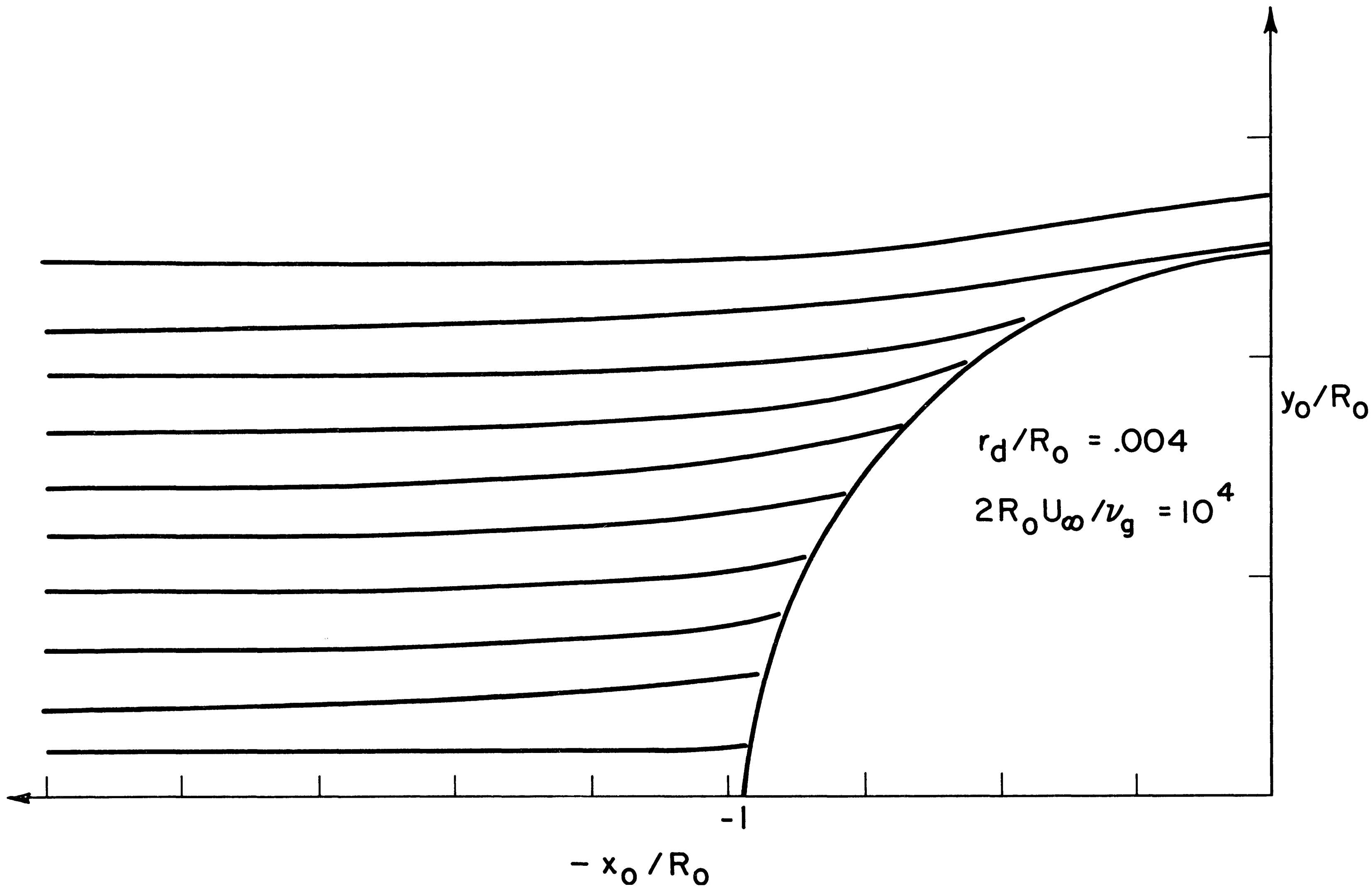

Figure 16-d. Drop trajectories around cylinder with r_d/R_o = .004 and $2R_oU_\infty/\nu_g = 10^4$.

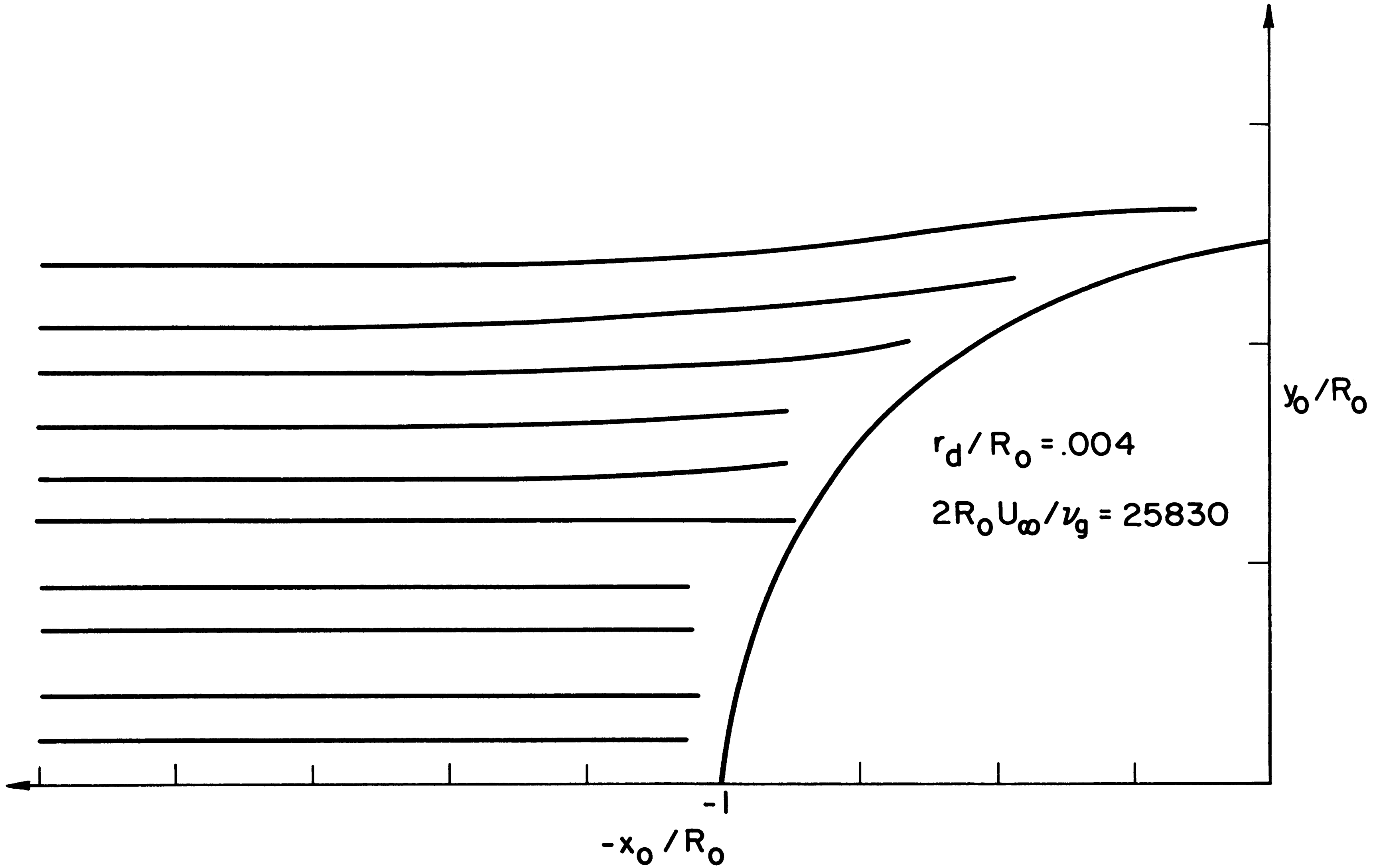

Figure 16-e. Drop trajectories around cylinder with $r_d/R_o = .004$ and $2R_oU_\infty/\nu_g = 25830$.

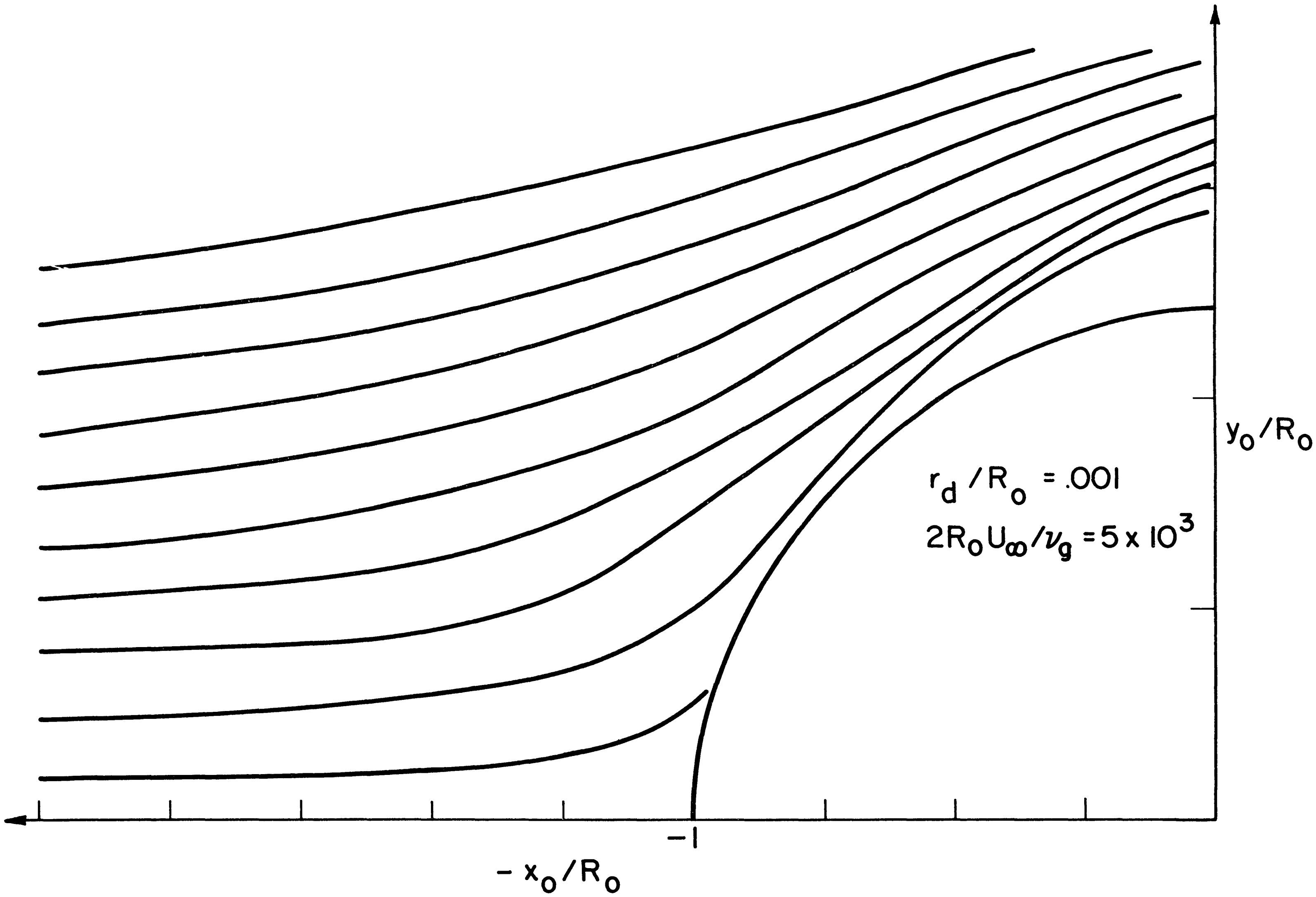

Figure 16-f. Drop trajectories around cylinder with r_d/R_o = .001 and $2R_oU_\infty/\nu_g$ = $5x10^3$.

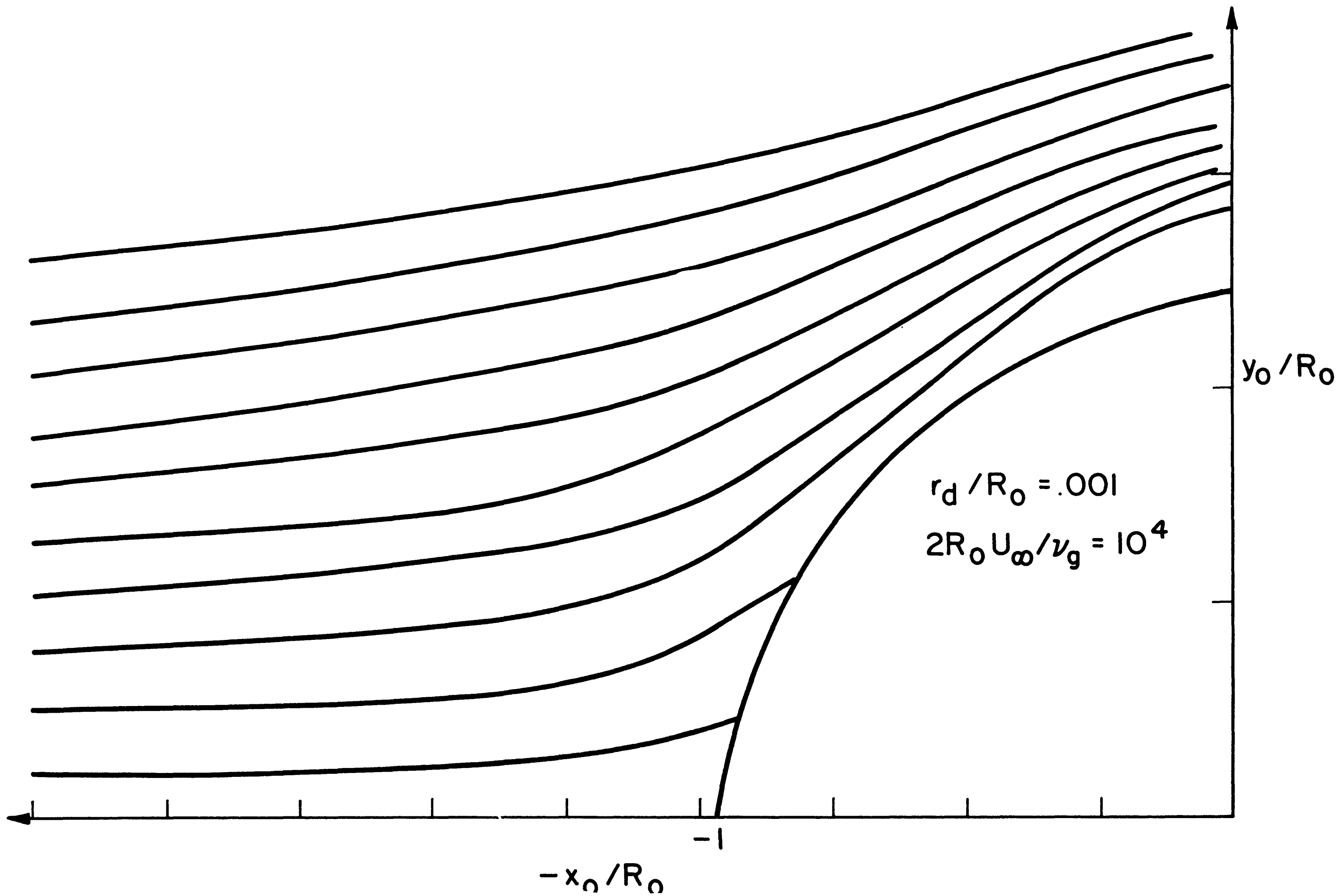

Figure 16-g. Drop trajectories around cylinder with $r_d/R_o = .001$ and $2R_oU_\infty/\nu_g = 10^4$.

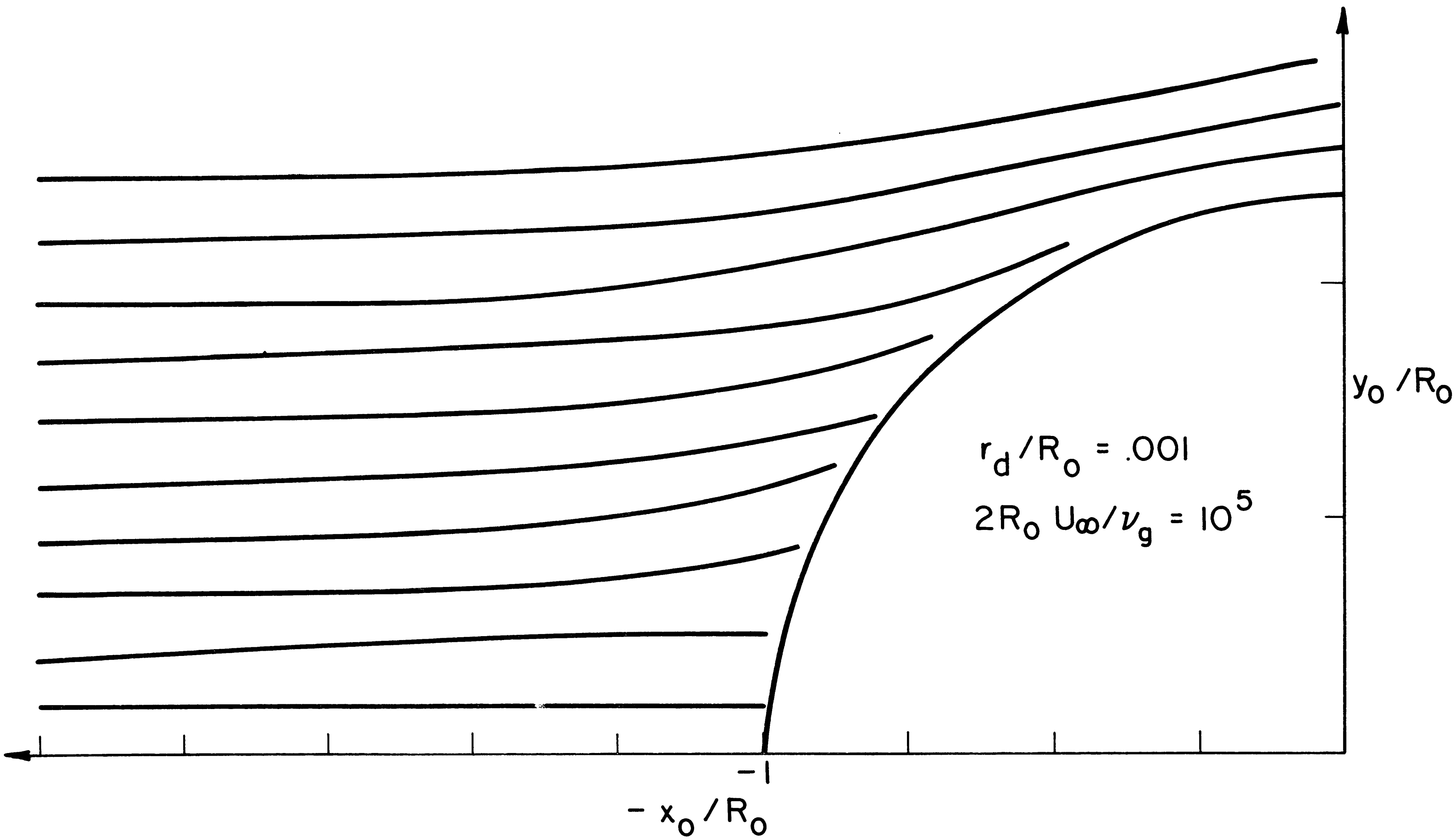

Figure 16-h. Drop trajectories around cylinder with r_d/R_o = .001 and $2R_oU_\infty/\nu_g = 10^5$.

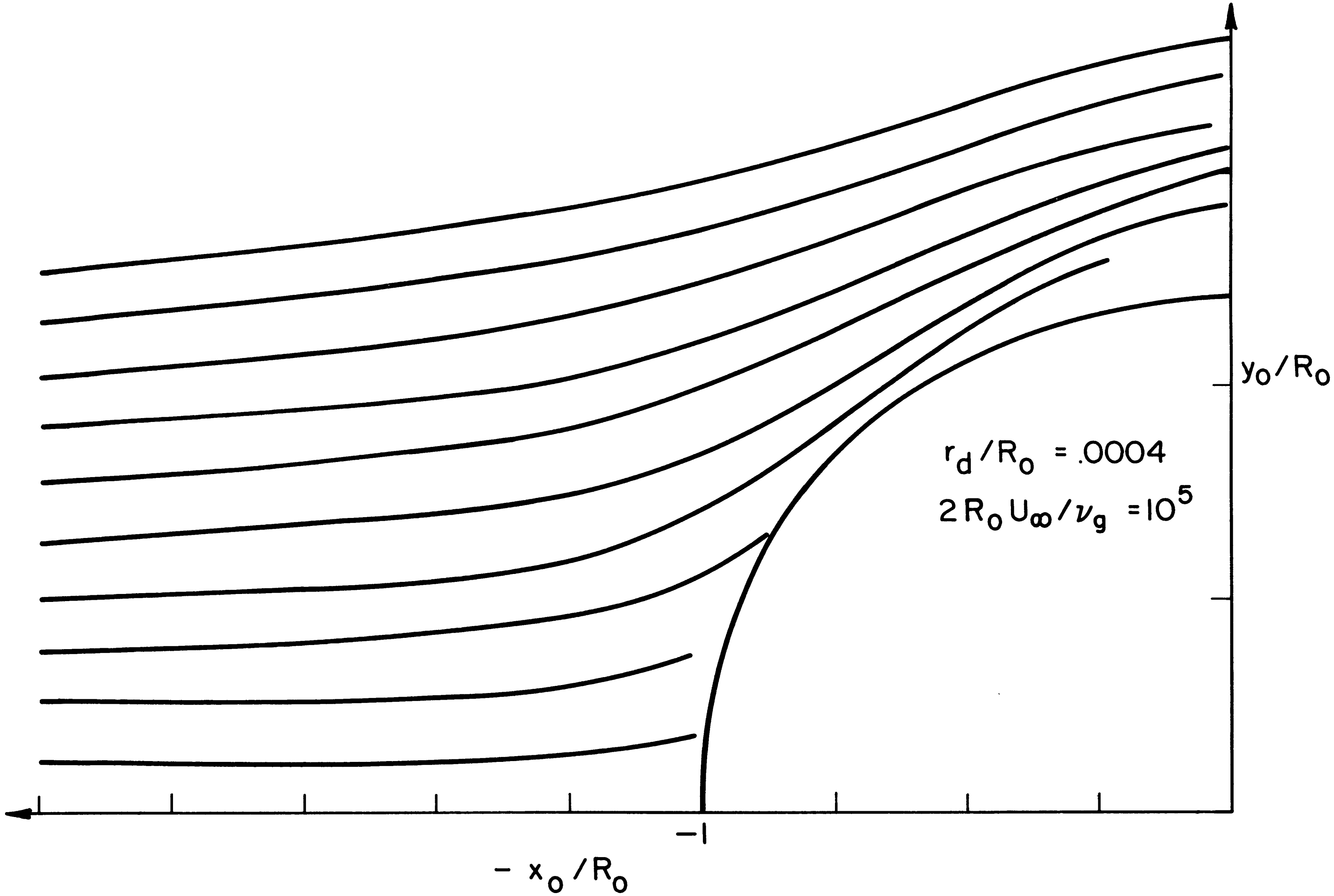

Figure 16-i. Drop trajectories around cylinder with $r_d/R_o = .0004$ and $2R_oU_\infty/\nu_g = 10^5$.

Representative ones are given in Table I.

An examination of Table I and Figure 16-e shows that for a value of the gas Reynolds number of 25830 and an $r_d/R_o = 0.004$ that the drop trajectories are practically straight and over 90% of the liquid drops which are heading toward the cylinder far upstream from it are intercepted by it. From Figure 27 the minimum value of the Reynolds number at which drops whose size is given by $r_d/R_o = 0.004$ will move in straight lines is seen to be 25830. Hence the dimensional arguments used in Chapter II to obtain a lower limit to the validity of the solution corresponding to straight drop trajectories is seen to be verified by the actual calculations.

These results for the drop trajectories were used in the computer program of Appendix XXVI to calculate the film thickness and local skin friction which are plotted in Figures 11-c through 11-i, and 13-b through 13-h, respectively. This program also calculates the terms of the expansions (52) by fitting a curve to the tabulated values of the quantities on the left. The α_{2n}'s and β_{2n}'s of these expansions so obtained were used in the computer program of Appendix XXV which calculated the solutions to the Eqs. (55). The results of these calculations are available for $0.01 \leq 0.1$ and for the same range of parameters as the drop trajectories, and have been used to calculate the local Nusselt numbers which are plotted in Figures 14-c through 14-i. An examination of the figures mentioned above shows that the physical quantities, film thickness, skin friction,

TABLE I

VALUES OF $\mathscr{M}$, J AND V_{φ}^{*}

$\tilde{x}$	$\dot{\mathscr{M}}$	J	V_{φ}^{*}
.040000	.038124	.952072	.040817
.080000	.076185	.949260	.081548
.120000	.114117	.944851	.122107
.160000	.151864	.937727	.162340
.200000	.189307	.931038	.203038
.240000	.226467	.921882	.242616
.280000	.263240	.909226	.281863
.320000	.299520	.898311	.321262
.360000	.335348	.881802	.359166
.400000	.370531	.867531	.397569
.440000	.405079	.851281	.435666
.480000	.438993	.830402	.471253
.520000	.472101	.811292	.507434
.560000	.504390	.790568	.543176
.600000	.535811	.767969	.577405
.640000	.566343	.740299	.609813
.680000	.595814	.715034	.643422
.720000	.624278	.690639	.675504
.760000	.651706	.659024	.704781
.800000	.677934	.630416	.734667
.840000	.703023	.598205	.763147
.880000	.726849	.566219	.790230
.920000	.749424	.532821	.815848
.960000	.770693	.496437	.839644
1.000000	.790555	.460724	.863966
1.040000	.809037	.423519	.886282
1.080000	.826112	.383390	.905594
1.120000	.841677	.342649	.925682
1.160000	.855731	.300477	.943274
1.200000	.868265	.255979	.960006
1.240000	.879254	.205461	.975866
1.280000	.888656	.142798	.990105
1.320000	.896417	.059399	1.001174
1.360000	.902464	-.055795	1.006724

and Nusselt number attain a maximum at a value of E near one. The peak values of the Nusselt number for a Prandtl number of 10 are shown in Figure 15. In general, it can be seen from the figures that the behavior of the film changes at this value of E, for example Figures 11-a and 11-b, show that the film thickness increases in a downstream direction when E is less than one, but decreases slightly in the downstream direction when E is greater than one. Figure 12 shows that there is also a change in the behavior of the velocity profiles in the neighborhood of E=1. For values of E greater than one there is an increased tendency for the velocity gradient to become very steep at the outer edge of the film. This may represent an unstable condition. The steeping is probably due to the fact that large values of E correspond to a large flux of momentum into the film. The flux becomes so rapid that the momentum does not have time to diffuse into the film near the stagnation point and there it has the effect of rapidly accelerating a small mass of fluid at the outer edge of the film. Since the momentum eventually penetrates into the film some distance downstream from the stagnation point this could account for the decrease in thickness of the film, since more and more of the liquid in the film is being speeded up. Further, since the drops don't penetrate deeply into the film at large E this could account for the decrease in the local Nusselt number as E increases beyond one. This can be seen from Figures 14-a and 14-b.

On the other hand, for $E < 1$ the velocity profiles tend to become increasingly linear as E decreases, (Figure 12), indicating that inertial effects are unimportant. This tends to verify the dimensional reasoning employed in Chapter II to eliminate the inertial terms in the momentum equation for $E \leq 0.1$.

It should be noted that the behavior of the solutions is generally the same for values of $E^2 < 0.01$ and values of $E^2 > 0.01$, (Figures 11-a through 11-i, 13-a through 13-b, and 14-a through 14-i), even though different terms were retained in the equations which describe these regimes.

However, there is an increasing tendency for the liquid film thickness to increase suddenly at some downstream point with decreasing values of E. The dimensionless film thickness, however, remains of the order of unity for all values of E which justifies the dimensional reasoning of Chapter II. This rapid increase in film thickness occurs at the point where the droplets no longer impinge on the cylinder for $E \leq 0.1$ as can be seen by a comparison of Figures 11-c through 11-i with Figures 16-b through 16-i.

This is probably due to the fact that at this point the momentum carried into the film by the drops rapidly falls to zero, and the gas shear stress is not capable of moving the heavy liquid and as a result the liquid tends to pile up. This makes it very likely that separation will tend to occur in the neighborhood of this point, or at least the film will tend to fall off on vertical cylinders.

It can be seen from Figures 14-c through 14-i that the heat transfer coefficient drops off rapidly in this region. For all values of E, (Figures 14-a through 14-i), the heat transfer coefficient is a maximum at the stagnation point and decreases in a downstream direction. As E becomes large the heat transfer coefficient tends to decrease less in the downstream direction, as can be seen from Figures 14-a and 14-b. A comparison of Figures 14-a with 14-b, 14-c with 14-d, and 14-e with 14-f shows that the Nusselt number increases with increasing Prandtl number as is usual. For values of E less than 0.1 the effects of drop size and gas Reynolds number based on the cylinder diameter have a strong influence on the heat transfer coefficient, friction factors, and film thickness since they influence the amount of liquid coming into the film. It can be seen from Figures 16-a through 16-i that the drop trajectories are more nearly straight at higher Reynolds numbers and larger drop sizes. Thus as can be seen from Table I larger amounts of mass and momentum are carried into the film with larger drop sizes and higher Reynolds numbers. The influence on the heat transfer (Figures 14-c through 14-i) skin friction (Figures 13-b through 13-h) and film thickness (Figures 11-c through 11-h) is what would be expected. They all increase with increasing values of gas Reynolds number and drop size. It can be seen from these figures that the drop size has a strong effect.

As noted experimentally,[1,9] and predicted quantitatively[29] there is a definite increase in heat transfer with the two-phase mixture above that, which would be obtained from a pure gas flow (Figures 14-a through 14-i).

No value for the drop diameter was reported for the experimental results of Reference (1) due to the difficulty of measuring this quantity. However the data of Reference (1) after being reduced in Appendix XXVII, are compared with the analytical results corresponding to $r_d/R_o = 0.004$* in Figure 17 and it can be seen that the agreement is close and well within the experimental scatter.

It can be seen from Figures 13-a through 13-h that the skin friction usually reaches a peak value somewhere around the 50° point on the cylinder. From Figure 13-a it can be seen that for values of $E > 1$ this peak moves so far to the rear that it does not show up in the analysis. On the other hand, when $E \leq 0.1$ the peak will move forward when the drop size and gas Reynolds numbers are small. A comparison of Figures 13-a through 13-h with Figures 14-a through 14-i shows that an increase in heat transfer is always accompanied by an increase in friction. Thus one must always pay the price of increased friction to obtain increased heat transfer.

*The diameter of the cylinder in the experiments was 1.5 in. For $r_d/R_o = 0.004$ this corresponds to a drop size of 152 microns. The manufacturer of the nozzle used in the experiments, reports a volume-median drop diameter of 167 microns for the nozzle under different operating conditions than those of the experiment.

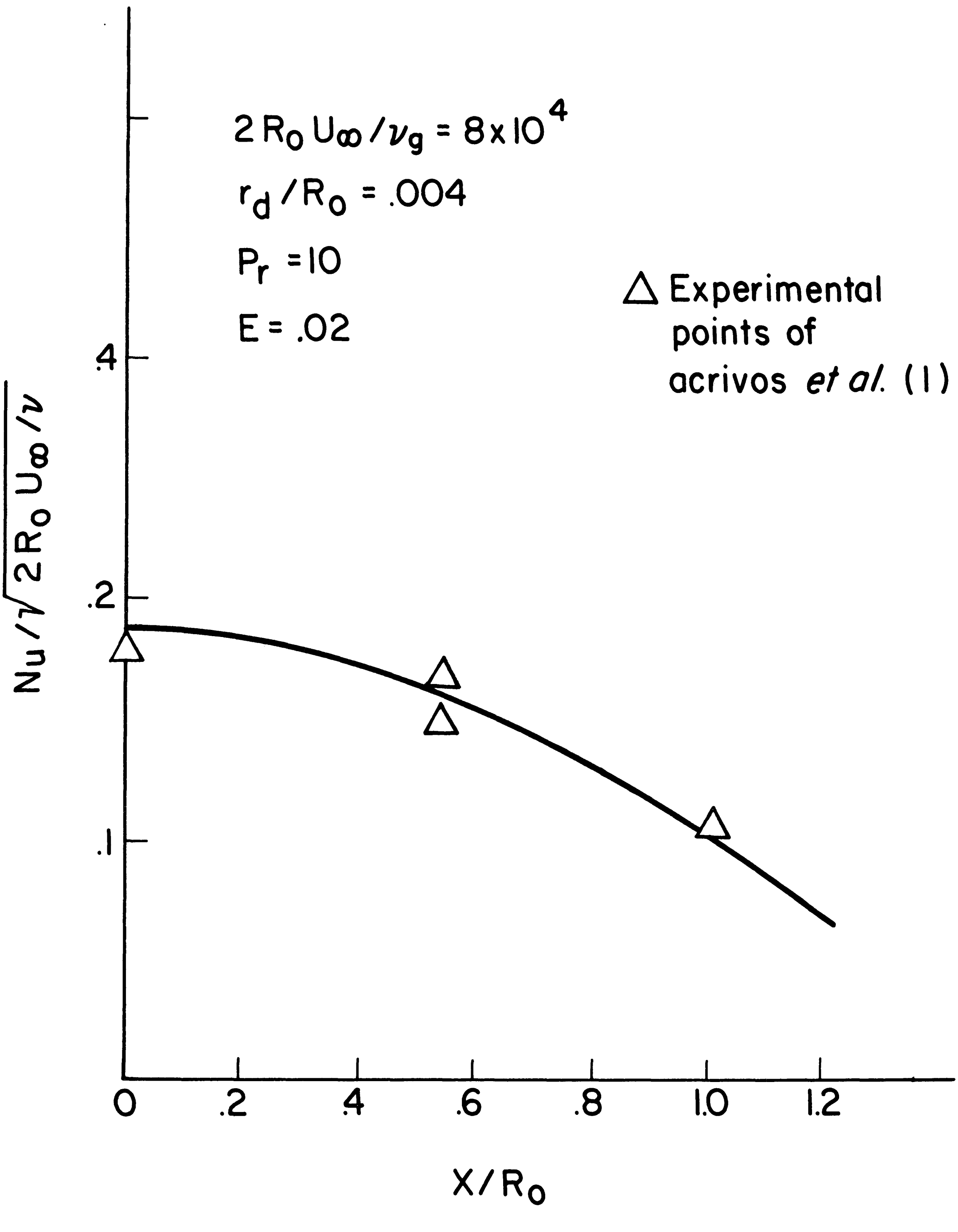

Figure 17. Comparison of calculated Nusselt number with experimental data for cylinder.

2. OSCILLATING FLAT PLATE

The solutions to Eqs. (93) through (95), (100) through (102), (106), (109), (115), (120), (124), and (129) have been obtained numerically by using the computer program listed in Appendix XXV. The results are available for $0.05 \leq X_e \leq 0.2$ and $P_r = 5$ and 10. These results can be used to calculate ψ^*, δ^*, and θ^* by using them in Eqs. (92), (99), (103) through (105), and (112) through (114), and then using these new quantities in the expansions (87). The solutions have been used to calculate the terms in the expansions (138) and (139) for wall shear stress and local Nusselt numbers. The first term of each of the expansions (87), and (137) through (139) represents the steady solution. The second term represents the harmonic components (i.e., it goes as $e^{i\omega t}$) and the third term is the sum of a part which represents the permanent alteration to the flow field due to the oscillations and at a time dependent part. The physical quantities for the steady solution are plotted in Figures 18-a through 20.

The harmonic components of the physical quantities are plotted in Figures 21-a through 23-c, and the permanent alterations in the physical quantities due to the oscillations are plotted in Figures 24 through 26-b.

Figures 18-a through 20 show that the steady (nonoscillating) components of the film thickness, skin friction, and Nusselt number increase with increasing X_e. Figure 20 shows that the steady component

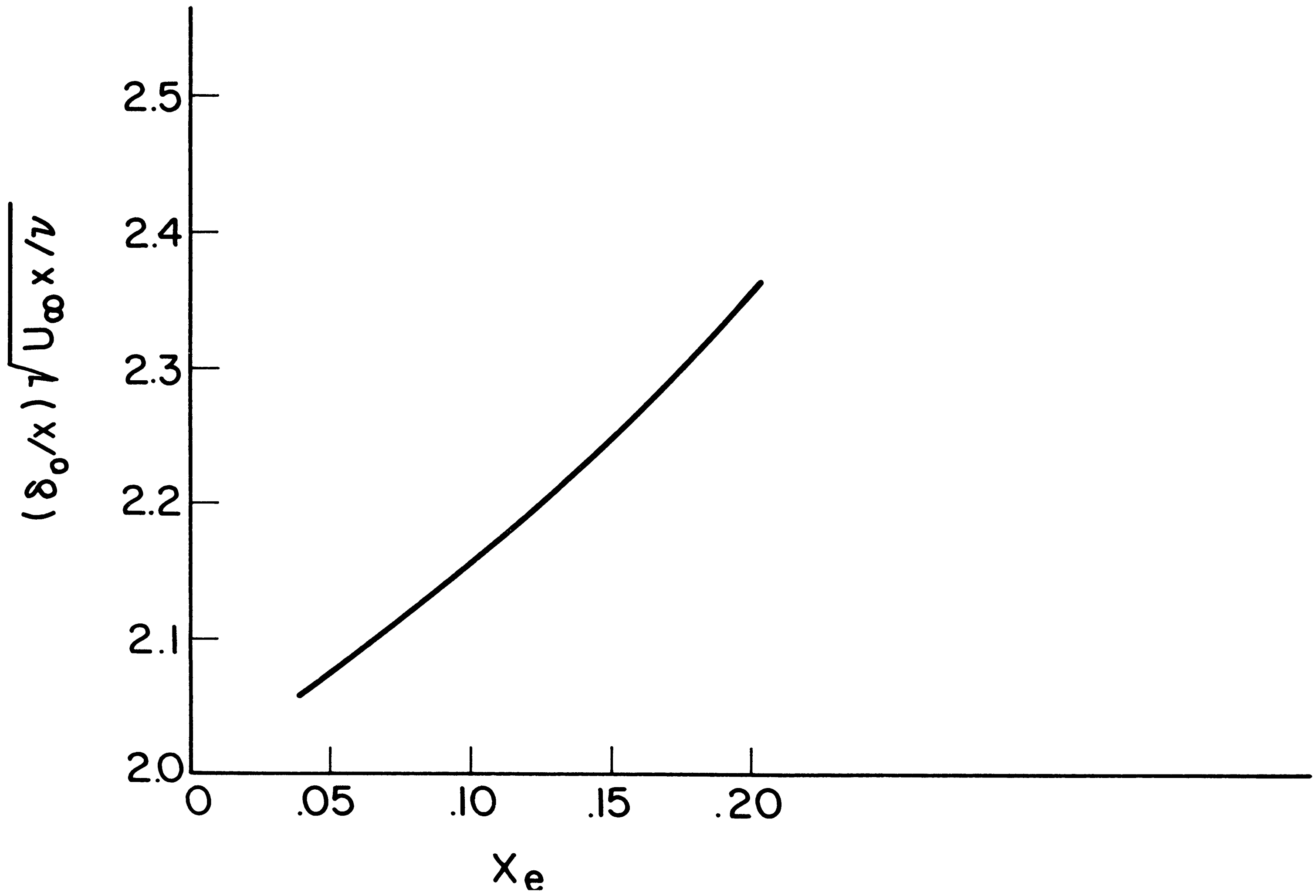

Figure 18-a. Local liquid film thickness versus volume fraction for steady flat plate.

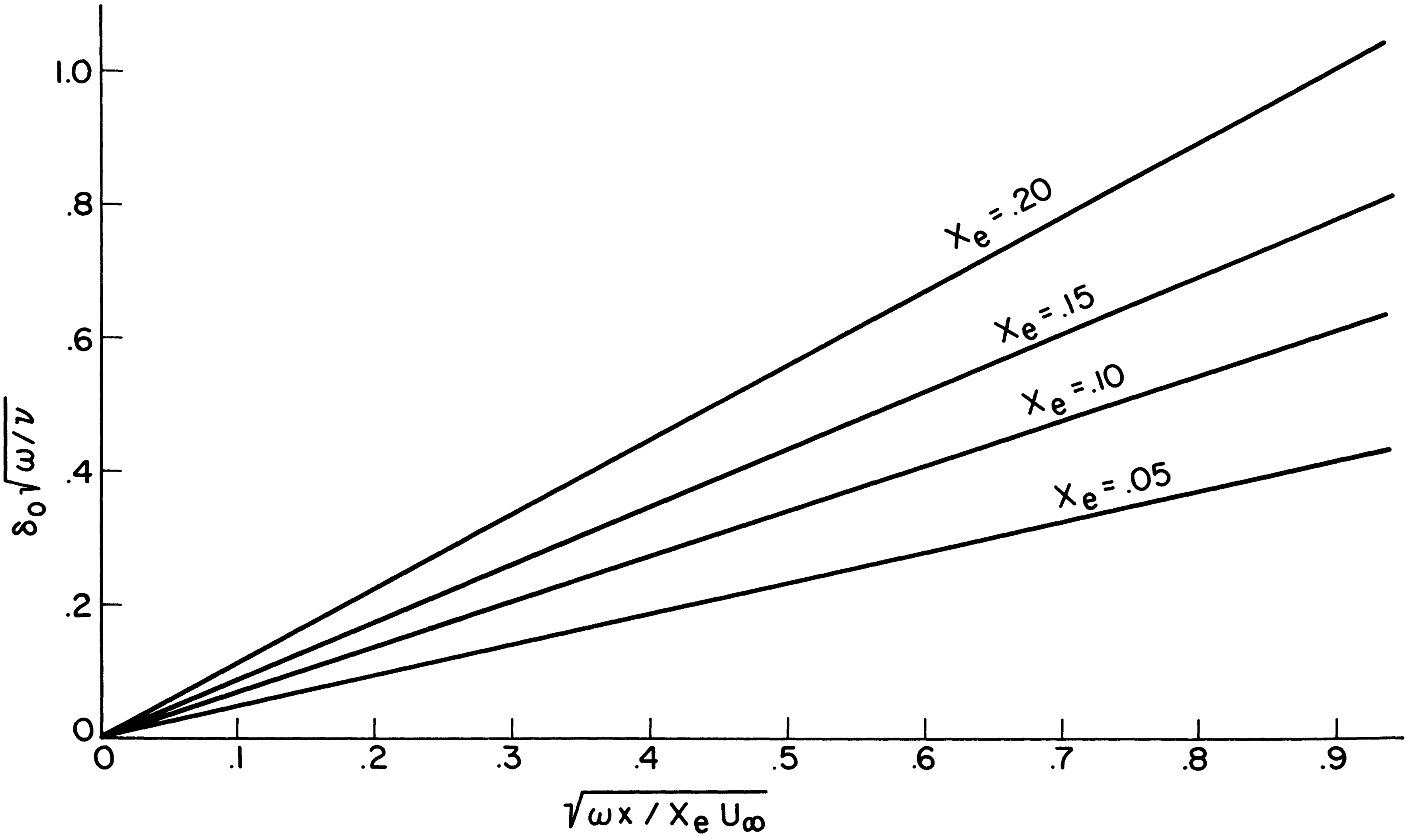

Figure 18-b. Replot of Figure 18-a in terms of oscillating variables.

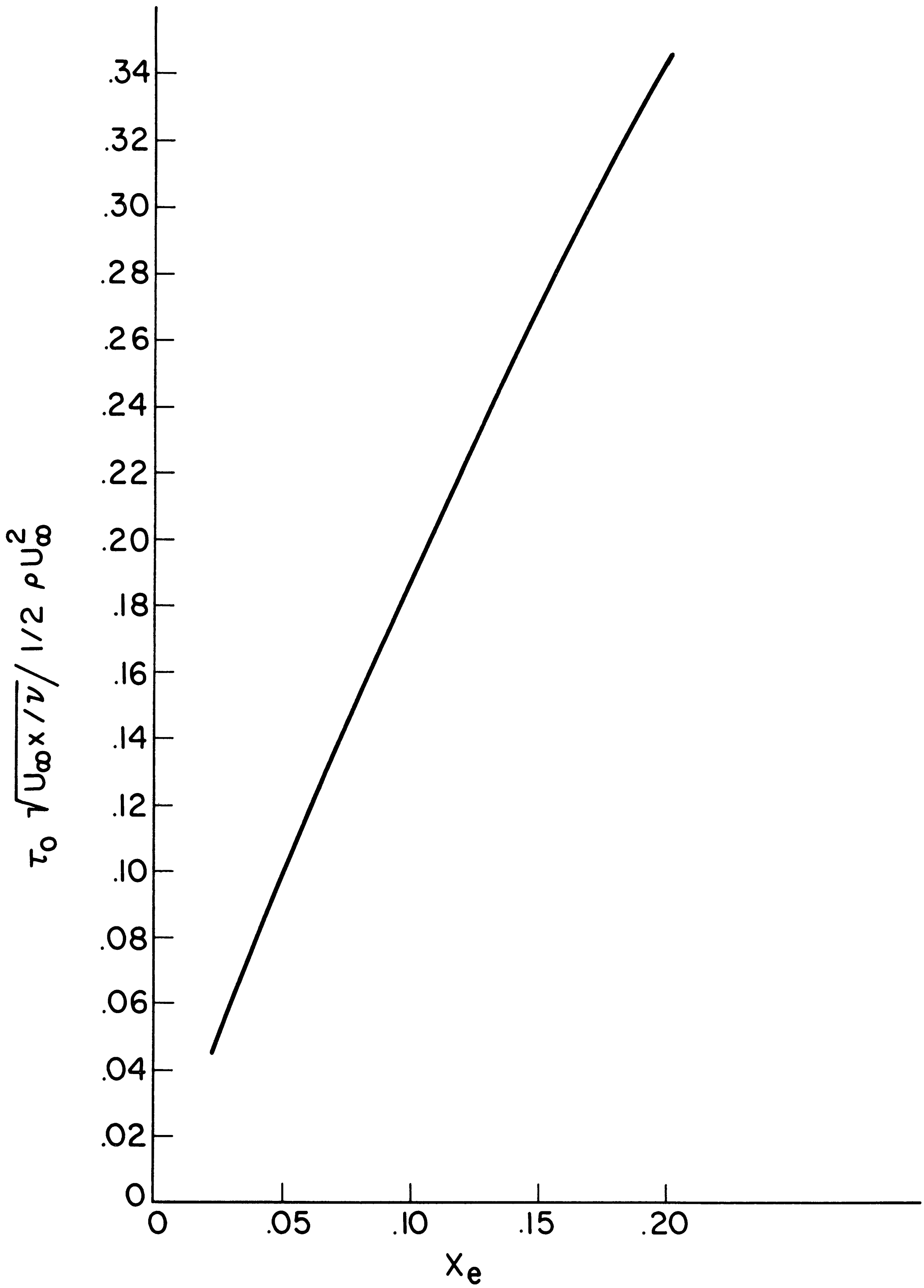

Figure 19. Local wall shear stress versus volume fraction for steady flat plate.

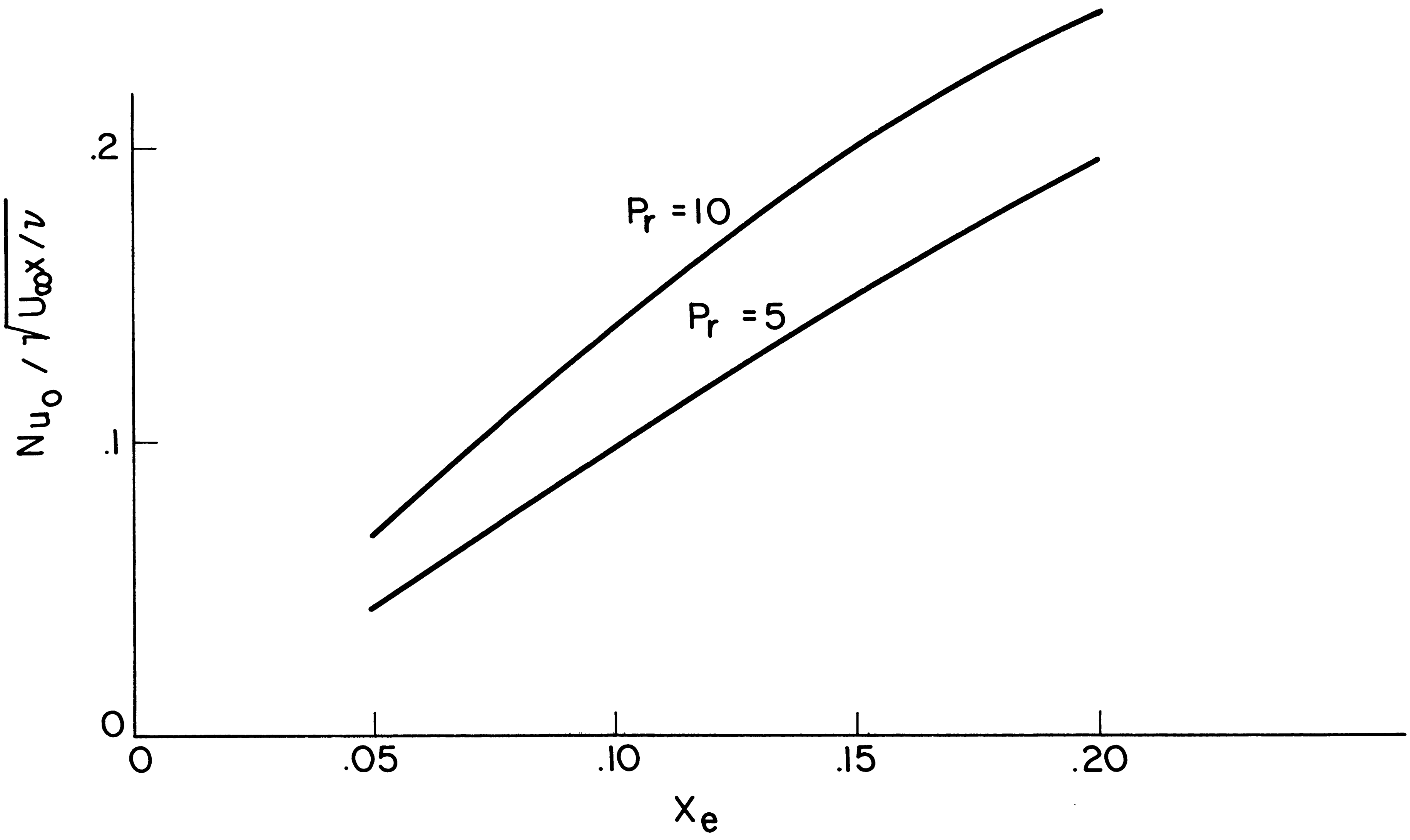

Figure 20. Local Nusselt number versus volume fraction for steady flat plate.

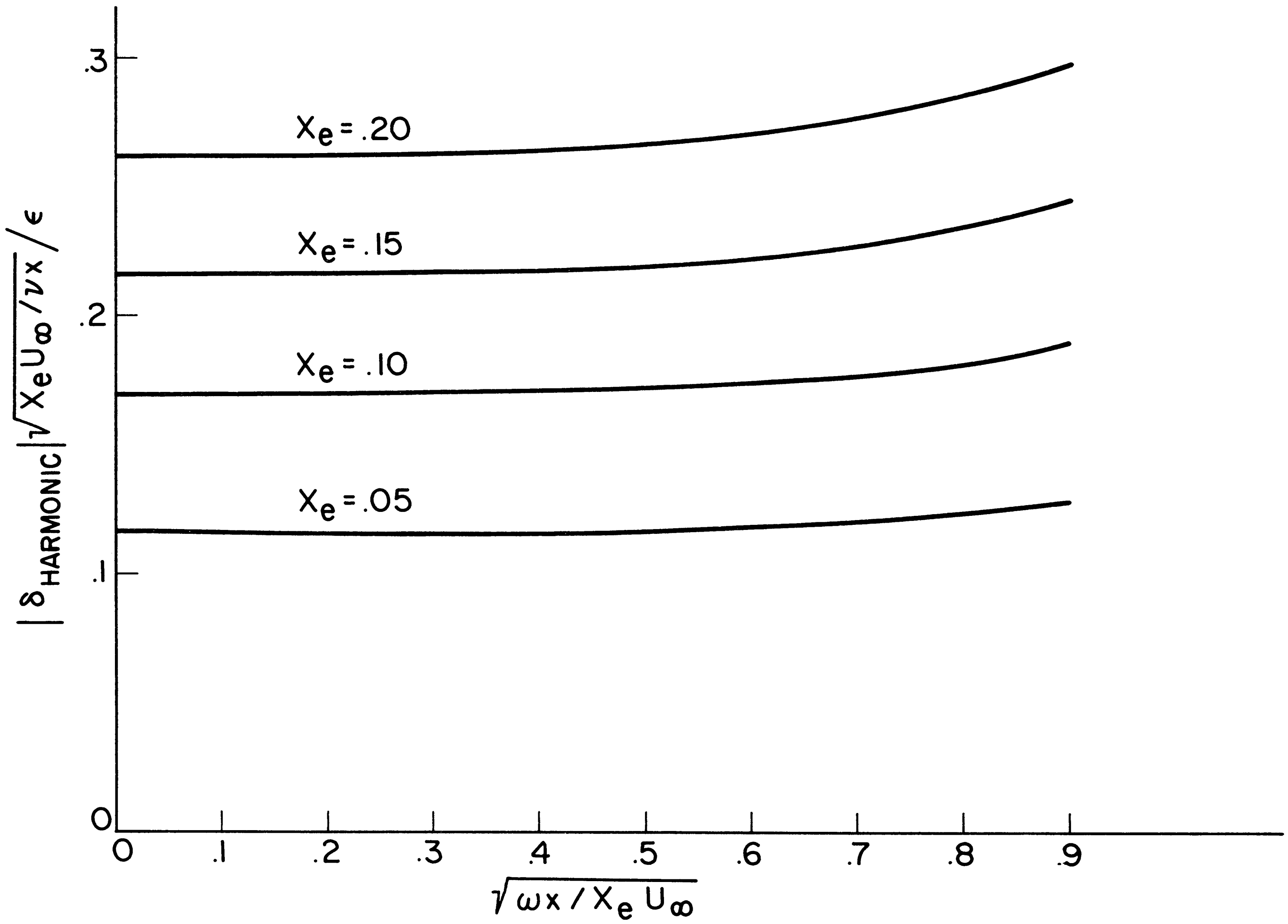

Figure 21-a. Local amplitude of liquid film thickness on oscillating flat plate.

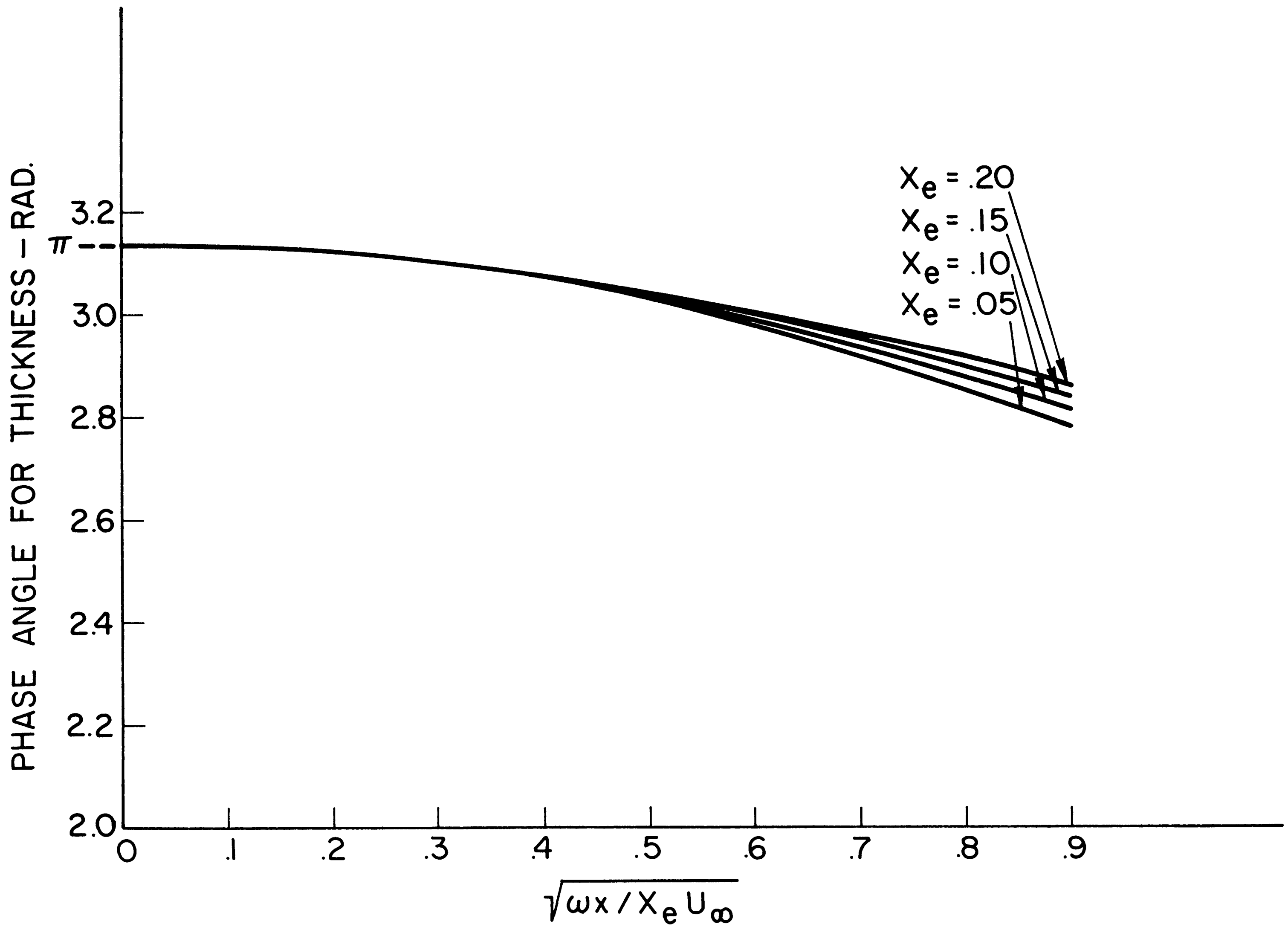

Figure 21-b. Local phase lag of liquid film thickness on oscillating flat plate.

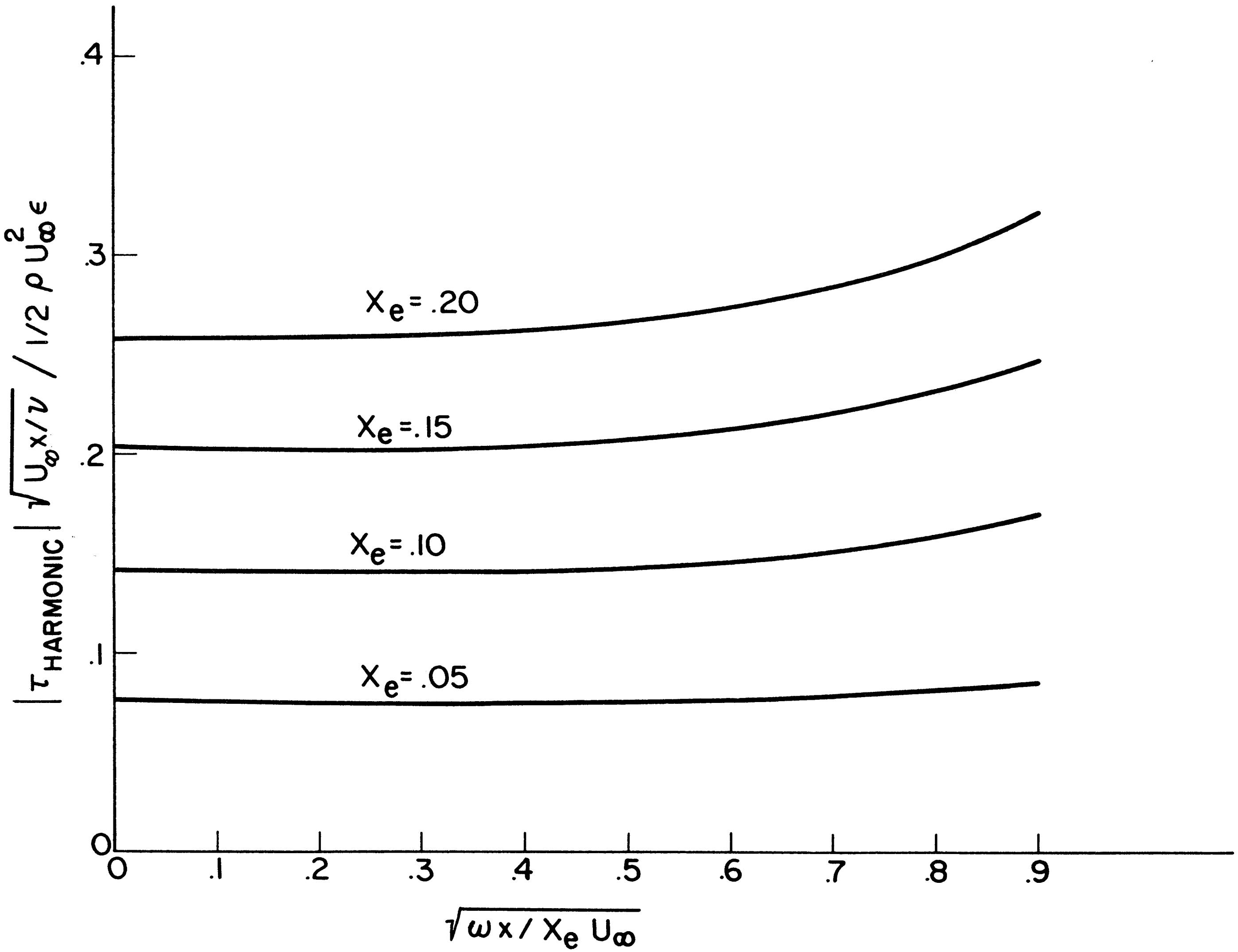

Figure 22-a. Local amplitude of wall shear stress on oscillating flat plate.

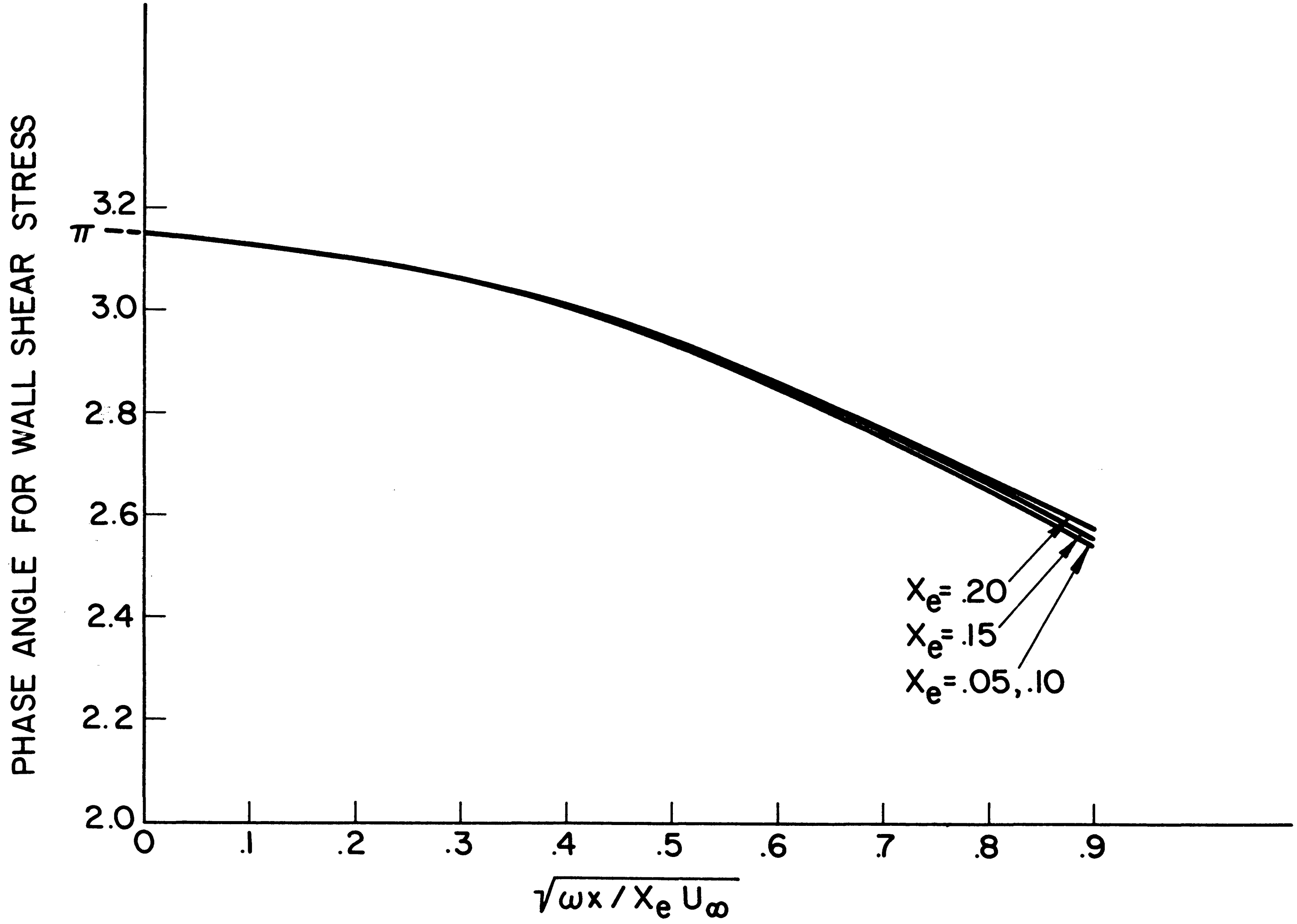

Figure 22-b. Local phase angle of wall shear stress on oscillating flat plate.

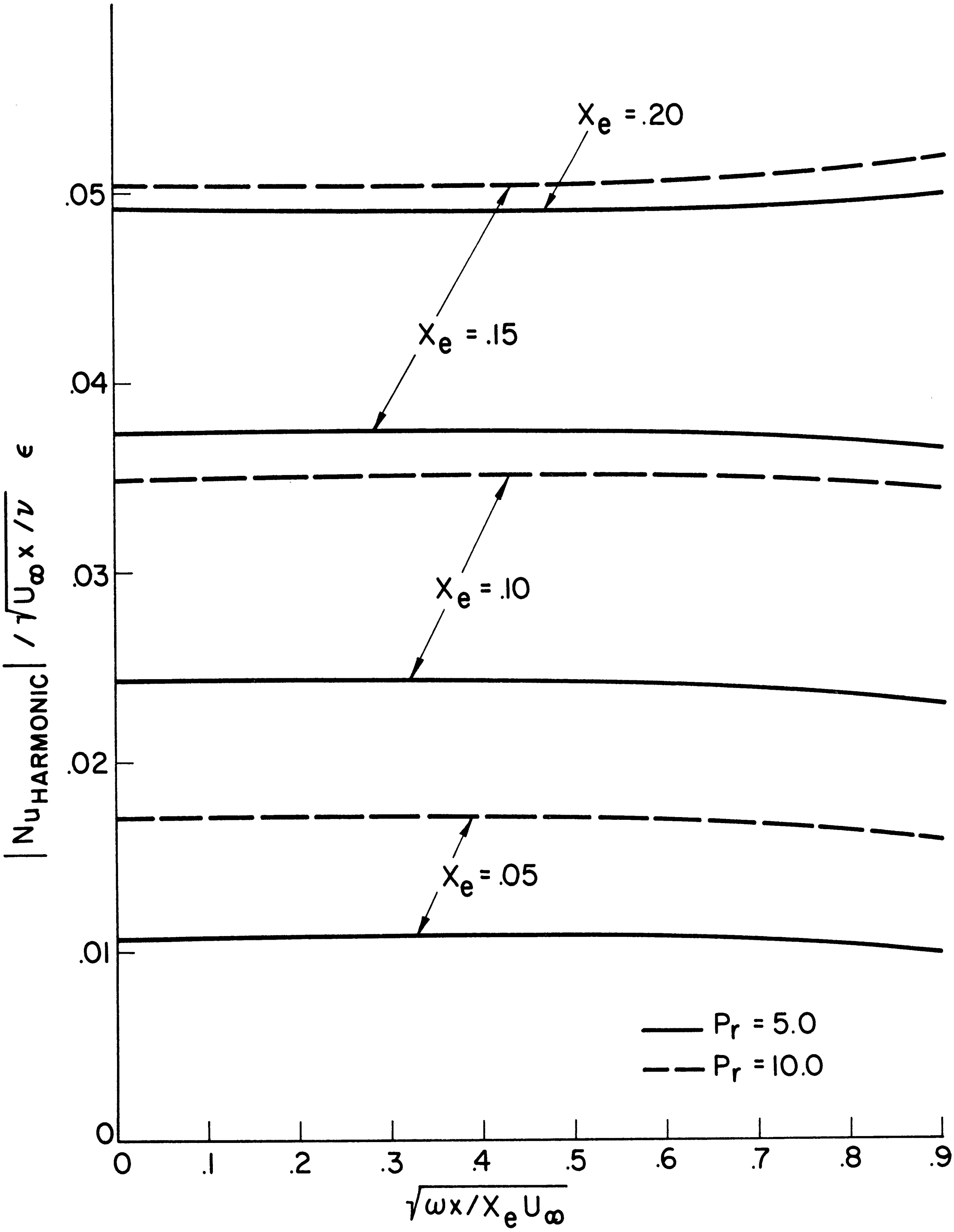

Figure 23-a. Local amplitude of Nusselt number on oscillating flat plate with P_r = 5.0 and 10.0.

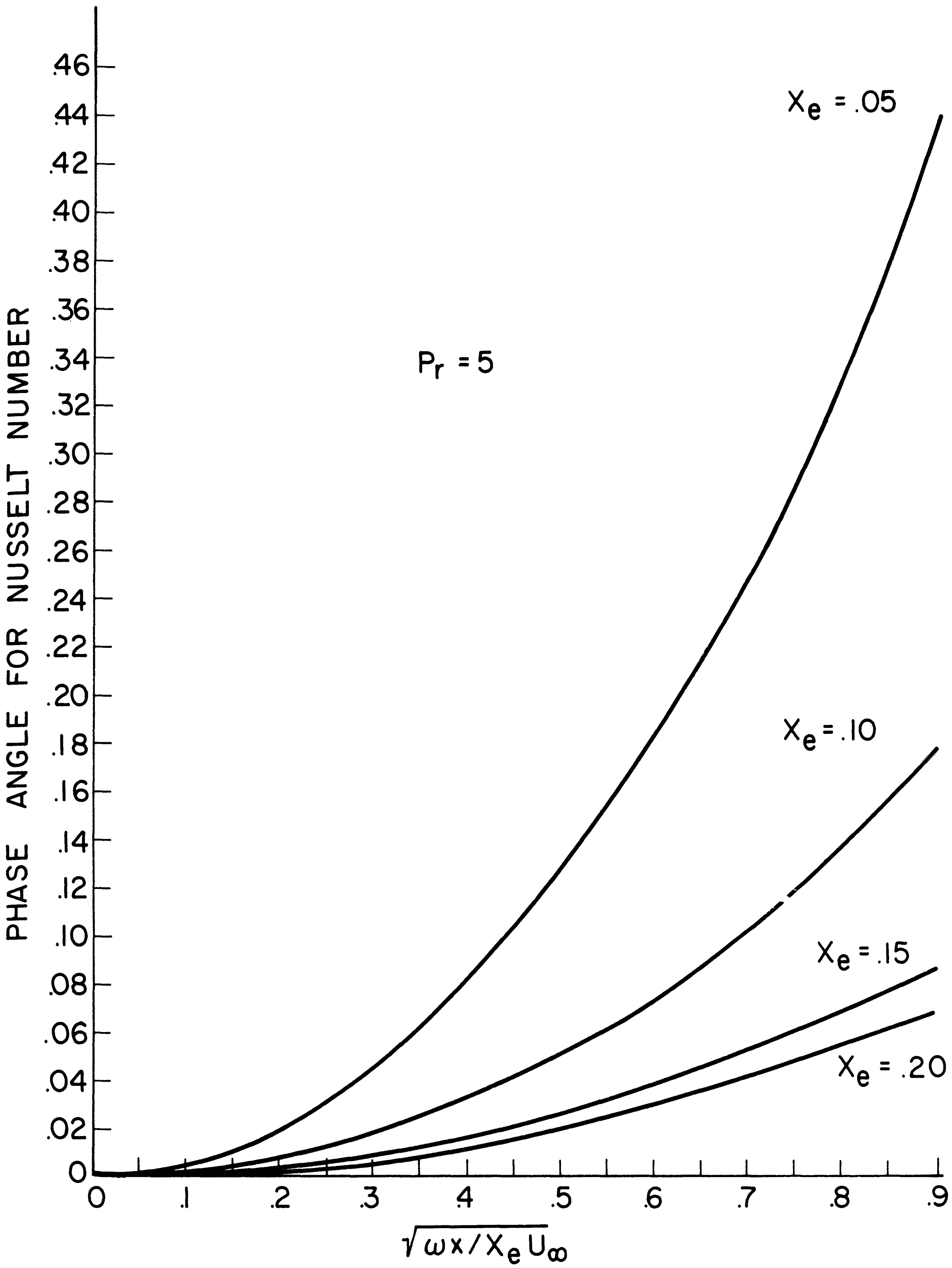

Figure 23-b. Local phase angle of Nusselt number on oscillating flat plate with $P_r = 5$.

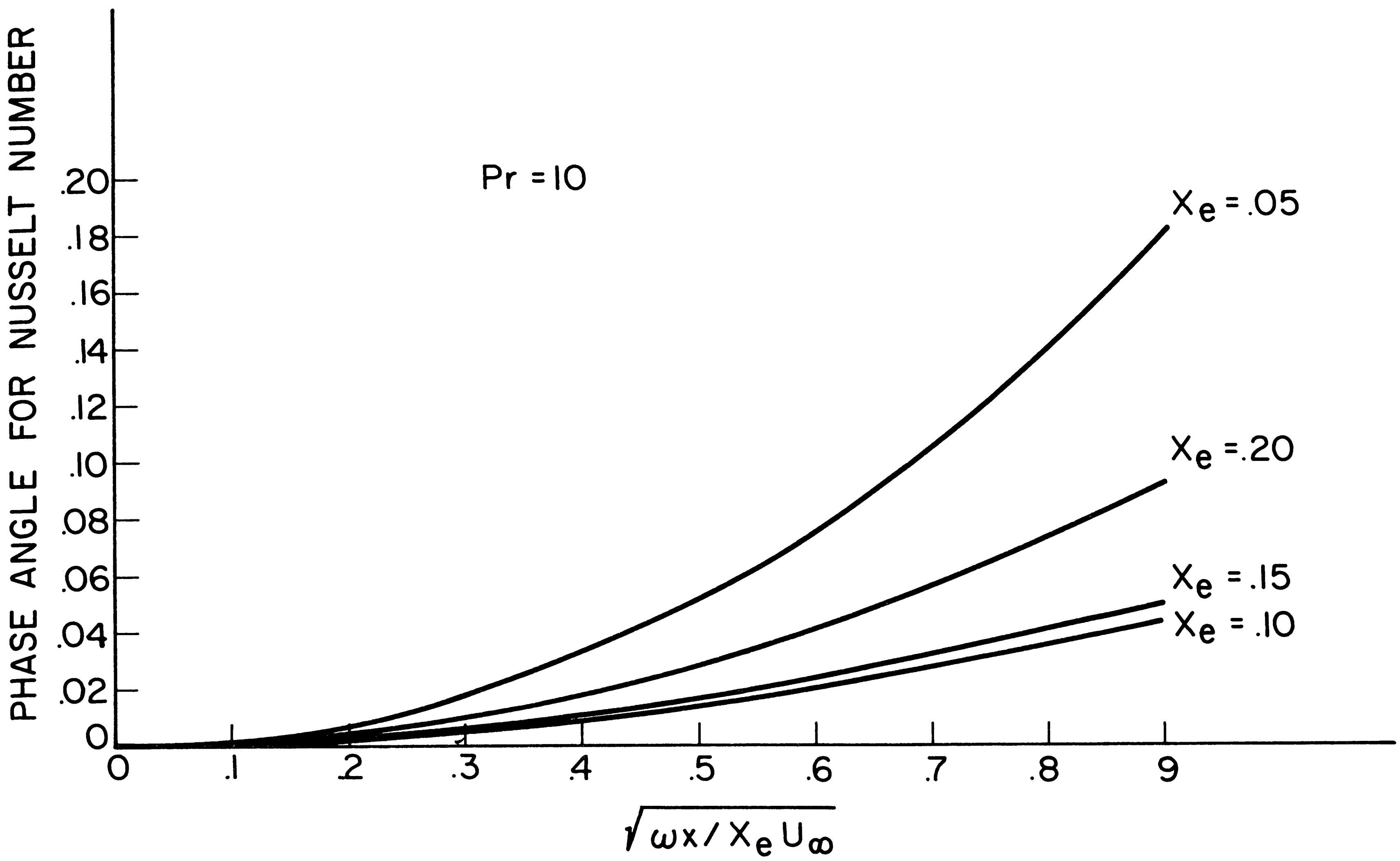

Figure 23-c. Local phase angle of Nusselt number on oscillating flat plate with P_r = 10.

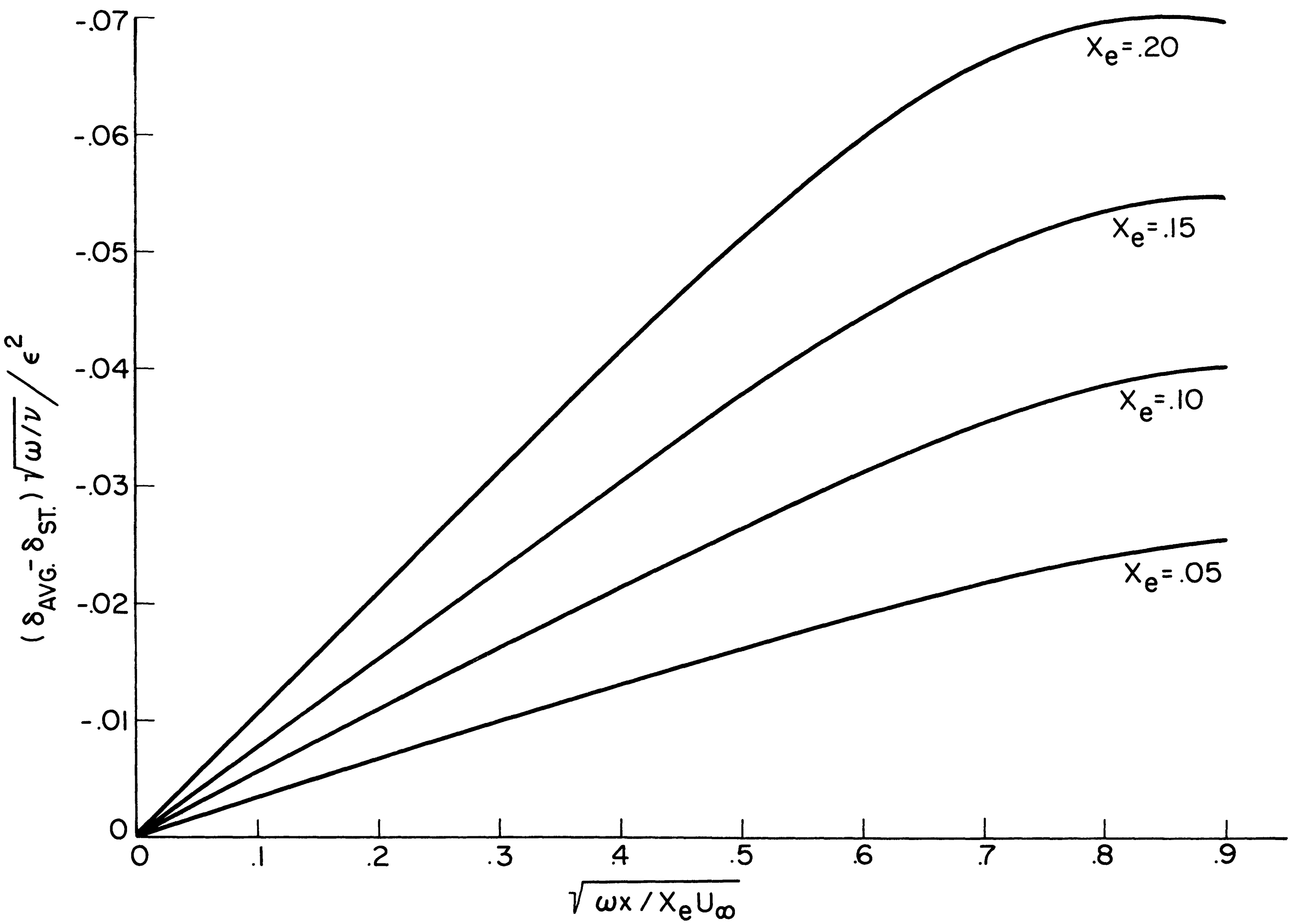

Figure 24. Permanent alteration in liquid film thickness due to oscillation of flat plate.

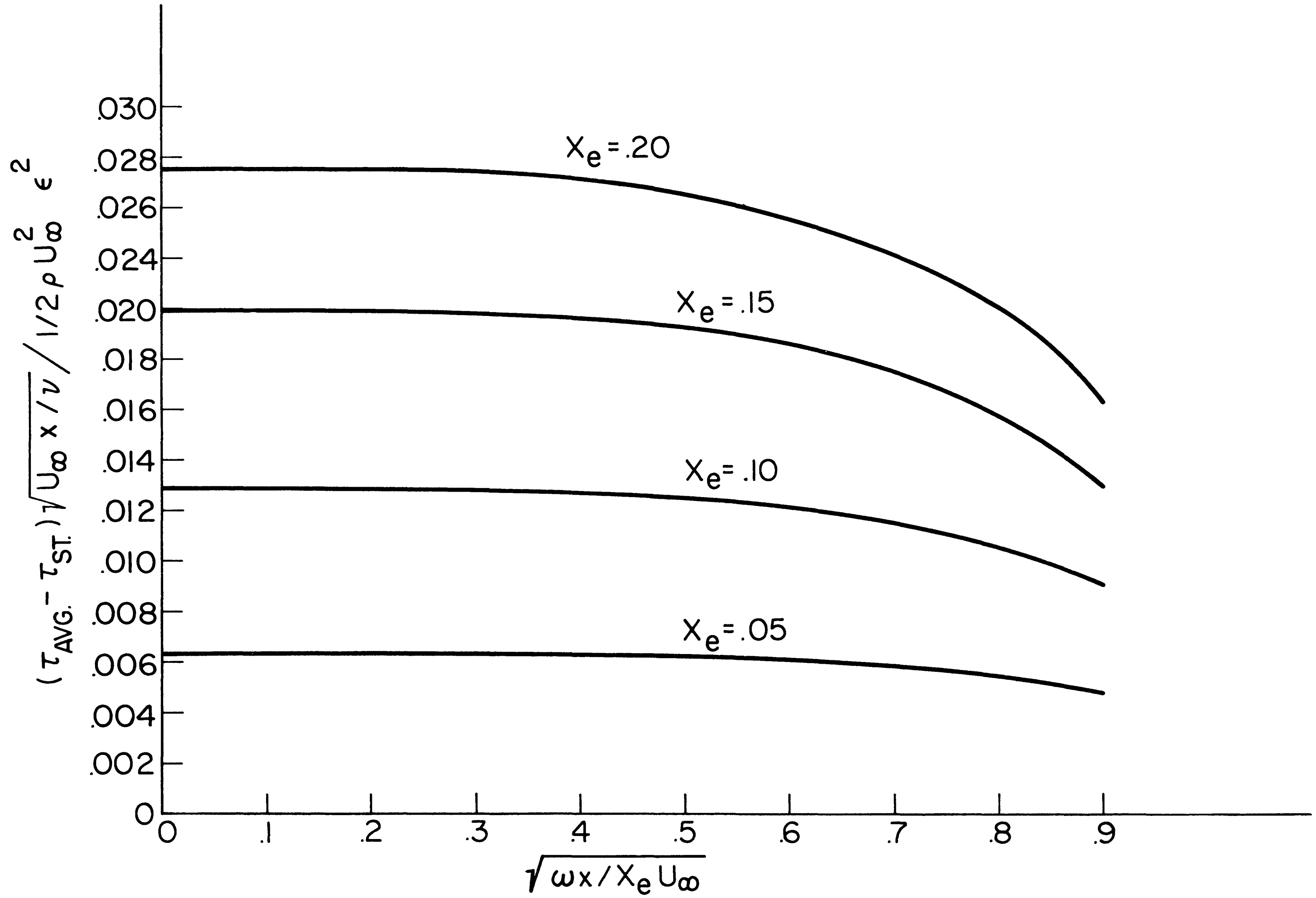

Figure 25. Permanent alteration in wall shear stress due to oscillation of flat plate.

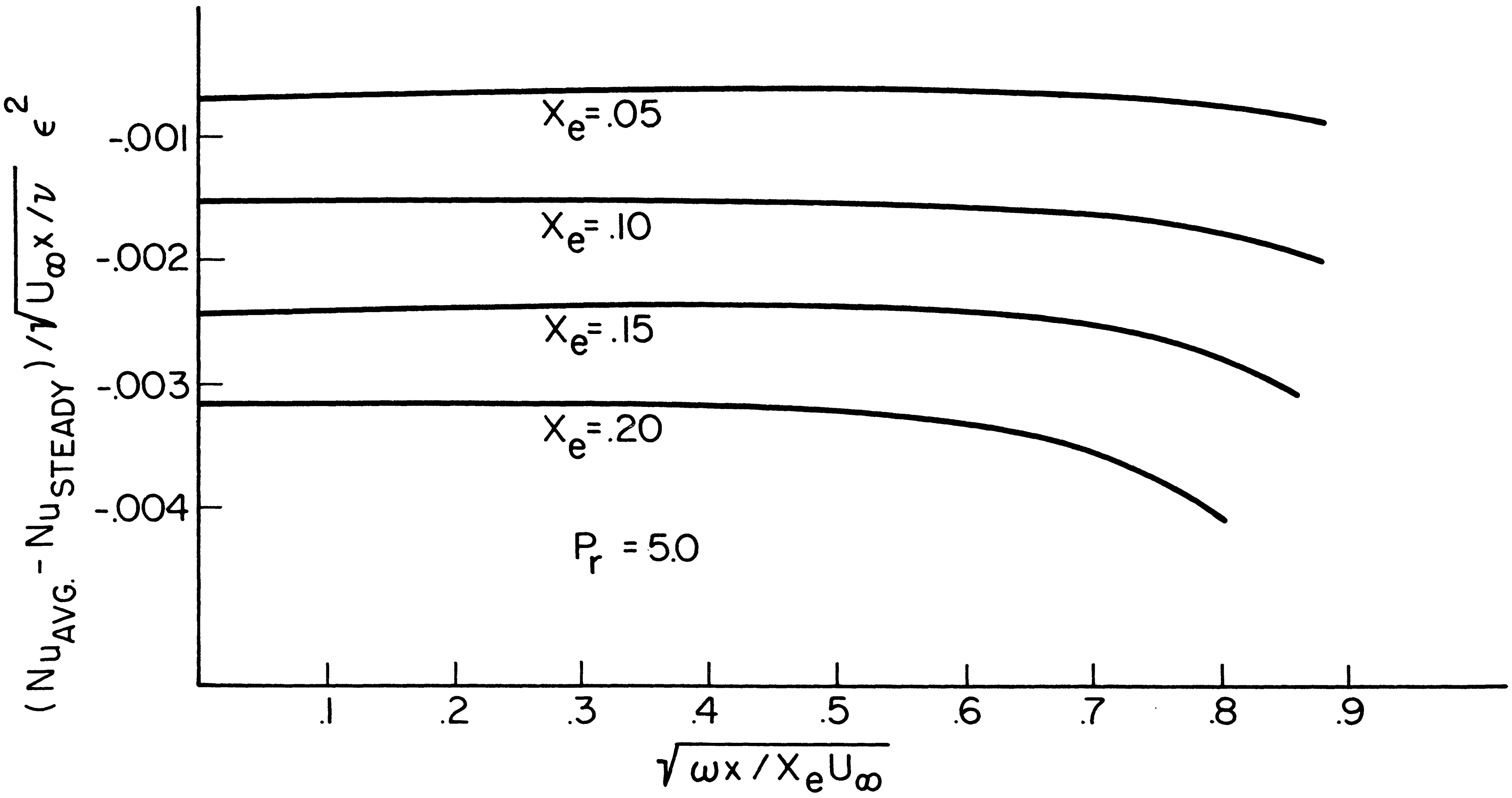

Figure 26-a. Permanent alteration in local Nusselt number due to oscillation of flat plate for P_r = 5.

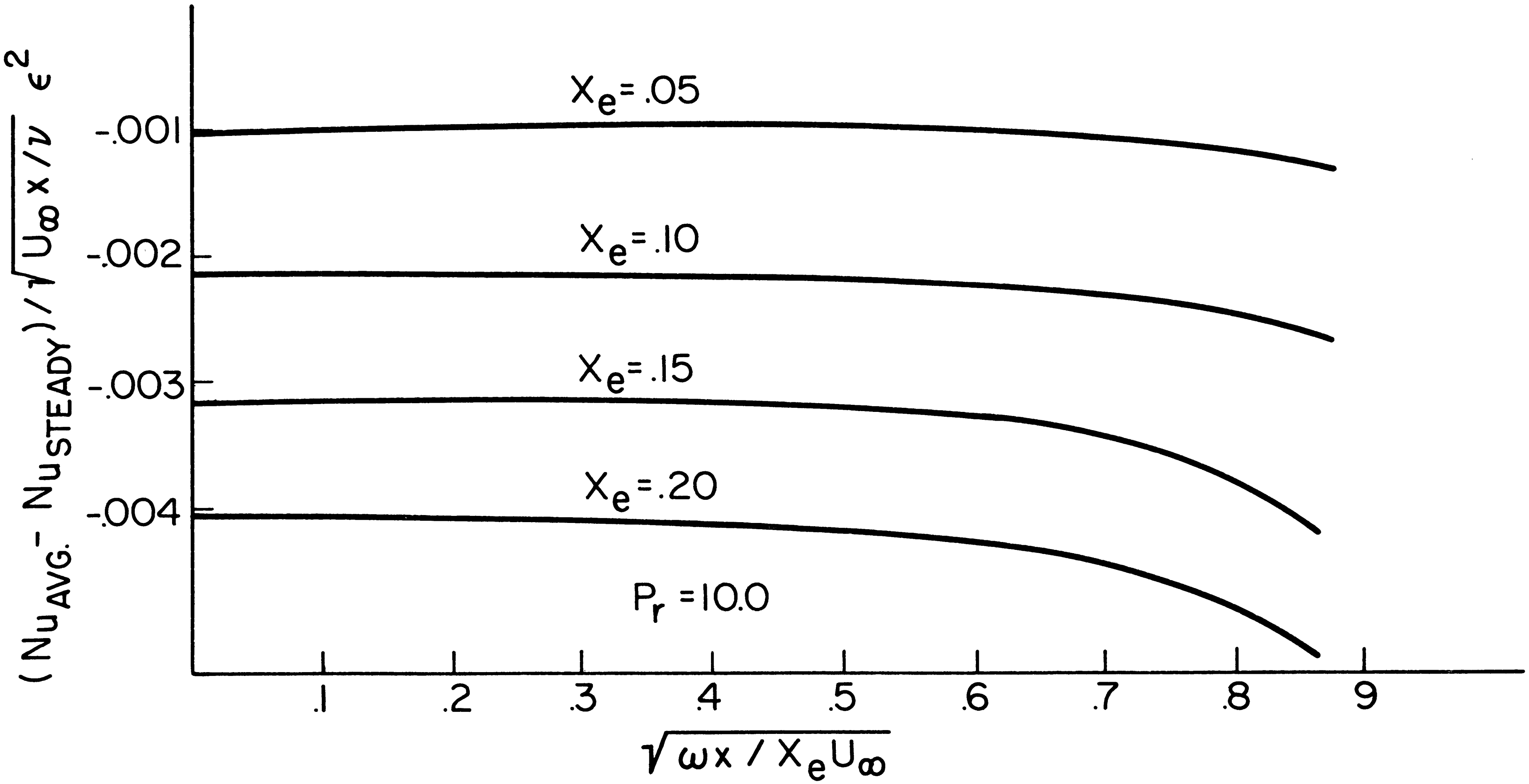

Figure 26-b. Permanent alteration in local Nusselt number due to oscillation of flat plate for $P_r = 10$.

of the Nusselt number increases with increasing Prandtl number.

It can be seen from Figures 21-a and 22-a that the amplitudes of the harmonic components of the film thickness remain relatively constant with increasing frequency (or distance along the plate) and then begin to increase somewhat, at larger values of frequency. However, Figure 23-a shows that although the amplitude of the harmonic component of the Nusselt remains relatively constant with frequency it may either increase or decrease at higher values of frequency depending on the volume fraction of the liquid in the free stream but not on the Prandtl number.

It can be seen from Figures 21-b, 22-b, 23-b, and 23-c that the phase lag of the harmonic components of film thickness, skin friction, and Nusselt number increases with increasing frequency and decreasing volume fraction of liquid in the free stream. The amplitudes of these quantities decrease with decreasing volume fraction of liquid (Figures 21-a, 22-a, and 23-a). Increasing Prandtl number increases both the amplitude and phase lag of the harmonic component of the Nusselt number (Figures 23-a through 23-c).

Examination of Figures 24 through 26-b shows that the oscillations result in an extremely small permanent alteration of the film thickness, shear stress, and Nusselt number. This is due to the fact that for small values of X_e the inertial terms in the momentum equation become relatively unimportant and so the equations are practically linear and linear equations cannot describe a physical situation in which there is a permanent alteration of any physical quantity. In this regard, one might anticipate that oscillating the cylinder would result in no permanent alterations of any physical quantities with E less than one, since the equations are also almost linear here.

Figure 25 shows that the permanent increase in the skin friction due to oscillations decreases with increasing frequency at the higher values of frequency and remains constant with change in frequency at the lower values (quasi-steady). It also increases with increasing amounts of liquid in the free stream (X_e).

Figures 25-a and 25-b show that there is a slight permanent decrease in the Nusselt number due to the oscillation which remains relatively constant with frequency. At higher frequencies this decrease becomes more pronounced. The decrease in the Nusselt number becomes larger with increase in the Prandtl number, and with increase in X_e.

CHAPTER V

CONCLUSION

A. Cylinder

Solutions are obtained for two ranges of the parameter

$$E \equiv X_e \sqrt{\frac{2R_o U_\infty}{\nu}} .$$

In case of steady flow the film thickness, skin friction, and Nusselt number have a maximum at E near unity. When E increases from unity, the liquid film thickness decreases toward the downstream with a decrease in the local Nusselt number and an increase in the velocity gradient at the outer edge of the film. This tends to unstabilize the flow in the liquid film. As E decreases from unity, the inertia effects become less important as indicated by the velocity profile which tends to become linear. The film thickness tends to increase more abruptly in the downstream direction. Because of the accompanying decrease in velocity the gravity force will probably become important in the region immediately downstream from the point at which this abrupt increase occurs on vertical cylinders.

Decreasing drop diameter and gas Reynolds number tend to move the point of sudden increase in film thickness toward the forward stagnation point. This is accompanied by a decrease in the local Nusselt number, skin friction, and film thickness.

The analytical predictions compared favorably with experimental results which indicated a substantial increase in local heat transfere rate over that of a single phase flow.

B. Oscillating Flat Plate

The steady (nonoscillating) components of the film thickness, skin friction, and Nusselt number increase with increasing X_e. The amplitudes of the harmonic components of the film thickness remain relatively constant with increasing frequency (or distance along the plate) and then begin to increase somewhat, at larger values of frequency. The amplitude of the harmonic component of the Nusselt number remains relatively constant with frequency. It may either increase or decrease at higher values of frequency depending on the volume fraction of the liquid in the free stream but not on the Prandtl number.

The phase lag of the harmonic components of film thickness, skin friction, and Nusselt number increase with increasing frequency and decreasing volume fraction of liquid in the free stream. The amplitudes of these quantities decrease with decreasing volume fraction of liquid. Increasing Prandtl number, increases both the amplitude and phase lag of the harmonic component of the Nusselt number.

Oscillations result in an extremely small permanent alteration of the film thickness, shear stress, and Nusselt number. This is due to the fact that for small values of X_e the inertial terms in the momentum equation become relatively unimportant. One may anticipate that oscillating the cylinder would result in no permanent alterations of any physical quantities with E less than one.

The permanent increase in the skin friction due to oscillations decreases with increasing frequency at the higher values of frequency and remains constant with change in frequency at the lower values (quasi-steady). It also increases with increasing amounts of liquid in the free stream (X_e).

There is a slight permanent decrease in the Nusselt number due to the oscillation which remains relatively constant with frequency. At higher frequencies this decrease becomes more pronounced. The decrease in the Nusselt number becomes larger with increase in the Prandtl number, and with increase in X_e.

BIBLIOGRAPHY

1. Acrivos, A., et al., "Research Investigation of Two-Component Heat Transfer," Aerospace Research Laboratories Report, ARL 64-116, July, 1964, under contract AF 33(657)-10930, Project 7063, Task 7063-01. Also film made during experiments at Marquardt Corporation.

2. Bellman, A., Perturbation Techniques in Mathematics, Physics, and Engineering, Holt, Rinehart, and Winston, Inc. (1964).

3. Bennett, J.A.R., Collier J.G., Pratl, H.R.C., and Thornton J.D., "Heat Transfer to Two-Phase Gas-Liquid Systems, Part I. Steam-Water Mixtures in the Liquid-Dispersed Region in an Annulus." Trans. Instr. Chem. Enginrs., 39, p. 113 (1961), AERE R-3159.

4. Bergman, N.R., "A Method for Numerically Calculating the Area-Distribution of Water Impingement on the Leading Edge of an Air-foil in a Cloud," NACA TN1397, August, 1947.

5. Bertoletti, S., et al., "Heat Transfer and Pressure Drop with Steam-Water Spray." C.I.S.E. R. 36, Milan, Italy, April, 1961.

6. Cicchetti, A., et al., "Two-Phase Cooling Experiments Pressure Drop, Heat Transfer and Burnout Measurements." Engin. Nuclear 7 (6), p. 407 (1960).

7. Collier, J.G., Pulling, O.J., "Heat Transfer to Two-Phase Gas-Liquid Systems." United Kingdom Atomic Energy Authority Research Group Report, AERE-R 3809, Chem. Eng. Div., Gerkshire (1962).

8. Davis, E.J., and David, M.M., "Heat Transfer to High-Quality Steam-Water Mixtures Flowing in Horizontal Rectangular Duct." Canadian Journ. of Chem. Engin., June, 1961, p. 99.

9. Elperin, I.T., "Heat Transfer of Two-Phase Flow with a Bundle of Tubes," Inzhereerno-Fizicheski Zhurnal, Vol. IV, No. 8, pp. 30-35, August, 1961.

10. Goldmann, K., "Burnout In Turbulent Flow—A Droplet Diffusion Model," Journal of Heat Transfer, Transactions of A.S.M.E. series C., pp. 158-162, May, 1961.

11. Groothuis, H., and Hendal, W.P.C., "Heat Transfer in Two-Phase Flow." Chem. Engin. Sci. 11, pp. 212-220 (1959).

BIBLIOGRAPHY (Continued)

12. Guibert, A., *et al.*, "Determination of the Rate, the Area, and the Destribution of Impingement of Water Drops on Various Airfoils from Trajectories Obtained on the Differential Analyzer," NACA RM 9A05, February, 1949.

13. Hodgman, C., ed., *C.R.C. Standard Mathematical Tables*, 12th Ed., Chemical Rubber Publishing Co. (1961).

14. Juyghe, J., *et al.*, "Heat Transfer by a Mixture of a Liquid and a Gas in Turbulent Forced Convection with Little Vaporization of the Liquid Phase." Comptes Rendus des Seances de l' Academiedes Sciences, *252*, pp. 3015-3017, May, 1961.

15. Isbin, H.S., *et al.*, "Heat Transfer to Steam-Water Flows." Proc. Heat Transfer and Fluid Mech. Inst., Stanford University Press, Stanford, U.S.A., p. 70 (1961).

16. Jahnke, E., Emde, F., *Tables of Functions*, Dover Publication (1943).

17. Kreith, F., *Principles of Heat Transfer*, International Text Book Co., Scranton (1958).

18. Landau-Lifshitz, *Fluid Mechanics*, Addison-Wesley Publishing Company, Inc. (1959).

19. Langmuir, I., and Blodget, K., "A Mathematical Investigation of Water Droplet Trajectories," AAF TR 5418, February, 1946.

20. Lighthill, M.J., "The Response of Laminar Skin Friction and Heat Transfer to Fluctuations in the Steam Velocity," Proc. Roy. Soc., Series A., *224*, pp. 1-23 (1954).

21. Lin, C.C., "Motion in the Boundary-Layer with a Rapidly Oscillating External Flow," Proc. 9th Int. Cong. Appl. Mech. Brussel, *4*, pp. 155-167 (1956).

22. Perroud, P., le la Harpe, "Heat Transfer by Liquids Entrained in a Turbulent Gas Stream." CEA No. 1422 (1960).

23. Rohsenow, W.H., and Choi, H., *Heat, Mass, and Momentum Transfer*, Prentice-Hall, Inc. (1961).

24. Rosenhead, L., *Laminar Boundary Layers*, Oxford Press (1963).

BIBLIOGRAPHY (Concluded)

25. Sach, P., and Long, R.A.K., "A Correlation for Heat Transfer in Stratified Two-Phase Flow with Vaporization." Int. Journal of Heat and Mass Transfer, 2 (3), pp. 222-230 (1961).

26. Sani, R.L., "Downward Boiling and Non-Boiling Heat Transfer in a Uniformly Heated Tube." University of California UCRL-9023, January, 1960.

27. Schrock, V.E., and Grossman, L.V., "Forced Convection Boiling Studies." Forced Convection Vaporization Project—Final Report 73308-UCX 2182, November, 1959, University of California at Berkeley.

28. Schlichting H., Boundary Layer Theory, McGraw-Hill Book Company, Inc., New York (1960).

29. Tifford, A.N., Ohio State University, "Exploratory Investigation of Laminar Boundary Layer Heat Transfer Characteristics of Gas Liquid-Spray Systems," Aerospace Research Laboratories Report, ARL 64-136, September, 1964.

30. Tribus, M., "A New Method for Calculating Water-Droplet Trajectories about Streamlined Bodies," University of Michigan, ERI Report on Project M992-E, December, 1951.

31. Van Dyke, M., Perturbation Methods in Fluid Mechanics, Academic Press (1964).

APPENDIXES

APPENDIX I

CALCULATIONS FOR DETERMINING THE MINIMUM VALUES OF X_e AND GAS REYNOLDS NUMBERS AT WHICH DROP TRAJECTORIES ARE STRAIGHT LINES

To find these minimum values we set:

$$\frac{3}{8}\,\frac{C_D}{(r_d/R_o)}\,\frac{\rho_g}{\rho} = .1 \tag{I.1}$$

$$N' = 10$$

and we have:

$$X_e \simeq \left(16\,\frac{r_d}{R_o}\right)^3 \tag{I.2}$$

For a given r_d/R_o this determines the minimum value of X_e and since the drag coefficient is a function of the Reynolds number, based on the drop diameter. (I.1) determines this and therefore the gas Reynolds number, and therefore (I.1) and (I.2) determine the minimum value of E^2.

These calculations have been carried out using the computer program listed in this Appendix.

The values of CD(J) corresponding to RED(J) used in this program are tabulated in Table II of Appendix XXV.

NOMENCLATURE FOR COMPUTER PROGRAM

CD(J) ~ tabulated value of C_D at RED(J)

EP(J) ~ $\dfrac{3}{8}\,\dfrac{CD(J)\rho_g/\rho}{(r_d/R_o)}$

ES	~	minimum value of E^2
ET	~	desired value of EP(J), .1
N	~	N'; = 10
R	~	r_d/R_o
RC	~	value of RE(J) at N' = 10 and EP(J) = .1 (minimum value)
RE(J)	~	$\frac{R_o U_\infty}{\nu} = \frac{R_o}{r_d} \times$ RED(J)
RED(J)	~	tabulated value of $r_d U_\infty/\nu_g$
RG	~	minimum value of $R_o U_\infty/\nu_g$
XE	~	value of X_e at N' = 10

```
$CCMPILE MAC, PRINT OBJECT                                          014456   07/26/65    11 33 28.5 PM

MAD (01 MAY 1965 VERSION) PROGRAM LISTING ... ... ...

          DIMENSION RE(100),RED(100),EP(100),CD(1CC)                  *001
          INTEGER J,JMAX                                              *002
          READ AND PRINT DATA                                         *003
          THROUGH SW2, FOR R=C.CO1,C.C005,R.G.RMAX                    *004
          THROUGH SW1, FOR J=1,1,J.G.JMAX                             *005       01
          RE(J)=RED(J)*7.52/R                                         *006       02
SW1       EP(J)=CD(J)*0.003/(8.*R)                                    *007       02
          XE=(1.6*N*R).P.3                                            *008       01
          WHENEVER EP(JMAX).L.ET .AND. XE.L.C.C3                      *009       01
          RC=TAB.(ET,EP(1),RE(1),1,1,4,JMAX,S)                        *010   01  01
          ES=(XE.P.2)*RC                                              *011   01  01
          RG=RC/7.52                                                  *012   01  01
          PRINT RESULTS R,RG,ES,RC,XE                                 *013   01  01
          OTHERWISE                                                   *014   01  01
          CONTINUE                                                    *015   01  01
SW2       END CF CONDITIONAL                                          *016   01  01
          END CF PROGRAM                                              *017
```

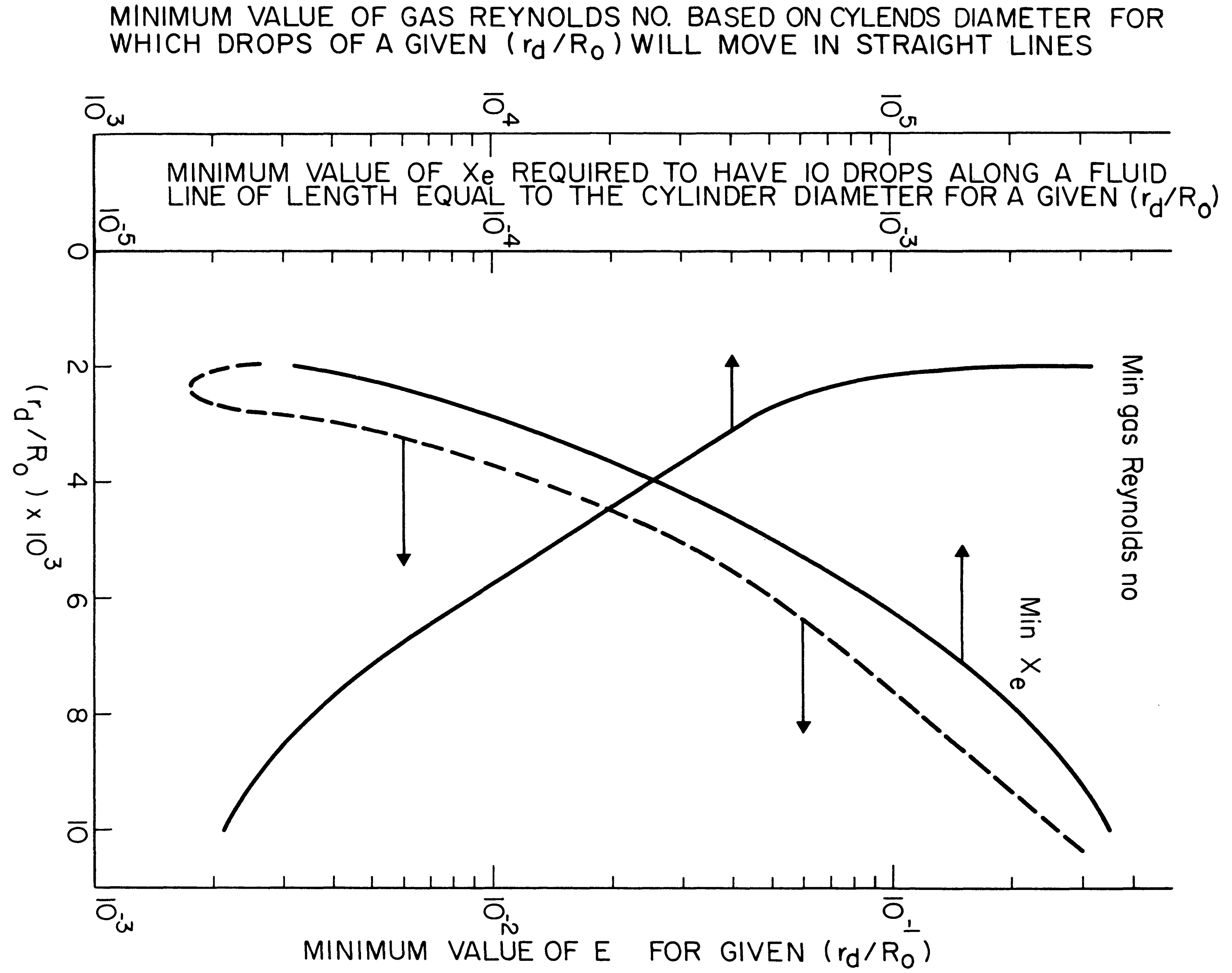

Figure 27. Minimum value of E^2 and gas Reynolds number for a straight drop trajectory.

APPENDIX II

ESTIMATION OF PARAMETERS FOR AIR/WATER MIXTURES

A. PROPERTY RATIOS* FOR AIR/WATER MIXTURES

$$\frac{\rho_g}{\rho} = \frac{.081}{62.4} = 1.3\text{x}10^{-3}$$

$$\frac{\nu_g}{\nu} = \frac{.145\text{x}10^{-3}}{1.93\text{x}10^{-5}} = 7.52$$

$$\sqrt{\frac{\nu_g}{\nu}} = \sqrt{7.52} = 2.74$$

$$\frac{C_{pg}}{C_p} = \frac{.25}{1} = .25$$

$$P_{r_g} = .7$$

B. CYLINDER PROBLEM

Evaluation of surface phenomena and pressure terms for single component gas boundary layer.

From Schlichting, p. 153, Figure 9.6:

$$\frac{\tau_g}{1/2\rho_g U_\infty^2} \sqrt{\frac{2U_\infty R_o}{\nu_g}} \leq \sqrt{2} \cdot 4.5$$

From Schlichting, p. 320, Figure 14.13:

$$\frac{N_{u_g}}{\sqrt{R_{e_g}}} \leq 1$$

For potential flow:

$$\frac{P_g}{1/2\rho_g U_\infty^2} \leq 2$$

---

*Kreith "Elements of Heat Transfer."

Hence for air/water mixtures one obtains:

$$\left[\sqrt{\frac{\nu_g}{\nu}}\,\frac{\rho_g}{\rho}\right]\left[\tau_g\sqrt{\frac{2U_\infty R_o}{\nu_g}}\Big/ 1/2\rho_g U_\infty^2\right] \approx 4.5\cdot\sqrt{2}\cdot 2.74\cdot 1.3\text{x}10^{-3}$$

$$\approx 2.2\text{x}10^{-3}$$

$$2\,\frac{\rho_g}{\rho}\left[\frac{P_g}{1/2\rho_g U_\infty^2}\right] \approx 2\cdot 1.3\text{x}10^{-3}\cdot 2 \approx 5.2\text{x}10^{-3}$$

$$\left[\frac{N_{u_g}}{\sqrt{\frac{2R_o U_\infty}{\nu_g}}\cdot P_{r_g}}\right]\left[\frac{C_{p_g}}{C_p}\,\frac{\rho_g}{\rho}\sqrt{\frac{\nu_g}{\nu}}\right] \approx \frac{1.0\cdot 0.25\cdot 1.3\text{x}10^{-3}\cdot 2.7}{.7}$$

$$\approx 1.2\text{x}10^{-3}$$

C. OSCILLATING FLAT PLATE PROBLEM

Evaluation of surface phenomena for single component gas boundary layer.

From Schlichting, p. 120.

$$\frac{\tau_g}{\rho_g U_\infty^2}\sqrt{\frac{U_\infty x}{\nu_g}} = .332$$

From Schlichting p. 305.

$$\left(\frac{q_g}{T-T_\infty}\right)\frac{x}{k_g}\Big/\sqrt{\frac{U_\infty x}{\nu_g}}\;\left(\frac{\mu_g C_{p_g}}{k_g}\right) = N_{u_g}\Big/\sqrt{\frac{U_\infty x}{\nu_g}}\;P_{r_g} \approx \frac{.332}{P_{r_g}^{1/3}}$$

Hence for air/water mixtures one has:

$$\left[\sqrt{\frac{\nu_g}{\nu}}\,\frac{\rho_g}{\rho}\right]\frac{\tau_g}{\rho_g U_\infty^2}\sqrt{\frac{U_\infty x}{\nu_g}} \approx .332\cdot 2.74\cdot 1.3\text{x}10^{-3} \approx 1.1\text{x}10^{-3}$$

$$\left[\left(\frac{q_g}{T-T_\infty}\right)\frac{x}{k_g}\Bigg/\sqrt{\frac{U_\infty x}{\nu_g}}\left(\frac{\mu_g C_{p_g}}{k_g}\right)\right]\left[\sqrt{\frac{\nu_g}{\nu}}\,\frac{\rho_g}{\rho}\,\frac{C_{p_g}}{C_p}\right]$$

$$\approx\left[\frac{.332}{(.7)^{2/3}}\right]\left[2.74\cdot 1.3\times10^{-3}\cdot .25\right]\approx 3.75\times10^{-4}$$

APPENDIX III

REDUCTION OF THE FIRST OF EQUATIONS (23)

Substitutions of Eqs. (26) and (27) into the first of Eqs. (23) yields:

$$a_o\left(1 + \frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4 + \frac{b_6}{6!}\xi^6 + \ldots\right)\left\{\left[\left(f_1'\xi - \frac{4}{3!}f_3'\xi^3 + \frac{6}{5!}f_5'\xi^5\right.\right.\right.$$

$$\left.- \frac{8}{7!}f_7'\xi^7 + \ldots\right)\left(f_1' - \frac{4}{2!}f_3'\xi^2 + \frac{6}{4!}f_5'\xi^4 - \frac{8}{6!}f_7'\xi^6 + \ldots\right) - \left(f_1\right.$$

$$\left.\left.- \frac{4}{2!}f_3\xi^2 + \frac{6}{4!}f_5\xi^4 - \frac{8}{6!}f_7\xi^6 + \ldots\right)\left(f_1''\xi - \frac{4}{3!}f_3''\xi^3 + \frac{6}{5!}f_5''\xi^5 - \frac{8}{7!}f_7''\xi^7 + \ldots\right)\right]\right\}$$

$$-a_o\left(b_2\xi + \frac{b_4}{3!}\xi^3 + \frac{b_6}{5!}\xi^5 + \ldots\right)\left(f_1'\xi - \frac{4}{3!}f_3'\xi^3 + \frac{6}{5!}f_5'\xi^5 - \frac{8}{7!}f_7'\xi^7 + \ldots\right)\Big]\Big\}$$

$$\left.+\ldots\right)^2 = \frac{1}{E^2}\left(f_1'''\xi - \frac{4}{3!}f_3'''\xi^3 + \frac{6}{5!}f_5'''\xi^5 - \frac{8}{7!}f_7'''\xi^7 + \ldots\right)$$

Collecting terms and rearranging yields:

$$\left(1 + \frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4 + \frac{b_6}{6!}\xi^6 + \ldots\right)\left\{\left[f_1'^2\xi - \left(\frac{4}{3!} + \frac{4}{2!}\right)f_1'f_3'\xi^3 + \left(\frac{6}{4!}f_5'f_1'\right.\right.\right.$$

$$\left.+ \frac{4\cdot 4}{2!\cdot 3!}f_3'^2 + \frac{6}{5!}f_5'f_1'\right)\xi^5 - \left(\frac{8}{6!}f_7'f_1' + \frac{6\cdot 4}{3!\cdot 4!}f_5'f_3' + \frac{6\cdot 4}{5!\cdot 2!}f_5'f_3'\right.$$

$$\left.+ \frac{8}{7!}f_7'f_1'\right)\xi^7 + \ldots - f_1f_1''\xi + \left(\frac{4}{3!}f_3''f_1 + \frac{4}{2!}f_3f_1''\right)\xi^3 - \left(\frac{6}{5!}f_5''f_1\right.$$

$$\left.+ \frac{4\cdot 4}{3!\cdot 2!}f_3''f_3 + \frac{6}{4!}f_1''f_5\right)\xi^5 + \left(\frac{8}{7!}f_7''f_1 + \frac{4\cdot 6}{5!\cdot 2!}f_5''f_3 + \frac{4\cdot 6}{3!\cdot 4!}f_3''f_5\right.$$

$$\left.\left.\left.+ \frac{8}{6!}f_1''f_7\right)\xi^7 + \ldots\right]\right\} - \left(b_2\xi + \frac{b_4}{3!}\xi^3 + \frac{b_6}{5!}\xi^5 + \ldots\right)\left\{\left[f_1'^2\xi^2 - \frac{2\cdot 4}{3!}f_1'f_3'\xi^4\right.\right.$$

$$\left.\left.+ \left(\frac{2\cdot 6}{5!}f_5'f_1' + \frac{4\cdot 4}{3!\cdot 3!}f_3'^2\right)\xi^6 + \ldots\right]\right\} = \frac{1}{a_oE^2}\left(f_1'''\xi - \frac{4}{3!}f_3'''\xi^3\right.$$

$$+ \frac{6}{5!} f_5''' \xi^5 - \frac{8}{7!} f_7''' \xi^7 + \ldots \Big)$$

$$\left(1 + \frac{b_2}{2!} \xi^2 + \frac{b_4}{4!} \xi^4 + \frac{b_6}{6!} \xi^6 + \ldots\right) \Big\{ \Big[\left(f_1'^2 - f_1 f_1''\right) \xi + \left(\frac{4}{3!} f_3'' f_1 + \frac{4}{2!} f_3 f_1'' - \frac{4 \cdot 4}{3!} f_1' f_3'\right)$$

$$\xi^3 + \left(\frac{6 \cdot 6}{5!} f_5' f_1' + \frac{4 \cdot 4}{2! \cdot 3!} f_3'^2 - \frac{6}{5!} f_5'' f_1 - \frac{4 \cdot 4}{3! \cdot 2!} f_3'' f_3 - \frac{6}{4!} f_1'' f_5\right) \xi^5 + \Big(- \frac{8 \cdot 8}{7!} f_7' f_1'$$

$$- \frac{4 \cdot 6 \cdot 8}{5! \cdot 3!} f_5' f_3' + \frac{8}{7!} f_7'' f_1 + \frac{4 \cdot 6}{5! \cdot 2!} f_5'' f_3 + \frac{4 \cdot 6}{3! \cdot 4!} f_3'' f_5 + \frac{8}{6!} f_1'' f_7\Big) \xi^7$$

$$+ \ldots \Big) - \left(b_2 + \frac{b_4}{3!} \xi^2 + \frac{b_6}{5!} \xi^4 + \ldots\right) \Big\{ \Big[f_1'^2 \xi^3 - \frac{2 \cdot 4}{3!} f_1' f_3' \xi^5 + \Big(\frac{2 \cdot 6}{5!} f_5' f_1'$$

$$+ \frac{4 \cdot 4}{3! \cdot 3!} f_3'^2\Big) \xi^7 + \ldots \Big] \Big\} = \frac{1}{a_o E^2} \Big(f_1''' \xi - \frac{4}{3!} f_3''' \xi^3 + \frac{6}{5!} f_5''' \xi^5$$

$$- \frac{8}{7!} f_7''' \xi^7 + \ldots \Big)$$

Now equating the coefficients of like powers of ξ we arrive at:

$$f_1'^2 - f_1 f_1'' = \frac{1}{a_o E^2} f_1''' .$$

$$-f_3'' f_1 - 3 f_3 f_1'' + 4 f_1 f_3' + \frac{3 b_2}{4} \left(f_1'^2 + f_1 f_1''\right) = \frac{1}{a_o E^2} f_3'''$$

$$\frac{6 \cdot 6}{5!} f_5' f_1' + \frac{4 \cdot 4}{3! \cdot 2!} f_3'^2 - \frac{6}{5!} f_5'' f_1 - \frac{4 \cdot 4}{3! \cdot 2!} f_3'' f_3 - \frac{6}{4!} f_1'' f_5 + \frac{b_2}{2!} \Big(\frac{4}{3!} f_3'' f_1$$

$$+ \frac{4}{2!} f_3 f_1'' - \frac{4 \cdot 4}{3!} f_1' f_3'\Big) + \frac{b_4}{4!} \left(f_1'^2 - f_1 f_1''\right) + \frac{2 \cdot 4 b_2}{3!} f_1' f_3'$$

$$- \frac{b_4}{3!} f_1'^2 = \frac{1}{a_o E^2} \frac{6}{5!} f_5'''$$

$$\frac{1}{a_o E^2} f_5''' = 6 f_5' f_1' + \frac{80}{3} f_3'^2 - f_5'' f_1 - \frac{80}{3} f_3'' f_3 - 5 f_1'' f_5 + \frac{20}{3} b_2 \big(f_3'' f_1$$

$$+ 3 f_3 f_1''\big) - \frac{5}{6} b_4 \left(f_1 f_1'' + 3 f_1'^2\right)$$

$$-\frac{1}{a_oE^2}\;\frac{8}{7!}\,f_7''' \;=\; -\frac{8\cdot 8}{7!}\,f_7'f_1' - \frac{4\cdot 6\cdot 8}{5!\cdot 3!}\,f_5'f_3' + \frac{8}{7!}\,f_7''f_1$$

$$+\frac{4\cdot 6}{5!\cdot 2!}\,f_5''f_3 + \frac{4\cdot 6}{3!\cdot 4!}\,f_3''f_5 + \frac{8}{6!}\,f_1''f_7 + \frac{b_2}{2!}\left(\frac{6\cdot 6}{5!}\,f_5'f_1' + \frac{4\cdot 4}{3!\cdot 2!}\,f_3'^2\right.$$

$$\left.-\frac{6}{5!}\,f_5''f_1 - \frac{4\cdot 4}{3!\cdot 2!}\,f_3''f_3 - \frac{6}{4!}\,f_1''f_5\right) + \frac{b_4}{4}\left(\frac{4}{3!}\,f_3''f_1 + \frac{4}{2!}\,f_3f_1''\right.$$

$$\left.-\frac{4\cdot 4}{3!}\,f_1'f_3'\right) + \frac{b_6}{6!}\,(f_1'^2 - f_1f_1'') - b_2\left(\frac{2\cdot 6}{5!}\,f_5'f_1' + \frac{4\cdot 4}{3!\cdot 3!}\,f_3'^2\right)$$

$$+\frac{2\cdot 4}{3!\cdot 3!}\,b_4f_1'f_3' - \frac{1}{5!}\,b_6f_1'^2$$

$$\frac{1}{a_oE^2}\,f_7''' \;=\; 8f_7'f_1' + 168f_5'f_3' - f_7''f_1 - 63f_5''f_3 - 105f_3''f_5 - 7f_1''f_7$$

$$-\frac{35}{20}\,b_2(18f_5'f_1' + 80f_3'^2 - 9f_5''f_1 - 240f_3''f_3 - 45f_1''f_5) - \frac{70}{4}\,b_4(f_3''f_1$$

$$+3f_3f_1'' + 4f_1'f_3') + \frac{7}{8}\,b_6(f_1f_1'' + 5f_1'^2)$$

Now if we define the operator L_n by:

$$L_n \;\equiv\; \frac{1}{a_oE^2}\frac{d^3}{d\eta^3} + f_1\frac{d^2}{d\eta^2} - (n-1)f_1'\frac{d}{d\eta} + n\,f_1'' \qquad n = 3,5,7,\ldots$$

Then we get upon collecting the results

$$\frac{1}{a_oE^2}\,f_1''' + f_1f_1'' - f_1'^2 \;=\; 0$$

$$L_3(f_3) \;=\; \frac{3b_2}{4}(f_1'^2 + f_1f_1'')$$

$$L_5(f_5) \;=\; \frac{80}{3}\,(f_3'^2 - f_3f_3'') + \frac{20}{3}b_2(f_3''f_1 + 3f_3f_1'') - \frac{5}{6}\,b_4(f_1f_1'' + 3f_1'^2)$$

$$L_7(f_7) \;=\; 21(8f_5'f_3' - 3f_5''f_3 - 5f_3''f_5) - \frac{7}{4}\,(18f_5'f_1' + 80[f_3'^2 - 3f_3f_3'']$$

$$-3f_5''f_1 - 15f_1''f_5) - \frac{35}{2}\,b_4(f_3''f_1 + 4f_1'f_3' + 3f_3f_1'') + \frac{7}{8}\,b_6(f_1f_1'' + 5f_1'^2)$$

And finally if we define H_n, n=3,5,7,....by:

$$H_3 = 0$$

$$H_5 = \frac{20}{3}[4(f_3'^2-f_3f_3'')+b_2(f_3''f_1+3f_3f_1'')]$$

$$H_7 = \frac{7}{4}[12(8f_5'f_3'-3f_5''f_3-5f_3''f_3)-b_2(18f_5'f_1' + 80[f_3'^2-3f_3f_3'']-9f_5''f_1-45f_1''f_5)-10b_4(f_3''f_1 +4f_1'f_3'+3f_3f_1'')]$$

We obtain:

$$L_n(f_n) = (-1)^{\frac{n+1}{2}} \frac{n}{n+1}[f_1f_1''+(n-2)f_1'^2]b_{n-1}+H_n$$

for $n = 3,5,7,\ldots$

APPENDIX IV

REDUCTION OF THE SECOND OF EQUATIONS (23)

Substitution of the expansions (26), (27), and (28) into the second of Eqs. (23) yields:

$$a_o\left(1 + \frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4 + \frac{b_6}{6!}\xi^6 + \dots\right)\left\{\left[\left(f_1'\xi - \frac{4}{3!} f_3'\xi^3 + \frac{6}{5!} f_5'\xi^5 - \frac{8}{7!}\xi^7 + \dots\right)\left(\frac{2}{2!} F_2\xi + \frac{4}{4!} F_4\xi^3 + \frac{6}{6!} F_6\xi^5 + \dots\right) - \left(F_o' + \frac{1}{2!} F_2'\xi^2 + \frac{1}{4!} F_4'\xi^4 + \frac{1}{6!} F_6'\xi^6 + \dots\right)\left(f_1 - \frac{4\cdot 3}{3!} f_3\xi^2 + \frac{6\cdot 5}{5!} f_5\xi^4 - \frac{8\cdot 7}{7!} f_7\xi^6 + \dots\right)\right]\right\} = \frac{1}{P_rE^2}\left(F_o'' + \frac{1}{2!} F_2''\xi^2 + \frac{1}{4!} F_4''\xi^4 + \frac{1}{6!} F_6''\xi^6 + \dots\right)$$

Collecting terms and rearranging gives:

$$a_o\left(1 + \frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4 + \frac{b_6}{6!}\xi^6 + \dots\right)\left\{\left[-f_1F_o' + \left(F_2f_1' + \frac{4}{2!} f_3F_o' - \frac{1}{2!} F_2'f_1\right)\xi^2 + \left(\frac{1}{3!} f_1'F_4 - \frac{4}{3!} f_3'F_2 - \frac{1}{4!} f_1F_4' + \frac{4}{2!\cdot 2!} f_3F_2' - \frac{6}{4!} f_5F_o'\right)\xi^4 + \left(\frac{1}{5!} f_1'F_6 - \frac{4}{3!\cdot 3!} f_3'F_4 + \frac{6}{5!} f_5'F_2 - \frac{1}{6!} f_1F_6' + \frac{4}{2!\cdot 4!} f_3F_4' - \frac{6}{4!\cdot 2!} f_5F_2' + \frac{8}{6!} f_7F_o'\right)\xi^6 + \dots\right]\right\} = \frac{1}{P_rE^2}\left(F_o'' + \frac{1}{2!} F_2''\xi^2 + \frac{1}{4!} F_4''\xi^4 + \frac{1}{6!} F_6''\xi^6 + \dots\right)$$

Upon equating the coefficients of like powers of ξ one obtains:

$$-f_1F_o' = \frac{1}{a_oP_rE^2} F_o''$$

$$f_1{}'F_2+2f_3F_o{}'-\frac{1}{2}f_1F_2{}'-\frac{b_2}{2}f_1F_o{}' \quad =$$

$$\frac{1}{a_oP_rE^2}\frac{1}{2}F_2{}''$$

$$\frac{1}{a_oP_rE^2}F_2{}''-2f_1{}'F_2+f_1F_2{}' \quad = \quad 4f_3F_o{}'-b_2f_1F_o{}'$$

$$\frac{1}{6}f_1{}'F_4-\frac{2}{3}f_3{}'F_2-\frac{1}{4!}f_1F_4{}'+f_3F_2{}'-\frac{1}{4}f_5F_o{}'$$

$$+\frac{b_2}{2}\left(F_2f_1{}'+2f_3F_o{}'-\frac{1}{2}f_1F_2{}'\right)+\frac{b_4}{4!}f_1F_o{}'$$

$$= \frac{1}{a_oP_rE^2}\frac{1}{4!}F_4{}''$$

$$\frac{1}{a_oP_rE^2}F_4{}''-4f_1{}'F_4+f_1F_4{}' \quad =$$

$$24f_3F_2{}'-16f_3{}'F_2-6f_5F_o{}'+6b_2(2f_1{}'F_2$$

$$-f_1F_2{}'+4f_3F_o{}')-b_4f_1F_o{}'$$

$$\frac{1}{5!}f_1{}'F_6-\frac{1}{3\cdot 3}f_3{}'F_4+\frac{6}{5!}f_5{}'F_2-\frac{1}{6!}f_1F_6{}'$$

$$+\frac{1}{2\cdot 3!}f_3F_4{}'-\frac{3}{4!}f_5F_2{}'+\frac{8}{6!}f_7F_o{}'+$$

$$\frac{b_2}{2}\left(\frac{1}{3!}f_1{}'F_4-\frac{4}{3!}f_3{}'F_2-\frac{1}{4!}f_1F_4{}'+f_3F_2{}'-\right.$$

$$\left.\frac{6}{4!}f_5F_o{}'\right)+\frac{b_4}{4!}\left(F_2f_1{}'+2f_3F_o{}'-\frac{1}{2}f_1F_2{}'\right)$$

$$-\frac{b_6}{6!}f_1F_o{}' \quad = \quad \frac{1}{a_oP_rE^2}\frac{1}{6!}F_6{}''$$

$$\frac{1}{a_oP_rE^2}F_6{}''-6f_1{}'F_6+f_1F_6{}' \quad = \quad 60f_3F_4{}'$$

$$-90f_5F_2{}'+8f_7F_o{}'-80f_3{}'F_4+36f_5{}'F_2$$

$$+15b_2(4f_1{}'F_4-f_1F_4{}'-16f_3{}'F_2+24f_3F_2{}'$$

$-6f_5F_o')+15b_4(2f_1'F_2-f_1F_2'+4f_3F_o')$

$-b_6f_1F_o'$

Now if the operator K_n is defined by:

$$K_n \equiv \frac{1}{a_oP_rE^2}\frac{d^2}{d\eta^2} + f_1\frac{d}{d\eta} - nf_1'; \; n=0,2,4,...$$

Then one obtains upon collecting the results:

$K_o(F_o) = 0$

$K_2(F_2) = (4f_3-b_2f_1)F_o'$

$K_4(F_4) = 8(3f_3F_2'-2f_3'F_2)+6b_2(2f_1'F_2-$

$f_1F_2')-(6f_5-24b_2f_3+b_4f_1)F_o'$

$K_6(F_6) = 20(3f_3F_4'-4f_3'F_4)-18(f_5F_2'-$

$2f_5'F_2)+15b_2(4f_1'F_4-f_1F_4'-16f_3'F_2$

$+24f_3F_2')+15b_4(2f_1'F_2-f_1F_2')+(8f_7$

$-90b_2f_5+60b_4f_3-b_6f_1)F_o'$

....

....

....

And finally if G_n; n = 0,2,4,6,...is defined by:

$G_o = 0$

$G_2 = (4f_3-b_2f_1)F_o'$

$G_4 = 8(3f_3F_2'-2f_3'F_2)+6b_2(2f_1'F_2-f_1F_2')-$

$(6f_5-24b_2f_3+b_4f_1)F_o'$

$G_6 = 20(3f_3F_4'-4f_3'F_4)-18(5f_5F_2'-2f_5'F_2)$

$+15b_2(4f_1'F_4-f_1F_4'-16f_3'F_2+24f_3F_2')+15b_2(2f_1'F_2$

$-f_1F_2')+(8f_7-90b_2f_5+60b_4f_3-b_6f_1)F_0'$

....

....

....

One obtains:

$K_n(F_n) \;=\; G_n$; for n=0,2,4,6,...

APPENDIX V

REDUCTION OF THE FIRST OF EQUATIONS (24)

Substitution of the expansions (26) and (27) into the first of Eqs. (24) and using the relations:

$$\text{SIN.}\,\xi = \xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} - \frac{\xi^7}{7!} + \dots$$

$$\text{COS.}\,\xi = 1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \dots$$

yields (if we denote in this appendix $f_1(1)$ by f_1; $f_1'(1)$ by f_1'; $f_3'(1)$ by f_3' etc.):

$$f_1''\xi - \frac{4}{3!} f_3''\xi^3 + \frac{6}{5!} f_5''\xi^5 - \frac{8}{7!} f_7''\xi^7 + \dots$$

$$= a_0E^2 \left(1 + \frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4 + \frac{b_6}{6!}\xi^6 + \dots\right) \left\{ \left[2a_0 \left(1 \right.\right.\right.$$

$$\left. + \frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4 + \frac{b_6}{6!}\xi^6 + \dots\right) \left(\xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} - \frac{\xi^7}{7!} + \dots\right)$$

$$\left. \left. - \left(f_1'\xi - \frac{4}{3!} f_3'\xi^3 + \frac{6}{5!} f_5'\xi^5 - \frac{8}{7!} f_7'\xi^7\right)\right]\right\} \left(1 - \frac{\xi^2}{2!}\right.$$

$$\left. + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \dots\right)$$

$$= a_0E^2 \left\{ \left[2a_0 \left(\xi + \left[\frac{b_2}{2!} - \frac{1}{3!}\right]\xi^3 + \left[\frac{1}{5!} - \frac{b_2}{2!\cdot 3!} + \frac{b_4}{4!}\right]\xi^5 + \left[-\frac{1}{7!} + \frac{b_2}{2!\cdot 5!}\right.\right.\right.\right.$$

$$\left.\left.\left.\left. - \frac{b_4}{4!\cdot 3!} + \frac{b_6}{6!}\right]\xi^7 + \dots\right) - \left(f_1'\xi - \frac{4}{3!} f_3'\xi^3 + \frac{6}{5!} f_5'\xi^5 - \frac{8}{7!} f_7'\xi^7 + \dots\right)\right]\right\}$$

$$\left\{ \left[1 + \frac{1}{2!}(b_2 - 1)\xi^2 + \left(\frac{1}{4!} + \frac{b_4}{4!} - \frac{b_2}{2!\cdot 2!}\right)\xi^4 + \left(\frac{b_6}{6!} - \frac{1}{6!} - \frac{b_4}{2!\cdot 4!} + \frac{b_2}{2!\cdot 4!}\right)\xi^6 + \dots \right] \right\}$$

$$= a_0E^2 \left\{ \left[(2a_0 - f_1')\xi + \left[2a_0\left(\frac{b_2}{2!} - \frac{1}{3!}\right) + \frac{4}{3!} f_3'\right]\xi^3 + \left[2a_0\left(\frac{1}{5!} - \frac{b_2}{2!\cdot 3!}\right.\right.\right.\right.$$

$$+ \frac{b_4}{4!}\Big) - \frac{6}{5!} f_5'\Big] \xi^5 + \Big[2a_o\Big(-\frac{1}{7!} + \frac{b_2}{2!\cdot 5!} - \frac{b_4}{4!\cdot 3!} + \frac{b_6}{6!}\Big) + \frac{8}{7!} f_7'\Big] \xi^7$$

$$+\ldots\Big\}\Big\{1 + \frac{1}{2!}(b_2-1)\xi^2 + \Big(\frac{1}{4!} + \frac{b_4}{4!} - \frac{b_2}{2!\cdot 2!}\Big)\xi^4 + \Big(\frac{b_6}{6!} - \frac{1}{6!} - \frac{b_4}{2!\cdot 4!}$$

$$+ \frac{b_2}{2!\cdot 4!}\Big)\xi^6 + \ldots\Big\} = a_oE^2\Big\{[2a_o-f_1']\xi + \frac{1}{3!}[2a_o(3b_2-1)+4f_3']\xi^3$$

$$+ \frac{1}{5!}[2a_o(1-10b_2+5b_4)-6f_5']\xi^5 + \frac{1}{7!}[2a_o(-1+21b_2-35b_4+7b_6)+8f_7']\xi^7$$

$$+\ldots\Big\}\Big\{1 + \frac{1}{2!}(b_2-1)\xi^2 + \frac{1}{4!}(b_4+1-6b_2)\xi^4 + \frac{1}{6!}(b_6-1-15b_4+15b_2)\xi^6$$

$$+\ldots\Big\} = a_oE^2\Big\{[2a_o-f_1']\xi + \frac{2}{3!}[3b_2a_o+(2f_3'-a_o)]\xi^3 + \frac{2}{5!}[5a_o(b_4$$

$$-2b_2)-(3f_5'-a_o)]\xi^5 + \frac{2}{7!}[7a_o(3b_2-5b_4+b_6)+(4f_7'-a_o)]\xi^7+\ldots\Big\}\Big\{1$$

$$+ \frac{1}{2!}(b_2-1)\xi^2 + \frac{1}{4!}(b_4+1-6b_2)\xi^4 + \frac{1}{6!}(b_6-1-15b_4+15b_2)\xi^6+\ldots\Big\}$$

$$= f_1''\xi - \frac{4}{3!} f_3''\xi^3 + \frac{6}{5!} f_5''\xi^5 - \frac{8}{7!} f_7''\xi^7 + \ldots$$

Equating coefficients of like powers of ξ one obtains:

$$f_1'' = a_oE^2[2a_o-f_1']$$

$$-\frac{4}{3!} f_3'' = a_oE^2\Big\{\frac{2}{3!}[3b_2a_o+(2f_3'-a_o)] + \frac{1}{2!}(b_2-1)[2a_o-f_1']\Big\}$$

$$f_3'' = -\frac{a_oE^2}{2}\{3b_2a_o+(2f_3'-a_o) + \frac{3}{2}(b_2-1)(2a_o-f_1')\}$$

$$\frac{6}{5!} f_5'' = a_oE^2\Big\{\frac{2}{5!}[5a_o(b_4-2b_2)-(3f_5'-a_o)] + \frac{1}{3!}(b_2-1)[3b_2a_o+(2f_3'-a_o)]$$

$$+ \frac{1}{4!}(2a_o-f_1')(b_4+1-6b_2)\Big\}$$

$$f_5'' = \frac{a_oE^2}{2}\left\{\left[\frac{2}{3}\,[5a_o(b_4-2b_2)-(3f_5'-a_o)]+\frac{20}{3}\,(b_2-1)[3b_2a_o+(2f_3'-a_o)]\right.\right.$$

$$\left.\left.+\frac{5}{3}\,(2a_o-f_1')(b_4+1-6b_2)\right]\right\}$$

$$-\frac{8}{7!}\,f_7'' = a_oE^2\left\{\left[\frac{2}{7!}\,[7a_o(3b_2-5b_4+b_6)+(4f_7'-a_o)]+\frac{1}{5!}\,(b_2-1)[5a_o(b_4\right.\right.$$

$$-2b_2)-(3f_5'-a_o)]+\frac{2}{4!\cdot 3!}\,(b_4+1-6b_2)[3b_2a_o+(2f_3'-a_o)]+\frac{1}{6!}\,(b_6-1$$

$$\left.\left.-15b_4+15b_2)[2a_o-f_1']\right]\right\}$$

$$f_7'' = -\frac{a_oE^2}{2}\left\{\left[\frac{1}{2}\,[7a_o(3b_2-5b_4+b_6)+4f_7'-a_o]+\frac{21}{2}\,(b_2-1)[5a_o(b_4\right.\right.$$

$$-2b_2)+a_o-3f_5']+\frac{35}{2}\,(b_4-6b_2+1)[a_o(3b_2-1)+2f_3']+\frac{7}{4}\,(b_6-1-15b_4$$

$$\left.\left.+15b_2)[2a_o-f_1']\right]\right\}$$

$$\frac{f_7''}{1/2a_oE^2}+2f_7'+\frac{7}{4}\,(4a_o-f_1')b_6-\frac{7}{4}\,(2a_o-f_1')+\frac{105}{4}\,(b_2-b_4)(2a_o-f_1')$$

$$+\frac{1}{2}\,a_o(21b_2-35b_4-1)+\frac{21}{2}\,(b_2-1)[a_o(5b_4-10b_2+1)-3f_5']+\frac{35}{2}\,(b_4-6b_2$$

$$+1)[a_o(3b_2-1)+2f_3']$$

Collecting the results and simplifying yields:

$$\frac{f_1''}{1/2a_oE^2}+2f_1'-4a_o = 0$$

$$\frac{f_3''}{1/2a_oE^2}+2f_3'+\frac{3}{2}\,(4a_o-f_1')b_2-\frac{3}{2}\,(2a_o-f_1')-a_o = 0$$

$$\frac{f_5''}{1/2 a_o E^2} + 2f_5' - \frac{5}{3}(4a_o - f_1')b_4 - \frac{5}{3}(2a_o - f_1') + 10b_2(2a_o - f_1') + \frac{2}{3}a_o(10b_2 - 1)$$

$$- \frac{20}{3}(b_2 - 1)[3b_2 a_o + (2f_3' - a_o)] = 0$$

$$\frac{f_7''}{1/2 a_o E^2} + 2f_7' + \frac{7}{4}(4a_o - f_1')b_6 - \frac{7}{4}(2a_o - f_1') + \frac{105}{4}(b_2 - b_4)(2a_o - f_1')$$

$$+ \frac{1}{2}a_o(21b_2 - 35b_4 - 1)$$

$$+ \frac{21}{2}(b_2 - 1)[a_o(5b_4 - 10b_2 + 1) - 3f_5'] + \frac{35}{2}(b_4 - 6b_2 + 1)[a_o(3b_2 - 1) + 2f_3'] = 0$$

....

....

....

If we define:

$$A_3 = -a_o$$

$$A_5 = \frac{2}{3}\{15b_2[2a_o - f_1'] + a_o(10b_2 - 1) + 10(b_2 - 1)[3b_2 a_o + 2f_3' - a_o]\}$$

$$A_7 = \frac{1}{4}\{105(b_2 - b_4)[2a_o - f_1'] + 2a_o(21b_2 - 35b_4 - 1) + 42(b_2 - 1)[a_o(5b_4$$

$$-10b_2 + 1) - 3f_5'] + 70(b_4 - 6b_2 + 1)[a_o(3b_2 - 1) + 2f_3']\}$$

....

....

....

Then we have:

$$\frac{2}{a_o E^2} f_1'' + 2f_1' - 4a_o = 0$$

$$\frac{2}{a_oE^2} f_n''+2f_n'+ \frac{2n}{n+1} \left\{(-1)^{\frac{n+1}{2}} b_{n-1}[4a_o-f_1']-2a_o+f_1'\right\}$$

$+A_n$ for $n = 3,5,7,\ldots$

APPENDIX VI

REDUCTION OF THE THIRD OF EQUATIONS (24)

Noting that:

$$\cos.\ \xi = 1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots$$

and substitution of the expansions (27) and (28) into the third of Equations (24) yields:

$$a_o\left(F_o + \frac{F_2}{2!}\xi^2 + \frac{F_4}{4!}\xi^4 + \frac{F_6}{6!}\xi^6 + \ldots\right)\left(1 + \frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4 + \frac{b_6}{6!}\xi^6 + \ldots\right)$$

$$\left(1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots\right) = -\frac{1}{P_r E^2}\left(F_o' + \frac{F_2'}{2!}\xi^2 + \frac{F_4'}{4!}\xi^4 + \frac{F_6'}{6!}\xi^6 + \ldots\right)$$

where we have written $F_o(1)$ as F_o; $F_o'(1)$ as F_o' etc.

Combining terms and simplifying we obtain:

$$\left[F_o + \frac{1}{2!}(F_2 + F_o b_2)\xi^2 + \frac{1}{4!}\left(F_4 + \frac{4!}{2!2!}F_2 b_2 + F_o b_4\right)\xi^4 + \frac{1}{6!}\left(F_6 + \frac{6!}{4!\cdot 2!}F_4 b_2 + \frac{6!}{4!2!}F_2 b_4 + F_o b_6\right)\xi^6 + \ldots\right]\left(1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots\right)$$

$$= -\frac{1}{a_o E^2 P_r}\left(F_o' + \frac{F_2'}{2!}\xi^2 + \frac{F_4'}{4!}\xi^4 + \frac{F_6'}{6!}\xi^6 + \ldots\right)$$

Equating coefficients of like powers of ξ yields:

$$-\frac{1}{a_o E^2 P_r} F_o' = F_o$$

$$-\frac{1}{a_o E^2 P_r} F_2' \frac{1}{2!} = \frac{1}{2!}(F_2 + F_o b_2) - \frac{1}{2!} F_o$$

$$-\frac{1}{a_oE^2P_r} F_2' = F_2+F_ob_2-F_o$$

$$-\frac{1}{a_oE^2P_r} F_4' \frac{1}{4!} = \frac{1}{4!}(F_4+6F_2b_2+F_ob_4)-\frac{1}{2!2!}(F_2+F_ob_2)+\frac{1}{4!}F_o$$

$$-\frac{1}{a_oE^2P_r} F_4' = F_4+F_ob_4+6F_2b_2-6(F_2+F_ob_2)+F_o$$

$$-\frac{1}{a_oE^2P_r} F_6' \frac{1}{6!} = \frac{1}{6!}(F_6+15b_2F_4+15F_2b_4+F_ob_6)$$

$$-\frac{1}{2!4!}(F_4+6F_2b_2+F_ob_4)+\frac{1}{2!4!}(F_2+F_ob_2)-\frac{1}{6!}F_o$$

$$-\frac{1}{a_oE^2P_r} F_6' = F_6+F_ob_6-F_o+15(b_2F_4+F_2b_4-F_4-6F_2b_2-F_ob_4+F_2+F_ob_2)$$

Now if we define:

$$B_2 = 0$$

$$B_4 = 6[b_2F_2-F_2-b_2F_o]$$

$$B_6 = 15[b_2F_4+b_4F_2-F_4-6b_2F_2-b_4F_o+F_2+b_2F_o]$$

....

....

We get upon collecting results:

$$\frac{1}{a_oE^2P_r} F_o'+F_o = 0$$

$$\frac{1}{a_oE^2P_r} F_n'+F_n+F_o\left[b_n+(-1)^{\frac{n}{2}}\right]+B_n \qquad n = 2,4,6,\ldots$$

APPENDIX VII

REDUCTION OF $u/U_\infty = (2\delta^*)^{-1} \dfrac{\partial\psi^*}{\partial\eta}$

Substitution of the expansions (26) and (27) into this equation yields:

$$\frac{u}{U_\infty} = \frac{1}{2}\left[\xi f_1' - \frac{4}{3!} f_3'\xi^3 + \frac{6}{5!} f_5'\xi^5 - \frac{8}{7!} f_7'\xi^7 + \ldots\right] x$$

$$\frac{1}{a_o}\left[1 + \frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4 + \frac{b_6}{6!}\xi^6 + \ldots\right]^{-1}$$

$$= \frac{1}{2a_o}\left[f_1'\xi - \frac{4}{3!} f_3'\xi^3 + \frac{6}{5!} f_5'\xi^5 - \frac{8}{7!} f_7'\xi^7 + \ldots\right] x$$

$$x\left[1 - \frac{b_2}{2!}\xi^2 - \frac{b_4}{4!}\xi^4 - \frac{b_6}{6!}\xi^6 + \left\{\frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4\right\}^2 - \left\{\frac{b_2}{2!}\xi^2\right\}^3 + \ldots\right]$$

$$= \frac{1}{2a_o}\left[f_1'\xi - \frac{4}{3!} f_3'\xi^3 + \frac{6}{5!} f_5'\xi^5 - \frac{8}{7!} f_7'\xi^7 + \ldots\right] x$$

$$x\left[1 - \frac{b_2}{2!}\xi^2 + \left\{\frac{b_2^2}{(2!)^2} - \frac{b_4}{4!}\right\}\xi^4 - \left\{\frac{b_2^3}{2^3} \quad \frac{2b_2b_4}{2!\cdot 4!} + \frac{b_6}{6!}\right\}\xi^6 + \ldots\right]$$

$$= \frac{1}{2a_o}\left[f_1'\xi - \frac{4}{3!} f_3'\xi^3 + \frac{6}{5!} f_5'\xi^5 - \frac{8}{7!} f_7'\xi^7 + \ldots\right]\left[1 - \frac{b_2}{2!}\xi^2 + \frac{1}{4!}(6b_2^2 - b_4)\xi^4 - \frac{1}{6!}(90b_2^3 - 15b_2b_4 + b_6)\xi^6 + \ldots\right]$$

If we now set:

$$\frac{u}{U_\infty} = u_1\xi - \frac{u_3}{3!}\xi^3 + \frac{u_5}{5!}\xi^5 - \frac{u_7}{7!}\xi^7 + \ldots$$

We get upon equating coefficients of like powers of ξ:

$$u_1 = \frac{1}{2a_o} f_1'$$

$$\frac{1}{3!}\ u_3 = \frac{1}{2a_o}\left[\frac{b_2}{2!}f_1' + \frac{4}{3!}f_3'\right] = \frac{1}{2a_o}\cdot\frac{1}{3!}(3b_2f_1'+4f_3')$$

$$u_3 = \frac{1}{2a_o}(4f_3'+3b_2f_1')$$

$$\frac{u_5}{5!} = \frac{1}{2a_o}\left[\frac{f_1'}{4!}(6b_2^2-b_4) + \frac{4}{3!\cdot 2!}b_2f_3' + \frac{6}{5!}f_5'\right]$$

$$u_5 = \frac{1}{2a_o}\ [6f_5'+40b_2f_3'+5(6b_2^2-b_4)f_1']$$

$$\frac{u_7}{7!} = \frac{1}{2a_o}\left[\frac{f_1'}{6!}(90b_2^3-15b_2b_4+b_6) + \frac{4}{3!\cdot 4!}f_3'(6b_2^2-b_4) + \frac{6b_2}{2!\cdot 5!}f_5'b_2 + \frac{8}{7!}f_7'\right]$$

$$u_7 = \frac{1}{2a_o}\ [8f_7'+126b_2f_5'+140(6b_2^2-b_4)f_3'+7(90b_2^3-15b_2b_4+b_6)f_1']$$

After collecting the results we get:

$$\frac{u}{U_\infty} = u_1\xi - \frac{u_3}{3!}\xi^3 + \frac{u_5}{5!}\xi^5 - \frac{u_7}{7!}\xi^7 + \ldots$$

$$u_1 = \frac{1}{2a_o}\ f_1'$$

$$u_3 = \frac{1}{2a_o}(4f_3'+3b_2f_1')$$

$$u_5 = \frac{1}{2a_o}\ \{6f_5'+40b_2f_3'+5(6b_2^2-b_4)f_1'\}$$

$$u_7 = \frac{1}{2a_o}\ \{8f_7'+126b_2f_5'+140(6b_2^2-b_4)f_3'+7(90b_2^3-15b_2b_4+b_6)f_1'\}$$

....

....

....

APPENDIX VIII

REDUCTION OF $\tau_w \sqrt{\dfrac{2U_\infty R_o}{\nu}} \Big/ 1/2\rho U_\infty^2 = \dfrac{1}{E^2\delta^{*2}} \dfrac{\partial^2\psi}{\partial\eta^2}$

Substituting the expansions (26) and (27) into this relation yields:

$$\frac{\tau_w\sqrt{\dfrac{2U_\infty R_o}{\nu}}}{1/2\rho U_\infty^2} = \frac{1}{a_o^2 E}\left[f_1''\xi - \frac{4}{3!}f_3''\xi^3 + \frac{6}{5!}f_5''\xi^5 - \frac{8}{7!}f_7''\xi^7 + \dots\right] x$$

$$x\left[1 + \frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4 + \frac{b_6}{6!}\xi^6 + \dots\right]^{-2}$$

$$= \frac{1}{a_o^2 E}\left[f_1''\xi - \frac{4}{3!}f_3''\xi^3 + \frac{6}{5!}f_5''\xi^5 - \frac{8}{7!}f_7''\xi^7 + \dots\right] x$$

$$x\left[1 - \frac{2b_2}{2!}\xi^2 - \frac{2b_4}{4!}\xi^4 - \frac{2b_6}{6!}\xi^6 + 3\left\{\frac{b_2}{2!}\xi^2 + \frac{b_4}{4!}\xi^4\right\}^2 - 4\left\{\frac{b_2}{2!}\xi^2\right\}^3 + \dots\right]$$

$$= \frac{1}{a_o^2 E}\left[f_1''\xi - \frac{4}{3!}f_3''\xi^3 + \frac{6}{5!}f_5''\xi^5 - \frac{8}{7!}f_7''\xi^7 + \dots\right]\left[1 - \frac{2b_2}{2!}\xi^2 + \left\{\frac{3b_2^2}{(2!)^2} - \frac{2b_4}{4!}\right\}\xi^4 - \left\{\frac{4b_2^3}{2^3} - \frac{6b_2b_4}{2!\,4!} + \frac{2b_6}{6!}\right\}\xi^6 + \dots\right]$$

$$= \frac{1}{a_o^2 E}\left[f_1''\xi - \frac{4}{3!}f_3''\xi^3 + \frac{6}{5!}f_5''\xi^5 - \frac{8}{7!}f_7''\xi^7 + \dots\right]\left[1 - \frac{2b_2}{2!}\xi^2 + \frac{2}{4!}\{9b_2^2 - b_4\}\xi^4 - \frac{2}{6!}\{180b_2^3 - 45b_2b_4 + b_6\}\xi^6 + \dots\right]$$

Now setting:

$$\tau_w\sqrt{\frac{2U_\infty R_o}{\nu}}\Big/ 1/2\rho U_\infty^2 = \tau_1\xi - \frac{1}{3!}\tau_3\xi^3 + \frac{1}{5!}\tau_5\xi^5 - \frac{1}{7!}\tau_7\xi^7 + \dots$$

Upon collecting coefficients of like powers of ξ one obtains:

$$\tau_1 = \frac{1}{a_o^2 E} f_1''$$

$$\frac{\tau_3}{3!} = \left[\frac{4}{3!} f_3'' + \frac{2b_2 f_1''}{2!}\right] \frac{1}{a_o{}^2E}$$

$$\tau_3 = (4f_3''+6b_2f_1'')\frac{1}{a_o{}^2E} = \frac{2}{a_o{}^2E}(2f_3''+3b_2f_1'')$$

$$\frac{\tau_5}{5!} = \frac{1}{a_o{}^2E}\left[\frac{6}{5!} f_5''+ \frac{2\cdot4}{2!\cdot3!} b_2f_3''+ \frac{2}{4!}\{9b_2^2-b_4\}f_1''\right]$$

$$\tau_5 = \frac{1}{a_o{}^2E}[6f_5''+80b_2f_3''+10(9b_2^2-b_4)f_1'']$$

$$= \frac{2}{a_o{}^2E}[3f_5''+40b_2f_3''+5(9b_2^2-b_4)f_1'']$$

$$\frac{\tau_7}{7!} = \frac{1}{a_o{}^2E}\left[\frac{8}{7!} f_7''+ \frac{2\cdot6}{2!\cdot5!} b_2f_5''+ \frac{4\cdot2}{3!\cdot4!}(9b_2^2-b_4)f_3''+ \frac{2}{6!}(180b_2^3 -45b_2b_4+b_6)f_1''\right]$$

$$\tau_7 = \frac{1}{a_o{}^2E}[8f_7''+252b_2f_5''+280(9b_2^2-b_4)f_3''+14(180b_2^3-45b_2b_4+b_6)f_1'']$$

$$= \frac{2}{a_o{}^2E}[4f_7''+126b_2f_5''+140(9b_2^2-b_4)f_3''+7(180b_2^3-45b_2b_4+b_6)f_1'']$$

And collecting the results one has:

$$\tau_1 = \frac{1}{a_o{}^2E} f_1''$$

$$\tau_3 = \frac{2}{a_o{}^2E}[2f_3''+3b_2f_1'']$$

$$\tau_5 = \frac{2}{a_o{}^2E}[3f_5''+40b_2f_3''+5(9b_2^2-b_4)f_1'']$$

$$\tau_7 = \frac{2}{a_o{}^2E}[4f_7''+126b_2f_5''+140(9b_2^2-b_4)f_3''+7(180b_2^3-45b_2b_4+b_6)f_1'']$$

....

....

....

APPENDIX IX

REDUCTION OF $N_u \Big/ \sqrt{\frac{2U_\infty R_o}{\nu}} = \frac{1}{E\delta^*} \frac{\partial\theta^*}{\partial\eta}$

Substituting the expansions (27) and (28) into this relation yields:

$$\frac{Nu}{\sqrt{\frac{2U_\infty R_o}{\nu}}} = \frac{1}{a_oE}\left[F_o' + \frac{F_2'}{2!}\xi^2 + \frac{F_4'}{4!}\xi^4 + \frac{F_6'}{6!}\xi^6 + \ldots\right]\left[1 - \frac{b_2}{2!}\xi^2\right.$$

$$\left. + \frac{1}{4!}(6b_2^2 - b_4)\xi^4 - \frac{1}{6!}(90b_2^3 - 15b_2b_4 + b_6)\xi^6 + \ldots\right]$$

Now setting:

$$\frac{Nu}{\sqrt{\frac{2U_\infty R_o}{\nu}}} = q_o + \frac{q_2}{2!}\xi^2 + \frac{q_4}{4!}\xi^4 + \frac{q_6}{6!}\xi^6 + \ldots$$

We get upon equating coefficients of like powers of ξ:

$$q_o = \frac{F_o'}{a_oE}$$

$$\frac{q_2}{2!} = \left[\frac{F_2'}{2!} - \frac{b_2F_o'}{2!}\right]\frac{1}{a_oE}$$

$$q_2 = [f_2' - b_2F_o']\frac{1}{a_oE}$$

$$\frac{q_4}{4!} = \left[\frac{F_4'}{4!} - \frac{b_2F_2'}{2!\ 2!} + \frac{1}{4!}(6b_2^2 - b_4)F_o'\right]\frac{1}{a_oE}$$

$$q_4 = [F_4' - 6b_2F_2' + (6b_2^2 - b_4)F_o']\frac{1}{a_oE}$$

$$\frac{q_6}{6!} = \frac{1}{Ea_o}\left[\frac{F_6'}{6!} - \frac{b_2F_4'}{2!\cdot 4!} + \frac{F_2'(6b_2^2 - b_4)}{2!\cdot 4!} - \frac{1}{6!}(90b_2^3 - 15b_2b_4 + b_6)F_o'\right]$$

$$q_6 = \frac{1}{Ea_o} \left[f_6' - 15b_2F_4' + 15(6b_2^2 - b_4)F_2' - (90b_2^3 - 15b_2b_4 + b_6)F_o'\right]$$

And upon collecting the results we have:

$$q_o = \frac{1}{a_oE} F_o'$$

$$q_2 = \frac{1}{a_oE} \{F_2' - b_2F_o'\}$$

$$q_4 = \frac{1}{a_oE} \{F_4' - 6b_2F_2' + (6b_2^2 - b_4)F_o'\}$$

$$q_6 = \frac{1}{a_oE} \{F_6' - 15b_2F_4' + 15(6b_2^2 - b_4)F_2' - (90b_2^3 - 15b_2b_4 + b_6)F_o'\}$$

....

....

....

APPENDIX X

REDUCTION OF THE ENERGY EQUATION FOR SMALL E TO A SERIES OF ORDINARY DIFFERENTIAL EQUATIONS

Substitution of the expansion (52) and (53) into Eq. (49) gives:

$$\sum_{n=0}^{\infty} \frac{F_{2n}''}{P_r E(2n)!} \xi^{2n} = 2\eta \sum_{n=1}^{\infty} \frac{F_{2n}}{(2n-1)!} \xi^{2n-1} \sum_{m=1}^{\infty} \frac{\beta_{2m}}{(2m-1)!} \xi^{2m-1}$$

$$- \eta^2 \sum_{n=0}^{\infty} \frac{F_{2n}'}{(2n)!} \xi^{2n} \sum_{m=0}^{\infty} \frac{\alpha_{2m}}{(2m)!} \xi^{2m}$$

Next rearranging the sums on the right, we get:

$$\sum_{k=0}^{\infty} \frac{F_{2k}''}{P_r E(2k)!} \xi^{2k} = 2\eta \sum_{k=2}^{\infty} \sum_{n=1}^{k-1} \frac{F_{2n}\beta_{2(k-n)}\xi^{2k-2}}{(2n-1)!(2k-2n-1)!}$$

$$- \eta^2 \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{F_{2n}'\alpha_{2(k-n)}}{(2n)!(2k-2n)!} \xi^{2k}$$

$$= 2\eta \sum_{k=1}^{\infty} \left[\sum_{n=1}^{k} \frac{F_{2n}\beta_{2(k-n+1)}}{(2n-1)!(2k-2n+1)!} \right] \xi^{2k}$$

$$- \eta^2 \sum_{k=0}^{\infty} \left[\sum_{n=0}^{k} \frac{F_{2n}'\alpha_{2(k-n)}}{(2n)!(2k-2n)!} \right] \xi^{2k}$$

Next introducing by binomial coefficients $\binom{m}{n}$ and using the relation:

$$\binom{m}{n} = \frac{m!}{(m-n)!n!}$$

we have:

$$\frac{1}{(2n-1)!(2k-2n+1)!} = \frac{1}{(2k)!}\binom{2k}{2n-1}$$

and:

$$\frac{1}{(2n)!(2k-2n)!} = \frac{1}{(2k)!}\binom{2k}{2n}$$

Using this in the above equation, we get:

$$\sum_{k=0}^{\infty} \frac{F''_{2k}}{P_r E(2k)!} \xi^{2k} =$$

$$2\eta \sum_{k=1}^{\infty} \frac{1}{(2k)!} \left[\sum_{n=1}^{k} F_{2n}\beta_{2(k-n+1)} \binom{2k}{2n-1} \right] \xi^{2k}$$

$$- \eta^2 \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left[\sum_{n=0}^{k} F'_{2n}\alpha_{2(k-n)} \binom{2k}{2n} \right] \xi^{2k}$$

Equating the coefficients of like powers of ξ gives:

$$\frac{F''_{2k}}{P_r E} = 2\eta \sum_{n=1}^{k} \binom{2k}{2n-1} F_{2n}\beta_{2(k-n+1)} (1-\delta_{ok})$$

$$- \eta^2 \sum_{n=0}^{k} \binom{2k}{2n} F'_{2n}\alpha_{2(k-n)}$$

or:

$$\frac{1}{P_r E} F''_{2k} + \eta^2 \alpha_o F'_{2k} - 4k\beta_{2\eta} F_{2k} =$$

$$2\eta(1-\delta_{ok}-\delta_{1k}) \sum_{n=1}^{k-1} \binom{2k}{2n-1} F_{2n}\beta_{2(k-n+1)}$$

$$- \eta^2 (1-\delta_{ok}) \sum_{n=0}^{k-1} \binom{2k}{2n} F_{2n}' \alpha_{2(k-n)}$$

$$= (1-\delta_{ok}-\delta_{1k}) \sum_{n=1}^{k-1} \left[2\eta \binom{2k}{2n-1} F_{2n}\beta_{2(k-n+1)} - \right.$$

$$\left. - \eta^2 \binom{2k}{2n} F_{2n}' \alpha_{2(k-n)} \right]$$

$$-(1-\delta_{ok})\eta^2 \alpha_{2k} F_o'$$

Next using (54) and defining:

$$K_k \equiv \frac{1}{P_r E \alpha_o} \frac{d^2}{d\eta^2} + \eta^2 \frac{d}{d\eta} - 4k\eta$$

$$G_k = \begin{cases} = 0 \;\cdot\; k = 0 \\ = -\eta^2 \alpha_2 F_o'; \; k = 1 \\ = \eta \sum_{n=1}^{k-1} \left[2\binom{2k}{2n-1} F_{2n}\beta_{2(k-n+1)} - \binom{2k}{2n}\eta F_{2n}' \alpha_{2(k-n)} \right] \\ \quad -\alpha_{2k}\eta^2 F_o'; \; k > 1 \end{cases}$$

We get:

$$K_k(F_{2k}) = \frac{1}{\alpha_o} G_k; \; k = 0,1,2,3....$$

APPENDIX XI

REDUCTION OF THE FIRST OF EQUATIONS (84)

Substitution of the first two expansions (87) into the first of Eqs. (84) yields:

$$\xi(\delta_o^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*+\dots)^2(\psi^*_{o\tau\eta}+\epsilon\psi^*_{1\tau\eta}+\epsilon^2\psi^*_{2\tau\eta}+\dots)+ 1/2\ (\delta_o^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*+$$

$$\dots)[(\psi^*_{o\xi\eta}+\epsilon\psi^*_{1\xi\eta}+\epsilon^2\psi^*_{2\xi\eta}+\dots)(\psi^*_{o\eta}+\epsilon\psi^*_{1\eta}+\epsilon^2\psi^*_{2\eta}+\dots)-(\psi^*_{o\eta\eta}+$$

$$\epsilon\psi^*_{1\eta\eta}+\epsilon^2\psi^*_{2\eta\eta}+\dots)(\psi^*_{o\xi}+\epsilon\psi^*_{1\xi}+\epsilon^2\psi^*_{2\xi}+\dots)]- 1/2\ [(\delta^*_{o\xi}+\epsilon\delta^*_{1\xi}+\epsilon^2\delta^*_{2\xi}+$$

$$\dots)(\psi^*_{o\eta}+\epsilon\psi^*_{1\eta}+\epsilon^2\psi^*_{2\eta}+\dots)+ 2\xi\ (\delta^*_o+\epsilon\delta^*_1+\epsilon^2\delta^*_2+\dots)(\delta^*_{o\tau}+\epsilon\delta^*_{1\tau}+$$

$$\epsilon^2\delta^*_{2\tau}+\dots)](\psi^*_{o\eta}+\epsilon\psi^*_{1\eta}+\epsilon^2\psi^*_{2\eta}+\dots)- \xi\eta\ (\delta_o^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*+\dots)(\delta^*_{o\tau}$$

$$+\epsilon\delta^*_{1\tau}+\epsilon^2\delta^*_{2\tau}+\dots)(\psi^*_{o\eta\eta}+\epsilon\psi^*_{1\eta\eta}+\epsilon^2\psi^*_{2\eta\eta}+\dots) = \epsilon/2X_e\ \xi(\delta^*_o+$$

$$\epsilon\delta_1^*+\epsilon^2\delta_2^*+\dots)^3 \cos\tau + \xi/X_e\ (\psi^*_{o\eta\eta\eta}+\epsilon\psi^*_{1\eta\eta\eta}+\epsilon^2\psi^*_{2\eta\eta\eta}+\dots)$$

By collecting terms we get:

$$\xi[\delta^{*2}_o+2\epsilon\delta^*_o\delta^*_1+\epsilon^2(\delta^{*2}_1+2\delta^*_2\delta^*_o)+\dots](\psi^*_{o\tau\eta}+\epsilon\psi^*_{1\tau\eta}+\epsilon^2\psi_{2\tau\eta}+\dots)+$$

$$1/2\ (\delta^*_o+\epsilon\delta^*_1+\epsilon^2\delta^*_2+\dots)[(\psi^*_{o\xi\eta}\psi^*_{o\eta}-\psi^*_{o\eta\eta}\psi^*_{o\xi})+\epsilon(\psi^*_{1\xi\eta}\psi^*_{o\eta}+$$

$$\psi^*_{o\xi\eta}\psi^*_{1\eta}-\psi^*_{o\eta\eta}\psi^*_{1\xi}-\psi^*_{o\xi}\psi^*_{1\eta\eta})+\epsilon^2(\psi^*_{2\xi\eta}\psi^*_{o\eta}+\psi^*_{1\xi\eta}\psi^*_{1\eta}+\psi^*_{o\xi\eta}\psi^*_{2\eta}-\psi^*_{2\eta\eta}\psi^*_{o\xi}$$

$$-\psi^*_{1\eta\eta}\psi^*_{1\xi}-\psi^*_{o\eta\eta}\psi^*_{2\xi})+\dots]- \frac{1}{2}\ (\delta^*_{o\xi}+\epsilon\delta^*_{1\xi}+\epsilon^2\delta^*_{2\xi}+\dots)[\psi^{*\,2}_{o\eta}+2\epsilon\psi^*_{1\eta}\psi^*_{o\eta}+$$

$$+\epsilon^2(\psi^{*\,2}_{1\eta}+2\psi^*_{o\eta}\psi^*_{2\eta})+\dots]-\xi[\delta^*_o\delta^*_{o\tau}+\epsilon(\delta^*_o\delta^*_{1\tau}+\delta^*_{o\tau}\delta^*_1)+\epsilon^2(\delta^*_{2\tau}\delta^*_o+\delta^*_{1\tau}\delta^*_1$$

$$+\delta^*_{0\tau}\delta^*_2)+\ldots](\psi^*_{0\eta}+\epsilon\psi^*_{1\eta}+\epsilon^2\psi^*_{2\eta}+\ldots)-\xi\eta[\delta^*_0\delta^*_{0\tau}+\epsilon(\delta^*_1\delta^*_{0\tau}+\delta^*_{1\tau}\delta^*_0)+$$

$$+\epsilon^2(\delta^*_2\delta^*_{0\tau}+\delta^*_1\delta^*_{1\tau}+\delta^*_0\delta^*_{2\tau})+\ldots][\psi^*_{0\eta\eta}+\epsilon\psi^*_{1\eta\eta}+\epsilon^2\psi^*_{2\eta\eta}+\ldots] = \frac{\epsilon}{2X_e}\,\xi\,[\delta^{*2}_0$$

$$+3\epsilon\delta^{*2}_0\delta^*_1+\epsilon^2(3\delta^{*2}_1\delta^*_0+3\delta^*_2\delta^{*2}_0)+\ldots]\cos\tau+\frac{\xi}{X_e}(\psi^*_{0\eta\eta\eta}+\epsilon\psi^*_{1\eta\eta\eta}+$$

$$+\epsilon^2\psi_{2\eta\eta\eta}+\ldots)$$

By equating the coefficients of like powers of ϵ one arrives at:

$$\frac{1}{X_e}\,\xi\,\psi^*_{0\eta\eta\eta} = \xi\delta^{*2}_0\psi^*_{0\tau\eta}+\frac{1}{2}\delta^*_0(\psi^*_{0\xi\eta}\psi^*_{0\eta}-\psi^*_{0\eta\eta}\psi^*_{0\xi})-\frac{1}{2}\delta^*_{0\xi}(\psi^*_{0\eta})^2-\xi\delta^*_0\delta^*_{0\tau}\psi^*_{0\eta}$$

$$-\xi\eta\delta^*_0\delta^*_{0\tau}\psi^*_{0\eta\eta}$$

$$\frac{1}{X_e}\,\xi\,\psi^*_{1\eta\eta\eta}+\frac{\xi}{2X_e}\delta^{*3}_0\cos\tau = \xi\delta^{*2}_0\psi^*_{1\tau\eta}+2\delta^*_0\delta^*_1\psi^*_{0\tau\eta}+\frac{1}{2}\delta^*_0(\psi^*_{1\xi\eta}\psi^*_{0\eta}+\psi^*_{0\xi\eta}\psi^*_{1\eta}-$$

$$\psi^*_{0\eta\eta}\psi^*_{1\xi}-\psi^*_{0\xi}\psi^*_{1\eta\eta})+\frac{1}{2}\delta^*_1(\psi^*_{0\xi\eta}\psi^*_{0\eta}-\psi^*_{0\eta\eta}\psi^*_{0\xi})-\frac{1}{2}\delta^*_{1\xi}(\psi^*_{0\eta})^2-\xi(\delta^*_{1\tau}\delta^*_0+\delta^*_{0\tau}\delta^*_1)\psi^*_{0\eta}$$

$$-\delta^*_{0\xi}\psi^*_{1\eta}\psi^*_{0\eta}-\xi\delta^*_0\delta^*_{0\tau}\psi^*_{1\eta}-\xi\eta(\delta^*_1\delta^*_{0\tau}+\delta^*_{1\tau}\delta^*_0)\psi^*_{0\eta\eta}-\xi\eta\delta^*_0\delta^*_{0\tau}\psi^*_{1\eta\eta}$$

$$\frac{1}{X_e}\,\xi\,\psi^*_{2\eta\eta\eta}+\frac{3\xi}{2X_e}\delta^{*2}_0\delta^*_1\cos\tau = \xi\delta^{*2}_0\psi^*_{2\tau\eta}+2\xi\delta^*_0\delta^*_1\psi^*_{1\tau\eta}+\xi(\delta^{*2}_1+2\delta^*_2\delta^*_0)\psi^*_{0\tau\eta}+$$

$$\frac{1}{2}\delta^*_0(\psi^*_{2\xi\eta}\psi^*_{0\eta}+\psi^*_{1\xi\eta}\psi^*_{1\eta}+\psi^*_{0\xi\eta}\psi^*_{2\eta}-\psi^*_{2\eta\eta}\psi^*_{0\xi}-\psi^*_{1\eta\eta}\psi^*_{1\xi}-\psi^*_{0\eta\eta}\psi^*_{2\xi})+\frac{1}{2}\delta^*_1(\psi^*_{1\xi\eta}\psi^*_{0\eta}$$

$$+\psi^*_{0\xi\eta}\psi^*_{1\eta}-\psi^*_{0\eta\eta}\psi^*_{1\xi}-\psi^*_{0\xi}\psi^*_{1\eta})+\frac{1}{2}\delta^*_2(\psi^*_{0\xi\eta}\psi^*_{0\eta}-\psi^*_{0\eta\eta}\psi^*_{0\xi})-\frac{1}{2}\delta^*_{2\xi}(\psi^*_{0\eta})^2-$$

$$-\xi(\delta^*_{2\tau}\delta^*_0+\delta^*_{1\tau}\delta^*_1+\delta^*_{0\tau}\delta^*_2)\psi^*_{0\eta}-\delta^*_{1\xi}\psi^*_{1\eta}\psi^*_{0\eta}-\xi(\delta^*_0\delta^*_{1\tau}+\delta^*_{0\tau}\delta^*_1)\psi^*_{1\eta}$$

$$-\frac{1}{2}\delta^*_{0\xi}(\psi^{*2}_{1\eta}+2\psi^*_{0\eta}\psi^*_{2\eta})-\xi\delta^*_0\delta^*_{0\tau}\psi^*_{2\eta}-\xi\eta(\delta^*_2\delta^*_{0\tau}+\delta^*_1\delta^*_{1\tau}+\delta^*_0\delta^*_{2\tau})\psi^*_{0\eta\eta}-\xi\eta(\delta^*_1\delta^*_{0\tau}$$

$$+\delta^*_{1\tau}\delta^*_0)\psi^*_{1\eta\eta}-\xi\eta\delta^*_0\delta^*_{0\tau}\psi^*_{2\eta\eta}$$

....

....

....

APPENDIX XII

REDUCTION OF THE SECOND OF EQUATIONS (85)

Substitution of the first two of the expansions (87) into the second of (85) yields:

$$(\delta_0^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*+\ldots)\Big[(X_e\xi\delta_{0\tau}^* + \frac{1}{2}\delta_{0\xi}^*)+\epsilon(X_e\xi\delta_{1\tau}^*+\frac{1}{2}\delta_{1\xi}^*)+\epsilon^2(X_e\xi\delta_{2\tau}^*+$$

$$\frac{1}{2}\delta_{2\xi}^*)+\ldots\Big]\Big[(\delta_0^*-X_e\psi_{0\eta}^*)+\epsilon(\delta_1^*-X_e\psi_{1\eta}^*)+\epsilon^2(\delta_2^*-X_e\psi_{2\eta}^*)+\ldots\Big] +$$

$$\frac{\epsilon}{2}(\delta_0^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*)\Big[(\delta_0^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*+\ldots)\Big\{(X_e\xi\delta_{0\tau}^*+\delta_{0\xi}^*)+\epsilon(X_e\xi\delta_{1\tau}^*$$

$$+\delta_{1\xi}^*)+\epsilon^2(X_e\xi\delta_{2\tau}^*+\delta_{2\xi}^*)+\ldots\Big\} - \frac{X_e}{2}(\delta_{0\xi}^*+\epsilon\delta_{1\xi}^*+\epsilon^2\delta_{2\xi}^*+\ldots)(\psi_{0\eta}^*$$

$$+\epsilon\psi_{1\eta}^*+\epsilon^2\psi_{2\eta}^*+\ldots)\Big]\sin\tau+\frac{\epsilon^2}{16}(\delta_0^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*+\ldots)^2(\delta_{0\xi}^*+$$

$$\epsilon\delta_{1\xi}^*+\epsilon^2\delta_{2\xi}^*+\ldots)(1-\cos 2\tau) = \xi\psi_{0\eta\eta}^*+\epsilon\xi\psi_{1\eta\eta}^*+\epsilon^2\xi\psi_{2\eta\eta}^*+\ldots$$

Upon collecting terms one obtains:

$$(\delta_0^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*+\ldots)\Big[\Big\{(X_e\xi\delta_{0\tau}^*+\frac{1}{2}\delta_{0\xi}^*)(\delta_0^*-X_e\psi_{0\eta}^*)\Big\} + \epsilon\Big\{(X_e\xi\delta_{1\tau}^*$$

$$+\frac{1}{2}\delta_{1\xi}^*)(X_e\delta_0^*-\psi_{0\eta}^*)+(X_e\xi\delta_{0\tau}^*+\frac{1}{2}\delta_{0\xi}^*)(\delta_1^*-X_e\psi_{1\eta}^*)\Big\} + \epsilon^2\Big\{(X_e\xi\delta_{2\tau}^*$$

$$+\frac{1}{2}\delta_{2\xi}^*)(\delta_0^*-X_e\psi_{0\eta}^*)+(X_e\xi\delta_{1\tau}^*+\frac{1}{2}\delta_{1\xi}^*)(\delta_1^*-X_e\psi_{1\eta}^*)+(X_e\xi\delta_{0\tau}^*+$$

$$\frac{1}{2}\delta_{0\xi}^*)(\delta_2^*-X_e\psi_{2\eta}^*)\Big\}+\ldots\Big] + \frac{\epsilon}{2}(\delta_0^*+\epsilon\delta_1^*+\ldots)\Big[\Big\{\delta_0^*(X_e\xi\delta_{0\tau}^*$$

$$+\delta_{0\xi}^*) - \frac{X_e}{2}\delta_{0\xi}^*\psi_{0\eta}^*\Big\} + \epsilon\Big\{\delta_1^*(X_e\xi\delta_{0\tau}^*+\delta_{0\xi}^*)+\delta_0^*(X_e\xi\delta_{1\tau}^*+\delta_{1\xi}^*)-\frac{X_e}{2}\delta_{1\xi}^*\psi_{0\eta}^*$$

$$- \frac{X_e}{2} \delta^*_{o\xi}\psi^*_{1\eta}\Big\}+\ldots\Big] \sin\tau + \frac{\epsilon^2}{16} (\delta_o^{*2}\delta^*_{o\xi}+\ldots)(1-\cos 2\tau) = \xi\psi^*_{o\eta\eta}$$

$$+ \epsilon\xi\psi_{1\eta\eta}+\epsilon^2\xi\psi_{2\eta\eta}+\ldots$$

And equating the coefficients of like powers of ϵ yields:

$$\delta_o^*(X_e\xi\delta^*_{o\tau}+ \frac{1}{2} \delta^*_{o\xi})(\delta_o^*-X_e\psi^*_{o\eta}) = \xi\psi^*_{o\eta\eta}$$

$$\delta_o^* \Big[(X_e\xi\delta^*_{1\tau}+ \frac{1}{2} \delta^*_{1\xi})(\delta^*_o-X_e\psi^*_{o\eta})+(X_e\xi\delta^*_{o\tau}+$$

$$\frac{1}{2} \delta^*_{o\xi})(\delta^*_1-X_e\psi^*_{1\eta})\Big] + \delta^*_1(X_e\xi\delta^*_{o\tau}+ \frac{1}{2} \delta^*_{o\xi})(\delta^*_o$$

$$-X_e\psi^*_{o\eta})+ \frac{1}{2} \Big[\delta_o^{*2}(X_e\xi\delta^*_{o\tau}+\delta^*_{o\xi})-$$

$$- \frac{X_e}{2} \delta^*_o\delta^*_{o\xi}\psi^*_{o\eta}\Big] \sin\tau = \xi\psi^*_{1\eta\eta}$$

or:

$$(\delta^*_o-X_e\psi^*_{o\eta}) \Big[\delta^*_o(X_e\xi\delta^*_{1\tau}+ \frac{1}{2} \delta^*_{1\xi})+\delta^*_1(X_e\xi\delta^*_{o\tau}$$

$$+ \frac{1}{2} \delta^*_{o\xi})\Big] + \delta^*_o(X_e\xi\delta^*_{o\tau}+ \frac{1}{2} \delta^*_{o\xi}) \Big[\delta^*_1-X_e\psi^*_{1\eta}$$

$$+ \frac{\delta^*_o}{2} \sin\tau\Big] + \frac{\delta^*_{o\xi}\delta^*_o}{4} (\delta^*_o-X_e\psi^*_{o\eta})\sin\tau = \xi\psi_{1\eta\eta}$$

$$\delta^*_o \Big[(X_e\xi\delta^*_{2\tau}+ \frac{1}{2} \delta^*_{2\xi})(\delta^*_o-X_e\psi^*_{o\eta})+(X_e\xi\delta^*_{1\tau} +$$

$$\frac{1}{2} \delta^*_{1\xi})(\delta^*_1-X_e\psi^*_{1\eta})+(X_e\xi\delta^*_{o\tau} + \frac{1}{2} \delta^*_{o\xi})(\delta^*_2-X_e\psi^*_{2\eta})\Big]$$

$$+\delta^*_1 \Big[(X_e\xi\delta^*_{1\tau}+ \frac{1}{2} \delta^*_{1\xi})(\delta^*_o-X_e\psi^*_{o\eta})+(X_e\xi\delta^*_{o\tau}+$$

$$+ \frac{1}{2} \delta^*_{o\xi})(\delta^*_1-X_e\psi^*_{1\eta})\Big] + \delta^*_2(X_e\xi\delta^*_{o\tau}+ \frac{1}{2} \delta^*_{o\xi})(\delta^*_o$$

$$-X_e\psi^*_{o\eta}) + \frac{1}{2}\left[2\delta^*_1\delta^*_o(X_e\xi\delta^*_{o\tau}+\delta^*_{o\xi})+\delta^{*2}_o(X_e\xi\delta^*_{1\tau}\right.$$

$$\left.+\delta^*_{1\xi}) - \frac{X_e}{2}(\delta^*_o\delta^*_{1\xi}+\delta^*_1\delta^*_{o\xi})\psi^*_{o\eta} - \frac{X_e}{2}\delta^*_o\delta^*_{o\xi}\psi^*_{1\eta}\right]\sin\tau$$

$$+\frac{\delta^{*2}_o}{16}\delta^*_{o\xi}(1-\cos 2\tau) = \xi\psi^*_{2\eta\eta}$$

or:

$$(\delta^*_o - X_e\psi^*_{o\eta})\left[\delta^*_o(X_e\xi\delta^*_{2\tau}+\frac{1}{2}\delta^*_{2\xi})+\delta^*_2(X_e\xi\delta^*_{o\tau}+\frac{1}{2}\delta^*_{o\xi})\right.$$

$$\left.+\delta^*_1(X_e\xi\delta^*_{1\tau}+\frac{1}{2}\delta^*_{1\xi})\right] + (\delta^*_1 - X_e\psi^*_{1\eta})\left[\delta^*_o(X_e\xi\delta^*_{1\tau}\right.$$

$$\left.+\frac{1}{2}\delta^*_{1\xi})+\delta^*_1(X_e\xi\delta^*_{o\tau}+\frac{1}{2}\delta^*_{o\xi})\right] + \delta^*_o(X_e\xi\delta^*_{o\tau}+\frac{1}{2}\delta^*_{o\xi})(\delta^*_2$$

$$-X_e\psi^*_{2\eta}) + \frac{1}{2}\left[2\delta^*_1\delta^*_o(X_e\xi\delta^*_{o\tau}+\delta^*_{o\xi})+\delta^{*2}_o(X_e\xi\delta^*_{1\tau}\right.$$

$$\left.+\delta^*_{1\xi}) - \frac{X_e}{2}(\delta^*_o\delta^*_{1\xi}+\delta^*_1\delta^*_{o\xi})\psi^*_{o\eta} - \frac{X_e}{2}\delta^*_o\delta^*_{o\xi}\psi^*_{1\eta}\right]\sin\tau$$

$$+\frac{\delta^{*2}_o}{16}\delta^*_{o\xi}(1-\cos 2\tau) = \xi\psi^*_{2\eta\eta}$$

APPENDIX XIII

SIMPLIFICATION OF FIRST- AND SECOND-ORDER MOMENTUM EQUATIONS

Substitution of (92) into Eq. (90-1) gives:

$$4a^2\xi^3\psi_{1\tau\eta} + a\xi(\overset{*}{\psi}_{1\xi\eta}2a\xi f_0' + 2af_0'\overset{*}{\psi}_{1\eta} - 2a\xi f_0''\overset{*}{\psi}_{1\xi}$$

$$-2af_0\overset{*}{\psi}_{1\eta\eta}) + \frac{1}{2}\overset{*}{\delta}_1(4a^2\xi f_0'^2 - 4a^2\xi f_0 f_0'') - 2a^2\xi\overset{*}{\delta}_{1\xi}f_0'^2$$

$$-a\xi f_0'\cdot 4a\xi^2\overset{*}{\delta}_{1\tau} - 8a^2\xi f_0'\psi_{1\eta} - \xi\eta(2a\xi)^2\overset{*}{\delta}_{1\tau}f_0''$$

$$= \frac{8a^3}{2X_e}\xi^4\cos\tau + \frac{1}{X_e}\xi\,\overset{*}{\psi}_{1\eta\eta\eta}$$

Rearranging we get:

$$-\frac{1}{X_e}\overset{*}{\psi}_{1\eta\eta\eta} + 4a^2\xi^2\overset{*}{\psi}_{1\tau\eta} + 2a^2(\xi f_0'\overset{*}{\psi}_{1\xi\eta} - f_0'\overset{*}{\psi}_{1\eta}$$

$$-\xi f_0''\overset{*}{\psi}_{1\xi} - f_0\overset{*}{\psi}_{1\eta\eta}) + 2a^2(f_0'^2 - f_0f_0'')\overset{*}{\delta}_1 - 2a^2\xi f_0'^2\overset{*}{\delta}_{1\xi}$$

$$-4a^2\xi^2(f_0' + \eta f_0'')\overset{*}{\delta}_{1\tau} = \frac{4a^3}{X_e}\xi^3\cos\tau$$

And collecting terms, we get:

$$-\frac{1}{2a^2X_e}\overset{*}{\psi}_{1\eta\eta\eta} - f_0\overset{*}{\psi}_{1\eta\eta} + f_0'(\xi\overset{*}{\psi}_{1\xi\eta} - \overset{*}{\psi}_{1\eta}) - \xi f_0''\overset{*}{\psi}_{1\xi}$$

$$+2\xi^2\overset{*}{\psi}_{1\tau\eta} + (f_0'^2 - f_0f_0'')\overset{*}{\delta}_1 - \xi f_0'^2\overset{*}{\delta}_{1\xi} - 2\xi^2(f_0' + \eta f_0'')\overset{*}{\delta}_{1\tau}$$

$$= \frac{2a}{X_e}\xi^3\cos\tau$$

Substitution of (92) into Eq. (90-2) gives:

$$4a^2\xi^3\psi^*_{2\tau\eta}+4a\xi^2\delta^*_1\psi^*_{1\tau\eta}+a\xi(2a\xi f'_o\psi^*_{2\xi\eta}+$$

$$\psi^*_{1\xi\eta}\psi^*_{1\eta} + 2af'_o\psi^*_{2\eta}-2af_o\psi^*_{2\eta\eta}-\psi^*_{1\eta\eta}\psi^*_{1\xi}$$

$$-2a\xi f''_o\psi^*_{2\xi}) + \frac{\delta^*_1}{2}(2a\xi f'_o\psi^*_{1\xi\eta}+2af'_o\psi^*_{1\eta} - 2af''_o\psi^*_{1\xi}$$

$$-2af_o\psi^*_{1\eta\eta}) + \frac{\delta^*_2}{2}(4a^2\xi f'^2_o-4a^2\xi f''_o f_o)-2a^2\xi^2 f'^2_o\delta^*_{2\xi}$$

$$-2\xi\delta^*_{1\tau}\delta^*_1\cdot a\xi\cdot f'_o-2a\xi f'_o\delta^*_{1\xi}\psi^*_{1\eta}-2a\xi^2\delta^*_{1\tau}\psi^*_{1\eta}$$

$$-\frac{1}{2}(2a)(\psi^*_{1\eta}+4a\xi f'_o\psi^*_{1\eta})-\xi\eta(\delta^*_1\delta^*_{1\tau}+2a\xi\delta^*_{2\tau})2a\xi f''_o$$

$$-\xi\eta(2a\xi)\delta^*_{1\tau}\psi^*_{1\eta\eta} = \frac{6a^2}{X_e}\xi^3\delta^*_1\cos\tau + \frac{1}{X_e}\xi\,\psi^*_{2\eta\eta\eta}$$

Upon rearranging we get:

$$2a^2\xi\Big[-\frac{1}{2a^2X_e}\psi^*_{2\eta\eta\eta} + 2\xi^2\psi^*_{2\tau\eta} + \xi f'_o\psi^*_{2\xi\eta}-f'_o\psi^*_{2\eta}$$

$$-f_o\psi^*_{2\eta\eta}-\xi f''_o\psi^*_{2\xi}+\delta^*_2(f'^2_o-f_of''_o)-\xi f'^2_o\delta^*_{2\xi}-$$

$$-2\xi^2(f'_o+\eta f''_o)\delta^*_{2\tau}\Big] = -4a\xi^2\delta^*_1\psi^*_{1\tau\eta}+$$

$$+\frac{6a^2}{X_e}\xi^3\delta^*_1\cos\tau - a\xi(\psi^*_{1\xi\eta}\psi^*_{1\eta}-\psi^*_{1\eta\eta}\psi^*_{1\xi})-$$

$$-a\delta^*_1(\xi f'_o\psi^*_{1\xi\eta}+f'_o\psi^*_{1\eta}-\xi f''_o\psi^*_{1\xi}-f_o\psi^*_{1\eta\eta})+$$

$$+2a\xi^2 f'_o\delta^*_1\delta^*_{1\tau}+2a\xi f'_o\delta_{1\xi}\psi^*_{1\eta}+2a\xi^2\delta_{1\tau}\psi^*_{1\eta}$$

$$+a\psi^{*2}_{1\eta}+2a\xi^2\eta f''_o\delta^*_1\delta^*_{1\tau}+2a\xi^2\eta\delta^*_{1\tau}\psi^*_{1\eta\eta}$$

And finally we get:

$$2a\xi\left[-\frac{1}{2a^2X_e}\psi^*_{2\eta\eta\eta}-f_o\psi^*_{2\eta\eta}+f_o'(\xi\psi^*_{2\xi\eta}-\psi^*_{2\eta})-\xi f_o''\psi^*_{2\xi}\right.$$

$$\left.+2\xi^2\psi^*_{2\tau\eta}+(f_o'^2-f_of_o'')\delta^*_2-\xi f_o'^2\delta^*_{2\xi}-2\xi^2(f_o'+\eta f_o'')\delta^*_{2\tau}\right]$$

$$=\frac{6a}{X_e}\ \xi^3\delta^*_1\cos\tau-4\xi^2\delta^*_1\psi^*_{1\tau\eta}-\xi\psi^*_{1\xi\eta}\psi^*_{1\eta}+\xi\psi^*_{1\eta\eta}\psi^*_{1\xi}$$

$$+\psi^{*\,2}_{1\eta}-\delta^*_1(\xi f_o'\psi^*_{1\xi\eta}+f_o'\psi^*_{1\eta}-\xi f_o''\psi^*_{1\xi}-f_o\psi^*_{1\eta\eta})$$

$$+2\xi^2(f_o'+\eta f_o'')\delta^*_1\delta^*_{1\tau}+2\xi f_o'\delta^*_{1\xi}\psi^*_{1\eta}+$$

$$+2\xi^2(\psi^*_{1\eta}+\eta\psi^*_{1\eta\eta})\delta^*_{1\tau}$$

APPENDIX XIV

SIMPLIFICATION OF THE FIRST- AND SECOND-ORDER MOMENTUM BOUNDARY CONDITIONS

Substituting Eq. (92) into Eq. (90-1) gives:

$$2a\xi(1-X_e f_o') \left[2a\xi(X_e\xi\delta^*_{1\tau} + \frac{1}{2}\ \delta^*_{1\xi}) + a\delta^*_1 + \right.$$

$$\left. + a^2\xi\sin\ \tau \right] + 2a^2\xi \left[\delta^*_1 - X_e\psi^*_{1\eta} + a\xi\sin\ \tau \right] =$$

$$\xi\ \psi^*_{1\eta\eta}$$

or:

$$\frac{1}{2a^2}\ \psi^*_{1\eta\eta} = (1-X_e f_o') \left[2X_e\xi^2\delta^*_{1\tau} + (\xi\delta^*_{1\xi} + \delta^*_1) + \right.$$

$$\left. + a\xi\sin\ \tau \right] + \delta^*_1 - X_e\psi^*_{1\eta} + a\xi\sin\ \tau$$

And use of (92) in Eq. (90-2) gives:

$$2a\xi(1-X_e f_o') \left[2a\xi(X_e\xi\delta^*_{2\tau} + 1/2\ \delta^*_{2\xi}) + a\delta^*_2 \right.$$

$$\left. + \delta^*_1(X_e\xi\delta^*_{1\tau} + 1/2\ \delta^*_{1\xi}) \right] + (\delta^*_1 - X_e\psi^*_{1\eta}) \left[2a\xi(X_e\xi\delta^*_{1\tau} \right.$$

$$\left. + \frac{1}{2}\ \delta^*_{1\xi}) + a\delta^*_1 \right] + 2a^2\xi(\delta^*_2 - X_e\psi^*_{2\eta}) + \frac{1}{2} \left[8a^2\xi\ \delta^*_1 \right.$$

$$+ 4a^2\xi^2\ (\delta^*_{1\xi} + X_e\xi\delta_{1\tau}) - X_e 2a^2\xi(\delta^*_{1\xi}\xi + \delta^*_1)f_o'$$

$$\left. - X_e 2a^2\xi\psi^*_{1\eta} \right]\ \sin\ \tau + \frac{a^3}{2}\ \xi^2(1-\cos 2\tau) = \xi\ \psi^*_{2\eta\eta}$$

Rearranging we get:

$$2a\xi(1-X_e f_0') \left[2a\xi^2 X_e \delta^*_{2\tau} + a(\xi\delta^*_{2\xi} + \delta^*_2) + \right.$$

$$\left. \delta^*_1 (X_e \xi \delta^*_{1\tau} + \frac{1}{2} \delta^*_{1\xi}) \right] + a(\delta^*_1 - X_e \psi^*_{1\eta}) \left[2X_e \xi^2 \delta^*_{1\tau} \right.$$

$$\left. + (\xi\delta^*_{1\xi} + \delta^*_1) \right] + \frac{1}{2} (2a)^2 \xi (\delta^*_2 - X_e \psi^*_{2\eta}) + a^2 \xi \left[(\xi\delta^*_{1\xi} \right.$$

$$+ \delta^*_1)(1 - X_e f_0') + (\delta^*_1 - X_e \psi^*_{1\eta}) + (\xi\delta^*_{1\xi} + 2\delta^*_1) +$$

$$\left. + 2X_e \xi^2 \delta^*_{1\tau} \right] \sin\tau + \frac{1}{2} a^3 \xi (1 - \cos 2\tau) = \xi \psi^*_{2\eta\eta}$$

and:

$$2a\xi(1-X_e f_0') \left[a(\xi\delta^*_{2\xi} + \delta^*_2) + 2X_e a\xi^2 \delta^*_{2\tau} + \right.$$

$$\left. \delta^*_1 (X_e \xi \delta^*_{1\tau} + \frac{1}{2} \delta^*_{1\xi}) + \frac{a}{2} (\xi\delta^*_{1\xi} + \delta^*_1) \sin\tau \right] +$$

$$a(\delta^*_1 - X_e \psi^*_{1\eta}) \left[(\xi\delta^*_{1\xi} + \delta^*_1) + 2X_e \xi^2 \delta^*_{1\tau} + a\xi \sin\tau \right]$$

$$+ \frac{1}{2} (2a)^2 \xi (\delta^*_2 - X_e \psi^*_{2\eta}) + \frac{1}{4} (2a)^2 \xi \left[(\xi\delta^*_{1\xi} + 2\delta^*_1) + \right.$$

$$\left. X_e 2\xi^2 \delta^*_{1\tau} \right] \sin\tau + \frac{1}{8} (2a)^2 a\xi^2 (1 - \cos 2\tau) = \xi \psi^*_{2\eta\eta}$$

APPENDIX XV

REDUCTION OF THE SECOND OF EQUATIONS (84)

Substitution of the expansions (87) into the second of Eqs. (84) yields:

$$\xi(\delta_0^{*2}+2\epsilon\delta_0^*\delta_1^*+\epsilon^2\{\delta_1^{*2}+2\delta_2^*\delta_0^*\}+\ldots)(\Theta_{0\tau}^*$$

$$+\epsilon\Theta_{1\tau}^*+\epsilon^2\Theta_{2\tau}^*+\ldots) + \frac{1}{2}(\delta_0^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*+\ldots)[(\Theta_{0\xi}^*$$

$$+\epsilon\Theta_{1\xi}^*+\epsilon^2\Theta_{2\xi}^*+\ldots)(\psi_{0\eta}^*+\epsilon\psi_{1\eta}^*+\epsilon^2\psi_{2\eta}^*+\ldots)-$$

$$-(\psi_{0\xi}^*+\epsilon\psi_{1\xi}^*+\epsilon^2\psi_{2\xi}^*+\ldots)(\Theta_{0\eta}^*+\epsilon\Theta_{1\eta}^*+\epsilon^2\Theta_{2\eta}^*+\ldots)]$$

$$-\xi\eta(\delta_0^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*+\ldots)(\delta_{0\tau}^*+\epsilon\delta_{1\tau}^*+\epsilon^2\delta_{2\tau}^*+\ldots$$

$$\ldots)(\Theta_{0\eta}^*+\epsilon\Theta_{1\eta}^*+\epsilon^2\Theta_{2\eta}^*+\ldots) = \frac{\xi}{XePr}(\Theta_{0\eta\eta}^*+\epsilon\Theta_{1\eta\eta}^*$$

$$+\epsilon^2\Theta_{2\eta\eta}^*+\ldots)$$

Upon collecting terms one obtains:

$$\xi(\delta_0^{*2}+2\epsilon\delta_0^*\delta_1^*+\epsilon^2\{\delta_1^{*2}+2\delta_0^*\delta_2^*\}+\ldots)(\Theta_{0\tau}^*+\epsilon\Theta_{1\tau}^*+\epsilon^2\Theta_{2\tau}^*+\ldots)$$

$$+\frac{1}{2}(\delta_0^*+\epsilon\delta_1^*+\epsilon^2\delta_2^*+\ldots)[(\Theta_{0\xi}^*\psi_{0\eta}^*-\psi_{0\xi}^*\Theta_{0\eta}^*)+\epsilon(\Theta_{1\xi}^*\psi_{0\eta}^*+\Theta_{0\xi}^*\psi_{1\eta}^*-$$

$$-\psi_{0\xi}^*\Theta_{1\eta}^*-\psi_{1\xi}^*\Theta_{0\eta}^*)+\epsilon^2(\psi_{2\eta}^*\Theta_{0\xi}^*+\psi_{1\eta}^*\Theta_{1\xi}^*+\Theta_{2\xi}^*\psi_{0\eta}^*-\Theta_{2\eta}^*\psi_{0\xi}^*$$

$$-\Theta_{1\eta}^*\psi_{1\xi}^*-\Theta_{0\eta}^*\psi_{2\xi}^*)+\ldots]-\xi\eta(\delta_0^*\delta_{0\tau}^*+\epsilon\{\delta_1^*\delta_{0\tau}^*+\delta_0^*\delta_{1\tau}^*\}$$

$$+\epsilon^2\{\delta_2^*\delta_{0\tau}^*+\delta_1^*\delta_{1\tau}^*+\delta_0^*\delta_{2\tau}^*\}+\ldots)(\Theta_{0\eta}^*+\epsilon\Theta_{1\eta}^*+\epsilon^2\Theta_{2\eta}^*+\ldots)$$

$$= \frac{\xi}{XePr}(\Theta_{0\eta\eta}^*+\epsilon\Theta_{1\eta\eta}^*+\epsilon^2\Theta_{2\eta\eta}^*+\ldots)$$

And finally equating the coefficients of like powers of ϵ we have:

$$\xi\delta_0^{*2}(\theta_{0\tau}^*) + \frac{1}{2}\,\delta_0^*(\theta_{0\xi}^*\psi_{0\eta}^* - \psi_{0\xi}^*\theta_{0\eta}^*) - \xi\eta\delta_0^*\delta_{0\tau}^*\theta_{0\eta}^* \quad = \quad \frac{\xi}{X_eP_r}\;\theta_{0\eta\eta}^*$$

$$2\xi\delta_0^*\delta_1^*\theta_{0\tau}^* + \xi\delta_0^{*2}\theta_{1\tau}^* + \frac{1}{2}\,\delta_0^*(\theta_{1\xi}^*\psi_{0\eta}^* + \theta_{0\xi}^*\psi_{1\eta}^* - \psi_{0\xi}^*\theta_{1\eta}^* - \psi_{1\xi}^*\theta_{0\eta}^*)$$

$$+\frac{1}{2}\;\delta_1^*(\theta_{0\xi}^*\psi_{0\eta}^* - \psi_{0\xi}^*\theta_{0\eta}^*) - \xi\eta\delta_0^*\delta_{0\tau}^*\theta_{1\eta}^* - \xi\eta(\delta_1^*\delta_{0\tau}^* + \delta_0^*\delta_{1\tau}^*)\theta_{0\eta}^*$$

$$= \frac{\xi}{X_eP_r}\;\theta_{1\eta\eta}^*$$

$$\xi\delta_0^{*2}\theta_{2\tau}^* + 2\xi\delta_0^*\delta_1^*\theta_{1\tau}^* + \xi(\delta_1^{*2} + 2\delta_0^*\delta_2^*)\theta_{0\tau}^*$$

$$+\frac{1}{2}\,\delta_0^*(\psi_{2\eta}^*\theta_{0\xi}^* + \psi_{1\eta}^*\theta_{1\xi}^* + \theta_{2\xi}^*\psi_{0\eta}^* - \theta_{2\eta}^*\psi_{0\xi}^* - \theta_{1\eta}^*\psi_{1\xi}^* - \theta_{0\eta}^*\psi_{2\xi}^*)$$

$$+\frac{1}{2}\;\delta_1^*(\theta_{1\xi}^*\psi_{0\eta}^* + \theta_{0\xi}^*\psi_{1\eta}^* - \psi_{0\xi}^*\theta_{1\eta}^* - \psi_{1\xi}^*\theta_{0\eta}^*) + \frac{1}{2}\,\delta_2^*(\theta_{0\xi}^*\psi_{0\eta}^* - \psi_{0\xi}^*\theta_{0\eta}^*)$$

$$-\xi\eta(\delta_2^*\delta_{0\tau}^* + \delta_1^*\delta_{1\tau}^* + \delta_0^*\delta_{2\tau}^*)\theta_{0\eta}^* - \xi\eta(\delta_1^*\delta_{0\tau}^* + \delta_0^*\delta_{1\tau}^*)\theta_{1\eta}^*$$

$$-\xi\eta\delta_0^*\delta_{0\tau}^*\theta_{2\eta}^* \quad = \quad \frac{\xi}{X_eP_r}\;\theta_{2\eta\eta}^*$$

APPENDIX XVI

REDUCTION OF THE THIRD OF CONDITIONS (85)

Substitution of the expansions (87) into the third of (85) yields:

$$(\Theta_0^* + \epsilon\Theta_1^* + \epsilon^2\Theta_2^* + \ldots)(\delta_0^* + \epsilon\delta_1^* + \epsilon^2\delta_2^*$$

$$+ \ldots)\ [(X_e\xi\delta_{0\tau}^* + 1/2\ \delta_{0\xi}^*) + \epsilon(X_e\xi\delta_{1\tau}^* + 1/2\ \delta_{1\xi}^*)$$

$$+ \epsilon^2(X_e\xi\delta_{2\tau}^* + 1/2\ \delta_{2\xi}^*) + \ldots] + \epsilon/4\ (\Theta_0^* + \epsilon\Theta_1^*$$

$$+ \ldots)(\delta_0^* + \epsilon\delta_1^* + \ldots)(\delta_{0\xi}^* + \epsilon\delta_{1\xi}^* + \ldots)\sin.\tau$$

$$= -\frac{1}{X_e P_r}\ \xi\ (\Theta_{0\eta}^* + \epsilon\Theta_{1\eta}^* + \epsilon^2\Theta_{2\eta}^* + \ldots)$$

$$= (\Theta_0^* + \epsilon\Theta_1^* + \epsilon^2\Theta_2^* + \ldots)[\delta_0^*(X_e\xi\delta_{0\tau}^* + 1/2\ \delta_{0\xi}^*)$$

$$+ \epsilon\{\delta_1^*(X_e\xi\delta_{0\tau}^* + 1/2\ \delta_{0\xi}^*) + \delta_0^*(X_e\xi\delta_{1\tau}^* + 1/2\ \delta_{1\xi}^*)\}$$

$$+ \epsilon^2\{\delta_2^*(X_e\xi\delta_{0\tau}^* + 1/2\ \delta_{0\xi}^*) + \delta_1^*(X_e\xi\delta_{1\tau}^* + 1/2\ \delta_{1\xi}^*) +$$

$$\delta_0^*(X_e\xi\delta_{2\tau}^* + 1/2\ \delta_{2\xi}^*)\} + \ldots] + \epsilon/4\ (\Theta_0^* +$$

$$\epsilon\Theta_1^* + \ldots)(\delta_0^*\delta_{0\xi}^* + \epsilon[\delta_1^*\delta_{0\xi}^* + \delta_0^*\delta_{1\xi}^*] + \ldots)\sin.\tau$$

And equating the coefficients of like powers of ϵ we get:

$$\Theta_0^*\delta_0^*(X_e\xi\delta_{0\tau}^* + 1/2\ \delta_{0\xi}^*) = -\frac{1}{X_e P_r}\ \xi\ \Theta_{0\eta}^*$$

$$\Theta_1^*\delta_0^*(X_e\xi\delta_{0\tau}^* + 1/2\ \delta_{0\xi}^*) + \Theta_0^*\ \{\delta_1^*(X_e\xi\delta_{0\tau}^* + 1/2\ \delta_{0\xi}^*) +$$

$$\delta_0^*(X_e\xi\delta_{1\tau}^* + 1/2\ \delta_{1\xi}^*)\} + 1/4\ \Theta_0^*\delta_0^*\delta_{0\xi}^*\ \sin\tau\ =\ -\frac{1}{X_eP_r}\ \xi\ \Theta_{1\eta}^*$$

$$\Theta_2^*\delta_0^*(X_e\xi\delta_{0\tau}^* + 1/2\ \delta_{0\xi}^*) + \Theta_1^*\ \{\delta_1^*(X_e\xi\delta_{0\tau}^* + 1/2\ \delta_{0\xi}^*)$$

$$+ \delta_0^*(X_e\xi\delta_{1\tau}^* + 1/2\ \delta_{1\xi}^*)\} + \Theta_0^*\{\delta_2^*(X_e\xi\delta_{0\tau}^* + 1/2\ \delta_{0\xi}^*)$$

$$+ \delta_1^*(X_e\xi\delta_{1\tau}^* + 1/2\ \delta_{1\xi}^*) + \delta_0^*(X_e\xi\delta_{2\tau}^* + 1/2\ \delta_{2\xi}^*)\}$$

$$+ \frac{1}{4}\ \{\Theta_1^*\delta_0^*\delta_{0\xi}^* + \Theta_0^*\ [\delta_0^*\delta_{0\xi}^* + \delta_0^*\delta_{1\xi}^*]\ \}\ \sin\tau$$

$$=\ -\frac{1}{XePr}\ \xi\ \Theta_{2\eta}^*$$

APPENDIX XVII

SIMPLIFICATION OF THE FIRST- AND SECOND-ORDER ENERGY EQUATIONS

Substitution of Eqs. (99) and (92) into Eq. (96-1) yields:

$$4a^2\xi^3\theta^*_{1\tau}+a\xi(2a\xi f'_o\theta^*_{1\xi}-2af_o\theta^*_{1\eta}-\psi^*_{1\xi}F'_o)$$

$$-\frac{1}{2}\ \delta^*_1\ 2af_oF'_o-\xi^2 2a\eta\delta^*_{1\tau}F'_o\ =\ \frac{\xi}{X_eP_r}\ \theta^*_{1\eta\eta}$$

And collecting terms we get:

$$\frac{1}{2a^2P_rX_e}\ \theta^*_{1\eta\eta}\ -\ 2\xi^2\theta^*_{1\tau}+f_o\theta^*_{1\eta}-\xi f'_o\theta^*_{1\xi}$$

$$=\ -\frac{1}{2a\xi}\ (\xi\psi^*_{1\xi}+\delta^*_1\ f_o+2\eta\xi^2\delta^*_{1\tau})F'_o$$

Substitution of Eqs. (99) and (92) into Eq. (96-2) gives:

$$4a^2\xi^3\theta^*_{2\tau}+4a\xi^2\delta^*_1\theta^*_{1\tau}+a\xi(\psi^*_{1\eta}\theta^*_{1\xi}+2a\xi f'_o\theta^*_{2\xi}$$

$$-2af_o\theta^*_{2\eta}-\theta^*_{1\eta}\psi^*_{1\xi}-F'_o\psi^*_{2\xi})\ +\ \frac{1}{2}\ \delta^*_1(2a\xi f'_o\theta^*_{1\xi}$$

$$-2af_o\theta^*_{1\eta}-F'_o\psi^*_{1\xi})-af_o\delta^*_2F'_o-\xi\eta(\delta^*_1\delta^*_{1\tau}$$

$$+2a\xi\delta^*_{2\tau})F'_o-\xi\eta^2 2a\delta^*_{1\tau}\theta^*_{1\eta}\ =\ \frac{\xi}{X_eP_r}\ \theta^*_{2\eta\eta}$$

Rearranging, we get:

$$\frac{1}{2a^2P_rX_e}\ \theta^*_{2\eta\eta}-2\xi^2\theta^*_{2\tau}+f_o\theta^*_{2\eta}-\xi f'_o\theta^*_{2\xi}\ =$$

$$\frac{1}{2a^2\xi}\ (4a\xi^2\delta^*_1\theta^*_{1\tau}+a\xi\psi^*_{1\eta}\theta^*_{1\xi}-a\xi\theta^*_{1\eta}\psi^*_{1\xi}-$$

$$a\xi F_o'\psi^*_{2\xi}+\frac{1}{2}\,2a\xi\delta^*_1 f_o'\theta^*_{1\xi}-af_o\delta^*_1\theta^*_{1\eta}\,-$$

$$-\frac{1}{2}\,\delta^*_1 F_o'\psi^*_{1\xi}-af_o\delta^*_2 F_o-\xi\eta[\delta^*_1\delta^*_{1\tau}+$$

$$2a\xi\delta^*_{2\tau}]F_o'-\eta\xi^2 2a\delta^*_{1\tau}\theta^*_{1\eta})$$

and:

$$\frac{1}{2a^2P_rX_e}\,\theta^*_{2\eta\eta}-2\xi^2\theta^*_{2\tau}+f_o\theta^*_{2\eta}-\xi f_o'\theta^*_{2\xi}\quad=$$

$$=\;\frac{1}{2a^2\xi}\,\{-a(\xi\psi^*_{2\xi}+f_o\delta^*_2+2\xi^2\eta\delta^*_{2\tau})F_o'-(\,\frac{1}{2}\,\delta^*_1\psi^*_{1\xi}$$

$$+\,\xi\eta\delta^*_1\delta^*_{1\tau})F_o'\;+\;a[4\xi^2\delta^*_1\theta^*_{1\tau}+(\psi^*_{1\eta}+f_o'\delta^*_1)\xi\theta^*_{1\xi}$$

$$-(\xi\psi^*_{1\xi}+f_o\delta^*_1+2\xi^2\eta\delta^*_{1\tau})\theta^*_{1\eta}]\}$$

APPENDIX XVIII

SIMPLIFICATION OF FIRST- AND SECOND-ORDER ENERGY BOUNDARY CONDITIONS

Using (99) and (92) in Eq. (97.1) we get:

$$\Theta_1^* \; 2a\xi a + F_o \left\{ \delta_1^* \; a + 2a \; \xi (X_e \xi \delta_{1\tau}^* + \frac{1}{2} \delta_{1\xi}^*) \right\}$$

$$+ \frac{1}{4} F_o \; 2a\xi 2a \; \sin \tau \; = \; - \frac{1}{X_e P_r} \; \xi \; \Theta_{1\eta}^*$$

and:

$$\xi \left\{ \frac{1}{2a^2 X_e P_r} \Theta_{1\eta}^* + \Theta_1^* \right\} + \frac{1}{2a} F_o \; \{(\delta_1^* + \xi \delta_{1\xi}^*) + 2X_e \xi^2 \delta_{1\tau}^*\} +$$

$$+ F_o \; \frac{1}{2} \; \xi \; \sin \tau \; = \; 0$$

Using (99) and (92) in Eq. (97.2) we get:

$$\Theta_2^* \; 2a\xi a + \Theta_1^* \; \{a\delta_1^* + 2a\xi^2 \; X_e \; \delta_{1\tau}^* + a \; \delta_{1\xi}^*\}$$

$$+ F_o \; \{\delta_2^* \; a + a\xi\delta_{2\xi}^* + 2aX_e\xi^2\delta_{2\tau}^* + \delta_1^* \; (X_e\xi\delta_{1\tau}^* + \frac{1}{2} \delta_{1\xi}^*)\}$$

$$+ \frac{1}{4} \; \{\Theta_1^* \; 2a\xi \; 2a + F_o[2a\delta_1^* + 2a\xi\delta_{1\xi}^*]\} \; \sin \tau$$

$$= - \frac{1}{X_e P_r} \; \xi \; \Theta_{2\eta}^*$$

or:

$$\xi \left\{ \frac{1}{2a^2 P_r X_e} \; \Theta_{2\eta}^* + \Theta_2^* \right\} + \Theta_1^* \frac{1}{2a} \; \{(\delta_1^* + \xi\delta_{1\xi}^*) + 2X_e\xi^2\delta_{1\tau}^*\}$$

$$+ F_o \frac{1}{2a} \; \left\{ (\delta_2^* + \xi\delta_{2\xi}^*) + 2X_e\xi^2\delta_{2\tau}^* + \frac{1}{a} \delta_1^* \; (X_e\xi\delta_{1\tau}^* + \frac{1}{2} \delta_{1\xi}^*) \right\}$$

$$+ \frac{1}{2} \xi \; \Theta_1^* \; \sin \tau + \frac{1}{2} F_o \frac{1}{2a} \; (\delta_1^* + \xi\delta_{1\xi}^*) \; \sin \tau \; = \; 0$$

APPENDIX XIX

REDUCTION OF FIRST-ORDER MOMENTUM PROBLEM TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Substitution of the expansion (103) and (104) into Eq. (88-1a) gives:

$$\sum_{n=1}^{\infty}\left\{\left[-\frac{1}{2a^2X_e} f_{1n}''' + nf_o'f_{1n}' - f_o'f_{1n}' - nf_o''f_{1n} - f_of_{1n}''\right]\right\}\xi^n$$

$$+\sum_{n=1}^{\infty}[(f_o'^2 - f_of_o'')a_{1n} - na_{1n}f_o'^2]\xi^n$$

$$+2i\sum_{n=1}^{\infty}[f_{1n}' - (f_o' + \eta f_o'')a_{1n}]\xi^{n+2}$$

$$= \frac{1}{X_e}\xi^3$$

or:

$$\sum_{n=1}^{\infty}\left\{\left[-\frac{1}{2a^2X_e} f_{1n}''' - f_of_{1n}'' + (n+1)f_o'f_{1n}' - nf_o''f_{1n} + [(1-n)f_o'^2 - f_of_o'']a_{1n}\right]\right\}\xi^n$$

$$+2i\sum_{n=3}^{\infty}\{f_{1(n-2)}' - (f_o' + \eta f_o'')a_{1(n-2)}\}\xi^n = \frac{\xi^3}{X_e}$$

And equating coefficients of like powers of ξ gives:

$$\frac{1}{2a^2X_e} f_{1n}''' + f_of_{1n}'' - (n-1)f_o'f_{1n}' + nf_o''f_{1n} + [(n-1)f_o'^2 + f_of_o'']a_{1n}$$

$$= \begin{cases} 0 & ; \quad n=1,2 \\ -\dfrac{1}{X_e}\delta_{3n} - 2i[(f_o' + \eta f_o'')a_{1(n-2)} - f_{1(n-2)}'] & ; \quad n=3,4,5,\ldots \end{cases} \qquad \text{(XIX.1)}$$

where δ_{mn} is the Kroneka Delta.*

Substitution of the expansions (103) and (104) into Eq. (89-1a) gives:

$$\sum_{n=1}^{\infty} f_{1n}\xi^{n} = \sum_{n=1}^{\infty} a_{1n}\xi^{n} - \frac{i}{2}\xi + 2i(X_e-1)\sum_{n=1}^{\infty}\frac{a_{1n}}{n+2}\xi^{n+2}$$

or:

$$\sum_{n=1}^{\infty} f_{1}\xi^{n} = \sum_{n=1}^{\infty} a_{1n}\xi^{n} - \frac{i}{2}\xi + 2i(X_e-1)\sum_{n=3}^{\infty}\frac{a_{1(n-2)}}{n}\xi^{n}$$

And equating the coefficients of like powers of ξ we get:

$$\eta = 1; \quad f_{1n}-a_{1n} = \begin{cases} -\frac{i}{2}\delta_{1n} & ; \quad n=1,2 \\ 2i(X_e-1)a_{1(n-2)}/n; & n=3,4,5,\ldots \end{cases} \qquad \text{(XIX.2)}$$

And finally substitution of the expansions (103) and (104) into Eq. (90-1a) gives:

$$\frac{1}{2}(2a)^2\left\{(1-X_e f_o')\left[\sum_{n=1}^{\infty}(2iX_e a_{1n}\xi^2+\{n+1\}a_{1n})\xi^n - \frac{i}{2}\xi\right] + \sum_{n=1}^{\infty}(a_{1n}-X_e f_{1n}')\xi^n - \frac{i}{2}\xi\right\} = \sum_{n=1}^{\infty} f_{1n}''\xi^n$$

or:

$$\sum_{n=1}^{\infty}\frac{1}{2a^2}f_{1n}''\xi^n = (1-X_e f_o')\left[\sum_{n=3}^{\infty}2iX_e a_{1(n-2)}\xi^n + \sum_{n=1}^{\infty}(n+1)a_{1n}\xi^n - \frac{i}{2}\xi\right] +$$

$$* \quad \delta_{mn} = \begin{cases} 1 & ; \quad m=n \\ 0 & ; \quad m\neq n \end{cases}$$

$$\sum_{n=1}^{\infty} (a_{1n}-X_e f'_{1n})\xi^n - \frac{i}{2}\xi$$

And equating the coefficients of like powers of ξ one obtains:

$$\eta=1 \quad : \quad \frac{1}{2a^2} f''_{1n} = \begin{cases} (1-X_e f'_o)\left[(n+1)a_{1n} - \frac{i}{2}\delta_{1n}\right] \quad : \quad n=1,2 \\ \qquad + a_1 - X_e f'_{1n} - \frac{i}{2}\delta_{1n} \\ (1-X_e f'_o)[(n+1)a_{1n}+2iX_e a_{1(n-2)}]+ \\ \qquad + a_{1n}-X_e f'_{1n} \quad : \quad n=3,4,5,\ldots \end{cases} \tag{XIX.3}$$

The boundary condition (91) (with n=1) shows that:

$$\eta = 0; \qquad f_{1n} = f'_{1n} = 0 \tag{XIX.4}$$

Examination of Eqs. (XIX.1) shows that f_{12} and a_{12} satisfy homogeneous equations with homogeneous boundary conditions which will be satisfied with $f_{12}, a_{12} = 0$.

Furthermore, this then makes the equations and boundary conditions for f_{14}, a_{14}, f_{16}, a_{16}, etc., homogeneous.

Hence:

$$f_{1n} = a_{1n} = 0 \quad ; \quad \text{for } n=2,4,6,\ldots$$

A further investigation shows that we must take

$$f_{11}, a_{11};\ f_{15}, a_{15};\ f_{19}, a_{19};\ \ldots$$

to be pure imaginary and f_{13}, a_{13}; f_{17}, a_{17};...to be real.

Hence introducing the operator L_n, defined by:

$$L_n \equiv \frac{1}{2a^2X_e}\frac{d^3}{d\eta^3} + f_o\frac{d^2}{d\eta^2} - (n-1)f'_o\frac{d}{d\eta} + nf''_o$$

one has:

$$
n=1,3,5,7,\quad
\begin{cases}
L_n(f_{1n}) + [(n-1)f_o'^2+f_of_o'']a_{1n} = -\dfrac{1}{X_e}\,\delta_{3n} \\[4pt]
\quad - 2i(1-\delta_{1n})[(f_o'+\eta f_o'')a_{1(n-2)}-f'_{1(n-2)}];\quad n=1,3,5,7,\dots \\[8pt]
\eta=1\begin{cases}
f_{1n} = a_{1n} - \dfrac{i}{2}\,\delta_{1n} + 2i(1-\delta_{1n})(X_e-1)a_{1(n-2)}/n \\[4pt]
\dfrac{1}{2a^2}\,f''_{1n} = (1-X_ef_o')\left[(n+1)a_{1n}-\dfrac{i}{2}\,\delta_{1n}+2i(1-\delta_{1n})X_ea_{1(n-2)}\right] \\[4pt]
\qquad + a_{1n} - X_ef'_{1n} - \dfrac{i}{2}\,\delta_{1n}
\end{cases} \\[8pt]
\eta=0\quad \{f_{1n} = f'_{1n} = 0
\end{cases}
$$

for $n=2,4,6,8,\dots$ $\quad f_{1n} = a_{1n} = 0$

for $n=1,5,9,\dots$ $\quad f_{1n},\ a_{1n} \sim$ pure imaginary

for $n=3,7,12,\dots$ $\quad f_{1n},\ a_{1n} \sim$ real

APPENDIX XX

REDUCTION OF FIRST-ORDER ENERGY PROBLEM TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Substitution of the expansions (103) through (105) into Eq. (96-1a) gives:

$$\sum_{n=0}^{\infty} \left\{ \frac{1}{2a^2 P_r X_e} F''_{1n} + f_o F'_{1n} \right\} \xi^n - \sum_{n=1}^{\infty} n\, f'_o\, F_{1n}\, \xi^n$$

$$- 2i \sum_{n=0}^{\infty} F_{1n}\, \xi^{n+2} = - F'_o \sum_{n=1}^{\infty} (f_{1n} n + f_o a_{1n} + 2\eta i \xi^2 a_{1n}) \xi^{n-1}$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{2a^2 P_r X_e} F''_{1n} + f_o F'_{1n} \right\} \xi^n - \sum_{n=1}^{\infty} n f'_o\, F_{1n}\, \xi^n$$

$$- 2i \sum_{n=2}^{\infty} F_{1(n-2)} \xi^n = -F_o \sum_{n=0}^{\infty} [(n+1) f_{1(n+1)} + f_o a_{1(n+1)}] \xi^n$$

$$- 2i\eta F'_o \sum_{n=2}^{\infty} a_{1(n-1)} \xi^n$$

And equating the coefficients of like powers of ξ, we get:

$$\frac{1}{2a^2 P_r X_e} F''_{1n} + f_o F'_{1n} - n f_o F_{1n} = -F'_o\, [(n+1) f_{1(n+1)} + f_o a_{1(n+1)}] + \begin{cases} 0 & ; \quad n = 0,1 \\ -2iF'_o \eta a_{1(n-1)} + 2iF_{1(n-2)}; & n = 2,3,4,\ldots \end{cases} \qquad \text{(XX.1)}$$

Substitution of the expansions (103) through (105) into Eq. (97-1a) gives:

$$\sum_{n=0}^{\infty}\left\{\left[\frac{1}{2a^2X_eP_r}\;F_{1n}'+F_{1n}\right\}\xi^n+F_o\sum_{n=1}^{\infty}\{(1+n)a_{1n}+\right.$$

$$\left.2iX_ea_{1n}\xi^2\}\xi^{n-1}-\frac{i}{2}F_o\;=\;0\right.$$

or:

$$\sum_{n=0}^{\infty}\left\{\left[\frac{1}{2a^2X_eP_r}\;F_{1n}'+F_{1n}\right\}\xi^n+F_o\sum_{n=0}^{\infty}\{(n+2)a_{1(n+1)}\}\xi^n\right.$$

$$+\;2iX_eF_o\sum_{n=2}^{\infty}a_{1(n-1)}\xi^n-\frac{i}{2}F_o\;=\;0$$

And equating coefficients of like powers of ξ we get:

$$\eta=1:\quad\frac{1}{2a^2X_eP_r}\;F_{1n}'+F_{1n}+F_o(n+2)a_{1(n+1)}+\begin{cases}-\frac{i}{2}\,\delta_{on}F_o & n=0,1\\[2ex] 2iX_eF_oa_{1(n-1)} & n=2,3,4,\ldots\end{cases}\qquad\text{(XX.2)}$$

and the boundary condition (98.1) implies:

$$\eta\;=\;0;\qquad F_{1n}\;=\;0\quad:\quad\text{for } n=0,1,2,\ldots\qquad\text{(XX.3)}$$

Since f_{12}, $a_{12} = 0$, inspection of Eqs. (XX.1) through (XX.3) shows that F_{11} satisfies homogeneous equations and boundary conditions. Therefore we may take $F_{11} = 0$. And since, the F_{1n}'s depend only on the $F_{1(n-2)}$'s and the $f_{1(n+1)}$, $f_{1(n-1)}$, $a_{1(n+1)}$, $a_{1(n-1)}$ explicitly, we have:

$$0\leq\eta\leq 1;\quad F_{1n}\;=\;0\quad:\quad\text{for } n=1,3,5,7,\ldots$$

A further inspection of Eqs. (XX.1) through (XX.3) shows that the F_{1n} must be pure imaginary quantities for n = 0,4,8,...and the F_{1n} must be real for n = 2,6,10,...

Thus, we have, if we define the operator K_n by:

$$K_n \equiv \frac{1}{2a^2 P_r X_e} \frac{d^2}{d\eta^2} + f_o \frac{d}{d\eta} - nf_o'$$

$$n=0,2,4,\ldots \begin{cases} 0 \le \eta \le 1: \{K_n(f_n) = -F_o'[(n+1)f_{1(n+1)} + f_o a_{1(n+1)} - 2i(1-\delta_{on})\eta a_{1(n-1)}] \\ \qquad\qquad + 2i(1-\delta_{on})F_{1(n-2)} \\ \eta=1: \left\{ \frac{1}{2a^2 X_e P_r} F_{1n}' + F_{1n} + F_o\{(2+n)a_{1(n+1)} + 2iX_e(1-\delta_{on})a_{1(n-1)} \right. \\ \qquad\qquad - \frac{i}{2}\delta_{on}\} = 0 \\ \eta=0: \{ \quad F_{1n} = 0 \end{cases}$$

$F_{1n} \equiv 0$: for $n=1,3,5,7,\ldots$

F_{1n} = pure imaginary quantity: for $n=0,4,8,\ldots$

F_{1n} = real quantity: for $n=2,6,10,\ldots$

APPENDIX XXI

REDUCTION OF THE SECOND-ORDER MOMENTUM PROBLEM TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Substitution of the expansions (103), (104), (112), and (113) into Eq. (88-2a), using the relation (111) to simplify and then equating the terms which are coefficients of $e^{2i\tau}$ with each other and the time independent terms with each other (since no other terms appear as can be easily seen by inspection) yields for the coefficients of $e^{2i\tau}$:

$$\xi \sum_{n=1}^{\infty} \left\{ \left[-\frac{1}{2a^2 X_e} f_{2n}''' - f_0 f_{2n}'' + (n-1) f_0' f_{2n}' - n f_0'' f_{2n} + [(1-n) f_0'^2 - f_0 f_0''] a_{2n} \right] \right\} \xi^n$$

$$+ 4i\xi \sum_{n=3}^{\infty} \{ f_{2(n-2)}' - (f_0' + \eta f_0'') a_{2(n-2)} \} \xi^n = \frac{3\xi^3}{2X_e} \sum_{n=1}^{\infty} a_{1n} \xi^n$$

$$+ \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{ -n f_{1n}' f_{1m}' + n f_{1n} f_{1m}'' + f_{1n}' f_{1m}' - a_{1m} (n f_0' f_{1n}'$$

$$+ f_0' f_{1n}' - n f_0'' f_{1n} - f_0 f_{1n}'') + 2n f_0' a_{1n} f_{1m}' \} \xi^{n+m} +$$

$$+ i\, \xi^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{ -2a_{1n} f_{1m}' + (\eta f_0'' + f_0') a_{1n} a_{1m} + a_{1n} (\eta f_{1m}''$$

$$+ f_{1m}') \} \xi^{n+m}$$

And for the time independent terms:

$$\xi \sum_{n=1}^{\infty} \left\{ \left[-\frac{1}{2a^2 X_e} \tilde{f}_{2n}''' - f_0 \tilde{f}_{2n}'' + (n-1) f_0' \tilde{f}_{2n}' - n f_0'' \tilde{f}_{2n} \right. \right.$$

$$+ [(1-n)f_o'^2 - f_o f_o'']\tilde{a}_{2n}\Big\} \xi^n = \frac{3\xi^3}{2X_e} \sum_{n=1}^{\infty} a_{1n}\xi^n +$$

$$+ \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{-nf_{1n}'\bar{f}_{1m}' + nf_{1n}\bar{f}_{1m}'' + f_{1n}'\bar{f}_{1m}' - \bar{a}_{1m}(nf_o'f_{1n}'$$

$$+ f_o'f_{1n}' - nf_o''f_{1n} - f_o f_{1n}'') + 2nf_o'\bar{a}_{1n}f_{1m}'\}\xi^{m+n}$$

$$+ i\,\xi^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{-2f_{1m}'\bar{a}_{1n} + (\eta f_o'' + f_o')a_{1m}\bar{a}_{1n} +$$

$$a_{1n}(\eta\bar{f}_{1m}'' + \bar{f}_{1m}')\}\xi^{m+n}$$

And equating coefficients of like powers of ξ we get from the above two equations:

$$\frac{1}{2a^2X_e} f_{2n}''' + f_o f_{2n}'' - (n-1)f_o'f_{2n}' + nf_o''f_{2n} =$$

$$- [(n-1)f_o'^2 + f_o f_o'']a_{2n} + H_n \qquad \text{(XXI.1)}$$

$$\frac{1}{2a^2X_e} \tilde{f}_{2n}''' + f_o\tilde{f}_{2n}'' - (n-1)f_o'\tilde{f}_{2n}' + nf_o''\tilde{f}_{2n} =$$

$$-[(n-1)f_o'^2 + f_o f_o'']\tilde{a}_{2n} + \tilde{H}_n \qquad \text{(XXI.2)}$$

Where for $n=1,3,5,7,\ldots$

$$H_n \equiv -\frac{3}{2X_e}(1-\delta_{1n})a_{1(n-2)} + 4i(1-\delta_{1n})[f_{2(n-2)}'$$

$$-(f_o' + \eta f_o'')a_{2(n-2)}] - \frac{1}{2}\{f_{11}f_{1n}'' - (n-1)f_{11}'f_{1n}'$$

$$+(1-\delta_{1n})nf_{11}''f_{1n} + [2(n-1)f_o'f_{11}' + f_o''f_{11} + f_o f_{11}'']a_{1n} \qquad \text{(XXI.3)}$$

$$(1-\delta_{1n})[nf_o''f_{1n}-(n-1)f_o'f_{1n}'+f_of_{1n}'']a_{11}\}-$$

$$-i(1-\delta_{1n})\{(\eta f_o''+f_o')a_{11}a_{1(n-2)}+(\eta f_{11}''-f_{11}')a_{1(n-2)}\}-P_n \qquad \text{(XXI.3)}$$

Where:

$$P_1 = 0$$

$$P_3 = 0$$

$$P_5 = \frac{1}{2}\{3f_{13}f_{13}''-2f_{13}'f_{13}'+[f_of_{13}''+$$

$$+3f_o''f_{13}+2f_o'f_{13}']a_{13}\}+i\{(\eta f_o''+f_o')a_{11}a_{13}+(\eta f_{13}''-f_{13}')a_{11}\}$$

....

....

....

for n=2,4,6,8,...:

$$H_n \equiv 0 \qquad \text{(XXI.4)}$$

follows from (108)

and for: n=1,5,...

$$\tilde{H}_n = -\frac{3}{2X_e}(1-\delta_{1n})a_{1(n-2)}-\frac{1}{2}\{f_{11}\bar{f}_{1n}''-(n-1)f_{11}'\bar{f}_{1n}'$$

$$+(1-\delta_{1n})nf_{11}''\bar{f}_{1n}+[2(n-1)f_o'f_{11}'+f_o''f_{11}+f_of_{11}'']\bar{a}_{1n}$$

$$+(1-\delta_{1n})[nf_o''f_{1n}-(n-1)f_o'f_{1n}'+f_of_{1n}'']\bar{a}_{11}\}-\tilde{P}_n \qquad \text{(XXI.5)}$$

where:

$$\tilde{P}_1 = 0$$

$$\tilde{P}_5 = \frac{1}{2}\{3f_{13}\bar{f}_{13}''-2\bar{f}_{13}'f_{13}+[f_of_{13}''+3f_o''f_{13}+$$

$+ 2f_o'f_{13}']\bar{a}_{13}\}+i\{(\eta\bar{f}_{11}''+3\bar{f}_{11}')a_{13}+$

$(\eta\bar{f}_{13}''+3\bar{f}_{13}')a_{11}\}$

....

....

For n=3,7,...we have from (108) $\tilde{H}_n$ = pure imaginary quantity.

And since we are actually seeking the real part of $\tilde{H}_n$ we set for n=3,7,...

$$\tilde{H}_n \equiv 0 \quad : \quad n=2,4,6,\ldots \qquad \text{(XXI.6)}$$

and for n=2,4,6,...it follows from (110) that

$$\tilde{H}_n \equiv 0 \quad : \quad n=2,4,6,\ldots \qquad \text{(XXI.7)}$$

Next substitution of the expansions (103), (104), (112), and (113) into Eq. (89-2a), using the relation (111) to simplify, and then equating the terms which are coefficients of $e^{2i\tau}$ with each other and the time independent terms with each other, (since no other terms appear) gives for the coefficients $e^{2i\tau}$:

$$\sum_{n=1}^{\infty} f_{2n}\xi^n = \sum_{n=1}^{\infty} a_{2n}\xi^n - \frac{i}{4}\sum_{n=1}^{\infty} a_{1n}+4i(X_e-1)\sum_{n=1}^{\infty} \frac{a_{2n}}{n+2}\xi^n$$

or:

$$\sum_{n=1}^{\infty} f_{2n}\xi^n = \sum_{n=1}^{\infty} \{a_{2n}-\frac{i}{4}a_{1n}\} + 4i(X_e-1)\sum_{n=3}^{\infty} \frac{a_{2(n-2)}}{n}\xi^n$$

and for the time independent terms:

$$\sum_{n=1}^{\infty} \tilde{f}_{2n}\xi^n = \sum_{n=1}^{\infty} \tilde{a}_{2n}\xi^n - \frac{i}{4}\sum_{n=1}^{\infty} \bar{a}_{1n}\xi^n$$

And equating the coefficients of like powers of ξ in these last two equations we get:

$$f_{2n} = a_{2n} - \frac{i}{4} a_{1n} + \begin{cases} 0 & n=1,2, \\ 4i(X_e-1)a_{2(n-2)}/n & n=3,4,5,7 \end{cases} \qquad \text{(XXI.8)}$$

$$\tilde{f}_{2n} = \tilde{a}_{2n} - \frac{i}{4} \bar{a}_{1n} \qquad n=1,2,3,\ldots \qquad \text{(XXI.9)}$$

And finally substitution of the expansions (103), (104), (112), and (113) into Eq. (90-2a), using the relation (111) to simplify and then equating the terms which are coefficients of $e^{2i\tau}$ with each other and the time independent terms with each other (since no other terms appear) gives for the coefficients of $e^{2i\tau}$:

$$2a\xi(1-X_e f_0') \Bigg[a\sum_{n=1}^{\infty} \{(n+1)a_{2n}+4iX_e a_{2n}\xi^2\}\xi^n$$

$$+ a\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n}(iX_e a_{1m}\xi + \frac{m}{2} a_{1m}\xi^{-1})\xi^{n+m}$$

$$- \frac{ia}{4}\sum_{n=1}^{\infty} (n+1)a_{1n}\xi^n \Bigg] + a^2 \Bigg[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{1n}$$

$$-X_e f_{1n}')\{(m+1)a_{1m}+2iX_e\xi^2\}\xi^{n+m}$$

$$- \frac{i}{2} \sum_{n=1}^{\infty} (a_{1n}-f_{1n}')\xi^{n+1} \Bigg] + \frac{1}{2}(2a)^2\xi \sum_{n=1}^{\infty} (a_{2n}-X_e f_{2n}')\xi^n$$

$$- \frac{i}{8}(2a)^2\xi \sum_{n=1}^{\infty} \{(n+2)a_{1n}+2iX_e a_{1n}\xi^2\}\xi^n$$

$$- \frac{(2a)^2}{16} \xi^2 = \xi \sum_{n=1}^{\infty} f_{2n}'' \xi^n$$

Simplifying we get:

$$\frac{1}{2}(2a)^2 \left\{ (1-X_e f_o') \sum_{n=1}^{\infty} [(n+1)(a_{2n} - \frac{i}{4} a_m) + \right.$$

$$4i(1-\delta_{1n})(1-\delta_{2n})a_{2(n-2)}]\xi^n +$$

$$+ (1-X_e f_o') \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{1n}\{(1-\delta_{1n})(1-\delta_{2n}) iX_e a_{1(m-2)}$$

$$+ \frac{1}{2} m\ a_{1m}\}\xi^{m+n-1} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{1n} - X_e f_{1n}') \ [(m+1)a_{1m}$$

$$+ 2iX_e(1-\delta_{1m})(1-\delta_{2m})a_{1(m-2)} - \frac{i}{2}\delta_{1m}]\xi^{n+m-1}$$

$$+ \sum_{n=1}^{\infty} (a_{2n} - X_e f_{2n}')\xi^n - \frac{i}{4} \sum_{n=1}^{\infty} [(n+2)a_{1n} +$$

$$\left. 2iX_e(1-\delta_{1n})(1-\delta_{2n})a_{1(n-2)}]\xi^n - \frac{\xi}{8} \right\} = \sum_{n=1}^{\infty} f_{2n}''\xi^n$$

or:

$$\frac{1}{2}(2a)^2 \left\{ \sum_{n=1}^{\infty} \{(1-X_e f_o') \ [(n+1)(a_{2n} - \frac{i}{4} a_{1n}) + \right.$$

$$(1-\delta_{1n})(1-\delta_{1m})4iX_e a_{2(n-2)}] - \frac{i}{2}(a_{1n} - X_e f_{1n}') +$$

$$(a_{2n} - X_e f_{2n}') - \frac{i}{4}[(n+2)a_{1n} + 2i(1-\delta_{1n})(1-\delta_{2n})X_e a_{1(n-2)}]$$

$$\left. - \frac{\delta_{1n}}{8} \right\}\xi^n +$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \{(1-X_e f_o')a_{1n} \ [(1-\delta_{1m})(1-\delta_{2m}) iX_e a_{1(m-2)}$$

$$+ \frac{1}{2} m\, a_{1m}] + (a_{1n} - X_e f_{1n}')[(m+1)a_{1m} +$$

$$\left. 2iX_e(1-\delta_{1m})(1-\delta_{2m})a_{1(m-2)}]\}\ \xi^{n+m-1} \right] = \sum_{n=1}^{\infty} f_{2n}'' \xi^n$$

and equating the coefficients of like powers of ξ gives:

$$\frac{1}{2}(2a)^2 \left\{ (1-X_e f_0')[(n+1)(a_{2n} - \frac{i}{4} a_{1n}) + \right.$$

$$+ 4i(1-\delta_{1n}-\delta_{2n})a_{2(n-2)}] - \frac{i}{2}(a_{1n} - X_e f_{1n}') + (a_{2n} - X_e f_{2n}')$$

$$\left. - \frac{i}{4}[(n+2)a_{1n} + 2i(1-\delta_{1n}-\delta_{2n})X_e a_{1(n-2)}] - \frac{\delta_{1n}}{8} + C_n \right\} = f_{2n}'' \qquad \text{(XXI.10)}$$

where:

$$\left.\begin{aligned}
C_1 &= (1-X_e f_0')\frac{a_{11}a_{11}}{2} + (a_{11} - X_e f_{11}')2a_{11} \\
C_3 &= (1-X_e f_0')\{a_{11}[\tfrac{3}{2} a_{13} + iX_e a_{11}] + \tfrac{1}{2} a_{13}a_{11}\} \\
&\quad + (a_{11} - X_e f_{11}')[4a_{13} + 2iX_e a_{11}] + 2a_{11}(a_{13} - X_e f_{13}') \\
C_5 &= (1-X_e f_0')\{a_{11}[\tfrac{5}{2} a_{15} + iX_e a_{13}] + \\
&\quad a_{13}[\tfrac{3}{2} a_{13} + iX_e a_{11}] + \tfrac{1}{2} a_{11}a_{15}\} + \\
&\quad (a_{11} - X_e f_{11}')[6a_{15} + 2iX_e a_{13}] + (a_{13} - X_e f_{13}')[4a_{13} \\
&\quad + 2iX_e a_{11}] + 2a_{11}(a_{15} - X_e f_{15}')
\end{aligned}\right\} \qquad \text{(XXI.11)}$$

....

....

And from (108) we have

$$C_n = 0 \quad : \quad \text{for} \quad n=2,4,6,\ldots \qquad \text{(XXI.12)}$$

And similarily one obtains for the time independent terms:

$$\frac{1}{2}(2a)^2\{(1-X_e f_o')(n+1)(\tilde{a}_{2n} - \frac{i}{4}\bar{a}_{1n}) - \frac{i}{2}(\bar{a}_{1n}-\bar{f}_{1n}')$$

$$+ (\tilde{a}_{2n}-X_e\tilde{f}_{2n}') - \frac{i}{4}[(n+2)\bar{a}_{1n}-2i(1-\delta_{1n}-\delta_{2n})X_e\bar{a}_{1(n-2)}]$$

$$+ \frac{\delta_{1n}}{8} + \tilde{C}_n\} = \tilde{f}_{2n}'' \qquad \text{(XXI.13)}$$

where:

$$\tilde{C}_1 = (1-X_e f_o')\frac{1}{2}\bar{a}_{11}a_{11}+2a_{11}(\bar{a}_{11}-X_e\bar{f}_{11}')$$

$$\tilde{C}_5 = (1-X_e f_o')\{\bar{a}_{11}[\frac{5}{2}a_{15}+iX_e a_{13}]+$$

$$\bar{a}_{13}[\frac{3}{2}a_{13}+iX_e a_{11}]+\frac{1}{2}a_{11}\bar{a}_{15}\}+$$

$$(\bar{a}_{11}-X_e\bar{f}_{11}')[6a_{15}+2iX_e a_{13}]+(\bar{a}_{13}-X_e\bar{f}_{13}')[4a_{13}$$

$$+ 2iX_e a_{11}]+(\bar{a}_{15}-X_e\bar{f}_{15}')2a_{11} \qquad \text{(XXI.14)}$$

....

....

And as before inspection of (110) shows that $\tilde{C}_3$, $\tilde{C}_1$, are pure imaginary and since we are only interested in the real part of $\tilde{f}_{2n}$ we can set:

$$\tilde{C}_n = 0 \quad : \quad \text{for } n=3,7,\ldots \qquad \text{(XXI.15)}$$

and (108) shows further that

$$\tilde{C}_n = 0 \quad : \quad \text{for} \quad n=2,4,6,\ldots \qquad \text{(XXI.16)}$$

The boundary condition (21) shows that we must have:

$$\left.\begin{array}{lll} \text{at } \eta = 0; & f_{2n} = \dfrac{df_{2n}}{d\eta} = 0 \\ \text{at } \eta = 0; & \tilde{f}_{2n} = \dfrac{d\tilde{f}_{2n}}{d\eta} = 0 \end{array}\right\} \qquad (XXI.17)$$

Inspection of Eqs. (XXI.1) through (XXI.17) and use of the condition (108) shows that a_{2n}, f_{2n} satisfy a homogeneous differential equation with homogeneous boundary conditions for n=2 and therefore a_{22}, $f_{22} = 0$. This then implies that the higher order f_{2n} for n=4,6,...satisfy a homogeneous equation with homogeneous boundary conditions and hence we have:

$$a_{2n},\ f_{2n} \equiv 0 \quad : \quad \text{for } n=2,4,6,\ldots$$

and similarly:

$$\tilde{a}_{2n},\ \tilde{f}_{2n} \equiv 0 \quad : \quad \text{for } n=2,4,6,\ldots$$

But since we are only interested in the real part of $\tilde{f}_{2n}$, $\tilde{a}_{2n}$ we see that these also satisfy homogeneous conditions for n=3,7,... Hence one has:

$$\tilde{a}_{2n},\ \tilde{f}_{2n} \equiv 0; \quad \text{for } n=3,7,\ldots$$

Thus we have upon introducing the operator L_n defined by (107):

For n=1,3,5,...:

$$0 \leq \eta \leq 1 \qquad L_n(f_{2n}) = -[(n-1)f_o'^2 + f_o f_o'']a_{2n} + H_n$$

$$\eta = 1 \left\{ \begin{aligned} f_{2n} &= a_{2n} - \frac{i}{4} a_{1n} + 4i(1-\delta_{1n})(X_e - 1) \frac{a_{2(n-2)}}{n} \\ \frac{1}{2a^2} f''_{2n} &= (1 - X_e f'_o)[(n+1)(a_{2n} - \frac{i}{4} a_{1n}) + \\ & 4i(1-\delta_{1n}) a_{2(n-2)}] - \frac{i}{2}(a_{1n} - X_e f'_{1n}) + a_{2n} \\ & - X_e f'_{2n} - \frac{i}{4}[(n+2) a_{1n} + 2i(1-\delta_{1n}) X_e a_{1(n-2)}] \\ & - \frac{\delta_{1n}}{8} + C_n \end{aligned} \right.$$

$$\eta = 0 \qquad f_{2n} = f'_{2n} = 0$$

and:

$$a_{2n},\ f_{2n}(\eta) \equiv 0 \quad : \quad \text{for } n=2,4,6,\ldots$$

For n=1,5,...

$$0 \leq \eta \leq 1 \qquad L_n(\tilde{f}_{2n}) = -[(n-1)f_o'^2 + f_o f''_o]\tilde{a}_{2n} + \tilde{H}_n$$

$$\eta = 1 \left\{ \begin{aligned} \tilde{f}_{2n} &= \tilde{a}_{2n} - \frac{i}{4} \bar{a}_{1n} \\ \frac{1}{2a^2} \tilde{f}''_{2n} &= (1 - X_e f'_o)(n+1)(\tilde{a}_{2n} - \frac{i}{4} \bar{a}_{1n}) - \frac{i}{2}(\bar{a}_{1n} \\ & - \bar{f}'_{1n}) + \tilde{a}_{2n} - X_e \tilde{f}'_{2n} - \frac{i}{4}[(n+2)\bar{a}_{1n} - 2i(1-\delta_{1n}) X_e \bar{a}_{1(n-2)}] \\ & + \frac{\delta_{1n}}{8} + \tilde{C}_n \end{aligned} \right.$$

$$\eta = 0 \qquad \tilde{f}_{2n} = \tilde{f}_{2n} = 0$$

and:

$$a_{2n},\ f_{2n}(\eta) \equiv 0; \quad \text{for } n=2,4,6,\ldots \quad \text{and} \quad n=3,7,\ldots$$

Where H_n, $\tilde{H}_n$, C_n and $\tilde{C}_n$ are given by (XXI.3), (XXI.5), (XXI.11) and (XXI.14) respectively.

APPENDIX XXII

REDUCTION OF THE SECOND-ORDER ENERGY PROBLEM TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Substitution of the expansions (103) through (105), and (112) through (114) into Eq. (96-2a), using the relation (111) to simplify, and then equating the terms which are coefficients of $e^{2i\tau}$ with each other and the time independent terms with each other (since no other terms appear as can easily be varified by inspection) gives for the coefficient of $e^{2i\tau}$:

$$\sum_{n=0}^{\infty}\left\{\left[\frac{1}{2a^2 PrXe} F''_{2n} + f_o F'_{2n} - f'_o n F_{2n}\right]\right\}\xi^n - i\, 2\cdot 2 \sum_{n=0}^{\infty} F_{2n}\, \xi^{n+2}$$

$$= -\sum_{n=1}^{\infty} F'_o(n\, f_{2n} + f_o\, a_{2n})\xi^{n-1} - 4i\, F'_o \sum_{n=1}^{\infty} \eta\, a_{2n}\, \xi^{n+1}$$

$$- \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m\, a_{1n} f_{1m} F'_o \xi^{n+m-2} - i\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \eta\, a_{1n} a_{1m} F'_o\, \xi^{n+m}$$

$$+ 2i\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{1n} F_{1m}\, \xi^{n+m+1} + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} [f'_{1n} + f'_o a_{1n}] m F_{1m} \xi^{n+m-1}$$

$$- \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} [f_{1n} n + f_o a_{1n}] F'_{1m} \xi^{n+m-1} - i\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \eta a_{1n} F'_{1m} \xi^{n+m+1}$$

or using (108), (110), (118) and rearranging:

$$\sum_{n=0}^{\infty}\left\{\left[\frac{1}{2a^2 X_e Pr} F''_{2n} + f_o F'_{2n} - n f'_o F_{2n}\right]\right\}\xi^n = 4i\sum_{n=0}^{\infty}(1-\delta_{on}-\delta_{1n}) F_{2(n-2)}\xi^n$$

$$-F_o'\sum_{n=0}^{\infty}\{(n+1)f_{2(n+1)}+f_o a_{2(n+1)}+4i\eta(1-\delta_{on})a_{2(n-1)}\}\xi^{n}$$

$$-\frac{1}{2}F_o'\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{ma_{1n}f_{1m}+2i\eta(1-\delta_{1m})a_{1n}a_{1(m-2)}\}\xi^{n+m-2}$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\{mF_{1m}(f_{1n}'+f_o'a_{1n})-F_{1m}'(f_{1n}n+f_o a_{1n})\}\xi^{n+m-1}$$

$$+i\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\{2a_{1n}F_{1m}-\eta a_{1n}F_{1m}'\}\xi^{n+m+1}$$

And for the time independent terms:

$$\sum_{n=0}^{\infty}\left\{\left[\frac{1}{2a^2P_rX_e}\tilde{F}_{2n}''+f_o\tilde{F}_{2n}'-nf_o'\tilde{F}_{2n}\right]\right\}\xi^{n}=-F_o'\sum_{n=1}^{\infty}(n\tilde{f}_{2n}+f_o\tilde{a}_{2n})\xi^{n-1}$$

$$-\frac{1}{2}F_o'\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{1n}\bar{f}_{1m}\,m\,\xi^{n+m-2}-i\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}F_o'\,\eta\bar{a}_{1n}a_{1m}\xi^{n+m}$$

$$+2i\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\bar{a}_{1n}F_{1m}\xi^{n+m+1}+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}[\bar{f}_{1n}'+f_o'\bar{a}_{1n}]mF_{1m}\xi^{n+m-1}$$

$$-\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}[\bar{f}_{1n}n+f_o\bar{a}_{1n}]F_{1m}'\xi^{m+n-1}-i\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\eta a_{1n}\bar{F}_{1m}'\xi^{n+m+1}$$

And using (108), (110), and (123) and rearranging we get:

$$\sum_{n=0}^{\infty}\left\{\left[\frac{1}{2a^2X_eP_r}\tilde{F}_{2n}''+f_o\tilde{F}_{2n}'-nf_o'\tilde{F}_{2n}\right]\right\}\xi^{n}=-F_o'\sum_{n=0}^{\infty}\{(n+1)\tilde{f}_{2(n+1)}+f_o\tilde{a}_{2(n+1)}\}\xi^{n}$$

$$-\frac{1}{2}F_o'\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{ma_{1n}\bar{f}_{1m}+2i\eta(1-\delta_{1m})\bar{a}_{1n}a_{1(m-2)}\}\xi^{n+m-2}$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\{mF_{1m}(\overline{f}'_{1n}+f'_o\overline{a}_{1n})-F'_{1m}(\overline{f}_{1n}n+f_o\overline{a}_{1n})\}\xi^{m+n-1}$$

$$+\,i\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\{2\overline{a}_{1n}F_{1m}-\eta a_{1n}\overline{F}'_{1m}\}\xi^{n+m+1}$$

And upon equating the coefficients of like powers of ξ and using (108), (110), (118), and (123), to simplify, one obtains:

$$\frac{1}{2a^2P_rX_e}F''_{2n}+f_oF'_{2n}-nf'_oF_{2n} = 4i(1-\delta_{on}-\delta_1)F_{2(n-2)}$$

$$-F'_o[(n+1)f_{2(n+1)}+f_oa_{2(n+1)}+4i(1-\delta_{on})\eta a_{2(n-1)}]+E_n \qquad \text{(XXII.1)}$$

where for $n = 0,2,4,\ldots$

$$E_o = -\frac{1}{2}F'_o\,a_{11}f_{11}-\frac{1}{2}F'_{10}(f_{11}+f_oa_{11})$$

$$E_2 = -\frac{1}{2}\{3a_{11}f_{13}+2i\eta a_{11}a_{11}+a_{13}f_{11}\}F'_o+\frac{1}{2}\{2F_{12}(f'_{11}+f_oa_{11})$$

$$-F'_{12}(f_{11}+f_oa_{11})-F'_{10}(3f_{13}+f_oa_{13})\}+ia_{11}(2F_{10}-\eta F'_{10}) \qquad \text{(XXII.2)}$$

$$E_4 = -\frac{1}{2}\{5a_{11}f_{15}+4i\eta a_{11}a_{13}+3a_{13}f_{13}+a_{15}f_{11}\}F'_o$$

$$+\frac{1}{2}\{4F_{14}(f'_{11}+a_{11}f'_o)-F'_{14}(f_{11}+f_oa_{11})+2F_{12}(f'_{13}+f'_oa_{13})-F'_{12}(f_{13}3+f_oa_{13})$$

$$-F'_{10}(5f_{15}+f_oa_{15})\}+ia_{11}(2F_{12}-\eta F'_{12})+ia_{13}(2F_{10}-\eta F'_{10})$$

....

....

and:

$$E_n = 0;\ \text{for } n = 1,3,5,\ldots \qquad \text{(XXII.3)}$$

$$\frac{1}{2a^2 X_e P_r}\tilde{F}''_{2n}+f_o\tilde{F}'_{2n}-nf'_o\tilde{F}_{2n} = -F'_o\{(n+1)\tilde{f}_{2(n+1)}+f_o\tilde{a}_{2(n+1)}\}+\tilde{E}_n \tag{XXII.4}$$

where for $n = 0,4,8,\ldots$

$$\tilde{E}_o = -\frac{1}{2}F'_o a_{11}\bar{f}_{11}-\frac{1}{2}F'_{10}(\bar{f}_{11}+f_o\bar{a}_{11})$$

$$\tilde{E}_4 = -\frac{1}{2}\{5a_{11}\bar{f}_{15}+3a_{13}\bar{f}_{13}+a_{15}\bar{f}_{11}\}F'_o+\frac{1}{2}\{4F_{14}(\bar{f}_{11}+\bar{a}_{11}f_o) \tag{XXII.5}$$

$$-F'_{14}(\bar{f}_{11}+f_o a_{11})+2F_{12}(\bar{f}'_{13}+f'_o\bar{a}_{13})-F'_{12}(3\bar{f}_{13}+f_o\bar{a}_{13})-F'_{10}(5\bar{f}_{15}$$

$$+f_o\bar{a}_{15})\}+i\bar{a}_{11}(2F_{12}+\eta F'_{12})+ia_{13}(2F_{10}-\eta\bar{F}'_{10})$$

....

....

and $\tilde{E}_n = 0$

for $n\neq 0,4,8,\ldots$

since we are taking the real parts and $\tilde{E}_2$, $\tilde{E}_6,\ldots$are pure imaginary and hence contribute nothing.

Next substitution of the expansion (103) through (105), and (112) through (114) into Eq. (97-2a), using the relation (111) to simplify and then equating the terms which are coefficients of $e^{2i\tau}$ with each other and the time independent terms with each other (since no other terms appear) gives for the coefficients of $e^{2i\tau}$:

$$\sum_{n=0}^{\infty}\frac{1}{2a^2P_rX_e}F'_{2n}+F_{2n}+\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}F_{1n}\{(1+m)a_{1m}$$

$$+2iX_e\xi^2 a_{1m}\}\xi^{n+m-1}$$

$$+ F_o \sum_{n=1}^{\infty} \{(1+n)a_{2n} + 4iX_e a_{2n}\xi^2\}\xi^{n-1}$$

$$+ F_o \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n}(iX_e\xi^2 a_{1m} + \frac{m}{2} a_{1m})\xi^{n+m-2}$$

$$- \frac{i}{4} \sum_{n=0}^{\infty} F_{1n}\xi^n - \frac{i}{4} F_o \sum_{n=1}^{\infty} (1+n)a_{1n}\xi^{n-1}$$

And using (108), (110), and (118) and rearranging we get:

$$\sum_{n=0}^{\infty} \left\{\left[\frac{1}{2a^2P_rX_e} F'_{2n}+F_{2n}\right\} + \sum_{n=0}^{\infty} [F_o\{(2+n)(a_{2(n+1)} - \frac{i}{4} a_{1(n+1)})+4iX_e\right.$$

$$(1-\delta_{on})a_{2(n-1)}\}$$

$$- \frac{i}{4} F_{1n}]\xi^n + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} F_{1n} \{(1+m)a_{1m}+2iX_e a_{1m}\xi^2\}\xi^{n+m-1}$$

$$+ F_o \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_1 (\frac{m}{2} a_{1m} + iX_e a_{1m}\xi^2)\xi^{n+m-2}$$

and upon equating the coefficients of like powers of ξ and using the relations (108), (110), and (118) we get:

$$\frac{1}{2a^2P_rX_e} F'_{2n} + F_{2n} + F_o \{(2+n)(a_{2(n+1)} - \frac{i}{4} a_{1(n+1)}) + 4iX_e(1-\delta_{on})a_{2(n-1)}\}$$

$$- \frac{i}{4} F_{1n} + B_n = 0 \qquad \text{(XXII.7)}$$

where for n=0,2,4,...

$$B_o = a_{11}F_{1o} + \frac{1}{2} a_{11}a_{11}F_o$$

$$B_2 = \frac{1}{2} F_{1o} (4a_{13}+2iX_e a_{11}) + \frac{1}{2} 2 a_{11}F_{12}$$

$$B_4 = \frac{1}{2} F_{1o} (6a_{15}+2iX_e a_{13}) + \frac{1}{2} F_{12}(4a_{13}+2iX_e a_{11}) + a_{11}F_{14} \qquad \text{(XXII.8)}$$

$$+ a_{11} (\frac{5}{2} a_{15}+iX_e a_{13})F_o + a_{13} (\frac{3}{2} a_{13}+iX_e a_{11})F_o + \frac{1}{2} a_{15}a_{11}F_o$$

....

....

and

$$B_n = 0 \quad \text{for} \quad n=1,3,5,\ldots \qquad \text{(XXII.9)}$$

and similarly for the time independent terms, we get:

$$\frac{1}{2a^2 P_r X_e} \tilde{F}'_{2n} + \tilde{F}_{2n} + F_o\{(2+n)(\tilde{a}_{2(n+1)} - \frac{i}{4} \bar{a}_{1(n+1)}\} - \frac{i}{4} \bar{F}_{1n} + \tilde{B}_n = 0 \qquad \text{(XXII.10)}$$

where for n=0,4,...

$$\left\{\begin{array}{l} \tilde{B}_o = a_{11}\bar{F}_{1o} + \frac{1}{2} a_{11}\bar{a}_{11}F_o \\ \tilde{B}_4 = \frac{1}{2} \bar{F}_{1o}(6a_{15}+2iX_e a_{13}) + \frac{1}{2} \bar{F}_{12}(4a_{13}+2iX_e a_{11}) \\ + a_{11}\bar{F}_{14} + \bar{a}_{11}(\frac{5}{2} a_{15}+iX_e a_{13})F_o + \bar{a}_{13}(\frac{3}{2} a_{13}+iX_e a_{11})F_o \\ + \frac{1}{2} a_{11}\bar{a}_{15}F_o \end{array}\right\} \qquad \text{(XXII.11)}$$

....

....

and

$$\tilde{B}_n = 0; \text{ for } n = 1,3,5,\ldots$$

but since we are taking real parts and $\tilde{B}_2$, $\tilde{B}_6$...are pure imaginary, we can set them equal to zero and we have:

$$\tilde{B}_n = 0 \quad \text{for} \quad n \neq 0,4,8,... \qquad \text{(XXII.12)}$$

and finally the boundary conditions (98-2) require that:

$$\left.\begin{aligned} \eta &= 0; \quad F_{2n} = 0 \\ \eta &= 0; \quad \tilde{F}_{2n} = 0 \end{aligned}\right\} \qquad \text{(XXII.13)}$$

Inspection of Eqs. (XXII.1) through (XXII.13) and use of Eqs. (108), (110), (118), and (123) shows that F_{21} satisfies a homogeneous differential equation with homogeneous boundary conditions for n=2 and therefore $F_{21}(\eta) \equiv 0$. This then implies that the higher order F_{2n} for n=3,5,7,...satisfy homogeneous equations with homogeneous boundary conditions and hence:

$$F_{2n}(\eta) \equiv 0; \qquad \text{for } n = 1,3,5,...$$

and similarly:

$$\tilde{F}_{2n}(\eta) \equiv 0; \qquad \text{for } n = 1,3,5,...$$

But since we are only interested in the real part of $\tilde{F}_{2n}$ we see that they also satisfy homogeneous conditions for n=2,6,...Hence we have:

$$\tilde{F}_{2n}(\eta) \equiv 0; \qquad \text{for } n \neq 0,4,8,...$$

Thus the relations of this appendix may be summarized introducing the operator K_n (defined below Eqs. (109)) by:

For n = 0,2,4,...

$$0 \leq \eta \leq 1 \quad K_n(F_{2n}) = 4i(1-\delta_{on})F_{2(n-2)} - F_o'[(n+1)f_{2(n+1)}$$

$$+ f_o a_{2(n+1)} + 4i(1-\delta_{on})\eta a_{2(n-1)}] + E_n$$

$$\eta = 1 \qquad \frac{1}{2a^2 P_r X_e} F_{2n}' + F_{2n} + F_o\{(2+n)(a_{2(n+1)} - \frac{i}{4} a_{1(n+1)})$$

$$+ 4iX_e(1-\delta_{on})a_{2(n-1)}\} - \frac{i}{4} F_{1n} + B_n = 0$$

$$\eta = 0 \qquad F_{2n} = 0$$

and

$$F_{2n}(\eta) \equiv 0; \quad \text{for} \quad n = 1,3,5,\ldots$$

for $n=0,4,\ldots$

$$0 \leq \eta \leq 1 \quad K_n(\tilde{F}_{2n}) = -F_o' \{(n+1)\tilde{f}_{2(n+1)} + f_o \tilde{a}_{2(n+1)}\} + E_n$$

$$\frac{1}{2a^2 P_r X_e} \tilde{F}_{2n}' + \tilde{F}_{2n} + F_o(2+n)(\tilde{a}_{2(n+1)} - \frac{i}{4} \bar{a}_{1(n+1)})$$

$$\eta = 1 \qquad - \frac{i}{4} \bar{F}_{1n} + \tilde{B}_n = 0$$

$$\eta = 0 \qquad \tilde{F}_{2n} = 0$$

and

$$\tilde{F}_{2n}(\eta) \equiv 0; \quad \text{for } n \neq 0,4,8,\ldots$$

Where E_n, $\tilde{E}_n$, B_n and $\tilde{B}_n$ are given by (XXII.2), (XXII.5), (XXII.8), and (XXII.11), respectively.

APPENDIX XXIII

REDUCTION OF THE FORMULAS FOR VELOCITY, SHEAR STRESS, AND NUSSELT NUMBER

Using the expansions (87) in the Eq. (133) gives:

$$\frac{u}{U_\infty} = X_e[\psi^*_{o\eta}+\epsilon\psi^*_{1\eta}+\epsilon^2\psi^*_{2\eta}+\ldots][\delta^*_o+\epsilon\delta^*_1+\epsilon^2\delta^*_2+\ldots]^{-1}$$

$$= \frac{X_e}{\delta^*_o}[\psi^*_{o\eta}+\epsilon\psi^*_{1\eta}+\epsilon^2\psi^*_{2\eta}+\ldots]\left[1+\epsilon\frac{\delta^*_1}{\delta^*_o}+\epsilon^2\frac{\delta^*_2}{\delta^*_o}+\ldots\right]^{-1}$$

$$= \frac{X_e}{\delta^*_o}[\psi^*_{o\eta}+\epsilon\psi^*_{1\eta}+\epsilon^2\psi^*_{2\eta}+\ldots]\left[1-\epsilon\frac{\delta^*_1}{\delta^*_o}-\epsilon^2\frac{\delta^*_2}{\delta^*_o}+\epsilon^2\frac{\delta^{*2}_1}{\delta^{*2}_o}+\ldots\right]$$

$$= \frac{X_e}{\delta^*_o}[\psi^*_{o\eta}+\epsilon\psi^*_{1\eta}+\epsilon^2\psi^*_{2\eta}]\left[1-\epsilon\frac{\delta^*_1}{\delta^*_o}-\epsilon^2\left(\frac{\delta^*_2}{\delta^*_o}-\frac{\delta^{*2}_1}{\delta^{*2}_o}\right)+\ldots\right] \equiv u^*_o+\epsilon u^*_1+\epsilon^2 u^*_2+\ldots$$

where:

$$u^*_o = X_e\frac{\psi^*_{o\eta}}{\delta^*_o}$$

$$u^*_1 = \frac{X_e}{\delta^*_o}\left[\psi^*_{1\eta}-\frac{\delta^*_1}{\delta^*_o}\psi^*_{o\eta}\right]$$

$$u^*_2 = \frac{X_e}{\delta^*_o}\left[\psi^*_{2\eta}-\frac{\delta^*_1}{\delta^*_o}\psi^*_{1\eta}-\left(\frac{\delta^*_2}{\delta^*_o}-\frac{\delta^{*2}_1}{\delta^{*2}_o}\right)\psi^*_{o\eta}\right]$$

$$= \frac{X_e}{\delta^*_o}\left[\psi^*_{2\eta}-\frac{\delta^*_2}{\delta^*_o}\psi^*_{o\eta}+\frac{\delta^*_1}{\delta^*_o}\left(\frac{\delta^*_1}{\delta^*_o}\psi^*_{o\eta}-\psi^*_{1\eta}\right)\right]$$

....

....

Using the expansions (87) in Eq. (134) gives:

$$\frac{\tau_w\sqrt{\frac{U_\infty x}{\nu}}}{X_e 1/2\rho U_\infty^2} = 2\xi[\psi^*_{o\eta\eta}+\epsilon\psi^*_{1\eta\eta}+\epsilon^2\psi^*_{2\eta\eta}+\dots][\delta^*_o+\epsilon\delta^*_1+\epsilon^2\delta^*_2+\dots]^{-2}$$

$$= \frac{2\xi}{\delta_o^{*2}}\,[\psi^*_{o\eta\eta}+\epsilon\psi^*_{1\eta\eta}+\epsilon^2\psi^*_{2\eta\eta}+\dots]\left[1+\frac{\epsilon\delta^*_1}{\delta^*_o}+\epsilon^2\frac{\delta^*_2}{\delta^*_o}+\dots\right]^{-2}$$

$$= \frac{2\xi}{\delta_o^{*2}}\,[\psi^*_{o\eta\eta}+\epsilon\psi^*_{1\eta\eta}+\epsilon^2\psi_{2\eta\eta}+\dots]\left[1-\frac{2\epsilon\delta^*_1}{\delta^*_o}-\frac{2\epsilon^2\delta^*_2}{\delta^*_o}+\right.$$

$$\left.+3\epsilon^2\frac{\delta_1^{*2}}{\delta_o^{*2}}+\dots\right]$$

$$= \frac{2\xi}{\delta_o^{*2}}\,[\psi^*_{o\eta\eta}+\epsilon\psi^*_{1\eta\eta}+\epsilon^2\psi^*_{2\eta\eta}+\dots]\left[1-\frac{2\epsilon\delta^*_1}{\delta^*_o}-\epsilon^2\left(\frac{2\delta^*_2}{\delta^*_o}-\right.\right.$$

$$\left.\left.-\frac{3\delta_1^{*2}}{\delta_o^{*2}}\right)+\dots\right] = \tau^*_o+\epsilon\,\tau^*_1+\epsilon^2\,\tau^*_2+\dots$$

$$\tau^*_o = \frac{2\xi\psi^*_{o\eta\eta}}{\delta_o^{*2}}$$

$$\tau^*_1 = \frac{2\xi}{\delta_o^{*2}}\left[\psi^*_{1\eta\eta}-\frac{2\delta^*_1}{\delta^*_o}\psi^*_{o\eta\eta}\right]$$

$$\tau^*_2 = \frac{2\xi}{\delta_o^{*2}}\left[\psi^*_{2\eta\eta}-\frac{2\delta^*_1}{\delta^*_o}\psi^*_{1\eta\eta}-\left(\frac{2\delta^*_2}{\delta^*_o}-\frac{3\delta_1^{*2}}{\delta_o^{*2}}\right)\psi^*_{o\eta\eta}\right]$$

$$= \frac{2\xi}{\delta_o^{*2}}\left[\psi^*_{2\eta\eta}-\frac{2\delta^*_2}{\delta^*_o}\psi^*_{o\eta\eta}-\frac{\delta^*_1}{\delta^*_o}\left(2\psi^*_{1\eta\eta}-\frac{3\delta^*_1}{\delta^*_o}\psi^*_{o\eta\eta}\right)\right]$$

And using the expansions (87) in Eq. (136) gives:

$$\frac{N_{ux}}{\sqrt{X_e}\cdot\sqrt{\frac{U_\infty x}{\nu}}} = \frac{\xi}{\delta^*_o}\,[\theta^*_{o\eta}+\epsilon\theta^*_{1\eta}+\epsilon^2\theta^*_{2\eta}+\dots]\left[1-\epsilon\frac{\delta^*_1}{\delta^*_o}-\right.$$

$$\left.\epsilon^2\left(\frac{\delta^*_2}{\delta^*_o}\quad\frac{\delta_1^{*2}}{\delta_o^{*2}}\right)+\dots\right] = q^*_o+\epsilon q^*_1+\epsilon^2 q^*_2+\dots$$

where:

$$q_o^* = \frac{\xi}{\delta_o^*} \Theta_{o\eta}^*$$

$$q_1^* = \frac{\xi}{\delta_o^*} \Theta_{1\eta}^* - \frac{\delta_1^* \xi}{\delta_o^{*2}} \Theta_{o\eta}^*$$

$$q_2^* = \frac{\xi}{\delta_o^{*2}} \Theta_{2\eta}^* - \frac{\delta_2^* \xi}{\delta_o^{*2}} \Theta_{o\eta}^* - \frac{\delta_1^* \xi}{\delta_o^{*2}} \left(\Theta_{1\eta}^* - \frac{\delta_1^*}{\delta_o^*} \Theta_{o\eta}^* \right)$$

APPENDIX XXIV

CALCULATION FOR EXPRESSING THE VELOCITY, WALL SHEAR STRESS, AND NUSSELT NUMBER EXPLICITLY IN TERMS OF THE SOLUTIONS TO THE ORDINARY DIFFERENTIAL EQUATION

Using $\psi_o^* = 2a\,\xi\,f_o$ and $\delta_o^* = 2a\,\xi$ in the Eqs. (140) we get:

$$u_o^* = X_e f_o'$$

$$u_1^* = \frac{X_e}{2a\xi}\left[\psi_{1\eta}^* - f_o'\delta_1^*\right]$$

$$u_2^* = \frac{X_e}{2a\xi}\left[\psi_{2\eta}^* - f_o'\delta_2^* + \frac{\delta_1^*}{2a\xi}\left(f_o'\delta_1^* - \psi_{1\eta}^*\right)\right]$$

Since:

$$\psi_1^* = \mathcal{R}e\ 2a\sum_{n=1}^{\infty} f_{1n}\xi^n\, e^{i\tau}$$

$$\delta_1^* = \mathcal{R}e\ 2a\sum_{n=1}^{\infty} a_{1n}\xi^n\, e^{i\tau}$$

we set:

$$u_1^* = \mathcal{R}e\sum_{n=0}^{\infty} g_{1n}\xi^n\, e^{i\tau}$$

and we have:

$$\sum_{n=0}^{\infty} g_{1n}\,\xi^n = X_e\sum_{n=1}^{\infty}\left(f_{1n}' - f_o'\,a_{1n}\right)\xi^{n-1}$$

$$= X_e\sum_{n=0}^{\infty}\left(f_{1(n+1)}' - f_o'\,a_{1(n+1)}\right)\xi^n$$

and equating coefficients:

$$g_{1n} = X_e(f'_{1(n+1)} - f'_o\, a_{1(n+1)}); \text{ for } n = 0,2,4,6,\ldots$$

since $f_{1n} = a_{1n} = 0$ for $n = 0,2,4,6,\ldots$

Since:

$$\psi_2^* = \mathcal{R}e\ 2a \sum_{n=1}^{\infty} (f_{2n}\, e^{2i\tau} + \tilde{f}_{2n})\xi^n$$

$$\delta_2^* = \mathcal{R}e\ 2a \sum_{n=1}^{\infty} (a_{2n}\, e^{2i\tau} + \tilde{a}_{2n})\xi^n$$

we set:

$$u_2^* = \mathcal{R}e \sum_{n=0}^{\infty} (g_{2n}\, e^{2i\tau} + \tilde{g}_{2n})\xi^n$$

and we have, using (111) and equating the coefficients of $e^{2i\tau}$:

$$\sum_{n=0}^{\infty} g_{2n}\, \xi^n = X_e \sum_{n=1}^{\infty} (f'_{2n} - f'_o\, a_{2n})\xi^n$$

$$- \frac{X_e}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n}\, (f'_{1m} - f'_o\, a_{1m})\xi^{n+m-2}$$

$$= X_e \sum_{n=0}^{\infty} (f'_{2(n+1)} - f'_o\, a_{2(n+1)})\xi^n - \frac{X_e}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n}(f'_{1m}-f'_o a_{1m})\xi^{n+m-2}$$

and equating coefficients of like powers of ξ, we get:

$$g_{2n} = X_e(f'_{2(n+1)} - f'_o\, a_{2(n+1)}-h_n) \quad n = 0,2,4,6\ldots$$

$$h_o = \frac{1}{2}\, a_{11}(f'_{11}-f'_o a_{11})$$

$$h_2 = \frac{1}{2}\, [a_{13}(f'_{11}-f'_o a_{11})+a_{11}(f'_{13}-f'_o a_{13})]$$

$$= \frac{1}{2}[a_{13}f'_{11}+a_{11}f'_{13} - 2a_{11}a_{13}f'_o]$$

$$h_4 = \frac{1}{2}[a_{15}(f'_{11}-f'_oa_{11})+a_{13}(f'_{13}-f'_oa_{13})$$

$$+ a_{11}(f'_{15}-f'_oa_{15})]$$

$$= \frac{1}{2}[a_{15}f'_{11}+a_{13}f'_{13}+a_{11}f'_{15}-f'_o(2a_{11}a_{15}+a_{13}a_{13})]$$

Since:

$$f_{1n},\ a_{1n},\ f_{2n},\ a_{2n} = 0;\ \text{for } n = 0,2,4,\ldots$$

and similarily for the time independent part:

$$\tilde{g}_{2n} = X_e\ (\tilde{f}'_{2(n+1)}-f'_o\tilde{a}_{2(n+1)}-\tilde{h}_n);\quad n = 0,4,8,\ldots$$

$$\tilde{h}_o = \frac{1}{2}\bar{a}_{11}(f'_{11}-f'_oa_{11})$$

$$\tilde{h}_4 = \frac{1}{2}[\bar{a}_{15}f'_{11}+\bar{a}_{13}f'_{13}+\bar{a}_{11}f'_{15}-f'_o(2\bar{a}_{11}a_{15}+\bar{a}_{13}a_{13})]$$

Since:

$$\tilde{f}_{2n},\ \tilde{a}_{2n} = 0;\ \text{for } n\neq 1,5,9,\ldots$$

Collecting the results, we get:

$$u_o* = X_ef'_o;\ u_1* = \mathcal{R}e\sum_{n=0}^{\infty} g_{1n}\xi^n e^{i\tau};\ u_2* = \mathcal{R}e\sum_{n=0}^{\infty}(g_{2n}e^{2i\tau}+\tilde{g}_{2n})\xi^n$$

$$g_{1n} = X_e(f'_{1(n+1)}-f'_o\ a_{1(n+1)})\quad n = 0,2,4,\ldots$$

$$g_{2n} = Xe(f'_{2(n+1)}-f'_oa_{2(n+1)}-h_n)\quad n = 0,2,4,\ldots$$

$$h_o = \frac{1}{2}a_{11}(f'_{11}-f'_oa_{11})$$

$$h_2 = \frac{1}{2}\{a_{13}f'_{11}+a_{11}f'_{13}-2a_{11}a_{13}f'_o\}$$

$$h_4 = \frac{1}{2}\{a_{15}f'_{11}+a_{13}f'_{13}+a_{11}f'_{15}-f'_o(2a_{11}a_{15}+a_{13}a_{13})\}$$

....

....

$$\tilde{g}_{2n} = X_e(\tilde{f}'_{2(n+1)} - f'_o\,\tilde{a}_{2(n+1)} - \tilde{h}_n) \qquad n=0,4,\ldots$$

$$\tilde{h}_o = \frac{1}{2}\,\bar{a}_{11}(f'_{11} - f'_o a_{11})$$

$$\tilde{h}_4 = \frac{1}{2}\,\{\bar{a}_{15}f'_{11} + \bar{a}_{13}f'_{13} + \bar{a}_{11}f'_{15} - f_o(2\bar{a}_{11}a_{15} + \bar{a}_{13}a_{13})\}$$

....

....

using the relations:

$$\psi_o{}^* = 2a\xi f_o \quad \text{and} \quad \delta_o{}^* = 2a\xi$$

in Eqs. (141), we get:

$$\tau_o{}^* = \frac{1}{a}\,f''_o$$

$$\tau_1{}^* = \frac{1}{2a^2\xi}\,[\psi^*_{1\eta\eta} - 2\,f''_o\,\delta_1{}^*]$$

$$\tau_2{}^* = \frac{1}{2a^2\xi}\left[\psi^*_{2\eta\eta} - 2f''_o\delta_2{}^* - \frac{\delta^*_1}{2a\xi}\,(2\psi^*_{1\eta\eta} - 3f''_o\delta_1{}^*)\right]$$

Setting:

$$\tau_1{}^* = \frac{1}{a}\,\mathcal{R}e\sum_{n=0}^{\infty} e_{1n}\xi^n e^{i\tau}$$

We get from using the expansion (103) and (104) in the second of the above equations: $e_{1n} = f''_{1(n+1)} - 2f''_o\,a_{1(n+1)}$: $n=0,2,4,\ldots$ since a_{1n}, $f_{1n} \equiv 0$ for $n = 0,2,4,\ldots$ and setting:

$$\tau_2{}^* = \mathcal{R}e\,\frac{1}{a}\sum_{n=0}^{\infty}(e_{2n}\,e^{2i\tau} + \tilde{e}_{2n})\xi^n$$

We get by using the expansions (112) and (113), equating first time dependent and time independent parts and then the coefficients of powers of ξ

$$e_{2n} = f''_{2(n+1)} - 2f''_0 a_{2(n+1)} - r_n; \quad n=0,2,4,\ldots$$

$$r_0 = \frac{1}{2} a_{11}(2f''_{11} - 3f''_0 a_{11})$$

$$r_2 = \frac{1}{2}[2a_{13}f''_{11} + 2a_{11}f''_{13} - 6a_{11}a_{13}f''_0]$$

$$r_4 = \frac{1}{2}[2a_{15}f''_{11} + 2a_{13}f''_{13} + 2a_{11}f''_{15} - 3f''_0(2a_{11}a_{15} + a_{13}a_{13})]$$

$$\tilde{e}_{2n} = \tilde{f}''_{2(n+1)} - 2f''_0 \tilde{a}_{2(n+1)} - \tilde{r}_n \quad n=0,4,\ldots$$

$$\tilde{r}_0 = \frac{1}{2}\bar{a}_{11}(2f''_{11} - 3f''_0 a_{11})$$

$$\tilde{r}_4 = \frac{1}{2}[2\bar{a}_{15}f''_{11} + 2\bar{a}_{13}f''_{13} + 2\bar{a}_{11}f''_{15} - 3f''_0(2\bar{a}_{11}a_{15} + \bar{a}_{13}a_{13})]$$

since

$$a_{1n},\ f_{1n},\ a_{2n},\ f_{2n} \equiv 0; \text{ for } n=0,2,4,6,\ldots$$

and

$$\tilde{a}_{2n},\ \tilde{f}_{2n} \equiv 0 \qquad \text{for } n \neq 1,5,9,\ldots$$

And we have also used Eqs. (111).

$$\tau_0^* = \frac{1}{a} f''_0; \quad \tau_1^* = \mathcal{R}e\, \frac{1}{a}\sum_{n=0}^{\infty} e_{1n}\xi^n e^{i\tau}; \quad \tau_2^* = \mathcal{R}e\, \frac{1}{a}\sum_{n=0}^{\infty}(e_{2n}e^{2i\tau} + \tilde{e}_{2n})\xi^n$$

$$e_{1n} = f''_{1(n+1)} - 2f''_0 a_{1(n+1)} \qquad n=0,2,4,\ldots$$

$$e_{2n} = f''_{2(n+1)} - 2f''_0 a_{2(n+1)} - r_n \qquad n=0,2,4,\ldots$$

$$r_0 = \frac{1}{2} a_{11}(2f''_{11} - 3f''_0 a_{11})$$

$$r_2 = \frac{1}{2}[2a_{13}f''_{11} + 2a_{11}f''_{13} - 6a_{11}a_{13}f''_0]$$

$$r_4 = \frac{1}{2}[2a_{15}f''_{11} + 2a_{13}f''_{13} + 2a_{11}f''_{15} - 3f''_0(2a_{11}a_{15} + a_{13}a_{13})]$$

$$\tilde{e}_{2n} = \tilde{f}''_{2(n+1)} - 2f''_0 \tilde{a}_{2(n+1)} - \tilde{r}_n \qquad n=0,4,\ldots$$

$$\tilde{r}_0 = \frac{1}{2}\bar{a}_{11}\left(2f''_{11}-3f''_0 a_{11}\right)$$

$$\tilde{r}_4 = \frac{1}{2}\left[2\bar{a}_{15}f''_{11}+2\bar{a}_{13}f''_{13}+2\bar{a}_{11}f''_{15}-3f''_0(2\bar{a}_{11}a_{15}+\bar{a}_{13}a_{13})\right]$$

....

Using the relations (91) and (99) in Eqs. (142) gives:

$$q_0{}^* = \frac{1}{2a}F'_0$$

$$q_1{}^* = \frac{1}{2a}\,\Theta^*_{1n} - \frac{1}{4a^2\xi}\,\delta_1{}^*\,F'_0$$

$$q_2{}^* = \frac{1}{2a}\,\Theta^*_{2n} - \frac{1}{4a^2\xi}\,\delta_2{}^*F'_0 - \frac{1}{4a^2\xi}\,\delta_1{}^*\left(\Theta^*_{1\eta} - \frac{\delta_1{}^*}{2a\xi}F'_0\right)$$

Since:

$$\delta_1{}^* = \mathcal{R}e\; 2a\sum_{n=1}^{\infty} a_{1n}\xi^n e^{i\tau}$$

$$\Theta_1{}^* = \mathcal{R}e\sum_{n=1}^{\infty} F_{1n}\xi^n e^{i\tau}$$

we set:

$$q_1{}^* = \mathcal{R}e\,\frac{1}{2a}\sum_{n=0}^{\infty} G_{1n}\xi^n e^{i\tau}$$

and we have

$$\sum_{n=0}^{\infty} G_{1n}\xi^n = \sum_{n=0}^{\infty} F'_{1n}\,\xi^n - F'_0\sum_{n=1}^{\infty} a_{1n}\,\xi^{n-1}$$

And equating coefficients of ξ:

$$G_{1n} = F'_{1n} - a_1(n+1)F'_0; \qquad n=0,2,4,\ldots$$

since $a_{1n} = 0$ for $n=0,2,4,\ldots$; $F_{1n} \equiv 0$ for $n=1,3,5,\ldots$

Since:

$$\delta_2^* = \mathcal{R}e\ 2a \sum_{n=1}^{\infty} (a_{2n}\, e^{2i\tau} + \tilde{a}_{2n})\ \xi^n$$

$$\theta_2^* = \mathcal{R}e \sum_{n=0}^{\infty} [F_{2n}\, e^{2i\tau} + \tilde{F}_{2n}]\ \xi^n$$

We set:

$$q_2^* = \mathcal{R}e\ \frac{1}{2a} \sum_{n=0}^{\infty} [G_{2n}\, e^{2i\tau} + \tilde{G}_{2n}]\xi^n$$

and we have, using (111) for the coefficients of $e^{2i\tau}$:

$$\sum_{n=0}^{\infty} G_{2n}\xi^n = \sum_{n=0}^{\infty} F_{2n}\xi^n - F_0' \sum_{n=1}^{\infty} a_{2n}\xi^{n-1}$$

$$- \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_{1n}\, F_{1m}'\ \xi^{n+m-1}$$

$$+ F_0' \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n}\, a_{1m}\ \xi^{n+m-2}$$

And equating the coefficients of like powers of ξ we get:

$$G_{2n} = F_{2n}' - a_{2(n+1)}F_0' - R_n \qquad n=0,2,4,\ldots$$

$$R_0 = \frac{1}{2}[a_{11}F_{10}' - a_{11}a_{11}F_0']$$

$$R_2 = \frac{1}{2}[a_{13}F_{10}' + a_{11}F_{12}' - 2F_0'\, a_{11}a_{13}]$$

$$R_4 = \frac{1}{2}[a_{15}F_{10}' + a_{13}F_{12}' + a_{11}F_{14}' - F_0'(2a_{11}a_{15}+a_{13}a_{13})]$$

....

....

Since a_{1n}, a_{2n} = 0 for n=0,2,4,...and F_{1n}, F_{2n} = 0 for n=1,3,5,...

And similarly for the time independent parts:

$$\tilde{G}_{2n} = \tilde{F}'_{2n} - \tilde{a}_{2(n+1)} F'_0 - \tilde{R}_n \qquad n=0,4,\ldots$$

$$\tilde{R}_0 = \frac{1}{2} [\bar{a}_{11}F'_{10} - \bar{a}_{11}a_{11}F'_0]$$

$$\tilde{R}_4 = \frac{1}{2} [\bar{a}_{15}F'_{10} + \bar{a}_{13}F'_{12} + \bar{a}_{11}F'_{14} - F'_0(2\bar{a}_{11}a_{15}+a_{13}\bar{a}_{13})]$$

Since:

$$\tilde{F}_{2n} \equiv 0 \quad \text{for} \quad n\neq 0,4,8,\ldots$$

$$\tilde{a}_{2n} \equiv 0 \quad \text{for} \quad n\neq 1,5,9,\ldots$$

Collecting the results:

$$q_0{}^* = \frac{1}{2a} F'_0$$

$$q_1{}^* = \mathcal{R}e \frac{1}{2a} \sum_{n=0}^{\infty} G_{1n}\xi^n e^{i\tau}$$

$$q_2{}^* = \mathcal{R}e \frac{1}{2a} \sum_{n=0}^{\infty} [G_{2n} e^{2i\tau} + \tilde{G}_{2n}]\xi^n$$

$$G_{1n} = F'_{1n} - a_1(n+1)F'_0 \qquad n=0,2,4,\ldots$$

$$G_{2n} = F'_{2n} - a_2(n+1)F'_0 - R_n \quad n=0,2,4,\ldots$$

$$R_0 = \frac{1}{2} [a_{11}F'_{10} - a_{11}a_{11}F'_0]$$

$$R_2 = \frac{1}{2} [a_{13}F'_{10} + a_{11}F'_{12} - 2F'_0 a_{11}a_{13}]$$

$$R_4 = \frac{1}{2} [a_{15}F'_{10} + a_{13}F'_{12} + a_{11}F'_{14} - F'_0(2a_{11}a_{15}+a_{13}a_{13})]$$

....

....

$$\tilde{G}_{2n} = \tilde{F}_{2n} - \tilde{a}_{2(n+1)}F_0' - \tilde{R}_n \qquad n=0,4,\ldots$$

$$\tilde{R}_0 = \frac{1}{2}[\bar{a}_{11}F_{10}' - \bar{a}_{11}a_{11}F_0']$$

$$\tilde{R}_4 = \frac{1}{2}[\bar{a}_{15}F_{10}' + \bar{a}_{13}F_{12}' + \bar{a}_{11}F_{14}' - F_0'(2\bar{a}_{11}a_{15}+\bar{a}_{13}a_{13})]$$

....

....

APPENDIX XXV

COMPUTER PROGRAMS INVOLVING INTEGRATION OF DIFFERENTIAL EQUATIONS

All the computer programs which involve the integration of differential equations by the Runge-Kutta method are listed in this appendix along with the nomenclature which is necessary for their understanding and for connecting them with the equations derived above. Each program is divided into blocks whose functions are indicated by titles.

NOTATION FOR COMPUTER PROGRAMS INVOLVING THE INTEGRATION OF DIFFERENTIAL EQUATIONS

Y(1)	~	calculated value of solution to differential equation
Y(2)	~	first derivitive of Y(1)
Y(3)	~	second derivitive of Y(1)
Y(4)	~	third derivitive of Y(1)
F(1) (U(1)), F(2) (U(2)), F(3)	~	(U(1), U(2) are used for F(1) and F(2) for cylinder problem with $.01 \leq E \leq .1$) Terms of the differential equations which have been rewritten as a series of first order equations of the form: Y(2) = F(1) Y(3) = F(2) Y(4) = F(3) (c.f. discussion in Chapter III)

CORRESPONDENCE BETWEEN NOMENCLATURE USED IN COMPUTER PROGRAM FOR CYLINDER PROBLEM WITH $E^2 > .01$ AND NOMENCLATURE USED IN TEXT

A(0)	~	a_0
A(K)	~	b_k; $k = 2,4,6,\ldots$
B(K)	~	B_k
C(K)	~	A_k
DE	~	δ^*
ES	~	E^2
FR	~	$\tau_w \sqrt{\frac{2R_0U_\infty}{\nu}} \Big/ \frac{1}{2}\rho U_\infty{}^2$
H(K,I)	~	$H_k(\eta)$; K,k $3,5,7,\ldots$
H(K,I)	~	$G_k(\eta)$; K,k $2,4,6,\ldots$
NU(J)	~	q_j
O	~	$N_u / \sqrt{\frac{2R_0U_\infty}{\nu}}$
P(I)	~	η
PR	~	P_r
RE	~	$U_\infty R_0/\nu$
TAU(J)	~	τ_j
U(I)	~	$f_1(\eta)$
UP(I)	~	$f_1'(\eta)$
UDP(I)	~	$f_1''(\eta)$
UTP(I)	~	$f_1'''(\eta)$
V(K,I)	~	$f_k(\eta)$
VP(K,I)	~	$f_k'(\eta)$
VDP(K,I)	~	$f_k''(\eta)$

(V(K,I), VP(K,I), VDP(K,I): K,k = $3,5,7,\ldots$)

V(K,I)	~	$F_k(\eta)$	k,K 0,2,4,6,...
VP(K,I)	~	$F'_k(\eta)$	
VDP(K,I)	~	$F''_k(\eta)$	
W(J,I)	~	$u_j(\eta)$	
X	~	η	
XE	~	X_e	
XI	~	ξ	

```
$COMPILE MAC, PRINT OBJECT                                        999595   06/30/65      7 28 50.9 AM

MAD (01 MAY 1965 VERSION) PROGRAM LISTING ... ... ...


             CYLENDER PROBLEM WITH E'SQUARED GREATER THAN 0.01
              *     *     *     *     *     *     *

                SET   UP
             -----------------------------------------------------------
             DIMENSION Y(3),F(3),D(3)                                                    *001
             EXECUTE SETRKD.(3,Y(1),F(1),D,X,STEP)                                       *002
             INTEGER J,K,M,N,I,T,EPS ,TS,TMAX,MMAX,NMAX                                  *003
             VECTOR VALUES FM1=$1H1/1H- *$                                               *004
             VECTOR VALUES FM2=$1H ,S4,54(1H-) *$                                        *005
             VECTOR VALUES FM3=$1H ,S4,F5.3,4F12.6 *$                                    *006
             VECTOR VALUES FM4=$1H , 4F12.6 *$                                           *007
             DIMENSION P(150),U(150),UP(150),UDP(150),UTP(150),B(11),C(11)               *008
           1 ,A(11),TAU(11),NU(11)                                                       *008
             DIMENSION R1(150),R2(150)                                                   *009
             DIMENSION V(1400,VS)                                                        *010
             DIMENSION VP(1400,PS)                                                       *011
             DIMENSION VDP(1400,DS)                                                      *012
             DIMENSION W(1400,TW)                                                        *013
             DIMENSION H(1400,HS)                                                        *014
             VECTOR VALUES VS=2,150,150                                                  *015
             VECTOR VALUES PS=2,150,150                                                  *016
             VECTOR VALUES DS=2,150,150                                                  *017
             VECTOR VALUES TW=2,1,150                                                    *018
             VECTOR VALUES HS=2,150,150                                                  *019
             READ AND PRINT DATA                                                         *020
             Z2=-1.00                                                                    *021
             ES=(XE.P.2)*2*RE                                                            *022
             YDP=4.0/(ES.P.0.50)                                                         *023
             TS=1                                                                        *024
             T=1                                                                         *025
SW16         WHENEVER T.E.1, EPS=2                                                       *026
             WHENEVER T.G.1, EPS=1                                                       *027
             -----------------------------------------------------------
               CALC. OF CORRECTED VALUES OF CONSTANT PARAMETER FROM
              ERRORS IN BOUNDARY COND'S
             -----------------------------------------------------------
             TA=AC                                                                       *028
             DA=0.001                                                                    *029
             DY=0.001                                                                    *030
             R1(0)=1.5                                                                   *031
             R1(1)=1.0                                                                   *032
             M=1                                                                         *033
SW2          Q1=CF1*(R1(M)/R1(M-1)-1)                                                    *034
             WHENEVER Q1.G.87.20,Q1=87.20                                                *035
             WHENEVER Q1.L.(-87.20),Q1=-87.20                                            *036
             DB=-DA*((2.35040/(EXP.(Q1)-EXP.(-Q1)))+1)                                   *037
             DA=DB                                                                       *038
             M=M+1                                                                       *039
             TA=TA+DA                                                                    *040
             AN=A(0)                                                                     *041
             AP=TA                                                                       *042
```

```
         ---------------------------------------------------------------
            CALC. OF CORR. VALUES OF INITIAL COND'S FROM PREVIOUS
          ERRORS IN BOUNDARY COND'S
         ---------------------------------------------------------------
          YX=YDP                                                          *043
SW1       R2(0)=1.5                                                       *044
          R2(1)=1.0                                                       *045
          N=1                                                             *046
SW4       Q2=CF2*(R2(N)/R2(N-1)-1)                                        *047
          WHENEVER Q2.G.87.20,Q2=87.20                                    *048
          WHENEVER Q2.L.(-87.20),Q2=-87.20                                *049
          DX=-DY*((2.35040/(EXP.(Q2)-EXP.(-Q2)))+1)                       *050
          DY=DX                                                           *051
          N=N+1                                                           *052
          YX=YX+DY                                                        *053
          WHENEVER TS.E.1                                                 *054
          Y(3)=YX                                                         *055      01
          Y(2)=0                                                          *056      01
          Y(1)=0                                                          *057      01
          OR WHENEVER TS.E.0                                              *058      01
          Y(2)=YX                                                         *059      01
          Y(1)=EPS-1                                                      *060      01
          END OF CONDITIONAL                                              *061      01
         ---------------------------------------------------------------
           INTEGRATION OF DIFF. EQ'S
         ---------------------------------------------------------------
SW3       I=0                                                             *062
          X=0                                                             *063
START     I=I+1                                                           *064
          P(I)=X                                                          *065
          WHENEVER T.E.1                                                  *066
          U(I)=Y(1)                                                       *067      01
          UP(I)=Y(2)                                                      *068      01
          UDP(I)=Y(3)                                                     *069      01
          UTP(I)=F(3)                                                     *070      01
          OTHERWISE                                                       *071      01
          V(T,I)=Y(1)                                                     *072      01
          VP(T,I)=Y(2)                                                    *073      01
          WHENEVER TS.E.1, VDP(T,I)=Y(3)                                  *074      01
          WHENEVER TS.E.0, VDP(T,I)=F(2)                                  *075      01
          END OF CONDITIONAL                                              *076      01
SW5       WHENEVER TS.E.1                                                 *077
          F(1)=Y(2)                                                       *078      01
          F(2)=Y(3)                                                       *079      01
          WHENEVER T.E.1, F(3)=-AP*(Y(1)*Y(3)-(Y(2).P.2))*ES              *080      01
          WHENEVER T.NE.1, F(3)=-AN*(U(I)*Y(3)-(T+1)*UP(I)*Y(2)+T*UDP(I   *081      01
        1 )*Y(1)-H(T,I)-((Z2).P.((T+1)/2))*(T/(T+1.))*(U(I)*UDP(I)+(T-2   *081
        2 )*(UP(I).P.2))*AP)*ES                                           *081
          OR WHENEVER TS.E.0                                              *082      01
          F(1)=Y(2)                                                       *083      01
          WHENEVER T.E.0                                                  *084      01
          F(2)=-MU*(U(I)*Y(2))                                            *085      02
          OTHERWISE                                                       *086      02
          F(2)=-MU*(U(I)*Y(2)-T*UP(I)*Y(1)-(2-EPS)*(H(T,I)-((Z2.P.(T/2    *087      02
        1 ))*(T+2)*V(T+1,I)+A(T)*U(I))*VP(0,I)))                          *087
          END OF CONDITIONAL                                              *088      02
          END OF CONDITIONAL                                              *089      01
CALC      S=RKDEQ.(0)                                                     *090
          WHENEVER S.E.1.0,TRANSFER TO SW5                                *091
          WHENEVER I.E.GRSIZ,TRANSFER TO SW6                              *092
```

```
              TRANSFER TO START                                                        *093
              ----------------------------------------------------------
               SELECT. OF WHICH BOUNDARY COND. TEST TO SUBJECT SOL'N TO
               DEPENDING ON WHETHER MOM. OR ENERGY EQU'S ARE BEING SOLVED
              ----------------------------------------------------------
SW6           WHENEVER TS.E.1, TRANSFER TO SW7                                         *094
              WHENEVER TS.E.0, TRANSFER TO SW8                                         *095
              ----------------------------------------------------------
              CALC. OF ERROR IN CONT. BOUNDARY COND.
              ----------------------------------------------------------
SW7           WHENEVER .ABS.(Y(1)-(1.0*EPS/(T+1.0))).G.CTOL  .AND. N.L.NMA             *096
            1 X                                                                        *096
              R2(N)=Y(1)-(EPS  /(T+1.))                                                *097     01
              TRANSFER TO SW4                                                          *098     01
              ----------------------------------------------------------
              CALC. OF ERROR IN MOM.  BOUNDARY COND.
              ----------------------------------------------------------
              OTHERWISE                                                                *099     01
              WHENEVER N.GE.NMAX, PRINT RESULTS R2(1)...R2(N)                          *100     01
              A(T-1)=AP                                                                *101     01
              C(1)=-4.0*(A.P.2)                                                        *102     01
              MBC = A*2*Y(2)  +C(T)-(2-EPS)*((Z2).P.((T-1)/2))*(2.*T/(T+1))            *103     01
            1 *A*(4*A-UP(I+1))*AP-(2.*T/(T+1))*A*(2.*A-UP(I+1))*(2-EPS)                *103
              END OF CONDITIONAL                                                       *104     01
              WHENEVER .ABS.(Y(3) + MBC*ES*0.5   ).G.MTOL  .AND. M.L.MMAX              *105
              R1(M)=Y(3)+MBC*ES*0.5                                                    *106     01
              TRANSFER TO SW2                                                          *107     01
              OTHERWISE                                                                *108     01
              WHENEVER M.GE.MMAX, PRINT RESULTS R1(1)...R1(M)                          *109     01
              ----------------------------------------------------------
                 STORAGE OF SOL'S AT LAST GRID PT.
              ----------------------------------------------------------
              WHENEVER T.E.1                                                           *110     01
              U(I+1)=Y(1)                                                              *111     02
              UP(I+1)=Y(2)                                                             *112     02
              UDP(I+1)=Y(3)                                                            *113     02
              UTP(I+1)=F(3)                                                            *114     02
              OTHERWISE                                                                *115     02
              V(T,I+1)=Y(1)                                                            *116     02
              VP(T,I+1)=Y(2)                                                           *117     02
              VDP(T,I+1)=Y(3)                                                          *118     02
              END OF CONDITIONAL                                                       *119     02
              END OF CONDITIONAL                                                       *120     01
              ----------------------------------------------------------
                CALC. OF TERMS TO BE USED IN HIGHER ORDER MOM. EQU'S
              ----------------------------------------------------------
              THROUGH SW14,FOR J=1,1,J.G.(GRSIZ+1)                                     *121
              WHENEVER T.E.1, H(3,J)=0                                                 *122        01
              WHENEVER T.E.3, H(5,J)=(80./3.)*((VP(3,J).P.2)-(V(3,J)*VDP(3,            *123        01
            1 J)))+(20./3.)*A(2)*((VDP(3,J)*U(J))+(3.*V(3,J)*UDP(J)))                  *123
SW14          WHENEVER T.E.5, H(7,J)=7.*(3.*(8.*VP(5,J)*VP(3,J)                        *124        01
            1  -5.*VDP(3,J)*V(5,J))-0.75*A(2)*(6.*VP(5,J)*UP(J)+(80./3.)*(             *124
            2 (VP(3,J).P.2)-3.*V(3,J)*VDP(3,J))-3.*VDP(5,J)*U(J)-15.*UDP(J             *124
            3 )*V(5,J))-2.5*A(4)*(VDP(3,J)*U(J)+4.*UP(J)*VP(3,J)+3.*V(3,J)             *124
            4 *UDP(J))-9.*VDP(5,J)*V(3,J))                                             *124
              WHENEVER T.E.1, C(3)=-(A.P.2)                                            *125
              WHENEVER T.E.3, C(5)=A*(10.*A(2)*(2.*A-UP(GRSIZ+1))+(2./3.)*             *126
            1 A*(10*A(2)-1.)-(20./3.)*(A(2)-1)*(3.*A(2)*A+2.*VP(3,GRSIZ+1)             *126
            2 -A))                                                                     *126
              WHENEVER T.E.5, C(7)=0.5*A*((105./2.)*(A(2)-A(4))*(2.*A-UP(G             *127
```

```
          1 RSIZ+1))+A*(21.*A(2)-35.*A(4)-1.)+21.*(A(2)-1.)*(A*(5.*A(4)-        *127
          2 10.*A(2)+1.)-3.*VP(5,GRSIZ+1))+35.*(A(4)-6.*A(2)+1.)*(A*(3.*        *127
          3 A(2)-1.)+2.*VP(3,GRSIZ+1)))                                         *127
            THROUGH SW12,FOR J=1,1,J.G.(GRSIZ+1)                                *128
            WHENEVER T.E.1, W(1,J)=UP(J)/(2.*A)                                 *129        01
            WHENEVER T.E.3, W(3,J)=(4.*VP(3,J)+3.*A(2)*UP(J))/(2.*A)            *130        01
            WHENEVER T.E.5, W(5,J)=(6.*VP(5,J)+40.*A(2)*VP(3,J)+5.*(6.*(A(2).   *131        01
          1 P.2)-A(4))*UP(J))/(2.*A)                                            *131
SW12        WHENEVER T.E.7, W(7,J)=(8.*VP(7,J)+126.*A(2)*VP(5,J)+140.*(6.*(A(   *132        01
          1 2).P.2)-A(4))*VP(3,J)+7.*(90.*(A(2).P.3)-15.*A(2)*A(4)+A(6))        *132
          2 *UP(J))/(2.*A)                                                      *132
            WHENEVER T.E.1, TAU(1)=UDP(1)/(ES*(A.P.2))                          *133
            WHENEVER T.E.3, TAU(3)=2.*(2.*VDP(3,1)+3.*A(2)*UDP(1))/(ES*(A.P.2   *134
          1 ))                                                                  *134
            WHENEVER T.E.5, TAU(5)=2.*(3.*VDP(5,1)+40.*A(2)*VDP(3,1)+5.*(9.*(   *135
          1 A(2).P.2)-A(4))*UDP(1))/(ES*(A.P.2))                                *135
            WHENEVER T.E.7, TAU(7)=2.*(4.*VDP(7,1)+126.*A(2)*VDP(5,1)+140.*(9   *136
          1 .*(A(2).P.2)-A(4))*VDP(3,1)+7.*(180.*(A(2).P.3)-45.*A(2)*A(4        *136
          2 )+A(6))*UDP(1))/(ES*(A.P.2))                                        *136
            -------------------------------------------------------------
             TEST LOC. TO DET. WHICH MOM. EQ. IS BEING SOLVED AND
             WHEN MOM. PROB. HAS BEEN COMP.
            -------------------------------------------------------------
SW20        T=T+2                                                               *137
            WHENEVER T.L.TMAX,TRANSFER TO SW16                                  *138
            -------------------------------------------------------------
             SET UP OF ENERGY PROB.
            -------------------------------------------------------------
            TS=0                                                                *139
            T=0                                                                 *140
            YX=YP                                                               *141
            MU=A*ES*PR                                                          *142
            EPS=2                                                               *143
            TMAX=TMAX-1                                                         *144
            TRANSFER TO SW1                                                     *145
            -------------------------------------------------------------
              CALC. OF ERROR IN ENERGY B.C.
            -------------------------------------------------------------
SW8         EBC=Y(1)+(2-EPS)*(V(0,I+1)*(A(T)+(Z2.P.(T/2)))+B(T))                *146
            WHENEVER .ABS.((Y(2)/MU)+EBC).G.ETOL .AND. N.L.NMAX                 *147
            R2(N)=(Y(2)/MU)+EBC                                                 *148    01
            TRANSFER TO SW4                                                     *149    01
            OTHERWISE                                                           *150    01
            WHENEVER N.GE.NMAX, PRINT RESULTS R2(1)...R2(N) ,T                  *151    01
            -------------------------------------------------------------
               STORAGE OF SOL'S AT LAST GRID PT.
            -------------------------------------------------------------
            V(T,I+1)=Y(1)                                                       *152    01
            VP(T,I+1)=Y(2)                                                      *153    01
            VDP(T,I+1)=F(2)                                                     *154    01
            END OF CONDITIONAL                                                  *155    01
            -------------------------------------------------------------
              CALC. OF TERMS TO BE USED IN HIGHER ORDER ENERGY EQU'S
            -------------------------------------------------------------
            THROUGH SW15, FOR J=1,1,J.G.(GRSIZ+1)                               *156
            WHENEVER T.E.0, H(2,J)=0                                            *157        01
            WHENEVER T.E.2, H(4,J)=8.*(3.*V(3,J)*VP(2,J)-2.*VP(3,J)*V(2,        *158        01
          1 J))+6.*A(2)*(2.*UP(J)*V(2,J)-U(J)*VP(2,J))+24.*A(2)*V(3,J)*V        *158
          2 P(0,J)                                                              *158
SW15        WHENEVER T.E.4, H(6,J)=20.*(3.*V(3,J)*VP(4,J)-4.*VP(3,J)*V(4        *159        01
```

```
1 ,J))-18.*(5.*V(5,J)*VP(2,J)-2.*VP(5,J)*V(2,J))+15.*A(2)*(4.*      *159
2 UP(J)*V(4,J)-U(J)*VP(4,J)-16.*VP(3,J)*V(2,J)+24.*V(3,J)*VP(2      *159
3 ,J))+15.*A(4)*(2.*UP(J)*V(2,J)-U(J)*VP(2,J))+30.*(-3.*A(2)*V      *159
4 (5,J)+2.*A(4)*V(3,J))*VP(0,J)                                     *159
  WHENEVER T.E.0, B(2)=0                                            *160
  WHENEVER T.E.2, B(4)=6.*(V(2,GRSIZ+1)*(A(2)-1.)-A(2)*V(0,GRS      *161
1 IZ+1))                                                            *161
  WHENEVER T.E.4, B(6)=15.*(V(4,GRSIZ+1)*A(2)+V(2,GRSIZ+1)*A(4      *162
1 )-V(4,GRSIZ+1)-6.*V(2,GRSIZ+1)*A(2)-V(0,GRSIZ+1)*A(4)+V(2,GR      *162
2 SIZ+1)+V(0,GRSIZ+1)*A(2))                                         *162
  WHENEVER T.E.0, NU(0)=VP(0,1)/A                                   *163
  WHENEVER T.E.2, NU(2)=(VP(2,1)-A(2)*VP(0,1))/A                    *164
  WHENEVER T.E.4, NU(4)=(VP(4,1)-6.*A(2)*VP(2,1)+(6.*(A(2).P.2)-A(4 *165
1 ))*VP(0,1))/A                                                     *165
  WHENEVER T.E.6, NU(6)=(VP(6,1)-15.*A(2)*VP(4,1)+15.*(6.*(A(2).P.2 *166
1 )-A(4))*VP(2,1)-(90.*(A(2).P.3)-15.*A(2)*A(4)+A(6))*VP(0,1))      *166
2 /A                                                                *166
 --------------------------------------------------------------
  TEST LOC. TO DET. WHICH ENERGY EQ. IS BEING SOLVED AND
   WHEN ENERGY PROB. IS COMPL.
 --------------------------------------------------------------
  T=T+2                                                             *167
  WHENEVER ETOL.L.0.001, ETOL=ETOL*(10.0.P.(T/2))                   *168
  YX=YP                                                             *169
  EPS=1                                                             *170
  WHENEVER T.L.TMAX, TRANSFER TO SW1                                *171
 --------------------------------------------------------------
      PRINT STATEMENTS AND MISC. CALC'S
 --------------------------------------------------------------
        PRINT FORMAT FM1                                            *172
        PRINT FORMAT FM2                                            *173
        THROUGH SW10, FOR K=1,1,K.G.(GRSIZ+1)                       *174
SW10    PRINT FORMAT FM3, P(K),U(K),V(3,K),V(5,K),V(7,K)            *175   01
        PRINT FORMAT FM1                                            *176
        PRINT FORMAT FM2                                            *177
        THROUGH SW17, FOR K=1,1,K.G.(GRSIZ+1)                       *178
SW17    PRINT FORMAT FM3, P(K),W(1,K),W(3,K),W(5,K),W(7,K)          *179   01
        PRINT FORMAT FM1                                            *180
        PRINT FORMAT FM2                                            *181
        THROUGH SW18, FOR K=1,1,K.G.(GRSIZ+1)                       *182
SW18    PRINT FORMAT FM3, P(K),UP(K),VP(3,K),VP(5,K),VP(7,K)        *183   01
        PRINT FORMAT FM1                                            *184
        PRINT FORMAT FM2                                            *185
        THROUGH SW19, FOR K=1,1,K.G.(GRSIZ+1)                       *186
SW19    PRINT FORMAT FM3, P(K),UDP(K),VDP(3,K),VDP(5,K),VDP(7,K)    *187   01
        PRINT FORMAT FM1                                            *188
        PRINT FORMAT FM2                                            *189
        THROUGH SW21, FOR K=1,1,K.G.(GRSIZ+1)                       *190
SW21    PRINT FORMAT FM3, P(K),V(0,K),V(2,K),V(4,K),V(6,K)          *191   01
        PRINT FORMAT FM1                                            *192
        PRINT FORMAT FM4, A,A(2),A(4),A(6)                          *193
        PRINT FORMAT FM4, TAU(1),TAU(3),TAU(5),TAU(7)               *194
        PRINT FORMAT FM4, NU(0),NU(2),NU(4),NU(6)                   *195
        PRINT FORMAT FM2                                            *196
        THROUGH SW30, FOR XI=0.,0.1,XI.G.1.5                        *197
        CE=A*(1.+(A(2)/2.)*(XI.P.2)+(A(4)/24.)*(XI.P.4)+(A(6)/720.)*  *198   01
      1 (XI.P.6))                                                   *198
        DE=DE*(ES.P.0.5)                                            *199   01
        FR=TAU(1)*XI-(TAU(3)/6.)*(XI.P.3)+(TAU(5)/120.)*(XI.P.5)-(TA  *200   01
      1 U(7)/5040.)*(XI.P.7)                                        *200
```

```
              FR=FR*(ES.P.0.5)                                                    *201          01
              O=NU(0)+(NU(2)/2.)*(XI.P.2)+(NU(4)/24.)*(XI.P.4)+(NU(6)/720.        *202          01
            1 )*(XI.P.6)                                                          *202
              O=O/(ES.P.0.5)                                                      *203          01
SW30          PRINT FORMAT FM4, XI,DE,O,FR                                        *204          01
             ------------------------------------------------------------
             ------------------------------------------------------------
SW11          END OF PROGRAM                                                      *205
```

The procedure used in the program for calculating drop trajectories is to use the Runge-Kutta method to integrate the equations of motion for a drop starting at particular values of $y^*_{d\infty}$ until the drop almost reaches the cylinder or until it entirely misses the cylinder. If the drop is going to hit the cylinder the calculated values of the solution are extended by linear interpolation to the required values at the cylinder. These values are then used to calculate the quantities of Eqs. (17*A.C.) which are needed to describe the conditions in the liquid film.

At each step of the integration the current value of the absolute value of the vector difference between the drop velocity and the gas velocity at the position of the drop is used to calculate an instantaneous Reynolds number based on the drop diameter. The drag coefficient corresponding to this Reynolds number is obtained by interpolation from the tabulated values of Table II, which are put into the program as "data" and were obtained from Figure 28 which is a smoothing of the values obtained from Reference 28. This value of the drag coefficient is then used in the equations of motion to calculate the new position of the drop.

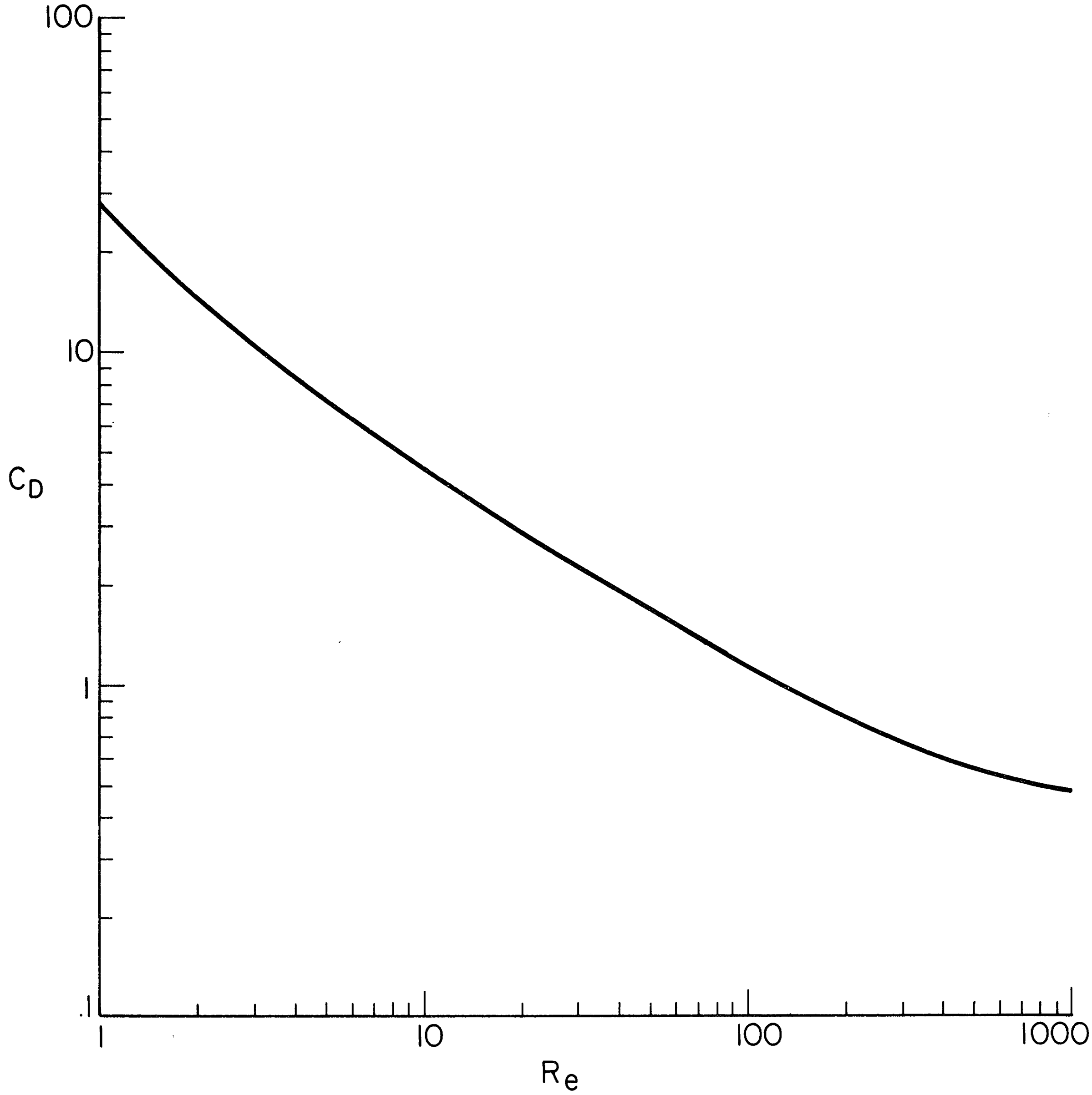

Figure 28. Values of drag coefficients for spheres used in calculating drop trajectories.

TABLE II

DRAG COEFFICIENTS FOR A SPHERE (28)

R_e	C_D
0.2	124.0
0.4	64.5
0.6	44.5
0.8	36.0
1	28.4
2	14.4
4	8.4
6	6.2
8	5.1
10	4.4
20	2.8
40	1.88
60	1.50
80	1.27
100	1.12
200	0.79
400	0.59
600	0.53
800	0.49
1000	0.48
2000	0.40

CORRESPONDENCE BETWEEN NOMENCLATURE USED IN COMPUTER PROGRAM FOR CALCULATING DROP TRAJECTORIES AND NOTATION USED IN TEXT

CD1	~	C_D
H	~	ΔV^*
MDOT	~	J
MF(N)	~	J (tabulated value)
MTOT	~	$\dot{m}$
P(I)	~	t^*
PH	~	φ_d
PHI(K)	~	φ_d (tabulated table)
R(I)	~	$\lvert\vec{r}_d^*\rvert$
RE1	~	$2r_d\Delta V^*/\nu_g$ (drop Reynolds number)
RINF	~	$2R_o U_\infty/\nu_g$
T	~	t^*
V1	~	V^*_{xo}
V2	~	V^*_{yo}
VPHI	~	V^*_φ
VXF	~	V^*_{xof}
VYF	~	V^*_{yof}
X1	~	x^*_d
X2	~	y^*_d
XBAR	~	$\tilde{x}$
XF	~	x^*_{df}

XF(K)	~	x^*_{df} (stored value)
YF	~	y^*_{df}
YF(K)	~	y^*_{df} (stored value)
YINF(K)	~	$y^*_{d\infty}$

```
$COMPILE MAD, PRINT OBJECT                                                            013991   07/23/65      3 49 47.3 AM

MAD (01 MAY 1965 VERSION) PROGRAM LISTING ... ... ...


              CALC. OF DROP TRAJECTORIES

                   *        *          *          *

              SET UP
              ------------------------------------------------------------
              VECTOR VALUES FM1=$1H1/1H- *$                                                      *001
              VECTOR VALUES FM2=$1H ,S4,48(1H-) *$                                               *002
              VECTOR VALUES FM3=$1H ,S4,4F12.6 *$                                                *003
              DIMENSION Y(4),F(4),D(4)                                                           *004
              EXECUTE SETRKD.(4,Y(1),F(1),D,T,STEP)                                              *005
              DIMENSION IMAGE(10000)                                                             *006
              EXECUTE PLOT2.(IMAGE,0.,-2.5,2.08335,0.)                                           *007
              DIMENSION RED(100),CD(100),P(1000),X1(1000),X2(1000),V1(1000),V2(1                 *008
            1 000),MF(1000),XF(1000),YF(1000),PHI(1000),VPHI(1000),YINF(10                       *008
            2 00),R(1000)                                                                        *008
              INTEGER JMAX,I,IMAX,K,L,KE,M,N                                                     *009
              INTEGER SW                                                                         *010
              INTEGER II,KI,IP,KP                                                                *011
              READ AND PRINT DATA                                                                *012
              K=0                                                                                *013
              SW=0                                                                               *014
              KI=0                                                                               *015
              ALPH=3.*ROH/(RR*8.)                                                                *016
              ------------------------------------------------------------
                      INITIAL CONDITIONS
              ------------------------------------------------------------
SW1           K=K+1
              WHENEVER KI.E.5, KI=0
              KI=KI+1
              Y(1)=-XINF
              Y(2)=INCY*(K-1,
              Y(3)=1.0
              Y(4)=0.
              I=0
              T=0
              ------------------------------------------------------------
               INTERGRATION OF DIFFERENTIAL EQUATIONS
              ------------------------------------------------------------
              II=0                                                                               *026
START         I=I+1                                                                              *027
              II=II+1                                                                            *028
              P(I)=T                                                                             *029
              X1(I)=Y(1)                                                                         *030
              X2(I)=Y(2)                                                                         *031
              V1(I)=Y(3)                                                                         *032
              V2(I)=Y(4)                                                                         *033
              R(I)=(Y(1).P.2+Y(2).P.2).P.0.5                                                     *034
              WHENEVER KI.E.KP .AND. II.E.IP                                                     *035
              EXECUTE PLOT3.($0$,Y(1),Y(2),1)                                                    *036      01
              II=0                                                                               *037      01
              OTHERWISE                                                                          *038      01
```

```
          CONTINUE                                                              *039   01
          END OF CONDITIONAL                                                    *040   01
SW5       F(1)=Y(3)                                                             *041
          F(2)=Y(4)                                                             *042
          H1=Y(3)-1.0+((Y(1).P.2-Y(2).P.2)/((Y(1).P.2+Y(2).P.2).P.2))           *043
          H2=Y(4)+(2.*Y(1)*Y(2)/((Y(1).P.2+Y(2).P.2).P.2))                      *044
          H=(H1.P.2.+H2.P.2.).P.0.5                                             *045
          RE1=RINF*(.ABS.H )*RR                                                 *046
          WHENEVER RE1.GE.1.0                                                   *047
          CD1=TAB.(RE1,RED(1),CD(1),1,1,4,JMAX,TS)                              *048   01
          OTHERWISE                                                             *049   01
          CD1=24.*(1.+(3./16.)*RE1)/RE1                                         *050   01
          END OF CONDITIONAL                                                    *051   01
          F(3)=-ALPH*CD1*H1*H                                                   *052
          F(4)=-ALPH*CD1*H2*H                                                   *053
          S=RKDEQ.(0)                                                           *054
          WHENEVER S.E.1.0, TRANSFER TO SW5                                     *055
          WHENEVER (Y(1).P.2+Y(2).P.2).LE.1.0 .OR. Y(1).GE.0., TRANSFER TO SW6  *056
          TRANSFER TO START                                                     -057
          ------------------------------------------------------------
          STORAGE OF SOL'S AT LAST GRID PT.
          ------------------------------------------------------------
SW6       P(I+1)=T                                                              *058
          X1(I+1)=Y(1)                                                          *059
          X2(I+1)=Y(2)                                                          *060
          V1(I+1)=Y(3)                                                          *061
          V2(I+1)=Y(4)                                                          *062
          R(I+1)=(Y(1).P.2+Y(2).P.2).P.0.5                                      *063
          ------------------------------------------------------------
          CALC. OF TANG. VEL. AND TOTAL MASS FLOW
          ------------------------------------------------------------
          WHENEVER R(I+1).LE.1.0                                                *064
          R(I)=(X1(I).P.2+X2(I).P.2).P.0.5                                      *065   01
          A=(X1(I-1)*X2(I)-X1(I)*X2(I-1))/(X1(I-1)-X1(I))                       *066   01
          B=(X2(I)-X2(I-1))/(X1(I)-X1(I-1))                                     *067   01
          C=(B.P.2-A.P.2+1.).P.0.5                                              *068   01
          XF=-(A*B+C)/(1.+(B.P.2))                                              *069   01
          YF=A+B*XF                                                             *070   01
          VXF=V1(I)                                                             *071   01
          VYF=V2(I)                                                             *072   01
          TAPH=YF/(.ABS.XF)                                                     *073   01
          PH=ATAN.(TAPH)                                                        *074   01
          PHI(K)=PH                                                             *075   01
          VPHI(K)=VXF*SIN.(PH)+VYF*COS.(PH)                                     *076   01
          YINF(K)=INCY*(K-1)                                                    *077   01
          XF(K)=XF                                                              *078   01
          YF(K)=YF                                                              *079   01
          ------------------------------------------------------------
            CALC. OF LAST DROP TO HIT CYLINDER
          ------------------------------------------------------------
          OR WHENEVER SW.E.0                                                    *080   01
          SW=1                                                                  *081   01
          KE=K                                                                  *082   01
          END OF CONDITIONAL                                                    *083   01
          ------------------------------------------------------------
           DET. OF LAST DROP IN PATH OF CYLINDER
          ------------------------------------------------------------
          WHENEVER X2(1).L.1.0, TRANSFER TO SW1                                 *084
          ------------------------------------------------------------
            PLOT STATEMENT
```

```
       ----------------------------------------------------------------
        PRINT FORMAT FM1                                                   *085
        PRINT COMMENT $                                                    *086
1               DROP TRAJECTORIES                                 $        *086
        EXECUTE PLOT4.(0,LABEL)                                            *087
       ----------------------------------------------------------------
           CALC OF MASS FLUX
       ----------------------------------------------------------------
        MF(KE-2)=0.                                                        *088
        MF(KE-1)=0.                                                        *089
        THROUGH SW8, FOR N=1,1,N.G. (KE-2)                                 *090
        MF(N)=INCY/(PHI(N+1)-PHI(N))                                       *091         01
       ----------------------------------------------------------------
                PRINT STATEMENTS
       ----------------------------------------------------------------
        PRINT RESULTS KE                                                   *092
        KE=KE+1                                                            *093
        PRINT FORMAT FM1                                                   *094
        PRINT FORMAT FM2                                                   *095
        THROUGH SW9, FOR M=1,1,XBAR.G.PHI(KE-2)                            *096
        XBAR=IPH*M                                                         *097         01
        MDOT=TAB.(XBAR,PHI(1),MF(1),1,1,4,KE-2,T6)                         *098         01
        VPHI=TAB.(XBAR,PHI(1),VPHI(1),1,1,4,KE-2,T7)                       *099         01
        MTOT=TAB.(XBAR,PHI(1),YINF(1),1,1,4,KE-2,T7)                       *100         01
        PUNCH FORMAT FM3, XBAR,MTOT,MDOT,VPHI                              *101         01
SW9     PRINT FORMAT FM3, XBAR,MTOT,MDOT,VPHI                              *102         01
        END OF PROGRAM                                                     *103
```

CORRESPONDENCE BETWEEN NOMENCLATURE USED IN COMPUTER PROGRAM FOR CYLINDER PROBLEM WITH $E \leq 0.1$ AND NOMENCLATURE USED IN TEXT

ALP(2*J)	~	α_{2j}
BC.(I,J)	~	$\binom{I}{J}$
BET(2*J)	~	β_{2j}
DEL(I)	~	$\tilde{\delta}(x)$ (tabulated value)
DELT	~	$\tilde{\delta}$
E	~	E
F.(N)	~	N!
G	~	G_k
NU	~	$N_u \Big/ \sqrt{\frac{2R_O U_\infty}{\nu}}$
P(I)	~	η
PD(J)	~	$\tilde{x}$ (tabulated value)
PR	~	P_r
RINF	~	$2R_O U_\infty/\nu_g$
RR	~	r_d/R_O
V(2*K,I)	~	$F_{2k}(\eta)$
VP(2*K,I)	~	$F'_{2k}(\eta)$
X	~	η
XI	~	$\tilde{x}$

```
$COMPILE MAD, PRINT OBJECT                                    013985   07/23/65      3 38 53.1 AM

MAD (01 MAY 1965 VERSION) PROGRAM LISTING ... ... ...


                 SOLUTION TO ENERGY EQUATION FOR SMALL  'E'

                   *       *      *        *         *         *
                     SET UP
               ------------------------------------------------------------
                INTEGER K,KMAX,I,J,N,NMAX,JMAX,IJ,IM,IN
                INTEGER II
                INTEGER T,TMAX
                INTEGER M
                DIMENSION ALP(50),BET(50),R(200),P(150),NO(75)
                DIMENSION DEL(200),PD(200)
                DIMENSION Y(3),U(3),D(3)
                EXECUTE SETRKD.(2,Y(1),U(1),D,X,STEP)
                VECTOR VALUES FM1=$1H1/1H-*$
                VECTOR VALUES FM2=$1H ,S4,82(1H-)*$
                VECTOR VALUES FM3=$1H ,S4,F5.3,S4,10F13.6 *$
                VECTOR VALUES FM4=$1H ,S4,I3,2F20.6 *$
                VECTOR VALUES FM5=$1H ,S4,10F13.6 *$
                VECTOR VALUES FM6=$1H ,S4,F5.3,F12.6*$
                VECTOR VALUES FM7=$1H ,S4,F5.3,I3*$
                DIMENSION V(2000,VS)
                DIMENSION VP(2000,PS)
                VECTOR VALUES VS=2,150,150
                VECTOR VALUES PS=2,150,150
                READ AND PRINT DATA
               ------------------------------------------------------------
                 DEFINITION OF F.
               ------------------------------------------------------------
                INTERNAL FUNCTION (IJ)
                ENTRY TO F.
                FACT=1
                WHENEVER IJ.E.0, FUNCTION RETURN FACT
                THROUGH LOOP, FOR II=1,1,II.G.IJ
 LOOP           FACT=FACT*II
                FUNCTION RETURN FACT
                END OF FUNCTION
               ------------------------------------------------------------
                 DEFINITION OF BC.
               ------------------------------------------------------------
                INTERNAL FUNCTION BC.(IM,IN)=F.(IM)/(F.(IN)*F.(IM-IN))
               ------------------------------------------------------------
                 INPUT FROM MOM. PROGRAM
               ------------------------------------------------------------
                T=0
 SW13           T=T+1
                READ FORMAT FM7, E,M
                THROUGH SW10, FOR J=1,1,J.G.M
 SW10           READ FORMAT FM6, PD(J),DEL(J)
                THROUGH SW1, FOR J=1,1,J.G.JMAX
 SW1            READ FORMAT FM4, NO(J),ALP(2*J-2),BET(2*J)
               ------------------------------------------------------------
                 CALC. OF INITIAL COND'S FROM ERROR IN B.C.
```

```
          ----------------------------------------------------------
          K=0                                                          *037
SW2       YX=1.0                                                       *038
          DY=0.001                                                     *039
          R(0)=1.5                                                     *040
          R(1)=1.0                                                     *041
          N=1                                                          *042
SW4       Q=CF*(R(N)/R(N-1)-1)                                         *043
          WHENEVER Q.G.87.2,Q=87.2                                     *044
          WHENEVER Q.L.(-87.2), Q=-87.2                                *045
          DX=-DY*((2.35040/(EXP.(Q)-EXP.(-Q)))+1)                      *046
          DY=DX                                                        *047
          N=N+1                                                        *048
          YX=YX+DY                                                     *049
          WHENEVER K.E.0                                               *050
          Y(1)=1.                                                      *051      01
          OTHERWISE                                                    *052      01
          Y(1)=0.                                                      *053      01
          END OF CONDITIONAL                                           *054      01
          Y(2)=YX                                                      *055
          ----------------------------------------------------------
          INTEGRATION OF DIFF. EQ'S
          ----------------------------------------------------------
SW3       I=0                                                          *056
          X=0.                                                         *057
START     I=I+1                                                        *058
          P(I)=X                                                       *059
          V(2*K,I)=Y(1)                                                *060
          VP(2*K,I)=Y(2)                                               *061
SW5       U(1)=Y(2)                                                    *062
          WHENEVER K.E.0                                               *063
          G=0.                                                         *064      01
          OR WHENEVER K.E.1                                            *065      01
          G=-ALP(2)*(X.P.2.)*VP(0,I)                                   *066      01
          OR WHENEVER K.G.1                                            *067      01
            G=                        -ALP(2*K)*(X.P.2.)*VP(0,I)       *068      01
          THROUGH SW6, FOR J=1,1,J.G.(K-1)                             *069      01
SW6       G=G+X*(2.*BC.(2*K,2*J-1)*V(2*J,I)*BET(2*(K-J+1))-BC.(2*K,2*J *070      01    01
        1 )*X*VP(2*J,I)*ALP(2*(K-J)))                                  *070
          END OF CONDITIONAL                                           *071      01
          U(2)=-PR*E*(ALP(0)*((X.P.2.)*Y(2)-4.*K*X*Y(1))-G)            *072
CALC      S=RKDEQ.(0)                                                  *073
          WHENEVER S.E.1.0, TRANSFER TO SW5                            *074
          WHENEVER I.E.GRSIZ, TRANSFER TO SW7                          *075
          TRANSFER TO START                                            *076
          ----------------------------------------------------------
           CALC. OF ERROR IN B.C.
          ----------------------------------------------------------
SW7       WHENEVER K.E.0                                               *077
          EBC=0.                                                       *078      01
          OTHERWISE                                                    *079      01
          EBC=0.                                                       *080      01
          THROUGH SW8, FOR J=0,1,J.G.(K-1)                             *081      01
SW8       EBC=EBC-BC.(2*K,2*J)*ALP(2*(K-J))*V(2*J,I+1)                 *082      01    01
          END OF CONDITIONAL                                           *083      01
          WHENEVER .ABS.(Y(2)/(PR*E)+ALP(0)*Y(1)-EBC).G.ETOL .AND. N.L.NMAX *084
          R(N)=Y(2)/(PR*E)+ALP(0)*Y(1)-EBC                             *085      01
          TRANSFER TO SW4                                              *086      01
          ----------------------------------------------------------
            STORAGE OF SOL'S AT LAST GRID PT.
```

```
        -----------------------------------------------------------
         OTHERWISE                                                      *087      01
         PRINT RESULTS N                                                *088      01
         V(2*K,I+1)=Y(1)                                                *089      01
         VP(2*K,I+1)=Y(2)                                               *090      01
         END OF CONDITIONAL                                             *091      01
        -----------------------------------------------------------
        -----------------------------------------------------------
         K=K+1                                                          *092
        -----------------------------------------------------------
        CALC. AND PRINT-OUT OF NUSSELT NO.
        -----------------------------------------------------------
         PRINT RESULTS E,RR,RINF,PR                                     *093
         THROUGH SW12, FOR J=1,1,((J+1)*INXI).G.XIMAX                   *094
         XI=J*INXI                                                      *095          01
         DELT=TAB.(XI,PD(1),DEL(1),I,1,4,M,T6)                          *096          01
         NU=0.                                                          *097          01
         THROUGH SW11, FOR I=0,1,I.G.KMAX                               *098          01
SW11     NU=NU+VP(2*I,1)*(XI.P.(2*I))/(F.(2*I)*DELT)                    *099          02
SW12     PRINT RESULTS XI,NU                                            *100          01
        -----------------------------------------------------------
         PRINT STATEMENTS
        -----------------------------------------------------------
         PRINT FORMAT FM1                                               *101
         PRINT FORMAT FM2                                               *102
         THROUGH SW9, FOR I=1,1,I.G.(GRSIZ+1)                           *103
SW9      PRINT FORMAT FM3, P(I),V(0,I),V(2,I),V(4,I),V(6,I),V(8,I)      *104          01
         PRINT FORMAT FM1                                               *105
         PRINT FORMAT FM2                                               *106
         PRINT FORMAT FM5, VP(0,1),VP(2,1),VP(4,1),VP(6,1),VP(8,1)      *107
         WHENEVER T.LE.TMAX, TRANSFER TO SW13                           *108
         END OF PROGRAM                                                 *109
```

Note: that all quantities which occur in program for the oscillating plate are either pure imaginary or real. The equations are written in this program so that the $i = \sqrt{-1}$ does not appear. It is to be understood that all quantities calculated in this program which correspond to pure imaginary numbers should be multiplied by i to get the quantities appearing in the text.

CORRESPONDENCE BETWEEN NOMENCLATURE USED IN COMPUTER PROGRAM FOR OSCILLATING PLATE WITH NOMENCLATURE IN TEXT

A(O) ~ a

A(J) ~ a_{1j}

A2(J) ~ a_{2j}

A2H(J) ~ $\tilde{a}_{2j}$

AD1 ~ $|\delta_1^*|$

AF1 ~ $|\tau_1^*|$

AN1 ~ $|q_1^*|$

ARD1 ~ φ_{δ_1}

ARF1 ~ φ_{τ_1}

ARN1 ~ φ_{q_1}

DELTA.(I,J) ~ δ_{ij}

E1(J) ~ e_{1j}

E2(J) ~ e_{2j}

EH(J) ~ $\tilde{e}_{2j}$

G1(J) ~ G_{1j}

G2(J)	~	G_{2_j}
GH(J)	~	$\tilde{G}_{2_j}$
H(I)	~	nonhomogeneous part of differential equations for momentum problem, terms appearing in nonhomogeneous part of differential equations for energy problem.
NU	~	$(N_{u_{avg.}} - N_{u_{steady}})/\sqrt{\frac{xU_\infty}{\nu}}$
P(I)	~	η
PB1	~	$(\delta/x)\sqrt{\frac{U_\infty x}{\nu}}$
PB2	~	$\delta_o\sqrt{\frac{\omega}{\nu}}$
PB3	~	$\tau_o\sqrt{\frac{U_\infty x}{\nu}} \Big/ \frac{1}{2}\rho U_\infty{}^2$
PI	~	$\pi/2$
PR	~	P_r
R	~	R_j, r_j; j = 0,2,4,... depending on which equation is being solved
RH	~	$\tilde{R}_j$, $\tilde{r}_j$; j = 0,2,4,... depending on which equation is being solved
RHS	~	nonhomogeneous part of differential equation for energy problem
SG	~	u_o^*
SG1	~	g_{10}
SG12	~	g_{12}
SG14	~	g_{14}
SG2	~	g_{20}
SG22	~	g_{22}
SG24	~	g_{24}
SGH	~	$\tilde{g}_{20}$

SGH4	~	$\tilde{g}_{24}$	
U(I)	~	$f_o(\eta)$	
UP(I)	~	$f_o'(\eta)$	
UDP(I)	~	$f_o''(\eta)$	
V(J,I)	~	$f_{1j}(\eta)$	
VP(J,I)	~	$f_{1j}'(\eta)$	j,J = 1,3,5,...
VDP(J,I)	~	$f_{1j}''(\eta)$	
V(J,I)	~	$F_{1j}(\eta)$	
VP(J,I)	~	$F_{1j}'(\eta)$	j,J = 0,2,4,...
VDP(J,I)	~	$F_{1j}''(\eta)$	
W(J,I)	~	$f_{2j}(\eta)$	
WP(J,I)	~	$f_{2j}'(\eta)$	j,J = 1,3,5,...
WDP(J,I)	~	$f_{2j}''(\eta)$	
W(J,I)	~	$F_{2j}(\eta)$	
WP(J,I)	~	$F_{2j}'(\eta)$	j,J = 0,2,4,...
WDP(J,I)	~	$F_{2j}''(\eta)$	
WH(J,I)	~	$\tilde{f}_{2j}(\eta)$	
WHP(J,I)	~	$\tilde{f}_{2j}'(\eta)$	j,J = 1,5,...
WHDP(J,I)	~	$\tilde{f}_{2j}''(\eta)$	
WH(J,I)	~	$\tilde{F}_{2j}(\eta)$	
WHP(J,I)	~	$\tilde{F}_{2j}'(\eta)$	j,J = 0,4,...
WHDP(J,I)	~	$F_{2j}''(\eta)$	
X	~	η	
XE	~	X_e	
Z(I)	~	$F_o(\eta)$	
ZP(I)	~	$F_o'(\eta)$	
ZDP(I)	~	$F_o''(\eta)$	

```
$COMPILE MAC, PRINT OBJECT                                                  014457   07/26/65     11 33 38.1 PM

MAD (01 MAY 1965 VERSION) PROGRAM LISTING ... ... ...

            PROGRAM FOR OSC. FLAT PLATE
          *    *    *     *     *     *     *

            SET   UP
         ------------------------------------------------------------
         DIMENSION Y(3),F(3),D(3)                                                  *001
         EXECUTE SETRKD.(3,Y(1),F(1),D,X,STEP)                                     *002
         INTEGER J,K,M,N ,T,EPS,GRSIZ,TAU,MP,NP ,I,T1,T2,NFE,NSE,TMAX              *003
       1 ,MMAX,NMAX,TS                                                             *003
         INTEGER DELTA.                                                            *004
       1 ,T3                                                                       *004
         VECTOR VALUES FM1=$1H1/1H- *$                                             *005
         VECTOR VALUES FM2=$1H ,S4,111(1H-)*$                                      *006
         VECTOR VALUES FM3=$1H ,S4,F5.3,S4,F10.6,2(S4,3F10.6), S4, 2F10.6*$        *007
         VECTOR VALUES FM4=$1H , F10.6, 2(S4,3F10.6), S4, 2F10.6 *$                *008
         DIMENSION P(150),U(150),UP(150),UDP(150),UTP(150),B(11),C(11)             *009
       1 ,A(11),A2(11),A2H(11),H(150),Z(150),ZP(150),ZDP(150)                      *009
         DIMENSION E1(6),E2(6),G1(6),G2(6),EH(6),GH(6)                             *010
         DIMENSION R1(50),R2(50)                                                   *011
         DIMENSION V(1400,VS)                                                      *012
         DIMENSION VP(1400,PS)                                                     *013
         DIMENSION VDP(1400,DS)                                                    *014
         DIMENSION W(1400,WT)                                                      *015
         DIMENSION WP(1400,PT)                                                     *016
         DIMENSION WDP(1400,DT)                                                    *017
         DIMENSION WH(1400,HT)                                                     *018
         DIMENSION WHP(1400,PH)                                                    *019
         DIMENSION WHDP(1400,DH)                                                   *020
         VECTOR VALUES VS=2,150,150                                                *021
         VECTOR VALUES PS=2,150,150                                                *022
         VECTOR VALUES DS=2,150,150                                                *023
         VECTOR VALUES WT=2,150,150                                                *024
         VECTOR VALUES PT=2,150,150                                                *025
         VECTOR VALUES DT=2,150,150                                                *026
         VECTOR VALUES HT=2,150,150                                                *027
         VECTOR VALUES PH=2,150,150                                                *028
         VECTOR VALUES DH=2,150,150                                                *029
         --------------------------------
             DEFINITION OF DELTA.
         --------------------------------
         INTERNAL FUNCTION (MP,NP)                                                 *030
         ENTRY TO DELTA.                                                           *031
         WHENEVER MP.E.NP                                                          *032
         FUNCTION RETURN 1                                                         *033      01
         OTHERWISE                                                                 *034      01
         FUNCTION RETURN 0                                                         *035      01
         END OF CONDITIONAL                                                        *036      01
         END OF FUNCTION                                                           *037
         --------------------------------
         INTEGER CAT ,CATM                                                         *038
         CAT=0                                                                     *039
SW31     CAT=CAT+1                                                                 *040
```

```
               READ AND PRINT DATA                                                   *041
               Z2=-1.00                                                              *042
               TS=1                                                                  *043
               TAU=0                                                                 *044
              ------------------------------------------------------------
                INDEX FOR MOM.EQU'S
              ------------------------------------------------------------
SW16           WHENEVER TAU.LE.T1                                                    *045
               T=2*TAU-1                                                             *046      01
               EPS=Z2.P.(TAU)                                                        *047      01
               OR WHENEVER TAU.LE.T2                                                 *048      01
               T=2*(TAU-T1)-1                                                        *049      01
               EPS=Z2.P.(TAU-T1)                                                     *050      01
               OR WHENEVER TAU.LE.T3                                                 *051      01
               T=4*(TAU-T2)-3                                                        *052      01
               EPS=Z2.P.(TAU-T2)                                                     *053      01
               END OF CONDITIONAL                                                    *054      01
              ------------------------------------------------------------
                 CALC. OF CORRECTED VALUES OF CONSTANT PARAMETER FROM
                ERRORS IN BOUNDARY COND'S
              ------------------------------------------------------------
               TA=ANO                                                                *055
               DA=0.001                                                              *056
               DY=0.001                                                              *057
               R1(0)=1.5                                                             *058
               R1(1)=1.0                                                             *059
               M=1                                                                   *060
SW2            Q1=CF1*(R1(M)/R1(M-1)-1)                                              *061
               WHENEVER Q1.G.87.20,Q1=87.20                                          *062
               WHENEVER Q1.L.(-87.20),Q1=-87.20                                      *063
               DB=-DA*((2.35040/(EXP.(Q1)-EXP.(-Q1)))+1)                             *064
               DA=DB                                                                 *065
               M=M+1                                                                 *066
               TA=TA+DA                                                              *067
               AN=A(0)                                                               *068
               AP=TA                                                                 *069
              ------------------------------------------------------------
                  CALC. OF CORR. VALUES OF INITIAL COND'S FROM PREVICUS
                ERRORS IN BOUNDARY COND'S
              ------------------------------------------------------------
               YX=YCP                                                                *070
SW1            R2(0)=1.5                                                             *071
               R2(1)=1.0                                                             *072
               N=1                                                                   *073
SW4            Q2=CF2*(R2(N)/R2(N-1)-1)                                              *074
               WHENEVER Q2.G.87.20,Q2=87.20                                          *075
               WHENEVER Q2.L.(-87.20),Q2=-87.20                                      *076
               DX=-DY*((2.35040/(EXP.(Q2)-EXP.(-Q2)))+1)                             *077
               DY=DX                                                                 *078
               N=N+1                                                                 *079
               YX=YX+DY                                                              *080
               WHENEVER TS.E.1                                                       *081
               Y(3)=YX                                                               *082      01
               Y(2)=0                                                                *083      01
               Y(1)=0                                                                *084      01
               OR WHENEVER TS.E.0                                                    *085      01
               Y(2)=YX                                                               *086      01
               Y(1)=DELTA.(F,TAU)                                                    *087      01
               END OF CONDITIONAL                                                    *088      01
              ------------------------------------------------------------
```

INTEGRATION OF DIFF. EQ'S

```
         ------------------------------------------------------------
SW3.     I=0                                                    *089
         X=0                                                    *090
START    I=I+1                                                  *091
         ------------------------------------------------------------
         P(I)=X                                                 *092
         WHENEVER TAU.E.0 .AND. TS.E.1                          *093
         U(I)=Y(1)                                              *094    01
         UP(I)=Y(2)                                             *095    01
         UDP(I)=Y(3)                                            *096    01
         OR WHENEVER TAU.E.0 .AND. TS.E.0                       *097    01
         Z(I)=Y(1)                                              *098    01
         ZP(I)=Y(2)                                             *099    01
         ZDP(I)=F(2)                                            *100    01
         OR WHENEVER TAU.LE.T1                                  *101    01
         V(T,I)=Y(1)                                            *102    01
         VP(T,I)=Y(2)                                           *103    01
         WHENEVER TS.E.1, VDP(T,I)=Y(3)                         *104    01
         WHENEVER TS.E.0, VDP(T,I)=F(2)                         *105    01
         OR WHENEVER TAU.LE.T2                                  *106    01
         W(T,I)=Y(1)                                            *107    01
         WP(T,I)=Y(2)                                           *108    01
         WHENEVER TS.E.1, WDP(T,I)=Y(3)                         *109    01
         WHENEVER TS.E.0, WDP(T,I)=F(2)                         *110    01
         OR WHENEVER TAU.LE.T3                                  *111    01
         WH(T,I)=Y(1)                                           *112    01
         WHP(T,I)=Y(2)                                          *113    01
         WHENEVER TS.E.1, WHDP(T,I)=Y(3)                        *114    01
         WHENEVER TS.E.0, WHDP(T,I)=F(2)                        *115    01
         END OF CONDITIONAL                                     *116    01
         WHENEVER TS.E.1, UTP(I)=F(3)                           *117
         ------------------------------------------------------------
         WHENEVER TS.E.1                                        *118
         F(1)=Y(2)                                              *119    01
         F(2)=Y(3)                                              *120    01
         WHENEVER TAU.E.0                                       *121    01
         F(3)=-2.*XE*(AP.P.2)*Y(1)*Y(3)                         *122    02
         OTHERWISE                                              *123    02
         F(3)=-2*(AN.P.2)*(U(I)*Y(3)-(T-1)*UP(I)*Y(2)+T*UDP(I)*Y(1)+(   *124    02
       1 (T-1)*(UP(I).P.2)+U(I)*UDP(I))*AP-H(I))*XE             *124
         END OF CONDITIONAL                                     *125    02
         OR WHENEVER TS.E.0                                     *126    01
         F(1)=Y(2)                                              *127    01
         WHENEVER TAU.E.0                                       *128    01
         RHS=0                                                  *129    02
         OR WHENEVER TAU.LE.T1                                  *130    02
         RHS=-ZP(I)*((T+1)*V(T+1,I)+U(I)*A(T+1)-2.*(1-DELTA.(0,T))*EP   *131    02
       1 S*I*STEP*A(T-1))-2.*(1-DELTA.(0,T))*EPS*V(T-2,I)       *131
         OR WHENEVER TAU.LE.T2                                  *132    02
         RHS=-ZP(I)*((T+1)*W(T+1,I)+U(I)*A2(T+1)+4.*(1-DELTA.(0,T))*E   *133    02
       1 PS*I*STEP*A2(T-1))+4.*(1-DELTA.(0,T))*EPS*W(T-2,I)+H(I)        *133
         OR WHENEVER TAU.LE.T3                                  *134    02
         RHS=-ZP(I)*((T+1)*WH(T+1,I)+U(I)*A2H(T+1))+H(I)        *135    02
         END OF CONDITIONAL                                     *136    02
         F(2)=-MU*(U(I)*Y(2)-T*UP(I)*Y(1)-RHS)                  *137    01
         END OF CONDITIONAL                                     *138    01
CALC     S=RKDEQ.(0)                                            *139
         WHENEVER S.E.1.0,TRANSFER TO SW5                       *140
         WHENEVER I.E.GRSIZ,TRANSFER TO SW6                     *141
```

```
         TRANSFER TO START                                               *142
        -------------------------------------------------------------
          SELECT. OF WHICH BOUNDARY COND. TEST TO SUBJECT SOL'N TO
          DEPENDING ON WHETHER MOM. OR ENERGY EQU'S ARE BEING SOLVED
        -------------------------------------------------------------
SW6      WHENEVER TS.E.1, TRANSFER TO SW7                                *143
         WHENEVER TS.E.0, TRANSFER TO SW8                                *144
        -------------------------------------------------------------
            CALC. OF ERROR IN CONTINUITY BOUNDARY CONDITION
        -------------------------------------------------------------
SW7      WHENEVER TAU.E.0                                                *145
         IBC=1.0                                                         *146      01
         OR WHENEVER TAU.LE.T1                                           *147      01
         WHENEVER T.E.1                                                  *148      01
         IBC=AP-0.5                                                      *149      02
         OTHERWISE                                                       *150      02
         IBC=   AP+2*(Z2.P.TAU)*(1-XE)*A(T-2)/T                          *151      02
         END OF CONDITIONAL                                              *152      02
         OR WHENEVER TAU.LE.T2                                           *153      01
         IBC=AP-0.25*EPS  *A(T)+(4.    *(1-DELTA.(1,T))*EPS*A2(T-2)      *154      01
       1 *(XE-1.)/(T*1.))                                                *154
         OR WHENEVER TAU.LE.T3                                           *155      01
         IBC=AP-0.25*A(T)                                                *156      01
         END OF CONDITIONAL                                              *157      01
         WHENEVER .ABS.(Y(1)-IBC).G.ITOL  .AND.N.L.NMAX                  *158
         R2(N)=Y(1)-IBC                                                  *159      01
         TRANSFER TO SW4                                                 *160      01
        -------------------------------------------------------------
             CALC. OF TERMS FOR MOM. BOUNDARY COND.
        -------------------------------------------------------------
         OTHERWISE                                                       *161      01
         WHENEVER TAU.E.0                                                *162      01
         A(0)=AP                                                         *163      02
         MBC=-(2.*   Y(3)/((2.*A).P.2))+1.-XE*Y(2)                       *164      02
         OR WHENEVER TAU.LE.T1                                           *165      02
         MBC=-(2.*   Y(3)/((2.*A).P.2))+(1.-XE*UP(I+1))*((T+1)*AP-2.*    *166      02
       1 XE*EPS*(1-DELTA.(1,T))*A(T-2)-DELTA.(1,T)*.5)+AP-XE*Y(2)-DEL    *166
       2 TA.(1,T)*.5                                                     *166
         OR WHENEVER TAU.E.(T1+1)                                        *167      02
         MBC=-(2.*   Y(3)/((2.*A).P.2))+(1.-XE*UP(I+1))*(2.*(AP+0.25     *168      02
       1  *A(1))-0.5*(A(1).P.2))+(AP-XE*Y(2))-(2.*A(1)-.5)*(A(1)-XE*V    *168
       2 P(1,I+1))+(3./4.)*A(1)-(1./8.)                                  *168
         OR WHENEVER TAU.LE.T2                                           *169      02
         MBC=-(2.*   Y(3)/((2.*A).P.2))+(1.-XE*UP(I+1))*((T+1)*(AP-0.    *170      02
       1 25*EPS  *A(T))+EPS*4.*XE*A2(T-2)-XE*A(1)*A(T-2) +(T+1)*0.5*E    *170
       2 PS*A(1)*A(T))+AP-XE*Y(2)+EPS*(2.*A(1)-.5)*(A(T)-XE*VP(T,I+1)    *170
       3 )+(A(1)-XE*VP(1,I+1))*(EPS*(T+1)*A(T)-2.*XE*A(T-2))-0.25*(      *170
       4 EPS* (T+2)*A(T)-2.*XE*A(T-2))+DELTA.(5,T)*(A(3)*((3./2.)*A(3    *170
       5 )-XE*A(1))+(A(3)-XE*VP(3,T+1))*(4.*A(3)-2.*XE*A(1))-XE*UP(I+    *170
       6 1)*A(3)*((3./2.)*A(3)-XE*A(1)))                                 *170
         OR WHENEVER TAU.E.(T2+1)                                        *171      02
         MBC=-(2.*   Y(3)/((2.*A).P.2))+(1.-XE*UP(I+1))*(2.*(AP-0.25     *172      02
       1  *A(1))+0.5*(A(1).P.2))+(AP-XE*Y(2))+(2.*A(1)-.5)*(A(1)-XE*V    *172
       2 P(1,I+1))-(3./4.)*A(1)+(1./8.)                                  *172
         OR WHENEVER TAU.LE.T3                                           *173      02
         MBC=-(2.*   Y(3)/((2.*A).P.2))+(1.-XE*UP(I+1))*((T+1)*(AP-0     *174      02
       1 .25*A(T))+XE*A(1)*A(T-2)+0.5*EPS*A(1)*A(T)*(T+1))+AP-XE*Y(2)    *174
       2 +(2.*A(1)-.5)*(A(T)-XE*VP(T,I+1))+(A(1)-XE*VP(1,I+1))*(EPS*(    *174
       3 T+1)*A(T)+2.*XE*A(T-2))-0.25  *((T+2)*A(T)+2.*EPS*XE*A(T-2))    *174
       4 +DELTA.(5,T)*(A(3)*((3./2.)*A(3)-XE*A(1))+(A(3)-XE*VP(3,I+1)    *174
```

```
5 )*(4.*A(3)-2.*XE*A(1))-XE*UP(I+1)*A(3)*((3./2.)*A(3)-XE*A(1)     *174
6 ))                                                              *174
  END OF CONDITIONAL                                              *175    02
  END OF CONDITIONAL                                              *176    01
 -------------------------------------------------------------
    CALC. OF ERROR IN MOM. BOUNDARY CONDITION
 -------------------------------------------------------------
  WHENEVER .ABS.MBC .G.MTOL  .AND. M.L.MMAX                       *177
  R1(M)=MBC                                                       *178    01
  TRANSFER TO SW2                                                 *179    01
 -------------------------------------------------------------
     STORAGE OF SOL'S AT LAST GRID PT.
 -------------------------------------------------------------
  OTHERWISE                                                       *180    01
  WHENEVER TAU.E.0                                                *181    01
  U(I+1)=Y(1)                                                     *182    02
  UP(I+1)=Y(2)                                                    *183    02
  UDP(I+1)=Y(3)                                                   *184    02
  OR WHENEVER TAU.LE.T1                                           *185    02
  A(T)=AP                                                         *186    02
  E1(T-1)=VDP(T,1)-2.*UDP(1)*A(T)                                 *187    02
  V(T,I+1)=Y(1)                                                   *188    02
  VP(T,I+1)=Y(2)                                                  *189    02
  VDP(T,I+1)=Y(3)                                                 *190    02
  OR WHENEVER TAU.LE.T2                                           *191    02
  A2(T)=AP                                                        *192    02
  E2(T-1)=WDP(T,1)-2.*UDP(1)*A2(T)-R                              *193    02
  W(T,I+1)=Y(1)                                                   *194    02
  WP(T,I+1)=Y(2)                                                  *195    02
  WDP(T,I+1)=Y(3)                                                 *196    02
  OR WHENEVER TAU.LE.T3                                           *197    02
  A2H(T)=AP                                                       *198    02
  EH(T-1)=WHDP(T,1)-2.*UDP(1)*A2H(T)-RH                           *199    02
  WH(T,I+1)=Y(1)                                                  *200    02
  WHP(T,I+1)=Y(2)                                                 *201    02
  WHDP(T,I+1)=Y(3)                                                *202    02
  END OF CONDITIONAL                                              *203    02
  UTP(I+1)=F(3)                                                   *204    01
  END OF CONDITIONAL                                              *205    01
 -------------------------------------------------------------
    CALC. OF TERMS TO BE USED IN HIGHER ORDER MOM. EQU'S
 -------------------------------------------------------------
  THROUGH SW14,FOR J=1,1,J.G.(GRSIZ+1)                            *206
  WHENEVER TAU.E.0                                                *207        01
  H(J)=0                                                          *208    01  01
  OR WHENEVER TAU.L.T1                                            *209    01  01
  H(J)=-2.0*(Z2.P.TAU)*((UP(J)+STEP*J*UDP(J))*A(T)-VP(T,J))       *210    01  01
1 -DELTA.(1,T)*(1./XE)                                            *210
  OR WHENEVER TAU.E.T1                                            *211    01  01
  H(J)=0.5*((UDP(J)* V(1,J)+U(J)*VDP(1,J))*A(1)+VDP(1,J)*V(1,J    *212    01  01
1 ))                                                              *212
  OR WHENEVER TAU.L.T2                                            *213    01  01
  H(J) =       0.5*(Z2.P.(TAU-T1))*((2*(T+1)*UP(J)*VF(1,J)+UDP    *214    01  01
1 (J)* V(1,J)+U(J)*VDP(1,J))*A(T+2)-((T+1)*UP(J)*VP(T+2,J)-(T+    *214
2 2)*UDP(J)* V(T+2,J)-U(J)*VDP(T+2,J))*A(1)+(T+2)*VDP(1,J)*V(T    *214
3 +2,J)+V(1,J)*VDP(T+2,J))+A(T)*(STEP*J*VDP(1,J)-VP(1,J)+A(1)*    *214
4 (STEP*J*UDP(J)+UP(J)))-4*(Z2.P.(TAU-T1))*(WP(T,J)-(UP(J)+STE    *214
5 P*J*UDP(J))*A2(T))-DELTA.(3,T)*(0.5*(3*V(3,J)*VDP(3,J)-2*(VP    *214
6 (3,J).P.2))-A(1)*(STEP*J*VDP(3,J)-VP(3,J))-A(1)*A(3)*(STEP*J    *214
7 *UDP(J)+UP(J))+0.5*(U(J)*VDP(3,J)+ 3*UDP(J)*V(3,J)+2.*UP(J)*    *214
```

```
      8 VP(3,J))*A(3))-0.5*EPS*(T+1)*VP(1,J)*VP(T+2,J)-(3./(2.*XE))*     *214
      9 A(T)                                                             *214
        OR WHENEVER TAU.E.T2                                             *215    01    01
        H(J)=-0.5*((UDP(J)* V(1,J)+U(J)*VDP(1,J))*A(1)+VDP(1,J)*V(1,     *216    01    01
      1 J))                                                              *216
        OR WHENEVER TAU.L.T3                                             *217    01    01
        H(J)=-3*A(T+2)/(2.*XE)-0.5*         ((2*(T+3)*UP(J)*VP(1,J)+U   *218    01    01
      1 DP(J)* V(1,J)+U(J)*VDP(1,J))*A(T+4)-((T+3)*UP(J)*VP(T+4,J)-(     *218
      2 T+4)*UDP(J)* V(T+4,J)-U(J)*VDP(T+4,J))*A(1)-(T+3)*               *218
      3 VP(1,J)*VP(T+4,J)+(T+4)*VDP(1,J)*V(                              *218
      4 T+4,J)+V(1,J)*VDP(T+4,J))+(A(1)*(STEP*J*VDP(3,J)+3*VP(3,J))-     *218
      5 A(3)*(STEP*J*VDP(1,J)+3.*VP(1,J)))-0.5*(3.*V(3,J)*VDP(3,J)-2     *218
      6 .*(VP(3,J).P.2)+(U(J)*VDP(3,J)+ 3.*UDP(J)*V(3,J)+2.*UP(J)*VP     *218
      7 (3,J))*A(3))                                                     *218
SW14    END OF CONDITIONAL                                               *219    01    01
        WHENEVER TAU.E.T1, R=-0.5*(2.*VDP(1,1)-3.*UDP(1)*A(1))*A(1)      *220
        WHENEVER TAU.E.(T1+1), R=0.5*(2.*A(3)*VDP(1,1)+2.*A(1)*VDP(3,1)-6 *221
      1 .*A(1)*A(3)*UDP(1))                                              *221
        WHENEVER TAU.E.(T1+2), R=-0.5*(2.*A(5)*VDP(1,1)-2.*A(3)*VDP(3,1)+ *222
      1 2.*A(1)*VDP(5,1)-3.*UDP(1)*(2.*A(1)*A(5)-(A(3).P.2)))            *222
        WHENEVER TAU.E.T2, RH=0.5*(2.*VDP(1,1)-3.*UDP(1)*A(1))*A(1)      *223
        WHENEVER TAU.E.(T2+1), RH=0.5*(2.*A(5)*VDP(1,1)+2.*A(3)*VDP(3,1)+ *224
      1 2.*A(1)*VDP(5,1)-3.*UDP(1)*(2.*A(1)*A(5)+(A(3).P.2)))            *224
       ---------------------------------------------------------------
         TEST LOC. TO DET. WHICH MOM. EQ. IS BEING SOLVED AND
         WHEN MOM. PROB. HAS BEEN COMP.
       ---------------------------------------------------------------
SW20    TAU=TAU+1                                                        *225
        WHENEVER TAU.LE.T3, TRANSFER TO SW16                             *226
       ---------------------------------------------------------------
         SET UP OF ENERGY PROB.
       ---------------------------------------------------------------
        T=0                                                              *227
        TAU=0                                                            *228
        TS=0                                                             *229
        MU=2.*(A.P.2)*PR*XE                                              *230
        YX=YP                                                            *231
        TRANSFER TO SW1                                                  *232
       ---------------------------------------------------------------
              CALC. OF TERMS FOR ENERGY BOUNDARY COND.
       ---------------------------------------------------------------
SW8     WHENEVER TAU.E.0                                                 *233
        EBC=0                                                            *234    01
        OR WHENEVER TAU.LE.T1                                            *235    01
        EBC=((2+T)*A(T+1)-0.5*DELTA.(0,T)-2.*EPS*(1-DELTA.(0,T))*XE*     *236    01
      1 A(T-1))*Z(I+1)                                                   *236
        OR WHENEVER TAU.E.(T1+1)                                         *237    01
        EBC=Z(T+1)*(2.*(A2(1)+0.25*A(1))-0.5*(A(1).P.2))-(A(1)-0.25)     *238    01
      1 *V(0,I+1)                                                        *238
        OR WHENEVER TAU.E.(T1+2)                                         *239    01
        EBC=Z(I+1)*(4.*(A2(3)-0.25*A(3))+2.*A(1)*A(3)+XE*(4.*A2(1)-(     *240    01
      1 A(1).P.2)))+(A(1)-0.25)*V(2,I+1)+0.5*(4.*A(3)-2.*XE*A(1))*V(     *240
      2 0,I+1)                                                           *240
        OR WHENEVER TAU.E.(T1+3)                                         *241    01
        EBC=Z(I+1)*(6.*(A2(5)+0.25*A(5))-3.*A(1)*A(5)-XE*(4.*A2(3)+A     *242    01
      1 (1)*A(3)))-(A(1)-0.25)*V(4,I+1)-0.5*(6.*A(5)+2.*XE*A(3))*V(0     *242
      2 ,I+1)+0.5*(4.*A(3)-2.*XE*A(1))*V(2,I+1)+A(3)*(1.5*A(3)-XE*A(     *242
      3 1))*Z(I+1)                                                       *242
        OR WHENEVER TAU.E.(T2+1)                                         *243    01
        EBC=Z(I+1)*(2.*(A2H(1)-0.25*A(1))+0.5*(A(1).P.2))+(A(1)-0.25     *244    01
```

```
      1 )*V(0,I+1)                                                          *244
        OR WHENEVER TAU.E.(T2+2)                                            *245     01
        EBC=Z(I+1)*(6.*(A2H(5)-0.25*A(5))+3.*A(1)*A(5)+XE*A(1)*A(3))        *246     01
      1 +(A(1)-0.25)*V(4,I+1)+0.5*(6.*A(5)+2.*XE*A(3))*V(0,I+1)+0.5*        *246
      2 (4.*A(3)-2.*XE*A(1))*V(2,I+1)+A(3)*(1.5*A(3)-XE*A(1))*Z(I+1)        *246
        END OF CONDITIONAL                                                  *247     01
       ------------------------------------------------------------
          CALC. OF ERROR IN ENERGY B.C.
       ------------------------------------------------------------
        WHENEVER .ABS.((Y(2)/MU)+Y(1)+EBC).G.ETOL .AND. N.L.NMAX            *248
        R2(N)=(Y(2)/MU)+Y(1)+EBC                                            *249     01
        TRANSFER TO SW4                                                     *250     01
       ------------------------------------------------------------
           STORAGE OF SOL'S AT LAST GRID PT.
       ------------------------------------------------------------
        OTHERWISE                                                           *251     01
        WHENEVER TAU.E.0                                                    *252     01
        Z(I+1)=Y(1)                                                         *253     02
        ZP(I+1)=Y(2)                                                        *254     02
        ZDP(I+1)=F(2)                                                       *255     02
        OR WHENEVER TAU.LE.T1                                               *256     02
        G1(T)=VP(T,1)-ZP(1)*A(T+1)                                          *257     02
        V(T,I+1)=Y(1)                                                       *258     02
        VP(T,I+1)=Y(2)                                                      *259     02
        VDP(T,I+1)=F(2)                                                     *260     02
        OR WHENEVER TAU.LE.T2                                               *261     02
        G2(T)=WP(T,1)-ZP(1)*A2(T+1)-R                                       *262     02
        W(T,I+1)=Y(1)                                                       *263     02
        WP(T,I+1)=Y(2)                                                      *264     02
        WDP(T,I+1)=F(2)                                                     *265     02
        OR WHENEVER TAU.LE.T3                                               *266     02
        GH(T)=WHP(T,1)-ZP(1)*A2H(T+1)-RH                                    *267     02
        WH(T,I+1)=Y(1)                                                      *268     02
        WHP(T,I+1)=Y(2)                                                     *269     02
        WHDP(T,I+1)=F(2)                                                    *270     02
        END OF CONDITIONAL                                                  *271     02
        END OF CONDITIONAL                                                  *272     01
       ------------------------------------------------------------
          CALC. OF TERMS TO BE USED IN HIGHER ORDER ENERGY EQU'S
       ------------------------------------------------------------
        THROUGH SW15, FOR J=1,1,J.G.(GRSIZ+1)                               *273
        WHENEVER TAU.E.T1, H(J)=0.5*ZP(J)*A(1)*V(1,J)+0.5*VP(0,J)*(V        *274        01
      1 (1,J)+U(J)*A(1))                                                    *274
        WHENEVER TAU.E.(T1+1), H(J)=-0.5*ZP(J)*(3.*A(1)*V(3,J)-2.*ST        *275        01
      1 EP*J*(A(1).P.2)+A(3)*A(1))+0.5*(2.*V(2,J)*(VP(1,J)+U(J)*A(1)        *275
      2 )-VP(2,J)*(V(1,J)+U(J)*A(1))-VP(0,J)*(3.*V(3,J)+U(J)*A(3)))-        *275
      3 A(1)*(2.*V(0,J)-STEP*J*VP(0,J))                                     *275
        WHENEVER TAU.E.(T1+2),H(J)=0.5*ZP(J)*(5.*A(1)*V(5,J)+4.*STEP*J*A(   *276        01
      1 1)*A(3)-3.*A(3)*V(3,J)+A(5)*V(1,J))-0.5*(4.*V(4,J)*(VP(1,J)+        *276
      2 A(1)*UP(J))-VP(4,J)*(V(1,J)+A(1)*U(J))-2.*V(2,J)*(VP(3,J)+A(        *276
      3 3)*UP(J))+VP(2,J)*(3.*V(3,J)+A(3)*U(J))-VP(0,J)*(5.*V(5,J)+A        *276
      4 (5)*U(J)))-A(1)*(2.*V(2,J)-STEP*J*VP(2,J))-A(3)*(2.*V(0,J)-S        *276
      5 TEP*J*VP(0,J))                                                      *276
        WHENEVER TAU.E.T2, H(J)=-0.5*ZP(J)*A(1)*V(1,J)-0.5*VP(0,J)*(        *277        01
      1 V(1,J)+A(1)*U(J))                                                   *277
SW15    WHENEVER TAU.E.(T2+1), H(J)=-0.5*ZP(J)*(5.*A(1)*V(5,J)+3.*A(3)*V(   *278        01
      1 3,J)+A(5)*V(1,J))+0.5*(4.*V(4,J)*(VP(1,J)+A(1)*UP(J))-VP(4,J        *278
      2 )*(V(1,J)+U(J)*A(1))+2.*V(2,J)*(VP(3,J)+UP(J)*A(3))-VP(2,J)*        *278
      3 (3.*V(3,J)+U(J)*A(3))-VP(0,J)*(5.*V(5,J)+U(J)*A(5)))+A(1)*(2        *278
      4 .*V(2,J)+STEP*J*VP(2,J))-A(3)*(2.*V(0,J)+STEP*J*VP(0,J))            *278
```

```
          WHENEVER TAU.E.T1, R=-0.5*A(1)*G1(0)                                        *279
          WHENEVER TAU.E.(T1+1), R=0.5*(A(1)*G1(2)+A(3)*G1(0))                        *280
          WHENEVER TAU.E.(T1+2), R=-0.5*(A(5)*G1(0)+A(1)*G1(4)-A(3)*G1(2))            *281
          WHENEVER TAU.E.T2, RH=0.5*A(1)*G1(0)                                        *282
          WHENEVER TAU.E.(T2+1), RH=0.5*(A(5)*G1(0)+A(1)*G1(4)+A(3)*G1(2))            *283
         ------------------------------------------------------------
          TEST LOC. TO DET. WHICH ENERGY EQ. IS BEING SOLVED AND
           WHEN ENERGY PROB. IS COMPL.
         ------------------------------------------------------------
          TAU=TAU+1                                                                   *284
          YX=YP                                                                       *285
          WHENEVER TAU.LE.T1                                                          *286
          T=2*TAU-2                                                                   *287      01
          EPS=Z2.P.TAU                                                                *288      01
          OR WHENEVER TAU.LE.T2                                                       *289      01
          T=2*(TAU-T1)-2                                                              *290      01
          EPS=Z2.P.(TAU-T1)                                                           *291      01
          OR WHENEVER TAU.LE.T3                                                       *292      01
          T=4*(TAU-T2)-4                                                              *293      01
          EPS=Z2.P.(TAU-T2)                                                           *294      01
          END OF CONDITIONAL                                                          *295      01
          WHENEVER TAU.LE.T3, TRANSFER TO SW1                                         *296
         ------------------------------------------------------------
             PRINT STATEMENTS AND MISC. CALC'S
         ------------------------------------------------------------
          PRINT FORMAT FM1                                                            *297
          PRINT FORMAT FM2                                                            *298
          THROUGH SW10, FOR K=1,1,K.G.(GRSIZ+1)                                       *299
SW10      PRINT FORMAT FM3, P(K),U(K),V(1,K),V(3,K),V(5,K),W(1,K),W(3,K),W(5,K)       *300           01
        1 ,WH(1,K),WH(5,K)                                                            *300
          PRINT FORMAT FM1                                                            *301
          PRINT FORMAT FM2                                                            *302
          THROUGH SW9,FOR K=1,1,K.G.(GRSIZ+1)                                         *303
          SG=UP(K)                                                                    *304           01
          SG1=VP(1,K)-UP(K)*A(1)                                                      *305           01
          SG12=VP(3,K)-UP(K)*A(3)                                                     *306           01
          SG14=VP(5,K)-UP(K)*A(5)                                                     *307           01
          SG2=WP(1,K)-UP(K)*A2(1)+0.5*A(1)*SG1                                        *308           01
          SG22=WP(3,K)-UP(K)*A2(3)-0.5*(A(1)*SG12+A(3)*SG1)                           *309           01
          SG24=WP(5,K)-UP(K)*A2(5)+0.5*(A(5)*SG1+A(1)*SG14-A(3)*SG12)                 *310           01
          SGH=WHP(1,K)-UP(K)*A2H(1)-0.5*A(1)*SG1                                      *311           01
          SGH4=WHP(5,K)-UP(K)*A2H(5)-0.5*(A(5)*SG1+A(1)*SG14+A(3)*SG12                *312           01
        1 )                                                                           *312
SW9       PRINT FORMAT FM3, P(K),SG,SG1,SG12,SG14,SG2,SG22,SG24,SGH,SGH4              *313           01
          PRINT FORMAT FM1                                                            *314
          PRINT FORMAT FM2                                                            *315
          THROUGH SW12, FOR K=1,1,K.G.(GRSIZ+1)                                       *316
SW12      PRINT FORMAT FM3, P(K),UP(K),VP(1,K),VP(3,K),VP(5,K),WP(1,K),WP(3,K),       *317           01
        1 WP(5,K),WHP(1,K),WHP(5,K)                                                   *317
          PRINT FORMAT FM1                                                            *318
          PRINT FORMAT FM2                                                            *319
          THROUGH SW13, FOR K=1,1,K.G.(GRSIZ+1)                                       *320
SW13      PRINT FORMAT FM3, P(K),Z(K),V(0,K),V(2,K),V(4,K),W(0,K),W(2,K),W(4,K)       *321           01
        1 ,WH(0,K),WH(4,K)                                                            *321
          PRINT FORMAT FM1                                                            *322
          PRINT FORMAT FM4, A(0),A(1),A(3),A(5),A2(1),A2(3),A2(5),A2H(1),A2H(5)       *323
          PRINT FORMAT FM4, UDP(1),E1(0),E1(2),E1(4),E2(0),E2(2),E2(4),EH(0),EH       *324
        1 (4)                                                                         *324
          PRINT FORMAT FM4, ZP(1),G1(0),G1(2),G1(4),G2(0),G2(2),G2(4),GH(0),GH(       *325
        1 4)                                                                          *325
```

```
         PRINT FORMAT FM2                                               *326
         PI=1.5708                                                      *327
         THROUGH SW30, FOR XI=.1,0.1,XI.G.0.9                           *328
         AD1=2.*A*XI*(.ABS.A(1))*(1.+(0.5*((A(3).P.2)+2.*A(5)*A(1))/(   *329   01
       1 A(1).P.2))*(XI.P.4))*(XE.P.0.5)                                *329
         AF1=(.ABS.E1(0))*(1.+(0.5*((E1(2).P.2)+2.*E1(4)*E1(0))/(E1(0   *330   01
       1 ).P.2))*(XI.P.4))*XE/A                                         *330
         AN1=(.ABS.G1(0))*(1.+(0.5*((G1(2).P.2)+2.*G1(4)*G1(0))/(G1(0   *331   01
       1 ).P.2))*(XI.P.4))*(XE.P.0.5)/(2.*A)                            *331
         ARD1=PI-ATAN.((A(1)+A(5)*(XI.P.4))/(A(3)*(XI.P.2)))            *332   01
         ARF1=PI-ATAN.((E1(0)+E1(4)*(XI.P.4))/(E1(2)*(XI.P.2)))         *333   01
         ARN1=PI-ATAN.((G1(0)+G1(4)*(XI.P.4))/(G1(2)*(XI.P.2)))         *334   01
         NU=(GH(0)+GH(4)*(XI.P.2))*(XE.P.0.5)/(2.*A)                    *335   01
SW30     PRINT FORMAT FM4, XI,AD1,AF1,AN1,ARD1,ARF1,ARN1,NU             *336   01
         PB1=2.*A*(XE.P.0.5)                                            *337
         PB2=2.*A                                                       *338
         PB3=UDP(1)*XE/A                                                *339
         PB4=  ZP(1)*(XE.P.0.5)/(2.*A)                                  *340
         PRINT FORMAT FM2                                               *341
         PRINT FORMAT FM4, PB1,PB2,PB3,PB4                              *342
         WHENEVER CAT.LE.CATM, TRANSFER TO SW31                         *343
SW11     END OF PROGRAM                                                 *344
```

APPENDIX XXVI

COMPUTER PROGRAM FOR CALCULATING FILM THICKNESS, LOCAL SKIN FRICTION, AND THE TERMS OF THE EXPANSIONS (52)

This program calculates the liquid film thickness, local skin friction, and the terms of the expansions (52) from the numerical values of $\dot{\mathcal{M}}$, J, and V_φ^* obtained from the program for drop trajectories.

Once $\tilde{\delta}$ is calculated, the product $\dot{\mathcal{M}}\ \tilde{\delta}$ and J $\tilde{\delta}$ are formed, and are tabulated. The expansions (52) are assumed to contain a finite number of terms and a system of linear equations is obtained by writing these expansions at various values of ξ which we denote by ξ_i.

Thus we have:

$$\dot{\mathcal{M}}\tilde{\delta}(\xi_i) = \sum_{n=0}^{m} \frac{\alpha_{2n}}{(2n)!}\ \xi_i^{2n} \quad ; \quad i = 0,1,2,\ldots m$$

This system of equations is written in matrix form and solved for the α_{2n}'s by inverting the matrix. Thus if we denote the matrix A by:

$$A = \left[\frac{\xi_i^{2n}}{(2n)!}\right]$$

the vector $\tilde{c}$ by

$$\tilde{c} = \begin{bmatrix} \dot{m}\tilde{\delta}(\xi_o) \\ \dot{m}\tilde{\delta}(\xi_1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dot{m}\tilde{\delta}(\xi_m) \end{bmatrix}$$

and the vector $\tilde{\alpha}$ by:

$$\tilde{\alpha} = \begin{bmatrix} \alpha_o \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{2m} \end{bmatrix}$$

We have:

$$\tilde{c} = A\,\tilde{\alpha}$$

And:

$$\tilde{\alpha} = A^{-1}\,\tilde{c}$$

CORRESPONDENCE BETWEEN NOTATION FOR COMPUTER PROGRAM FOR CALCULATING FILM THICKNESS, LOCAL SKIN FRICTION, AND TERMS OF THE EXPANSION (52), AND NOTATIONS USED IN THE TEXT

AG(2K-1)	~	γ_{2k-1}
DEL(M)	~	$\tilde{\delta}$
DR	~	ρ_g/ρ
E	~	E
F.(N)	~	N!

FF(M)	~	$\tau_w \sqrt{\frac{2R_o U_\infty}{\nu}} \Big/ \frac{1}{2} \rho U_\infty{}^2$
GAM	~	γ
MDOT	~	$\dot{m}$
P(M)	~	$\tilde{x}$
R	~	τ_{eff}
RINF	~	$2R_o U_\infty / \nu_g$
RR	~	r_d / R_o
VR	~	$\sqrt{\frac{\nu_g}{\nu}}$
X	~	$\tilde{x}$
Z.(x)	~	Z(x)

```
$COMPILE MAD, PRINT OBJECT                                              013997   07/23/65     3 54 38.9 AM

MAD (01 MAY 1965 VERSION) PROGRAM LISTING ... ... ...

           *        *        *        *        *        *

             SET UP
           ------------------------------------------------------------
            INTEGER I                                                              *001
            INTEGER T,TMAX                                                         *002
            INTEGER IMAX ,SM                                                       *003
            INTEGER M,K,N,IP                                                       *004
            DIMENSION Q(8),D(8),AG(20)                                             *005
            DIMENSION C(100),D(100)                                                *006
            DIMENSION FF(200),A(200),B(200),ALP(200),BET(200),CV(200),P(200)       *007
            DIMENSION XBAR(500),MDOT(500),J(500),VPHI(500)                         *008
            DIMENSION DEL(200)                                                     *009
            DIMENSION C1(4000,CS)                                                  *010
            DIMENSION C2(4000,CT)                                                  *011
            VECTOR VALUES CS=2,1,60                                                *012
            VECTOR VALUES CT=2,1,60                                                *013
            VECTOR VALUES FM1=$1H1/1H-*$                                           *014
            VECTOR VALUES FM2=$1H ,S4,53(1H-)*$                                    *015
            VECTOR VALUES FM3=$1H ,S4,F5.3,2F12.6*$                                *016
            VECTOR VALUES FM4=$1H ,S4,I3,2F20.6*$                                  *017
            VECTOR VALUES FM5=$1H ,S4,4F12.6*$                                     *018
            VECTOR VALUES FM6=$1H ,S4,F5.3,F12.6*$                                 *019
            VECTOR VALUES FM7=$1H ,S4,F5.3,I3*$                                    *020
            READ AND PRINT DATA                                                    *021
            THROUGH SW9, FOR K=1,1,K.G.KMAX                                        *022
SW9         READ FORMAT FM5,  XBAR(K),MDOT(K),J(K),VPHI(K)                         *023
            AG(1)=1.2326                                                           *024
            AG(3)=0.7244                                                           *025
            AG(5)=1.0320                                                           *026
            AG(7)=0.5792+7.*0.1829+(70.*0.0076/3.)                                 *027
            AG(9)=0.5399+12.*0.1520+(126.*0.0572/5.)+84.*0.0607-280.*0.0           *028
          1 308                                                                    *028
            AG(11)=0.5100+(55.*0.1323/3.)+66.*0.0742+220.*0.0806+462.*0.           *029
          1 1164-1540.*0.1796+(15400.*0.0516/3.)                                   *029
            THROUGH SW4, FOR K=1,1,K.G.6                                           *030
SW4         PRINT RESULTS AG(2*K-1)                                                *031
           ------------------------------------------------------------
             DEFINITION OF F.
           ------------------------------------------------------------
            INTERNAL FUNCTION F.(IJ)=GAMMA.(IJ+1)                                  *032
           ------------------------------------------------------------
             DEFINITION OF Z.
           ------------------------------------------------------------
            INTERNAL FUNCTION (IX)                                                 *033
            ENTRY TO Z.                                                            *034
            SZ=AG(1)*IX                                                            *035
            THROUGH SW5, FOR K=2,1,K.G.6                                           *036
SW5         SZ=SZ-(((-1.).P.K)*(2*K)*AG(2*K-1)*(IX.P.(2*K-1))/F.(2.*K-1.           *037
          1 ))                                                                     *037
            FUNCTION RETURN SZ                                                     *038
```

```
        END OF FUNCTION                                                  *039
        ------------------------------------------------------------
          CALC. OF DELTA AND RELATED QUANTITIES
        ------------------------------------------------------------
        T=0                                                              *040
        E=0.                                                             *041
SW10    T=T+1                                                            *042
        E=E+0.02                                                         *043
        K1=DR/E                                                          *044
        K2=4.*K1*VR                                                      *045
        M=0                                                              *046
SW1     M=M+1                                                            *047
        X=M*INCX                                                         *048
        N=IP*M                                                           *049
        P(M)=X                                                           *050
        Q    =(4./3.)*K1*SIN.(2*X)                                       *051
        R    =K2*Z.(X)+J(N)*VPHI(N)                                      *052
        S=MDOT(N)                                                        *053
        WHENEVER (.ABS.(1.-(27./2.)*((Q.P.2)*S/(R.P.3)))).G.1.           *054
        M=M-1                                                            *055     01
        TRANSFER TO SW15                                                 *056     01
        OTHERWISE                                                        *057     01
        CONTINUE                                                         *058     01
        END OF CONDITIONAL                                               *059     01
        GAM=ARCCOS.(-(1.-(27./2.)*((Q.P.2)*S/(R.P.3))))                  *060
        DEL(M)=(R/(3.*Q))*(2.*COS.(GAM/3.)-1.)                           *061
        ------------------------------------------------------------
           CALC. OF LOCAL SKIN FRICTION
        ------------------------------------------------------------
        FF(M)=2.*E*R    +3.*E*Q    *DEL(M)                               *062
        ------------------------------------------------------------
          CALC. OF ALPHA SUB. 2J AND BETA SUB. 2J
        ------------------------------------------------------------
        C(M)=DEL(M)*J(N)                                                 *063
        D(M)=DEL(M)*MDOT(N)                                              *064
        WHENEVER X.L.XMAX, TRANSFER TO SW1                               *065
        THROUGH SW3, FOR K=1,1,K.G.IMAX                                  *066
        WHENEVER K.E.1                                                   *067          01
        X=P(1)                                                           *068     01   01
        A(1)=C(1)                                                        *069     01   01
        B(1)=D(1)                                                        *070     01   01
        OR WHENEVER K.E.IMAX                                             *071     01   01
        X=P(M)                                                           *072     01   01
        A(K)=C(M)                                                        *073     01   01
        B(K)=D(M)                                                        *074     01   01
        OTHERWISE                                                        *075     01   01
        X=X+(P(M)-P(1))/(IMAX-1.)                                        *076     01   01
        A(K)=TAB.(X,P(1),C(1),1,1,4,M,T6)                                *077     01   01
        B(K)=TAB.(X,P(1),D(1),1,1,4,M,T6)                                *078     01   01
        END OF CONDITIONAL                                               *079     01   01
        THROUGH SW3, FOR I=1,1,I.G.IMAX                                  *080          01
        C1(K,I)=(X.P.(2*I-2))                                            *081          02
SW3     C2(K,I)=(X.P.(2*I-1))                                            *082          02
        S1=SLE.(IMAX,60,C1(1,1),ALP(1),A(1),CV,0)                        *083
        PRINT RESULTS S1                                                 *084
        S2=SLE.(IMAX,60,C2(1,1),BET(1),B(1),CV,0)                        *085
        PRINT RESULTS S2                                                 *086
        ------------------------------------------------------------
          PRINT STATEMENTS
        ------------------------------------------------------------
```

```
            PRINT RESULTS RINF                                  *087
            PRINT RESULTS RR                                    *088
            PRINT RESULTS E,M                                   *089
            PUNCH FORMAT FM7, E,M                               *090
            PRINT FORMAT FM1                                    *091
            PRINT FORMAT FM2                                    *092
            THROUGH SW7, FOR K=1,1,K.G.M                        *093
            PUNCH FORMAT FM6, P(K),DEL(K)                       *094      01
SW7         PRINT FORMAT FM3, P(K),DEL(K),FF(K)                 *095      01
            PRINT FORMAT FM1                                    *096
            PRINT FORMAT FM2                                    *097
            THROUGH SW8, FOR K=1, 1,K.G.IMAX                    *098
            ALP(K)=ALP(K)*F.(2.*K-2.)                           *099      01
            BET(K)=BET(K)*F.(2.*K-1.)                           *100      01
            PUNCH FORMAT FM4, K,ALP(K),BET(K)                   *101      01
SW8         PRINT FORMAT FM4, K,ALP(K),BET(K)                   *102      01
            WHENEVER T.LE.TMAX, TRANSFER TO SW10                *103
END         END OF PROGRAM                                      *104
```

APPENDIX XXVII

REDUCTION OF EXPERIMENTAL DATA FROM REFERENCE (1)

From Reference 1 we shall consider the data corresponding to the SC-6 nozzle with an air velocity of 110 ft/sec and a water flow rate of 0.083 lb/sec, given in Table II-B, page 48. The reported value of $2R_oU_\infty/\nu_g$ was 8×10^4.

The measured total value of flow using a drop collecting device and performing a numerical integration was 0.061 lb/sec. The actual total flow obtained from their rotameters was 0.083 lb/sec. The measured value of the flow impinging on the cylinder using the drop collecting device was 0.002 lb/sec. We therefore take as a corrected value for the total flow impinging on the cylinder 0.002 x (0.083/0.061) or 0.00272 lb/sec. On page 24, Reference 1, is reported that for an air velocity of 110 ft/sec the drop velocity was 85 ft/sec. From Reference 23, Table E-2, ν = 0.045 ft^2/hr at the drop temperature of 57°. The diameter of their cylinder was 1.5 in.

Hence:

$$\frac{2R_oU_\infty}{\nu} = \frac{1.5}{12} \times \frac{3600}{.046} \times 85 = 8.30 \times 10^5$$

Since the cylinder is 2 in. in length, the mean liquid flux impinging on the cylinder

$$= \frac{\text{actual flow impinging on the cylinder}}{\text{area of cylinder}}$$

$$= \frac{.00272}{1.5x2} \times 144 = .13 \text{ lb/sec ft}^2$$

Hence the effective mass of liquid per unit volume in the free stream is:

$$\frac{.13}{85} = 1.52x10^{-3} \text{ lb/ft}^3$$

and the volume fraction of liquid in the free stream, X_e, is

$$X_e = \frac{1.52}{62.4} \times 10^{-3} = 2.44x10^{-5}$$

Hence:

$$\sqrt{\frac{2R_o U_\infty}{\nu}} = \sqrt{8.30x10^5} = 910$$

$$E = X_e \sqrt{\frac{2R_o U_\infty}{\nu}} = .022$$

From Reference 23, Table E-2, at a wall temperature of 78°, $k = .35$ Btu/hr-ft°F.

Hence:

$$\frac{2R_o}{k\sqrt{\frac{2R_o U_\infty}{\nu}}} = \frac{1.5}{12x.35x910} = 3.94$$

Table II-B (from Reference 1)

Position	h average Btu/ft^2-hr-°F	$NU\sqrt{\frac{2R_o U_\infty}{\nu}}$
0°	451	0.178
30°	372	0.147
60°	270	0.1065
330°	448	0.168