BOUNDARY LAYER ANALYSIS OF TWO-PHASE (LIQUID-GAS) FLOW OVER
A CIRCULAR CYLINDER AND OSCILLATING FLAT PLATE

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NOMENCLATURE

Notation for Cylinder Problem

Latin Letters

$a_0$ = constant appearing in the expansion for $\delta^*$

$A_n$ = function of the $f_{2n-1}'s$ and their derivatives and the $b_{2n}'s$ appearing in the ordinary differential equations for the momentum problem, $n = 1, 3, 5, ...$

$b_{2n}$ = constants appearing in the expansion for $\delta^*$; $n = 0, 1, 2, ...$

$B_n$ = function of $F_{2n}'s$ and $b_{2n}'s$ appearing in the ordinary differential equation for the energy problem; $n = 0, 2, 4$

$C_D$ = drag coefficient for liquid drops

$C_p$ = specific heat at constant pressure of liquid

$C_{pg}$ = specific heat at constant pressure of gas

$D^2$ = partial differential operator in momentum equations

$D_{xy}^2$ = $D^2$, rewritten in terms of $x, y$

$ar{D}^2$ = dimensionless version of $D^2$

$	ilde{D}^2$ = dimensionless version of $D^2$

$E^2$ = $2X_e^2 R_0 U_\infty / \nu$

$f_{2n-1}$ = $n = 1, 2, ..., function of \eta in expansion for \psi^*$

$F_{2n}$ = $n = 0, 1, 2, ..., function of \eta in expansion for \Theta^*$

$G_n$ = $n = 0, 2, 4, ..., function of f_{2n-1}'s and F_{2n}'s appearing in the differential equations for F_{2n}$

$H_n$ = $n = 1, 3, 5, ..., function of b_{2n}'s f_{2n-1}'s and their derivatives appearing in differential equations for f_{2n-1}'s$
NOMENCLATURE (Continued)

\( \hat{\mathbf{i}} \) = unit vector in \( x_0 \) direction

\( \hat{\mathbf{j}} \) = unit vector in \( y_0 \) direction

\( J \) = \(- \frac{x_e}{x_{e\infty}} V_r^* \)

\( k \) = thermal conductivity of the liquid

\( k_g \) = thermal conductivity of the gas

\( K_n \) = linear operator for energy equations

\( L_n \) = linear operator for momentum equations

\( \mathcal{N} \) = \(- \int_0^x \frac{x_e}{x_{e\infty}} V_r^* \, d\tilde{x} \)

\( \hat{n} \) = unit outward drawn normal to the cylinder

\( N' \) = number of liquid drops lying along a fluid line whose length is equal to the radius of the cylinder

\( N_o \) = number of drops per unit volume

\( N_u \) = Nusselt number of the liquid

\( N_{ug} \) = Nusselt number of the gas

\( P \) = pressure in the liquid

\( P_g \) = pressure in the gas

\( \overline{P} \) = \( P/1/2 \rho U_{\infty}^2 \)

\( \tilde{P} \) = \( P/1/2 \rho^2 U_{\infty}^2 \)

\( P_g^* \) = \( P_g/1/2 \rho_g U_{\infty}^2 \)

\( P_r \) = Prandtl Number of liquid

\( P_{rg} \) = Prandtl Number of gas

\( q_w \) = heat flux at cylinder wall
\( \dot{q}_n \) = terms of expansion for \( N_u/ \sqrt{\frac{2U_\infty R_\infty}{v}} \)

\( \dot{q}_g \) = heat flux into the gas at boundary of liquid film

\( r \) = radial coordinate

\( r_d \) = radius of drop

\( \vec{r}_d \) = position vector of drop

\( R_o \) = radius of cylinder

\( R_e \) = \( 2R_o U_\infty/v \)

\( s \) = surface of liquid film

\( t \) = time

\( T \) = temperature of liquid

\( T_w \) = temperature of cylinder wall

\( T_\infty \) = temperature of flow system at infinity

\( t^* \) = \( t U_\infty/R_o \)

\( u \) = \( v_\varphi \); velocity of liquid in film in azimuthal direction

\( u_n^* \) = terms in expansion for \( u/U_\infty \)

\( u_g^* \) = \( x_\circ \) component of gas velocity

\( u_g^* \) = \( u_g/U_\infty \)

\( U_\infty \) = velocity of liquid and gas at infinity

\( v_\varphi \) = velocity of liquid in film in azimuthal direction

\( v_r \) = velocity of liquid in film in radial direction

\( v_g \) = \( y_\circ \) component of gas velocity

\( v_g^* \) = \( V_g/U_\infty \)

\( v_\varphi \) = azimuthal component of drop velocity at edge of liquid film
NOMENCLATURE (Continued)

\[ V_r \] = radial component of drop velocity at edge of liquid film

\[ V^*_\phi \] = \[ V_\phi / U_\infty \]

\[ V^*_r \] = \[ V_r / U_\infty \]

\[ \vec{V} = \hat{i} V_{xo} + \hat{j} V_{yo} \]

\[ V_{xo} \] = drop velocity in \( x_o \) direction

\[ V_{yo} \] = drop velocity in \( y_o \) direction

\[ V^*_{xo} \] = \[ V_{xo} / U_\infty \]

\[ V^*_{yo} \] = \[ V_{yo} / U_\infty \]

\[ \vec{V}_g = \hat{i} u_g + \hat{j} v_g \]

\[ \vec{V}^*_g = \vec{V}_g / U_\infty \]

\[ \vec{V}^* = \vec{V} / U_\infty \]

\[ V_{yo}^f \] = drop velocity in \( y_o \) direction at edge of liquid film

\[ V_{xo}^f \] = drop velocity in \( x_o \) direction at edge of liquid film

\[ V^*_{yo} \] = \[ V_{yo}^f / U_\infty \]

\[ V^*_{xo} \] = \[ V_{xo}^f / U_\infty \]

\[ \Delta V^* = \sqrt{(V^*_{xo} - u^*_g)^2 + (V^*_{yo} - v^*_g)^2} \]

\[ x = \wp: \text{ distance along surface of cylinder divided by } R_o \]

\[ x = \tilde{x} \]

\[ x = x \]

\[ x_o \] = coordinate in direction of flow at infinity

\[ x_d \] = \( x_o \) position of drop

\[ x_o^* \] = \( x_o / R_o \)
### NOMENCLATURE (Continued)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$x_d^*$</td>
<td>$x_d/R_o$</td>
</tr>
<tr>
<td>$x_{d_f}$</td>
<td>$x_o$ position of drop at surface of liquid film</td>
</tr>
<tr>
<td>$x_{d_f}^*$</td>
<td>$x_{d_f}/R_o$</td>
</tr>
<tr>
<td>$X_e$</td>
<td>volume fraction of liquid outside of film</td>
</tr>
<tr>
<td>$X_{e_\infty}$</td>
<td>volume fraction of liquid at infinity</td>
</tr>
<tr>
<td>$y$</td>
<td>coordinate in radial direction: $\frac{r}{R_o} - 1$</td>
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<tr>
<td>$\bar{y}$</td>
<td>$y/2X_{e_\infty}$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$y/2\sqrt{\frac{2R_oU_\infty}{v}}$</td>
</tr>
<tr>
<td>$y_o$</td>
<td>coordinate perpendicular to flow at infinity</td>
</tr>
<tr>
<td>$y_d$</td>
<td>$y_o$ coordinate of drop</td>
</tr>
<tr>
<td>$y_{d_\infty}$</td>
<td>$y_o$ coordinate of drop at infinity</td>
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<tr>
<td>$y_{d_f}$</td>
<td>$y_o$ coordinate of drop at surface of film</td>
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<tr>
<td>$y_{o}^*$</td>
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<tr>
<td>$y_d^*$</td>
<td>$y_d/R_o$</td>
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<tr>
<td>$y_{d_\infty}^*$</td>
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<tr>
<td>$y_{d_f}^*$</td>
<td>$y_{d_f}/R_o$</td>
</tr>
<tr>
<td>$Z(\tilde{x})$</td>
<td>dimensionless gas shear stress term</td>
</tr>
</tbody>
</table>

#### Greek Letters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{2n}$</td>
<td>terms in expansion for $\tilde{\delta J}$</td>
</tr>
<tr>
<td>$\beta_{2n}$</td>
<td>terms in expansion for $\tilde{\delta }\tilde{\mathcal{H}}$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\cos^{-1} \left( -\frac{1}{1 - \frac{24\rho g}{E^2}} \frac{(\sin 2\tilde{x})^2}{\tau_{\text{eff}}^3} \right)$</td>
</tr>
<tr>
<td>$\gamma_n$</td>
<td>$n = 1, 3, 5, \ldots$, terms in series expansion of $Z(\tilde{x})$</td>
</tr>
</tbody>
</table>
NOMENCLATURE (Continued)

δ = thickness of liquid film

δ* = δ(ξ)/2R₀ Xₑ∞

δ = δ(ξ)/2R₀ Xₑ∞

ζ = \frac{\delta}{2} \sqrt{\frac{2R₀ U_{∞}}{\nu}}

\hat{\delta} = \delta/R₀

ξ = function arising from integration of stream function

η = y/\delta

θ = angle between tangent to cylinder and tangent to surface of liquid film

θ* = \frac{T - T_{∞}}{T_{w} - T_{∞}}

μ = viscosity of liquid

μ₉ = viscosity of gas

ν = μ/ρ; kinematic viscosity of liquid

ν₉ = μ₉/ρ₉; kinematic viscosity of gas

ξ = x/R₀

ρ = density of liquid

ρ₉ = density of gas

τᵣᵣ = radial component of shear stress in liquid film

τᵣᵣ = azimuthal and radial components of shear stress in liquid film

τᵣᵣ = azimuthal component of shear stress in liquid film

τ₉ = shear stress exerted by gas on surface of film

τ₉ = shear stress exerted by liquid on surface of cylinder
NOMENCLATURE (Continued)

\[ \tau_n = n = 1, 3, 5, \ldots, \text{terms in expansion for } \tau_w \sqrt{\frac{2U_\infty R_0}{v}} / \sqrt{1/2pU_\infty^2} \]

\[ \tau_{\text{eff}} = \left\{ \frac{h}{E} \frac{\rho g}{\rho} \sqrt{\frac{v g}{v}} Z(x) - \frac{x_e}{x_e^\infty} V_r V_\phi \right\} \]

\[ \varphi = \text{azimuthal coordinate} \]

\[ \varphi_d = \text{azimuthal coordinate of drop at surface of film} \]

\[ \psi = \text{stream function in liquid film} \]

\[ \psi^* = \psi / U_\infty R_0 X_e^\infty \]

\[ \sim \psi = \psi / U_\infty R_0 X_e^\infty \]

\[ \overline{\psi} = \psi / U_\infty R_0 X_e^\infty \]

Special notation

\[ \nu_o = \uparrow \frac{\partial}{\partial x_o} + \downarrow \frac{\partial}{\partial y_o} \]

\[ \nu_d = \uparrow \frac{\partial}{\partial x_d} + \downarrow \frac{\partial}{\partial y_d} \]

\[ \nu^* = R_0 \nu_o \]

\[ \nu^*_d = R_0 \nu_d \]

Subscripts

\[ d = \text{drop} \]

\[ g = \text{gas} \]

\[ i = \text{integer} \]

\[ j = \text{integer} \]

\[ \infty = \text{upstream conditions} \]

\[ m = \text{integer} \]

\[ n = \text{integer} \]
NOMENCLATURE (Continued)

Notation for Oscillating Flat Plate

Latin Letters

\[ a, a_n, \tilde{a}_n, \tilde{a}_{2n} \] = coefficients in expansion for \( \delta^* \) (constants)

\[ A \] = area of control surface

\[ A_{c.s.} \] = area of control surface

\[ B_n, B_n \] = terms appearing in momentum boundary conditions

\[ C_n, C_n \] = terms appearing in energy boundary conditions

\[ c_p \] = specific heat at constant pressure of liquid

\[ c_{pg} \] = specific heat at constant pressure of gas

\[ e_{1n}, e_{2n}, \tilde{e}_{2n} \] = terms appearing in expansion for \( \tau_n^* \); \( n = 0, 1, 2, \ldots \)

\[ E_n, E_n \] = nonhomogeneous terms appearing in differential equations

\[ f_0, f_1, f_{2n}, \tilde{f}_{2n} \] = coefficients in expansions for \( \psi^* \) (functions of \( \eta \))

\[ F_0, F_1, F_{2n}, \tilde{F}_{2n} \] = coefficients in expansions for \( \Theta^* \) (functions of \( \eta \))

\[ \tilde{e}_{1n}, \tilde{e}_{2n}, \tilde{e}_{2n} \] = terms of expansion of \( u_n^* \)

\[ G_{1n}, G_{2n}, \tilde{G}_{2n} \] = terms of expansion for \( \tilde{e}_n^* \)

\[ h_n, \tilde{h}_n \] = terms appearing in expansion for \( u_n^* \)

\[ H_n, \tilde{H}_n \] = nonhomogeneous terms appearing in the differential equations

\[ i = \sqrt{-1} \]

\[ k \] = thermal conductivity of liquid

\[ k_g \] = thermal conductivity of gas
NOMENCLATURE (Continued)

$K_n$ = differential operator for energy equations

$L_n$ = differential operator for momentum equations

$N_{ux}$ = Nusselt number

$N_{ug}$ = Nusselt number of gas

$Pr$ = Prandtl number of liquid

$Pr_g$ = Prandtl number of gas

$P_n, \tilde{P}_n$ = nonhomogeneous terms appearing in momentum equations

$q_w$ = heat flux at plate

$q_g$ = heat flux into gas at edge of film

$q_n^*$ = coefficient of $\epsilon$ in expansion for local-Nusselt number

$r_n, \tilde{r}_n$ = terms appearing in expansion for $\tau_n^*$

$R_n, \tilde{R}_n$ = terms appearing in expansion for $q_n^*$

$s'$ = upper surface of liquid film

$s''$ = vertical boundary of control surface

$t$ = time

$T$ = temperature

$T_w$ = temperature of plate

$T_\infty$ = temperature at infinity

$u$ = $x$-component of velocity in film

$u_n^*$ = coefficient of $\epsilon$ in expansion for $u/U_\infty$

$U$ = $U_\infty(1 + \frac{\epsilon}{2} \sin \omega t)$

$U_\infty$ = velocity of liquid drops and gas at infinity
NOMENCLATURE (Continued)

\( v \) = y-component of velocity in film

\( v(x)_{c.s.} \) = x-component of velocity of edge of liquid film

\( v(y)_{c.s.} \) = y-component of velocity of edge of liquid film

\( v^\perp_{c.s.} \) = normal component of velocity of edge of liquid film

\( x_1 \) = x-distance in x-y coordinates

\( x \) = coordinate direction along plate in coordinate system fixed to plate

\( X_e \) = volume fraction of liquid in free stream

\( y \) = coordinate direction perpendicular to plate in coordinate system fixed to plate

\( Z_1, Z_2 \) = arbitrary complex quantities

reek Letters

\( \alpha_n \) = arbitrary complex quantities with \( \alpha_n \) imaginary for \( n = 0, 0.5, 1, \ldots \), \( \alpha_n \) real for \( n = 2, 3, \ldots \), \( \alpha_n = 0 \) for \( n = 1, 3, 5, \ldots \)

\( \delta \) = thickness of liquid film

\( \delta^* \) = \( \delta \sqrt{\frac{\omega}{\nu X_e}} \)

\( \delta^*_n \) = \( n = 0, 1, 2, \ldots \), coefficient of \( \epsilon^n \) in expansion for \( \delta^* \)

\( \epsilon \) = dimensionless amplitude of oscillation

\( \eta \) = \( y/\delta \)

\( \Theta^* \) = \( (T - T_\infty)/(T_W - T_\infty) \)

\( \Theta^*_n \) = coefficient of \( \epsilon^n \) in expansion for \( \Theta^* \)

\( \lambda \) = equal to \( \pm 1 \)

\( \mu \) = viscosity of liquid
NOMENCLATURE (Concluded)

\( \mu_g \) = viscosity of gas  \\
\( \nu \) = kinematic viscosity of liquid  \\
\( \nu_g \) = kinematic viscosity of gas  \\
\( \xi = \sqrt{\frac{\omega X}{X e U_\infty}} \)  \\
\( \rho \) = density of liquid  \\
\( \rho_g \) = density of gas  \\
\( \tau = \omega t \)  \\
\( \tau_{xy} \) = component of liquid stress tensor  \\
\( \tau_g \) = shear stress exerted by gas at edge of film  \\
\( \tau_w \) = shear stress at plate  \\
\( \tau_n^* \) = coefficient of \( \epsilon^n \) in expansion for \( \tau_w \)  \\
\( \varphi \) = slope of liquid film and phase angle  \\
\( \psi \) = stream function  \\
\( \psi^* = \frac{\psi}{U_\infty X e} \sqrt{\frac{\omega}{X e \nu}} \)  \\
\( \psi_n^* \) = coefficient of \( \epsilon^n \) in expansion for \( \psi^* \)  \\
\( \omega \) = frequency of oscillation of plate

Subscripts

avg. = time average value  \\
g = gas  \\
w = surface of plate  \\
n = integer  \\
steady = steady state value  \\
\( \infty \) = upstream conditions
ABSTRACT

An analysis is carried out of a two phase (gas/liquid) flow over a circular cylinder and over an oscillating flat plate. The flow is assumed two dimensional and gravity, vaporization of the liquid phase, and compressibility are ignored. The liquid is assumed to be in the form of small drops far upstream from the body. The liquid film which forms on the surface of the body due to drop impingement is analyzed extensively. In the case of the cylinder the analysis is started from the full incompressible Navier-Stokes and Energy equations in the film which are simplified by using dimensional arguments. The order of magnitude of the physical properties of the fluids is taken to be those of air/water mixtures. The analysis is carried out for two ranges of a parameter appearing in the problem. For low values of the parameter the liquid drop trajectories deviate appreciably from straight lines and are obtained numerically. For the flat plate the boundary layer approximations are assumed to hold a priori in the film. The plate is taken to be oscillating sinusoidally in its plane with small amplitude and small frequency; and the drops are assumed to move in straight lines.

After appropriate changes of variable the solutions to the governing equations and boundary conditions are carried out by a series expansion technique which results in a series of ordinary differential equations which have been solved numerically on a 7090 computer.

The solutions are used to calculate velocity and temperature profiles in the film and also such physical quantities as local film thickness, local Nusselt Number, and local skin friction.

The analysis shows that in general there is a significant increase in heat transfer as well as skin friction over what would be obtained from a single component gas flow. In the case of the flat plate only a very small permanent change in the heat transfer was found due to the oscillations. In the case of the cylinder a peaking in the heat transfer, film thickness and skin friction was found with respect to the parameter $E^2$ (the product of the volume fraction of the liquid in the free stream, squared, and the diameter Reynolds number based on liquid properties). The peaking occurs at a value of this parameter of about unity. The theoretical prediction compares favorably with the experimental results of Acrivos et al. (ARL-64-116).
CHAPTER I
INTRODUCTION

For some years now there has been a growing interest in the artificial creation of two phase flows for the purpose of significantly increasing the attainable level of surface heat transfer rates from solid bodies. Up until recently much of the interest has come from the expanding nuclear power field with much emphasis on systems in which solid particles, such as graphite dust, have been added to a coolant flow.

A large number of papers and reports have also appeared dealing with two-phase (gas-liquid) hydrodynamics and heat transfer phenomena. Among them are a large number on forced convective flows of steam/water mixtures. Data have been presented for channels of circular, rectangular and annular cross-section with the flow vertically upwards, downwards and horizontal over a wide range of values of steam quality, mass velocity and heat flux. In addition details are also given of investigations with refrigerants and with two-component systems both with and without appreciable vaporization of the liquid phase.

A number of empirical correlations have been suggested and some analysis has been carried out.

An analysis has also been carried out for two-phase solid/liquid flows.
However due to the fact that relatively recent Russian liquid injection tests in flow across a tube bank have yielded increases in the measured heat transfer rates of an order of magnitude.\textsuperscript{9} Research programs have currently been undertaken to investigate the heat transfer characteristics of external gas/liquid flow systems. Tests have been conducted at the Marquandt Corporation on a system which consisted of an air stream containing water droplets flowing over a heated vertical cylinder.\textsuperscript{1} An intensification of the heat transfer coefficient from 2.5 to nine times was noted. An exploratory analytic investigation was carried out by Tifford on the boundary layer characteristics of gas/liquid spray systems, with particular emphasis on heat transfer characteristics.\textsuperscript{29}

Experiments on single component systems have shown that oscillations of the system may result in significantly increased heat transfer performance. Analytical two dimensional studies\textsuperscript{20,21,24,28} however have not confirmed these results and it is believed that this disagreement between analysis and experiment may be due to the fact that the increased heat transfer is due to three dimensional effects. Nevertheless it may be possible that the two dimensional effects of oscillations may be augmented considerably by the increased mass which results in a two component system.

In any event the use of two-phase heat transfer systems with oscillations is strongly encouraged by the following two important factors:
1. For a given wall temperature, which is often limited by the physical properties of material, a higher heat-transfer rate can be obtained.

2. For a given heat-transfer surface area, which is often limited by the space and other considerations, a large increase in heat-transfer rate can be obtained.

It is anticipated that a detailed analytical study would provide significant information on the dependence of the augmentative effects upon the choice of operating conditions for such a gas/liquid heat transfer system and would aid in the understanding of the physical mechanisms involved.

The present study has therefore been undertaken in order to make some progress toward developing such detailed analytical investigations. Since however the physical situation is quite complex and changes considerably with the geometry and operating conditions the study was limited to two specific systems each in a certain range of operating conditions.

Thus the present work is to study the momentum and heat transfer in a two-component (gas/liquid) steady flow over a circular cylinder and over a flat plate oscillating sinusoidally in its plane.

Analytical results are obtained through the use of the perturbation technique. These results include velocity and temperature profiles, local friction factors, and heat transfer coefficients.
Numerical computations are performed by means of the Runge-Cutta method for a gas/liquid system which corresponds to air-water mixture.

For low values of Reynolds Numbers and small drop sizes it has been found that the liquid drop trajectories deviate appreciably from straight lines.\textsuperscript{30} Analytical and numerical studies have been carried out to determine drop trajectories around bodies in connection with icing problems by several investigators. Tribus\textsuperscript{30} has calculated trajectories about cylinders and airfoils on a differential analyzer. Langmuir\textsuperscript{19} has considered the cylinder analytically, and Bergran\textsuperscript{4} and Guibert \textit{et al.}\textsuperscript{17} have studied airfoils numerically.

It was disclosed that the presence of the liquid in the gas stream caused a significant increase in both the heat transfer and friction. The oscillations (in the case of the flat plate) caused only very small time average changes. Theoretical predictions on heat transfer and friction for the steady two phase flow over a circular cylinder agree qualitatively with the qualitative results of Tifford.\textsuperscript{29}
CHAPTER II

ANALYSIS

A. General Considerations

The physical system to be studied consists of a two-phase (gas/liquid) flow over a heated surface. Two cases are to be treated: A steady flow over a circular cylinder and a steady flow over a flat plate oscillating in its own plane.

The experiments conducted at Marquandt Corporation indicate that when a streaming gas containing liquid drops impinges on a solid body a liquid film is formed on the surface of the body and is caused to flow in the downstream direction.

The following assumptions are imposed on the solution:

(1) The liquid drops contained in the streaming gas are uniformly distributed in the gas and move with the velocity of the gas far upstream.

(2) The flow is taken to be two dimensional and laminar.

(3) Due to inertial effects some of the drops in the free stream splash on the surface of the liquid film. The incoming drops contribute their mass, momentum and heat content to the film. However, the unsteady effects due to random drop impingement are negligible.

(4) There are a sufficient number of drops in the free stream so that only the time average effect of drop impingement need be considered.
(5) It is disclosed, from Marquadt's experiments that waves formed on the surface of the liquid layer and that under certain conditions splashing occurred at this surface. These latter two effects are also neglected.

(6) On top of the liquid film a gas boundary layer forms which joins the flow in the region of the body to the external flow field. The usual boundary layer assumptions apply in this region.

(7) Surface tension effects on the surface of the liquid layer are neglected.

(8) The effects of compressibility and heat generated by dissipation can be ignored. The effects of gravity are also neglected.

(9) The ratio of the gas/liquid properties will be taken to be of the order of those of air and water.

(10) All fluid properties will be taken as constant. (e.g., the liquid and the gas are separately assumed incompressible).

(11) No appreciable vaporization occurs.

B. Gas/Liquid Flow Over an Infinite Circular Cylinder

1. FORMULATION

With these assumptions, the following governing differential equations and boundary conditions are obtained for the two component (gas/liquid) flow about the infinite circular cylinder for three regions of the flow field: (1) The liquid film; (2) the gas/liquid free stream
flow; (3) the gas/liquid boundary layer.

a. Equations and Boundary Conditions for the Liquid Film

Consider the element of volume shown in Figure 1 bounded by \(d\phi\), \(d\delta\) and \(ds\). The law of conservation of momentum about \(O\) yields:

\[
- \tau_{r\phi}(R_o+\delta,\phi)[R_o+\delta]^2 d\phi + \tau_{\phi\phi}(R_o+\delta,\phi)[R_o+\delta]d\delta
+ \tau_g(0,\phi)[R_o+\delta]\cos\theta ds + P_g(0,\phi)[R_o+\delta]\sin\theta ds
\]

\[
= \rho[u_{\phi}(R_o+\delta,\phi)]^2[R_o+\delta] d\delta
- \rho u_r(R_o+\delta,\phi)u_{\phi}(R_o+\delta,\phi)[R_o+\delta]^2 d\phi
- X_e(R_o+\delta,\phi) \rho u_{\phi}(R_o+\delta,\phi)[R_o+\delta][u_{\phi}(R_o+\delta,\phi)\sin\theta - u_r(R_o+\delta,\phi)\cos\theta] ds
\]

where terms in the square of small quantities have been neglected.

A force balance in the radial direction yields upon neglecting the squares of small quantities:

\[
- \tau_{rr}(R_o+\delta,\phi)[R_o+\delta]d\phi + \tau_{r\phi}(R_o+\delta,\phi)ds - P_g(0,\phi)\cos\theta ds
+ \tau_g(0,\phi)\sin\theta ds = \rho u_{\phi}(R_o+\delta,\phi)V_r(R_o+\delta,\phi) d\delta
- \rho[(u_r(R_o+\delta,\phi)]^2[R_o+\delta]d\phi
- X_e(R_o+\delta,\phi)\rho u_{\phi}(R_o+\delta,\phi)[u_{\phi}(R_o+\delta,\phi)\sin\theta - u_r(R_o+\delta,\phi)\cos\theta] ds
Figure 1. Coordinates and control volume for liquid film on cylinder.
And the law of conservation of mass is:

\[ \rho \nabla \varphi (R_o + \delta, \varphi) d\delta - \rho \nabla \tau (R_o + \delta, \varphi)[R_o + \delta] d\varphi = \]

\[ X_e(R_o + \delta, \varphi) \rho [V_\varphi(R_o + \delta, \varphi) \sin \theta - V_\tau(R_o + \delta, \varphi) \cos \theta] ds \]

From elementary calculus we have the relations:

\[ (ds)^2 = \left[ (R_o + \delta)^2 + \left( \frac{d\delta}{d\varphi} \right)^2 \right] (d\varphi)^2 \]

\[ \tan \varphi = \frac{1}{R_o + \delta} \frac{d\delta}{d\varphi} \]

\[ ds \cos \theta = [R_o + \delta] d\varphi \]

\[ ds \sin \theta = d\delta \]

After dividing the conservation laws by \( d\varphi \) and using the above relations between the differentials one obtains:

\[ -\tau_\varphi (R_o + \delta) + \tau_{\varphi \varphi} \frac{d\delta}{d\varphi} + (R_o + \delta) \tau_g + P_g \frac{d\delta}{d\varphi} \]

\[ = \rho \nabla \varphi^2 \frac{d\delta}{d\varphi} - \rho \nabla \tau \nabla \varphi (R_o + \delta) - \rho X_e \nabla \varphi \left[ V_\varphi \frac{d\delta}{d\varphi} \right] - (R_o + \delta) V_\varphi \]

\[ \tau_{r \varphi} \frac{d\delta}{d\varphi} - (R_o + \delta) \tau_{rr} - (R_o + \delta) P_g + \tau_g \frac{d\delta}{d\varphi} \]

\[ = \rho \nabla \tau \nabla \varphi \frac{d\delta}{d\varphi} - \rho \nabla \tau (R_o + \delta) - \rho X_e \nabla \tau \left[ V_\varphi \frac{d\delta}{d\varphi} \right] - (R_o + \delta) V_\tau \]

\[ \frac{d\delta}{d\varphi} - \rho (R_o + \delta) \nabla \tau = \rho X_e \left[ V_\varphi \frac{d\delta}{d\varphi} - (R_o + \delta) V_\tau \right] \]

\[ \text{st}\quad r = R_o + \delta \]

\[ \varphi \leq \varphi \]
Using the last of these two relations to simplify the first two, one arrives at:

\[
\begin{aligned}
- \tau_{r\phi}(R_0 + \delta) + \tau_{\phi\phi} \frac{d\delta}{d\phi} + (R_0 + \delta)\tau_g + \rho \frac{d\delta}{d\phi} & = \\
\rho X_e(v_r - v_\phi) \left[ v_\phi \frac{d\delta}{d\phi} + (R_0 + \delta)v_r \right] & \\
- \tau_{rr}(R_0 + \delta) + \tau_{r\phi} \frac{d\delta}{d\phi} - \rho g (R_0 + \delta) + \tau_g \frac{d\delta}{d\phi} & = \\
& \quad r = R_0 + \delta \\
0 & \leq \phi \\
\rho v_\phi \frac{d\delta}{d\phi} - \rho v_r (R_0 + \delta) = \rho X_e \left[ v_\phi \frac{d\delta}{d\phi} - v_r (R_0 + \delta) \right]
\end{aligned}
\]

Within the film the liquid is assumed to satisfy the incompressible Navier-Stokes equations and the continuity equation. These equations and the relations of the components of the stress tensor to the velocity field are:

\[
\begin{aligned}
\frac{\partial v_r}{\partial r} + \frac{v_\phi}{r} & = - \frac{1}{\rho} \frac{\partial P}{\partial r} + \\
\left( \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \phi^2} + \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{2}{r^2} \frac{\partial v_\phi}{\partial \phi} - \frac{v_r}{r^2} \right) & = 0 \\
\frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} & = - \frac{1}{\rho r} \frac{\partial P}{\partial \phi} + \\
\left( \frac{1}{r^2} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial v_\phi}{\partial r} + \frac{2}{r^2} \frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2} \right) & = 0
\end{aligned}
\]
\[ \begin{align*}
\tau_{rr} &= -P + 2\mu \frac{\partial v_r}{\partial r} \\
\tau_{\varphi\varphi} &= -P + 2\mu \left( \frac{1}{r} \frac{\partial v_{\varphi}}{\partial \varphi} + \frac{v_r}{r} \right) \\
\tau_{r\varphi} &= \mu \left( \frac{1}{r} \frac{\partial v_r}{\partial \varphi} + \frac{\partial v_{\varphi}}{\partial r} - \frac{v_{\varphi}}{r} \right)
\end{align*} \]

Since the equation of continuity Eq. (3) may be rewritten as:

\[ \frac{\partial}{\partial r} (rv_r) + \frac{\partial v_{\varphi}}{\partial \varphi} = 0 \]

it will be satisfied identically if one introduces the usual form of the stream function \( \psi \) for cylindrical coordinates. That is:

\[ v_r = -\frac{1}{r} \frac{\partial \psi}{\partial \varphi}; \quad v_{\varphi} = \frac{\partial \psi}{\partial r} \]

Now since:

\[ d\psi = \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial \varphi} d\varphi \]

we have:

\[ \frac{d\psi}{ds} = \frac{\partial \psi}{\partial r} \left. \right|_{r=R_0+\delta} \cdot \frac{dr}{ds} + \frac{\partial \psi}{\partial \varphi} \left. \right|_{r=R_0+\delta} \cdot \frac{d\varphi}{ds} \]

\[ = \frac{\partial \psi}{\partial r} \left. \right|_{r=R_0+\delta} \cdot \frac{d\delta}{ds} + \frac{\partial \psi}{\partial \varphi} \left. \right|_{r=R_0+\delta} \cdot \frac{d\varphi}{ds} \]

\[ \frac{d\psi}{ds} \frac{ds}{d\varphi} = \frac{\partial \psi}{\partial r} \left. \right|_{r=R_0+\delta} \cdot \frac{d\delta}{d\varphi} \frac{ds}{d\varphi} + \frac{\partial \psi}{\partial \varphi} \left. \right|_{r=R_0+\delta} \]
\[ \frac{\partial \psi}{\partial \phi} \Bigg|_{\text{along } s} = \frac{\partial \psi}{\partial r} \Bigg|_{r=R_0+\delta} + \frac{\partial \psi}{\partial \phi} \Bigg|_{r=R_0+\delta} \]

\[ \frac{\partial \psi}{\partial \phi} \Bigg|_{\text{along } s} = v_\phi \frac{d\delta}{d\phi} - (R_0+\delta)v_r \quad \text{at } r = R_0+\delta \quad (6) \]

Now the vanishing of the velocity at the wall of the cylinder requires that:

\[ v_r = v_\phi = 0 \quad \text{at } r = R_0 \quad ; \quad 0 \leq \phi \quad (7) \]

in order to satisfy this condition it is sufficient to take:

\[ \psi = \frac{\partial \psi}{\partial r} = 0 \quad \text{at } r = R_0 \quad ; \quad 0 \leq \phi \quad (7*) \]

Introducing (5) into (2), yields:

\[
\frac{\partial \psi}{\partial r} \frac{\partial^2 \psi}{\partial r \partial \phi} - \frac{\partial \psi}{\partial \phi} \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial \phi} \frac{\partial \psi}{\partial r} = -\frac{1}{\rho} \frac{\partial \rho}{\partial \phi}
\]

\[ + \nu \left( \frac{r}{r^3} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial r \partial \phi^2} - \frac{2}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \quad (2*) \]

\[
\frac{\partial \psi}{\partial r} \frac{\partial^2 \psi}{\partial \phi^2} - \frac{\partial \psi}{\partial \phi} \frac{\partial^2 \psi}{\partial r \partial \phi} + r \left( \frac{\partial \psi}{\partial r} \right)^2 + \frac{1}{r} \left( \frac{\partial \psi}{\partial \phi} \right)^2 = \frac{1}{\rho r^2} \frac{\partial \rho}{\partial r}
\]

\[ + \nu \left( \frac{1}{r} \frac{\partial^3 \psi}{\partial \phi^3} + r \frac{\partial^3 \psi}{\partial r \partial \phi^2} + \frac{\partial^3 \psi}{\partial r^2 \partial \phi} + \frac{2}{r} \frac{\partial \psi}{\partial \phi} \right) \]

Using (6) in the last of Eqs. (1), one obtains:

\[ \frac{\partial \psi}{\partial \phi} = \rho X_e \left[ v_\phi \frac{d\delta}{d\phi} - v_r (R_0+\delta) \right] \quad \text{for } r = R_0+\delta \]
or integrating both sides with respect to $\psi$ along $s$, gives:

$$
\psi - \psi\bigg|_{\phi=0} = \rho \int_{\phi=0}^{\phi} X_e \left[ V_\phi \frac{d\delta}{d\phi} - V_r(R_0 + \delta) \right] d\phi ; \quad r = R_0 + \delta
$$

but since $\psi$ must be an odd function of $\phi$, $\psi\bigg|_{\phi=0} = 0$ and one has:

$$
\psi = \rho \int_{\phi=0}^{\phi} X_e \left[ V_\phi \frac{d\delta}{d\phi} - V_r(R_0 + \delta) \right] d\phi : \text{at } r = R_0 + \delta \quad \text{1*)}
$$

Now using Eqs. (4) in the first two of Eqs. (1) and substituting Eqs. (5) into the resulting equations, rewriting Eqs. (2*) and collecting Eqs. (1*) and (7*) yields:

$$
\begin{align*}
\frac{\partial \psi}{\partial r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial \phi} \right) - \frac{1}{r} \frac{\partial \psi}{\partial \phi} \frac{\partial^2 \psi}{\partial r^2} &= - \frac{1}{\rho} \frac{1}{r} \frac{\partial P}{\partial \phi} + \nu \frac{\partial}{\partial r} \frac{\partial^2 \psi}{\partial r^2} \\
\frac{1}{r} \frac{\partial \psi}{\partial \phi} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial \phi} \right) - \frac{\partial \psi}{\partial r} \left( \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right) &= - \frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \frac{1}{r} \frac{\partial}{\partial \phi} \frac{\partial^2 \psi}{\partial r^2}
\end{align*}
$$

$$
\begin{align*}
\nu_e &= \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial^2}{\partial \phi^2} \\
v_r &= - \frac{1}{r} \frac{\partial \psi}{\partial \phi} ; \quad v_\phi = \frac{\partial \psi}{\partial r}
\end{align*}
$$

\text{8*)}
\[
\begin{aligned}
\mu \left[ \frac{\partial^2 \psi}{\partial \varphi^2} (R_0 + \delta)^{-1} - (R_0 + \delta)^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right] + 2 \frac{\partial \delta}{\partial r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \right) - \rho \frac{\partial \delta}{\partial \varphi} \\
+ (R_0 + \delta) \tau_g + p_g \frac{\partial \delta}{\partial \varphi} = \rho \chi_e \left[ \frac{\partial \psi}{\partial r} - V_\varphi \right] \left[ V_\varphi \frac{\partial \delta}{\partial \varphi} - (R_0 + \delta)V_r \right]
\end{aligned}
\]

\[
\begin{aligned}
P(R_0 + \delta) = \mu \left[ \frac{\partial^2 \psi}{\partial \varphi^2} (R_0 + \delta)^{-2} \frac{\partial \delta}{\partial r} - (R_0 + \delta) \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \right) \frac{\partial \delta}{\partial \varphi} \right] \\
+ 2(R_0 + \delta) \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \right) - p_g (R_0 + \delta) + \tau_g \frac{\partial \delta}{\partial \varphi} = \rho \chi_e \left[ \frac{\partial \psi}{\partial r} (R_0 + \delta) + V_r \right] \left[ V_\varphi \frac{\partial \delta}{\partial \varphi} - (R_0 + \delta)V_r \right]
\end{aligned}
\]

\[
\begin{aligned}
\psi = \rho \int_0^{\varphi} \chi_e \left[ V_\varphi \frac{\partial \delta}{\partial \varphi} - (R_0 + \delta)V_r \right] d\varphi
\end{aligned}
\]

These equations i.e. Eqs. (8), (9), and (10) would completely determine the flow in the liquid film if the gas conditions (\(P_g\) and \(\tau_g\)) and the liquid drop conditions (\(X_e, V_\varphi\) and \(V_r\)) were known at the edge of the liquid film. Thus the hydrodynamic problem in the film is completely determined by these relations.*

Since the hydrodynamic equations are decoupled from the energy equations by the assumption of constant density and viscosity, the flow problem can be solved without consideration of the energy and the temperature field within the film can be solved for once the velocity field is known.

*Note that Eqs. (8) constitute a fourth-order system in \(r\) and that (9) and (10) specify five boundary conditions. But the thickness of the layer is unknown and the additional condition is necessary to determine this.
With the assumptions that (1) the effects due to kinetic energy changes across this volume are negligible, and (2) the temperature of the drops is not significantly reduced from that of the free stream when they pass through the gas boundary layer, the energy balance produces:

\begin{align*}
  & k \frac{\partial T}{\partial r} (R_0 + \delta, \varphi) \left[ R_0 + \delta \right] \delta \varphi - k \frac{\partial T}{\partial \varphi} (R_0 + \delta, \varphi) \left[ R_0 + \delta \right]^{-1} \delta \delta \\
  & + q_g \delta s = - \rho C_p T(R_0 + \delta, \varphi) v_\varphi(R_0 + \delta, \varphi) \delta \delta \\
  & + \rho C_p T(R_0 + \delta, \varphi) v_r(R_0 + \delta, \varphi) \left[ R_0 + \delta \right] \delta \varphi \\
  & + X_e(R_0 + \delta, \varphi) C_p \rho T_\infty \left[ V_\varphi(R_0 + \delta, \varphi) \sin \Theta - \\
  & - V_r(R_0 + \delta, \varphi) \cos \Theta \right] \delta s
\end{align*}

Dividing through by \( \delta \varphi \), using the relations given above between \( \delta s \), \( \delta \varphi \) and \( \sin \Theta \) and the last of Eqs. (1) one gets:

\begin{align*}
  & k \frac{\partial T}{\partial r} (R_0 + \delta) - k(R_0 + \delta)^{-1} \frac{\partial T}{\partial \varphi} \frac{\partial \delta}{\partial \varphi} + q_g \left[ (\delta + R_0)^2 + \left( \frac{\delta}{\delta \varphi} \right)^2 \right]^{1/2} \\
  & \text{at} \quad r = R_0 + \delta, \quad 0 \leq \varphi \\
  & = X_e C_p \rho(T_\infty - T) \left[ V_\varphi \frac{\partial \delta}{\partial \varphi} - (R_0 + \delta) V_r \right] \\
  & \text{(11)}
\end{align*}

Under the above restrictions the energy equation is:

\begin{align*}
  \frac{1}{r} \left( \rho C_p \frac{\partial T}{\partial r} + \rho C_p \frac{\partial T}{\partial \varphi} \right) = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) \\
  \text{with the boundary condition at the wall:} \\
  T = T_w \quad \text{at} \quad r = R_0, \quad 0 \leq \varphi \quad \text{(13)}
\end{align*}
Thus Eqs. (11), (12) and (13) determine the temperature field inside the liquid film completely once \( \dot{u}_g \) and the velocity field are known. Now the order of magnitude of the various terms in the governing equations is to be investigated. The following new variables are introduced:

\[
y = \frac{r}{R_0} - 1; \quad x = \varphi; \quad \delta = \delta/R_0
\]

and Eqs. (8) through (13) become:

\[
\begin{cases}
\psi_y \frac{\partial}{\partial y} \left[ (y+1) \psi_x \right] - (y+1) \psi_x \psi_y = -\frac{R_0^2}{\rho} (y+1) P_x + \nu \frac{\partial}{\partial y} D_{xy} \psi \\
(y+1) \psi_x \frac{\partial}{\partial y} \left[ (y+1) \psi_x \right] - \psi_y \left[ (y+1) \psi_{xx} + (y+1) \psi_y \right] = \\
-\frac{R_0^2}{\rho} P_y - \nu(y+1) \frac{\partial}{\partial x} D_{xy} \psi
\end{cases}
\]

\[
D_{xy} = \frac{1}{1+y} \frac{\partial}{\partial y} (1+y) \frac{\partial}{\partial x} + \frac{1}{(1+y)^2} \frac{\partial^2}{\partial x^2}
\]

\[
v_\varphi = \frac{1}{R_0} \frac{\partial \psi}{\partial y}
\]

\[
(y+1)^{-1} \left[ \psi_y T_x - \psi_x T_y \right] = \frac{k}{\rho c_p} \left[ T_{yy} + (1+y) T_{xx} + (1+y) T_y \right]
\]

at \( y = 0 \)

\[
0 \leq x \quad \psi = \psi_y = 0; \quad T = T_w
\]

*The subscript notation for partial derivatives is being used. Thus

\[
\psi_y = \frac{\partial \psi}{\partial y}, \quad \psi_x = \frac{\partial \psi}{\partial x}, \quad \psi_{yy} = \frac{\partial^2 \psi}{\partial y^2}, \text{ etc.}
\]
\[
\begin{align*}
\nu \{ \psi_{xx}(1+\hat{\delta})^{-1} - (1+\hat{\delta}) \frac{\partial}{\partial y} \left[ (y+1)^{-1} \psi_y \right] + 2\hat{\delta}_x \frac{\partial}{\partial y} \left[ (y+1)^{-1} \psi_x \right] \} \\
= \frac{R_e^2}{\rho} \tau_g(1+\hat{\delta}) + \frac{R_e^2}{\rho} (P_g-P) \frac{\partial \delta}{\partial x} = \xi_x R_o \left[ \frac{1}{R_o} \psi_y - V_\phi \right] \frac{\partial V_\delta}{\partial x} - (1+\hat{\delta})V_r \\
= \frac{R_e^2}{\rho} (P_g-P)(1+\hat{\delta}) - \nu \{ \psi_{xx}(1+\hat{\delta})^{-2} \frac{\partial \delta}{\partial x} - (1+\hat{\delta}) \frac{\partial}{\partial y} \left[ (1+y)^{-1} \frac{\partial \psi}{\partial y} \right] \frac{\partial \delta}{\partial x} \}
+ 2(1+\hat{\delta}) \frac{\partial}{\partial y} \left[ (1+y)^{-1} \frac{\partial \psi}{\partial x} \right] + \frac{R_e^2}{\rho} \tau_g \frac{\partial \delta}{\partial x} = \\
= \frac{R_e^2}{\rho} \xi_x \left[ \frac{1}{R_o} \psi_x(1+\hat{\delta})^{-1} + V_r \right] \left[ V_\phi \frac{\partial \delta}{\partial x} - (1+\hat{\delta})V_r \right] \\
\psi \frac{1}{R_o} = \int_x^\infty \xi_x \left[ V_\phi \frac{\partial \delta}{\partial x} - V_x(1+\hat{\delta}) \right] dx \\
k \frac{\partial T}{\partial y} (1+\hat{\delta}) - k(1+\hat{\delta})^{-1} \frac{\partial T}{\partial x} \frac{\partial \delta}{\partial x} + R_o \frac{\partial \delta}{\partial x} \left[ (1+\hat{\delta})^2 + \delta_x^2 \right]^{1/2} \\
= \xi_x R_o C_p \rho(T_{\infty}-T) \left[ V_\phi \frac{\partial \delta}{\partial x} - (1+\hat{\delta})V_r \right]
\end{align*}
\]

Now interest is focused on the asymptotic behavior of these equations as \( \xi_x \rightarrow 0 \) and \( \frac{2R_o U_{\infty}}{\nu} \rightarrow \infty \) that is when \( \xi_x \ll 1 \); \( 2R_o U_{\infty}/\nu \gg 1 \).

For large Reynolds numbers and small volume fractions of the liquid in the free stream, these equations will behave differently depending on whether the parameter \( E^2 \equiv \xi_x^2 \) \( 2R_o U_{\infty}/\nu \) is less than or greater than 1.

We want to make these equations dimensionless in such a way that all the coefficients of the parameters appearing in the resulting equations are of order 1 in order to determine which terms can be considered negligibly small. But this requires that the terms be made
dimensionless in different ways depending upon whether $E^2$ is less or greater than 1.* Therefore these two cases have to be treated separately.

Case a) $E^2 \geq 1$

Under these circumstances the numerical results show that:

$$\psi = O(U_\infty R_0 X_{e_0}) \quad \text{and} \quad y, \delta = O(2X_{e_0})$$

Therefore the dimensionless quantities:

$$\bar{\psi} = \psi/U_\infty R_0 X_{e_0}; \quad \bar{x} = x; \quad \bar{y} = y/2X_{e_0};$$

$$\bar{\delta} = \delta/2X_{e_0} = \delta/2R_0 X_{e_0}; \quad \bar{v}_\varphi = v_\varphi/U_\infty;$$

$$\bar{v}_r = v_r/U_\infty; \quad \bar{P} = P/1/2 \rho U_\infty^2$$

are introduced and Eqs. (14), (15) and (16) become

$$-\bar{\psi} \frac{\partial}{\partial \bar{y}} \left[ (1+2X_{e_0} \bar{y})^{-1} \bar{\psi} \right] - (1+2X_{e_0} \bar{y})^{-1} \bar{\psi} \bar{\psi}_{\bar{y}\bar{y}} =$$

$$-2(1+2X_{e_0} \bar{y})^{-1} \bar{\psi}_x + \frac{1}{E^2} \frac{\partial}{\partial \bar{y}} \bar{P}^2 \bar{\psi}$$

$$\left[ (1+2X_{e_0} \bar{y})^{-1} \frac{\partial}{\partial \bar{y}} \left[ (1+2X_{e_0} \bar{y})^{-1} \bar{\psi}_x \right] - \bar{\psi}_{\bar{y}} \left( (1+2X_{e_0} \bar{y})^2 \bar{\psi}_{\bar{x}\bar{x}} \right) \right.$$

$$\left. + \frac{(1+2X_{e_0} \bar{y})^{-1}}{2X_{e_0}} \bar{\psi}_y \right] 4X_{e_0}^2 = -2\bar{P}_{\bar{y}} - \frac{4X_{e_0}}{E^2} (1+2X_{e_0} \bar{y}) \frac{\partial}{\partial \bar{x}} \bar{\psi}^2$$

*It can be verified from the numerical results that all the dimensionless terms are of the order 1.
\[ 
\bar{D}^2 = (1+2X_{e_\infty} \bar{y}) \frac{\partial}{\partial \bar{y}} \left( 1+2X_{e_\infty} \bar{y} \right) \frac{\partial}{\partial \bar{y}} + \\
+ (1+2X_{e_\infty} \bar{y}) \ 4X_{e_\infty} \frac{\partial}{\partial \bar{x}} \frac{\partial}{\partial \bar{x}} \ \
(1+2X_{e_\infty} \bar{y})^{-1} \left[ \frac{\bar{y}}{\bar{y}} \frac{T_{\bar{x}}}{\bar{y}} - \frac{\bar{y}}{\bar{y}} \frac{T_{\bar{y}}}{\bar{y}} \right] = \frac{1}{P_U^2 \bar{y}^2} \left[ T_{\bar{y}} \right] + \\
+ 2X_{e_\infty} (1+2X_{e_\infty} \bar{y})^{-1} \frac{T_{\bar{y}}}{\bar{y}} + 4X_{e_\infty} (1+2X_{e_\infty} \bar{y})^{-2} \frac{T_{\bar{y}}}{\bar{y}} \\
\]
\[
\begin{align*}
\left[ \frac{X_e}{X_{e\infty}} E^2 P_r (T_\infty - T) \left[ \bar{V}_r^* 2X_{e\infty} \frac{\delta}{\delta X} - (1 + 2X_{e\infty} \delta) V_r^* \right] \right] \\
\bar{y} = 0 \quad 0 \leq \bar{x} \quad \bar{\psi} = \bar{\psi}_y = 0 \quad T = T_w
\end{align*}
\]

Now taking the limit as \(X_{e\infty} \to 0\) and retaining all terms relating to the two-phase region one gets:

at \(0 \leq \bar{y} \leq \delta\), \(0 \leq \bar{x}\)

\[
\begin{align*}
\bar{\psi} \frac{\partial \bar{\psi}}{\partial \bar{y}} - \bar{\psi} \frac{\partial \bar{\psi}}{\partial \bar{x}} &= -2 \frac{\bar{P}}{\bar{P}_r} + \frac{1}{E^2} \frac{\partial \bar{P}}{\partial \bar{y}} \\
\frac{\partial \bar{P}}{\partial \bar{y}} &= 0 \\
\bar{\psi}_y \frac{\partial \bar{x}}{\partial \bar{y}} - \bar{\psi}_x \frac{\partial \bar{y}}{\partial \bar{y}} &= \frac{1}{P_r E^2} \frac{T_r}{\bar{y}} \\
-\bar{\psi} \frac{\partial \bar{y}}{\partial \bar{y}} &= E^2 \left( \frac{X_e}{X_{e\infty}} \left[ 2V^* - \frac{\bar{\psi}}{\bar{y}} \right] V_r^* + \frac{-2 \tau e_g}{X_{e\infty} \rho_{\infty}} \right)
\end{align*}
\]

at \(\bar{y} = \delta\), \(0 \leq \bar{x}\)

\[
\begin{align*}
P - P_g + 2X_{e\infty} \tau e_g \delta X &= 0 \\
\bar{\psi} &= - \int_{0}^{x} \frac{X_e}{X_{e\infty}} V_r^* d\bar{x} \\
T_r + \frac{2X_{e\infty} R_g}{k} q_g &= - \frac{X_e}{X_{e\infty}} P_r E^2 (T_\infty - T) V_r^*
\end{align*}
\]

at \(\bar{y} = 0\), \(0 \leq \bar{x}\)

\(\bar{\psi} = \bar{\psi}_y = 0 \quad T = T_w\)

Where the pressure term in the first of Eqs. (15a) has been eliminated by substituting the second equation into it.

Case b) \(E^2 \leq 1\)

Under these circumstances the numerical results show that:

\[
\psi = O\left( U_{\infty} R_g X_{e\infty} \right) \quad \text{and} \quad \delta, y = 0 \left( \frac{2}{2R_{\infty} U_g} \right)
\]
Therefore the dimensionless quantities:

$$
\tilde{\psi} = \psi/U_\infty R_0 x_\infty; \tilde{x} = x; \tilde{y} = \frac{y}{z} \sqrt{\frac{2 R_0 U_\infty}{v}};
$$

$$
\tilde{\delta} = \frac{\delta}{2} \sqrt{\frac{2 R_0 U_\infty}{v}}; \quad \tilde{V}_\phi = \frac{V_\phi}{U_\infty}; \quad \tilde{V}_r = \frac{V_r}{U_\infty};
$$

$$
P = \frac{V_0}{\sqrt{2 / 2 \rho U_\infty^2}}; \quad R_e = \frac{2 R_0 U_\infty}{v}
$$

are introduced and Eqs. (14), (15) and (16) now become:

$$
E^2 \left( \frac{\tilde{\psi}}{y} \frac{\partial}{\partial y} \left[ \left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right)^{-1} \tilde{\psi}_x \right] - \left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right)^{-1} \tilde{\psi}_y \tilde{\psi}_{yy} \right) =
$$

$$
= -2 \left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right)^{-1} \tilde{F}_x + E \frac{\partial}{\partial y} \tilde{D}^2 \tilde{\psi}
$$

$$
\frac{4 X e_\infty k^2}{(1 + \frac{2}{\sqrt{R_e}} \tilde{y})^{-1} \tilde{\psi}_x \frac{\partial}{\partial y} \left[ \left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right)^{-1} \tilde{\psi}_x \right] - \tilde{\psi}_y \left[ \left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right)^{-2} \tilde{\psi}_{xx} + \frac{\sqrt{R_e}}{2} \left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right)^{-1} \tilde{\psi}_y \right]}
$$

$$
= -2 \tilde{F}_y - \frac{4 X e_\infty k^2}{(1 + \frac{2}{\sqrt{R_e}} \tilde{y})^{-1} \tilde{\psi}_x \frac{\partial}{\partial y} \tilde{D}^2 \tilde{\psi}
$$

$$
\tilde{D}^2 = \left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right)^{-1} \frac{\partial}{\partial y} \left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right) \frac{\partial}{\partial y} + \left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right)^{2} \frac{1}{R_e} \frac{\partial^2}{\partial x^2}
$$

$$
\left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right)^{-1} \left[ \tilde{\psi} \tilde{\psi}_x - \tilde{\psi}_x \tilde{\psi}_y \right] = \frac{1}{P_r E} \left[ \tilde{T}_{yy} \right] +
$$

$$
+ \left(1 + \frac{2}{\sqrt{R_e}} \tilde{y} \right)^{-2} \frac{1}{R_e} \tilde{T}_{xx}
$$
\[
\begin{align*}
&\left\{-\left(1+\frac{2}{\sqrt{Re}}\right)\frac{\partial^2}{\partial y^2}\left[(1+\frac{2}{\sqrt{Re}}y)\tilde{\psi}\right] + \frac{4}{Re}\left[\tilde{\psi}_{xx}(1+\frac{2}{\sqrt{Re}}\tilde{\delta})^{-1}\right]
\right. \\
&\left. + 2\frac{2}{\sqrt{Re}}\frac{\partial}{\partial y}\left[(1+\frac{2}{\sqrt{Re}}y)\tilde{\psi}\right] + \frac{2}{x_{\varepsilon}\sqrt{1+2\rho U_{\infty}}}\tilde{\tau}(1+\frac{2}{\sqrt{Re}}\tilde{\delta})\right)
\right. \\
&\left. + \frac{2}{\sqrt{Re}}(P-P_{g})\tilde{\delta}_x \right] = \frac{2}{x_{\varepsilon}} \left[ \tilde{\psi}_y - 2V_{\phi}^* \right] \left[ \frac{2}{\sqrt{Re}} \tilde{\delta}_x \tilde{V}_{\phi}^* \right.
\right. \\
&\left. + (1+\frac{2}{\sqrt{Re}}\tilde{\delta})V_r^* \right]
\end{align*}
\]

at 
\[y = \delta, \quad 0 \leq \tilde{x} \]

\[
\begin{align*}
&\left. -x_{\varepsilon}^e \left[ \tilde{\psi}_{xx} \frac{2}{\sqrt{Re}} \tilde{\delta}_x \left(1+\frac{2}{\sqrt{Re}}\tilde{\delta}\right)^2 - \left(1+\frac{2}{\sqrt{Re}}\tilde{\delta}\right)\frac{\partial}{\partial y}\left(1+\frac{2}{\sqrt{Re}}\tilde{\delta}\right) \tilde{\psi}_y \right] \tilde{\psi}_x \right)
\right. \\
&\left. + 2(1+\frac{2}{\sqrt{Re}}\tilde{\delta})\frac{\partial}{\partial y}\left[(1+\frac{2}{\sqrt{Re}}y)\tilde{\psi}_x\right] \right)
\right. \\
&\left. = x_{\varepsilon}^e \left[ \frac{\tilde{\psi}_y}{x_{\varepsilon}} \left(1+\frac{2}{\sqrt{Re}}\tilde{\delta} \right)^{-1} + V_r^* \right] \left[ \frac{2}{\sqrt{Re}} \tilde{\delta}_x \tilde{V}_{\phi}^* \right.
\right. \\
&\left. - (1+\frac{2}{\sqrt{Re}}\tilde{\delta})V_r^* \right]
\right. \\
&\left. \tilde{\psi} = \int_{0}^{\tilde{x}} \frac{\tilde{\psi}_x}{x_{\varepsilon}} \left[ V_{\phi}^* \frac{2}{\sqrt{Re}} \tilde{\delta}_x - V_r^* \left(1+\frac{2}{\sqrt{Re}}\tilde{\delta}\right) \right] \tilde{dx}
\right. \\
&\left. \frac{\partial T}{\partial y} - \frac{4}{Re} \left(1+\frac{2}{\sqrt{Re}}\tilde{\delta}\right) T_{\infty} \tilde{x} \right)
\right. \\
&\left. + \frac{2Ro\delta_0}{1+2Re} \left[ \left(1+\frac{2}{\sqrt{Re}}\tilde{\delta}\right)^2 + \frac{4}{Re} \tilde{\delta}_x \right] \right)
\right. \\
&\left. = EPr \frac{\tilde{\psi}_x}{x_{\varepsilon}} \left( T_{\infty} - T \right) \left[ V_{\phi}^* \frac{2}{\sqrt{Re}} \tilde{\delta}_x - (1+\frac{2}{\sqrt{Re}}\tilde{\delta})V_r^* \right]
\right. \\
&\left. \text{at} \quad \tilde{y} = 0; \quad 0 \leq \tilde{x}; \quad \tilde{\psi} = \tilde{\psi}_{\tilde{y}} = 0; \quad T = T_w
\right.
\]
Now taking the limit as \( R_e \to \infty \) and \( X_{e_{\infty}} \to 0 \) and retaining all those
terms relating to the two-phase region yields:

\[
\begin{align*}
E_e \left[ \frac{\psi_{xy}}{\psi_{x}} - \frac{\psi_{x}}{\psi_{yy}} \right] &= -2P_x + E \psi_{yy} \\
\frac{\partial \psi}{\partial y} &= 0 \\
\psi_{y}^{T} - \psi_{x}^{T} &= \frac{1}{PrE} T_{xy}^{yy} \\
- \psi_{yy}^{T} + \frac{2\tau_{g}}{X_{e_{\infty}}^{1/2} \rho U^2} \\
&= - \frac{X_{e}}{X_{e_{\infty}}} \left[ E \psi_{y}^{*} - 2V_{\psi}^{*} \right] v_{r}^{*} \\
P - P_{g} + \frac{2}{R_e} \Delta T_{g} &= 0 \\
\psi &= - \int_{0}^{X_{e}} \frac{X_{e}}{X_{e_{\infty}}} v_{r}^{*} dx \\
\frac{\partial T}{\partial y} + \frac{2R_{d} \tau_{g}}{k \Delta T_{g}} &= E Pr \frac{X_{e}}{X_{e_{\infty}}} (T - T_{\infty}) v_{r}^{*}
\end{align*}
\]

at \( \psi = 0 \); \( 0 \leq x \) \( \{ \psi = \psi_{y} = 0 ; T = T_{w} \} \)

Where the pressure term has been eliminated in the first of Eqs. (15b) by use of the second Eq. (15b).

In order to proceed further it is necessary to bring in the phenomenon which occurs in the two-phase region.

b. Consideration of the Two-Phase Region

The two-phase region may be thought of as consisting of a boundary
layer (on top of the liquid film) and a free stream flow outside of the boundary layer.* However due to the presence of the liquid drops there is the possibility of shear stresses being exerted on the gas outside of the boundary layer even though the mean velocity gradients are not steep.

In any case we can conclude that the gas and liquid interact with one another only in the following ways:

BI 1) The gas and liquid can interact at the surface of the liquid film by exerting a mutual shearing stress on one another. This is characterized by the term $\tau_g$ appearing above.

BI 2) There is a pressure coupling between the gas and liquid at the surface of the liquid film. This is characterized by the term $P_g$.

BI 3) There is a heat transfer by conduction at the surface of the film from the liquid to the gas (or the opposite direction). This is characterized by the term $\dot{q}_g$.

DI 1) The gas and the liquid drops can exert mutual frictional effects on one another. The drops due to their inertia tend to move in straight lines. The gas on the other hand tends to flow around the cylinder. A difference in velocity therefore develops between the gas and the liquid drops in the neighborhood of the cylinder, and as a result there is a mutual shearing stress between the two of them.

*And, of course, the region of the wake and separated flow which will not be considered here.
DI 2) The gas can exert pressure forces on the liquid drops.*

Thus BI 1) through BI 3) are interactions which occur at the surface of the liquid film. We may anticipate that under certain conditions these interactions will have only a small effect on hydrodynamic and heat transfer behavior of the film. Because of the large difference in density between the gas and the liquid we will expect that the interaction BI 1) will have only a small effect on the film. We further expect that the interaction BI 2) will be dominated by the momentum carried into the film by the liquid drops provided that the velocity of the drops and the volume fraction of the liquid in the free stream is sufficiently high. And finally we might anticipate that the interaction BI 3) will be dominated by the enthalpy flux into the film due to the drops. Thus for two given fluids we anticipate, for sufficiently high velocities and amount of liquid, that the presence of the gas will have a negligible effect on the film at least as far as the interactions BI 1) through BI 3) are concerned. At the other extreme the liquid film will be moving at a very low velocity, will have a relatively uniform pressure, and there will be only a small temperature drop across it relative to the gas (since the thermal conductivity of the liquid is much higher than for the gas). Under these conditions we will expect that the presence of the film has only a small effect on the gas boundary layer. It is further possible that these two domains will over lap. Thus there is a

*Note that we have already neglected the heat transfer effects between the drops and the gas.
domain where BI 1) through BI 3) are negligible for the behavior of the film, a domain where BI 1) through BI 3) are negligible for the gas boundary layer, and a region where the coupling between the two is mutually weak (i.e., the above two domains overlap).* But of course the lower boundary of the domain where the gas has only a small effect on the film (BI 1) through BI 3)) will depend also upon the dynamics of the drops (i.e., DI 1) and DI 2) since if the drops are strongly deflected by the gas stream little momentum, mass, and enthalpy can be carried into the film by the drops. Therefore before proceeding further with the interactions which occur at the surface of the film we will consider the interactions which effect the drop trajectories (i.e., DI 1) and DI 2)). In order to do this we should divide DI 1) and DI 2) up a little differently. We distinguish now between the total drag force exerted on the drops by the gas due to the difference in velocity between them and the force exerted on the drops by the gas due to the pressure gradients in the gas. The drops will be taken to be spherical and relatively small. Then the equations of motion for a single drop can be written as**:

\[
\frac{4}{3} \pi r_d^3 \rho \frac{dV}{dt} = -\frac{4}{3} \pi r_d^3 \nu \rho g - \frac{1}{2} C_D \pi r_d^3 \rho g |V - V_g| |V - V_g| \]

*Note that even if at the lower end of the domain where the gas influences the film there is a fairly large influence exerted on the gas by the liquid only a small error will be introduced if we replace the interaction on the film by that which would result if only the gas was present.

**| denotes the absolute value.
\[ v_0 = i \frac{\partial}{\partial x_0} + j \frac{\partial}{\partial y_0}; \quad \dot{\psi} = i \dot{V}_{x_0} + j \dot{V}_{y_0} = \frac{\dot{r}_d}{dt} \]

\[ \dot{r}_d = i x_d + j y_d; \quad \dot{V}_g = i \dot{u}_g + j \dot{v}_g \]

\[ V \rightarrow i U_\infty \text{ for } \dot{r}_d \rightarrow \infty \]

where \( C_D \) is of course a function of \(|\vec{V} - \vec{V}_g|\).

If we introduce the notation:

\[ \dot{v}_d = i \frac{\partial}{\partial x_d} + j \frac{\partial}{\partial y_d} \]

then it follows from elementary mechanics that:

\[ \frac{d\dot{V}}{dt} = \dot{V} \cdot \dot{v}_d \]

We introduce the dimensionless quantities:

\[ x_0^* = x_0/R_0; \quad y_0^* = y_0/R_0; \quad x_d^* = x_d/R_0; \quad y_d^* = \frac{x_d}{x_0} \frac{y_d}{y_0} \]

\[ v_d^* = R_0 \dot{v}_d; \quad \dot{v}^* = \dot{V}/U_\infty; \quad \dot{V}_{x_0}^* = \dot{V}_{x_0}/U_\infty; \quad \dot{V}_g^* = \dot{V}_g/U_\infty \]

\[ \dot{u}_g^* = \dot{u}_g/U_\infty; \quad \dot{p}_g^* = \dot{p}_g/\frac{1}{2} \rho_\infty U_\infty^2 \quad \text{etc.} \]

and we get:

\[ \dot{V}^* \cdot \dot{v}_d^* \dot{V}^* = -2 \frac{\dot{p}_g^*}{\rho_\infty^* V_\infty^*} - \frac{3}{8} C_D \frac{1}{(r_d/R_0)} \rho_\infty \frac{\dot{V}^* \cdot \dot{V}_g^*}{(V^* - \dot{V}_g^*)} \]

Now the drag forces exerted by the gas on the drops must equal minus

the drag force exerted by the drops on the gas. Therefore if \( N_0 \) is

the number of drops per unit volume then on the average (i.e., we
assume that there are a sufficient number of drops per unit volume so that only the average effect influences the overall gas motion the force exerted by the drops on the gas per unit volume is:

\[ + \frac{1}{2} N_0 C_D \pi r_d^2 \rho_g \left| \vec{v}_{g}^* - \vec{v}_{g}^* \right| (\vec{v}_{g}^* - \vec{v}_{g}^*) \]

in the region outside the gas boundary layer where the average velocity gradients are not steep the frictional effects which are due to these gradients can be neglected, and the equation of motion for the gas can be written as

\[ \rho_g \vec{v}_{g}^* \cdot \vec{v}_o \vec{v}_{g}^* = - \vec{v}_o^* \vec{P}_g + \frac{1}{2} N_0 C_D \pi r_d^2 \rho_g \left| \vec{v} - \vec{v}_g \right| (\vec{v} - \vec{v}_g) \]

Now it is obvious that \( N_0 \) is related to the volume fraction of the liquid \( X_e \) by the following formula:

\[ N_0 \left( \frac{4}{3} \right) \pi r_d^3 = X_e \]

Using this and the dimensionless quantities already defined the equation of motion for the gas becomes:

\[ \vec{v}_{g}^* \cdot \vec{v}_o \vec{v}_{g}^* = - 2 \vec{v}_o^* \vec{P}_g + \frac{3}{8} \frac{C_D}{(r_d/R_o)} X_e \left| \vec{v}_{g}^* - \vec{v}_{g}^* \right| (\vec{v}_{g}^* - \vec{v}_{g}^*) \]

From Eq. (17) it might be anticipated that under certain conditions the drop trajectories may be straight lines (i.e., their motion is not appreciably influenced by the motion of the gas).

We are now in a position to determine the lower limit of the domain where the presence of the gas has a negligible effect on the liquid film.
In order to do this it is necessary to consider the ratio of properties of the gas and the liquid specifically. We will limit ourselves in what follows to air and water since these are the substances of principal interest.

From the first of Eqs. (15a) and (15b) it can be concluded that the shear stress on the surface of the liquid film will be negligibly small if:

$$2 \frac{X_e}{X_{eo}} \frac{V^* V_r}{X_e \rho U_{io}^2} = \frac{2 \tau_g}{X_{eo} \rho U_{io}^2} = \frac{1}{E} \left[ \frac{\nu_g \rho_g}{\nu \rho} \right] \left[ \tau_g \sqrt{\frac{2 U_{o} R_o}{\nu_g}} \right] \frac{1}{2 \rho_g U_{io}^2} \quad (19)$$

By substituting the second of Eqs. (15a) and (15b) into the first of Eqs. (14a) and(14b), respectively, it can be concluded that the pressure forces exerted on the liquid film by the gas are negligibly small whenever:

$$E^2 > 1 \quad ; \quad 2F = 2(P_g - 2X_{eo} \tau_g \delta_{x})/\frac{1}{2 \rho U_{io}^2} =$$

$$2 \frac{\rho_g}{\rho} \left[ \frac{P_g}{1/2 \rho U_{io}^2} - \frac{2X_{eo}^2}{E} \frac{\delta_x}{X} \sqrt{\frac{\nu_g}{\nu}} \left[ \tau_g \left[ \frac{2U_{o} R_o}{\nu_g} \right] \frac{1}{1/2 \rho_g U_{io}^2} \right] \right] \ll 1 \quad (20a)$$

or whenever:

$$E^2 > 1 \quad ; \quad 2F = 2(P_g - \frac{2}{\sqrt{Re}} \tau_g \delta_{x})/\frac{1}{2 \rho U_{io}^2} =$$

$$2 \frac{\rho_g}{\rho} \left[ \frac{P_g}{1/2 \rho U_{io}^2} - \frac{2}{\sqrt{Re}} \frac{\delta_x}{X} \sqrt{\frac{\nu_g}{\nu}} \left[ \tau_g \left[ \frac{2U_{o} R_o}{\nu_g} \right] \frac{1}{1/2 \rho_g U_{io}^2} \right] \right] \ll E \quad (20b)$$

And finally from the last of Eqs. (15a) and (15b) it can be concluded that the gas conduction effects are negligible whenever:
\[
\frac{X_e}{X_{e\infty}} (T - T_{\infty}) V_r^* \gg \frac{\dot{q} g}{k P_r E} =
\]

\[
\left[ \frac{N_{ug}}{2 R_0 U_{\infty}} \right] \left[ \frac{C_p \rho g}{C_p \rho} \sqrt{\frac{v_g}{v}} \right] \frac{(T - T_{\infty})}{E}
\]

that is whenever:

\[
\left[ \frac{N_{ug}}{2 R_0 U_{\infty}} \right] \left[ \frac{C_p \rho g}{C_p \rho} \sqrt{\frac{v_g}{v}} \right] \ll EV_r^* \frac{X_e}{X_{e\infty}}
\]

These inequalities are seen to depend upon the quantities \( \frac{X_e}{X_{e\infty}} V_r^* \) and \( V_\varphi^* \) which in turn depend upon the liquid drop trajectories. If the drop trajectories were straight lines we would have from Figures 1 and 2:

\[
\left\{ \begin{array}{l}
X_e = X_{e\infty} \\
V_r^* = - \cos \bar{x} = - \cos \bar{\xi} \\
V_\varphi^* = \sin \bar{x} = \sin \bar{\xi}
\end{array} \right.
\]

We now proceed to investigate under what conditions Eq.(22) hold with sufficient accuracy. Now if the drops are uniformly distributed then the number of drops per unit length of line in the plain of the flow and perpendicular to the velocity at infinity is \( N_0 \). And the number \( N' \) lying along the line segment \( \overline{ab} \) in Figure 3 is \( N_0 \). \( R_0 \) or since:

\[
N_0 = \frac{X_e}{\left( \frac{4}{3} \right) \pi r_d^3}
\]

*(14) and (21).*
Figure 2. Coordinates for dynamical equations of liquid droplet.
Figure 3. Sketch for linear drop density.
we have:

\[ N' = \left( \frac{4}{3} \pi \right)^{1/3} \frac{x_e^{1/3}}{\left( \frac{r_d}{R_o} \right)^{1/3} \rho g R_o} \]  \hfill (23)

Now the number \( N' \) must be large enough so that the assumption that the unsteady effects due to drop impingement on the liquid film can be neglected, is valid. This fixes the minimum value of \( N' \). From Eq. (17) one can see that if the dimensionless quantities are of the order unity then the particle trajectories will be straight if \( \frac{\rho g}{\rho} \) and \( 3/8 c_D \left( \frac{r_d}{R_o} \right) \) are both much less than 1. But if the drops did not influence the gas appreciably then \( \gamma_o^{*} P^* \) would certainly be of order 1 and even if the presence of the drops could influence the gas pressure field by a factor of 100 the ratio \( \frac{\rho g}{\rho} \) for air/water mixtures is of the order of \( 10^{-3} \) so that this term is almost certainly negligible.

On the other hand the coefficient of \( 3/8 c_D \left( \frac{r_d}{R_o} \right) \frac{\rho g}{\rho} \) might conceivably be small almost everywhere if the liquid drops actually caused the gas to follow their trajectories. This is what we might expect if there were a large mass of liquid present. But with a large amount of liquid present (i.e., large \( x_e \)) we could take \( \frac{r_d}{R_o} \) large and still satisfy (23) and with large \( \frac{r_d}{R_o} \), \( 3/8 c_D \frac{1}{(r_d/R_o)} \frac{\rho g}{\rho} \) will certainly be small. We are interested rather in the other extreme when \( x_e \) is small. It is in this case that (23) requires that \( \frac{r_d}{R_o} \) be small and therefore \( 3/8 c_D \frac{1}{(r_d/R_o)} \) will be large not only from the \( \frac{r_d}{R_o} \) in the denominator but also from the fact that \( c_D \) increases considerably
with a decrease in the drop Reynolds number and therefore (for a fixed Reynolds number based on the cylinder diameter) with decreasing \( r_d/R_0 \).

Now a comparison of Eqs. (17) and (18) show that the ratio of the drag force per unit volume acting on the gas to the drag force per unit volume acting on the drops is given by:

\[
\frac{X_e}{\left(\frac{\rho_g}{\rho}\right)}
\]

We anticipate (and the numerical results below verify) that this minimum value of \( X_e^\ast \) is somewhat less than \( (\rho_g/\rho) \) so that the gas velocity field at the minimum value of \( X_e \) is less affected by the liquid than the liquid velocity by the gas. Hence we conclude that at the minimum value of \( X_e \) for which the particle trajectories are straight lines the gas can not be affected much by the liquid if the liquid is not affected much by the gas. Therefore we anticipate the gas to behave as potential flow and the coefficient of \( 3/8C_D \frac{1}{(r_d/R_0)} \frac{\rho_g}{\rho} \) must be small. Of course the minimum value of \( X_e \) depends also on the gas Reynolds number. That is for lower gas Reynolds numbers the minimum value of \( X_e \) must be larger in order to satisfy (23) and still make \( 3/8C_D \frac{1}{(r_d/R_0)} \frac{\rho_g}{\rho} \) small. But the numerical results obtained below for air/water mixtures show that for reasonable values of \( (r_d/R_0) \) the minimum value of \( X_e \) is always such for all gas Reynolds numbers that we can not expect to strong an effect on the gas

*I.e., minimum value of \( X_e \) for which the particle trajectories are straight and there is a sufficient number of liquid drops.
by the liquid. In order to obtain numerical values for the minimum value of \( X_e \) the minimum value of \( E^2 \) and the required \( r_d/R_o \) for which the particle trajectories are straight lines for a given value of the gas Reynolds number we first choose a reasonable minimum value of \( N' \) of ten. And we have the following conditions:

\[
\text{Drop Reynolds No.} = \left( \frac{r_d}{R_o} \right) \times \text{Gas Reynolds No.}
\]

\[
C_D = \text{Function of Drop Reynolds No.}^*
\]

\[
3/8 \frac{C_D}{(r_d/R_o)} \frac{\rho_g}{\rho} \leq 0.1
\]

\[
N' = \frac{X_e^{1/3}}{\left( \frac{4}{3\pi} \right)^{1/3} \left( \frac{r_d}{R_o} \right)} \geq 10
\]

These relations serve to determine the desired minimum values and the results** are shown in Figure 7 for air/water mixtures. We have therefore shown that under reasonable restrictions we may expect the particle trajectories to be approximately straight. We will therefore impose this restriction temporarily on the analysis. And therefore require that Eqs.(22) hold. We now return to the consideration of the interaction which occurs at the surface of the liquid film and to what one might expect the minimum conditions to be for air/water mixtures for these interactions to have only a small effect on the liquid film.

*See Schlichting, p. 16.
**These were obtained from the computer program in Appendix I.
From Eq. (22) one gets:

\[-2 \frac{X_e}{X_{e\infty}} \frac{v^*}{r} \frac{v^*}{r} = 2 \cos \theta \sin \theta = \sin 2\theta = \sin 2x\]

hence

\[-2 \frac{X_e}{X_{e\infty}} \frac{v^*}{r} \frac{v^*}{r} = O(1)\]

and the inequality (19) becomes

\[E \gg \left[ \left( \frac{v_g}{\nu} \right) \left( \frac{\rho_g}{\rho} \right) \right] \left[ \frac{2U_{cR}}{v_g} \frac{1/2}{\rho_gU_{\infty}^2} \right] \]

(19*)

Now using the inequality (19*) in the inequalities (20a) and (20b) and remembering that the limit as \(X_e \to 0\) and \(R_e \to \infty\) is being taken the inequalities (20a) and (20b) become:

\[E^2 \geq 1 \quad 2P = \frac{2 \rho_g}{\rho} \left[ \frac{p_g}{1/2 \rho_gU_{\infty}^2} \right] \ll 1 \]

(20a*)

\[E^2 \leq 1 \quad 2P = \frac{2 \rho_g}{\rho} \left[ \frac{p_g}{1/2 \rho_gU_{\infty}^2} \right] \ll E \]

(20b*)

and finally the inequality (21) becomes:

\[\left\{ \frac{N_{ug}}{2R_0U_{\infty}} \sqrt{\frac{\nu_g}{\nu}} \right\} \left[ \frac{C_{pg}}{C_p} \frac{\rho_g}{\rho} \sqrt{\frac{v_g}{\nu}} \right] \ll E \]

(21*)

one can now estimate the minimum value of \(E\) at which the effects of the gas on the liquid film are small for air/water mixtures by evaluating the terms in the inequalities (19*) through (21*) for a single component gas boundary layer (i.e., evaluating them from the relations for single
component flows over a cylinder). It should be noted that a low values of \( E^2 \) the presence of the liquid does not modify the gas flow very much since the liquid in the film (which is thin) is moving at a low velocity so it is relatively stationary as far as the gas boundary layer is concerned. Low values of \( E^2 \) also correspond to the range of parameters where the presence of the drops doesn't modify the potential flow very much.

Hence the inequalities (19*) through (21*) become using the numerical values obtained in Appendix II:

\[
E \gg 2.2 \times 10^{-2} \quad 19^{*A/W}
\]

\[
E^2 \geq 1 \quad 20a^{*A/W}
\]

\[
E^2 \leq 1 \quad 20b^{*A/W}
\]

\[
E \gg 1.2 \times 10^{-3} \quad 21^{*A/W}
\]

Hence it can be concluded that for a value of \( E \geq .1 \) (or \( E^2 \geq .01 \)) the terms in Eqs. (14a) through (16a) and Eqs. (14b) through (16b) which result from the presence of the gas can be neglected. Therefore using the relations (22) and the inequalities (19*\( A/W \)) through (21*\( A/W \)) we arrive at the following formulation:

\[
E^2 \geq 1 \quad \text{(straight drop trajectories \& conditions of Appendix I)}
\]
\[ 0 \leq y \leq \delta \]
\[ \begin{align*}
\left\{ \begin{array}{l}
\psi_y \psi_{yy} - \psi_x \psi_{xy} = \frac{1}{E^2} \psi_{yy} \\
\psi_y T_X - \psi_x T_Y = \frac{1}{P_x E^2} T_{YY}
\end{array} \right. \\
\end{align*} \]
\[ 0 \leq x \]
\[ 14a^* \]

\[ \bar{y} = \delta \]
\[ \begin{align*}
\psi_{yy} &= E^2 (2 \sin \bar{x} - \psi_{y}) \cos \bar{x} \\
\psi &= \sin \bar{x} \\
T_{yy} &= P_x E^2 (T_{\infty} - T) \cos \bar{x}
\end{align*} \]
\[ 0 \leq \bar{x} \]
\[ 15a^* \]

\[ \bar{y} = 0 \]
\[ \begin{align*}
\psi &= \psi_y = 0 \quad ; \quad T = T_w
\end{align*} \]
\[ 0 \leq \bar{x} \]
\[ 16a^* \]

\[ \gamma \leq E^2 \leq 1 \text{ (straight drop trajectories in conditions of Appendix I apply)} \]

\[ 0 \leq \gamma \leq \delta \]
\[ \begin{align*}
\left\{ \begin{array}{l}
\psi_y \psi_{yy} - \psi_x \psi_{xy} = \frac{1}{E^2} \psi_{yy} \\
\psi_y T_X - \psi_x T_Y = \frac{1}{P_x E^2} T_{YY}
\end{array} \right. \\
\end{align*} \]
\[ 0 \leq \gamma \leq \delta \]
\[ 14b^* \]

\[ \gamma = \delta \]
\[ \begin{align*}
\psi_{yy} &= \left[ 2 \sin \gamma - E \psi_{y} \right] \cos \gamma \\
\psi &= \sin \gamma \\
T_{yy} &= P_x (T_{\infty} - T) \cos \gamma
\end{align*} \]
\[ 0 \leq \gamma \]
\[ 15b^* \]

\[ \gamma = 0 \]
\[ \begin{align*}
\psi &= \psi_y = 0 \quad ; \quad T = T_w
\end{align*} \]
\[ 0 \leq \gamma \]
\[ 16b^* \]

We now notice that Eqs. (14a*) through (16a*) and Eqs. (14b*) through (16b*) are essentially the same. It is only necessary to use the relations \( \gamma = yE \); \( \delta = \delta E \) to go from one to the other and thus there is a single formulation of the problem for \( E^2 \geq 0.01 \) and the condition of straight particle trajectories. And therefore only the Eqs. (14a*) through (16a*)
will be needed to solve the problem.

Let us now consider the case when \( E^2 \leq .01 \). For reasonable values of the Reynolds number this implies that \( X_e \) is small.\(^*\) Hence there is only a small range of parameters for which one might expect the particle trajectories to be straight and still have a sufficient number of drops to have a continuum. On the other hand, as argued above, one might expect that the presence of the liquid film does not influence the gas boundary layer very much and one may therefore evaluate the gas/liquid interaction terms in Eqs. (14a) through (14b) by assuming that the gas boundary layer and the pressure field in the region of the cylinder are the same as for single component gas flow about a cylinder. Also since we are considering the conditions when \( E \leq .1 \) we may take the limit as \( E \to 0 \). With one exception! In the energy relations the product \( E \cdot P_r \) appears, and since the Prandtl number is of order ten we must retain these terms. It seems rather unlikely that a film will form for values of \( E^2 \leq 10^{-4} \) hence only the range \( .01 \leq E \leq .1 \) will be considered. From the inequality \( (21*A/w) \) one sees that one may also neglect the gas conduction term for this range of parameters. From Schlichting, p. 153 (26) we have for single component flow around a cylinder:

\[
\frac{\tau_g}{1/2 \rho_g U_{\infty}^2} \left[ 2U_{\infty} R_0 \right] \sqrt{\frac{2U_{\infty} R_0}{v}} = 8 \left\{ \gamma_1 \sum_{n=2}^{\infty} \frac{(-1)^n(2n)}{(2n-1)!} \gamma(2n-1) \right\}
\]

\(^*\)For air/water mixtures if the gas Reynolds number based on the cylinder diameter is 1000 than for \( E^2 = .01 \), \( X_e = 10^{-3} \) larger Reynolds numbers and smaller values of \( E^2 \) correspond to lower values of \( X_e \).
Where the $\gamma_1$'s are the $f''_1(0)$'s there, and are listed in Table 9.1. It also follows from the general theory of potential flow that:

$$
-\frac{\partial}{\partial x} \left[ \frac{P_g}{1/2 \rho g U^2} \right] = 2 \sin 2x
$$

And finally after noting as in the derivation of the inequalities (20a*) and (20b*) from (20a) and (20b) that $P_\infty = \frac{P_g}{\rho} \frac{P_g}{1/2 \rho g U^2}$ Eqs. (14b) through (16b) become:

$$0.01 \leq E \leq 0.1$$

$$0 \leq \psi \leq \delta$$

$$\begin{align*}
\psi_{yy} &= -\frac{4 \rho g}{E \rho} \sin 2x \\
0 \leq x & \quad \text{subject to} \\
\psi &= \frac{8 \rho g}{E \rho} \left[ \frac{V}{y} Z(x) - 2 \frac{X_e}{X_{\infty}} \psi^* \varphi^* \right] \\
\psi &= \delta \\
0 \leq x &
\end{align*}$$

$$\begin{align*}
\psi &= -\int_0^{X_e} \frac{X_e}{X_{\infty}} \psi^* dx \\
T_y &= \frac{E \rho}{X_{\infty}} \left( T - T_\infty \right) \psi^*
\end{align*}$$

$$\begin{align*}
\psi &= 0 \\
0 \leq x &
\end{align*}$$

$$Z(x) = \gamma_1 x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} \gamma_1^{2n-1} x^{2n-1}$$

Of course $V^*_\psi$, $V^*_r$ and $X_e/X_{\infty}$ are still unknown. These must be obtained by integrating Eqs. (17) and (18) and using suitable boundary conditions.
i.e.,
\[
\begin{align*}
V & \rightarrow \hat{1} \quad x_d^2 + y_d^2 \rightarrow \infty \\
\gamma^* g & \rightarrow \hat{1} \quad x_o^2 + y_o^2 \rightarrow \infty \\
\gamma^* g \cdot \hat{n} & = 0 \quad x_o^2 + y_o^2 = R_o^2
\end{align*}
\]

where \( \hat{n} \) is the normal to the cylinder. This in general constitutes a difficult task. However for sufficiently small \( X_e \) the presence of the liquid drops don't influence the gas motion very much. By combining Eqs. (17) and (18) we get:

\[
\gamma^* g \cdot \gamma^* g = -2(1-X_e) \gamma^* g \frac{X_e}{\rho g/\rho} \gamma^* g \cdot \gamma^* g \gamma^* g \gamma^* g
\]

Hence for \( X_e \leq 1 \frac{\rho g}{\rho} \) one would expect the gas flow to be essentially a potential flow. Under these circumstances Eq. (17) can be solved for the components of the liquid velocity at the surface of the cylinder by replacing \( u^* g \) and \( v^* g \) by the corresponding values for potential flow.

Several numerical procedures have been developed for integrating this equation in connection with icing problems on airfoils. See for example References.\(^{19,30}\)

We now formulate the problem of drop trajectories for the case when the gas flow can be considered a potential flow to a sufficient approximation i.e., when \( X_e \leq 1 \frac{\rho g}{\rho} \).

We first recall from potential flow theory that the velocity components outside of the boundary layer and away from the region of the wake are given to a fair approximation by:

---

*This approximation is probably good for larger values of \( X_e \), if the drops are large enough so that their trajectories are fairly straight.*
\[
\begin{align*}
\dot{u}_g^* &= 1 - \frac{x_d^* \cdot y_d^*}{(x_d^2 + y_d^2)^2} \\
\dot{v}_g^* &= -\frac{2x_d^* \cdot y_d^*}{(x_d^2 + y_d^2)^2}
\end{align*}
\]

The situation is as shown in Figures 2 and 4. Next \( P_g^* \) must be of the order of unity and since \( \rho_g / \rho \) is very small we are entirely justified in neglecting the pressure force term in Eq. (17). Now if one introduces the dimensionless time defined by:

\[
t^* = \frac{t}{R_o U_\infty}
\]

one may write

\[
\nabla \cdot \nabla_d^* \dot{v}^* = \ddot{v}^*
\]

where the dot denotes differentiation with respect to \( t^* \). We also have:

\[
\begin{align*}
\dot{v}^*_{y_0} &= \dot{y}_d^* \\
\dot{v}^*_{x_0} &= \dot{x}_d^*
\end{align*}
\]

Equation (17) may now be rewritten in component form as:

\[
\begin{align*}
\dot{y}_d^* &= \frac{\dot{v}^*_{y_0}}{
\dot{x}_d^* &= \frac{\dot{v}^*_{x_0}}{
\dot{v}_{x_0}^* &= \frac{3/8 \cdot C_D}{(r_d/R_o)} \cdot \frac{\rho_g}{\rho} \cdot (\dot{v}_{x_0}^* - \dot{u}_g^*) \cdot \dot{w}^* \\
\dot{v}_{y_0}^* &= \frac{3/8 \cdot C_D}{(r_d/R_o)} \cdot \frac{\rho_g}{\rho} \cdot (\dot{v}_{y_0}^* - \dot{v}_g^*) \cdot \dot{w}^*
\end{align*}
\]
\[ \Delta V^* = (v_{x_0}^* - u_g^*)^2 + (v_{y_0}^* - v_g^*)^2 \]

\[ u_g^* = 1 - \frac{x_d^* - y_d^*}{(x_d^* + y_d^*)^2} \]

\[ v_g^* = -\frac{2x_d^* y_d^*}{(x_d^* + y_d^*)^2} \]

This constitutes a system of four first order ordinary differential equations in \( y_d, x_d^*, v_{x_0}^*, \) and \( v_{y_0}^* \). We must therefore specify four boundary conditions. For our purposes we take these as:

at \( t^* = 0 \):

\[ x_d^* \to -\infty \]

\[ y_d^* \to y_{d_{\infty}} \]

\[ x_d^* \to 1 \]

\[ y_d^* \to 0 \]

These are the equations for an individual drop. We must now relate the solutions of these equations to the quantities at the edge of the liquid film which are of interest to us. In order to do this we consider the control surface in Figure 4 bounded on two sides by drop trajectories, on the third side by the liquid film and on the fourth side by a line perpendicular to the particle trajectories at a distance far enough from the cylinder so that they are parallel. Then the particles only cross the control surface at the latter two sides (i.e., they enter at the upstream side and leave at the downstream side). Now from the previous developments we have that to the order of approximation of the quantities appearing in the liquid film equations the total flux of
Liquid drops far upstream from cylinder (assumed uniformly distributed.)

Control surface

Drop trajectories

$V_{y_{of}}$, $V_{r}$, $V_{x_{of}}$, $\phi_{d}$, $y_{df}$, $x_{of}$, $y_{o}$

Figure 4: Control volume for the determination of physical quantities at the surface of liquid film from the trajectory data.
liquid across the downstream surface (which is the flux of liquid into
the film between $\varphi = 0$ and $\varphi = \varphi_d$) is given by:

$$\int_{0}^{\varphi_d} X_e V_r R_o \, d\varphi$$

The flux through the downstream side is given by:

$$X_e U_\infty y_{d_\infty}$$

and since these must be equal to one another we have:

$$-\int_{0}^{\varphi_d} X_e V_r R_o \, d\varphi = X_e U_\infty y_{d_\infty}$$

$$-\int_{0}^{\varphi_d} \left(\frac{X_e}{X_e_\infty}\right) V_r^* \, d\varphi = y_{d_\infty}^*$$

differentiating both sides with respect to $y_{d_\infty}^*$ we get:

$$\frac{d}{dy_{d_\infty}^*} \left[ -\int_{0}^{\varphi_d} \left(\frac{X_e}{X_e_\infty}\right) V_r^* \, d\varphi \right] = 1$$

$$= \frac{d}{d\varphi} \left[ -\int_{0}^{\varphi_d} \left(\frac{X_e}{X_e_\infty}\right) V_r^* \, d\varphi \right] \frac{d\varphi_d}{dy_{d_\infty}^*}$$

$$= -\left(\frac{X_e}{X_e_\infty}\right) V_r^* \frac{d\varphi_d}{dy_{d_\infty}^*}$$

*This is essentially equivalent to the approximation that the film is thin.
Next we note that:

\[ \varphi_d = \tan^{-1} \frac{y_{df}^*}{x_{df}^*} \]

\[ \cos \varphi_d = -x_{df}^* \]

\[ \sin \varphi_d = y_{df}^* \]

\[ \cos \varphi_d \, d \varphi_d = dy_{df}^* \]

\[ d \varphi_d = - \frac{dy_{df}^*}{x_{df}^*} \]

Hence:

\[ -(x_e/x_{e\infty})y_r^* = x_{df}^* \frac{dy_{df}^*}{dy_{d\infty}^*} \]

and we may write from elementary analytic geometry at the edge of the liquid film

\[ v_{\varphi}^* = v_{x_{df}}^* \sin \varphi_d + v_{y_{df}}^* \cos \varphi_d \]

and remembering that we have set \( \zeta = \varphi \), the results may be summarized as follows:

\[
\begin{bmatrix}
\zeta = \tan^{-1} \frac{y_{df}^*}{x_{df}^*} \\
-x \int_0^{x_e/x_{e\infty}} v_r^* \, dx = y_{d\infty}^*
\end{bmatrix} \quad (7**A.C.)
\]
\[
\begin{align*}
- \left( \frac{x_e}{x_{e_\infty}} \right) v_r^* &= - x_d_f^* \frac{\partial y_f^*}{\partial y_{d_\infty}^*} \\
\psi^* &= - v_{x_{o_f}} x_d_f^* + v_{y_{d_f}} y_d_f^*
\end{align*}
\]

We now notice that all the quantities on the right of the above relations can be obtained by integrating Eq. (17*) for different values of \( y_{d_\infty}^* \). The quantities on the left are precisely those which are needed for the Eqs. (14c) through (16c). The calculations have been carried out numerically. These are discussed in Appendix XXV.

We will consider here only the cases when (1) \( E^2 \geq .01 \) and the drop trajectories are straight as formulated in (14a*) through (16a*) or equivalently in (14b*) through (16b*) (since these are the same equations), and when (2) \( .01 \leq E \leq .1 \) and the external gas flow field is not influenced by the motion of the drops as formulated in (14c) through (16c) and (17*) and (17*A.C.). In the next section we therefore turn to the solution of Eqs. (14a*) through (16a*), and (14c) through (16c).

2. SOLUTION

An examination of Eqs. (14a*) through (16a*) (or equivalently Eqs. (14b*) through (16b*)) shows that the position of a boundary of the domain of the problem is unknown and must be determined by the problem. That is the thickness \( \delta(x) \) is an unknown function. Since it is inconvenient to solve the problem in this form we will map the domain of the problem into one whose boundaries are known and in the process reintroduce
\( \delta(x) \) as a dependent variable (on the same footing as \( \psi \)). In order to accomplish this let us introduce the following variables:

\[
\eta \equiv \overline{y}/\delta = \overline{\psi}/\delta = y/\delta R_0
\]

\[
\xi = \overline{x} = x/R_0
\]

\[
\psi = \overline{\psi} = \psi / U_\infty R_0 X_e
\]

\[
\delta^* = \delta / \delta = \delta / E R_0 X_e
\]

\[
\theta^* = \frac{T - T_\infty}{T_w - T_\infty}
\]

Then we have:

\[
\frac{\partial}{\partial \overline{y}} = \left( \frac{\partial \eta}{\partial \overline{y}} \right) \frac{\partial}{\partial \eta} + \left( \frac{\partial \xi}{\partial \overline{y}} \right) \frac{\partial}{\partial \xi} = \frac{1}{\delta} \frac{\partial}{\partial \eta}
\]

\[
\frac{\partial}{\partial \overline{x}} = \left( \frac{\partial \eta}{\partial \overline{x}} \right) \frac{\partial}{\partial \eta} + \left( \frac{\partial \xi}{\partial \overline{x}} \right) \frac{\partial}{\partial \xi} = \frac{1}{\delta} \frac{\partial}{\partial \eta} \frac{\partial \delta}{\partial \xi} \frac{\partial}{\partial \eta}
\]

\[
\frac{\partial^2}{\partial \overline{x} \partial \overline{y}} = \frac{1}{\delta} \frac{\partial^2}{\partial \eta \partial \xi} - \frac{1}{\delta^2} \frac{\partial \delta}{\partial \xi} \frac{\partial}{\partial \eta} \frac{\partial \delta}{\partial \eta}
\]

Now using these in the first of Eqs. (14a*) we get:

\[
\frac{1}{\delta^2} \frac{\partial \psi^*}{\partial \eta} \frac{\partial^2 \psi^*}{\partial \eta \partial \xi} - \frac{1}{\delta^3} \frac{\partial \xi}{\partial \xi} \frac{\partial \psi^*}{\partial \eta} \left[ \frac{\partial}{\partial \eta} \left( \eta \frac{\partial \psi^*}{\partial \eta} \right) \right]
\]

\[- \frac{1}{\delta^2} \frac{\partial \psi^*}{\partial \xi} \frac{\partial^2 \psi^*}{\partial \eta \partial \xi} + \frac{1}{\delta^3} \frac{\partial \xi}{\partial \xi} \eta \frac{\partial \psi^*}{\partial \eta} \frac{\partial^2 \psi^*}{\partial \eta^2} = \frac{1}{E \delta^3} \frac{\partial^3 \psi^*}{\partial \eta^3}
\]
which becomes after some rearrangement:

\[
\delta^* \left[ \frac{\partial \psi^*}{\partial \eta} \frac{\partial^2 \psi^*}{\partial \eta \partial \xi} - \frac{\partial \psi^*}{\partial \xi} \frac{\partial^2 \psi^*}{\partial \eta^2} \right] - \frac{d \delta^*}{d \xi} \left( \frac{\partial \psi^*}{\partial \eta} \right)^2 = \frac{1}{E^2} \frac{\partial^3 \psi^*}{\partial \eta^3}
\]

and we get from the second of Eqs. (14a*):

\[
\delta^* \left[ \frac{\partial \psi^*}{\partial \eta} \frac{\partial \theta^*}{\partial \xi} - \frac{\partial \theta^*}{\partial \eta} \frac{\partial \psi^*}{\partial \xi} \right] = \frac{1}{P_r E^2} \frac{\partial^2 \theta^*}{\partial \eta^2}
\]

and finally since \( \eta = 1 \) when \( \theta = \delta^* \) and \( \eta = 0 \) when \( \theta = 0 \) Eqs. (14a*) through (16a*) become:

\[
0 \leq \eta \leq 1 \quad 0 \leq \xi
\]

\[
\left\{ \begin{array}{l}
\delta^* \left[ \frac{\partial \psi^*}{\partial \eta} \frac{\partial^2 \psi^*}{\partial \eta \partial \xi} - \frac{\partial \psi^*}{\partial \xi} \frac{\partial^2 \psi^*}{\partial \eta^2} \right] - \frac{d \delta^*}{d \xi} \left( \frac{\partial \psi^*}{\partial \eta} \right)^2 = \frac{1}{E^2} \frac{\partial^3 \psi^*}{\partial \eta^3} \\
\frac{\partial^2 \psi^*}{\partial \eta^2} = E^2 \delta^* \cos \xi \left[ 2 \sin \xi - \frac{1}{\delta^*} \frac{\partial \psi^*}{\partial \eta} \right] \quad 23) \\
\eta = 1 \\
\psi^* = \sin \xi \quad 24) \\
\frac{\partial \theta^*}{\partial \eta} = -P_r E^2 \theta^* \delta^* \cos \xi \quad 25) \\
\eta = 0 \\
0 \leq \xi \quad \psi^* = \frac{\partial \psi^*}{\partial \eta} = 0 \quad ; \quad \theta^* = 1
\end{array} \right.
\]

The same results could also have been obtained from Eqs. (14b*) through (16b*). The problem is now completely formulated except that we have not specified the boundary conditions in the \( \xi \) direction but
these follow from the symmetry of the problem. That is \( \delta^* \) and \( \theta^* \) must have even symmetry while \( \psi^* \) must have odd symmetry.

Because of these symmetries we seek a solution to the problem in the following form:

\[
\psi^* = f_1(\eta) \xi - \sum_{n=2}^{\infty} \frac{(-1)^n(2n)}{(2n-1)!} f_{(2n-1)}(\eta) \xi^{2n-1} \tag{26}
\]

\[
\delta^* = a_0 \left[ 1 + \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!} \xi^{2n-1} \right] \tag{27}
\]

\[
\theta^* = F_0(\eta) + \sum_{n=1}^{\infty} \frac{F_{2n}(\eta)}{(2n)!} \xi^{2n} \tag{28}
\]

If we substitute Eqs. (26) and (27) into the first Eq. (23) and equate coefficients of like powers of \( \xi \) after a lengthy calculation which is carried out in Appendix III we arrive at:

\[
\frac{1}{a_0 E^2} f_1'''' + f_1 f_1'' - f_1^4 = 0 \tag{29}
\]

\[
L_n(f_n) = \left( -1 \right)^{n+1} \frac{n+1}{n+1} \left[ f_1 f_1'' + (n-2)f_1^2 \right] b_{n-1} + H_n \quad n = 3, 5, 7, \ldots
\]

Where:

\[
L_n = \frac{1}{a_0 E^2} \frac{d^3}{d\eta^3} + f_1 \frac{d^2}{d\eta^2} - (n+1)f_1' \frac{d}{d\eta} + nf_1''
\]

And:

\[
H_3 = 0
\]
\[ H_5 = \frac{20}{3} \left[ 4(f_{\delta}^2 - f_{\alpha}f_{\gamma}) + b_2(f_{\delta}f_{\gamma} + 3f_{\alpha}f_{\beta}) \right] \]

\[ H_7 = \frac{7}{4} \left[ 12(8f_{\delta}f_{\alpha} - 3f_{\alpha}f_{\gamma} - 5f_{\delta}f_{\gamma}) - b_2(18f_{\delta}f_{\gamma} + 80f_{\gamma}^2 - 3f_{\alpha}f_{\gamma}) \right. \\
\left. - 9 f_{\delta}f_{\gamma} - 45f_{\alpha}f_{\gamma} - 10 b_4(f_{\delta}^2 + 4f_{\alpha}f_{\gamma} + 3f_{\gamma}^2) \right] \]

Upon substituting the expansion (26), (27), and (28) into the second of Eqs. (23) and equating coefficients of like powers of \( \frac{1}{P} \) we get after some calculation which is performed in Appendix IV:

\[ K_n(F_n) = G_n \quad n = 0, 2, 4, 6 \ldots \quad (30) \]

Where:

\[ K_n = \frac{1}{a_o^2P} \frac{d^2}{d\eta^2} + f_{1j} \frac{d}{d\eta} - nf_{1j} \]

\[ G_0 = 0 \]

\[ G_2 = (4f_{\delta} - b_2f_{\gamma})F_{10}' \]

\[ G_4 = 8(3f_{\delta}F_{12} - 2f_{\gamma}F_{12}) + 6b_2(2f_{12} - f_{1}F_{12}) \]

\[ - (6f_{\delta} - 24b_2f_{\gamma} + b_4f_{1})F_{10}' \]

\[ G_6 = 20(3f_{\delta}F_{14} - 4f_{\gamma}F_{14}) - 18(5f_{\delta}F_{12} - 2f_{1}F_{12}) \]
\[ + 15b_2(4f_1F_2 - f_1F_4 - 16f_3F_2 + 24f_3F_6) + \]
\[ + 15b_4(2f_1F_2 - f_1F_4) + (8f_3 - 90b_2f_5 + 60b_4f_3 \]
\[-bf_1F_0 \]
\[
\]
\[
\]
\[
\]

Next we get after substituting the expansions (26) and (27) into the first of Eqs. (24), noting that:

\[
\sin \xi = 1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots
\]
\[
\cos \xi = \frac{\xi^3}{3!} - \frac{\xi^5}{5!} + \frac{\xi^7}{7!} + \ldots
\]

and equating coefficients of like powers of \(\xi\) (see Appendix V for the calculations):

\[
\frac{2}{a_0E^2} f''_n (1) + 2f'_n (1) + 4a_0 = 0
\]
\[
\frac{2}{a_0E^2} f''_n (1) + 2f'_n (1) + \frac{2n}{n+1} \left\{ (-1)^{n+1} \frac{1}{2} b_{n-1} \left[ 4a_0 - f_1 (1) \right] - 3n \right\}
\]
\[-2a_0 + f_1 (1) \] + \(A_n \quad n = 3, 5, 7 \]

where:

\[ A_3 = -a_0 \]
\[
A_5 = \frac{2}{3} \left\{ 15b_2 \left[ 2a_o - f_1(1) \right] + a_o(10b_2 - 1) \\
+ 10(b_2-1) \left[ 3b_2a_o + 2f_3(1) - a_o \right] \right\}
\]

\[
A_7 = \frac{1}{4} \left\{ 105(b_2 - b_4) \left[ 2a_o - f_1(1) \right] + 2a_o(21b_2 - 35b_4 - 1) + 42(b_2-1) \left[ a_o(5b_4 - 10b_2 + 1) - 3f_5(1) \right] + \\
- 70(b_4 - 6b_2 + 1) \left[ a_o(3b_2-1) + 2f_3(1) \right] \right\}
\]

\[
\frac{1}{a_o E^2 P_r} F'_0(1) + F_0(1) = 0
\]

\[
\frac{1}{a_o E^2 P_r} F'_n(1) + F_n(1) + F_0(1) \left[ b_n + (-1)^n \right] + B_n ; n = 2, 4, 6, \ldots
\]

Where:

\[
B_2 = 0
\]

\[
B_4 = 6 \left[ b_2 F_2(1) - F_2(1) - b_2 F_0(1) \right]
\]

\[
B_6 = 15 \left[ b_2 F_4(1) + b_4 F_2(1) - F_4(1) - 6b_2 F_2(1) - b_4 F_0(1) + F_2(1) + b_2 F_0(1) \right]
\]

Next we turn to the third of Eqs. (24). We substitute in the expansion (27) and (28) and equate coefficients of \( \xi \). After the calculation performed in Appendix VI we arrive at:

32)
The second of Eqs. (24) yields upon comparison of:

\[
\sin \xi = - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} \xi^{2n-1}
\]

with the expansion (26):

\[
\begin{align*}
    f_1(1) &= 1 \\
    f_n(1) &= \frac{1}{n+1}, \quad n = 3, 5, 7, \ldots
\end{align*}
\]

And finally to satisfy the conditions (25) we must have

\[
\begin{align*}
    f_n(0) &= \frac{df_n(0)}{d\eta} = 0 \quad ; \quad n = 1, 3, 5, 7, \ldots \\
    F_0(0) &= 1 \\
    F_n(0) &= 0 \quad n = 2, 4, 6, \ldots
\end{align*}
\]

We have now completely formulated the problem in terms of a set of simultaneous ordinary differential equations with their appropriate boundary conditions. These equations can be solved successively, i.e., the second equation involves the solution to the first but not of the third, fourth, etc. We also note that the equations in \( f_n \) are of the third order but that there are four boundary conditions. The fourth boundary condition is necessary to determine the \( a_0, b_2, b_4, \) etc. which are to be determined by the problem.

Before summarizing the results of this section we will derive explicit formulas for the velocity, shear stress, and heat transfer coefficients
in terms of the solutions to the equations derived above.

We first recall that the velocity in the x direction (i.e., in the ϕ direction) \( v_\phi \) is given terms of the stream function in Eqs. (14) by:

\[
v_\phi = \frac{1}{R_o} \frac{\partial \psi}{\partial y}
\]

If we now change the symbol from \( v_\phi \) to \( u \) (i.e., \( v_\phi = u \)) and introduce the new variables of this section (i.e., \( \psi^*, \eta, \delta^* \)) we get:

\[
\frac{u}{U_\infty} = \frac{1}{2\delta^*} \frac{\partial \psi^*}{\partial \eta} \tag{35}
\]

Also the wall shear stress in the ϕ direction is given by:

\[
\tau_w = \frac{\mu}{R_o} \left. \frac{\partial^2 \psi}{\partial y^2} \right|_{y = 0}
\]

Introducing the variables \( \psi^*, \eta, \) and \( \delta^* \) we get:

\[
\tau_w = \frac{\mu U_\infty R_o X_e}{4 R_o X_e^2} \left. \frac{1}{\delta^*} \frac{\partial^2 \psi^*}{\partial \eta^2} \right|_{\eta = 0}
\]

or:

\[
\tau_w \sqrt{\frac{2 U_\infty R_o}{\nu}} \left. \frac{1}{1/2 \rho U_\infty^2} \frac{\partial^2 \psi^*}{\partial \eta^2} \right|_{\eta = 0} \tag{36}
\]

*Note that from the last of Eq. (4) \( \tau R_\psi = \mu \left( \frac{1}{r} \frac{\partial v_\psi}{\partial \psi} + \frac{\partial v_\psi}{\partial r} - \frac{v_\phi}{r} \right) \) and from (7) \( v_r = v_\phi = 0 \) at \( r = R_o \) and therefore \( \frac{\partial v_r}{\partial \psi} = 0 \) at \( r = R_o \) hence \( \tau R_\psi \)

\[
(R_\psi, \phi) = \mu \left. \frac{\partial v_\phi}{\partial r} \right|_{r = R_o} = \mu \left. \frac{\partial v_\phi}{\partial y} \right|_{y = 0} = \mu \left. \frac{\partial^2 \psi}{\partial y^2} \right|_{y = 0}
\]
Finally the heat flux at the wall \( q_w \) is given by:

\[
q_w = -k \frac{\partial T}{\partial y} \bigg|_{r=R_o} = -\frac{k}{R_o} \frac{\partial T}{\partial y} \bigg|_{y=0}
\]

and substitution of \( \theta^*, \eta, \) and \( \delta^* \) yields:

\[
q_w = -\frac{k(T_w-T_\infty)}{2R_oX_e} \frac{1}{\delta^*} \frac{\partial \theta^*}{\partial \eta} \bigg|_{\eta=0}
\]

or introducing the Nusselt number \( N_u \equiv q_w 2R_o/(T_\infty-T_w)k \) we get:

\[
\sqrt{\frac{2U R_o}{\nu}} N_u = \frac{1}{\delta^*} \frac{\partial \theta^*}{\partial \eta} \bigg|_{\eta=0}
\]

Now upon substitution of the expansions (26) and (27) we get from Eq. (35) after a calculation performed in Appendix VII

\[
\frac{u}{U_\infty} = u_1 \xi - \frac{u_3}{3!} \xi^3 + \frac{u_5}{5!} \xi^5 - \frac{u_7}{7!} \xi^7 + \ldots.
\]

Where:

\[
u_1 = \frac{1}{2a_o} f_1
\]

\[
u_3 = \frac{1}{2a_o} (4f'_3 + 3b_2 f'_1)
\]

\[
u_5 = \frac{1}{2a_o} \left\{ 6f'_5 + 40b_2 f'_3 + 5(6b_2^2 - b_4)f'_1 \right\}
\]

\[
u_7 = \frac{1}{2a_o} \left\{ 8f'_7 + 126b_2 f'_5 + 140(6b_2^2 - b_4)f'_3 + \ldots \right\}
\]
\[ + 7(90b_2^3 - 15b_2b_4 + b_6)f_1 \] 
\[ \frac{2U_bR_0}{v} \sqrt{\frac{1}{l/2\rho U_c^2}} = \tau_1 \xi - \frac{\tau_2}{3!} \xi^3 + \frac{\tau_5}{5!} \xi^5 - \frac{\tau_7}{7!} \xi^7 + \ldots \quad 39 \]

After substituting the expansions (26) and (27) into Eq. (36) the calculation of Appendix VIII yields:

\[ \tau_1 = \frac{1}{a_0^2 E_0} f_1''(0) \]
\[ \tau_3 = \frac{2}{a_0^2 E_0} \left[ 2f_3''(0) + 3b_2f_1''(0) \right] \]
\[ \tau_5 = \frac{2}{a_0^2 E_0} \left[ 3f_5''(0) + 40b_2f_3''(0) + 5(9b_2^2 - b_4)f_1''(0) \right] \]
\[ \tau_7 = \frac{2}{a_0^2 E_0} \left[ 4f_7''(0) + 126b_2f_5''(0) + 140(9b_2^2 - b_4)f_3''(0) + 7(180b_2^3 - 45b_2b_4 + b_6)f_1''(0) \right] \]

And finally the calculation of Appendix IX yields after substituting the expansions (27) and (28) into Eq. (37):
\[
\frac{N_u}{\sqrt{2\ell B^2_0}} = q_0 + \frac{q_2}{2!} \xi^2 + \frac{q_4}{4!} \xi^4 + \frac{q_6}{6!} \xi^6 + \ldots \quad (40)
\]

where:

\[
q_0 = \frac{1}{a_0 E} F'_0(0)
\]

\[
q_2 = \frac{1}{a_0 E} \left[ F'_2(0) - b_2 F'_0(0) \right]
\]

\[
q_4 = \frac{1}{a_0 E} \left[ F'_4(0) - 6b_2 F'_2(0) + (6b_2^2 - b_4) F'_0(0) \right]
\]

\[
q_6 = \frac{1}{a_0 E} \left[ F'_6(0) - 15b_2 F'_4(0) + 15(6b_2^2 - b_4) F'_2(0) - (90b_2^3 - 15b_2b_4 + b_6) F'_0(0) \right]
\]

We now summarize the results of this section:

\[
\begin{align*}
\frac{1}{a_0 E^2} f''_1 + f'_1 f''_1 - f'_1^2 &= 0 \, ; \, 0 \leq \eta \leq 1 \\
\frac{2}{a_0 E^2} f''_1(1) + 2f'_1(1) - 4a_0 &= 0 \\
\begin{cases}
f_1(l) = 1 \\
f_1(0) = 0 \\
f'_1(0) = 0
\end{cases}
\end{align*}
\]
\[ L_n(f_n) = (-1)^{n/2} \frac{n}{n+1} \left[ f_{1}f''_{1} + (n-2)f_{1}f'_{1} \right] b_{n-1} + \]
\[ + H_n(f_1, f_2, \ldots, f_{n-2}, f_1, \ldots, f'_{n-2}, f''_{1}, \ldots, f''_{n-2}, b_2, \ldots b_{n-3}) \]
\[ L_n = \frac{1}{a_0 E^2} \left( \frac{d^2}{d \eta^2} f_1 + \frac{d}{d \eta} (n+1) f_1 \right) + n f''_{1} \]
\[ = \frac{2}{a_0 E^2} f''_{n}(1) + 2 f'_{n}(1) + \frac{2n}{n+1} \left[ (-1)^{n/2} b_{n-1} \left[ 4a_0 - f'_{1}(1) \right] - 2a_0 + f_{1}(1) \right] \]
\[ + A_n \left[ f_{1}(1), \ldots, f'_{n-2}(1), a_0, b_2, \ldots b_{n-3} \right] \]
\[ f_{n}(1) = \frac{1}{n+1} \]
\[ f_{n}(0) = 0 \]

\[ f_{n}'(0) = 0 \]

\[ K_n(F_n) = G_n(f_1, \ldots, f_{n+1}, F_0, \ldots, F_{n-2}, a_0, b_2, \ldots b_n) \]
\[ K_n = \frac{1}{a_0 E^2 Pr} \left( \frac{d^2}{d \eta^2} f_1 + \frac{d}{d \eta} f_1 - nf_{1} \right) ; \quad 0 \leq \eta \leq 1 \]
\[ = \frac{1}{a_0 E^2 Pr} F_{n}(1) + F_{n}(1) = \epsilon(n) \left[ F_{0}(1) \left[ b_n + \left( -1 \right)^{n/2} \right] \right] + \]
\[ + B_n \left[ F_{0}(1), \ldots, F_{n-2}(1), a_0, b_2, \ldots b_{n-2} \right] \]
\[ F_{n}(0) = 1 - \epsilon(n) \]
\[ \epsilon(n) = \begin{cases} 0 & n = 0 \\ 1 & n \neq 0 \end{cases} \]

We now notice that the differential Eq. in (41) is of the third order but it contains the unknown parameter \( a_0 \). There are thus three boundary
conditions plus a fourth condition to determine $a_0$. Thus the problem for $f_1$ and $a_0$ is completely specified and can in principle be solved. The situation is exactly the same for each of the Eqs. in (42) and they thus determine the pair $f_n$, $b_{n-1}$ uniquely. It should be noted however that the solution to (41) and the solutions to the preceding Eqs. in (42) must first be found before any of the Eqs. (42) can be solved but this presents no problem since after the solution to (41) is determined the Eqs. of (42) can be solved successively. Finally the Eqs. of (43) are of second order and there are two boundary conditions. These depend of course on the solutions of Eqs. (41) and (42) and the solutions to the Eqs. (43) corresponding to lower value of $n$. Thus these equations can be successively solved once the solutions to (41) and (42) have been obtained. Having obtained the solutions to these equations the liquid film thickness (Eq. (27)) the liquid stream function (Eq. (26)) temperature (Eq. (28)) and velocity (Eq. (38)) can be computed. Also the local Nusselt number and shear stress can be obtained from Eq. (40) and (39), respectively. These equations have been solved numerically. The actual calculations are discussed below. Before considering these solutions we turn to the case when $0.01 \leq E \leq .1$ and the presence of the liquid has no effect on the gas (i.e., there is potential flow upstream of the cylinder).

Integrating the first of Eqs. (14c) and using the first two boundary conditions (16c) we get:
\[ \psi = - \frac{2}{3} \frac{l}{E} \rho \frac{g}{\rho} \sin 2xy \psi + \xi(x) \psi^2 \]  

Using this in the first and second of Eqs. (15c) we get:

\[ - \frac{l}{E} \rho \frac{g}{\rho} \sin 2x \delta + \xi(x) \psi = \frac{\delta}{E} \rho \frac{\sqrt{v_{\phi} v}}{v} Z(x) - 2 \frac{X_e}{X_e} \psi^* \psi^* \]

and:

\[ - \frac{2}{3} \frac{l}{E} \rho \frac{g}{\rho} \sin 2x \delta \psi^2 + \xi(x) \psi^2 = \int_0^{X_e} \frac{X_e}{X_{e\infty}} v_r^* \psi^* \Delta \]

or:

\[ \frac{l}{3} \frac{1}{E} \frac{\rho g}{\rho} \sin 2x \delta \psi^2 + \left[ \left( \frac{l}{E} \rho \frac{\sqrt{v_{\phi} v}}{v} Z(x) - \frac{X_e}{X_{e\infty}} v_r^* \psi^* \right) \right] \psi^2 + \int_0^{X_e} \frac{X_e}{X_{e\infty}} v_r^* \psi^* \Delta = 0 \]  

and:

\[ \xi(x) = \left[ \left( \frac{l}{E} \rho \frac{\sqrt{v_{\phi} v}}{v} Z(x) - \frac{X_e}{X_{e\infty}} v_r^* \psi^* \right) \right] + \frac{2}{3} \frac{l}{E} \rho \frac{g}{\rho} \sin 2xy \delta \]

Next introducing the notation:

\[ \tau_{\text{eff.}} = \left[ \left( \frac{l}{E} \rho \frac{\sqrt{v_{\phi} v}}{v} Z(x) - \frac{X_e}{X_{e\infty}} v_r^* \psi^* \right) \right] \]

\[ \xi = - \int_0^{X_e} \frac{X_e}{X_{e\infty}} v_r^* \psi^* \Delta \]

the solution to Eq. (45) can be written as:

\[ \delta = \frac{E}{4(\rho g/\rho)} \frac{\tau_{\text{eff.}}}{\sin 2x} (2 \cos \frac{\gamma}{3} - 1) \]
where:
\[
\gamma = \cos^{-1}\left\{ -1 - \frac{2 \left( \frac{\rho g}{\rho} \right)^2 \sin^2 2\chi \frac{\tau}{\tau_{\text{eff}}}^3}{\frac{\tau}{\tau_{\text{eff}}}^3} \right\}
\]

it should be noted that \( \gamma + 4\pi \) and \( \gamma + 2\pi \) would also be solutions to (45) but \( \gamma + 2\pi \) corresponds to negative values of \( \delta \) and \( \gamma + 4\pi \) correspond to values of \( \delta \) which increase with increasing \( \tau_{\text{eff}} \) which can be ruled out on physical grounds.

Substitution of Eq. (44) into the second of Eqs. (44c) yields:
\[
\begin{aligned}
\frac{\theta_{\overset{\cdot}{y}}}{\overset{\cdot}{y}} &= - 2Pr \frac{\rho g}{\rho} \sin 2\chi \frac{\gamma^2}{\theta_{\overset{\cdot}{x}}} \frac{\theta_{\overset{\cdot}{y}}}{\theta_{\overset{\cdot}{x}}} + \frac{1}{3} Pr \frac{\rho g}{\rho} \cos 2\chi \frac{\gamma^2}{\theta_{\overset{\cdot}{y}}} \frac{\theta_{\overset{\cdot}{y}}}{\theta_{\overset{\cdot}{y}}}
+ Pr E \left[ 2\gamma \xi \frac{\theta_{\overset{\cdot}{x}}}{\theta_{\overset{\cdot}{x}}} - \gamma^2 \xi' \frac{\theta_{\overset{\cdot}{y}}}{\theta_{\overset{\cdot}{y}}} \right]
\end{aligned}
\]

where we have introduced:
\[
\theta^* = \frac{T - T_\infty}{T_\infty - T_\infty}
\]

We next note that for air/water mixtures
\[
Pr \lesssim 10
\]
\[
\frac{\rho g}{\rho} \lesssim 1.3 \times 10^{-3}
\]
\[
2 Pr \frac{\rho g}{\rho} \lesssim 2.6 \times 10^{-2}
\]

Hence we will neglect the first two terms on the right of the energy equation and we have:
\[
\frac{\theta_{\overset{\cdot}{y}}}{\overset{\cdot}{y}} = Pr E \left[ 2\gamma \xi \frac{\theta_{\overset{\cdot}{x}}}{\theta_{\overset{\cdot}{x}}} - \gamma^2 \xi' \frac{\theta_{\overset{\cdot}{y}}}{\theta_{\overset{\cdot}{y}}} \right] \tag{47}
\]
Next we introduce the variables:

\[ \eta = \frac{\eta'}{\delta} = \frac{\gamma}{\delta} \]

\[ \xi = \frac{\xi'}{\delta} \]

and we have:

\[ \frac{\partial}{\partial y} = \frac{1}{\delta} \frac{\partial}{\partial \eta} \]

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{1}{\delta} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} \frac{\partial}{\partial \eta} \]

and Eq. (47) now becomes:

\[ \Theta^* \eta \eta = P_r E \delta \left\{ 2\eta \left[ \frac{\eta'}{\delta^2 \xi} \right] \Theta^* \xi - \eta^2 \Theta^* \eta \frac{\partial}{\partial \xi} \left[ \frac{\eta'}{\delta^2 \xi} \right] \right\} \quad (48) \]

Now if we define \( J(x) = J(\xi) \) by:

\[ J = -\frac{X_e}{X_{e\infty}} \frac{\eta'}{\xi} \]

It follows from the equation preceding (45):

\[ \xi \delta^2 = \int_0^\xi Jd\xi + \frac{2}{3} E \frac{\rho \xi}{\rho} \left( \sin 2\xi \right) \delta^3 \]

Now using this in Eq. (48) and neglecting terms of the order of \( 2P_r \frac{\rho \xi}{\rho} \)

we get:

\[ \Theta^* \eta \eta = P_r E \delta \left\{ 2\eta \Theta^* \xi \int_0^\xi Jd\xi - \eta^2 \Theta^* \eta J \right\} \quad (49) \]

and from the last condition (15c) and (16c) we have:
\[ \theta_\eta^* = -P_1 E \delta J \theta^* \quad \text{at } \eta = 1 \quad \text{(50)} \]

\[ \theta^* = 1 \quad \text{at } \eta = 0 \quad \text{(51)} \]

We now notice that \( \delta \) must be an even function of \( \xi \), \( J \) must be an even function of \( \xi \) and \( \int_0^\xi J d\xi \) must be an odd function of \( \xi \). We therefore set:

\[
\delta J = \sum_{n=0}^{\infty} \frac{\alpha_{2n}}{(2n)!} \xi^{2n} \quad \text{(52)}
\]

\[
\int_0^\xi J d\xi = \sum_{n=1}^{\infty} \frac{\beta_{2n}}{(2n)!} \xi^{2n-1} \quad \text{(52)}
\]

and:

\[ \theta^* = \sum_{n=0}^{\infty} \frac{F_{2n}}{(2n)!} \xi^{2n} \quad \text{(53)} \]

where as usual we take \( 0! = 1 \).

Next differentiating the second of Eqs. (52) with respect to \( \xi \) and setting \( \xi = 0 \) we get upon comparison with the first Eq. (52)

\[ \beta_2 = \alpha_0 \quad \text{(54)} \]

Substitution of Eqs. (53) and (52) into Eq. (49) and equating the coefficients of like powers of \( \xi \) yields after the calculation performed in Appendix X:

\[ 0 \leq \eta \leq 1 \quad K_k(F_{2k}) = g_k \frac{1}{\alpha_0} \quad k = 0, 1, 2, \ldots \quad \text{(55)} \]
Where:

\[ K_k \equiv \frac{1}{F \alpha} \frac{d^2}{d\eta^2} + \eta^2 \frac{d}{d\eta} - 4k\eta \]

\[ G_k = \begin{cases} 
0 & k = 0 \\
-\alpha \eta^2 F'_0 & k = 1 \\
\eta \sum_{n=1}^{k-1} \binom{2k}{2n-1} F_{2n} \theta_{2(k-n+1)} - \binom{2k}{2n} \eta^{2n} \alpha \theta(k-n) & k > 1 \\
-\alpha \eta^2 F'_0 & k > 1
\end{cases} \]

In obtaining these equations the binomial coefficients which are written as

\[ \binom{m}{n} \]

have been introduced and the relation

\[ \binom{m}{n} = \frac{m!}{(m-n)!n!} \]

has been used.

Substitution of the expansion (52) and (53) into the boundary condition (50) yields after equating the coefficients of like powers of \( \xi \)

\[ \eta = 1 ; \quad \frac{1}{F} F_{2k} = - \sum_{n=0}^{k} \binom{2k}{2n} \alpha \theta(k-n) F_{2n} \]

and the boundary condition (51) requires that
\[ \eta = 0 \quad \begin{vmatrix} F_{2n} = 1 & n = 0 \\ = 0 & n > 0 \end{vmatrix} \tag{57} \]

Now the heat flux at the wall \( q_w \) is given by:

\[ q_w = -\frac{k}{R_o} \frac{\partial T}{\partial y} \bigg|_{y=0} \]

and substitution of \( \theta^* \), \( \eta \), and \( \delta \) yields:

\[ q_w = -\frac{k}{2R_o} \frac{T_W - T_\infty}{\sqrt{\frac{2R_o U_\infty}{\nu}}} \frac{1}{\delta} \frac{\partial \theta^*}{\partial \eta} \bigg|_{\eta=0} \]

and the normalized Nusselt number is:

\[ \frac{N_u}{\sqrt{2U_\infty R_o}} = \frac{1}{\delta} \frac{\partial \theta^*}{\partial \eta} \bigg|_{\eta=0} \]

or substituting in the expansion \((53)\) yields

\[ \frac{N_u}{\sqrt{2U_\infty R_o}} = \frac{1}{\delta} \sum_{n=0}^{\infty} \frac{R^'_{2n} (0)}{(2n)!} \frac{R^2_{2n}}{2n} \]

Next the wall shear stress can be written as in the derivations of Eqs. \((36)\):

\[ \tau_w = \frac{\mu}{R_o^2} \frac{\partial^2 \psi}{\partial y^2} \bigg|_{y=0} \]

and introducing \( \psi \) and \( \psi \) we get:

\[ *C.f. \ derivation \ of \ Eq. \ (37). \]
\[ \tau_w \sqrt{\frac{2U_\infty R_0}{\nu}} = E \frac{\partial \psi}{\partial \gamma} \bigg|_{\gamma = 0} \]

and using Eqs. (44) and (46) we have

\[ \tau_w \sqrt{\frac{2U_\infty R_0}{\nu}} = 2E \tau_{\text{eff.}} + 4 \frac{\rho g}{\rho} \delta \sin 2\chi \]

Next we have:

\[ u = \frac{1}{R_0} \frac{\partial \psi}{\partial y} \]

and introducing \( \tilde{\psi} \) and \( \tilde{\gamma} \) we get:

\[ \frac{u}{U_\infty} = \frac{E}{2} \frac{\tilde{\psi}}{\tilde{\gamma}} \]

using Eqs. (44) and (46) we arrive at:

\[ \frac{u}{U_\infty} = E \tau_{\text{eff.}} \tilde{\gamma} + \frac{\rho g}{\rho} (\sin 2\chi) (2\delta - \tilde{y}) \tilde{y} \]

and finally summarizing the results of this section we have:

\[ \delta = \frac{E}{4(\rho_0/\rho)} \frac{\tau_{\text{eff.}}}{(\sin 2\chi)} \left( 2 \cos \frac{\gamma}{3} - 1 \right) \]

\[ \gamma = \cos^{-1} \left\{ -1 - \frac{2u(\rho_0/\rho)^{1/2}}{E^2} \left( \frac{\sin 2\chi}{\tau_{\text{eff.}}} \right) \gamma' \right\} \]

\[ \frac{u}{U_\infty} = E \tau_{\text{eff.}} \tilde{\gamma} + \frac{\rho g}{\rho} (\sin 2\chi) (2\delta - \tilde{y}) \tilde{y} \]

\[ \tau_w \sqrt{\frac{2U_\infty R_0}{\nu}} = 2E \tau_{\text{eff.}} + 4 \frac{\rho g}{\rho} \delta \sin 2\chi \]

*C.f. derivation of Eq. (35).*
\[
\frac{N_u}{\sqrt{\frac{2U_\infty R_0}{\nu}}} = \frac{1}{\alpha_o} \sum_{n=0}^{\infty} \frac{F_{2n}(0)}{(2n)!} \xi^{2n}
\]

Where the \( F_{2n} \)'s are determined by:

\[
\begin{align*}
0 \leq \eta \leq 1 ; & \quad K_k(F_{2k}) = \frac{1}{\alpha_o} G_k \\
\eta = 1 ; & \quad \frac{1}{P_rE} F_{2k}'' = \sum_{n=0}^{k} \binom{2}{k} \frac{\alpha_{2(k-n)}F_{2n}}{(2n)!} \\
\eta = 0 ; & \quad F_{2n} \begin{cases} = 1 & n = 0 \\ = 0 & n > 0 \end{cases} \\
\end{align*}
\]

\[
J = - \frac{X_e v^*}{X_{e\infty}}
\]

\[
\dot{\mathbf{m}} = - \int_{0}^{x} \frac{X_e v^*}{X_{e\infty}} \mathbf{v}_r \mathrm{d}x
\]

\[
\frac{\partial \xi}{\partial J} = \sum_{n=0}^{\infty} \frac{\alpha_{2n}}{(2n)!} \xi^{2n}
\]

\[
\frac{\partial \xi}{\partial \mathbf{m}} = \sum_{n=1}^{\infty} \frac{\beta_{2n}}{(2n)!} \xi^{2n-1}
\]

\[
\tau_{\text{eff}} = \frac{1}{\rho g} \sqrt{\frac{\mu g}{\rho}} Z(x) + J V_\phi^*
\]

\[
G_k = \begin{cases} 
0 & \\
- \alpha_2 \eta^2 F'_0 & \\
\eta \sum_{n=1}^{k-1} \left[ \binom{2}{2n} \frac{\beta_{2(k-n)+1} - \binom{2k}{2n}}{(2n)!} \eta F'_{2n} \frac{\alpha_{2(k-n)}}{2k} \right] \\
- \alpha_2 \eta^2 F'_0
\end{cases}
\]
\[ K_k = \frac{1}{P_e \sigma_0} \frac{d^2}{d\eta^2} \eta^2 \frac{d}{d\eta} - 4k\eta \]

\[ Z(x) = \gamma_1 x - \sum_{n=2}^{\infty} \frac{(-1)^n (2n)}{(2n-1)!} \gamma_n (2n-1) \gamma_{2n-1} x^{2n-1} \]

<table>
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<tr>
<th>$\gamma_1$</th>
<th>$\gamma_3$</th>
<th>$\gamma_5$</th>
<th>$\gamma_7$</th>
<th>$\gamma_9$</th>
<th>$\gamma_{11}$</th>
</tr>
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<td>1.0320</td>
<td>2.036833</td>
<td>0.280140</td>
<td>67.637501</td>
</tr>
</tbody>
</table>

These formulas can now be used to calculate the desired physical quantities once $J$, $\dot{\gamma}$, and $V_\phi^*$ are known as functions of $x$. These quantities have been obtained numerically (see Appendix XXV) and the results are tabulated for various values of the Reynolds number and drop size. The numerical calculations are described in Appendix XXVI.

Now the problem of the oscillating flat plate will be discussed.

C. Semi-Infinite Oscillating Flat Plate in a Streaming Gas/Liquid Flow with Oscillations Parallel to the Flow

1. FORMULATION

Now a semi-infinite oscillating flat plate which is maintained at a constant temperature $T_w$ in a streaming gas which contains liquid drops will be considered. The plate is taken to be parallel to the velocity of the gas far upstream from it and to be oscillating sinusoidally in its plane with a small amplitude $U_\infty \frac{\epsilon}{2}$. Now we will use the flow model described above in the section entitled "Preliminary Consideration" and assume from the outset that the boundary layer assumptions hold with
sufficient accuracy within the liquid film. The boundary layer approximations will hold with good accuracy in the film if it is sufficiently thin. But since the liquid capture area of the film in this case is only proportional to its thickness, whereas in the case of the cylinder it is equal to the cylinder cross sectional area one can be sure that the film will be thin. Next since the pressure field due to the presence of the plate is only a second order effect and the gas flow outside of the boundary layer is essentially parallel, it will further be assumed that the drop trajectories are straight lines and the volume fraction $X_e$ of the liquid outside the film is everywhere the same.

The liquid film will now be considered. We set up an $x, y$ coordinate system which is fixed to the plate. The velocity of the free stream $U(t)$ relative to these coordinates is then $U_\infty$ minus the velocity of oscillation of the plate which is taken as $-\frac{\epsilon}{2} U_\infty \sin \omega t$.

Thus:

$$U(t) = U_\infty (1 + \frac{\epsilon}{2} \sin \omega t) \tag{59}$$

Attention is now focused upon the control volume of Figure 5 which is bounded by the liquid free surface $S'$, the surface perpendicular to the plate $S''$ and by the plate itself. Its cross sectional area is $A_{c.s}$. If the $x$ and $y$ components of the velocity of the surface $S'$ are $v_{c.s}^{(x)}$
Figure 5. Coordinates and control volume for liquid layer on flat plate.
and \( V_{c.s.}^{(y)} \), respectively, then they must be related to \( \delta \) by:

\[
V_{c.s.}^{(y)} = \frac{\partial \delta}{\partial t} + V_{c.s.}^{(x)} \frac{\partial \delta}{\partial x}
\]

and they must be related to the velocity of the control surface perpendicular to itself \( V_{c.s.} \) by:

\[
V_{c.s.} \perp = - V_{c.s.}^{(x)} \cos \varphi + V_{c.s.}^{(y)} \sin \varphi
\]

\[
= - V_{c.s.}^{(x)} \cos \varphi + V_{c.s.}^{(x)} \frac{\partial \delta}{\partial x} \sin \varphi + \frac{\partial \delta}{\partial t} \sin \varphi
\]

\[
= \left[ V_{c.s.}^{(x)} \left( \frac{\partial \delta}{\partial x} - \cotan \varphi \right) + \frac{\partial \delta}{\partial t} \right] \sin \varphi
\]

but

\[
\frac{\partial \delta}{\partial x} = \cotan \varphi
\]

and:

\[
V_{c.s.} \perp = \frac{\partial \delta}{\partial t} \sin \varphi
\]

60)

Thus \( V_{c.s.} \perp \) is dependent only on \( \frac{\partial \delta}{\partial t} \) and independent of what motion we may apply to the control surface along its length which is as it should be. The law of conservation of mass for the control volume may be written as:

*Note that:

\[
\frac{\partial \delta}{\partial t} = V_{c.s.}^{(y)} = \frac{\partial \delta}{\partial t} + \frac{\partial \delta}{\partial S} \frac{\partial S}{\partial t} = \frac{\partial \delta}{\partial t} + \frac{\partial \delta}{\partial x} \frac{dx}{dt} \frac{\partial S}{\partial t}
\]

\[
= \frac{\partial \delta}{\partial t} + \frac{\partial \delta}{\partial x} \frac{dx}{dt}
\]
\[
\frac{d}{dt} \int\int_{A_{c.s.}} \rho \, dA + \int_{s'} \rho X_e \left[ -U \cos \varphi - \frac{1}{V_{c.s.}} \right] \, ds' + \int_{s''} \rho \, u \, ds'' = 0
\]

And remembering that an accelerated coordinate system is being used the law of conservation of x-direction momentum may be written as:

\[
\frac{d}{dt} \int\int_{A_{c.s.}} \rho \, u \, dA + \int_{s'} \rho X_e U \left[ -U \cos \varphi - \frac{1}{V_{c.s.}} \right] \, ds' + \int_{s''} \rho \, u \, ds'' = -\int_{0}^{x} \tau_{x,y}(x_1,0,t) \, dx_1 + \int\int_{A_{c.s.}} \rho \, \frac{du}{dt} \, dy + \int_{s'} \tau_g \, \sin \varphi \, ds'
\]

where the pressure forces have been neglected since it is assumed that the boundary layer approximations apply. And finally an energy balance on the control volume is performed. In doing this, we will neglect a priori the effects due to kinetic energy changes, and assume that the temperature of the drops is not significantly reduced from that of the free stream when they pass through the gas boundary layer.

With these approximations one obtains:

\[
\frac{d}{dt} \int\int_{A_{c.s.}} \rho \, C_p T \, dA + \int_{s'} \rho X_e C_p T_\infty \left[ -U \cos \varphi - \frac{1}{V_{c.s.}} \right] \, ds'
\]
\[ + \int_{s''} \rho C_p T u \, ds'' = \int_0^x k \left. \frac{\partial T}{\partial y} \right|_{y=0} \, dx_1 + \int_{s'} q_s \, ds' \quad \text{(63)} \]

We next notice that:

\[ \frac{\partial \delta}{\partial x} = \cotan \varphi \]

\[ ds' = \sqrt{1 + \left(\frac{\partial \delta}{\partial x}\right)^2} \, dx = \sqrt{1 + \cotan^2 \varphi} \, dx = \frac{dx}{\sin \varphi} \]

\[ dA = dy \, dx_1 \]

\[ ds'' = dy \]

and by using these and Eqs. (60) in Eqs. (61) through (63), one obtains:

\[ \frac{d}{dt} \int_0^x \int_0^x \delta(x_1, t) \, dy \, dx_1 - X_e \int_0^x \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] \, dx_1 \]

\[ + \int_0^\delta u(x, y, t) \, dy = 0 \quad \text{(61A)} \]

\[ \rho X_e \int_0^x \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] \, dx_1 + \int_0^\delta \gamma \, dx_1 \]

\[ = \rho \frac{d}{dt} \int_0^x \int_0^x u \, dy \, dx_1 + \rho \int_0^\delta u^2(x, y, t) \, dy \]
\[
- \rho \int_0^x \int_0^y \frac{\partial u}{\partial t} \ dy + \int_0^x \tau_{xy}(x_1, t) \ dx_1 \tag{62A}
\]

\[
\rho X_e C_p T \int_0^x \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] \ dx_1 + \int_0^x \frac{q_{g}}{\sin \phi} \ dx_1
\]

\[
= \rho C_p \int_0^x T(x, y, t) \ u(x, y, t) \ dy + \rho C_p \frac{\partial}{\partial t} \int_0^x T \ dy \ dx_1
\]

\[- k \int_0^x \frac{\partial T}{\partial y} \bigg|_{y=0} \ dx_1 \tag{63A} \]

Now differentiating (61A) through (63A) with respect of \( x \) yields:

\[
\frac{\partial \delta}{\partial t} \int_0^y \ dy - X_e \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \int_0^x \frac{\partial u}{\partial x} \ dy + u(x, \delta, t) \frac{\partial \delta}{\partial x} = 0
\]

\[
\rho X_e U \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \tau_g = \rho \frac{\partial}{\partial t} \int_0^x u \ dy
\]

\[
= \rho \int_0^x \delta(x, t) \ dy + \rho u^2(x, \delta, t) \frac{\partial \delta}{\partial x} + \tau_{xy}(x, 0, t)
\]

\[- \rho \int_0^x \frac{\partial u}{\partial t} \ dy \]

\[
\rho X_e C_p T \int_0^x \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \frac{q_{g}}{\sin \phi} = \rho C_p \int_0^x \frac{\partial (u T)}{\partial x} \ dy
\]

\[
+ \rho C_p T(x, \delta, t) u(x, \delta, t) \frac{\partial \delta}{\partial x} + \rho C_p \frac{\partial}{\partial t} \int_0^x T \ dy
\]

\[- k \frac{\partial T}{\partial y} \ (x, 0, t) \]
or performing the integrations with respect to $t$:

$$\left[ \frac{\partial \delta}{\partial t} + u(x, \delta, t) \frac{\partial \delta}{\partial x} \right] = X_{e} \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] - \int_{0}^{\delta(x, t)} \frac{\partial u}{\partial x} \, dy$$

$$\rho X_{e} U \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] - \rho u(x, \delta, t) \left[ \frac{\partial \delta}{\partial t} + u(x, \delta, t) \frac{\partial \delta}{\partial x} \right] + \tau_{g} =$$

$$\rho \int_{0}^{\delta(x, t)} 2 u \frac{\partial u}{\partial x} \, dy + \rho \int_{0}^{\delta(x, t)} \frac{\partial u}{\partial t} \, dy + \tau_{x, y}(x, 0, t) - \rho \int_{0}^{\delta(x, t)} \frac{\partial u}{\partial t} \, dy$$

$$\rho X_{e} C_{p} T_{\infty} \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] - \rho C_{p} T(x, \delta, t) \left[ \frac{\partial \delta}{\partial t} + u(x, \delta, t) \frac{\partial \delta}{\partial x} \right] +$$

$$+ \frac{q_{g}}{\sin \varphi} = \rho C_{p} \int_{0}^{\delta(x, t)} \left[ u \frac{\partial T}{\partial x} + \frac{\partial T}{\partial t} \right] \, dy + \rho C_{p} \int_{0}^{\delta(x, t)} \frac{\partial u}{\partial x} \, dy$$

$$- k \frac{\partial T}{\partial y} (x, 0, t)$$

and finally using the first equation to eliminate $\left[ \frac{\partial \delta}{\partial t} + u(x, \delta, t) \frac{\partial \delta}{\partial x} \right]$ from the last two we obtain:

$$\rho X_{e} \left[ U - u(x, \delta, t) \right] \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \tau_{g} = + \tau_{x, y}(x, 0, t)$$

$$\rho \int_{0}^{\delta(x, t)} \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right] \, dy + \rho \int_{0}^{\delta(x, t)} u \frac{\partial u}{\partial x} \, dy - \rho u(x, \delta, t) \int_{0}^{\delta(x, t)} \frac{\partial u}{\partial x} \, dy$$

$$- \rho \int_{0}^{\delta(x, t)} \frac{\partial u}{\partial t} \, dy$$

$$\rho X_{e} C_{p} \left[ T_{\infty} - T(x, \delta, t) \right] \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \frac{q_{g}}{\sin \varphi} = - k \frac{\partial T}{\partial y} (x, 0, t)$$
\[
\rho C_p \int_0^\delta \left[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] dy + \frac{\partial u}{\partial x} \int_0^\delta \frac{\partial u}{\partial x} dy - \rho C_p T(x, \delta) \int_0^\delta \frac{\partial u}{\partial x} dy
\]

In order to proceed further the continuity equation is now introduced, i.e.:

\[
\frac{\partial u}{\partial x} = - \frac{\partial v}{\partial y}
\]

By using this in the last two integrals on the right of the last two equations and performing an integration by parts one obtains:

\[
\rho \chi_e \left[ U - u(x, \delta, t) \right] \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \tau_y = \delta(x, t) \int_0^\delta \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] dy + \tau_{xy}(x, 0, t) - \rho \int_0^\delta \frac{\partial u}{\partial t} dy \quad 64)
\]

\[
\rho C_p \chi_e \left[ T_\infty - T(x, \delta, t) \right] \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \frac{a g}{\sin \varphi} = \delta(x, t) \int_0^\delta \rho C_p \left[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] dy - k \frac{\partial T}{\partial y}(x, 0, t) \quad 65)
\]

Since the liquid in the film is assumed to satisfy the boundary layer approximations, (and since for flow over a flat plate it is consistent with these to neglect the pressure forces) since we are using an accelerated coordinate system, the governing differential equations become:

\[
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = \frac{\partial u}{\partial t} + \mu \frac{\partial^2 u}{\partial y^2} \quad 66)
\]

\[
\rho C_p \left[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = k \frac{\partial^2 T}{\partial y^2} \quad 67)
\]
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \] 

68)

and the shear stress \( \tau_{xy}(x,0,t) \) is given by:

\[ \tau_{xy}(x,0,t) = -\mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \]

69)

Using Eqs. (66) and (67), and (69) in Eqs. (64) and (65) and performing the integrations we obtain:

\[
\begin{bmatrix}
\rho x_e \left[ U - u \right] \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \tau_x = \mu \frac{\partial u}{\partial y} \\
\rho x_e c_p \left[ T_w - T \right] \left[ \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \right] + \frac{q_g}{\sin \varphi} = k \frac{\partial T}{\partial y}
\end{bmatrix}
\]

70)

Next we introduce the stream function \( \psi \) defined by:

\[ u = \frac{\partial \psi}{\partial y} ; \quad v = -\frac{\partial \psi}{\partial x} \]

71)

which satisfies equation (68) identically.

Now the boundary conditions at the plate are:

\[ T = T_w ; \quad u = v = 0 ; \quad y = 0 ; \quad 0 < x \]

the last two of which will be satisfied by taking:

\[ \psi = \frac{\partial \psi}{\partial y} = 0 ; \quad y = 0, \quad 0 < x \]

72)

Using (71) in Eq. (61A) gives:

\[
\frac{d}{dt} \int_0^x \int_0^x \delta(x_1,t) \ dy \ dx_1 - \rho x_e \int_0^x \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x} \ dx_1
\]
\[
\delta(x,t) + \int_0^x \frac{\partial \psi}{\partial y} \, dy = 0
\]

and performing the integrations and using the boundary condition (72) gives

\[
\frac{d}{dt} \int_0^x \delta(x_1,t) \, dx_1 - X_e \int_0^x \frac{\partial \delta}{\partial t} \, dx_1 - X_e \delta(x,t)U + \psi(x,\delta,t) = 0
\]

or

\[
\psi = X_e \delta + (X_e - 1) \int_0^x \frac{\partial \delta}{\partial t} \, dx_1 \tag{73}
\]

Now using Eqs. (71) and (59) in Eqs. (66), (67), (70) and (73) yields:

\[
\begin{bmatrix}
\frac{\partial^2 \psi}{\partial t \partial y} + \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \frac{U_0 \omega}{2} \cos \omega t + \nu \frac{\partial^3 \psi}{\partial y^3} \\
\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} - \nu \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} = \frac{k}{\rho C_p} \frac{\partial^2 \psi}{\partial y^2}
\end{bmatrix}
\tag{74}
\]

\[
\psi = X_e \delta U_\infty (1 + \frac{\epsilon}{2} \sin \omega t) + (X_e - 1) \int_0^x \frac{\partial \delta}{\partial t} \, dx_1
\]

\[
X_e \left[ \frac{\partial \psi}{\partial y} - U_\infty (1 + \frac{\epsilon}{2} \sin \omega t) \right] \left[ \frac{\partial \delta}{\partial t} + U_\infty (1 + \frac{\epsilon}{2} \sin \omega t) \frac{\partial \delta}{\partial x} \right]
\]

\[
\frac{\tau}{\rho} = -\nu \frac{\partial^2 \psi}{\partial y^2}
\]

\[
X_e \left[ T_\infty - T \right] \left[ \frac{\partial \delta}{\partial t} + U_\infty (1 + \frac{\epsilon}{2} \sin \omega t) \frac{\partial \delta}{\partial x} \right] + \frac{q_{\delta}}{\rho C_p \sin \varphi}
\]

\[
= \frac{k}{\rho C_p} \frac{\partial T}{\partial y}
\]

\[
\tag{75}
\]
\[ y = 0 \quad \left\{ \begin{array}{l}
T = T_w; \\
\psi = \frac{\partial \psi}{\partial y} = 0
\end{array} \right. \]  \hspace{1cm} (76)

These equations and boundary conditions would completely determine the problem of the liquid film if \( \tau_g \) and \( q_g \) were known. Before discussing these, the following dimensionless variables are introduced:

\[ \eta = \frac{y}{\delta} \]
\[ \xi = \sqrt{\frac{\omega x}{X_e U_\infty}} \]
\[ \psi^* = \frac{\psi}{U_\infty X_e} \sqrt{\frac{\omega}{X_e \nu \psi}} \]
\[ \xi^* = \xi \sqrt{\frac{\omega}{v X_e}} \]
\[ \tau = \omega t \]
\[ \Theta^* = \frac{T - T_\infty}{T_w - T_\infty} \]

It can be verified from the results of the calculations which this analysis leads to, that the terms of the Eqs. (73) through (76) expressed in terms of these new variables are of order unity. Now employing the chain rule, we get:

\[ \frac{\partial}{\partial x} = \left( \frac{\partial \eta}{\partial x} \right)_{y,t} \frac{\partial}{\partial \eta} + \left( \frac{\partial \xi}{\partial x} \right)_{y,t} \frac{\partial}{\partial \xi} \]
\[ \frac{\partial}{\partial t} = \left( \frac{\partial \eta}{\partial t} \right)_{x,y} \frac{\partial}{\partial \eta} + \left( \frac{\partial \xi}{\partial t} \right)_{x,y} \frac{\partial}{\partial \xi} \]
\[ \frac{\partial}{\partial y} = \left( \frac{\partial \eta}{\partial y} \right) \frac{\partial}{\partial \eta} \]

or

\[
\begin{align*}
\frac{\partial}{\partial x} &= \frac{\omega}{2X_e U_\infty} \left\{ \frac{1}{\delta} \frac{\partial}{\partial \delta} - \frac{1}{\delta^2} \frac{\partial \delta^*}{\partial \delta} \right\} \\
\frac{\partial}{\partial t} &= \omega \left\{ \frac{\partial}{\partial t} - \frac{1}{\delta^*} \frac{\partial \delta^*}{\partial \delta} \right\} \\
\frac{\partial}{\partial y} &= \sqrt{\frac{\omega}{vX_e}} \frac{1}{\delta^*} \frac{\partial}{\partial \eta}
\end{align*}
\]

and:

\[
\frac{\partial^2}{\partial x \partial y} = \frac{1}{2} \frac{\omega}{X_e U_\infty} \sqrt{\frac{\omega}{vX_e}} \left\{ \frac{1}{\delta^*} - \frac{\partial^2}{\partial t \partial \eta} \right\}
\]

\[
\frac{\partial^2}{\partial t \partial y} = \omega \left\{ \frac{1}{\delta^*} \frac{\partial^2}{\partial t \partial \eta} - \frac{1}{\delta^2} \frac{\partial \delta^*}{\partial \delta} \right\}
\]

\[
\frac{\partial^3}{\partial y^3} = \frac{\omega}{vX_e} \sqrt{\frac{\omega}{vX_e}} \frac{1}{\delta^3} \frac{\partial^3}{\partial \eta^3}
\]

Rearranging the second of Eqs. (75) one obtains:

\[
\begin{align*}
\left( U_\infty - \frac{\partial \psi}{\partial y} \right) \left( \frac{\partial \delta}{\partial t} + U_\infty \frac{\partial \delta}{\partial x} \right) + \frac{\varepsilon}{2} U_\infty \sin \omega t \left\{ \frac{\partial \delta}{\partial t} + 2U_\infty \frac{\partial \delta}{\partial x} \right\} \\
- \frac{\partial \delta}{\partial x} \frac{\partial \psi}{\partial y} + \frac{\varepsilon^2}{4} \left( \frac{1}{2} \frac{\partial^2 \delta}{\partial x^2} \right) \frac{1}{2} (1 - \cos 2\omega t) \frac{\partial \delta}{\partial x} = \frac{\nu}{X_e} \frac{\partial^2 \psi}{\partial y^2}
\end{align*}
\]

at \( y = 5 \)

where the identity:

\[ \sin^2 \omega t = \frac{1}{2} (1 - \cos 2\omega t) \]
has been used.

Using the relations (77) and (79) in this, one obtains:

\[
\begin{align*}
\text{at } \eta = 1 \quad & (1 - \frac{X_e}{\delta^*} \frac{\partial \psi^*}{\partial \eta}) \left( X_e \frac{\partial \delta^*}{\partial \tau} \frac{\partial}{\partial \xi} \frac{1}{2} \frac{\partial \delta^*}{\partial \xi} \right) + \\
& + \frac{\varepsilon}{2} \sin \tau \left( X_e \frac{\partial \delta^*}{\partial \tau} + \frac{\partial \delta^*}{\partial \xi} - \frac{X_e}{2\delta^*} \frac{\partial \delta^*}{\partial \xi} \frac{\partial \psi^*}{\partial \eta} \right) \\
& + \frac{\varepsilon^2}{16} (1 - \cos 2\tau) \frac{\partial \delta^*}{\partial \xi} = \frac{1}{\delta^*} \frac{\partial^2 \psi^*}{\partial \eta^2} - \\
& - \frac{\tau_g}{X_e \rho \gamma U^2} \left( \frac{U_\infty \gamma}{\nu_g} \left[ \begin{matrix} \nu_g \\ \rho_g \end{matrix} \right] \right)
\end{align*}
\]

Now the order of magnitude of the last term on the right of this equation can be estimated by evaluating it for a single component gas boundary layer and evaluating the property ratios for a particular liquid/gas combination. Since the principle interest is in air/water mixtures, we will limit ourselves to these. Thus from Appendix II we have:

\[
\frac{\tau_g}{\rho \gamma U^2} \left( \frac{U_\infty \gamma}{\nu} \left[ \begin{matrix} \nu_g \\ \rho_g \end{matrix} \right] \right) = 1.1 \times 10^{-3}
\]

This term will be taken to be negligible if it is less than 0.1 since the term \( \frac{1}{2} \frac{\partial \delta^*}{\partial \xi} \) \( \geq 1 \) for all values of \( X_e \) as shown by the numerical calculations. And the above results shown for air/water mixtures will be less than 0.1 if:

\( X_e > 1.1 \times 10^{-2} \)
We therefore limit ourselves to values of $X_e$ greater than this, and neglect the gas shear stress term, and we have:

\[
\delta^* (\delta^* - X_e \psi^* \eta) (X_e \delta^*_T + 1/2 \delta^*_x) + \frac{\epsilon}{2} \delta^* \left[ X_e (\delta^*_T \delta^*_T + \delta^*_x \delta^*_x) - \frac{X_e}{2} \delta^*_x \psi^* \eta \right] \sin \tau + \frac{\epsilon \tau}{16} \delta^*_x^2 \delta^*_x (1 - \cos 2\tau) = \xi^* \psi^* \eta
\]

............... For $\eta = 1$ ; $0 < \xi^* \leq 80$

Next using the relations (77) and (79) in the third of Eqs. (75), one obtains:

at $\eta = 1$

\[
(T - T_\infty) \left[ X_e \delta^*_x \frac{\delta^*_x}{\delta T} + \frac{1}{2} (1 + \frac{\epsilon}{2} \sin \tau) \frac{\partial \delta^*_x}{\partial \xi} \right] = - \frac{1}{X_e \rho_c} \frac{\xi^*}{\delta^*_x} \frac{\partial (T - T_\infty)}{\partial \eta}
\]

\[
+ \frac{1}{X_e} \left[ \left( \frac{\rho g}{T - T_\infty} \right) \frac{x}{k g} \right] \sqrt{\frac{U_{\infty} x}{g}} \left( \frac{1}{\gamma \rho g} \frac{C_p g}{k g} \right) \left( \frac{\gamma \rho g C_p}{\rho \gamma C_p} \right) \frac{(T - T_\infty)}{\sin \phi}
\]

We again estimate the order of magnitude of the last term in this equation for air/water mixture by calculating its value from the relations for a single component gas boundary layer. From Appendix II its value is then approximately:

\[
\frac{1}{X_e} \times 3.75 \times 10^{-4} (T - T_\infty)
\]

*The subscript notation is used here to indicate partial derivatives i.e.,

$\psi^*_x \equiv \frac{\partial \psi^*}{\partial \xi}$; $\delta^*_x \equiv \frac{\partial \delta^*_x}{\partial \xi}$; $\psi^*_\eta \equiv \frac{\partial \psi^*}{\partial \eta}$; $\psi^*_\eta^2 \equiv \frac{\partial^2 \psi^*}{\partial \eta^2}$; $\delta^*_T \equiv \frac{\partial \delta^*_T}{\partial \tau}$; etc.
and thus it is negligible compared with the term:

\[
\frac{1}{2} \frac{\partial \delta^*}{\partial \xi} (T - T_\infty)
\]

for \( X_e > 10 \) which is the minimum value to be considered. Hence this term is neglected and we have

\[
\begin{align*}
\text{at } \eta = 1 & ; \quad 0 < \xi : \psi^* & = \frac{-1}{X_e P_r} \frac{\partial \delta^*}{\partial \eta} \\
& = \frac{X_e \xi \frac{\partial \delta^*}{\partial \tau} + 1/2 \left( 1 + \frac{\varepsilon}{2} \sin \tau \right) \frac{\partial \delta^*}{\partial \tau}}{\text{Eq. (81)}}
\end{align*}
\]

Using the relations (77) and (78) in the first of Eqs. (75) yields:

\[
\psi^* = \delta^* (1 + \frac{\varepsilon}{2} \sin \tau) + 2 (X_e - 1) \int_0^\xi \frac{\partial \delta^*}{\partial \tau} \xi d \xi ; \text{ at } \eta = 1 ; \quad 0 < \xi
\]

\[\text{Eq. (82)}\]

And using the relations (77), (78), and (79) in the first and second of Eqs. (74) gives:

\[
X_e \left[ \delta^* \frac{\partial \delta^*}{\partial \tau} \xi + \frac{1}{2} \frac{\partial \delta^*}{\partial \tau} \xi \frac{\partial^2 \delta^*}{\partial \eta^2} \right] - \left[ \frac{1}{2\xi} \frac{\partial \delta^*}{\partial \xi} \frac{\partial \delta^*}{\partial \eta} + \delta^* \delta^* \frac{\partial \delta^*}{\partial \tau} \right] \frac{\partial \psi^*}{\partial \eta} - \eta \delta^* \frac{\partial \delta^*}{\partial \tau} \frac{\partial^2 \psi^*}{\partial \eta^2} \right] = \text{Eq. (82)}
\]

\[\text{Eq. (82)}\]

\[\frac{\varepsilon}{2} \delta^* \cos \tau + \frac{\partial^3 \psi^*}{\partial \eta^3} \]

*Note that:

\[
\sin \varphi = \frac{1}{\sqrt{1 + \cot^2 \varphi}} = \frac{1}{\sqrt{1 + \left( \frac{\partial \delta}{\partial x} \right)^2}} \approx 1
\]
\[
X e Pr \left\{ 5 \delta^* \frac{\partial \theta^*}{\partial \xi} + \frac{1}{2} \delta^* \left[ \frac{\partial^2 \psi^*}{\partial \xi^2} \frac{\partial \psi^*}{\partial \eta} - \frac{\partial \psi^*}{\partial \xi} \frac{\partial^2 \psi^*}{\partial \eta^2} \right] \right\} - \\
- 5 \delta^* \frac{\partial \psi^*}{\partial \eta} \frac{\partial \theta^*}{\partial \eta} \right\} = 5 \frac{\partial^2 \theta^*}{\partial \eta^2}
\]

The boundary conditions (76) are in terms of (77):

\[
\eta = 0 ; \quad 0 < \xi : \quad \theta^* = 1, \quad \psi^* = \frac{\partial \psi^*}{\partial \eta} = 0 \quad 83)
\]

And finally collecting Eqs. (80) through (83) we have:

\[
\left\{ \begin{array}{l}
X e \left\{ 5 \delta^* \frac{\partial^2 \psi^*}{\partial \tau^2 \partial \eta} + \frac{1}{2} \delta^* \left[ \frac{\partial^2 \psi^*}{\partial \xi^2} \frac{\partial \psi^*}{\partial \eta} - \frac{\partial \psi^*}{\partial \xi} \frac{\partial^2 \psi^*}{\partial \eta^2} \right] \right\} - \\
\frac{1}{2} \left[ \frac{\partial \delta^*}{\partial \xi} \frac{\partial \psi^*}{\partial \eta} + 2 \delta^* \frac{\partial \delta^*}{\partial \tau} \right] \frac{\partial \psi^*}{\partial \eta} - \eta \delta^* \frac{\partial^2 \psi^*}{\partial \tau^2 \partial \eta^2} \right\} = \\
\frac{\epsilon}{2} \delta^* \frac{\partial^3 \psi^*}{\partial \tau^3} + \frac{\partial \psi^*}{\partial \eta^3} \\
X e Pr \left\{ 5 \delta^* \frac{\partial \theta^*}{\partial \tau} + \frac{1}{2} \delta^* \left[ \frac{\partial \theta^*}{\partial \xi} \frac{\partial \psi^*}{\partial \eta} - \frac{\partial \psi^*}{\partial \xi} \frac{\partial \theta^*}{\partial \eta} \right] \right\} - \\
- \delta^* \frac{\partial \psi^*}{\partial \eta} \frac{\partial \theta^*}{\partial \eta} \right\} = 5 \frac{\partial^2 \theta^*}{\partial \eta^2}
\right\} 84)
\]

\[
\eta = 1 \quad 0 < \xi
\]

\[
\psi^* = \delta^* \left( 1 + \frac{\epsilon}{2} \sin \tau \right) + 2(X e - 1) \int_0^{\xi} \frac{\partial \delta^*}{\partial \xi} \xi d \xi \\
\delta^* \left( \delta^* - X e \psi^* \right) \left( X e \delta^* \frac{\partial \psi^*}{\partial \tau} + \frac{1}{2} \delta^* \right) + \frac{\epsilon}{2} \delta^* \left[ \delta^* \left( X e \delta^* \right) + \\
+ \frac{\delta^*}{\xi} - \frac{X e}{2} \delta^* \frac{\partial \psi^*}{\partial \xi} \right] \sin \tau + \frac{\epsilon^2}{16} \delta^* \frac{\partial^2 \psi^*}{\partial \xi^2} (1 - \cos 2\tau) = \\
\delta^* \psi^* \left( 1 - \cos \tau \right) \eta^2 \eta
\]

85)
Equations (84) through (86) completely determine $\theta^*, \delta^*, \psi^*$ within the liquid film. The solution to this boundary value problem is now considered.

2. SOLUTION

First we remember that the amplitude of the oscillations is taken to be small and expand the solutions in powers of $\varepsilon$. Thus:

$$
\begin{align*}
\psi^* &= \psi_0^* + \varepsilon \psi_1^* + \varepsilon^2 \psi_2^* + \ldots \\
\delta^* &= \delta_0^* + \varepsilon \delta_1^* + \varepsilon^2 \delta_2^* + \ldots \\
\theta^* &= \theta_0^* + \varepsilon \theta_1^* + \varepsilon^2 \theta_2^* + \ldots
\end{align*}
$$

87)  

We now substitute the first two of the assumed expansions (87) into the first of Eqs. (84), the coefficients of like powers of $\varepsilon$ are equated, and after a rather lengthy calculation carried out in Appendix XI, we get:

$$
\begin{align*}
\varepsilon^0: \quad &\xi^2_{\delta 0} \psi_0^* \psi_{\theta \theta} + \frac{1}{2} \delta_0^* \left( \psi_0^* \psi_0^* - \psi_0^* \psi_0^* \right) - \frac{1}{2} \delta_0^* \left( \psi_0^* \psi_0^* \right)^2 \\
&- \xi^2_{\delta 0} \delta_0^* \psi_1^* \psi_{\theta \theta} - \xi^2_{\delta 0} \delta_0^* \psi_0^* \psi_{\theta \theta} = \frac{1}{X_e} \xi^2 \psi_0^* \psi_{\theta \theta}
\end{align*}
$$

88-0)
\[\begin{align*}
\varepsilon_1 & : \quad \xi^o o_{11}^o \psi_{11}^o + 2 \delta_0 o_{11}^o \psi_{11}^o + \frac{1}{2} \delta_0 o_{11}^o \left( \psi_{11}^o \psi_{11}^o + \psi_{11}^o \psi_{11}^o \right) \\
& \quad - \psi_{11}^o \psi_{11}^o - \psi_{11}^o \psi_{11}^o) + \frac{1}{2} \delta_2 o_{11}^o \left( \psi_{11}^o \psi_{11}^o - \psi_{11}^o \psi_{11}^o \right) \\
& \quad - \frac{1}{2} \delta_1 o_{11}^o \left( \psi_{11}^o \psi_{11}^o \right) - \xi^o o_{11}^o \left( \delta_1 o_{11}^o + \delta_1 o_1^o + \delta_1 o_1^o \right) \psi_{11}^o - \delta_0 o_{11}^o \psi_{11}^o \psi_{11}^o \\
& \quad - \delta_0 o_{11}^o \delta_0 o_{11}^o - \delta_0 o_{11}^o \delta_0 o_{11}^o - \delta_0 o_{11}^o \delta_0 o_{11}^o \\
& \quad = \frac{\xi_{\psi}^o}{2 \chi_e} \delta_1 o_{11}^o \cos \tau + \frac{1}{\chi_e} \xi^o \psi_{11}^o \tag{88-1} \end{align*}\]
\[ \frac{3\delta}{2x_e} \delta^* \sin \tau + \frac{1}{x_e} \xi \psi^* \eta \eta \eta \] (88-2)

.................................

.................................

.................................

Next substitution of the first two of the expansions (87) into the
first condition (85) gives upon equating the coefficients of like powers
of \( \epsilon \):

\[
\begin{align*}
\epsilon^0 : \psi_0^* &= \delta_0^* + 2(x_e - 1) \int_0^\xi \frac{\partial \delta_0^*}{\partial \tau} \xi \, d\xi \\
\epsilon^1 : \psi_1^* &= \delta_1^* + \frac{1}{2} \delta_0^* \sin \tau + 2(x_e - 1) \int_0^\xi \frac{\partial \delta_1^*}{\partial \tau} \xi \, d\xi \\
\epsilon^2 : \psi_2^* &= \delta_2^* + \frac{1}{2} \delta_1^* \sin \tau + 2(x_e - 1) \int_0^\xi \frac{\partial \delta_2^*}{\partial \tau} \xi \, d\xi
\end{align*}
\] (89-0)

at \( \eta = 1 \)

.................................

.................................

Substitution of the first two of the expansions (87) into the
second condition (85) and equating the coefficients of like powers of
\( \epsilon \) gives after the calculation performed in Appendix XII:

\[
\begin{align*}
\epsilon^0 : \delta_0^* (x_e \xi \delta_{0 T}^* + \frac{1}{2} \delta_{0 \xi}^*) (\delta_0^* - x_e \psi_0^*) &= \xi \psi_0^* \\
\epsilon^1 : (\delta_0^* - x_e \psi_0^*) \left[ \delta_0^* (x_e \xi \delta_{1 T}^* + \frac{1}{2} \delta_{1 \xi}^*) + \delta_1^* (x_e \xi \delta_{0 T}^*) \right]
\end{align*}
\] (90-0)
\[
\begin{align*}
\text{at } \eta = 1 & \quad \eta \in \mathbb{R}^+ \\
\psi_1^* & = \frac{\delta_1^*}{2} \sin \tau \quad + \quad \psi_0^*(X_0^* \delta_0^*_{1T} + \frac{1}{2} \delta_2^*_{1T} + \frac{1}{2} \delta_0^*_{1T}) \\
& \quad + \quad \delta_0^*(X_0^* \delta_0^*_{1T} + \frac{1}{2} \delta_2^*_{1T} + \frac{1}{2} \delta_0^*_{1T}) \\
& \quad + \quad \delta_0^*(X_0^* \delta_0^*_{1T} + \frac{1}{2} \delta_2^*_{1T} + \frac{1}{2} \delta_0^*_{1T}) \\
\end{align*}
\]

\[
\begin{align*}
\psi_2^* & = \frac{\delta_2^*}{2} \sin \tau \\
& \quad + \quad \psi_0^*(X_0^* \delta_0^*_{1T} + \frac{1}{2} \delta_2^*_{1T} + \frac{1}{2} \delta_0^*_{1T}) \\
& \quad + \quad \delta_0^*(X_0^* \delta_0^*_{1T} + \frac{1}{2} \delta_2^*_{1T} + \frac{1}{2} \delta_0^*_{1T}) \\
\end{align*}
\]

and the boundary conditions (86) show that one must have:

\[
\begin{align*}
\text{at } \eta = 0 \quad \psi_n^* & = \frac{\partial \psi_n^*}{\partial \eta} = 0 \quad ; \quad n = 0, 1, 2 \ldots 
\end{align*}
\]

Now Eq. (88-0) with the boundary condition (89-0), (90-0), and (91) (with \(n = 0\)) will be satisfied by:

\[
\psi_0^* = 2a^* f_0(\eta) \quad ; \quad \delta_0^* = 2a^* 
\]
since (88-0) through (90-0) involve the time $\tau$ only implicitly. Substitution of (92) into (88-0) through (90-0) and (91) show that $f_o$ and $a$ must satisfy the following equations and boundary conditions:

$$0 < \eta \leq 1 : \quad \frac{1}{2a^2 X_\epsilon} f_o^{'''} + f_o f_o^{''} = 0 \quad \text{(93)}$$

$$\eta = 1 : \begin{cases} f_o = 1 \\ \frac{1}{2a^2} f_o^{''} = 1 - X_\epsilon f_o^{'} \end{cases} \quad \text{(94)}$$

$$\eta = 0 : \quad f_o = f_o^{'} = 0 \quad \text{(95)}$$

This as can easily be seen represents the steady state solution.

Now (92) is used to simplify the Eqs. (88-1) through (90-1) and (88-2) through (90-2). First substitutions of (92) into Eqs. (88-1) and (88-2) yields after a calculation performed in Appendix XIII:

$$\epsilon^1: \quad \frac{1}{2a^2 X_\epsilon} \psi_1^* \eta \eta - f_o \psi_1^* + f_o^{'} \left( \frac{1}{2} \psi_1^* \eta \eta + \psi_1^* \right) - \frac{1}{2} f_o^{'''} \psi_1^*$$

$$+ 2 \xi^2 \psi_1^* \eta + \left( f_o^{'''} - f_o f_o^{''} \right) \delta^*_1 - \frac{1}{2} f_o^{'''} \delta^*_1 - 2 \xi^2 \left( f_o^{'} + \eta f_o^{''} \right) \delta^*_1$$

$$= \frac{2a}{X_\epsilon} \xi^3 \cos \tau \quad \text{(88-1a)}$$

$$\epsilon^2: \quad 2a \xi \left[ \frac{1}{2a^2 X_\epsilon} \psi_2^* \eta \eta - f_o \psi_2^* + f_o^{'} \left( \frac{1}{2} \psi_2^* \eta \eta + \psi_2^* \right) - \frac{1}{2} f_o^{'''} \psi_2^* \right]$$

$$+ 2 \xi^2 \psi_2^* \eta + \left( f_o^{'''} - f_o f_o^{''} \right) \delta^*_2 - \xi f_o^{'''} \delta^*_2 - 2 \xi^2 \left( f_o^{'} + \eta f_o^{''} \right) \delta^*_2$$
\[
\psi^*_{1\eta} = \frac{6a}{X_e} \delta^*_1 \cos \tau - 4f^o_{1\tau} \psi^*_{1\eta} - \frac{5}{4} \psi^*_{1\eta} \psi^*_{1\eta} + \frac{5}{4} \psi^*_{1\eta} \psi^*_{1\eta} \\
+ 2f_{1\tau} \psi^*_{1\eta} - \delta^*_{1\tau} + f'_{1\tau} \psi^*_{1\eta} - \delta^*_{1\tau} + f''_{1\tau} \psi^*_{1\eta} - f_{1\tau} \psi^*_{1\eta} \\
+ \eta \psi^*_{1\eta} \delta^*_{1\tau} + 2f_{1\tau} (f'_{1\tau} + \eta f''_{1\tau}) \psi^*_{1\eta} + 2f_{1\tau} \psi^*_{1\eta} + 2f^o_{1\tau} (\psi^*_{1\eta} + \eta \psi^*_{1\eta}) \\
\]

Substitution of Eqs. (92) into Eqs. (89-1) and (89-2) yields:

\[
\eta = 1 \begin{cases} \\
\psi^*_1 = \delta^*_1 + \frac{1}{2} \sin \tau + 2 (X_e f'_{1\tau}) \int_0^\tau \frac{\delta^*_{1\tau}}{d\tau} \psi^{\tau}_{1\tau} \\
\psi^*_2 = \delta^*_2 + \frac{1}{2} \sin \tau + 2 (X_e f'_{1\tau}) \int_0^\tau \frac{\delta^*_{1\tau}}{d\tau} \psi^{\tau}_{1\tau} \\
\end{cases} \]

And finally using Eq. (92) in Eqs. (90-1) and (90-2) gives after some calculation performed in Appendix XIV:

\[
\eta = 1 \begin{cases} \\
\psi^*_1 = \frac{1}{2a} \psi^*_{1\eta} = (1 - X_e f'_{1\tau}) \left[ 2X_e \delta^*_{1\tau} + \left( \frac{5}{4} \delta^*_{1\tau} + \delta^* \right) + \frac{1}{2} \sin \tau \right] + \delta^* - X_e \psi^*_{1\eta} + \frac{1}{2} \sin \tau \\
\psi^*_2 = 2a \psi^*_{1\eta} = 2a \left( 1 - X_e f'_{1\tau} \right) \left[ a \left( \delta^*_{1\tau} + \delta^* \right) + 2X_e a \delta^*_{1\tau} \right] + \frac{1}{2} \sin \tau \\
\end{cases} \]
Next substitution of the expansions (87) into the second of Eqs. (84) and equating coefficients of like powers of $\varepsilon$ yields after a calculation performed in Appendix XV:

$$
\varepsilon^0 : \quad \theta^* \frac{\delta^*}{\theta^*} + \frac{1}{2} \theta^* \left( \psi^* \theta^*_1 \theta \eta - \psi^* \theta^*_1 \theta^* \right) - \varepsilon \frac{\delta^*}{\theta^*} \theta^* \left( \theta^*_1 \theta \eta - \theta^*_1 \theta^* \right) = \\
= \frac{\delta^*}{X_e P_r} \theta^* \eta \eta \eta 96-0)
$$

$$
\varepsilon^1 : \quad 2\frac{\delta^*}{\theta^*} \theta^* \theta^*_1 \theta^* + \frac{1}{2} \theta^* \left( \theta^*_1 \theta^* \theta \eta - \theta^* \theta^*_1 \theta^* \right) + \frac{1}{2} \theta^* \left( \theta^*_1 \theta^* \theta \eta - \theta^* \theta^*_1 \theta^* \right) \eta \eta \\
- \theta^* \left( \psi^*_1 \theta^* \eta \eta - \psi^*_1 \theta^* \eta \eta \right) - \frac{1}{2} \left( \theta^* \theta^*_1 \theta^* \theta \eta - \theta^* \theta^*_1 \theta^* \theta \eta \right) \eta \eta \\
- \frac{\delta^*}{\theta^*} \theta^* \left( \delta^* \theta^*_1 \theta^* \eta \eta - \delta^* \theta^*_1 \theta^* \eta \eta \right) \eta \eta = \\
= \frac{\delta^*}{P_r X_e} \theta^* \eta \eta 96-1)
$$

$$
\varepsilon^2 : \quad \frac{\delta^*}{\theta^*} \theta^* \eta \eta + \frac{1}{2} \theta^* \left( \psi^*_1 \theta^* \theta \eta - \psi^*_1 \theta^* \theta \eta \right) + \frac{1}{2} \theta^* \left( \theta^*_1 \theta^* \theta \eta - \theta^* \theta^*_1 \theta^* \theta \eta \right) \eta \eta \\
+ \frac{1}{2} \left( \theta^* \theta^*_1 \theta^* \theta \eta - \theta^* \theta^*_1 \theta^* \theta \eta \right) \eta \eta \\
- \theta^* \left( \psi^*_1 \theta^* \eta \eta - \psi^*_1 \theta^* \eta \eta \right) - \frac{1}{2} \left( \theta^* \theta^*_1 \theta^* \theta \eta - \theta^* \theta^*_1 \theta^* \theta \eta \right) \eta \eta \\
- \delta^* \theta^* \left( \delta^* \theta^*_1 \theta^* \eta \eta - \delta^* \theta^*_1 \theta^* \eta \eta \right) \eta \eta \\
= \frac{\delta^*}{P_r X_e} \theta^* \eta \eta 96-2)
$$
After substituting the expansions (87) into the third condition (85), equating coefficients of like powers of \( \varepsilon \) and performing the calculation of Appendix XVI, one obtains:

\[
\varepsilon^0: \quad \delta_0^* \delta_0^* \left( X_e \delta_0^* + \frac{1}{2} \delta_0^* \right) = - \frac{1}{X_e P_r} \delta_0^* \theta_0^* \eta \quad 97-0
\]

\[
\varepsilon^1: \quad \theta_0^* \delta_0^* \left( X_e \delta_0^* + \frac{1}{2} \delta_0^* \right) + \delta_0^* \left( X_e \delta_1^* + \frac{1}{2} \delta_1^* \right) + \frac{1}{4} \delta_0^* \delta_0^* \delta_0^* \sin \tau = - \frac{1}{X_e P_r} \delta_0^* \eta \quad 97-1
\]

\[
\varepsilon^2: \quad \theta_2^* \delta_0^* \left( X_e \delta_0^* + \frac{1}{2} \delta_0^* \right) + \theta_1^* \left( X_e \delta_1^* + \frac{1}{2} \delta_1^* \right) + \delta_0^* \left( X_e \delta_2^* + \frac{1}{2} \delta_2^* \right) + \delta_1^* \left( X_e \delta_1^* + \frac{1}{2} \delta_1^* \right) + \delta_0^* \left( X_e \delta_2^* + \frac{1}{2} \delta_2^* \right) + \frac{1}{4} \left[ \theta_1^* \delta_0^* \delta_1^* + \theta_0^* \left( \delta_1^* \delta_1^* + \delta_0^* \delta_1^* \right) \right] \sin \tau = - \frac{1}{X_e P_r} \delta_0^* \eta \quad 97-2
\]

\-----------------------------

And finally from the first boundary condition of (86), one must have:

\[
\varepsilon^0: \quad \theta_0^* = 1 \quad 98-0
\]

\[
\varepsilon^1: \quad \theta_1^* = 0 \quad 98-1
\]
Using (92) to simplify (96-0) and (97-0), one obtains:

\[
\frac{1}{2 a^2 \bar{X}_e p_r} \theta^*_{\eta \eta} = \frac{1}{3} \theta^*_{00} + \frac{1}{2} \theta^*_{0} - \frac{1}{3} \theta^*_{r r} + \theta^*_{00} + \theta^*_{0} = 0
\]

\[
\eta = 1 \quad \frac{1}{2 a^2 \bar{X}_e p_r} \theta^*_{0} + \theta^*_{0} = 0
\]

Since the time is involved only implicitly these equations will be satisfied along with (98-0) by taking:

\[
\theta^*_{0} = F_0(\eta)
\]

And \( F_0 \) is a solution of the following boundary value problem:

\[
0 \leq \eta \leq 1 \quad \frac{1}{2 a^2 p_r X_e} F''_0 + \frac{1}{3} F'_0 F'_0 = 0  \quad 100)
\]

\[
\eta = 1 \quad \frac{1}{2 a^2 p_r X_e} F'_0 + F_0 = 0  \quad 101)
\]

\[
\eta = 0 \quad F_0 = 1  \quad 102)
\]

Now Eqs. (99) and (92) are used to simplify Eqs. (96-1) and (96-2). This gives, after a calculation performed in Appendix XVII:

\[
\epsilon^2: \quad \frac{1}{2 a^2 p_r X_e} \theta^*_1 \eta \eta - \frac{1}{2} \theta^*_1 r^*_1 r^*_1 + \frac{1}{2} \theta^*_1 r^*_1 \eta = \frac{1}{3} F'_0 \theta^*_1 r^*_1
\]
\[-\frac{1}{2a^2} \left( \frac{\psi^*}{\psi_{1\xi}} + \frac{\delta^*}{\delta_{1\xi}} f_0 + 2\eta \frac{\delta^*}{\delta_{1\xi}} \right) F'_0 \]

\[\epsilon^2 : \frac{1}{2a^2 \rho X_e} \left[ \begin{array}{l}
\theta^* - 2\eta \frac{\delta^*}{\delta_{1\xi}} + f_0 \theta^* - \xi f'_0 \theta^* \\
- \left( \frac{1}{2} \frac{\delta^*}{\delta_{1\xi}} + \frac{\delta^*}{\delta_{1\xi}} \right) F'_0 + a \left[ 4 \frac{\delta^*}{\delta_{1\xi}} \theta^* + \left( \frac{\psi^*}{\psi_{1\xi}} + f'_0 \theta^* \right) \theta^* \left( \frac{\psi^*}{\psi_{1\xi}} + f'_0 \theta^* \right) \right] \end{array} \right] \]

And using Eqs. (99) and (92) to simplify Eqs. (97-1) and (97-2)

yields, after a calculation performed in Appendix XVIII:

\[ \epsilon^1 : \left[ \frac{1}{2a^2 \rho X_e \rho} \left( \frac{\psi^*}{\psi_{1\xi}} + \theta^* \right) \right] + \frac{1}{2a^2} F_0 \left[ \frac{\delta^*}{\delta_{1\xi}} + 2X_e \frac{\delta^*}{\delta_{1\xi}} \right] \]

\[ + \frac{1}{2} \frac{F_0}{\rho} \sin \tau = 0 \]  

97-1a)

\[ \epsilon^2 : \left[ \frac{1}{2a^2 \rho X_e \rho} \left( \frac{\psi^*}{\psi_{1\xi}} + \theta^* \right) \right] + \frac{1}{2a^2} \left[ \frac{\delta^*}{\delta_{1\xi}} + 2X_e \frac{\delta^*}{\delta_{1\xi}} \right] \]

\[ + \frac{1}{2a} F_0 \left[ \frac{\delta^*}{\delta_{1\xi}} + 2X_e \frac{\delta^*}{\delta_{1\xi}} \right] + \frac{1}{2} \frac{\delta^*}{\delta_{1\xi}} \sin \tau + \frac{1}{2} \frac{1}{2a^2} F_0 \left( \frac{\psi^*}{\psi_{1\xi}} + \frac{\delta^*}{\delta_{1\xi}} \right) \sin \tau \]

\[ = 0 \]  

97-2a)

The first order equations now may be solved, i.e., Eq. (88-1a) with the boundary conditions (89-1a) (90-1a) and (91) with (n = 1) for the momentum problem and Eq. (96-1a) with the boundary conditions (97-1a) and (98-1) for the energy problem:
Solutions are sought to these equations in the form:

\[ \chi^* = \Re \sum_{n=1}^{\infty} f_{1n}(\eta) \xi^n e^{i\tau} \quad (103) \]

\[ \delta^* = \Re \sum_{n=1}^{\infty} a_{1n} \xi^n e^{i\tau} \quad (104) \]

\[ \theta^* = \sum_{n=0}^{\infty} F_{1n}(\eta) \xi^n e^{i\tau} \quad (105) \]

where \( \Re \) denotes the real part and \( f_{1n} \), \( a_{1n} \) and \( F_{1n} \) are complex quantities.

A bar denotes the complex conjugate. Thus the complex conjugate of \( a_{1n} \) is \( \overline{a_{1n}} \).

Upon substituting Eqs. (105) and (104) in Eq. (88-1a) through (90-1a) and (91)(with \( n = 1 \)) and equating coefficients of like powers of \( \xi \), it is found from the calculation of Appendix XIX that the \( f_{1n} \)'s and the \( a_{1n} \)'s; \( n = 1, 3, 5, \ldots \) must satisfy:

\[
0 \leq \eta \leq 1
\left\{ \begin{array}{l}
I_n(f_{1n}) + \left[ (n-1)f_0'^2 + f_0''f_0' \right] a_{1n} = -\frac{1}{X_e} \delta_{1n} \\
-2i(1-\delta_{1n}) \left[ (f_0' + \eta f_0'') a_{1(n-2)} - f_{1(n-2)}' \right] \\
\end{array} \right.
\]

\[
\eta = 1
\left\{ \begin{array}{l}
f_{1n} = a_{1n} - \frac{i}{2} \delta_{1n} + 2i(1-\delta_{1n}) (X_e-1) a_{1(n-2)} \\
\end{array} \right. \quad n = 1, 3, 5, 7, \ldots
\]

\[
\eta = 0
\left\{ \begin{array}{l}
f_{1n} = f_{1n}' = 0 \\
\end{array} \right.
\]

\[ \left[ \begin{array}{l}
\frac{1}{2a} f''_{1n} = (1-X_e f_0') \left[ (n+1)a_{1n} - \frac{i}{2} \delta_{1n} + \\
2i(1-\delta_{1n}) X_e a_{1(n-2)} \right] + a_{1n} - X_e f_{1n} - \frac{i}{2} \delta_{1n} \\
\end{array} \right. \quad (106) \]
Where:

\[ I_n = \frac{1}{2a^2 X_e} \frac{d^3}{d\eta^3} + f_0 \frac{d^2}{d\eta^2} - (n-1) f_0' \frac{d}{d\eta} + n f_0'' \]  \quad (107) \]

and \( \delta_{mn} \) is the "Kronecker Delta" defined by:

\[ \delta_{mn} = \begin{cases} 1 ; & m = n \\ 0 ; & m \neq n \end{cases} \]

and the \( f_n \)'s and \( a_n \)'s have the following restrictions:

\[ f_n \quad \text{for} \quad n = 2, 4, 6, 8, ... \]

\[ f_n, a_n \quad \text{pure imaginary quantity for} \quad n = 1, 5, 9, 13, ... \]

\[ f_n, a_n \quad \text{real quantity for} \quad n = 3, 7, 11, ... \]

by substituting Eqs. (103), (104) and (105) into Eqs. (96-1a), (97-1a) and (98-1), and equating coefficients of like powers of \( \xi \), it is found from the calculation of Appendix XX that the \( F_n \)'s; \( n = 0, 2, 4, ... \) must satisfy:

\[ \begin{align*}
0 \leq \eta < 1 &: \\
K_n(F_n) = -F_0' \left[ (n+1)f_1(n+1) + f_0 a_1(n+1) \right] \\
\frac{1}{2a^2 X_e F'} F_n' + F_n + F_0 \left[ (2+n) a_1(n+1) + \right] \\
\eta = 1 &: \\
2i X_e (1-\delta \eta_{1,n}) a_1(n-1) - \frac{1}{2} \delta \eta_{0,n} = 0 \\
\eta = 0 &: \\
F_n = 0
\end{align*} \quad (109) \]
Where:

\[ K_n = \frac{1}{2s^2 p^2 X_e} \frac{d^2}{d\eta^2} + \int_0^\infty \frac{d}{d\eta} - n f_0 \]

and the \( F_{1n} \)'s have the following restrictions:

\[
\begin{align*}
F_{1n} &= 0 \quad \text{for} \quad n = 1, 3, 5, 7, \ldots \\
F_{1n} &= \text{pure imaginary quantity for} \quad n = 0, 4, 8, \ldots \\
F_{1n} &= \text{real quantity for} \quad n = 2, 6, 10, \ldots
\end{align*}
\]

Before proceeding to the second order equations it will be convenient to develop a certain relation between complex quantities. Let us consider two complex variables \( Z_1 \) and \( Z_2 \) and let us evaluate the product:

\[(\text{Re} Z_1 e^{i\tau})(\text{Re} Z_2 e^{i\tau})\]

we first note that:

\[
\text{Re} Z_1 e^{i\tau} = \frac{1}{2} (Z_1 e^{i\tau} + \overline{Z_1} e^{-i\tau})
\]

\[
= \frac{1}{2} (Z_1 e^{i\tau} + \overline{Z_1} e^{-i\tau})
\]

Hence:

\[
(\text{Re} Z_1 e^{i\tau})(\text{Re} Z_2 e^{i\tau}) = \frac{1}{4} (Z_1 e^{i\tau} + \overline{Z_1} e^{-i\tau}) (\overline{Z_2} e^{i\tau} + Z_2 e^{-i\tau})
\]

\[
+ \frac{1}{4} (Z_1 \overline{Z_2} e^{2i\tau} + \overline{Z_1} Z_2 e^{-2i\tau}) +
\]

\[
\frac{1}{4} (Z_1 Z_2 + Z \overline{Z}) = \frac{1}{2} \text{Re}(Z_1 Z_2) e^{2i\tau} + \frac{1}{4} (Z_1 Z_2 + \overline{Z_1} \overline{Z_2})
\]
and:

\[
\left( \frac{\Re Z_1 e^{i\tau}}{Z_2 e^{i\tau}} \right) \left( \frac{\Re Z_2 e^{i\tau}}{Z_1 e^{i\tau}} \right) = \frac{1}{2} \left( \frac{\Re(Z_1 Z_2 e^{2i\tau} + Z_1 Z_2)}{Z_1 Z_2} \right)
\]

Next keeping this result in mind inspection of the right hand sides of Eqs. (88-2a), (89-2a), and (90-2a), and of Eqs. (96-2a) and (97-2a) shows that one might anticipate the solutions of the second order equations to be of the form:

\[
\psi_2^* = \Re \sum_{n=1}^{\infty} \left[ f_{2n}(\eta) e^{2i\tau} + \tilde{f}_{2n}(\eta) \right] \xi^n
\]

\[
\delta_2^* = \Re \sum_{n=1}^{\infty} \left[ a_{2n} e^{2i\tau} + \tilde{a}_{2n} \right] \xi^n
\]

\[
\theta_2^* = \Re \sum_{n=0}^{\infty} \left[ f_{2n}(\eta) e^{2i\tau} + \tilde{f}_{2n}(\eta) \right] \xi^n
\]

Upon substituting the expansions (112), (113), (103), and (104) into Eqs. (88-2a), (89-2a), (90-2a) and (91) (with \( n = 2 \)) and using the relation (111) to simplify, it is found that two types of terms appear; those with a time dependence of the form \( e^{2i\tau} \), and those which are independent of time. It is therefore necessary to equate the coefficients of \( e^{2i\tau} \) with each other and the terms which are independent of time with each other. These calculations are performed in Appendix XXI.

Then (Appendix XXI) the coefficients of like powers of \( \xi \) are equated in each of these expressions. It is then found from these calculations that the \( f_{2n} \)'s, and the \( a_{2n} \)'s for \( n = 1, 3, 5, ... \) must satisfy:
\[
0 \leq \eta \leq 1 \\
L_n(2n) = - \left[ (n-1)f''_0 + f''_0 \right] a_{2n} + \frac{1}{n} \eta \left( a_{2n} - \frac{i}{4} a_{1n} \right) + \frac{4i(1-\eta)}{\eta} \left( X_e \right) a_{2(n-2)} \\
\frac{1}{2a^2} f''_{2n} = (1 - X_e f'_0) \left[ (n+1) a_{2n} - \frac{i}{4} a_{1n} \right] + \\
\left\{ \begin{array}{l}
\eta = 1 \\
\frac{4i(1-\eta)}{\eta} a_{2(n-2)} - \frac{i}{2} (a_{1n} - X_e f'_1) + a_{2n} \\
- X_e f'_2n - \frac{i}{4} \left[ (n+2) a_{1n} + 2i (1-\eta) X_e a_{1(n-2)} \right] \\
- \frac{\delta_{1n}}{\delta} + C_n
\end{array} \right. \\
\eta = 0 \\
f_{2n} = f'_{2n} = 0
\]

Where:
\[
H_n = -\frac{5}{2X_e} (1-\eta) a_{1(n-2)} + 4i(1-\eta) \left[ f''_0 n_{2(n-2)} \right] - \left( f'_0 + \eta f''_0 \right) a_{2(n-2)} - \frac{1}{2} \left\{ f'_{11} a_{1n} - (n-1)f'_0 f''_{11} \right\} \\
+ (1-\eta) n^2 f''_{11} a_{1n} + \left\{ 2(n-1) f'_0 f''_{11} + f'_0 f'_1 + f''_{11} \right\} a_{1n} \\
+ (1-\eta) \left[ n f''_0 a_{1n} - (n-1)f'_0 f''_{11} + f'_0 f''_{1n} \right] a_{1n} \left\{ \begin{array}{l}
n=1,3,5,... \\
\left( \eta f''_0 + f'_0 \right) a_{11} a_{1(n-2)} + (\eta f''_{11} f''_{11} + f''_{11} f''_{11}) a_{1n} \\
f''_{11} a_{1(n-2)} \right\}
\end{array} \right. \\
- \frac{\delta_{1n}}{\delta} + C_n
\]

\[
\begin{align*}
\text{and} \\
P_1 &= 0 \\
P_3 &= 0
\end{align*}
\]
\[ P_5 \equiv \frac{1}{2} \left\{ 3f_{13}' f_{13}'' - 2f_{13}' f_{13}' + \left[ f_0' f_{13}' + \right. \right. \\
+ 3f_0'' f_{13}' + 2 f_0' f_{13}' a_{13} \bigg] \left( \eta f_{13}'' + f_{13}' a_{11} \right) a_{13} + \\
+ (\eta f_{13}'' - f_{13}' a_{11}) \bigg\} \]

\[ C_1 = (1 - X_e f_0') \frac{a_{12} a_{11}}{2} + 2a_{11} (a_{11} - X_e f_{11}') \]
\[ C_3 = (1 - X_e f_0') \left\{ a_{11} \left[ \frac{3}{2} a_{13} + i X_e a_{11} \right] + \frac{1}{2} a_{11} a_{13} \right\} \\
+ (a_{11} - X_e f_{11}') \left[ 4a_{13} + 2 i X_e a_{11} \right] + 2a_{11} (a_{13} - X_e f_{13}') \]

\[
\begin{align*}
C_5 & = (1 - X_e f_0') \left\{ a_{11} \left[ \frac{5}{2} a_{15} + i X_e a_{13} \right] + a_{13} \left[ \frac{3}{2} a_{13} \\
+ i X_e a_{11} \right] + \frac{1}{2} a_{11} a_{15} \right\} + (a_{11} - X_e f_{11}') \left[ 6 a_{15} + \\
+ 2 i X_e a_{13} \right] + (a_{13} - X_e f_{13}') \left[ 4a_{13} + 2 i X_e a_{11} \right] \\
+ 2 a_{11} (a_{15} - X_e f_{15}')
\end{align*}
\]

And that for \( n = 2, 4, 6, \ldots \):

\[ a_{2n} = f_{2n} = 0 ; \text{ for } n = 2, 4, 6, \ldots \]
And comparison of Eqs. (108) with Eqs. (115) through (117) shows that

\[
\begin{align*}
\text{a}_{2n}, f_{2n} & = \text{real quantities for } n = 1, 5, \ldots \\
\text{a}_{2n}, f_{2n} & = \text{pure imaginary quantities for } n = 3, 7, \ldots
\end{align*}
\]

(119)

The calculations also show that for \( n = 1, 5, \ldots \) the \( \frac{\hat{v}}{a_{2n}} \)'s and the \( \frac{a}{a_{2n}} \)'s must satisfy:

\[
0 \leq \eta \leq 1
\]

\[
\left\{ \begin{array}{l}
L_n(f_{2n}) = - \left[ (n-1)f_0 f_0'' + f_0 f_0'' \right] \frac{\hat{v}}{a_{2n}} + \hat{v} \\
\frac{\hat{v}}{a_{2n}} = \frac{\hat{v}}{a_{2n}} - \frac{i}{4} \frac{\hat{v}}{a_{2n}} \\
\frac{1}{2} \frac{\hat{v}}{a_{2n}} = (1 - X_e f_{2n}) (n+1) \left( \frac{\hat{v}}{a_{2n}} - \frac{i}{4} \frac{\hat{v}}{a_{2n}} \right) - \frac{i}{2} (\frac{\hat{v}}{a_{2n}})
\end{array} \right.
\]

\[
\eta = 1
\]

\[
\left\{ \begin{array}{l}
\frac{f_0'}{1n} + \frac{\hat{v}}{a_{2n}} - X_e \frac{\hat{v}}{a_{2n}} - \frac{i}{4} \left[ \frac{(n+2)\hat{a}_{1n}}{\hat{a}_{1n}} - 2i(1 - \delta \frac{1}{2n}) \right] \\
\frac{\delta_{1n}}{\delta_{1n}} + \frac{\hat{v}}{a_{2n}}
\end{array} \right\} + \frac{\delta_{1n}}{\delta_{1n}} + \frac{\hat{v}}{a_{2n}}
\]

\[
\eta = 0
\]

\[
\left\{ \begin{array}{l}
\frac{\hat{v}}{a_{2n}} = \frac{\hat{v}}{a_{2n}} = 0
\end{array} \right\}
\]

Where:

\[
\frac{\hat{v}}{a_{2n}} = - \frac{3}{2X_e} (1 - \delta \frac{1}{1n})a_{1n}(n-2) - \frac{i}{2} \left[ f_{11} f_{1n} - (n-1)f_{11} f_{1n} \right]
\]

\[
+ (1 - \delta \frac{1}{1n}) n f_{11} f_{1n} + \left[ 2(n-1)f_0 f_1 f_{1n} + f_0 f_1 f_{1n} + f_0 f_{11} f_{1n} \right] \frac{\hat{a}_{1n}}{a_{1n}}
\]

\[
+ (1 - \delta \frac{1}{1n}) \left[ n f_0^2 f_{1n} - (n-1)f_0 f_{1n} + f_0 f_{1n} \right] \frac{\hat{a}_{1n}}{a_{1n}} - \frac{\hat{p}}{a_{1n}}
\]

\[
n = 1, 5, \ldots
\]

\[
\frac{\hat{p}}{a_{1n}} = 0
\]

(121)
\[
\begin{align*}
\bar{\rho}_s &= \frac{1}{2} \left[ 3 f_{13} f'_{13} - 2 f'_{0 \rightarrow 13} f_{13} + \left[ f_{0 f''_{13}}ight. \right. \\
&+ 2 f''_{0 \rightarrow 13} + 2 f'_{0 \rightarrow 13} \left. a_{13} \right] \left. \right] + \left. \left( \eta f''_{11} + 3 f'_{11} \right) a_{13} \right. \\
&\left. + (\eta f''_{13} + 3 f'_{13}) a_{11} \right]
\end{align*}
\]

And:

\[
\begin{align*}
\bar{c}_1 &= (1 - X_e f'_{0}) \frac{1}{2} \bar{a}_{11} a_{11} + 2 a_{11} \left( a_{11} - X_e f'_{11} \right) \\
\bar{c}_5 &= (1 - X_e f'_{0}) \left[ a_{11} \left[ \frac{5}{2} a_{15} + i X_e a_{13} \right] + \\
&\bar{a}_{13} \left[ \frac{3}{2} a_{13} + i X_e a_{11} \right] + \frac{1}{2} a_{11} \bar{a}_{15} \right] + (\bar{a}_{11} - X_e f'_{11}) \left[ 6 a_{15} + 2 i X_e a_{13} \right] + (\bar{a}_{13} - X_e f'_{13}) \left[ 4 a_{13} \right] \\
&\bar{a}_{11} \left[ 2 i X_e a_{11} \right] + (\bar{a}_{15} - X_e f'_{15}) 2 a_{11}
\end{align*}
\]

\[122)\]

And for \( n \neq 1,5, \ldots \)

\[
\eta_{a_{2n}} = \eta_{f_{2n}}(\eta) = 0 \; \; ; \; \; n \neq 1,5,9, \ldots
\]

And:

\[123)\]

\[
\eta_{a_{2n}}, \eta_{f_{2n}} \text{ real}
\]
Next substituting the expansions (103) through (105) and (112) through (114) into Eqs. (96-2a), (97-2), and using the relation (111) to simplify, it is again found that the only type of time dependent terms which appear are those whose time dependence is of the form $e^{2\pi \tau}$. Therefore the coefficients of $e^{2\pi \tau}$ are equated to one another and the time independent terms are equated to one another. After carrying out these operations in Appendix XXII, the coefficients of like powers of $\xi$ are equated in each of the resulting expressions.

From this is found that the $F_{2n}$'s for $n = 0, 2, 4, \ldots$ must satisfy:

$$
0 \leq \eta < 1 \left\{ \begin{array}{l}
K_n(F_{2n}) = 4 i (l-\delta_{on})F_{2(n-2)} - F'_0 \left[ (n+1) f_{2(n+1)} \\
+ f_0 a_{2(n+1)} + 4 i (l-\delta_{on}) \eta a_{2(n-1)} \right] + E_n
\end{array} \right.
$$

$$
\eta = 1 \left\{ \begin{array}{l}
\frac{1}{2}X_e F'_{2n} [F'_{2n} + F_{2n} + F_0 \left\{ (2+n) (a_{2(n+1)} - \frac{i}{4} a_{1(n+1)} \right\} \\
+ 4 i X_e(1-\delta_{on}) a_{2(n-1)} \right\} - \frac{i}{4} F_{1n} + E_n = 0
\end{array} \right.
$$

$$
\eta = 0 \left\{ \begin{array}{l}
F_{2n} = 0
\end{array} \right.
$$

Where:

$$
E_0 = -\frac{1}{2} a_{11} F'_1 f_{11} - \frac{1}{2} F'_0 (f_{11} + a_{11} f_0)
$$

$$
E_2 = -\frac{1}{2} \left\{ 3 a_{11} f_{13} + 2 i a_{11} a_{11} \eta + a_{13} f_{11} \right\} F'_0
$$

$$
+ \frac{1}{2} \left\{ 2 f_{12} (f_{11} + f_0 a_{11}) - F'_1 (f_{11} + f_0 a_{11}) \right\}
$$
\[ E_4 = - \frac{1}{2} \left\{ 5a_{11} f_{15} + 4i \eta a_{11} a_{13} + 3a_{13} f_{13} + a_{15} f_{11} \right\} F'_0 + \frac{1}{2} \left\{ 4F_{14} \left( f'_{11} + a_{11} f'_0 \right) - F_{14} \left( f_{11} + f'_0 a_{11} \right) + 2F_{12} \left( f_{13} + f'_0 a_{13} \right) - F_{12} \left( 3f_{13} + a_{13} f'_0 \right) - F'_{10} \left( 5f_{15} + f'_0 a_{15} \right) \right\} \]

\[ + i a_{11} \left( 2F_{12} - \eta F'_{12} \right) + i a_{13} \left( 2F_{10} - \eta F'_{10} \right) \]

........................................................................

And:

\[ B_0 = a_{11} F_{10} + \frac{1}{2} a_{11} a_{11} F_0 \]

\[ B_2 = \frac{1}{2} F_{10} \left( 4a_{13} + 2i X_e a_{11} \right) + a_{11} F_{12} + a_{11} \left( \frac{3}{2} a_{13} + i X_e a_{11} \right) F_0 + \frac{1}{2} a_{11} a_{13} F_0 \]

\[ B_4 = \frac{1}{2} F_{10} \left( 6a_{15} + 2i X_e a_{13} \right) + \frac{1}{2} F_{12} \left( 4a_{13} + 2i X_e a_{13} \right) \]

\[ + F_{14} a_{11} + a_{11} \left( \frac{5}{2} a_{15} + i X_e a_{13} \right) F_0 + a_{13} \left( \frac{3}{2} a_{13} + i X_e a_{11} \right) F_0 + \frac{1}{2} a_{11} a_{15} F_0 \]

........................................................................
And for $n = 1, 3, 5, \ldots$ we find that:

$$F_{2n}(\eta) = 0 \quad ; \quad \text{for } n = 1, 3, 5, \ldots \quad 127$$

A comparison of Eqs. (108), (110) and (119) with Eqs. (124) through (126) shows that:

$$
\begin{align*}
F_{2n} & = \text{real quantity for } n = 0, 4, 8, \ldots \\
F_{2n} & = \text{pure imaginary quantity for } n = 2, 6, \ldots
\end{align*}
$$

The calculation of Appendix XXII further shows that the $F_{2n}$ must satisfy for $n = 0, 4, \ldots$:

$$
0 \leq \eta \leq 1 \quad \begin{cases}
K_n(F_{2n}) = - F_0 \left[ (n+1) F_{2(n+1)} + f_o a_{2(n+1)} \right] + E_n \\
\frac{1}{2a^2 F_{2n} X_e} \left( F_{2n}' + F_{2n} + F_0(2+n)(a_{2(n+1)} - \frac{i}{4} a_{1(n+1)}) \right) \\
- \frac{i}{4} F_{1n} + B_n = 0
\end{cases}
\quad \text{for } n=0, 4, \ldots \quad 129$$

Where:

$$
\begin{align*}
E_0 &= \left[ - \frac{1}{2} F_0' a_{11} (\bar{f}_{11} - \frac{1}{2} F_{10}' (\bar{f}_{11} + f_o \bar{a}_{11}) \right] \\
E_4 &= \left[ - \frac{1}{2} \left( 5a_{11} \bar{f}_{15} + 3a_{13} \bar{f}_{13} + a_{15} \bar{f}_{11} \right) F_0' + \right. \\
&\left. \frac{1}{2} \left( 4F_{14} (\bar{f}_{11} + a_{11}f_o') - F_{14} (\bar{f}_{11} + f_o a_{11}) + 2F_{12}(\bar{f}_{13} \right)
\end{align*}
$$
\[
\begin{align*}
0 \leq \eta < 1 & \quad \left\{ + f'_0 \bar{a}_{13} - F_{12} (3f_{13} + f'_0 \bar{a}_{13}) - F_{10} (5f_{15} + f'_0 \bar{a}_{15}) \right. \\
& \quad + \left. i \bar{a}_{11} (2F_{12} + \eta F_{12}) + i a_{13} (2F_{10} - \eta F_{10}) \right. \\
& \quad \vdots \\
& \quad \vdots
\end{align*}
\]

and:
\[
\begin{align*}
\eta = 1 & \quad \left\{ \begin{array}{l}
\mathcal{B}_0 = a_{11} \bar{F}_{10} + \frac{1}{2} a_{11} \bar{a}_{11} F_0 \\
\mathcal{B}_4 = \frac{1}{2} \bar{F}_{10} (6a_{15} + 2 i X_e a_{13}) + \frac{1}{2} \bar{F}_{12} (2a_{13} + 2 i X_e a_{13}) F_0 \\
+ \bar{a}_{13} (3 a_{13} + i X_e a_{11}) F_0 + \frac{1}{2} a_{11} \bar{a}_{15} F_0 \end{array} \right.
\end{align*}
\]

And for \( n \neq 0, 4, 8, \ldots \) we find that:
\[
\mathcal{F}_{2n}(\eta) = 0 \quad \text{for} \quad n \neq 0, 4, 8, \ldots
\]

And the \( \mathcal{F}_{2n} \)'s are real otherwise.

The problem has now been completely formulated in terms of a set of simultaneous ordinary differential equations with their appropriate boundary conditions. These equations can be solved successively, i.e., once \( f_0 \) and \( a \) are known one can solve for \( f_{11} \) and \( a_{11} \) etc. Then once these are known we can solve for \( f_{13} \) and \( a_{13} \), etc. Once the solutions to the momentum equations are known (\( f_{mn} \)'s, \( a_{mn} \)'s), one can solve
successively in the same way for the solution to the energy equation (i.e., the $F_{mn}$'s). The remarks at the end of Section II-B, about the general nature of the equations and their solutions are also appropriate here. The discussion will not be repeated now.

It now remains to derive explicit formulae for the velocity, shear stress and heat transfer coefficients in terms of the solutions to the equations derived above.

From the first of Eqs. (71) one has:

$$u = \frac{\partial \psi}{\partial y}$$

And using the dimensionless quantities defined in (77) we obtain:

$$\frac{u}{U_\infty} = \frac{X_e}{\delta_*} \frac{\partial \psi^*}{\partial \eta}$$

133)

Next the wall shear stress is given by:

$$\tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} = \mu \frac{\partial^2 \psi}{\partial y^2} \bigg|_{y=0}$$

And using the dimensionless quantities of (77) we have:

$$\frac{\tau_w \sqrt{\frac{U_\infty x}{v}}}{\frac{1}{2} \rho U_\infty^2} = X_e \frac{2 \xi}{\delta_*^2} \frac{\partial^2 \psi^*}{\partial \eta^2} \bigg|_{\eta=0}$$

134)

And finally the heat flux at the wall is given by:

$$q_w = -k \frac{\partial T}{\partial y} \bigg|_{y=0}$$

135)
and using (77) we get:

\[
\frac{N_{u_x}}{U_{\infty} x} \equiv \frac{a_w x}{k(T_{\infty} - T_w)} \sqrt{\frac{U_{\infty} x}{\nu}} = \sqrt{X_e \left( \frac{\delta^*}{\delta^* \eta} \right)}_{\eta=0} \tag{136}
\]

Substitution of the expressions (87) into the relations (133), (134), and (136) yields after the calculation of Appendix XXIII:

\[
\frac{\tau}{U_{\infty} x} = \sqrt{\frac{U_{\infty} x}{\nu}} = X_e \left( \tau_0^* + \epsilon \tau_1^* + \epsilon^2 \tau_2^* + \ldots \right) \tag{138}
\]

\[
\frac{N_{u_x}}{U_{\infty} x} = \sqrt{X_e \left( q_0^* + \epsilon q_1^* + \epsilon^2 q_2^* + \ldots \right)} \tag{139}
\]

Where:

\[
0 \leq \eta \leq 1 \begin{align*}
\tau_0^* &= \frac{X_e \psi_{0 \eta}}{\delta^*_0} \\
\tau_1^* &= \frac{X_e \psi_{1 \eta}}{\delta^*_0} \\
\tau_2^* &= \frac{X_e \psi_{2 \eta}}{\delta^*_0} \\
\tau_3^* &= \frac{X_e \psi_{3 \eta}}{\delta^*_0}
\end{align*} \tag{140}
\]

\[
\eta = 0 \begin{align*}
\tau_0^* &= \frac{2 \xi \psi_{0 \eta \eta}}{\delta^*_0} \\
\tau_1^* &= \frac{2 \xi \psi_{1 \eta \eta}}{\delta^*_0} \\
\tau_2^* &= \frac{2 \xi \psi_{2 \eta \eta}}{\delta^*_0}
\end{align*} \tag{141}
\]
Now using the relations (91), (99), (103), (104), (105), (112),
(113), and (114) with the conditions (108), (110), (119), (123), (127)
and (132) in the Eqs. (140) through (142) we get after performing the
calculations of Appendix XXIV:

\[
\begin{align*}
\mathcal{u}_0^* &= X_e f_0' \\
\mathcal{u}_1^* &= \mathcal{R} \sum_{n=0}^{\infty} \mathcal{g}_1 n^n e^{i\tau} \\
\mathcal{u}_2^* &= \mathcal{R} \sum_{n=0}^{\infty} \left( \mathcal{g}_2 n e^{i\tau} + \mathcal{g}_2 n \right) n^n
\end{align*}
\]

where for $n = 0, 2, 4, \ldots$.
\[
\begin{align*}
g_{1n} & = X_c (f'_1(n+1) - f'_0 a_1(n+1)) \\
g_{2n} & = X_c (f'_2(n+1) - f'_0 a_2(n+1) - h_n) \\
g_{2n} & = X_c (\tilde{f}'_2(n+1) - f'_0 \tilde{a}_2(n+1) - \tilde{h}_n) \\
0 \leq \eta \leq 1 \quad \text{and:} \quad 144)
\end{align*}
\]

\[
\begin{align*}
h_0 & = \frac{1}{2} a_{11} (f_{11} - f'_0 a_{11}) \\
h_2 & = \frac{1}{2} \left[ a_{13} f_{11} + a_{11} f'_{13} - 2 a_{11} a_{13} f'_0 \right] \\
h_4 & = \frac{1}{2} \left[ a_{15} f_{11} + a_{13} f_{13} + a_{11} f'_{15} - f'_0 (2a_{11} a_{15} + a_{13} a_{13}) \right]
\end{align*}
\]

\[
\begin{align*}
h^0 & = \frac{1}{2} \bar{a}_{11} (\bar{f}_{11} - \bar{f}'_0 a_{11}) \\
h^4 & = \frac{1}{2} \left[ \bar{a}_{15} f_{11} + \bar{a}_{13} f_{13} + \bar{a}_{11} f'_{15} - f'_0 (2\bar{a}_{11} a_{15} + a_{13} \bar{a}_{13}) \right]
\end{align*}
\]

and:

\[
\begin{align*}
h_n & = 0 \quad \text{for} \quad n = 1, 3, 5, \ldots \\
h^0 & = 0 \quad \text{for} \quad n \neq 0, 4, 5, \ldots
\end{align*}
\]
\[ \tau_0^* = \frac{1}{a} f''_0 \]

\[ \tau_1^* = \text{Re} \left( \frac{1}{a} \sum_{n=0}^{\infty} e^{i \tau} e_{1n} \frac{i}{a} e_{1n}^n \right) \]

\[ \tau_2^* = \text{Re} \left( \frac{1}{a} \sum_{n=0}^{\infty} (e_{2n} e^{21T} + \bar{e}_{2n}) \frac{i}{a} e_{2n}^n \right) \]

\[ \eta = 0 \]

Where:

\[ e_{1n} = f''_{1(n+1)} - 2f''_0 a_{1(n+1)} \]

\[ e_{2n} = f''_{2(n+1)} - 2f''_0 a_{2(n+1)} - r_n \]

\[ \bar{e}_{2n} = f''_{2(n+1)} - 2f''_0 \bar{a}_{2(n+1)} - \bar{r}_n \]

and:

\[ r_0 = \frac{1}{2} a_{11} \left( 2r_{11}'' - 3f''_0 a_{11} \right) \]

\[ r_2 = \frac{1}{2} \left[ 2a_{13} f_{11}'' + 2a_{11} f_{13}'' - 6a_{11} a_{13} f''_0 \right] \]

\[ r_4 = \frac{1}{2} \left[ 2a_{15} f_{11}'' + 2a_{13} f_{15}'' + 2a_{11} f_{15}'' - 3f''_0 \right. \]

\[ \left. (2a_{11} a_{15} + a_{13} a_{13}) \right] \]

\[ \bar{r}_0 = \frac{1}{2} \bar{a}_{11} \left( 2f_{11}'' - 3f''_0 a_{11} \right) \]
\[ r_{\frac{4}{4}} = \frac{1}{2} \left[ 2\bar{a}_{15} f_{11}'' + 2\bar{a}_{13} f_{13}'' + 2\bar{a}_{11} f_{15}'' - 3f_0'' \right] \]

\[ \left( 2\bar{a}_{11} a_{15} + \bar{a}_{13} a_{13} \right) \]

\[ r_n = 0 \quad \text{for} \quad n = 1, 3, 5, \ldots \]

\[ \varphi r_n = 0 \quad \text{for} \quad n \neq 0, 4, 8, \ldots \]

\[ q_0^* = \frac{1}{2a} F_0' \]

\[ q_1^* = \Re \left\{ \frac{1}{2a} \sum_{n=0}^{\infty} G_{1n} \xi^n e^{i\tau} \right\} \]

\[ q_2^* = \Re \left\{ \frac{1}{2a} \sum_{n=0}^{\infty} \left[ G_{2n} e^{2i\tau} + \tilde{G}_{2n} \right] \xi^n \right\} \]

\[ \text{Where:} \]

\[ G_{1n} = F_{1n}' - a_1(n+1)F_0' \]

\[ G_{2n} = F_{2n}' - a_2(n+1)F_0' - R_n \]

\[ \tilde{G}_{2n} = F_{2n}' - a_2(n+1)F_0' - \tilde{R}_n \]

\[ R_0 = \frac{1}{2} \left[ a_{11} F_{10}' - a_{11} a_{11} F_0' \right] \]

\[ R_2 = \frac{1}{2} \left[ a_{13} F_{10}' + a_{11} F_{12}' - 2a_{11} a_{13} F_0' \right] \]
\[ R_4 = \frac{1}{2} \left[ a_{15} F_{10} + a_{13} F_{12} + a_{11} F_{14} - (2a_{11} a_{15} + \right. \\
\left. a_{13} a_{13} F_0') \right] \]

\[ \widetilde{R}_4 = \frac{1}{2} \left[ \bar{a}_{11} F_{10} - a_{11} \bar{a}_{11} F_0' \right] \]

\[ \tilde{R}_4 = \frac{1}{2} \left[ a_{15} F_{10} + \bar{a}_{13} F_{12} + \bar{a}_{11} F_{14} - (2\bar{a}_{11} a_{15} + \right. \\
\left. a_{13} \bar{a}_{13} F_0') \right] \]

and:

\[ R_n = 0 \quad \text{for} \quad n = 1, 3, 5, \ldots \]

\[ \tilde{R}_n = 0 \quad \text{for} \quad n \neq 0, 4, 8, \ldots \]

It is convenient to have explicit formulas for the amplitude and phase of the first order harmonics (i.e., the coefficient of \( \varepsilon \) to the first power) of the various quantities defined above. To accomplish this we note that they all can be put into the form:

\[
Z = Q_0 \xi^\lambda \sum_{n=0}^{\infty} \alpha_n \xi^n e^{i\tau}
\]
Where:

\[ \alpha_n \text{ imaginary } \quad \text{for } \quad n = 0, 4, 8, \ldots \]

\[ \alpha_n \text{ real } \quad \text{for } \quad n = 2, 6, \ldots \]

\[ \alpha_n = 0 \quad \text{for } \quad n = 1, 3, 5, 7, \ldots \]

and:

\[ \lambda = 0 \text{ or } 1 \]

Then:

\[ Z = \Re \xi^\lambda \left| \sum_{n=0}^{\infty} \alpha_n \xi^n \right| e^{i(\tau + \varphi')} \]

\[ = - \Im \xi^\lambda \left| \sum_{n=0}^{\infty} \alpha_n \xi^n \right| e^{i(\tau + \varphi' + \frac{\pi}{2})} \]

\[ = \xi^\lambda \left| \sum_{n=0}^{\infty} \alpha_n \xi^n \right| \sin. \left( \tau + \varphi' + \frac{\pi}{2} \right) \]

\[ = \left| \xi^\lambda \sum_{n=0}^{\infty} \alpha_n \xi^n e^{i\tau} \right| \sin. \left( \tau + \varphi' + \frac{\pi}{2} \right) \]

\[ = |Z| \sin. \left( \tau - \varphi \right) \]

Where:

\[ \varphi' = \arg. \left( \sum_{n=0}^{\infty} \alpha_n \xi^n \right) = \tan^{-1} \left[ \frac{\Im \sum_{n=0}^{\infty} \alpha_n \xi^n}{\Re \sum_{n=0}^{\infty} \alpha_n \xi^n} \right] \]

\[ |Z| = \xi^\lambda \left| \sum_{n=0}^{\infty} \alpha_n \xi^n \right| \]
and:

\[ \varphi = - (\varphi' - \frac{\pi}{2}) \]

Hence:

\[ |Z| = \xi^\lambda \left[ \sum_{n=0}^{\infty} \alpha_n \xi^n \sum_{m=0}^{\infty} \overline{\alpha}_m \xi^m \right] \]

\[ = \xi^\lambda \left\{ \alpha_0 \overline{\alpha}_0 + (\alpha_2 \overline{\alpha}_0 + \alpha_0 \overline{\alpha}_2) \xi^2 + (\alpha_4 \overline{\alpha}_0 + \alpha_2 \overline{\alpha}_4 + \alpha_6 \overline{\alpha}_0) \xi^4 + \ldots \right\}^{1/2} \]

\[ = \xi^\lambda \left| \alpha_0 \right| \left[ 1 + \left( \frac{\left| \alpha_2 \right|^2}{\left| \alpha_0 \right|^2} + \frac{2 \alpha_4 \overline{\alpha}_0}{\left| \alpha_0 \right|^2} \xi^4 + \ldots \right) \right]^{1/2} \] 

Hence the amplitude is given by:

\[ |Z| = \left| \alpha_0 \right| \xi^\lambda \left[ 1 + \frac{1}{2} \left( \frac{\left| \alpha_2 \right|^2}{\left| \alpha_0 \right|^2} + \frac{2 \alpha_4 \overline{\alpha}_0}{\left| \alpha_0 \right|^2} \xi^4 + \ldots \right) \right] \] \quad (150)

and:

\[ \varphi' = \tan^{-1} \left\{ -\frac{\alpha_0 + \alpha_4 \xi^4 + \ldots}{\alpha_2 \xi^2 + \ldots} \right\} = \tan^{-1} \left( \frac{-\frac{1}{\xi^2} \left\{ \frac{\alpha_0}{\alpha_2} + \frac{\alpha_4}{\alpha_2} \xi^4 + \ldots \right\}}{ } \right) \]

and therefore:

\[ \varphi' = -\frac{\pi}{2} - \tan^{-1} \left( \frac{-\frac{1}{\xi^2} \left\{ \frac{\alpha_0}{\alpha_2} + \frac{\alpha_4}{\alpha_2} \xi^4 + \ldots \right\}}{ } \right) \] \quad (151)

Now comparison of (149) with (104), (143), (145), and (147) shows that we may write:

\[ u_1^* = |u_1^*| \sin.(\tau - \varphi_1) \]
\[ \tau_1^* = |\tau_1^*| \sin (\tau - \varphi_{\tau_1}) \]
\[ q_1^* = |q_1^*| \sin (\tau - \varphi_{q_1}) \]
\[ \delta_1^* = |\delta_1^*| \sin (\tau - \varphi_{\delta_1}) \]

Where:
\[ |u_1^*| = |g_{10}| \left\{ 1 + \frac{1}{2} \left[ \frac{|g_{12}|^2}{|g_{10}|^2} + \frac{2g_{14}g_{10}}{|g_{10}|^2} \right] \xi^4 + \ldots \right\} \]
\[ \varphi_{u_1} = -\frac{\pi}{2} - \tan^{-1} \left\{ \frac{1}{\xi^2} \left[ \frac{g_{14}}{g_{12}} \xi^4 + \frac{g_{10}}{g_{12}} \xi^4 + \ldots \right] \right\} \]
\[ |\tau_1^*| = \frac{1}{a} |e_{10}| \left\{ 1 + \frac{1}{2} \left[ \frac{|e_{12}|^2}{|e_{10}|^2} + \frac{2e_{14}e_{10}}{|e_{10}|^2} \right] \xi^4 + \ldots \right\} \]
\[ \varphi_{\tau_1} = -\frac{\pi}{2} - \tan^{-1} \left\{ \frac{1}{\xi^2} \left[ \frac{e_{10}}{e_{12}} + \frac{e_{14}}{e_{12}} \xi^4 + \ldots \right] \right\} \]
\[ |\delta_1^*| = \frac{|G_{10}|}{2a} \left\{ 1 + \frac{1}{2} \left[ \frac{|G_{12}|^2}{|G_{10}|^2} + \frac{2G_{14}G_{10}}{|G_{10}|^2} \right] \xi^4 + \ldots \right\} \]
\[ \varphi_{q_1} = -\frac{\pi}{2} - \tan^{-1} \left\{ \frac{1}{\xi^2} \left[ \frac{G_{10}}{G_{12}} + \frac{G_{14}}{G_{12}} \right] \xi^4 + \ldots \right\} \]
\[ |\delta_1^*| = 2a \xi |a_{11}| \left\{ 1 + \frac{1}{2} \left[ \frac{|a_{13}|^2}{|a_{11}|^2} + \frac{2a_{15}a_{13}}{|a_{11}|^2} \right] \xi^4 + \ldots \right\} \]
\[ \varphi_{\delta_1} = -\frac{\pi}{2} - \tan^{-1} \left\{ \frac{1}{\xi^2} \left[ \frac{a_{13}}{a_{11}} + \frac{a_{15}}{a_{13}} \right] \xi^4 + \ldots \right\} \]
CHAPTER III

NUMERICAL PROCEDURES

For both the flat plate problem and the cylinder problem with $E^2 > .01$, there is the task of solving successively a system of ordinary differential equations. That is, the equations under consideration are such that the solution to the $n$th equation depends on that of the $n$-1st equation, $n$-2nd equation, etc., but not on that of the $n$+1st, $n$+2nd, etc. The momentum equations (which of course must be solved before the energy equation) are of the third order. They each involve however an arbitrary parameter which must be determined by an additional boundary condition imposed on each of the equations. Thus there are two boundary conditions at $\eta=0$ (independent variable in the notation) and two at $\eta=1$.

In the case of the cylinder problem with $E^2 \leq .01$, only the energy equation is formulated in terms of a system of constitutive ordinary differential equations. In all cases however, once the solutions to the momentum equations are obtained the consecutive ordinary differential equations resulting from the energy equation can be solved. These are of second order with two boundary conditions; one at $\eta=0$ and one at $\eta=1$.

The Runge-Cutta method which has been used to solve all these equations is a numerical procedure which performs a marching solution
to a system of $n$ simultaneous first order equations. That is, it calculates numerically the value of the solution at any desired number of points.

In order to apply this method to equations of the type considered here, some additional procedures must be used with it.

First the $n$'th order equations which are to be solved must be reduced to a system of $n$ first order equations. This is easy to do and an illustrative example will further clarify the procedure.

**Example:**

Consider the third order differential equation:

$$x'' + ax' + bx' = f(x)$$

and let us define:

$$u_1 = x$$
$$u_2 = x'$$
$$u_3 = x''$$

It then follows that

$$u_1' = u_2$$
$$u_2' = u_3$$
$$u_3' = f(u_1) - au_3 - bu_2$$

which is the desired decomposition.

Next for a differential equation of order $n$ the first $n-1$ derivatives must be specified at a single point ($\eta=0$ in this case) before the Runge-Cutta method can be used to calculate the solution to the equation.
In order to accomplish this the derivatives at \( \eta=0 \) which are unknown are first guessed at. The solution is carried out to \( \eta=1 \) by the Runge-Cutta method. Now the desired solution \( f_d \) must satisfy a condition of the form:

\[
G(f_d(1), f_d'(1), \ldots, f_d^{(n-1)}(1)) = 0
\]

at this point. The calculated solution \( f_c \) is then substituted into these relations and the errors \( R \):

\[
R = G(f_c(1), f_c'(1), \ldots, f_c^{(n-1)}(1)) - G(f_d(1), f_d'(1), \ldots, f_d^{(n-1)}(1))
\]

are noted. On the basis of these errors the initial guesses are modified. Since many equations had to be solved it was desirable to have an efficient scheme for iterating on the initial conditions. This was accomplished by incorporating a procedure which could remember how much of a percentage change in \( R \) was caused by the previous change in the initial conditions and using this information to calculate what would be the most suitable change in the initial conditions. For example, suppose it is known that the previous change in the initial conditions decreased \( R \) by only a small percentage, then it would be desirable to make the next change in the initial conditions in the same direction as the previous one (since it decreased \( R \) and that is what is required) but it should be much larger (since the previous change in the initial conditions only caused a small percentage change in \( R \) and we want \( R \) to go to zero as rapidly as possible).

In order to make these ideas quantitative consider the equation:
\[
\frac{\Delta I_{N+1}}{\Delta I_N} + 1 = -\frac{\text{CSCH}(R_N/R_{N-1} - 1)}{\text{CSCH}(1)}
\]

which is sketched in Figure 6. Where \(\Delta I_N\) is the \(N\)'th increment in the initial conditions, \(R_N\) is the value of the error in the boundary condition after the initial conditions were incremented by \(\Delta I_N\). \(R_{N-1}\) is the error that existed before this, and \(\Delta I_{N+1}\) is the new value of the increment. First it is noted that if \(R_N/R_{N-1}\) is negative then that means that \(\Delta I_N\) was made too large and it should be decreased by a certain percentage which is less than one (otherwise the error would be increased over the previous one).

Hence:

\[-1 < \frac{\Delta I_{N+1}}{\Delta I_N} < 0\]

is the proper change for \(\Delta I\) to decrease \(R_N\) in this case. That this function accomplishes this, can be verified from Figure 6. Now also the more negative \(R_N/R_{N-1}\) is the closer \(\Delta I_{N+1}\) should be to \(\Delta I_N\) in order to bring the error back closer to zero.

Next it should be noted that \(0 \leq R_N/R_{N-1} < 1\) means that the previous change in \(\Delta I_N\) accomplished a reduction in \(R_N\) and therefore the next change in \(\Delta I_N\) should be in the same direction as the previous one. But if \(R_N/R_{N-1}\) is close to 1 then the change in \(\Delta I_N\) was not to effect in reducing \(R_N\) and hence \(\Delta I_{N+1}\) should be much larger than \(\Delta I_N\) and similarly if \(R_N/R_{N-1}\) is close to zero then it was very effective and \(\Delta I_{N+1}\) need now only be a small change compared with \(\Delta I_N\).
Figure 6. Plot of $\Delta I_{n+1}/\Delta I_n$ versus $R_n/R_{n-1}$ for iterative procedure.
in order to bring the error close to 0. Finally notice that if $r_n/r_{n+1}$ is greater than 1, then the change, $\Delta I_n$, was in the wrong direction and $\Delta I_{n+1}$ should be in the opposite direction; i.e., $\Delta I_{n+1}/\Delta I_n$ should be negative. Note also that it is now necessary to change $\Delta I_n$ by such an amount that it not only compensates for the previous $\Delta I_n$ moving the error further away from 0 but that it also decreases $r_n$. It should further be noted that $r_n/r_{n-1}$ close to one means that $\Delta I_n$ didn't change $r_n$ very much, and a large change in $\Delta I_{n+1}$ will be necessary and conversely. That this will all be accomplished by the given function can easily be verified from Figure 6. It has been found that this function works very well in practice and that it usually requires less than ten iterations to get the desired solution when it is employed.

It is still necessary to discuss how the parameters appearing in the momentum equations are to be determined. It can easily be seen that the same method as is used for the initial conditions can be used here. That is, an initial guess can be made for the value of the parameters. Then, the error in the boundary conditions can be checked and a correction can be made using the same function as described above. In the case of these equations, however, iterations must be performed on both an initial condition and the values of a parameter in order to satisfy the two boundary conditions at $\eta=1$. This presents no problem since for a given assumed value for the initial condition,
iterations can be performed on the parameter until one of the boundary conditions at $\eta=1$ is satisfied. When this is accomplished, the error in the other boundary condition can be noted, the function described above can be used to calculate a new value for the initial condition, and so on until both boundary conditions are satisfied.

In the case of the energy equations only a single iteration loop need be performed.

Thus the procedure for both the oscillating flat plate problem and the circular cylinder is to solve each of the equations successively using the iteration procedure described above and storing the solutions to be used in the higher order equations.

The Runge-Cutta method is also used for integrating the equations for the drop trajectories, but since this is an initial value problem no special difficulties are presented here.

There calculations are all performed on a 7090 computer using the "MAD" language. The individual programs are listed in Appendix XXV.
CHAPTER IV
DISCUSSION AND RESULTS

A. Convergence of Solutions

The computer programs listed in the appendix were run on the IBM 7090 digital computer located at The University of Michigan Computing Center. For those programs (Appendix XXV) which involved the integration of differential equations by the Runge-Kutta method it was necessary to determine the proper number of subdivisions into which the domain should be divided in order to get accurate answers (see Runge-Kutta Subroutine in Michigan Executive System). This was accomplished experimentally on the machine by calculating the solutions using various numbers of subdivisions. The results of these experiments for the differential equations describing the liquid film on the cylinder with \( E^2 > 0.01 \) are shown in Figures 7-a and 7-b. It can be seen from these that increasing the number of subdivisions past 100 results in no significant change in the solution. Of course, a point will be reached when round off errors become excessive and the solutions will begin to diverge again. But this could not be demonstrated, since it would involve excessive machine time to reach this point. It should be noted that the higher order terms are tabulated in these figures since the lower order terms were found to reach a stable value at a much smaller number of subdivisions.
Figure 7-a. Convergence of function \( f_5 \) with respect to number of subdivisions used for numerical calculations in cylinder problem.
Figure 7-b. Convergence of function $F_4$ with respect to number of subdivisions used for numerical calculations in cylinder problem.
Figure 7-c. Convergence of series for liquid film thickness with respect to the number of terms retained in cylinder problem.
Figure 7-d. Convergence of series for liquid velocity with respect to the number of terms retained in cylinder problem.
Figure 7-e. Convergence of series for Nusselt number with respect to the number of terms retained in cylinder problem.
Similarly the results of experiments on the equations for drop trajectories are shown in Figures 8-a and 8-b, and those for the energy equation in the liquid film with $E \leq .1$ are shown in Figure 9-a. The results for the equations describing the liquid film on the oscillating flat plate are shown in Figures 10-a through 10-d. In the last two cases, again only the higher order terms are shown since the lower order ones become stable at a lower number of subdivisions. In the case of the flat plate, it can be seen that a hundred subdivisions will give very satisfactory answers.

It is in general impossible to prove convergence of the expansions that have been employed in obtaining solutions to the partial differential equations. In the case of the cylinder problem for both large and small $E$ it is hoped that a convergent series was obtained. In the case of the flat plate only an asymptotic series was sought, $^{31}$ (a coordinate expansion plus a parameter expansion in Van Dyke's terminology). The best one can hope to do is to show that increasing the number of terms retained in the expansion has a small effect on the solution after a certain number of terms.

This has been carried out for the expansions used in the cylinder problem (for both large and small $E$) and the results are shown in Figures 7-c through 7-e for $E^2 \geq .01$ and in Figure 9-b and 9-c for $E \leq .1$. Because of the extreme complexity of the solutions for the oscillating flat plate problem this could not reasonably be done.
Figure 8-a. Convergence of solution for $y_{df}^*$ with respect to size of increment used in integrating drop trajectories.
Figure 8-b. Convergence of solution for $\phi_d$ with respect to size of increment used in integrating drop trajectories.

$r_d/R_o = 0.004$
$2R_0 U_\infty / \nu_g = 10^4$
$\gamma_{d\infty}^* = 0.500$
Figure 9-a. Convergence of function $F_6$ with respect to number of subdivisions for cylinder problem with $E \leq .1$. 

$E = .04$
$\frac{r_d}{R_0} = .004$
$2R_0 U_\infty / \nu_g = 10^4$
$P_f = 10$
Figure 9-b. Convergence of function $F_{6}'$ with respect to number of subdivisions for cylinder problem with $E \leq .1$. 

- $E = .04$
- $r_d/R_0 = .004$
- $2R_0 U_\infty / \nu_g = 10^4$
- $Pr = 10$
Figure 9-c. Convergence of solutions with respect to number of terms retained in expansion for cylinder problem with $E \leq .1$. 

- $2 = 1/2$
- $E = 0.04$
- $r_d/R_o = 0.004$
- $2R_oU_\infty/\nu_g = 10^4$
- $P_r = 10$
Figure 10-a. Convergence of function $f_{15}$ with respect to number of subdivisions used for numerical calculations in flat plate problem.
Figure 10-b. Convergence of function $\tilde{f}_{25}$ with respect to number of subdivisions used for numerical calculations in flat plate problem.
Figure 10-c. Convergence of function $F_{24}$ with respect to number of subdivisions used for numerical calculations in flat plate problem.
Figure 10-d. Convergence of function $\tilde{F}_{24}$ with respect to number of subdivisions used for numerical calculations in flat plate problem.
B. Numerical Results

1. THE CIRCULAR CYLINDER

The solutions to Eqs. (41) through (43) have been obtained numerically by means of the computer program listed in Appendix XV. The results are available for values of $E^2$ between .01 and 100 and Prandtl numbers between 5 and 13.

These results can be used to calculate $\psi^*$, $\delta^*$, and $\Theta^*$ by using Eqs. (26) through (28). They have also been used to calculate the terms of Eqs. (38) through (40) for velocity, skin friction and Nusselt numbers respectively. The liquid film thickness has been plotted in Figures 11-a and 11-b for various values of $E$. The local wall shear stress is plotted in Figure 13-a and the local Nusselt number has been plotted in Figures 14-a, 14-b, and 15. Typical velocity profiles are plotted in Figure 12. For $E \leq 0.1$ it is first necessary to calculate the liquid drop trajectories before any of the physical quantities such as skin friction can be found in the film. The calculation of the drop trajectories has been carried out by using the computer program listed in Appendix XXV to integrate Eqs. (17*). This program also calculates the quantities of Eqs. (17*A.C.) which must be known to solve the governing equations for the film. Typical drop trajectories are shown in Figures 16-a through 16-1. The quantities of (17*A.C.) were tabulated for values of $r_d/R_o$ ranging between 0.0004 and 0.01 and values of $2R_oU_*/v_g$ between $10^3$ and $10^5$. 
Figure 11-a. Local liquid film thickness for cylinder with $E^2 \geq 0.01$. 
Figure 11-b. Local liquid film thickness for cylinder with $E^2 \geq 0.01$. 
Figure 11-c. Local liquid film thickness for cylinder with \( E \leq 0.1, \frac{r_d}{R_0} = .004 \) and \( 2R_0 U_\infty / \nu_g = 10^3 \).
Figure 11-d. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_o = 0.004$ and $2R_oU_\infty/\nu_g = 5 \times 10^3$. 
Figure 11-e. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_o = .004$ and $2R_o U_\infty / \nu_g = 10^4$. 

$r_d/R_o = .004$

$2R_o U_\infty / \nu_g = 10^4$
Figure 11-f. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_o = .001$ and $2R_oU_\infty/\nu_g = 10^5$. 
Figure 11-g. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_o = .001$ and $2R_o U_\infty / \nu_g = 10^4$. 
Figure 11-h. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_o = 0.001$ and $2R_o U_\infty / \nu_g = 5 \times 10^3$. 

$r_d / R_o = 0.001$

$2R_o U_\infty / \nu_g = 5 \times 10^3$
Figure 11-1. Local liquid film thickness for cylinder with $E \leq 0.1$, $r_d/R_0 = 0.0004$ and $2R_0U_\infty/\nu_g = 10^5$. 

$\frac{r_d}{R_0} = 0.0004$

$2R_0U_\infty/\nu_g = 10^5$
Figure 13-a. Local wall shear stress for cylinder with $E^2 \geq .01$. 
Figure 13-b. Local wall shear stress for cylinder with $E \leq 0.1$, 
$r_d/R_0 = .004$ and $2R_0 U_\infty/\nu_g = 10^3$. 
Figure 13-c. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_o = 0.004$ and $2R_o U_\infty/\nu_g = 5 \times 10^3$. 

\[ r_d/R_o = 0.004 \]
\[ 2R_o U_\infty/\nu_g = 5 \times 10^3 \]
Figure 13-d. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_o = 0.004$ and $2R_o U_\infty / \nu = 10^4$. 
Figure 13-e. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_0 = 0.001$ and $2R_0U_\infty/\nu_g = 5 \times 10^3$. 
Figure 13-f. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_0 = .001$ and $2R_0 U_\infty / \nu_g = 10^4$. 
Figure 13-g. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_o = .001$ and $2R_oU_\infty/\nu_g = 10^5$. 

\[ \frac{\tau_w}{\sqrt{2R_o U_\infty}} = \frac{1}{2} \rho U_\infty^2 \]

\[ x/R_o \]

\[ r_d/R_o = 0.001 \]

\[ 2R_o U_\infty/\nu_g = 10^5 \]
Figure 13-h. Local wall shear stress for cylinder with $E \leq 0.1$, $r_d/R_o = 0.0004$ and $2R_o U_\infty / \nu_g = 10^5$. 

$r / R = 0.0004$

$2R_o U_\infty / \nu_g = 10^5$
Figure 14-a. Local Nusselt number for cylinder with \( Pr = 5.0 \) and \( E > 0.01 \).
Figure 14-b. Local Nusselt number for cylinder with $Pr = 13.4$ and $E \geq 0.01$. 
Figure 14-c. Local Nusselt number for cylinder with $\text{Pr} = 5.0$, $E \leq 0.1$, $r_d/R_o = 0.04$ and $2R_o U_\infty/\nu_g = 5 \times 10^3$. 

$$r_d/R_o = 0.004$$
$$2R_o U_\infty/\nu_g = 5 \times 10^3$$
$$\text{Pr} = 5.0$$
Figure 14-d. Local Nusselt number for cylinder with Pr = 10, $E \leq 0.1$, $r_d/R_o = .004$, and $2R_o U_\infty/\nu_g = 5 \times 10^3$. 

$r/R = .004$ 
$2R_o U_\infty/\nu_g = 5 \times 10^3$ 
$Pr = 10$
Figure 14-e. Local Nusselt number for cylinder with $Pr = 5$, $E \leq 0.1$, $r_d/R_0 = 0.004$ and $2R_0\omega U_0/\nu_g = 10^4$. 

$\frac{r_d}{R_0} = 0.004$

$2R_0 \frac{U_0}{\nu_g} = 10^4$

$Pr = 5$
Figure 14-f. Local Nusselt number for cylinder with $P_r = 10$, $E \leq 0.1$, $r_d/R_o = 0.004$ and $2R_o U_\infty / \nu_g = 10^4$. 

$\frac{Nu}{1/2 R_o U_\infty / \nu}$ vs $X/R_o$.

$E = 0.08$

$E = 0.06$

$E = 0.04$

$E = 0.02$

$r_d/R_o = 0.004$

$2R_o U_\infty / \nu_g = 10^4$

$P_r = 10$
Figure 14-g. Local Nusselt number for cylinder with $Pr = 10$, $E \leq .1$, $r_d/R_o = .001$ and $2R_o U_\infty / \nu_g = 5 \times 10^3$. 
Figure 14-h. Local Nusselt number for cylinder with $P_r = 10$, $E \leq .1$, $r_d/R_o = .001$ and $2R_o U_\infty / \nu_g = 10^4$. 
\[ \frac{r_d}{R_o} = 0.001 \]
\[ 2R_o U_w / \nu_g = 10^5 \]
\[ P_r = 10 \]

Figure 14-1. Local Nusselt number for cylinder with \( P_r = 10 \), \( E \leq 0.1 \), \( r_d / R_o = 0.001 \) and \( 2R_o U_w / \nu_g = 10^5 \).
Figure 15. Peak values of local Nusselt number for cylinder with $Pr = 10$. 
Figure 16-a. Drop trajectories around cylinder with $r_d/R_o = 0.01$ and $2R_o U_\infty / \nu_g = 10^3$. 
Figure 16-b. Drop trajectories around cylinder with $r_d/R_o = .004$ and $2R_o U_\infty / v_g = 10^3$. 
Figure 16-c. Drop trajectories around cylinder with $r_d/R_o = .004$ and $2R_o U_\infty/\nu_g = 5 \times 10^3$. 
Figure 16-d. Drop trajectories around cylinder with \( r_d/R_o = 0.004 \) and \( 2R_o U_\infty/\nu_g = 10^4 \).
Figure 16-e. Drop trajectories around cylinder with $r_d/R_0 = 0.004$ and $2R_0U_\infty/\nu_g = 25830$. 
Figure 16-f. Drop trajectories around cylinder with $r_d/R_o = .001$ and $2R_o U_\infty/\nu_g = 5 \times 10^3$. 
Figure 16-g. Drop trajectories around cylinder with $r_d/R_o = .001$ and $2R_oU_0/v_g = 10^4$. 

$y_0/R_o$ 

$-x_0/R_o$
Figure 16-h. Drop trajectories around cylinder with \( r_d/R_o = .001 \)
and \( 2R_oU_\infty/\nu_g = 10^5 \).
Figure 16-1. Drop trajectories around cylinder with $r_d/R_o = 0.0004$ and $2R_o U_\infty / \nu_g = 10^5$. 
Representative ones are given in Table I.

An examination of Table I and Figure 16-e shows that for a value of the gas Reynolds number of 25830 and an $r_d/R_o = 0.004$ that the drop trajectories are practically straight and over 90% of the liquid drops which are heading toward the cylinder far upstream from it are intercepted by it. From Figure 27 the minimum value of the Reynolds number at which drops whose size is given by $r_d/R_o = 0.004$ will move in straight lines is seen to be 25830. Hence the dimensional arguments used in Chapter II to obtain a lower limit to the validity of the solution corresponding to straight drop trajectories is seen to be verified by the actual calculations.

These results for the drop trajectories were used in the computer program of Appendix XXVI to calculate the film thickness and local skin friction which are plotted in Figures 11-c through 11-i, and 13-b through 13-h, respectively. This program also calculates the terms of the expansions (52) by fitting a curve to the tabulated values of the quantities on the left. The $\alpha_m$'s and $\beta_m$'s of these expansions so obtained were used in the computer program of Appendix XXV which calculated the solutions to the Eqs. (55). The results of these calculations are available for $0.01 \leq 0.1$ and for the same range of parameters as the drop trajectories, and have been used to calculate the local Nusselt numbers which are plotted in Figures 14-c through 14-i. An examination of the figures mentioned above shows that the physical quantities, film thickness, skin friction,
# Table I

Values of $\mathcal{N}$, J and $V_\Phi$

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and Nusselt number attain a maximum at a value of E near one. The peak values of the Nusselt number for a Prandtl number of 10 are shown in Figure 15. In general, it can be seen from the figures that the behavior of the film changes at this value of E, for example Figures 11-a and 11-b, show that the film thickness increases in a downstream direction when E is less than one, but decreases slightly in the downstream direction when E is greater than one. Figure 12 shows that there is also a change in the behavior of the velocity profiles in the neighborhood of E=1. For values of E greater than one there is an increased tendency for the velocity gradient to become very steep at the outer edge of the film. This may represent an unstable condition. The steeping is probably due to the fact that large values of E correspond to a large flux of momentum into the film. The flux becomes so rapid that the momentum does not have time to diffuse into the film near the stagnation point and there it has the effect of rapidly accelerating a small mass of fluid at the outer edge of the film. Since the momentum eventually penetrates into the film some distance downstream from the stagnation point this could account for the decrease in thickness of the film, since more and more of the liquid in the film is being speeded up. Further, since the drops don't penetrate deeply into the film at large E this could account for the decrease in the local Nusselt number as E increases beyond one. This can be seen from Figures 14-a and 14-b.
On the other hand, for \( E < 1 \) the velocity profiles tend to become increasingly linear as \( E \) decreases, (Figure 12), indicating that inertial effects are unimportant. This tends to verify the dimensional reasoning employed in Chapter II to eliminate the inertial terms in the momentum equation for \( E \leq 0.1 \).

It should be noted that the behavior of the solutions is generally the same for values of \( E^2 < 0.01 \) and values of \( E^2 > 0.01 \), (Figures 11-a through 11-i, 13-a through 13-b, and 14-a through 14-i), even though different terms were retained in the equations which describe these regimes.

However, there is an increasing tendency for the liquid film thickness to increase suddenly at some downstream point with decreasing values of \( E \). The dimensionless film thickness, however, remains of the order of unity for all values of \( E \) which justifies the dimensional reasoning of Chapter II. This rapid increase in film thickness occurs at the point where the droplets no longer impinge on the cylinder for \( E \leq 0.1 \) as can be seen by a comparison of Figures 11-c through 11-i with Figures 16-b through 16-i.

This is probably due to the fact that at this point the momentum carried into the film by the drops rapidly falls to zero, and the gas shear stress is not capable of moving the heavy liquid and as a result the liquid tends to pile up. This makes it very likely that separation will tend to occur in the neighborhood of this point, or at least the film will tend to fall off on vertical cylinders.
It can be seen from Figures 14-c through 14-i that the heat transfer coefficient drops off rapidly in this region. For all values of E, (Figures 14-a through 14-i), the heat transfer coefficient is a maximum at the stagnation point and decreases in a downstream direction. As E becomes large the heat transfer coefficient tends to decrease less in the downstream direction, as can be seen from Figures 14-a and 14-b. A comparison of Figures 14-a with 14-b, 14-c with 14-d, and 14-e with 14-f shows that the Nusselt number increases with increasing Prandtl number as is usual. For values of E less than 0.1 the effects of drop size and gas Reynolds number based on the cylinder diameter have a strong influence on the heat transfer coefficient, friction factors, and film thickness since they influence the amount of liquid coming into the film. It can be seen from Figures 16-a through 16-i that the drop trajectories are more nearly straight at higher Reynolds numbers and larger drop sizes. Thus as can be seen from Table I larger amounts of mass and momentum are carried into the film with larger drop sizes and higher Reynolds numbers. The influence on the heat transfer (Figures 14-c through 14-i) skin friction (Figures 13-b through 13-h) and film thickness (Figures 11-c through 11-h) is what would be expected. They all increase with increasing values of gas Reynolds number and drop size. It can be seen from these figures that the drop size has a strong effect.
As noted experimentally,\textsuperscript{1,9} and predicted quantitatively\textsuperscript{29} there is a definite increase in heat transfer with the two-phase mixture above that, which would be obtained from a pure gas flow (Figures 14-a through 14-i).

No value for the drop diameter was reported for the experimental results of Reference (1) due to the difficulty of measuring this quantity. However the data of Reference (1) after being reduced in Appendix XXVII, are compared with the analytical results corresponding to \(r_d/R_0 = 0.004\)\textsuperscript{*} in Figure 17 and it can be seen that the agreement is close and well within the experimental scatter.

It can be seen from Figures 13-a through 13-h that the skin friction usually reaches a peak value somewhere around the 50° point on the cylinder. From Figure 13-a it can be seen that for values of \(E > 1\) this peak moves so far to the rear that it does not show up in the analysis. On the other hand, when \(E \leq 0.1\) the peak will move forward when the drop size and gas Reynolds numbers are small. A comparison of Figures 13-a through 13-h with Figures 14-a through 14-i shows that an increase in heat transfer is always accompanied by an increase in friction. Thus one must always pay the price of increased friction to obtain increased heat transfer.

\textsuperscript{*}The diameter of the cylinder in the experiments was 1.5 in. For \(r_d/R_0 = 0.004\) this corresponds to a drop size of 152 microns. The manufacturer of the nozzle used in the experiments, reports a volume-median drop diameter of 167 microns for the nozzle under different operating conditions than those of the experiment.
$2R_0 U_\infty / \nu_g = 8 \times 10^4$

$r_d / R_0 = 0.004$

$P_r = 10$

$E = 0.02$

$\bigtriangleup$ Experimental points of acrivos et al. (1)

Figure 17. Comparison of calculated Nusselt number with experimental data for cylinder.
2. OSCILLATING FLAT PLATE

The solutions to Eqs. (93) through (95), (100) through (102), (106), (109), (115), (120), (124), and (129) have been obtained numerically by using the computer program listed in Appendix XXV. The results are available for $0.05 \leq X_e \leq 0.2$ and $Pr = 5$ and 10. These results can be used to calculate $\psi^*$, $\delta^*$, and $\Theta^*$ by using them in Eqs. (92), (99), (103) through (105), and (112) through (114), and then using these new quantities in the expansions (87). The solutions have been used to calculate the terms in the expansions (138) and (139) for wall shear stress and local Nusselt numbers. The first term of each of the expansions (87), and (137) through (139) represents the steady solution. The second term represents the harmonic components (i.e., it goes as $e^{i\omega t}$) and the third term is the sum of a part which represents the permanent alteration to the flow field due to the oscillations and a time dependent part. The physical quantities for the steady solution are plotted in Figures 18-a through 20.

The harmonic components of the physical quantities are plotted in Figures 21-a through 23-c, and the permanent alterations in the physical quantities due to the oscillations are plotted in Figures 24 through 26-b.

Figures 18-a through 20 show that the steady (nonoscillating) components of the film thickness, skin friction, and Nusselt number increase with increasing $X_e$. Figure 20 shows that the steady component
Figure 18-a. Local liquid film thickness versus volume fraction for steady flat plate.
Figure 18-b. Replot of Figure 18-a in terms of oscillating variables.
Figure 19. Local wall shear stress versus volume fraction for steady flat plate.
Figure 20. Local Nusselt number versus volume fraction for steady flat plate.
Figure 21-a. Local amplitude of liquid film thickness on oscillating flat plate.
Figure 21-b. Local phase lag of liquid film thickness on oscillating flat plate.
Figure 22-a. Local amplitude of wall shear stress on oscillating flat plate.
Figure 22-b. Local phase angle of wall shear stress on oscillating flat plate.
Figure 23-a. Local amplitude of Nusselt number on oscillating flat plate with $P_r = 5.0$ and $10.0$. 
Figure 23-b. Local phase angle of Nusselt number on oscillating flat plate with $Pr = 5$. 
Figure 23-c. Local phase angle of Nusselt number on oscillating flat plate with $Pr = 10$. 
Figure 24. Permanent alteration in liquid film thickness due to oscillation of flat plate.
Figure 25. Permanent alteration in wall shear stress due to oscillation of flat plate.
Figure 26-a. Permanent alteration in local Nusselt number due to oscillation of flat plate for $Pr = 5$. 

\[
\left( \frac{Nu_{AVG.} - Nu_{STEADY}}{\sqrt{\nu x / \epsilon U_\infty}} \right)
\]
Figure 26-b. Permanent alteration in local Nusselt number due to oscillation of flat plate for $Pr = 10$. 
of the Nusselt number increases with increasing Prandtl number.

It can be seen from Figures 21-a and 22-a that the amplitudes of the harmonic components of the film thickness remain relatively constant with increasing frequency (or distance along the plate) and then begin to increase somewhat, at larger values of frequency. However, Figure 23-a shows that although the amplitude of the harmonic component of the Nusselt remains relatively constant with frequency it may either increase or decrease at higher values of frequency depending on the volume fraction of the liquid in the free stream but not on the Prandtl number.

It can be seen from Figures 21-b, 22-b, 23-b, and 23-c that the phase lag of the harmonic components of film thickness, skin friction, and Nusselt number increases with increasing frequency and decreasing volume fraction of liquid in the free stream. The amplitudes of these quantities decrease with decreasing volume fraction of liquid (Figures 21-a, 22-a, and 23-a). Increasing Prandtl number increases both the amplitude and phase lag of the harmonic component of the Nusselt number (Figures 23-a through 23-c).

Examination of Figures 24 through 26-b shows that the oscillations result in an extremely small permanent alteration of the film thickness, shear stress, and Nusselt number. This is due to the fact that for small values of $X_0$ the inertial terms in the momentum equation become relatively unimportant and so the equations are practically linear and linear equations cannot describe a physical situation in which there
is a permanent alteration of any physical quantity. In this regard, one might anticipate that oscillating the cylinder would result in no permanent alterations of any physical quantities with $E$ less than one, since the equations are also almost linear here.

Figure 25 shows that the permanent increase in the skin friction due to oscillations decreases with increasing frequency at the higher values of frequency and remains constant with change in frequency at the lower values (quasi-steady). It also increases with increasing amounts of liquid in the free stream ($X_e$).

Figures 25-a and 25-b show that there is a slight permanent decrease in the Nusselt number due to the oscillation which remains relatively constant with frequency. At higher frequencies this decrease becomes more pronounced. The decrease in the Nusselt number becomes larger with increase in the Prandtl number, and with increase in $X_e$. 
CHAPTER V

CONCLUSION

A. Cylinder

Solutions are obtained for two ranges of the parameter

\[ E = X_e \sqrt{\frac{2R_0 U_0}{v}}. \]

In case of steady flow the film thickness, skin friction, and Nusselt number have a maximum at \( E \) near unity. When \( E \) increases from unity, the liquid film thickness decreases toward the downstream with a decrease in the local Nusselt number and an increase in the velocity gradient at the outer edge of the film. This tends to unstabilize the flow in the liquid film. As \( E \) decreases from unity, the inertia effects become less important as indicated by the velocity profile which tends to become linear. The film thickness tends to increase more abruptly in the downstream direction. Because of the accompanying decrease in velocity the gravity force will probably become important in the region immediately downstream from the point at which this abrupt increase occurs on vertical cylinders.

Decreasing drop diameter and gas Reynolds number tend to move the point of sudden increase in film thickness toward the forward stagnation point. This is accompanied by a decrease in the local Nusselt number, skin friction, and film thickness.
The analytical predictions compared favorably with experimental results which indicated a substantial increase in local heat transfer rate over that of a single phase flow.

B. Oscillating Flat Plate

The steady (nonoscillating) components of the film thickness, skin friction, and Nusselt number increase with increasing $X_0$. The amplitudes of the harmonic components of the film thickness remain relatively constant with increasing frequency (or distance along the plate) and then begin to increase somewhat, at larger values of frequency. The amplitude of the harmonic component of the Nusselt number remains relatively constant with frequency. It may either increase or decrease at higher values of frequency depending on the volume fraction of the liquid in the free stream but not on the Prandtl number.

The phase lag of the harmonic components of film thickness, skin friction, and Nusselt number increase with increasing frequency and decreasing volume fraction of liquid in the free stream. The amplitudes of these quantities decrease with decreasing volume fraction of liquid. Increasing Prandtl number, increases both the amplitude and phase lag of the harmonic component of the Nusselt number.

Oscillations result in an extremely small permanent alteration of the film thickness, shear stress, and Nusselt number. This is due to
the fact that for small values of $X_e$ the inertial terms in the momentum equation become relatively unimportant. One may anticipate that oscillating the cylinder would result in no permanent alterations of any physical quantities with $E$ less than one.

The permanent increase in the skin friction due to oscillations decreases with increasing frequency at the higher values of frequency and remains constant with change in frequency at the lower values (quasi-steady). It also increases with increasing amounts of liquid in the free stream ($X_e$).

There is a slight permanent decrease in the Nusselt number due to the oscillation which remains relatively constant with frequency. At higher frequencies this decrease becomes more pronounced. The decrease in the Nusselt number becomes larger with increase in the Prandtl number, and with increase in $X_e$. 
BIBLIOGRAPHY


APPENDIXES
APPENDIX I

CALCULATIONS FOR DETERMINING THE MINIMUM VALUES OF $X_e$ AND GAS
REYNOLDS NUMBERS AT WHICH DROP TRAJECTORIES ARE STRAIGHT LINES

To find these minimum values we set:

$$\frac{3}{8} \frac{C_D}{(r_d/R_o)} \frac{\rho g}{\rho} = .1 \tag{I.1}$$

$$N' = 10$$

and we have:

$$X_e \approx \left(16 \frac{r_d}{R_o}\right)^3 \tag{I.2}$$

For a given $r_d/R_o$ this determines the minimum value of $X_e$ and since
the drag coefficient is a function of the Reynolds number, based on
the drop diameter. (I.1) determines this and therefore the gas
Reynolds number, and therefore (I.1) and (I.2) determine the minimum
value of $E^2$.

These calculations have been carried out using the computer pro-
gram listed in this Appendix.

The values of $CD(J)$ corresponding to RED(J) used in this program
are tabulated in Table II of Appendix XXV.

NOMENCLATURE FOR COMPUTER PROGRAM

$CD(J) \sim$ tabulated value of $C_D$ at RED(J)

$EP(J) \sim \frac{3}{8} \frac{CD(J)\rho g/\rho}{(r_d/R_o)}$
ES ~ minimum value of $E^2$

ET ~ desired value of $EP(J)$, .1

N ~ $N'; = 10$

R ~ $r_d / R_o$

RC ~ value of $RE(J)$ at $N' = 10$ and $EP(J) = .1$ (minimum value)

$RE(J) \sim \frac{R_o U_o}{\nu} = \frac{R_o}{r_d} \times RED(J)$

$RED(J) \sim$ tabulated value of $r_d U_o / \nu_g$

RG ~ minimum value of $R_o U_o / \nu_g$

XE ~ value of $X_e$ at $N' = 10$
DIMENSION RE(100),RD(100),EP(100),CD(100)  *001
INTEGER J,JMAX  *002
READ AND PRINT DATA  *003
THROUGH SW2, FOR R=C,C11,C.CDS,R.G.RMAX  *004
THROUGH SW1, FOR J=1,1,J,G,JMAX  *005
RE(J)=RED(J)*7.52/R  *006
SW1  *007
EP(J)=CD(J)*C.C3C3/(R.*R)  *008
XE=(1.6*R)+R.P  *009
WHENEVER (EP(JMAX)).LT.E,AND, XE.L.C3  *010
RC=TAE(E,EP(J),RE(J),1,1,4,JMAX,S)  *011
ES=(XE.P.2)*RC  *012
RC=RC/7.52  *013
PRINT RESULTS R,RC,ES,RE,RC,XE  *014
OTHERWISE  *015
CONTINUE  *016
SW2  *017
END CF CONDITIONAL
END CF PROGRAM
Figure 27. Minimum value of $E^2$ and gas Reynolds number for a straight drop trajectory.
APPENDIX II

ESTIMATION OF PARAMETERS FOR AIR/WATER MIXTURES

A. PROPERTY RATIOS* FOR AIR/WATER MIXTURES

\[
\frac{\rho_g}{\rho} = \frac{0.081}{62.4} = 1.3 \times 10^{-3}
\]

\[
\frac{\nu_g}{\nu} = \frac{0.145 \times 10^{-3}}{1.93 \times 10^{-5}} = 7.52
\]

\[
\frac{\sqrt{\nu_g}}{\sqrt{\nu}} = \sqrt{7.52} = 2.74
\]

\[
\frac{C_{pg}}{C_p} = \frac{0.25}{1} = 0.25
\]

\[P_{rg} = 0.7\]

B. CYLINDER PROBLEM

Evaluation of surface phenomena and pressure terms for single component gas boundary layer.

From Schlichting, p. 153, Figure 9.6:

\[
\frac{\tau_g}{1/2 \rho_g U_\infty^2} \sqrt{\frac{2U_B}{\nu_g}} \leq \sqrt{2} \cdot 4.5
\]

From Schlichting, p. 320, Figure 14.13:

\[
\frac{N_{ug}}{\sqrt{Re_g}} \leq 1
\]

For potential flow:

\[
\frac{P_g}{1/2 \rho_g U_\infty^2} \leq 2
\]

*Kreith "Elements of Heat Transfer."
Hence for air/water mixtures one obtains:

\[
\left[ \frac{v_g \rho_g}{\nu \rho} \right] \left[ \frac{1 + \sqrt{\frac{2U_{\infty} \rho}{v_g}}}{1/2 \rho_g U_{\infty}^2} \right] \approx 4.5 \cdot 2 \cdot 2.74 \cdot 1.3 \times 10^{-3}
\]

\[ \approx 2.2 \times 10^{-3} \]

\[ \frac{2 \rho_g}{\rho} \left[ \frac{P_g}{1/2 \rho_g U_{\infty}^2} \right] \approx 2 \cdot 1.3 \times 10^{-3} \cdot 2 \approx 5.2 \times 10^{-3} \]

\[
\left[ \frac{\frac{N_u g}{2R_0 U_{\infty}^2}}{\nu_g} \right] \left[ \frac{C_{pg} \rho_g}{C_p \rho} \sqrt{\frac{v_g}{\nu}} \right] \approx \frac{1.0 \cdot 0.25 \cdot 1.3 \times 10^{-3} \cdot 2.7}{.7}
\]

\[ \approx 1.2 \times 10^{-3} \]

C. OSCILLATING FLAT PLATE PROBLEM

Evaluation of surface phenomena for single component gas boundary layer.

From Schlichting, p. 120.

\[
\frac{\tau_g}{\rho_g U_{\infty}^2} \sqrt{\frac{U_{\infty}}{v_g}} = .332
\]

From Schlichting p. 305.

\[
\left( \frac{q_g}{\tau - T_{\infty}} \right) \text{kg} / \sqrt{\frac{U_{\infty} \nu}{v_g}} = N_u g \left( \frac{\mu g C_{pg}}{k_g} \right) \approx N_u g \left( \frac{U_{\infty}}{v_g} \right) \text{Pr}_g \approx .332 \frac{1}{\text{Pr}_g^{1/3}}
\]

Hence for air/water mixtures one has:

\[
\left[ \frac{v_g \rho_g}{\nu \rho} \right] \frac{\tau_g}{\rho_g U_{\infty}^2} \sqrt{\frac{U_{\infty}}{v_g}} \approx .332 \cdot 2.74 \cdot 1.3 \times 10^{-3} \approx 1.1 \times 10^{-3}
\]
\[
\left[ \frac{q_g}{T - T_\infty} \right] \frac{x}{k_g} \sqrt{\frac{U_{\text{ex}}}{\nu_g} \left( \frac{\mu_g C_p g}{k_g} \right)} \left[ \frac{\nu_g}{\nu} \frac{\rho_g}{\rho} C_p g \right] \\
\approx \left[ \frac{.332}{(.7)^{2/3}} \right] \left[ 2.74 \cdot 1.3 \times 10^{-3} \cdot .25 \right] \approx 3.75 \times 10^{-4}
\]
APPENDIX III

REDUCTION OF THE FIRST OF EQUATIONS (23)

Substitutions of Eqs. (26) and (27) into the first of Eqs. (23)
yields:

\[
a_0 \left( 1 + \frac{b_2}{2!} \xi^2 + \frac{b_4}{4!} \xi^4 + \frac{b_6}{6!} \xi^6 + \ldots \right) \left[ f_1'' - \frac{4}{3!} f_1' f_3 + \frac{6}{5!} f_5' \xi^5 \right.
\]
\[
- \frac{8}{7!} f_7' \xi^7 + \ldots \right) \left( f_1' - \frac{4}{2!} f_3' \xi^2 + \frac{6}{4!} f_5' \xi^4 - \frac{8}{6!} f_7' \xi^6 + \ldots \right)
\]
\[
- \frac{4}{2!} f_3' \xi^2 + \frac{6}{4!} f_5' \xi^4 - \frac{8}{6!} f_7' \xi^6 + \ldots \right) \left( f_1'' - \frac{4}{3!} f_3' \xi^2 + \frac{6}{5!} f_5' \xi^4 - \frac{8}{7!} f_7' \xi^6 + \ldots \right) - \left( f_1' \right.
\]
\[
- \frac{8}{7!} f_7' \xi^7 + \ldots \right) \left( f_1' - \frac{4}{3!} f_3' \xi^2 + \frac{6}{5!} f_5' \xi^4 - \frac{8}{7!} f_7' \xi^6 + \ldots \right)
\]
\[
- a_0 \left( b_2 \xi^2 + \frac{b_4}{3!} \xi^3 + \frac{b_6}{5!} \xi^5 + \ldots \right) \left( f_1'' - \frac{4}{3!} f_3' \xi^2 + \frac{6}{5!} f_5' \xi^4 - \frac{8}{7!} f_7' \xi^6 + \ldots \right)
\]
\[
+ \ldots \right)^2 = \frac{1}{E^2} \left( f_1''' - \frac{4}{3!} f_3'' \xi^3 + \frac{6}{5!} f_5'' \xi^5 - \frac{8}{7!} f_7''' \xi^7 + \ldots \right)
\]

Collecting terms and rearranging yields:

\[
\left( 1 + \frac{b_2}{2!} \xi^2 + \frac{b_4}{4!} \xi^4 + \frac{b_6}{6!} \xi^6 + \ldots \right) \left[ f_1'' - \left( \frac{4}{3!} + \frac{4}{2!} \right) f_1' f_3' \xi^2 + \left( \frac{6}{4!} \right) f_5' \xi^5 \right.
\]
\[
+ \frac{4}{2!} f_3' \xi^2 + \frac{6}{4!} f_5' \xi^4 \right) \xi^5 \left( \frac{8}{5!} f_7' f_1' \xi^7 + \frac{4}{5!} f_3 f_2 + \frac{6}{5!} \right) f_3 \xi^3 + \frac{6}{5!} f_5' \xi^5 \right.
\]
\[
+ \frac{8}{7!} f_7' f_1' \xi^7 + \ldots \left( f_1'' \xi^2 + \left( \frac{4}{3!} + \frac{4}{2!} \right) f_1' f_3 + \frac{6}{5!} \right) f_5' \xi^5 + \frac{4}{3!} f_3' \xi^3 + \frac{4}{6!} f_5' \xi^5 \right.
\]
\[
+ \frac{8}{7!} f_7' f_1' \xi^7 + \ldots \right) \left( b_2 \xi^2 + \frac{b_4}{3!} \xi^3 + \frac{b_6}{5!} \xi^5 + \ldots \right) \left[ f_1''' - \left( \frac{4}{3!} + \frac{4}{2!} \right) f_1' f_3' \xi^2 + \left( \frac{6}{4!} \right) f_5' \xi^5 \right.
\]
\[
+ \left( \frac{2}{5!} + \frac{4}{3!} \right) f_5' \xi^5 \right) \xi^6 + \ldots \right) = \frac{1}{a_0 E^2} \left( f_1''' - \frac{4}{3!} f_3'' \xi^3 \right.
\]

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\[
\begin{align*}
+ \frac{6}{5!} f_5''' & \cdot \frac{5}{5} \cdot \frac{8}{7!} f_7'''' \cdot \frac{7}{5} \cdot \cdots \\
(1+ \frac{b_2}{2!} \cdot \frac{2}{2!} + \frac{b_4}{4!} \cdot \frac{4}{2!} + \frac{b_6}{6!} \cdot \frac{6}{2!} + \cdots ) \left[ f_1''^2 \cdot f_1'' + \left( \frac{4}{3!} f_3'' f_1'+ \frac{4}{2!} f_3 f_1'' - \frac{4}{3!} f_1' f_3' \right) \cdot f_5'' + \left( \frac{4}{7!} f_7' f_1'' + \frac{4}{2!} \cdot \frac{2}{3!} \cdot \frac{2}{2!} f_3'' f_1'- \frac{4}{3!} \cdot \frac{2}{2!} \cdot \frac{2}{2!} f_3'' f_3' - \frac{4}{4!} f_3'' f_3' - \frac{4}{3!} f_3 f_1'' \cdot \frac{4}{5!} f_1'' f_7' \right) \cdot f_5'' + \left( \frac{4}{3!} f_3'' f_3' + \frac{4}{2!} f_3 f_3'' + \frac{4}{3!} f_3 f_5'' + \frac{4}{6!} f_7' f_1'' \right) \cdot f_5'' + \left( \frac{4}{2!} f_3'' f_3' + \frac{4}{3!} f_3 f_3'' + \frac{4}{5!} f_3'' f_5'' \right) \cdot f_5'' + \left( \frac{4}{6!} f_7' f_1'' \right) \cdot f_5'' + \left( \frac{4}{7!} f_7' f_1'' \right) \cdot f_5'' + \cdots \right] = \frac{1}{a_0 E^2} \left[ f_1'''' + \frac{1}{2!} f_3'' f_1' + \frac{1}{3!} f_3 f_3'' + \frac{1}{5!} f_5'' f_5'' + \cdots \right] = \frac{1}{a_0 E^2} \left( f_1'''' + \frac{1}{2!} f_3'' f_1' + \frac{1}{3!} f_3 f_3'' + \frac{1}{5!} f_5'' f_5'' + \cdots \right) = \frac{1}{a_0 E^2} \left( f_1'''' + \frac{1}{2!} f_3'' f_1' + \frac{1}{3!} f_3 f_3'' + \frac{1}{5!} f_5'' f_5'' + \cdots \right)
\end{align*}
\]

Now equating the coefficients of like powers of \( f \) we arrive at:

\[
\begin{align*}
f_1''^2 & = \frac{1}{a_0 E^2} f_1'''
\end{align*}
\]
\[- \frac{1}{a_0 E^2} \frac{8}{7!} f_7'' = - \frac{8 \cdot 8}{7!} f_7' f_1'' - \frac{4 \cdot 8 \cdot 8}{5! \cdot 3!} f_5 f_3' + \frac{8}{7!} f_7'' f_1 + \frac{4 \cdot 6}{5! \cdot 2!} f_5'' f_3 + \frac{4 \cdot 6}{3! \cdot 4!} f_3'' f_5 + \frac{8}{6!} f_1'' f_7 + \frac{b_2}{2} \left( \frac{6 \cdot 6}{5!} f_5 f_1' + \frac{4 \cdot 4}{3! \cdot 2!} f_3' f_3 \right) \]

\[- \frac{6}{5!} f_5'' f_1' - \frac{4 \cdot 4}{3! \cdot 2!} f_3'' f_3 - \frac{6}{4!} f_1'' f_5 \] + \frac{b_4}{4} \left( \frac{4}{3!} f_3'' f_1 + \frac{4}{3!} f_3' f_3 \right) \]

\[- \frac{4 \cdot 4}{3!} f_1' f_3' \] + \frac{b_2}{6!} (f_1'' - f_1 f_1''') - b_2 \left( \frac{2 \cdot 6}{5!} f_5 f_1' + \frac{4 \cdot 4}{3! \cdot 3!} f_3' f_3' \right) \]

\[- \frac{2 \cdot 4}{3! \cdot 3!} b_4 f_1' f_3' - \frac{1}{5!} b_6 f_1''^2 \]

\[\frac{1}{a_0 E^2} f_7'' = 8 f_7' f_1' + 168 f_5 f_3' - f_7'' f_1 - 63 f_5' f_3 - 105 f_3'' f_5 - 7 f_1 f_7' \]

\[- \frac{35}{20} b_2 (18 f_5 f_1' + 80 f_3' f_1' - 9 f_5 f_1 - 240 f_3 f_3' - 145 f_1 f_5' - 70 b_4 (f_3'' f_1 + 3 f_3') + 7 f_6 (f_1 f_1' + 5 f_1''^2) \]

Now if we define the operator \( L_n \) by:

\[L_n = \frac{1}{a_0 E^2} \frac{d^3}{d\eta^3} + f_1 \frac{d^2}{d\eta^2} - (n-1) f_1' \frac{d}{d\eta} + n f_1'' \quad n = 3, 5, 7, \ldots \]

Then we get upon collecting the results

\[\frac{1}{a_0 E^2} f_1'' + f_1 f_1'' - f_1''^2 = 0 \]

\[L_3(f_3) = \frac{3b_2}{4} (f_1' f_1' + f_1 f_1'' ) \]

\[L_5(f_5) = \frac{80}{3} (f_3^2 f_3' + f_3 f_3'') + \frac{20}{3} b_2 (f_3'' f_3 + 3 f_3 f_1'') - \frac{5}{6} b_4 (f_1 f_1'' + 3 f_1''^2) \]

\[L_7(f_7) = 21 (18 f_5 f_3' - 3 f_5 f_3 f_3' + 240 f_3 f_3' f_3) - \frac{7}{4} (18 f_5 f_1' + 80 f_3 f_3' - 145 f_1 f_5) \]

\[-3 f_5 f_1' - 5 f_1' f_5 - \frac{35}{2} b_4 (f_3 f_1 + 4 f_1' f_3' + 3 f_3' f_1') + \frac{7}{8} b_6 (f_1 f_1'' + 5 f_1''^2) \]
And finally if we define $H_n$, $n=3,5,7,\ldots$ by:

$$H_3 = 0$$

$$H_5 = \frac{20}{3} \left[ 4(f_3''-2f_3f_3') + b_2(f_3''f_1 + 3f_3f_1') \right]$$

$$H_7 = \frac{7}{4} \left[ 12(8f_5'f_3' - 3f_5''f_3 - 5f_3''f_3') - b_2(18f_5'f_1' \
+ 80(f_3'' - 3f_3f_3'') - 9f_5''f_1' - 45f_1''f_5') - 10b_4(f_3''f_1 \
+ 4f_1'f_3' + 3f_3f_1') \right]$$

We obtain:

$$L_n(f_n) = (-1)^{\frac{n+1}{2}} \frac{n}{n+1} \left[ f_1f_1'' + (n-2)f_1'' \right] b_{n-1} + H_n$$

for $n = 3,5,7,\ldots$. 
APPENDIX IV

REDUCTION OF THE SECOND OF EQUATIONS (23)

Substitution of the expansions (26), (27), and (28) into the
second of Eqs. (25) yields:

\[
\begin{align*}
&\frac{a_0}{2!} \left(1 + \frac{b_2}{2!} \xi^2 + \frac{b_4}{4!} \xi^4 + \frac{b_6}{6!} \xi^6 + \ldots\right) \left[ f_1 \xi^3 - \frac{4}{3} f_3 \xi^3 + \frac{6}{5} f_5 \xi^5 \right. \\
&- \left. \frac{8}{7!} f_7 \xi^7 + \ldots \right] \left( \frac{2}{2!} \frac{F_2}{F_1} + \frac{4}{4!} \frac{F_4}{F_1} + \frac{6}{6!} \frac{F_6}{F_1} \xi^3 + \ldots \right) - \left( F_0 + \frac{1}{2!} \frac{F_2}{F_1} \xi^2 \right. \\
&\left. + \frac{1}{4!} \frac{F_4}{F_1} \xi^4 + \ldots \right) \\
&\left. \left( f_1 - \frac{4}{3} f_3 + \frac{6}{5} f_5 - \frac{8}{7!} f_7 \xi^7 \xi^8 + \ldots \right) \right] = \frac{1}{F_1 P E^2} \left( r_0'' + \frac{1}{2} f_2'' \xi^2 + \frac{1}{4!} f_4'' \xi^4 + \frac{1}{6!} f_6'' \xi^6 + \ldots \right)
\end{align*}
\]

Collecting terms and rearranging gives:

\[
\begin{align*}
\frac{a_0}{2!} \left(1 + \frac{b_2}{2!} \xi^2 + \frac{b_4}{4!} \xi^4 + \frac{b_6}{6!} \xi^6 + \ldots\right) \left[ -f_1 F_0' + \left( F_2 f_1' + \frac{4}{2!} f_3 F_0' \right. \\
&\left. - \frac{1}{2!} F_2 f_1' \xi^2 + \left( \frac{1}{3!} \frac{f_1'}{f_1} \frac{F_4}{F_1} - \frac{4}{3!} f_3' F_2^2 - \frac{1}{4!} f_1 F_4' + \frac{4}{2!} f_3 F_2' \right. \right. \\
&\left. \left. - \frac{6}{5!} \frac{f_5 F_0'}{f_5} \xi^4 + \left( \frac{1}{5!} f_1 F_6 - \frac{4}{3!} f_3 F_4^2 + \frac{6}{5!} f_5 F_2^2 - \frac{1}{6!} f_1 F_8' \right. \right. \\
&\left. \left. + \frac{4}{2!} \frac{f_3 F_4'}{f_3} - \frac{6}{5!} \frac{f_5 F_2'}{f_5} + \frac{8}{6!} \frac{f_7 F_0'}{f_7} \right) \xi^6 + \ldots \right] \right] = \frac{1}{F_1 P E^2} \left( F_0'' \\
&+ \frac{1}{2!} F_2'' \xi^2 + \frac{1}{4!} F_4'' \xi^4 + \frac{1}{6!} F_6'' \xi^6 + \ldots \right)
\end{align*}
\]

Upon equating the coefficients of like powers of \( \xi \) one obtains:

\[- f_1 F_0' = \frac{1}{a_0 P E^2} F_0'' \]
\[
\begin{align*}
\frac{1}{a_0^2 p_1^2 E^2} \left( \frac{1}{2} f_{1F_2}^2 - \frac{b_2}{2} f_{1F_0}^2 \right) & = \\
\frac{1}{a_0^2 p_1^2 E^2} F_2'' & = 4 f_{3F_0}^2 - b_2 f_{1F_0}^2 \\
\frac{1}{6} f_{1F_4}^2 & \frac{2}{3} f_{3F_2} - \frac{1}{4!} f_{1F_4}^2 + f_{3F_2}^2 - \frac{1}{4} f_{5F_0}^2 + \\
& \frac{b_2}{2} (2 f_{1F_1} + 2 f_{3F_0}^2 - \frac{1}{2} f_{1F_2}^2) + \frac{b_4}{4!} f_{1F_0}^2 \\
& = \frac{1}{a_0^2 p_1^2 E^2} \frac{1}{4!} F_4'' \\
\frac{1}{a_0^2 p_1^2 E^2} F_4'' - 4 f_{1F_4}^2 + f_{1F_4}^2 & = \\
24 f_{3F_2}^2 - 16 f_{3F_2} - 6 f_{5F_0}^2 + 6 b_2 (2 f_{1F_2}^2 - f_{1F_2}^2 + 4 f_{3F_0}^2) - b_4 f_{1F_0}^2 \\
\frac{1}{5!} f_{1F_8}^2 - \frac{1}{3!} f_{3F_4}^2 + 6 f_{5F_2}^2 - \frac{1}{6!} f_{1F_6}^2 + \\
\frac{b_2}{2} (\frac{1}{5!} f_{1F_4}^2 - \frac{1}{3!} f_{3F_2}^2 - \frac{1}{4!} f_{1F_4}^2 + f_{3F_2}^2 - \\
\frac{6}{4!} f_{5F_0}^2) + \frac{b_4}{4!} (2 f_{1F_1} + 2 f_{3F_0}^2 - \frac{1}{2} f_{1F_2}^2) \\
- \frac{b_2}{6!} f_{1F_0}^2 & = \frac{1}{a_0^2 p_1^2 E^2} \frac{1}{6!} F_8'' \\
\frac{1}{a_0^2 p_1^2 E^2} F_8'' - 6 f_{1F_8} + f_{1F_6} & = 60 f_{3F_4}^2 \\
- 90 f_{5F_2}^2 + 8 f_{7F_0}^2 - 80 f_{3F_4} + 36 f_{5F_2}^2 + \\
+ 15 b_2 (4 f_{1F_4} - f_{1F_4}^2 - 16 f_{3F_2}^2 + 2 f_{3F_2}^2) 
\end{align*}
\]
\(-6f_5 F_0') + 15b_4 (2f_1' F_2 - f_1 F_2' + 4f_3 F_0') - \frac{b_6 f_1 F_0'}{2}

Now if the operator K\(_n\) is defined by:

\[
K_n = \frac{1}{a_0 \rho E} \frac{d^2}{d \eta^2} + f_1 \frac{d}{d \eta} - n f_1'; \ n = 0, 2, 4, \ldots
\]

Then one obtains upon collecting the results:

\[
K_0(F_0) = 0
\]

\[
K_2(F_2) = (4f_3 - b_2 f_1) F_0'
\]

\[
K_4(F_4) = 8(3f_3 F_2' - 2f_3' F_2) + 6b_2 (2f_1' F_2 - f_1 F_2') - (6f_5 - 24b_2 f_3 + 6b_4 f_1) F_0'
\]

\[
K_6(F_6) = 20(3f_3 F_4' - 4f_3' F_4) - 18(f_5 F_2' - 2f_5' F_2) + 15b_2 (4f_1' F_4 - f_1 F_4' - 16f_3' F_2 + 24f_3 F_2') + 15b_4 (2f_1' F_2 - f_1 F_2') + (8f_7 - 90b_2 f_5 + 60b_4 f_3 - b_6 f_1) F_0'
\]

And finally if G\(_n\); \ n = 0, 2, 4, 6, \ldots is defined by:

\[
G_0 = 0
\]

\[
G_2 = (4f_3 - b_2 f_1) F_0'
\]

\[
G_4 = 8(3f_3 F_2' - 2f_3' F_2) + 6b_2 (2f_1' F_2 - f_1 F_2') - (6f_5 - 24b_2 f_3 + b_4 f_1) F_0'
\]

\[
G_6 = 20(3f_3 F_4' - 4f_3' F_4) - 18(5f_5 F_2' - 2f_5' F_2)
\]
+15b_{2}(4f_{1}'F_{4}-f_{1}F_{4}'-16f_{3}'F_{2}+24f_{3}F_{2}')+15b_{2}(2f_{1}'F_{2}-f_{1}F_{2}')+(8f_{7}+90b_{2}f_{5}+60b_{4}f_{3}-b_{8}f_{1})F_{0}^{'}

\hline
\hline
\hline

One obtains:

\[ K_{n}(F_{n}) = c_{n}; \text{ for } n=0,2,4,6,\ldots \]
APPENDIX V

REDUCTION OF THE FIRST OF EQUATIONS (24)

Substitution of the expansions (26) and (27) into the first of Eqs. (24) and using the relations:

\[
\sin \xi = \xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} - \frac{\xi^7}{7!} + \ldots
\]

\[
\cos \xi = 1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots
\]

yields (if we denote in this appendix \(f_1(l)\) by \(f_1\); \(f_1'(l)\) by \(f_1'\); \(f_3'(l)\) by \(f_3'\) etc.):

\[
f_1'' \xi - \frac{4}{3!} f_3' \xi^3 + \frac{6}{5!} f_5'' \xi^5 - \frac{8}{7!} f_7'' \xi^7 + \ldots
\]

\[
= a_0 B^2 \left(1 + \frac{b_2}{2!} \xi^2 + \frac{b_4}{4!} \xi^4 + \frac{b_6}{6!} \xi^6 + \ldots\right) \left[2a_0 \left(1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots\right)
\right.

\[
+ \left. \left(\xi^3 - \frac{\xi^5}{5!} - \frac{7}{7!} \xi^7 + \ldots\right) \left(1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots\right)\right]
\]

\[
- \left(f_1' \xi - \frac{4}{3!} f_3' \xi^3 + \frac{6}{5!} f_5' \xi^5 - \frac{8}{7!} f_7' \xi^7 + \ldots\right)\left(1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots\right)
\]

\[
= a_0 B^2 \left[2a_0 \left(\xi + \frac{b_2}{2!} \xi^2 - \frac{1}{3!} \right) \xi^3 + \left[\frac{1}{5!} - \frac{b_2}{2! \cdot 3!} + \frac{b_4}{4!}\right] \xi^5 + \left[-\frac{1}{7!} + \frac{b_2}{2! \cdot 5!}\right]
\]

\[
- \left[\frac{b_4}{4! \cdot 3!} + \frac{b_6}{6!}\right] \xi^7 + \ldots\right) - \left(f_1' \xi - \frac{4}{3!} f_3' \xi^3 + \frac{6}{5!} f_5' \xi^5 - \frac{8}{7!} f_7' \xi^7 + \ldots\right)\right]
\]

\[
\left\{1 + \frac{1}{2!} (b_2 - l) \xi^2 + \frac{b_4}{4!} \xi^4 - \frac{b_6}{6!} \xi^6 + \left(\frac{b_6}{6!} - \frac{1}{2! \cdot 4!} + \frac{b_2}{2! \cdot 4!}\right) \xi^8 + \ldots\right\}
\]

\[
= a_0 B^2 \left[(2a_0 - f_1') \xi + \left[2a_0 \left(b_2 - \frac{1}{3!}\right) \right] \xi^3 + \left[2a_0 \left(\frac{1}{5!} - \frac{b_2}{2! \cdot 3!}\right) \right] \xi^5 + \ldots\right]
\]

\[
= a_0 B^2 \left[(2a_0 - f_1') \xi + \left[2a_0 \left(b_2 - \frac{1}{3!}\right) \right] \xi^3 + \left[2a_0 \left(\frac{1}{5!} - \frac{b_2}{2! \cdot 3!}\right) \right] \xi^5 + \ldots\right]
\]

\[
= a_0 B^2 \left[(2a_0 - f_1') \xi + \left[2a_0 \left(b_2 - \frac{1}{3!}\right) \right] \xi^3 + \left[2a_0 \left(\frac{1}{5!} - \frac{b_2}{2! \cdot 3!}\right) \right] \xi^5 + \ldots\right]
\]

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\[+ \frac{b_4}{4!} \bigg] + \frac{6}{5!} f_5' \bigg] \xi^5 \bigg[ 2a_0 \bigg( \frac{-1}{7!} + \frac{b_2}{2! \cdot 5!} - \frac{b_4}{4! \cdot 3!} + \frac{b_6}{6!} \bigg) + \frac{8}{7!} \xi^7 \bigg] \xi^7 + \ldots \bigg] \left\{ 1 + \frac{1}{2!} \xi^2 + \frac{1}{4!} \left( \frac{1}{5} - \frac{b_4}{2! \cdot 2!} \right) \xi^4 \left( \frac{b_6}{6!} - \frac{1}{6!} - \frac{b_4}{2! \cdot 4!} \right) \xi^6 + \ldots \right\} = a_0 E^2 \left[ [2a_0 - f_1'] \xi^5 + \frac{1}{3!} \left[ 2a_0 (3b_2 - 1) + 4f_3' \right] \xi^3 \right. \]

\[+ \frac{1}{5!} \left[ 2a_0 (1 - 10b_2 + 5b_4) - 6f_5' \right] \xi^5 + \frac{1}{7!} \left[ 2a_0 (-1 + 21b_2 - 35b_4 + 7b_6) + 8f_7' \right] \xi^7 \]

\[+ \ldots \bigg] \left\{ 1 + \frac{1}{2!} \left( b_2 - 1 \right) \xi^2 + \frac{1}{4!} \left( b_4 + 1 - b_6 \right) \xi^4 + \frac{1}{6!} \left( b_6 - 1 - 15b_4 + 15b_2 \right) \xi^6 + \ldots \right\} = a_0 E^2 \left[ [2a_0 - f_1'] \xi^5 + \frac{2}{3!} \left[ 3b_2 a_0 + (2f_3' - a_0) \right] \xi^3 + \frac{2}{5!} \left[ 5a_0 (b_4 - 2b_2) - (3f_5' - a_0) \right] \xi^5 + \frac{2}{7!} \left[ 7a_0 (3b_2 - 5b_4 + b_6) + (4f_7' - a_0) \right] \xi^7 + \ldots \right] \left\{ 1 + \frac{1}{2!} \left( b_2 - 1 \right) \xi^2 + \frac{1}{4!} \left( b_4 + 1 - b_6 \right) \xi^4 + \frac{1}{6!} \left( b_6 - 1 - 15b_4 + 15b_2 \right) \xi^6 + \ldots \right\} \]

\[= f_1'' \xi^5 - \frac{4}{3!} f_3'' \xi^3 + \frac{6}{5!} f_5'' \xi^5 - \frac{8}{7!} f_7'' \xi^7 + \ldots \]

Equating coefficients of like powers of \( \xi \) one obtains:

\[f_1'' = a_0 E^2 [2a_0 - f_1']\]

\[- \frac{4}{3!} f_3'' = a_0 E^2 \left[ \frac{2}{3!} [3b_2 a_0 + (2f_3' - a_0)] + \frac{1}{2!} (b_2 - 1) [2a_0 - f_1'] \right]\]

\[f_3'' = - \frac{a_0 E^2}{2} \left[ 3b_2 a_0 + (2f_3' - a_0) + \frac{3}{2} (b_2 - 1)(2a_0 - f_1') \right]\]

\[\frac{6}{5!} f_5'' = a_0 E^2 \left[ \frac{2}{5!} [5a_0 (b_4 - 2b_2) - (3f_5' - a_0)] + \frac{1}{3!} (b_2 - 1) [3b_2 a_0 + (2f_3' - a_0)] \right. \]

\[+ \frac{1}{4!} (2a_0 - f_1')(b_4 + 1 - 6b_2) \right\]
\[ f_5'' = \frac{a_0 E^2}{2} \left\{ \frac{2}{3} \left[ 5a_0(b_4-2b_2)-(3f_5'-a_0) \right] + \frac{29}{3} (b_2-1)[3b_8a_0+(2f_3'-a_0)] + \frac{5}{3} (2a_0-f_1')(b_4+1-6b_2) \right\} \]

\[ - \frac{8}{7!} f_7'' = \frac{a_0 E^2}{2} \left\{ \frac{2}{7!} \left[ 7a_0(3b_2-5b_4+b_8)+(4f_7'-a_0) \right] + \frac{1}{5!} (b_2-1)[5a_0(b_4-2b_2)-(3f_5'-a_0)] + \frac{2}{4!} (b_4+1-6b_2)[3b_8a_0+(2f_3'-a_0)] + \frac{1}{6!} (b_8-1-15b_4) \left[ 2a_0-f_1' \right] \right\} \]

\[ f_7'' = -\frac{a_0 E^2}{2} \left\{ \frac{1}{2} \left[ 7a_0(3b_2-5b_4+b_8)+(4f_7'-a_0) \right] + \frac{21}{2} (b_2-1)[5a_0(b_4-2b_2)+a_0-3f_5'] + \frac{35}{2} (b_4-6b_2+1)[a_0(3b_2-1)+2f_3'] + \frac{7}{4} (b_8-1-15b_4) \left[ 2a_0-f_1' \right] \right\} \]

\[ \frac{f_7''}{1/2a_0E^2} + 2f_7' + \frac{7}{4} (4a_0-f_1')b_8 - \frac{7}{4} (2a_0-f_1') + \frac{105}{4} (b_2-b_4)(2a_0-f_1') \]

\[ + \frac{1}{2} a_0(2b_2-35b_4+1) + \frac{21}{2} (b_2-1)[a_0(5b_4-10b_2+1)-3f_5'] + \frac{35}{2} (b_4-6b_2 \]

\[ + 1)[a_0(3b_2-1)+2f_3'] \]

Collecting the results and simplifying yields:

\[ \frac{f_1''}{1/2a_0E^2} + 2f_1'-4a_0 = 0 \]

\[ \frac{f_3''}{1/2a_0E^2} + 2f_3' - \frac{3}{2} (4a_0-f_1')b_2 - \frac{3}{2} (2a_0-f_1')a_0 = 0 \]
\[
\frac{f_5''}{1/2a_0E^2} + 2f_5' - \frac{5}{3} (4a_0-f_1')b_4 - \frac{5}{3} (2a_0-f_1') + 10b_2(2a_0-f_1') + \frac{2}{3} a_0(10b_2-1)
- \frac{20}{3} (b_2-1)(3b_2a_0 + (2f_3'-a_0)) = 0
\]

\[
\frac{f_7''}{1/2a_0E^2} + 2f_7' + \frac{7}{4} (4a_0-f_1')b_4 - \frac{7}{4} (2a_0-f_1') + \frac{105}{4} (b_2-b_4)(2a_0-f_1')
+ \frac{1}{2} a_0(21b_2-35b_4-1)
+ \frac{21}{2} (b_2-1)(a_0(5b_4-10b_2+1)-3f_5') + \frac{35}{2} (b_4-6b_2+1)[a_0(3b_2-1)+2f_3'] = 0
\]

If we define:

\[
A_3 = -a_0
\]

\[
A_5 = \frac{2}{3} \{15b_2[2a_0-f_1'] + a_0(10b_2-1) + 10(b_2-1)(3b_2a_0 + 2f_3'-a_0)\}
\]

\[
A_7 = \frac{1}{4} \{105(b_2-b_4)[2a_0-f_1'] + 2a_0(21b_2-35b_4-1) + 42(b_2-1)[a_0(5b_4-10b_2+1)-3f_5'] + 70(b_4-6b_2+1)[a_0(3b_2-1)+2f_3']\}
\]

Then we have:

\[
\frac{2}{a_0E^2} f_1'' + 2f_1' - 4a_0 = 0
\]
\[ \frac{2}{a_0 b_0} f_0'' + 2 f_0' + \frac{2n}{n+1} \left\{ \frac{n+1}{2} \left[ (-1)^n b_{n-1} [4e_0 - f_1'] - 2e_0 + f_1' \right] \right\} \]

\[ + A_n \text{ for } n = 3, 5, 7, \ldots \]
APPENDIX VI

REDUCTION OF THE THIRD OF EQUATIONS (24)

Noting that:
\[
\cos \xi = 1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots
\]

and substitution of the expansions (27) and (28) into the third of Equations (24) yields:

\[
a_0 \left( F_0 + \frac{F_2}{2!} \xi^2 + \frac{F_4}{4!} \xi^4 + \frac{F_6}{6!} \xi^6 + \ldots \right) \left( 1 + \frac{b_2}{2!} \xi^2 + \frac{b_4}{4!} \xi^4 + \frac{b_6}{6!} \xi^6 + \ldots \right)
\]

\[
\left( 1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots \right) = - \frac{1}{a_0 E^2 Pr} \left( F_0' + \frac{F_2'}{2!} \xi^2 + \frac{F_4'}{4!} \xi^4 + \frac{F_6'}{6!} \xi^6 + \ldots \right)
\]

where we have written \( F_0(1) \) as \( F_0 \); \( F_0'(1) \) as \( F_0' \) etc.

Combining terms and simplifying we obtain:

\[
\left[ F_0 + \frac{1}{2!} (F_2 + F_0 b_2) \xi^2 + \frac{1}{4!} \left( F_4 + \frac{4!}{2!} F_2 b_2 + F_0 b_4 \right) \xi^4 \right. \\
\left. + \frac{1}{6!} \left( F_6 + \frac{6!}{4!} F_2 b_2 + \frac{6!}{4!} F_4 b_4 + F_0 b_6 \right) \xi^6 + \ldots \right] \left( 1 - \frac{\xi^2}{2!} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} + \ldots \right)
\]

\[
= - \frac{1}{a_0 E^2 Pr} \left( F_0' + \frac{F_2'}{2!} \xi^2 + \frac{F_4'}{4!} \xi^4 + \frac{F_6'}{6!} \xi^6 + \ldots \right)
\]

Equating coefficients of like powers of \( \xi \) yields:

\[
- \frac{1}{a_0 E^2 Pr} F_0' = F_0
\]

\[
- \frac{1}{a_0 E^2 Pr} F_2' \frac{1}{2!} = \frac{1}{2!} (F_2 + F_0 b_2) - \frac{1}{2!} F_0
\]

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\[ \frac{1}{a_0E^2_{Fr}} \quad F_2' = F_2 + F_{0} b_2 - F_0 \]

\[ \frac{1}{a_0E^2_{Fr}} \quad F_4' \quad \frac{1}{4!} = \frac{1}{4!} \left( F_4 + 6F_2 b_2 + F_0 b_4 \right) - \frac{1}{2!2!} \left( F_2 + F_0 b_2 \right) + \frac{1}{4!} F_0 \]

\[ \frac{1}{a_0E^2_{Fr}} \quad F_4' = F_4 + F_0 b_4 + 6F_2 b_2 - 6(F_2 + F_0 b_2) + F_0 \]

\[ \frac{1}{a_0E^2_{Fr}} \quad F_6' \quad \frac{1}{6!} = \frac{1}{6!} \left( F_6 + 15b_2 F_4 + 15F_2 b_4 + F_0 b_6 \right) - \frac{1}{2!4!} \left( F_4 + 6F_2 b_2 + F_0 b_4 \right) + \frac{1}{2!4!} \left( F_2 + F_0 b_2 \right) - \frac{1}{6!} F_0 \]

\[ \frac{1}{a_0E^2_{Fr}} \quad F_6' = F_6 + F_0 b_6 - F_0 + 15(b_2 F_4 + F_0 b_4 - F_4 - 6F_2 b_2 - F_0 b_4 + F_2 + F_0 b_2) \]

Now if we define:

\[ B_2 = 0 \]

\[ B_4 = 6 \left[ b_2 F_2 - F_2 - b_2 F_0 \right] \]

\[ B_6 = 15 \left[ b_2 F_4 + b_4 F_2 - F_4 - 6 b_2 F_2 - b_4 F_0 + F_2 + b_2 F_0 \right] \]

We get upon collecting results:

\[ \frac{1}{a_0E^2_{Fr}} \quad F_0' + F_0 = 0 \]

\[ \frac{1}{a_0E^2_{Fr}} \quad F_n' + F_n + F_0 \left[ b_n + (-1)^\frac{n}{2} \right] + B_n \quad n = 2, 4, 6, \ldots \]
APPENDIX VII

REDUCTION OF \( \frac{u}{U_\infty} = (28)^{-1} \frac{\partial \Psi^*}{\partial \eta} \)

Substitution of the expansions (26) and (27) into this equation yields:

\[
\frac{u}{U_\infty} = \frac{1}{2} \left[ f_1' - \frac{4}{3!} f_3' \xi^3 + \frac{6}{5!} f_5' \xi^5 - \frac{8}{7!} f_7' \xi^7 + \ldots \right] \times \\
\frac{1}{a_0} \left[ 1 + \frac{b_2}{2!} \xi^2 + \frac{b_4}{4!} \xi^4 + \frac{b_6}{6!} \xi^6 + \ldots \right]^{-1}
\]

\[
= \frac{1}{2a_0} \left[ f_1' \xi^2 - \frac{4}{3!} f_3' \xi^3 + \frac{6}{5!} f_5' \xi^5 - \frac{8}{7!} f_7' \xi^7 + \ldots \right] \times \\
x \left[ 1 - \frac{b_2}{2!} \xi^2 - \frac{b_4}{4!} \xi^4 - \frac{b_6}{6!} \xi^6 + \left( \frac{b_2}{2!} \xi^2 + \frac{b_4}{4!} \xi^4 \right)^2 \\
- \left( \frac{b_2}{2!} \xi^2 \right)^3 + \ldots \right]
\]

\[
= \frac{1}{2a_0} \left[ f_1' \xi^2 - \frac{4}{3!} f_3' \xi^3 + \frac{6}{5!} f_5' \xi^5 - \frac{8}{7!} f_7' \xi^7 + \ldots \right] \times \\
x \left[ 1 - \frac{b_2}{2!} \xi^2 + \left( \frac{b_2}{2!} \xi^2 \right) - \frac{b_4}{4!} \xi^4 \right] - \left( \frac{2b_2 b_4}{2! \cdot 4!} \xi^6 + \frac{b_6}{6!} \xi^6 + \ldots \right)
\]

\[
= \frac{1}{2a_0} \left[ f_1' \xi^2 - \frac{4}{3!} f_3' \xi^3 + \frac{6}{5!} f_5' \xi^5 - \frac{8}{7!} f_7' \xi^7 + \ldots \right] \left[ 1 - \frac{b_2}{2!} \xi^2 \\
+ \frac{1}{4!} (6b_2 - b_4) \xi^4 - \frac{1}{6!} (9b_2^2 - 15b_2 b_4 + b_6) \xi^6 + \ldots \right]
\]

If we now set:

\[
\frac{u}{U_\infty} = u_1 \xi - \frac{u_3}{3!} \xi^3 + \frac{u_5}{5!} \xi^5 - \frac{u_7}{7!} \xi^7 + \ldots
\]

We get upon equating coefficients of like powers of \( \xi \):

\[
u_1 = \frac{1}{2a_0} \ f_1'
\]
\[ \frac{1}{3!} u_3 = \frac{1}{2a_0} \left[ \frac{b_2}{2!} f_1' + \frac{4}{3!} f_3' \right] = \frac{1}{2a_0} \cdot \frac{1}{3!} (3b_2 f_1' + 4f_3') \]

\[ u_3 = \frac{1}{2a_0} (4f_3' + 3b_2 f_1') \]

\[ \frac{u_5}{5!} = \frac{1}{2a_0} \left[ \frac{f_1'}{4!} (6b_2\frac{b_4}{2} + \frac{4}{3! \cdot 2!} b_2 f_3' + \frac{6}{5!} f_5') \right] \]

\[ u_5 = \frac{1}{2a_0} [6f_5' + 4b_2 f_3' + 5(6b_2\frac{b_4}{2}) f_1'] \]

\[ \frac{u_7}{7!} = \frac{1}{2a_0} \left[ \frac{f_1'}{6!} (90b_2 - 15b_2 b_4 + b_6) + \frac{4}{3! \cdot 2!} f_3' (3b_2 - b_4) + \frac{6b_2}{2! \cdot 5!} f_5' b_2 + \frac{8}{7!} f_7' \right] \]

\[ u_7 = \frac{1}{2a_0} [8f_7' + 12b_2 f_5' + 140(6b_2 - b_4) f_3' + 7(90b_2 - 15b_2 b_4 + b_6) f_1'] \]

After collecting the results we get:

\[ \frac{u}{u_0} = u_1\frac{3}{3!} + u_3\frac{5}{5!} + u_5\frac{5}{7!} + u_7\frac{7}{7!} + \ldots \]

\[ u_1 = \frac{1}{2a_0} f_1' \]

\[ u_3 = \frac{1}{2a_0} (4f_3' + 3b_2 f_1') \]

\[ u_5 = \frac{1}{2a_0} [6f_5' + 4b_2 f_3' + 5(6b_2\frac{b_4}{2}) f_1'] \]

\[ u_7 = \frac{1}{2a_0} [8f_7' + 12b_2 f_5' + 140(6b_2 - b_4) f_3' + 7(90b_2 - 15b_2 b_4 + b_6) f_1'] \]
APPENDIX VIII

REDUCTION OF \( \tau_w \sqrt{\frac{2U_0 R_0}{\nu}} \left/ \frac{1}{\sqrt{2\rho U_0}} \right. \) = \( \frac{1}{\frac{\varepsilon_2^2 \varepsilon_2}{2}} \) \( \frac{\delta^2 \psi}{\partial \eta^2} \)

Substituting the expansions (26) and (27) into this relation yields:

\[
\frac{2U_0 R_0}{\nu \left/ \frac{1}{\sqrt{2\rho U_0}} \right.} = \frac{1}{a_0^2 E} \left[ f_1'' \xi^2 - \frac{4}{3!} f_3'' \xi^3 + \frac{6}{5!} f_5'' \xi^5 - \frac{8}{7!} f_7'' \xi^7 + \ldots \right] \times
\left[ 1 + \frac{b_2}{2!} \frac{\xi^2}{\xi^4} + \frac{b_4}{4!} \frac{\xi^4}{\xi^6} + \ldots \right]^{-2}
\]

\[
= \frac{1}{a_0^2 E} \left[ f_1'' \xi^2 - \frac{4}{3!} f_3'' \xi^3 + \frac{6}{5!} f_5'' \xi^5 - \frac{8}{7!} f_7'' \xi^7 + \ldots \right] \times
\left[ 1 - \frac{2b_2}{2!} \frac{\xi^2}{\xi^4} - \frac{2b_4}{4!} \frac{\xi^4}{\xi^6} - \frac{2b_6}{6!} \frac{\xi^6}{\xi^8} + 3 \left( \frac{b_2}{2!} \frac{\xi^2}{\xi^4} + \frac{b_4}{4!} \frac{\xi^4}{\xi^6} \right)^2 \right.
\left. - \frac{b_2}{2!} \frac{\xi^2}{\xi^4} \right] \left[ 1 + \frac{b_2}{2!} \frac{\xi^2}{\xi^4} \right] \left[ 1 - \frac{2b_2}{2!} \frac{\xi^2}{\xi^4} \right]^2
\]

\[
= \frac{1}{a_0^2 E} \left[ f_1'' \xi^2 - \frac{4}{3!} f_3'' \xi^3 + \frac{6}{5!} f_5'' \xi^5 - \frac{8}{7!} f_7'' \xi^7 + \ldots \right] \left[ 1 - \frac{2b_2}{2!} \frac{\xi^2}{\xi^4} \right. \\
\left. + \frac{3b_2^2}{2!} - \frac{2b_4}{4!} \frac{\xi^4}{\xi^6} - \frac{6b_2 b_4}{2! 4!} \frac{\xi^6}{\xi^8} + \frac{2b_6}{6!} \frac{\xi^6}{\xi^8} \right] \left[ 1 - \frac{2b_2}{2!} \frac{\xi^2}{\xi^4} \right]^2
\]

\[
= \frac{1}{a_0^2 E} \left[ f_1'' \xi^2 - \frac{4}{3!} f_3'' \xi^3 + \frac{6}{5!} f_5'' \xi^5 - \frac{8}{7!} f_7'' \xi^7 + \ldots \right] \left[ 1 - \frac{2b_2}{2!} \frac{\xi^2}{\xi^4} \right. \\
\left. + \frac{2}{4!} (9b_2 - b_4) \frac{\xi^4}{\xi^6} - \frac{2}{6!} (18b_2 - 45b_2 b_4 + b_6) \frac{\xi^6}{\xi^8} \right] \left[ 1 - \frac{2b_2}{2!} \frac{\xi^2}{\xi^4} \right]^2
\]

Now setting:

\[
\tau_w \sqrt{\frac{2U_0 R_0}{\nu}} \left/ \frac{1}{\sqrt{2\rho U_0}} \right. = \tau_1 \xi^2 - \frac{1}{3!} \tau_3 \xi^3 + \frac{1}{5!} \tau_5 \xi^5 - \frac{1}{7!} \tau_7 \xi^7 + \ldots
\]

Upon collecting coefficients of like powers of \( \xi \) one obtains:

\[
\tau_1 = \frac{1}{a_0^2 E} f_1''
\]

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\[
\tau_3 = \frac{1}{\frac{4}{3!} f_3'' + \frac{2b_2f_1''}{2!}} \cdot \frac{1}{a_0^2E} \cdot \frac{1}{a_0^2E} \cdot \frac{1}{a_0^2E} (2f_3'' + 3b_2f_1'')
\]

\[
\tau_5 = \frac{1}{a_0^2E} \left[ \frac{6}{5!} f_5'' + \frac{2 \cdot 4}{2! \cdot 3!} b_2f_3'' + \frac{2}{4!} (9b_2^2 - b_4) f_1'' \right]
\]

\[
\tau_7 = \frac{1}{a_0^2E} \left[ \frac{8}{7!} f_7'' + \frac{2 \cdot 6}{2! \cdot 5!} b_2f_5'' + \frac{4 \cdot 2}{3! \cdot 4!} (9b_2^2 - b_4) f_3'' + \frac{2}{6!} (180b_2^3 - 45b_2b_4 + b_6) f_1'' \right]
\]

\[
\tau_7 = \frac{1}{a_0^2E} \left[ 8f_7'' + 252b_2f_5'' + 280(9b_2^2 - b_4) f_3'' + 14(180b_2^3 - 45b_2b_4 + b_6) f_1'' \right]
\]

And collecting the results one has:

\[
\tau_1 = \frac{1}{2a_0 E} f_1''
\]

\[
\tau_3 = \frac{2}{a_0^2E} [2f_3'' + 3b_2f_1'']
\]

\[
\tau_5 = \frac{2}{a_0^2E} [3f_5'' + 40b_2f_3'' + 5(9b_2^2 - b_4)f_1'']
\]

\[
\tau_7 = \frac{2}{a_0^2E} [4f_7'' + 126b_2f_5'' + 140(9b_2^2 - b_4)f_3'' + 7(180b_2^3 - 45b_2b_4 + b_6)f_1'']
\]

\[
\sum_{n=3}^{\infty} \sum_{m=3}^{\infty} \sum_{k=3}^{\infty} ...
\]
APPENDIX IX

\[ \frac{\text{Nu}}{\sqrt{\frac{2u_0 R_0}{\nu}}} = \frac{1}{\frac{\partial \theta^*}{\partial \eta}} \]

Substituting the expansions (27) and (28) into this relation yields:

\[ \frac{\text{Nu}}{\sqrt{\frac{2u_0 R_0}{\nu}}} = \frac{1}{a_0 E} \left[ F'_0 + \frac{F'_2}{2!} \xi^2 + \frac{F'_4}{4!} \xi^4 + \frac{F'_6}{6!} \xi^6 + \ldots \right] \left[ 1 - \frac{b_2}{2!} \xi^2 \right.
\n\n\n+ \frac{1}{4!} (6b_2^2 - b_4) \xi^4 - \frac{1}{6!} (90b_2^3 - 15b_2b_4 + b_6) \xi^6 + \ldots \left] \right.
\]

Now setting:

\[ \frac{\text{Nu}}{\sqrt{\frac{2u_0 R_0}{\nu}}} = q_0 + \frac{q_2}{2!} \xi^2 + \frac{q_4}{4!} \xi^4 + \frac{q_6}{6!} \xi^6 + \ldots \]

We get upon equating coefficients of like powers of \( \xi \):

\[ q_0 = \frac{F'_0}{a_0 E} \]
\[ q_2 = \left[ \frac{F'_2}{2!} - \frac{b_2 F'_0}{2!} \right] \frac{1}{a_0 E} \]
\[ q_4 = \left[ \frac{F'_4}{4!} - \frac{b_2 F'_2}{2!} + \frac{1}{4!} (6b_2^2 - b_4) F'_0 \right] \frac{1}{a_0 E} \]
\[ q_6 = \left[ F'_6 - 6b_2 F'_4 + (6b_2^2 - b_4) F'_0 \right] \frac{1}{a_0 E} \]

\[ \frac{q_6}{6!} = \frac{1}{E a_0} \left[ \frac{F'_6}{6!} - \frac{b_2 F'_4}{2! \cdot 4!} + \frac{F'_2 (6b_2^2 - b_4)}{2! \cdot 4!} - \frac{1}{6!} (90b_2^3 - 15b_2b_4 + b_6) F'_0 \right] \]
\[ q_6 = \frac{1}{E_a} \left[ f_3' - 15b_2F_2' + 15(6b_2^2 - b_4)F_2' - (9b_2^3 - 15b_2b_4 + b_6)F_0' \right] \]

And upon collecting the results we have:

\[ q_0 = \frac{1}{a_0E} F_0' \]

\[ q_2 = \frac{1}{a_0E} \{ F_2' - b_2F_0' \} \]

\[ q_4 = \frac{1}{a_0E} \{ F_4' - 6b_2F_2' + (6b_2^2 - b_4)F_0' \} \]

\[ q_6 = \frac{1}{a_0E} \{ F_6' - 15b_2F_4' + 15(6b_2^2 - b_4)F_2' - (9b_2^3 - 15b_2b_4 + b_6)F_0' \} \]
APPENDIX X

REDUCTION OF THE ENERGY EQUATION FOR SMALL $E$ TO A SERIES OF
ORDINARY DIFFERENTIAL EQUATIONS

Substitution of the expansion (52) and (53) into Eq. (49) gives:

$$
\sum_{n=0}^{\infty} \frac{F_{2n}}{P_{1,E}(2n)!} \xi^{2n} = 2\eta \sum_{n=1}^{\infty} \frac{F_{2n}}{(2n-1)!} \xi^{2n-1} \sum_{m=1}^{\infty} \frac{\beta_{2m}}{(2m-1)!} \xi^{2m-1}
$$

$$
- \eta^2 \sum_{n=0}^{\infty} \frac{F_{2n}}{(2n)!} \xi^{2n} \sum_{m=0}^{\infty} \frac{\alpha_{2m}}{(2m)!} \xi^{2m}
$$

Next rearranging the sums on the right, we get:

$$
\sum_{k=0}^{\infty} \frac{F_{2k}}{P_{1,E}(2k)!} \xi^{2k} = 2\eta \sum_{k=2}^{\infty} \frac{\beta_{2k} \xi^{2k-2}}{(2n-1)!}(2k-2n-1)! 
$$

$$
- \eta^2 \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{F_{2n} \alpha_{2(k-n)}}{(2n)!}(2k-2n)! \xi^{2k}
$$

$$
= 2\eta \sum_{k=1}^{\infty} \left[ \sum_{n=1}^{k} \frac{F_{2n} \beta_{2(k-n+1)}}{(2n-1)!}(2k-2n+1)! \right] \xi^{2k}
$$

$$
- \eta^2 \sum_{k=0}^{\infty} \left[ \sum_{n=0}^{k} \frac{F_{2n} \alpha_{2(k-n)}}{(2n)!}(2k-2n)! \right] \xi^{2k}
$$

Next introducing by binomial coefficients \( \binom{m}{n} \) and using the relation:

$$
\binom{m}{n} = \frac{m!}{(m-n)!n!}
$$

we have:

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\[
\frac{1}{(2n-1)!(2k-2n+1)!} = \frac{1}{(2k)!} \binom{2k}{2n-1}
\]

and:

\[
\frac{1}{(2n)!(2k-2n)!} = \frac{1}{(2k)!} \binom{2k}{2n}
\]

Using this in the above equation, we get:

\[
\sum_{k=0}^{\infty} \frac{F_{2k}''}{P_{rE}(2k)!} \xi^k = 2\eta \sum_{k=1}^{\infty} \frac{1}{(2k)!} \left[ \sum_{n=1}^{k} \frac{F_{2n'\beta}(k-n+1)}{(2n-1)!} \binom{2k}{2n-1} \right] \xi^k
\]

\[
- \eta^2 \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left[ \sum_{n=0}^{k} \frac{F_{2n'\alpha'}(k-n)}{2n} \binom{2k}{2n} \right] \xi^k
\]

Equating the coefficients of like powers of \( \xi \) gives:

\[
\frac{F_{2k}''}{P_{rE}} = 2\eta \sum_{n=1}^{k} \binom{2k}{2n-1} F_{2n'\beta}(k-n+1) (1-\delta_{0k})
\]

\[- \eta^2 \sum_{n=0}^{k} \binom{2k}{2n} F_{2n'\alpha'}(k-n)\]

or:

\[
\frac{1}{P_{rE}} F_{2k}'' + \eta^2 \alpha_0 F_{2k} - \frac{\eta_k \beta_2 \eta}{2} F_{2k} = 2\eta (1-\delta_{0k}-\delta_{1k}) \sum_{n=1}^{k-1} \binom{2k}{2n-1} F_{2n'\beta}(k-n+1)
\]
\[ - \eta^2 (1-\delta_{ok}) \sum_{n=0}^{k-1} \binom{2k}{2n} F_{2n} \alpha(z(k-n)) \]

\[ = (1-\delta_{ok} - \delta_{ik}) \sum_{n=1}^{k-1} \left[ \begin{array}{c} 2 \eta \binom{2k}{2n-1} F_{2n} \alpha_2(k-n+1) - \\
- \eta^2 \binom{2k}{2n} F_{2n} \alpha_2(k-n) \end{array} \right] \]

\[-(1-\delta_{ok}) \eta^2 \alpha_2 F_0' \]

Next using (54) and defining:

\[ K_k = \frac{1}{Pr\mathcal{E}x_0} \frac{d^2}{d\eta^2} + \eta^2 \frac{d}{d\eta} - 4k\eta \]

\[ G_k = \begin{cases} 
0 & k = 0 \\
- \eta^2 \alpha_2 F_0' & k = 1 \\
\eta \sum_{n=1}^{k-1} \left[ \binom{2k}{2n-1} F_{2n} \alpha_2(k-n+1) - \binom{2k}{2n} F_{2n} \alpha_2(k-n) \right] & k > 1
\end{cases} \]

We get:

\[ K_k(F_{2k}) = \frac{1}{\alpha_0} G_k, \quad k = 0,1,2,3,... \]
APPENDIX XI

REDUCTION OF THE FIRST OF EQUATIONS (84)

Substitution of the first two expansions (87) into the first of

Eqs. (84) yields:

\[
\xi (8_0^*+e8_1^*+e^28_2^*+\ldots)^2(\phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots)+1/2 \left( 8_0^*+e8_1^*+e^28_2^*+\ldots \right)\left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right)-\left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right) \left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right) - \left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right) \left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right) - \frac{1}{2} \left( 8_0^*+e8_1^*+e^28_2^*+\ldots \right) \left( 8_0^*+e8_1^*+e^28_2^*+\ldots \right)
\]

\[
\xi (8_0^*+e8_1^*+e^28_2^*+\ldots)+2\xi (8_0^*+e8_1^*+e^28_2^*+\ldots)
\]

\[
\xi (8_0^*+e8_1^*+e^28_2^*+\ldots) \left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right) = \frac{e}{2\chi_e} \xi (8_0^*+e8_1^*+e^28_2^*+\ldots)^\lambda \cos \tau + \xi (8_0^*+e8_1^*+e^28_2^*+\ldots) \left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right)
\]

By collecting terms we get:

\[
\xi (8_0^*+e8_1^*+e^28_2^*+\ldots)^2(\phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots)+1/2 \left( 8_0^*+e8_1^*+e^28_2^*+\ldots \right)\left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right)-\left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right) \left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right) - \left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right) \left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right) - \frac{1}{2} \left( 8_0^*+e8_1^*+e^28_2^*+\ldots \right) \left( 8_0^*+e8_1^*+e^28_2^*+\ldots \right)
\]

\[
\xi (8_0^*+e8_1^*+e^28_2^*+\ldots)+2\xi (8_0^*+e8_1^*+e^28_2^*+\ldots)
\]

\[
\xi (8_0^*+e8_1^*+e^28_2^*+\ldots) \left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right) = \frac{e}{2\chi_e} \xi (8_0^*+e8_1^*+e^28_2^*+\ldots)^\lambda \cos \tau + \xi (8_0^*+e8_1^*+e^28_2^*+\ldots) \left( \phi_{0_{\lambda}}^*+e\phi_{1_{\lambda}}^*+e^2\phi_{2_{\lambda}}^*+\ldots \right)
\]
By equating the coefficients of like powers of $\epsilon$ one arrives at:

$$\frac{1}{X_0} \xi \eta = 55_{0*} \psi_{0*} + \frac{1}{2} 58_{0*} (\psi_{0*} \psi_{0*} - \psi_{0*} \psi_{0*}) + \frac{1}{2} 58_{2*} (\psi_{2*} \psi_{2*} - \psi_{2*} \psi_{2*}) - \xi_0 \xi_{0*} \psi_{0*} \psi_{0*} \psi_{0*}$$

$$-\xi_0 \xi_{0*} \psi_{0*} \psi_{0*}$$

$$\frac{1}{X_0} \xi \eta + \frac{3}{2} \eta_{0*} \cos \tau = 55_{0*} \psi_{1*} + 55_{0*} \psi_{0*} + \frac{1}{2} 58_{0*} (\psi_{0*} \psi_{0*} + \psi_{0*} \psi_{0*}) + \frac{1}{2} 58_{2*} (\psi_{2*} \psi_{2*} + \psi_{2*} \psi_{2*}) - \xi_0 \xi_{0*} \psi_{0*} \psi_{0*} \psi_{0*}$$

$$-\xi_0 \xi_{0*} \psi_{0*} \psi_{0*}$$

$$\frac{1}{X_0} \xi \eta + \frac{3}{2} \eta_{0*} \cos \tau = 55_{0*} \psi_{1*} + 55_{0*} \psi_{0*} + \frac{1}{2} 58_{0*} (\psi_{0*} \psi_{0*} + \psi_{0*} \psi_{0*}) + \frac{1}{2} 58_{2*} (\psi_{2*} \psi_{2*} + \psi_{2*} \psi_{2*}) - \xi_0 \xi_{0*} \psi_{0*} \psi_{0*} \psi_{0*}$$

$$-\xi_0 \xi_{0*} \psi_{0*} \psi_{0*}$$

$$\frac{1}{X_0} \xi \eta + \frac{3}{2} \eta_{0*} \cos \tau = 55_{0*} \psi_{1*} + 55_{0*} \psi_{0*} + \frac{1}{2} 58_{0*} (\psi_{0*} \psi_{0*} + \psi_{0*} \psi_{0*}) + \frac{1}{2} 58_{2*} (\psi_{2*} \psi_{2*} + \psi_{2*} \psi_{2*}) - \xi_0 \xi_{0*} \psi_{0*} \psi_{0*} \psi_{0*}$$

$$-\xi_0 \xi_{0*} \psi_{0*} \psi_{0*}$$

$$\frac{1}{X_0} \xi \eta + \frac{3}{2} \eta_{0*} \cos \tau = 55_{0*} \psi_{1*} + 55_{0*} \psi_{0*} + \frac{1}{2} 58_{0*} (\psi_{0*} \psi_{0*} + \psi_{0*} \psi_{0*}) + \frac{1}{2} 58_{2*} (\psi_{2*} \psi_{2*} + \psi_{2*} \psi_{2*}) - \xi_0 \xi_{0*} \psi_{0*} \psi_{0*} \psi_{0*}$$

$$-\xi_0 \xi_{0*} \psi_{0*} \psi_{0*}$$

$$\frac{1}{X_0} \xi \eta + \frac{3}{2} \eta_{0*} \cos \tau = 55_{0*} \psi_{1*} + 55_{0*} \psi_{0*} + \frac{1}{2} 58_{0*} (\psi_{0*} \psi_{0*} + \psi_{0*} \psi_{0*}) + \frac{1}{2} 58_{2*} (\psi_{2*} \psi_{2*} + \psi_{2*} \psi_{2*}) - \xi_0 \xi_{0*} \psi_{0*} \psi_{0*} \psi_{0*}$$

$$-\xi_0 \xi_{0*} \psi_{0*} \psi_{0*}$$
APPENDIX XII

REDUCTION OF THE SECOND OF EQUATIONS (85)

Substitution of the first two of the expansions (87) into the second of (85) yields:

\[
\left( \delta_0^* + c b_1^* + c^2 b_2^* + \ldots \right) \left[ \left( X_e \delta_0^* + \frac{1}{2} \delta_0^* \right) + c \left( X_e \delta_1^* + X_e \delta_2^* + \ldots \right) \right] + \frac{1}{2} \delta_2^* + \ldots \] 

\[
+ \left( \delta_0^* - X_e \psi_0^* \right) + c \left( \delta_1^* - X_e \psi_1^* \right) + c^2 \left( \delta_2^* - X_e \psi_2^* \right) + \ldots \] 

\[
\frac{c^2}{2} \left( \delta_0^* + b_1^* + c b_2^* + \ldots \right) \left[ \left( X_e \delta_0^* + b_0^* \right) + c \left( X_e \delta_1^* + b_1^* \right) \right] + \frac{1}{2} \delta_2^* + \ldots \] 

\[
- \frac{X_e}{2} \left( \delta_0^* + b_1^* + c b_2^* + \ldots \right) \left( \psi_0^* + c \psi_1^* + c^2 \psi_2^* + \ldots \right) \sin \tau + \frac{c^2}{16} \left( \delta_0^* + b_1^* + c b_2^* + \ldots \right)^2 \left( \delta_0^* + b_1^* + c b_2^* + \ldots \right) \left( 1 - \cos 2\tau \right) = \delta_0^* \psi_0^* + \delta_1^* \psi_1^* + \delta_2^* \psi_2^* + \ldots
\]

Upon collecting terms one obtains:

\[
\left( \delta_0^* + b_1^* + c b_2^* + \ldots \right) \left[ \left( X_e \delta_0^* + \frac{1}{2} \delta_0^* \right) \left( \delta_0^* - X_e \psi_0^* \right) \right] + c \left[ \left( X_e \delta_1^* + \frac{1}{2} \delta_1^* \right) \left( \delta_1^* - X_e \psi_1^* \right) \right] + c^2 \left[ \left( X_e \delta_2^* + \frac{1}{2} \delta_2^* \right) \left( \delta_2^* - X_e \psi_2^* \right) \right] + \ldots \]

\[
+ \frac{1}{2} \delta_2^* \left( \delta_0^* - X_e \psi_0^* \right) + c \left( \delta_1^* - X_e \psi_1^* \right) + c^2 \left( \delta_2^* - X_e \psi_2^* \right) + \ldots \] 

\[
+ \frac{1}{2} \delta_0^* \left( \delta_2^* - X_e \psi_2^* \right) \right] \right] + \frac{c}{2} \left( \delta_0^* + b_1^* + \ldots \right) \left[ \left( X_e \delta_0^* + b_0^* \right) \delta_0^* + \delta_1^* \left( X_e \delta_1^* + b_1^* \right) \right] \left( \delta_0^* - \frac{X_e}{2} \delta_1^* \psi_0^* \right) \]

\[
- \frac{X_e}{2} \delta_0^* \psi_0^* \]
\[ -\frac{X_e}{2} \delta_o^* \psi_{\eta} \left( \ldots \right) \sin \tau + \frac{e^2}{16} \left( \delta_o^* \delta_o^* \ldots \right) (1 - \cos 2\tau) = \delta^*_\psi_{\eta} \]

\[ + e \delta^*_{\psi_{\eta}} + e^2 \psi_{\eta} \ldots \]

And equating the coefficients of like powers of \( e \) yields:

\[ \delta_o^* (X_e \delta_o^* + \frac{1}{2} \delta_o^*) (\delta_o^* - X_e \psi_{\eta}) = \delta^*_\psi_{\eta} \]

\[ \delta_o^* \left[ (X_e \delta_1^* + \frac{1}{2} \delta_1^*) (\delta_o^* - X_e \psi_{\eta}) + (X_e \delta_o^* + \frac{1}{2} \delta_o^*) \right] \]

\[ \frac{1}{2} \delta_o^* (\delta_1^* - X_e \psi_{\eta}) \left. \right] + \delta_1^* (X_e \delta_o^* + \frac{1}{2} \delta_o^*) (\delta_o^* - X_e \psi_{\eta}) \]

\[ + \frac{1}{2} \left[ \delta_o^* (X_e \delta_o^* + \delta_o^* - X_e \psi_{\eta}) \right] \sin \tau = \delta^*_\psi_{\eta} \]

or:

\[ (\delta_o^* - X_e \psi_{\eta}) \left[ \delta_o^* (X_e \delta_1^* + \frac{1}{2} \delta_1^*) + \delta_1^* (X_e \delta_o^* + \frac{1}{2} \delta_o^*) \right] \]

\[ + \frac{1}{2} \delta_o^* \left. \right] \left[ \delta_o^* (X_e \delta_o^* + \frac{1}{2} \delta_o^*) \right] \delta_1^* - X_e \psi_{\eta} \]

\[ + \frac{\delta_o^*}{2} \sin \tau + \frac{\delta_o^* \delta_o^*}{4} (\delta_o^* - X_e \psi_{\eta}) \sin \tau = \delta^*_\psi_{\eta} \]

\[ \delta_o^* \left[ (X_e \delta_2^* + \frac{1}{2} \delta_2^*) (\delta_o^* - X_e \psi_{\eta}) + (X_e \delta_1^* + \frac{1}{2} \delta_1^*) \right] \]

\[ \frac{1}{2} \delta_1^* (\delta_1^* - X_e \psi_{\eta}) + (X_e \delta_o^* + \frac{1}{2} \delta_o^*) (\delta_2^* - X_e \psi_{\eta}) \]

\[ + \delta_1^* \left[ (X_e \delta_o^* + \frac{1}{2} \delta_o^*) (\delta_o^* - X_e \psi_{\eta}) + (X_e \delta_o^* + \frac{1}{2} \delta_o^*) \right] \]
\[-X^*_e \psi^*_\eta + \frac{1}{2} \left[ 28^*_b \delta^*_o (X^*_e \delta^*_o \delta^*_T + \delta^*_o) + \delta^*_o (X^*_e \delta^*_b T \right] \\
+ \delta^*_b \left( \delta^*_o \delta^*_b + \delta^*_o \right) \psi^*_\eta - \frac{X^*_e}{2} \delta^*_o \delta^*_b \psi^*_\eta \right] \sin \tau \\
+ \frac{\delta^*_o}{16} \delta^*_o (1 - \cos 2\tau) = \xi \psi^*_\eta \]

or:

\[(\delta^*_o - X^*_e \psi^*_\eta) \left[ \delta^*_o (X^*_e \delta^*_b T + \frac{1}{2} \delta^*_b) + \delta^*_o (X^*_e \delta^*_o T + \frac{1}{2} \delta^*_o) \right] \\
+ \delta^*_b (X^*_e \delta^*_b T + \frac{1}{2} \delta^*_b) \right] + (\delta^*_o - X^*_e \psi^*_\eta) \left[ \delta^*_o (X^*_e \delta^*_b T \\
+ \frac{1}{2} \delta^*_o) + \delta^*_o (X^*_e \delta^*_o T + \frac{1}{2} \delta^*_o) \right] + \delta^*_o (X^*_e \delta^*_o T + \frac{1}{2} \delta^*_o) (\delta^*_o \\
- X^*_e \psi^*_\eta) + \frac{1}{2} \left[ 28^*_b \delta^*_o (X^*_e \delta^*_o \delta^*_T + \delta^*_o) + \delta^*_o (X^*_e \delta^*_b T \right] \\
+ \delta^*_b \left( \delta^*_o \delta^*_b + \delta^*_o \right) \psi^*_\eta - \frac{X^*_e}{2} \delta^*_o \delta^*_b \psi^*_\eta \right] \sin \tau \\
+ \frac{\delta^*_o}{16} \delta^*_o (1 - \cos 2\tau) = \xi \psi^*_\eta \]
APPENDIX XIII

SIMPLIFICATION OF FIRST- AND SECOND-ORDER MOMENTUM EQUATIONS

Substitution of (92) into Eq. (90-1) gives:

\[ 4a_0 \dot{\xi}^3 \psi_{1\eta} + a_0^* (\psi_{1\xi}^* - 2a_0 \psi_{1\xi}^* - 2a_0 f_{0\xi}^* - 2a_0 f_{0\xi}^* \psi_{1\xi}^*) \]

\[-2a_0 \psi_{1\xi}^* + \frac{1}{2} \xi (4a_0^2 f_{0\xi}^* - 4a_0^2 f_{0\xi}^* f_{0\xi}^*) - 2a_0^2 \xi \psi_{1\xi}^* \]

\[-a_0 f_{0\xi}^* - 4a_0^2 \psi_{1\xi}^* - 2a_0^2 f_{0\xi}^* \psi_{1\xi}^* f_{0\xi}^* \] 

\[= \frac{8a_0^3}{2X_e} \xi^3 \cos \tau + \frac{1}{X_e} \xi \psi_{1\eta} \]

Rearranging we get:

\[-\frac{1}{X_e} \psi_{1\eta}^* + 4a_0^2 \psi_{1\eta}^* + 2a_0^2 (f_{0\xi}^* \psi_{1\xi}^* - f_{0\xi}^* \psi_{1\xi}^*) \]

\[-f_{0\xi}^* (f_{0\xi}^* - f_{0\xi}^*) + 2a_0^2 (f_{0\xi}^* - f_{0\xi}^*) \psi_{1\xi} - 2a_0^2 f_{0\xi}^* f_{0\xi}^* \]

\[-4a_0^2 \psi_{1\xi}^* - 2(f_{0\xi}^* f_{0\xi}^*) \psi_{1\xi} = \frac{4a_0^3}{X_e} \xi^3 \cos \tau \]

And collecting terms, we get:

\[-\frac{1}{2a_0} \psi_{1\eta}^* - f_{0\xi}^* \psi_{1\eta} + f_{0\xi}^* (\xi \psi_{1\xi}^* - \psi_{1\xi}^*) - f_{0\xi}^* \psi_{1\xi}^* \]

\[+ 2\xi \psi_{1\xi}^* + (f_{0\xi}^* f_{0\xi}^*) \xi f_{0\xi}^* f_{0\xi}^* f_{0\xi}^* - 2\xi (f_{0\xi}^* + f_{0\xi}^*) \psi_{1\xi}^* \]

\[= \frac{2a_0}{X_e} \xi^3 \cos \tau \]

Substitution of (92) into Eq. (90-2) gives:
\[
4a^2 \delta^2 \psi_2 \eta + 4a^2 \delta^2 \psi_1 \psi_1 \eta + a \xi (2a^2 f_0^* \psi_2 \eta +

\psi_1 \psi_1 \eta + 2a f_0^* \psi_2 \eta - 2a f_0^* \psi_2 \eta - \psi_1 \psi_1 \eta \]

\[
-2a \xi f_0^* \psi_2 \eta + \frac{b_1}{2} (2a \xi f_0^* \psi_2 \eta + 2a \xi f_0^* \psi_2 \eta - 2a \xi f_0^* \psi_2 \eta

-2a f_0^* \psi_2 \eta + \frac{b_2}{2} (4a^2 f_0^2 - 4a^2 f_0^* f_0) - 2 - 2a^2 f_0^* f_0^* - 2a \delta^2 \psi_2 \eta \]

\[
-2a \delta f_1 \xi \psi_2 \eta - 2a \delta f_1 \xi \psi_2 \eta - 2a^2 \delta \psi_2 \eta \]

\[
- \frac{1}{2} (2a) (\psi_2 \eta + 4a \xi f_0^* \psi_2 \eta) - \xi (2a \delta f_0^* - 2a \delta f_0^2) - 2a \xi f_0^* \eta

- \xi (2a \xi) \delta^2 \psi_2 \eta = \frac{6a^2}{X_e} \delta^2 \xi \cos \tau + \frac{1}{X_e} \xi \psi_2 \eta

Upon rearranging we get:

\[
2a^2 \xi \left[ - \frac{1}{2a} \psi_2 \eta \eta + 2a \delta^2 \psi_2 \eta + \xi f_0^* \psi_2 \eta - f_0^* \psi_2 \eta

- f_0^* \psi_2 \eta - f_0^* \psi_2 \eta + 2a (f_0^2 - f_0^* f_0^*) - f_0^2 \delta^2 \xi

- 2a \xi f_0^* f_0^* \delta^2 \xi \right] = -4a^2 \delta \psi_1 \psi_1 \eta\]

\[
+ \frac{6a^2}{X_e} \delta^2 \xi \cos \tau - a \xi (\psi_1 \psi_1 \psi_1 \psi_1 \eta - \psi_1 \psi_1 \psi_1 \psi_1 \eta - \psi_1 \psi_1 \psi_1 \psi_1 \eta + \psi_1 \psi_1 \psi_1 \psi_1 \eta +

- \psi_1 \psi_1 \psi_1 \psi_1 \eta + 2a \xi f_0^* f_0^* \psi_2 \eta - 2a \xi f_0^* f_0^* \psi_2 \eta

+ 2a \xi f_0^* f_0^* \psi_2 \eta + 2a \xi f_0^* f_0^* \psi_2 \eta + 2a \xi f_0^* f_0^* \psi_2 \eta

\]

And finally we get:
\[
2a \xi \left[ -\frac{1}{2a^2 x_e} \psi^* \eta - f_0 \psi^* \eta + f_0' (\xi \psi^* \eta - \psi^* \eta) - 2f_0'' \psi^* \eta \right] \\
+ 2 f_0 \psi^* \eta + (f_0'^2 - f_0 f_0'') \delta^2 - \frac{1}{2} f_0'^2 \delta^2 - 2 f_0 + (\eta f_0'') \delta^2 \right] \\
= \frac{6a}{x_e} \xi \beta^h \cos \tau - 4 \xi^2 \beta^h \psi^* \xi - 5 \psi^* \psi^* + 5 \psi^* \eta \psi^* \\
+ \psi^* - \beta^h (\xi f_0' \psi^* + f_0' \psi^* - f_0' \psi^* - f_0 \psi^* \eta) \\
+ 2 f_0 (f_0' + f_0'') \delta^2 \beta^h + 2 f_0 \delta^2 \beta^h \psi^* \\
+ 2 \xi^2 (\psi^* \eta \psi^* \eta) \delta^2 \beta^h 
\]
APPENDIX XIV

SIMPLIFICATION OF THE FIRST- AND SECOND-ORDER MOMENTUM BOUNDARY CONDITIONS

Substituting Eq. (92) into Eq. (90-1) gives:

\[ 2a \delta (1 - X_e f_0') \left[ 2a \delta (X_e \delta_2^* + \frac{1}{2} \delta_1^*) + a \delta_2^* + \right. \]

\[ + a^2 \delta \sin \tau \left. \right] + 2a^2 \delta \left[ \delta_1^* - X_e \psi_1^* + a \delta \sin \tau \right] = \psi_1^* \eta \]

or:

\[ \frac{1}{2a^2} \psi_1^* \eta = (1 - X_e f_0') \left[ 2X_e \delta_2^* \delta_2^* + (\delta_1^* \delta_1^* + \delta_1^*) + \right. \]

\[ + a \delta \sin \tau \left. \right] + \delta_1^* - X_e \psi_1^* + a \delta \sin \tau \]

And use of (92) in Eq. (90-2) gives:

\[ 2a \delta (1 - X_e f_0') \left[ 2a \delta (X_e \delta_2^* + \frac{1}{2} \delta_2^*) + a \delta_2^* \right. \]

\[ + \delta_1^* (X_e \delta_2^* + \frac{1}{2} \delta_2^*) + (\delta_1^* - X_e \psi_1^*) \left. \right] \left[ 2a \delta (X_e \delta_2^* \right. \]

\[ + \frac{1}{2} \delta_1^* \left. \right] + 2a^2 \delta \left( \delta_2^* - X_e \psi_2 \eta \right) + \frac{1}{2} \left[ \delta_1^* \delta_1^* \right. \]

\[ + 4a^2 \delta_1^* (\delta_1^* + X_e \delta_2^* \left. \right] - X_e 2a^2 \delta \left( \delta_1^* \delta_1^* + \delta_1^* \right) f_0' \]

\[ - X_e 2a^2 \psi_1^* \eta \sin \tau + \frac{a^3}{2} \right] \eta^2 (1 - \cos 2 \tau) = \psi_1^* \eta \]

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Rearranging we get:

\[ 2a^2 \Delta (1-\chi_0 \chi_0') \left[ a^2 \Delta \delta^* \eta + a (\delta^* \delta + \delta^* \delta) + \right. \]

\[ \delta^* \left( \chi_0 \delta^* \eta + \frac{1}{2} \delta^* \delta \right) + a (\delta^* \chi_0 \chi_1^* \eta) \left[ 2a^2 \Delta \delta^* \eta + \right. \]

\[ + \delta^* (1-\chi_0 \chi_0') \left( \delta^* \chi_0 \chi_1^* \eta \right) + \left( \delta^* \delta + \delta^* \delta \right) + \]

\[ + \frac{1}{2} (2a)^2 \delta^* (\delta^* - \chi_0 \chi_1^* \eta) + a^2 \delta^* \left( \delta^* \delta + \delta^* \delta \right) + \]

\[ + 2a^2 \delta^* \sin \tau + \frac{1}{2} a^2 \sin \left( 1-\cos 2\tau \right) = \delta^* \psi_\eta \]

and:

\[ 2a^2 \Delta (1-\chi_0 \chi_0') \left[ a (\delta^* \delta + \delta^* \delta) + 2a^2 \Delta \delta^* \eta + \right. \]

\[ \delta^* \left( \chi_0 \delta^* \eta + \frac{1}{2} \delta^* \delta \right) + a (\delta^* \chi_0 \chi_1^* \eta) \left[ 2a^2 \Delta \delta^* \eta + \right. \]

\[ + a (\delta^* \chi_0 \chi_1^* \eta) \left[ \delta^* (\delta^* \delta + \delta^* \delta) + 2a^2 \delta^* \sin \tau \right] + \]

\[ + \frac{1}{2} (2a)^2 \delta^* (\delta^* - \chi_0 \chi_1^* \eta) + \frac{1}{4} (2a)^2 \delta^* \left( \delta^* \delta + \delta^* \delta \right) + \]

\[ + \chi_0 \delta^* \sin \tau + \frac{1}{6} (2a)^2 a^2 \left( 1-\cos 2\tau \right) = \delta^* \psi_\eta \]
APPENDIX XV

REDUCTION OF THE SECOND OF EQUATIONS (84)

Substitution of the expansions (87) into the second of Eqs. (84)
yields:

\[ \xi (b_0^2 + 2c_0^2 + \ldots + c^2(b_1^2 + \ldots + b_0^2 + \ldots) + \ldots) (\varphi_0^* + e\varphi_1^* + \ldots) + \frac{1}{2} (b_0^* + c_0^* + \ldots) [ (\varphi_0^* + e\varphi_1^* + \ldots) + (\varphi_0^* + e\varphi_1^* + \ldots) - (\varphi_0^* + e\varphi_1^* + \ldots) (\varphi_0^* + e\varphi_1^* + \ldots)] - \xi (b_0^* + c_0^* + \ldots) (\varphi_0^* + e\varphi_1^* + \ldots) \]

\[ \ldots (\varphi_0^* + e\varphi_1^* + \ldots) = \frac{\xi}{\chi e^{Pr}} (\varphi_0^* + e\varphi_1^* + \ldots) + e^2 \theta_2 \eta^* + \ldots) \]

Upon collecting terms one obtains:

\[ \xi (b_0^2 + 2c_0^2 + \ldots + c^2(b_1^2 + \ldots + b_0^2 + \ldots) + \ldots) (\varphi_0^* + e\varphi_1^* + \ldots) + \frac{1}{2} (b_0^* + c_0^* + \ldots) [ (\varphi_0^* + \varphi_1^* - \varphi_0^* \varphi_1^*)) + e(\varphi_1^* \varphi_0^* + \theta_0^* \psi_1^*) - \varphi_0^* \varphi_1^* - \psi_0^* \varphi_1^* + e^2(\psi_0^* \varphi_1^* + \psi_1^* \varphi_0^* + \theta_0^* \varphi_1^* + \theta_1^* \psi_0^*) - \theta_1^* \psi_0^* - \theta_0^* \psi_1^*) + \ldots - \xi (b_0^* + c_0^* + \ldots) (b_0^2 + \ldots + b_0^2 + \ldots) + \ldots + e^2 (b_0^2 + \ldots + b_0^2 + \ldots) + \ldots + \ldots = \frac{\xi}{\chi e^{Pr}} (\varphi_0^* + e\varphi_1^* + \ldots) + e^2 \theta_2 \eta^* + \ldots) \]
And finally equating the coefficients of like powers of \( \varepsilon \) we have:

\[
\delta_0^2 \Phi_0^* + \frac{1}{2} \delta_0 (\Phi_0^* \psi_{01}^* - \psi_{05}^* \Phi_{01}^*) - \xi_0 \delta_0 \Theta_{01}^* = \frac{\xi}{\chi \varepsilon Pr} \Phi_{01}^* \psi_{01}^* 
\]

\[
2 \delta_0 \delta_1 \Phi_{01}^* + \delta_0^2 \Theta_{11}^* + \frac{1}{2} \delta_0 (\Phi_1^* \psi_{11}^* + \Phi_{05}^* \psi_{11}^* - \Phi_1^* \phi_{11}^* - \phi_5^* \psi_{11}^*) 
\]

\[
+ \frac{1}{2} \delta_1 (\Phi_0^* \psi_{01}^* - \psi_{05}^* \Phi_{01}^*) - \xi_0 \delta_0 \Theta_{11}^* - \xi_0 (\delta_1 \delta_0^* + \delta_0 \delta_1^*) \Phi_{01}^* 
\]

\[
= \frac{\xi}{\chi \varepsilon Pr} \Phi_{11}^* 
\]

\[
\delta_0 \Phi_2^* + 2 \delta_0 \delta_1 \Theta_{11}^* + \delta_0^2 (\delta_1^2 + 2 \delta_0 \delta_2) \Phi_{01}^* 
\]

\[
+ \frac{1}{2} \delta_0 (\psi_{01}^* \phi_{05}^* + \psi_{11}^* \phi_{15}^* + \phi_{21}^* \psi_{01}^* - \phi_{11}^* \psi_{05}^* - \phi_{11}^* \phi_{15}^* - \phi_{11}^* \psi_{11}^*) 
\]

\[
+ \frac{1}{2} \delta_1 (\Phi_0^* \psi_{01}^* + \Phi_{05}^* \psi_{11}^* - \psi_{05}^* \phi_{11}^*) + \frac{1}{2} \delta_2 (\Phi_0^* \phi_{05}^* - \psi_{05}^* \phi_{11}^*) 
\]

\[
- \xi_0 (\delta_2 \delta_0^* + \delta_0 \delta_2^* + \delta_0 \delta_1^*) \Theta_{01}^* - \xi_0 (\delta_1 \delta_0^* + \delta_0 \delta_1^*) \Phi_{01}^* 
\]

\[
- \xi_0 \delta_0 \Theta_{01}^* = \frac{\xi}{\chi \varepsilon Pr} \Theta_{01}^* 
\]
APPENDIX XVI

REDUCTION OF THE THIRD OF CONDITIONS (85)

Substitution of the expansions (87) into the third of (85) yields:

\[(\varphi_0^* + \epsilon \varphi_1^* + \epsilon^2 \varphi_2^* + \ldots) (\delta_0^* + \epsilon \delta_1^* + \epsilon^2 \delta_2^* + \ldots) \]

\[+ \ldots (\delta_0^* \delta_0^* + \delta_1^* \delta_1^* + \ldots) + \epsilon/4 (\varphi_0^* \varphi_1^* + \varphi_0^* \varphi_0^* + \varphi_1^* \varphi_1^* + \ldots) \sin \tau \]

\[= - \frac{1}{XePr} \xi (\varphi_0^* \varphi_1^* + \varphi_0^* \varphi_0^* + \varphi_1^* \varphi_1^* + \ldots) \]

\[= (\varphi_0^* \varphi_1^* + \varphi_0^* \varphi_0^* + \varphi_1^* \varphi_1^* + \ldots) (\delta_0^* \delta_0^* + \delta_1^* \delta_1^* + \delta_2^* \delta_2^* + \ldots) \sin \tau \]

And equating the coefficients of like powers of \(\epsilon\) we get:

\[\varphi_0^* \delta_0^* (x e \delta_0^* + 1/2 \delta_0^*) = - \frac{1}{XePr} \xi \varphi_0^* \]
\[ \Phi_{10}(X_0 \xi_{0T} + 1/2 \xi_{05}) + \Phi_{01}(X_0 \xi_{1T} + 1/2 \xi_{15}) + \\
\xi_{0}(X_0 \xi_{2T} + 1/2 \xi_{25}) \right) + 1/4 \Phi_{00} \xi_{005} \sin \tau = - \frac{1}{XePr} \xi_{1\eta} \]

\[ + \Phi_{20}(X_0 \xi_{20} + 1/2 \xi_{05}) + \Phi_{11}(X_0 \xi_{1T} + 1/2 \xi_{15}) + \\
\xi_{0}(X_0 \xi_{2T} + 1/2 \xi_{25}) \right) + \Phi_{20}(X_0 \xi_{20} + 1/2 \xi_{05}) \]

\[ + \xi_{1}(X_0 \xi_{2T} + 1/2 \xi_{25}) + \xi_{0}(X_0 \xi_{2T} + 1/2 \xi_{25}) \right) \sin \tau \]

\[ = - \frac{1}{XePr} \xi_{1\eta} \]
APPENDIX XVII

SIMPLIFICATION OF THE FIRST- AND SECOND-ORDER ENERGY EQUATIONS

Substitution of Eqs. (99) and (92) into Eq. (96-1) yields:

\[ 4a_5^2 \xi^* \Theta_{1T} + a_5^* (2a_5^* \psi_{1T}^* - 2a_5^* \Theta_{2T} - \psi_{1T}^* F_0^*) \]

\[ - \frac{1}{2} \delta_{1}^* 2a_0 F_0^* - 2a_{2T} \psi_{1T} F_0^* = \frac{5}{X_e^* P_r} \Theta^* \Theta_{1T} \]

And collecting terms we get:

\[ \frac{1}{2a_5^*} \Theta_{1T} - 2a_{1T} \psi_{1T} + 2a_{2T} \psi_{1T} - \psi_{1T}^* F_0^* \]

Substitution of Eqs. (99) and (92) into Eq. (96-2) gives:

\[ 4a_5^2 \xi^* \Theta_{1T} + 4a_5^2 \psi_{1T}^* + a_5^* (\psi_{1T}^* \psi_{1T}^* + 2a_5^* \psi_{1T}^* \psi_{1T}^*) \]

\[ - 2a_0 \psi_{1T} - 2a_{2T} \psi_{1T} - \psi_{1T}^* \psi_{1T}^* F_0^* + \frac{1}{2} \delta_{1}^* (2a_5^* \psi_{1T}^* \psi_{1T}^* \psi_{1T}^* \psi_{1T}^*) \]

\[ - 2a_0 \psi_{1T} - 2a_{2T} \psi_{1T} - a_0 \psi_{1T}^* \psi_{1T}^* - \psi_{1T}^* \psi_{1T}^* \psi_{1T}^* \psi_{1T}^* \psi_{1T}^* \psi_{1T}^* \]

\[ + 2a_5^2 \psi_{1T}^* F_0^* - 2a_{2T} \psi_{1T}^* \psi_{1T}^* = \frac{5}{X_e^* P_r} \Theta^* \Theta_{1T} \]

Rearranging, we get:

\[ \frac{1}{2a_5^*} \Theta_{1T} - 2a_{1T} \psi_{1T} + 2a_{2T} \psi_{1T} - \psi_{1T}^* F_0^* = \]

\[ \frac{1}{2a_5^*} (4a_5^2 \psi_{1T}^* + a_5^* \psi_{1T}^* \psi_{1T}^* - a_5^* \psi_{1T}^* \psi_{1T}^* - \psi_{1T}^* \psi_{1T}^* \psi_{1T}^* \psi_{1T}^* \psi_{1T}^* \psi_{1T}^*) \]

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\[
\begin{align*}
&= a^* F_0 \psi_2 + \frac{1}{2} 2 a^* \delta_1 f_0^* \delta_1 - a^* \delta_1^* \psi_1 \\
&= -\frac{1}{2} \delta_1^* F_0^* - a^* \delta_2^* F_0 - \eta \delta_1^* \\
&= a^* \delta_2^* F_0 - \eta \delta_1^* \psi_1^* \}
\end{align*}
\]

and:

\[
\begin{align*}
&\frac{1}{2a^2 F_0 x_0^*} \psi_0^* \eta \psi_2^* + f_0^* \eta \psi_2^* - f_0^* \psi_2^* = \\
&= \frac{1}{2a^2 \delta_1^*} \left\{ -a (\delta_1^* \psi_2^* + f_0^* \delta_2^* + 2 \delta_1^* \eta \delta_2^* ) F_0^* - \left( \frac{1}{2} \delta_1^* \psi_2^* \\
&+ \eta \delta_1^* \psi_0^* \right) F_0^* + a \left[ 4 \delta_1^* \eta \delta_1^* \psi_0^* + (\psi_0^* + f_0^* \delta_1^* ) \delta_1^* \right] \\
&- (\delta_1^* \psi_2^* + f_0^* \delta_2^* + 2 \delta_1^* \eta \delta_2^*) \psi_1^* \right\}
\end{align*}
\]
Simplification of First- and Second-Order Energy Boundary Conditions

Using (99) and (92) in Eq. (97.1) we get:

\[ \Theta_1^* = 2\alpha a + F_0 \left\{ \delta_1^* a + 2a \xi \left( X_e \delta_1^{*2} + \frac{1}{2} \delta_{15}^* \right) \right\} \]

\[ + \frac{1}{4} F_0 \frac{2 \alpha \xi a \sin \tau}{X_e \xi} = - \frac{1}{X_e \xi} \xi \Theta_1^* \]

and:

\[ \xi \left\{ \frac{1}{2a^2 X_e \xi} \Theta_1^* + \Theta_2^* \right\} + \frac{1}{2a} F_0 \left\{ (\delta_1^* + \xi \delta_{15}^*) + 2X_e \xi \delta_{15}^* \right\} + \frac{1}{2} \xi \sin \tau = 0 \]

Using (99) and (92) in Eq. (97.2) we get:

\[ \Theta_2^* = 2\alpha a + \Theta_1^* \left\{ a \delta_1^* + 2a \xi^2 X_e \delta_1^{*2} + a \delta_{15}^* \right\} \]

\[ + F_0 \left\{ (\delta_2^* a + \alpha \xi \delta_{25}^*) + 2a X_e \xi \delta_{25}^{*2} + \delta_1^* \left( X_e \xi \delta_1^{*2} + \frac{1}{2} \delta_{15}^* \right) \right\} \]

\[ + \frac{1}{4} \left\{ \Theta_1^* 2a \xi a + F_0 \left[ 2a \delta_1^* + 2a \delta_{15}^* \right] \right\} \sin \tau \]

\[ = - \frac{1}{X_e \xi} \xi \Theta_2^* \]

or:

\[ \xi \left\{ \frac{1}{2a^2 X_e \xi} \Theta_2^* + \Theta_2^* \right\} + \Theta_1^* \frac{1}{2a} \left\{ (\delta_1^* + \xi \delta_{15}^*) + 2X_e \xi \delta_{15}^* \right\} \]

\[ + F_0 \frac{1}{2a} \left\{ (\delta_2^* + \xi \delta_{25}^*) + 2X_e \xi \delta_{25}^{*2} + \frac{1}{a} \delta_1^* \left( X_e \xi \delta_1^{*2} + \frac{1}{2} \delta_{15}^* \right) \right\} \]

\[ + \frac{1}{2} \xi \Theta_1^* \sin \tau + \frac{1}{2} F_0 \frac{1}{2a} (\delta_1^* + \xi \delta_{15}^*) \sin \tau = 0 \]
APPENDIX XIX

REDUCTION OF FIRST-ORDER MOMENTUM PROBLEM TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Substitution of the expansion (103) and (104) into Eq. (88-1a) gives:

\[ \sum_{n=1}^{\infty} \left\{ -\frac{1}{2a^2 x_e} f''_{in} + nf'_{0, in} - f'_{0, in} - nf''_{0, in} - f''_{0, in} \right\} \xi^n \]

\[ + \sum_{n=1}^{\infty} \left[ \left( f''_{0, in} - f'_{0, in} \right) a_{in} - n a_{in} f_{0, in} \right] \xi^n \]

\[ + 21 \sum_{n=1}^{\infty} \left[ f'_{in} - \left( f'_{0, in} \right) a_{in} \right] \xi^{n+2} \]

\[ = \frac{1}{x_e} \xi^3 \]

or:

\[ \sum_{n=1}^{\infty} \left\{ -\frac{1}{2a^2 x_e} f''_{in} - f'_{0, in} + (n+1) f'_{0, in} - nf''_{0, in} + \left[ (1-n) f''_{0, in} - f''_{0, in} \right] a_{in} \right\} \xi^n \]

\[ + 21 \sum_{n=3}^{\infty} \left[ f'_{1(n-2)} - \left( f'_{0, in} \right) a_{1(n-2)} \right] \xi^n = \frac{3}{x_e} \]

And equating coefficients of like powers of \( \xi \) gives:

\[ \frac{1}{2a^2 x_e} f''_{in} + f_{0, in} + (n-l) f_{0, in} + nf''_{0, in} + [(n-l) f''_{0, in} - f''_{0, in}] a_{in} \]

\[ = \left\{ \begin{array}{l} 0 \quad ; \quad n=1,2 \\
-\frac{1}{x_e} 5_{3(n-2)} \left( f'_{0, in} - f_{1(n-2)} \right) a_{1(n-2)} \quad ; \quad n=3,4,5,\ldots \end{array} \right. \]

(XIX.1)
where $\delta_{mn}$ is the Kronecker Delta.*

Substitution of the expansions (103) and (104) into Eq. (89-1a) gives:

\[
\sum_{n=1}^{\infty} f_{1n}^n = \sum_{n=1}^{\infty} a_{1n}^n \xi^n - \frac{i}{2} \xi + 2i(X_e - 1) \sum_{n=1}^{\infty} \frac{a_{1n}}{n+2} \xi^{n+2}
\]

or:

\[
\sum_{n=1}^{\infty} f_{1n}^n = \sum_{n=1}^{\infty} a_{1n}^n \xi^n - \frac{i}{2} \xi + 2i(X_e - 1) \sum_{n=3}^{\infty} \frac{a_{1(n-2)}}{n} \xi^n
\]

And equating the coefficients of like powers of $\xi$ we get:

\[
\eta = 1; \quad f_{1n-a1n} = \begin{cases} 
-\frac{i}{2} \delta_{1n} & ; \quad n=1,2 \\
2i(X_e - 1)a_{1(n-2)}/n & ; \quad n=3,4,5,\ldots
\end{cases} \tag{XIX.2}
\]

And finally substitution of the expansions (103) and (104) into Eq. (90-1a) gives:

\[
\frac{1}{2} (2a)^2 \left[ (1-X_e f'_o) \left[ \sum_{n=1}^{\infty} (2iX_e a_{1n}) \xi^n \right] \right.
\]

\[
- \frac{i}{2} \xi + \sum_{n=1}^{\infty} (a_{1n} - X_e f_{1n}) \xi^n - \frac{i}{2} \xi \right] = \sum_{n=1}^{\infty} f_{1n}^n \xi^n
\]

or:

\[
\sum_{n=1}^{\infty} \frac{1}{2a_e^2} f_{1n}^n \xi^n = (1-X_e f'_o) \left[ \sum_{n=3}^{\infty} 2iX_e a_{1(n-2)} \xi^n \right.
\]

\[
+ \sum_{n=1}^{\infty} (n+1) a_{1n} \xi^n - \frac{i}{2} \xi \right] +
\]

* $\delta_{mn} = 1$ ; $m=n$

$\delta_{mn} = 0$ ; $m \neq n$
\[
\sum_{n=1}^{\infty} \left( a_{1n} - X_e f_1^n \right) \xi^n - \frac{1}{2} \xi
\]

And equating the coefficients of like powers of \( \xi \) one obtains:

\[
\eta = 1 : \frac{1}{2a^2} f_1'' = \begin{cases} 
(1-X_e f_0') \left[ (n+1)a_{1n} - \frac{1}{2} \delta_{1n} \right] : n=1,2 \\
+ a_{1-1} X_e f_1' - \frac{1}{2} \delta_{1n} \\
(1-X_e f_0')[(n+1)a_{1n} + 21X_e a_1(n-2)] + \\
a_{1n} - X_e f_1' : n=3,4,5,...
\end{cases}

\text{(XIX.3)}

The boundary condition (91) (with \( n=1 \)) shows that:

\[
\eta = 0; \quad f_1 = f_1' = 0
\]

\text{(XIX.4)}

Examination of Eqs. (XIX.1) shows that \( f_{12} \) and \( a_{12} \) satisfy homogeneous equations with homogeneous boundary conditions which will be satisfied with \( f_{12}, a_{12} = 0 \).

Furthermore, this then makes the equations and boundary conditions for \( f_{14}, a_{14}, f_{15}, a_{15}, \text{etc.} \), homogeneous.

Hence:

\[
f_1 = a_{1n} = 0 \quad; \quad n=2,4,6,...
\]

A further investigation shows that we must take

\[
f_{11}, a_{11}; f_{15}, a_{15}; f_{19}, a_{19}; ...
\]

to be pure imaginary and \( f_{13}, a_{13}; f_{17}, a_{17}; \text{...} \) to be real.

Hence introducing the operator \( L_n \), defined by:

\[
L_n \equiv \frac{1}{2a^2 X_e} \frac{d^3}{d\eta^3} + f_0' \frac{d^2}{d\eta^2} - (n-1)f_0 \frac{d}{d\eta} + nf_0''
\]
one has:

\[
\begin{align*}
\ln(f_{in}) + \left[ (n-1)f''_0 + f'_0 \right] a_{1n} &= -\frac{1}{\chi_e} \delta_{\eta n} \\
-2i(1-\delta_{1n})[(f'_0 + \eta f''_0) a_{1(n-2)} - f'_1(n-2)]; \quad n=1,3,5,7,\ldots \\
\begin{cases}
\eta = 1 \\
\frac{1}{2a_2} f''_{in} = (1-\chi_e f'_0) \left[ (n+1) a_{1n} - \frac{i}{2} \delta_{1n} + 2i(1-\delta_{1n}) \chi_e a_{1(n-2)} \right] \\
\quad + a_{1n} - \chi_e f'_0 - \frac{i}{2} \delta_{1n} \\
\eta = 0 \quad [f_{in} = f'_in = 0]
\end{cases}
\end{align*}
\]

for \( n=2,4,6,8,\ldots \) \( f_{in} = a_{1n} = 0 \)

for \( n=1,5,9,\ldots \) \( f_{in}, a_{1n} \sim \text{pure imaginary} \)

for \( n=3,7,12,\ldots \) \( f_{1n}, a_{1n} \sim \text{real} \)
APPENDIX XX

REDUCTION OF FIRST-ORDER ENERGY PROBLEM TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Substitution of the expansions (103) through (105) into Eq. (96-la) gives:

\[
\sum_{n=0}^{\infty} \left[ \frac{1}{2a^2PrX_e} F_{1n} + f'_oF_{1n} \right] \xi^n - \sum_{n=1}^{\infty} n f'_o F_{1n} \xi^n = 0
\]

\[
-2i \sum_{n=0}^{\infty} F_{1n} \xi^{n+2} = -F'_0 \sum_{n=1}^{\infty} (f_{1n} + f'_o a_{1n} + 2n \xi^2 a_{1n}) \xi^{n-1}
\]

\[
= \sum_{n=0}^{\infty} \left[ \frac{1}{2a^2PrX_e} F''_{1n} + f'_o F_{1n} \right] \xi^n - \sum_{n=1}^{\infty} n f'_o F_{1n} \xi^n
\]

\[
-2i \sum_{n=2}^{\infty} F_{1(n-2)} \xi^n = -F'_0 \sum_{n=0}^{\infty} [(n+1)f_{1(n+1)} + f'_o a_{1(n+1)}] \xi^n
\]

\[
-2i \eta F'_o \sum_{n=2}^{\infty} a_{1(n-1)} \xi^n
\]

And equating the coefficients of like powers of \( \xi \), we get:

\[
\frac{1}{2a^2PrX_e} F''_{1n} + f'_o F_{1n} - n f'_o F_{1n} = -F'_0 [(n+1)f_{1(n+1)} + f'_o a_{1(n+1)}] + \begin{cases} 0 & ; \quad n = 0,1 \\ -2i F_1 \eta a_{1(n-1)} + 2i F_1(n-2) ; \quad n = 2,3,4,\ldots \end{cases} \quad (XX.1)
\]

Substitution of the expansions (103) through (105) into Eq. (97-la) gives:
\[ \sum_{n=0}^{\infty} \left\{ \frac{1}{2a^2 Xe Pr} \cdot \left( F_1' + F_1n \right) \right\} \xi^n + F_0 \sum_{n=1}^{\infty} \{(1+n)a_1n\} \]

2iXe a_1n \xi^{n-1} - \frac{i}{2} F_0 = 0

or:

\[ \sum_{n=0}^{\infty} \left\{ \frac{1}{2a^2 Xe Pr} \cdot \left( F_1' + F_1n \right) \right\} \xi^n + F_0 \sum_{n=0}^{\infty} \{(n+2)a_1(n+1)\} \xi^n \]

\[ + 2iXe F_0 \sum_{n=2}^{\infty} a_1(n-1) \xi^n - \frac{i}{2} F_0 = 0 \]

And equating coefficients of like powers of \( \xi \) we get:

\[ \eta = 1: \left( \frac{1}{2a^2 Xe Pr} \cdot F_1' + F_1n + F_0(n+2)a_1(n+1) \right) + \left\{ \frac{i}{2} \delta_{n0} F_0 \right\} = 0, n=0,1 \]

\[ 2iXe F_0 a_1(n-1) = 0, n=2,3,4,... \]

(XX.2)

and the boundary condition (98.1) implies:

\[ \eta = 0; \quad F_1n = 0 : \text{for } n=0,1,2,... \quad \text{(XX.3)} \]

Since \( f_{12}, a_{12} = 0 \), inspection of Eqs. (XX.1) through (XX.3) shows that \( F_{11} \) satisfies homogeneous equations and boundary conditions. Therefore we may take \( F_{11} = 0 \). And since, the \( F_{1n} \)'s depend only on the \( F_1(n-2)'s \) and the \( f_1(n+1), f_1(n-1), a_1(n+1), a_1(n-1) \) explicitly, we have:

\[ 0 \leq \eta \leq 1; \quad F_1n = 0 : \text{for } n=1,3,5,7,... \]

A further inspection of Eqs. (XX.1) through (XX.3) shows that the \( F_{1n} \) must be pure imaginary quantities for \( n = 0,4,8,... \) and the \( F_{1n} \) must be real for \( n = 2,6,10,... \).
Thus, we have, if we define the operator $K_n$ by:

$$
K_n = \frac{1}{2a^2PrXe} \frac{d^2}{d\eta^2} + f_0 \frac{d}{d\eta} - nf'_0
$$

$$
0 < \eta < 1: \{K_n(f_n) = -f'_0[n(n+1)f_1(n+1)+f_0a_1(n+1)-2i(1-\delta_{on})\eta a_1(n-1)] + 2i(1-\delta_{on})f_1(n-2)
$$

$$
\eta=1: \left\{ \frac{1}{2a^2XePr} F_{1n}^1 + F_{1n} + F_0((2+n)a_1(n+1)+2iXe(1-\delta_{on})a_1(n-1) - \frac{i}{2} \delta_{on}) = 0
$$

$$
\eta=0: \{ F_{1n} = 0
$$

$F_{1n} = 0$ : for $n=1,3,5,7,...$

$F_{1n}$ = pure imaginary quantity: for $n=0,4,8,...$

$F_{1n}$ = real quantity: for $n=2,6,10,...$
APPENDIX XXI

REDUCTION OF THE SECOND-ORDER MOMENTUM PROBLEM TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Substitution of the expansions (103), (104), (112), and (113) into Eq. (88-2a), using the relation (111) to simplify and then equating the terms which are coefficients of \( e^{2\alpha t} \) with each other and the time independent terms with each other (since no other terms appear as can be easily seen by inspection) yields for the coefficients of \( e^{2\alpha t} \):

\[
\xi \sum_{n=1}^{\infty} \left\{ -\frac{1}{2a^2\chi_e} f''_{2n} - f_o f''_{2n} + (n-1) f'_o f'_{2n} - nf'_o f''_{2n} + [(1-n) f'_o - f'_o] a_{2n} \right\} \xi^n
\]

\[+ 4i\xi \sum_{n=3}^{\infty} \left\{ f'_{2(n-2)} - (f'_o + \eta f'_o) a_{2(n-2)} \right\} \xi^n = \frac{3\xi^3}{2\chi_e} \sum_{n=1}^{\infty} a_{1n} \xi^n
\]

\[+ \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{ -nf'_i f'_{lm} + nf'_i f''_{lm} + f'_i f'_{lm} - a_{lm} (n f'_o f'_{lm}
\]

\[+ f'_o f'_{lm} - nf'_o f''_{lm} - f'_o f''_{lm}) + 2nf'_o a_{1mf'_{lm}} \} \xi^{n+m} +
\]

\[+ i \xi^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{ -2a_{1mf'_{lm}} (\eta f'_o + f'_o) a_{1n} a_{1m} + a_{1n} (\eta f''_{lm}
\]

\[+ f'_{lm}) \} \xi^{n+m}
\]

And for the time independent terms:

\[
\xi \sum_{n=1}^{\infty} \left\{ -\frac{1}{2a^2\chi_e} f''_{2n} - f_o f''_{2n} + (n-1) f'_o f'_{2n} - nf'_o f''_{2n} \right\} \xi^n
\]

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\[ + \left[ (1-n)f_0^2 + f_0 f_n^n \right] a_2n \] \[ \xi^n = \frac{3n^3}{2X_e} \sum_{n=1}^{\infty} a_{1n} \xi^n + \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ \begin{array}{l}
-nf_n f_{1m} + nf_{1m} f_n f_{1m} - a_{1m} \left( n f_0 f_n f_{1m} \right) + f_0 f_n f_{1m} - 2nf_0 a_{1m} f_{1m} \right] \xi^{m+n} \\
+ i \xi^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ -2f_{1m} a_{1n} + (\eta f_0 + f_0) a_{1m} a_{1n} + a_{1n} (\eta f_{1m} + f_{1m}) \right] \xi^{m+n} 
\]

And equating coefficients of like powers of \( \xi \) we get from the above two equations:

\[ \frac{1}{2aX_e} f_{2n}^{2} + f_{0} f_{2n} - (n - 1)f_0 f_{2n} + nf_0 f_{2n} = \]
\[ - [(n - 1)f_0^2 + f_0 f_n^n] a_{2n} H_n \] \hspace{1cm} (XXI.1)

\[ \frac{1}{2aX_e} f_{2n}^{2} + f_{0} f_{2n} - (n - 1)f_0 f_{2n} + nf_0 f_{2n} = \]
\[ - [(n - 1)f_0^2 + f_0 f_n^n] a_{2n} + \tilde{H}_n \] \hspace{1cm} (XXI.2)

Where for \( n = 1, 3, 5, 7, \ldots \)

\[ H_n = - \frac{3}{2X_e} \left( 1 - \delta_{1n} \right) a_1 (n - 2) + \frac{1}{2} \left( 1 - \delta_{1n} \right) \left[ f_{12} (n - 2) \right] \]
\[ (f_0 + \eta f_0) a_2 (n - 2) - \frac{1}{2} \left[ f_{1f} f_{1n} (n - 1) + f_{0} f_{1f} + f_{2f} f_{1f} + f_0 f_{1f} \right] \]
\[ + (1 - \delta_{1n}) nf_{1f} f_{1n} + [2(n - 1)f_0 f_{1f} + f_{2f} f_{1f} + f_0 f_{1f}] a_{1n} \] \hspace{1cm} (XXI.3)
\( (1 - \delta_{1n})[nf''_{1n} - (n-1)f''_{0f^1n} + f''_{0f^1n}]a_{11} - \\
- i(1 - \delta_{1n})\{(\eta^f_{0f^1} + f'_{0f^1})a_{11}a_{1}(n-2)\} + (\eta''_{f_{11}} - f''_{11})a_{1}(n-2)\} - P_n \)  

**Where:**

\[
\begin{align*}
P_1 &= 0 \\
P_5 &= \frac{1}{2} (3f_{13}f''_{13} - 2f_{13}f_{13} + [f_{0f^1}^f + f_{0f^1}^g]a_{13} + [f_{0f^1}^f + f_{0f^1}^g]a_{13}) \\
&+ 3f''_{0f^1} + 2f_{0f^1}f_{13} a_{13} + i((\eta^f_{0f^1} + f'_{0f^1})a_{11}a_{13} + (\eta''_{f_{11}} - f''_{11})a_{11})
\end{align*}
\]

For \( n = 2, 4, 6, 8, \ldots \):

\[
H_n = 0 \quad (XXI.4)
\]

follows from (108)

and for: \( n = 1, 5, \ldots \):

\[
\begin{align*}
H_n &= - \frac{3}{2x_{0f}} (1 - \delta_{1n})a_{1}(n-2) - \frac{1}{2} (f''_{11}f''_{1n} - (n-1)f'_{11}f'_{1n} \\
&+ (1 - \delta_{1n})nf''_{1n}f''_{1n} + (2n-1)f'_{0f^1}f'_{11} + f''_{0f^1}f''_{11})a_{1n} \\
&+ (1 - \delta_{1n})[nf'_{0f^1}f'_{1n} - (n-1)f'_{0f^1}f'_{1n} + f''_{0f^1}f''_{1n}]a_{11} - P_n
\end{align*}
\]

**Where:**

\[
\begin{align*}
\tilde{P}_1 &= 0 \\
\tilde{P}_5 &= \frac{1}{2} (3f_{13}f''_{13} - 2f_{13}f_{13} + [f_{0f^1}^f + 3f_{0f^1}^g]f_{13} + \\
&+ 3f''_{0f^1} + 2f_{0f^1}f_{13} a_{13} + i((\eta^f_{0f^1} + f'_{0f^1})a_{11}a_{13} + (\eta''_{f_{11}} - f''_{11})a_{11}) - \tilde{P}_n
\end{align*}
\]

where:
\[ + 2f_0 f_{13} \bar{\xi}_{13} + i((\bar{\eta} f_{11} + 3f_{11}) a_{13} + \\
(\eta f_{13} + 3f_{13}) a_{11}) \]

For \( n = 3, 7, \ldots \) we have from (108) \( \tilde{\mathbb{H}}_n = \) pure imaginary quantity.

And since we are actually seeking the real part of \( \tilde{\mathbb{H}}_n \) we set for \( n = 3, 7, \ldots \)

\[ \tilde{\mathbb{H}}_n = 0 : n = 2, 4, 6, \ldots \quad (XXI.6) \]

and for \( n = 2, 4, 6, \ldots \) it follows from (110) that

\[ \tilde{\mathbb{H}}_n = 0 : n = 2, 4, 6, \ldots \quad (XXI.7) \]

Next substitution of the expansions (103), (104), (112), and (113) into Eq. (89-2a), using the relation (111) to simplify, and then equating the terms which are coefficients of \( e^{2i\tau} \) with each other and the time independent terms with each other, (since no other terms appear) gives for the coefficients \( e^{2i\tau} \):

\[ \sum_{n=1}^{\infty} f_{2n} \xi^n = \sum_{n=1}^{\infty} a_{2n} \xi^n - \frac{1}{4} \sum_{n=1}^{\infty} a_{1n} + 4i(Xe-1) \sum_{n=1}^{\infty} a_{2n} \frac{\xi^n}{n+2} \]

or:

\[ \sum_{n=1}^{\infty} f_{2n} \xi^n = \sum_{n=1}^{\infty} [a_{2n} - \frac{1}{4} a_{1n}] + 4i(Xe-1) \sum_{n=3}^{\infty} a_{2(n-2)} \frac{\xi^n}{n} \]

and for the time independent terms:

\[ \sum_{n=1}^{\infty} \tilde{f}_{2n} \xi^n = \sum_{n=1}^{\infty} \tilde{a}_{2n} \xi^n - \frac{1}{4} \sum_{n=1}^{\infty} \tilde{a}_{1n} \ xi^n \]
And equating the coefficients of like powers of $\xi$ in these last two equations we get:

\[
f_{2n} = a_{2n} - \frac{i}{4} a_{1n} \begin{cases} 0 & n=1,2, \quad \ldots \\ \frac{1}{2} (X_{e} - 1) a_{2(n-2)}/n & n=3,4,5,7 \end{cases} \quad \text{(XXI.8)}
\]

\[
\tilde{f}_{2n} = \tilde{a}_{2n} - \frac{i}{4} \tilde{a}_{1n} \quad n=1,2,3, \ldots \quad \text{(XXI.9)}
\]

And finally substitution of the expansions (103), (104), (112), and (113) into Eq. (90-2a), using the relation (111) to simplify and then equating the terms which are coefficients of $e^{2i\tau}$ with each other and the time independent terms with each other (since no other terms appear) gives for the coefficients of $e^{2i\tau}$:

\[
2a\xi(1-X_{e} f'_{0}) \left[ a \sum_{n=1}^{\infty} ((n+1)a_{2n} + iX_{e} a_{2n} \xi^{2})\xi^{n} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n}(iX_{e} a_{1m} \xi^{2} + \frac{m}{2} a_{1m} \xi^{2})\xi^{n+m} - \frac{i a}{4} \sum_{n=1}^{\infty} (n+1) a_{1n} \xi^{n} \right] + a^{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{1n} \xi^{m} + X_{e} f'_{in})(m+1)a_{1m} + 2iX_{e} \xi^{2} \xi^{n+m} - \frac{1}{2} \sum_{n=1}^{\infty} (a_{1n} - f'_{in})\xi^{n+1} + \frac{1}{2} (2a)^{2} \sum_{n=1}^{\infty} (a_{2n} - X_{e} f'_{2n})\xi^{n} - \frac{1}{6} (2a)^{2} \sum_{n=1}^{\infty} ((n+2)a_{1n} + 2iX_{e} a_{1n} \xi^{2})\xi^{n} - \frac{(2a)^{2}}{16} \xi^{2} = \xi \sum_{n=1}^{\infty} f''_{2n} \xi^{n}
\]
Simplifying we get:

\[
\frac{1}{2} (2a)^2 \left\{ (1-Xe^{f_0'}) \sum_{n=1}^{\infty} [(n+1)(a_{2n} - \frac{1}{4} a_m)]^n + \
4i(1-\delta_{1n})(1-\delta_{2n})a_2(n-2)\xi^n + \
+ (1-Xe^{f_1'}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{1n}((1-\delta_{1n})(1-\delta_{2n})iXe a_1(n-2) \
+ \frac{1}{2} m a_{1m})\xi^{m+n-1} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{1n}Xe^{f_{1n}}) [(m+1)a_{1m} \
+ 2iXe(1-\delta_{1m})(1-\delta_{2m})a_1(n-2) - \frac{i}{2} \delta_{1m}]\xi^{n+m-1} \
+ \sum_{n=1}^{\infty} (a_{2n}Xe^{f_{2n}})\xi^n - \frac{1}{4} \sum_{n=1}^{\infty} [(n+2)a_{1n}^+ \
2iXe(1-\delta_{1n})(1-\delta_{2n})a_1(n-2)\xi^n - \frac{\xi}{8} \right\} = \sum_{n=1}^{\infty} f_{2n}^n \xi^n
\]

or:

\[
\frac{1}{2} (2a)^2 \left\{ \sum_{n=1}^{\infty} ((1-Xe^{f_0'}) [(n+1)(a_{2n} - \frac{1}{4} a_m)]^n + \
(1-\delta_{1n})(1-\delta_{1m})4iXe a_2(n-2) - \frac{1}{2} (a_{1n}Xe^{f_{1n}})^+ \
(a_{2n}Xe^{f_{2n}})^- - \frac{1}{4} [(n+2)a_{1n}^+ + 2i(1-\delta_{1n})(1-\delta_{2n})Xe a_1(n-2)] \
- \frac{\delta_{1n}}{\xi} \right\} \xi^n + \
+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} ((1-Xe^{f_0'})a_{1n} [(1-\delta_{1m})(1-\delta_{2m})iXe a_1(n-2))
\]
\[
\frac{1}{2} m a_{lm} + (a_{1n} - X_{e f_{1n}}) [(m+1) a_{lm} + \\
2i X_e (1-\delta_{lm})(1-\delta_{mn}) a_1(m-2)] \xi^{n + m - 1} = \sum_{n=1}^{\infty} f_{2n}^{\prime n} 
\]

and equating the coefficients of like powers of \( \xi \) gives:
\[
\frac{1}{2} (2a)^2 \left\{ (1-X_{e f_0}) [(n+1)(a_{2n} - \frac{1}{4} a_{1n}) + \\
+ 4i(1-\delta_{ln}-\delta_{mn}) a_{2(n-2)} - \frac{i}{2} (a_{1n} - X_{e f_1}) + (a_{2n} - X_{e f_2})] \\
- \frac{i}{4} [(n+2)a_{1n} + 2i(1-\delta_{ln}-\delta_{mn}) X_e a_1(n-2)] - \frac{5}{8} n + c_n \right\} = f_{2n}^{\prime n} \tag{XXI.10}
\]

where:
\[
C_1 = (1-X_{e f_0}) \frac{a_{11} a_{11}}{2} + (a_{11} - X_{e f_{11}}) 2a_{11} \\
C_3 = (1-X_{e f_0}) (a_{11}[\frac{3}{2} a_{13} + i X_e a_{11}] + \frac{1}{2} a_{13} a_{11}) \\
+ (a_{11} - X_{e f_{11}}) [4a_{13} + 2i X_e a_{11}] + 2a_{11} (a_{13} - X_{e f_{13}}) \\
C_5 = (1-X_{e f_0}) (a_{11}[\frac{5}{2} a_{15} + i X_e a_{13}] + \\
a_{13} [\frac{3}{2} a_{13} + i X_e a_{11}] + \frac{1}{2} a_{11} a_{13}] + \\
(a_{11} - X_{e f_{11}}) [6a_{15} + 2i X_e a_{13}] + (a_{13} - X_{e f_{13}}) [4a_{13} + 2i X_e a_{11}] + 2a_{11} (a_{15} - X_{e f_{15}}) \tag{XXI.11}
\]
And from (108) we have

\[ C_n = 0 : \text{ for } n=2,4,6,... \]  \hspace{1cm} (XXI.12)

And similarly one obtains for the time independent terms:

\[ \frac{1}{2} (2a)^2 ((1-X_\theta f')^\dagger) (n+1) (\tilde{a}_{2n} - \frac{i}{4} \bar{a}_{1n}) - \frac{i}{2} (\bar{a}_{1n} - \bar{f}_{1n}) \]

\[ + (\tilde{a}_{2n} - X_\theta \tilde{f}_{2n}) - \frac{i}{4} [(n+2) \bar{a}_{1n} - 2i (1 - 6_{1n} - 6_{2n}) X_\theta \bar{a}_{1}(n-2)] \]

\[ + \frac{\delta_{1n}}{8} + \tilde{C}_n \right] = \tilde{f}_{2n} \]  \hspace{1cm} (XXI.13)

where:

\[ \tilde{C}_1 = (1-X_\theta f') \frac{1}{2} \bar{a}_{11} a_{11} + 2 a_{11} (\bar{a}_{11} - X_\theta \bar{f}_{11}) \]

\[ \tilde{C}_5 = (1-X_\theta f') [\bar{a}_{11} \left( \frac{3}{2} a_{15} + i X_\theta a_{13} \right) + \bar{a}_{13} \left( \frac{3}{2} a_{13} + i X_\theta a_{11} \right) + \frac{1}{2} a_{11} \bar{a}_{13}] \]

\[ (\bar{a}_{11} - X_\theta \bar{f}_{11}) [6 a_{15} + 2i X_\theta a_{13}] + (\bar{a}_{13} - X_\theta \bar{f}_{13}) [4 a_{13} \]

\[ + 2i X_\theta a_{11}] + (\bar{a}_{15} - X_\theta \bar{f}_{15}) 2 a_{11} \]  \hspace{1cm} (XXI.14)

And as before inspection of (110) shows that \( \tilde{C}_3, \tilde{C}_1, \) are pure imaginary and since we are only interested in the real part of \( \tilde{f}_{2n} \) we can set:

\[ \tilde{C}_n = 0 : \text{ for } n=3,7,... \]  \hspace{1cm} (XXI.15)

and (108) shows further that

\[ \tilde{C}_n = 0 : \text{ for } n=2,4,6,... \]  \hspace{1cm} (XXI.16)
The boundary condition (21) shows that we must have:

\[
\begin{align*}
\text{at } \eta = 0; \quad f_{2n} &= \frac{df_{2n}}{d\eta} = 0 \\
\text{at } \eta = 0; \quad \tilde{f}_{2n} &= \frac{df_{2n}}{d\eta} = 0
\end{align*}
\]

(XXI.17)

Inspection of Eqs. (XXI.1) through (XXI.17) and use of the condition (108) shows that \(a_{2n}, f_{2n}\) satisfy a homogeneous differential equation with homogeneous boundary conditions for \(n=2\) and therefore \(a_{22}, f_{22} = 0\). This then implies that the higher order \(f_{2n}\) for \(n=4,6,...\) satisfy a homogeneous equation with homogeneous boundary conditions and hence we have:

\[a_{2n}, f_{2n} \equiv 0 \quad : \quad \text{for } n=2,4,6,...\]

and similarly:

\[\tilde{a}_{2n}, \tilde{f}_{2n} \equiv 0 \quad : \quad \text{for } n=2,4,6,...\]

But since we are only interested in the real part of \(\tilde{f}_{2n}, \tilde{a}_{2n}\) we see that these also satisfy homogeneous conditions for \(n=3,7,...\)

Hence one has:

\[\tilde{a}_{2n}, \tilde{f}_{2n} \equiv 0 \quad : \quad \text{for } n=3,7,...\]

Thus we have upon introducing the operator \(L_n\) defined by (107):

For \(n=1,3,5,...\):

\[0 \leq \eta \leq 1 \quad L_n(f_{2n}) = -[(n-1)f_0^2 + f_0 f''']a_{2n} + H_n\]
\[ f_{2n} = a_{2n} - \frac{i}{4} a_{1n} + 4i(1-5i_n)(x_e-1) \frac{a_{2(n-2)}}{n} \]

\[ \eta = 1 \]

\[ \frac{1}{2a_{2n}^2} f_{2n}^{\eta} = (1-x_{e_{0}}')[(n+1)(a_{2n} - \frac{i}{4} a_{1n})+4i(1-5i_n)a_{2(n-2)}] - \frac{i}{2} (a_{1n} - x_{e_{0}}'a_{1n}) + a_{2n} \]

\[ - x_{e}f_{2n}' - \frac{i}{4} [(n+2)a_{1n} + 2i(1-5i_n)x_{e}a_{1}(n-2)] \]

\[ - \frac{\delta_{1n}}{8} + c_{n} \]

\[ \eta = 0 \]

\( f_{2n} = f_{2n}' = 0 \)

and:

\( a_{2n}, f_{2n}(\eta) \equiv 0 : \text{for } n=2,4,6,... \)

For \( n=1,5,... \)

\[ 0 \leq \eta \leq 1 \]

\[ I_n(\tilde{f}_{2n}) = - [(n-1)f_{o}^{12} + f_{o}^{12} f_{o}^{12}]a_{2n} + \tilde{h}_{n} \]

\[ \eta = 1 \]

\[ \tilde{f}_{2n} = \tilde{a}_{2n} - \frac{i}{4} \tilde{a}_{1n} \]

\[ \frac{1}{2a_{2n}^2} \tilde{f}_{2n}^{\tilde{\eta}} = (1-x_{e_{0}}')[(n+1)(\tilde{a}_{2n} - \frac{i}{4} \tilde{a}_{1n}) - \frac{i}{2} (\tilde{a}_{1n}) - \tilde{f}_{1n} + \tilde{a}_{2n} - x_{e}f_{2n}' - \frac{i}{4} [(n+2)\tilde{a}_{1n} - 2i(1-5i_n)x_{e}a_{1}(n-2)] + \frac{\delta_{1n}}{8} + \tilde{c}_{n} \]

\[ \eta = 0 \]

\( \tilde{f}_{2n} = \tilde{f}_{2n}' = 0 \)

and:

\( a_{2n}, \tilde{f}_{2n}(\eta) \equiv 0; \text{for } n=2,4,6,... \) and \( n=5,7,... \)

Where \( H_{n}, \tilde{h}_{n}, C_{n} \) and \( \tilde{C}_{n} \) are given by (XXI.3), (XXI.5), (XXI.11) and (XXI.14) respectively.
APPENDIX XXII

REDUCTION OF THE SECOND-ORDER ENERGY PROBLEM TO A SYSTEM OF
ORDINARY DIFFERENTIAL EQUATIONS

Substitution of the expansions (103) through (105), and (112)
through (114) into Eq. (96-2a), using the relation (111) to simplify,
and then equating the terms which are coefficients of \(e^{2\alpha\tau}\) with each
other and the time independent terms with each other (since no other
terms appear as can easily be verified by inspection) gives for the
coefficient of \(e^{2\alpha\tau}\):

\[
\sum_{n=0}^{\infty} \left\{ \frac{1}{2a^2PrX_0} \frac{F_{2n}'' + f_0 F_{2n}'}{F_{2n}'' + f_0 F_{2n}' - f_0' n F_{2n}} \right\} \xi^n - i \sum_{n=0}^{\infty} \frac{2n+2}{F_{2n}} \xi^n = -\sum_{n=1}^{\infty} F_0' (n f_{2n} + f_0 a_{2n}) \xi^{n-1} - \frac{1}{4} \sum_{n=1}^{\infty} \eta a_{2n} \xi^{n+1}
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ a_{1n} F_{1m} F_0' \xi^{n+m-2} \right] - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \eta a_{1n} F_{1m} F_0' \xi^{n+m+2} + 2i \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[ f_{1n} F_{1m} F_0' \xi^{n+m+1} \right] + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[ f_{1n}' F_{1m} F_0' \xi^{n+m+1} \right]
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[ f_{1n}' F_{1m} F_0' \xi^{n+m+1} \right] - i \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \eta a_{1n} F_{1m} F_0' \xi^{n+m+1}
\]

or using (108), (110), (118) and rearranging:

\[
\sum_{n=0}^{\infty} \left\{ \frac{1}{2a^2PrX_0} \frac{F_{2n}'' + f_0 F_{2n}'}{F_{2n}'' + f_0 F_{2n}' - f_0' n F_{2n}} \right\} \xi^n = 4i \sum_{n=0}^{\infty} \left( 1 - \delta_{0n} - \delta_{1n} \right) F_2(n-2) \xi^n
\]

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\[- \frac{F_0'}{2} \sum_{n=0}^{\infty} \left( (n+1)f_2(n+1) + f_0 a_2(n+1) + 4i \eta (1 - \delta_0 n) a_2(n-1) \right) \xi^n \]

\[- \frac{1}{2} \frac{F_0'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( m a_1 f_{1m} + 2i \eta (1 - \delta_1 m) a_1 a_1(m-2) \right) \xi^{n+m-2} \]

\[+ \frac{i}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left( m F_{1m} (f_{1n} + f_0 a_1 n) - F_{1m} (f_{1n} + f_0 a_1 n) \right) \xi^{n+m-1} \]

\[+ \frac{i}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left( 2a_1 f_{1m} - \eta a_1 f_{1m} \right) \xi^{n+m+1} \]

And for the time independent terms:

\[\sum_{n=0}^{\infty} \left[ \frac{1}{2} F_{2n} + f_0 f_{2n} - \eta f_{2n} \right] \xi^n = - \frac{F_0'}{2} \sum_{n=1}^{\infty} \left( (n+1) f_2(n+1) + f_0 a_2(n+1) \right) \xi^n \]

\[- \frac{1}{2} \frac{F_0'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_1 f_{1m} \xi^{n+m-2} - \frac{i}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} F_{1m} \eta a_1 a_1 \xi^{n+m} \]

\[+ 2i \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_1 f_{1m} \xi^{n+m+1} + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left( f_{1n} + f_0 a_1 n \right) m F_{1m} \xi^{n+m-1} \]

\[+ \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left( f_{1n} + f_0 a_1 n \right) F_{1m} \xi^{n+m-1} - \frac{i}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \eta a_1 f_{1m} \xi^{n+m+1} \]

And using (108), (110), and (123) and rearranging we get:

\[\sum_{n=0}^{\infty} \left[ \frac{1}{2} a_2 f_{2n} + f_0 f_{2n} - \eta f_{2n} \right] \xi^n = - \frac{F_0'}{2} \sum_{n=0}^{\infty} \left( (n+1) f_2(n+1) + f_0 a_2(n+1) \right) \xi^n \]

\[- \frac{1}{2} \frac{F_0'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( m a_1 f_{1m} + 2i \eta (1 - \delta_1 m) a_1 a_1(m-2) \right) \xi^{n+m-2} \]
\[
\begin{align*}
+ \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (mF'_{1m}(F_{1n}+f_0a_{1n})-F'_{1m}(F_{1n}+f_0a_{1n}))\xi^{m+n-1} \\
+ \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (2a_{1n}F_{1m}-\eta a_{1n}F_{1m})\xi^{n+m+1}
\end{align*}
\]

And upon equating the coefficients of like powers of \( \xi \) and using

(108), (110), (118), and (123), to simplify, one obtains:

\[
\frac{1}{2a_{1}^{2}} F_{0}' F_{2m} + f_{0} F_{0}' F_{2m} - nf_{0}' F_{2m} = 4i(1-\varepsilon_{0n}-\varepsilon_{1})F_{0}(n-2)
\]

- \( F_{0}'[(n+1)f_{2}(n+1)+f_{0}a_{2}(n+1)+4i(1-\varepsilon_{0n})\eta a_{2}(n-1)]+E_{n} \) (XXII.1)

where for \( n = 0, 2, 4, \ldots \)

\[
\begin{align*}
E_{0} & = -\frac{1}{2} F_{0}' a_{11}f_{11} - \frac{1}{2} F_{0}' (f_{11}+f_{0}a_{11}) \\
E_{2} & = -\frac{1}{2} (5a_{11}f_{13}+2i\eta a_{11}a_{11}+a_{13}f_{11})F_{0}' + \frac{1}{2} (2F_{12}(f_{11}+f_{0}a_{11})
\end{align*}
\]

- \( F_{12}(f_{11}+f_{0}a_{11})-F_{10}(3f_{13}+f_{0}a_{13})+ia_{11}(2F_{10}-\eta F_{10}) \) (XXII.2)

\[
\begin{align*}
E_{4} & = -\frac{1}{2} (5a_{11}f_{15}+4i\eta a_{11}a_{13}+3a_{13}f_{13}+a_{15}f_{11})F_{0}' \\
& + \frac{1}{2} [4F_{14}(f_{11}+a_{11}F_{0}')] - F_{14}(f_{11}+f_{0}a_{11})+2F_{12}(f_{13}+f_{0}a_{13})-F_{12}(f_{13}+f_{0}a_{13})
\end{align*}
\]

- \( F_{10}'(5f_{15}+f_{0}a_{15})+ia_{11}(2F_{12}-\eta F_{12})+ia_{13}(2F_{10}-\eta F_{10}) \)

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and:

\[
E_{n} = 0; \text{ for } n = 1, 3, 5, \ldots \] (XXII.3)
\[
\frac{1}{2a^2x_\text{Pr}} \tilde{F}_\text{en} + f'_0 \tilde{F}_\text{en} - n^2 \tilde{F}_\text{en} = -F'_0 ((n+1)\tilde{f}_2(n+1)+f'_0 \tilde{a}_2(n+1)) + \tilde{E}_\text{en} \tag{XXII.4}
\]

where for \( n = 0, 4, 8, \ldots \)

\[
\tilde{E}_0 = -\frac{1}{2} F'_0 a_{11} \tilde{q}_{11} - \frac{1}{2} F_{10} (\tilde{q}_{11} + f'_0 \tilde{a}_{11})
\]

\[
\tilde{E}_4 = -\frac{1}{2} \left( 5a_{11} \tilde{q}_{15} + 3a_{13} \tilde{q}_{13} + a_{15} \tilde{q}_{11} \right) F'_0 + \frac{1}{2} \left( 4F_{14} (\tilde{q}_{11} + a_{11} f'_0) \right. 
\]

\[-F'_4 (\tilde{q}_{11} + f'_0 \tilde{a}_{11}) + 2F_{12} (\tilde{q}_{12} + f'_0 \tilde{a}_{13}) - F_{12} (3\tilde{q}_{13} + f'_0 \tilde{a}_{13}) - F_{10} (5\tilde{q}_{15} + f'_0 \tilde{a}_{15}) \right) + a_{11} (2F_{12} + \eta F_{12}) + a_{13} (2F_{10} - \eta F_{10})
\]

\[\tilde{E}_n = 0 \quad \text{for} \ n \neq 0, 4, 8, \ldots \]

since we are taking the real parts and \( \tilde{E}_2, \tilde{E}_6, \ldots \) are pure imaginary and hence contribute nothing.

Next substitution of the expansion (103) through (105), and (112) through (114) into Eq. (97-2a), using the relation (111) to simplify and then equating the terms which are coefficients of \( e^{\pm i\tau} \) with each other and the time independent terms with each other (since no other terms appear) gives for the coefficients of \( e^{\pm i\tau} \):

\[
\sum_{n=0}^{\infty} \frac{1}{2a^2x_\text{Pr}x_e} F_{\text{en}} + F_{\text{en}} + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} F_{1n} ((1+m)a_{1m} \\
+ 2i x_e a_{1m}) \xi^{n+m-1}
\]
\[ + F_0 \sum_{n=1}^{\infty} \left\{ (l+n)a_{2n} + 4iX_e a_{2n} \xi^2 \right\} \xi^{n-1} \]
\[ + F_0 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n} (iX_e \xi^2 a_{1m} + \frac{m}{2} a_{1m}) \xi^{n+m-2} \]
\[ - \frac{i}{4} \sum_{n=0}^{\infty} F_{1n} \xi^n - \frac{i}{4} F_0 \sum_{n=1}^{\infty} (l+n)a_{1n} \xi^{n-1} \]

And using (108), (110), and (118) and rearranging we get:

\[ \sum_{n=0}^{\infty} \left[ \frac{1}{2a^2 F_{1n} + F_{2n}} \right] + \sum_{n=0}^{\infty} \left[ F_0 \left\{ (2+n)(a_2(n+1) - \frac{i}{4} a_1(n+1)) + 4iX_e (1-\delta_{on}) a_2(n-1) \right\} \right] \]
\[ - \frac{i}{4} F_{1n} \xi^n + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} F_{1n} \left\{ (1+m)a_{1m} + 2iX_e a_{1m} \xi^2 \right\} \xi^{n+m-1} \]
\[ + F_0 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n} \left( \frac{m}{2} a_{1m} + iX_e a_{1m} \xi^2 \right) \xi^{n+m-2} \]

and upon equating the coefficients of like powers of \( \xi \) and using the relations (108), (110), and (118) we get:

\[ \frac{1}{2a^2 F_{1n} + F_{2n}} F_{2n} + F_{2n} + F_0 \left\{ (2+n)(a_2(n+1) - \frac{i}{4} a_1(n+1)) + 4iX_e (1-\delta_{on}) a_2(n-1) \right\} \]
\[ - \frac{i}{4} F_{1n} + B_n = 0 \quad (XXII.7) \]

where for \( n=0, 2, 4, \ldots \)
\[ B_0 = a_{11} F_{10} + \frac{1}{2} a_{11} a_{11} F_0 \]

\[ B_2 = \frac{1}{2} F_{10} \left( 4a_{13} + 2iX_e a_{11} \right) + \frac{1}{2} 2 a_{11} F_{12} \]

\[ B_4 = \frac{1}{2} F_{10} \left( 6a_{15} + 2iX_e a_{13} \right) + \frac{1}{2} F_{12} \left( 4a_{13} + 2iX_e a_{11} \right) + a_{11} F_{14} \]

\[ + a_{11} \left( \frac{5}{2} a_{15} + iX_e a_{13} \right) F_0 + a_{13} \left( \frac{7}{2} a_{13} + iX_e a_{11} \right) F_0 + \frac{1}{2} a_{15} a_{11} F_0 \]

(XXII.8)

and

\[ B_n = 0 \quad \text{for} \quad n=1,3,5,\ldots \]  

(xxii.9)

and similarly for the time independent terms, we get:

\[ \frac{1}{2a^2 E} F_{en} + \frac{1}{2} F_{en} + F_0 ((2+n)(\bar{a}_{2(n+1)} - \frac{i}{4} \bar{a}_{1(n+1)}) - \frac{i}{4} \bar{F}_{1n} + B_n = 0 \]

(xxii.10)

where for \( n=0,4,\ldots \)

\[ B_0 = a_{11} F_{10} + \frac{1}{2} a_{11} a_{11} F_0 \]

\[ B_4 = \frac{1}{2} F_{10} (6a_{15} + 2iX_e a_{13}) + \frac{1}{2} F_{12} (4a_{13} + 2iX_e a_{11}) \]

\[ + a_{11} F_{14} + a_{11} \left( \frac{5}{2} a_{15} + iX_e a_{13} \right) F_0 + \bar{a}_{13} \left( \frac{7}{2} a_{13} + iX_e a_{11} \right) F_0 \]

\[ + \frac{1}{2} a_{15} a_{11} F_0 \]

(xxii.11)

and

\[ \bar{B}_n = 0; \quad \text{for} \quad n=1,3,5,\ldots \]
but since we are taking real parts and \( \tilde{B}_2, \tilde{B}_6 \ldots \) are pure imaginary, we can set them equal to zero and we have:

\[
\tilde{B}_n = 0 \quad \text{for} \quad n \neq 0, 4, 8, \ldots \quad (XXII.12)
\]

and finally the boundary conditions (98.2) require that:

\[
\begin{align*}
\eta = 0; & \quad F_{2n} = 0 \\
\eta = 0; & \quad \tilde{F}_{2n} = 0
\end{align*} \quad (XXII.13)
\]

Inspection of Eqs. (XXII.1) through (XXII.13) and use of Eqs. (108), (110), (118), and (123) shows that \( F_{21} \) satisfies a homogeneous differential equation with homogeneous boundary conditions for \( n=2 \) and therefore \( F_{21}(\eta) \equiv 0 \). This then implies that the higher order \( F_{2n} \) for \( n=3, 5, 7, \ldots \) satisfy homogeneous equations with homogeneous boundary conditions and hence:

\[
F_{2n}(\eta) \equiv 0; \quad \text{for} \quad n = 1, 3, 5, \ldots
\]

and similarly:

\[
\tilde{F}_{2n}(\eta) \equiv 0; \quad \text{for} \quad n = 1, 3, 5, \ldots
\]

But since we are only interested in the real part of \( \tilde{F}_{2n} \) we see that they also satisfy homogeneous conditions for \( n=2, 6, \ldots \). Hence we have:

\[
\tilde{F}_{2n}(\eta) \equiv 0; \quad \text{for} \quad n \neq 0, 4, 8, \ldots
\]

Thus the relations of this appendix may be summarized introducing the operator \( K_n \) (defined below Eqs. (109)) by:

For \( n = 0, 2, 4, \ldots \)
\[ 0 \leq \eta \leq 1 \quad K_n(F_{2n}) = 4i(1-\delta_{on})F_{2(n-2)} - F_o'((n+1)F_{2(n+1)} \]
\[ + f_o a_{2(n+1)} + 4i(1-\delta_{on})\eta a_{2(n-1)} + E_n \]
\[ \eta = 1 \quad \frac{1}{2a^2 p_x X_e} \bar{F}_{2n} + F_{2n} + F_o((2+n)(a_{2(n+1)} - \frac{i}{4} a_{1(n+1)})) \]
\[ + 4i X_e (1-\delta_{on})a_{2(n-1)} - \frac{i}{4} F_{in} + B_n = 0 \]
\[ \eta = 0 \quad F_{2n} = 0 \]

and

\[ F_{2n}(\eta) = 0; \quad \text{for} \quad n=1,3,5,... \]

for \( n=0,4,... \)

\[ 0 \leq \eta \leq 1 \quad \tilde{K}_n(F_{2n}) = -F_o'((n+1)\tilde{F}_{2(n+1)} + f_o a_{2(n+1)}) + E_n \]
\[ \frac{1}{2a^2 p_x X_e} \tilde{F}_{2n} + \tilde{F}_{2n} + F_o((2+n)(\tilde{a}_{2(n+1)} - \frac{i}{4} \tilde{a}_{1(n+1)})) \]
\[ \eta = 1 \quad -\frac{i}{4} \tilde{F}_{in} + \tilde{B}_n = 0 \]
\[ \eta = 0 \quad \tilde{F}_{2n} = 0 \]

and

\[ \tilde{F}_{2n}(\eta) = 0; \quad \text{for} \quad n\neq 0,4,8,... \]

Where \( E_n, \tilde{E}_n, B_n \) and \( \tilde{B}_n \) are given by (XXII.2), (XXII.5), (XXII.8), and (XXII.11), respectively.
APPENDIX XXIII

REDUCTION OF THE FORMULAS FOR VELOCITY, SHEAR STRESS, AND NUSSELT NUMBER

Using the expansions (87) in the Eq. (153) gives:

\[
\frac{u}{U_\infty} = X_e \left[ \psi_{\eta*} + \varepsilon \psi_{1*} + \varepsilon^2 \psi_{2*} + \ldots \right] \left[ b_0^* + \varepsilon b_1^* + \varepsilon^2 b_2^* + \ldots \right]^{-1}
\]

\[
= \frac{X_e}{b_0^*} \left[ \psi_{\eta*} + \varepsilon \psi_{1*} + \varepsilon^2 \psi_{2*} + \ldots \right] \left[ 1 + \varepsilon \frac{b_1^*}{b_0^*} + \varepsilon^2 \frac{b_2^*}{b_0^*} + \ldots \right]^{-1}
\]

\[
= \frac{X_e}{b_0^*} \left[ \psi_{\eta*} + \varepsilon \psi_{1*} + \varepsilon^2 \psi_{2*} + \ldots \right] \left[ 1 - \varepsilon \frac{b_1^*}{b_0^*} - \varepsilon^2 \frac{b_2^*}{b_0^*}
+ \varepsilon^2 \frac{b_1^*}{b_0^*} + \ldots \right]
\]

\[
= \frac{X_e}{b_0^*} \left[ \psi_{\eta*} + \varepsilon \psi_{1*} + \varepsilon^2 \psi_{2*} + \ldots \right] \left[ 1 - \varepsilon \frac{b_1^*}{b_0^*} - \varepsilon^2 \left( \frac{b_2^*}{b_0^*} - \frac{b_1^*}{b_0^*} \right)
+ \ldots \right] \equiv u_0^* + \varepsilon u_1^* + \varepsilon^2 u_2^* + \ldots
\]

where:

\[
u_0^* = \frac{X_e}{b_0^*} \psi_{\eta*}
\]

\[
u_1^* = \frac{X_e}{b_0^*} \left[ \psi_{1*} - \frac{b_1^*}{b_0^*} \psi_{\eta*} \right]
\]

\[
u_2^* = \frac{X_e}{b_0^*} \left[ \psi_{2*} - \frac{b_1^*}{b_0^*} \psi_{1*} - \left( \frac{b_2^*}{b_0^*} - \frac{b_1^*}{b_0^*} \right) \psi_{\eta*} \right]
\]

\[
= \frac{X_e}{b_0^*} \left[ \psi_{2*} - \frac{b_1^*}{b_0^*} \psi_{1*} + \frac{b_1^*}{b_0^*} \left( \psi_{1*} - \psi_{\eta*} \right) \right]
\]

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\]
Using the expansions (87) in Eq. (134) gives:

\[
\tau_{\times} = \frac{\psi_{\eta}^*}{N_{vX}} \left[ e^{\psi_{\eta}^*} e^{\psi_{\eta}^*} + e^{\psi_{\eta}^*} \right] \left[ 1 - e^{\sum_{i=1}^{n} \theta_{\eta_{i}}} \right]^{-2}
\]

\[
= \frac{2 \varepsilon}{\delta_{0}^*} \left[ \psi_{\eta}^* + e^{\psi_{\eta}^*} + e^{2 \psi_{\eta}^*} + \ldots \right] \left[ 1 + \epsilon e^{\delta_{0}^*} + e^{2 \delta_{0}^*} + \ldots \right]^{-2}
\]

\[
= \frac{2 \varepsilon}{\delta_{0}^*} \left[ \psi_{\eta}^* + e^{\psi_{\eta}^*} + e^{2 \psi_{\eta}^*} + \ldots \right] \left[ 1 - 2 \epsilon \delta_{0}^* - 2 \epsilon e^{\delta_{0}^*} + \epsilon^2 \delta_{0}^* + \ldots \right]^{-2}
\]

\[
+ 3 \varepsilon^2 \frac{\delta_{0}^*}{\delta_{0}^*} + \ldots
\]

\[
= \frac{2 \varepsilon}{\delta_{0}^*} \left[ \psi_{\eta}^* + e^{\psi_{\eta}^*} + e^{2 \psi_{\eta}^*} + \ldots \right] \left[ 1 - 2 \epsilon \delta_{0}^* - \epsilon^2 \left( \frac{\delta_{0}^*}{\delta_{0}^*} \right) - 3 \epsilon^2 \frac{\delta_{0}^*}{\delta_{0}^*} + \ldots \right] = \tau_{\times} + \epsilon \tau_{1}^* + \epsilon^2 \tau_{2}^* + \ldots
\]

\[
\tau_{0}^* = \frac{2 \varepsilon}{\delta_{0}^*} \psi_{\eta}^*
\]

\[
\tau_{1}^* = \frac{2 \varepsilon}{\delta_{0}^*} \left[ \psi_{\eta}^* - \frac{\delta_{0}^*}{\delta_{0}^*} \psi_{\eta}^* \right]
\]

\[
\tau_{2}^* = \frac{2 \varepsilon}{\delta_{0}^*} \left[ \psi_{\eta}^* - \frac{\delta_{0}^*}{\delta_{0}^*} \psi_{\eta}^* - \left( \frac{\delta_{0}^*}{\delta_{0}^*} - 3 \frac{\delta_{0}^*}{\delta_{0}^*} \right) \psi_{\eta}^* \right]
\]

\[
= \frac{2 \varepsilon}{\delta_{0}^*} \left[ \psi_{\eta}^* - \frac{\delta_{0}^*}{\delta_{0}^*} \psi_{\eta}^* - \frac{\delta_{0}^*}{\delta_{0}^*} \psi_{\eta}^* \right]
\]

And using the expansions (87) in Eq. (136) gives:

\[
\frac{N_{vX}}{N_{vX}^*} = \frac{\epsilon}{\delta_{0}^*} \left[ \theta_{\eta}^* + e^{\theta_{\eta}^*} + e^{2 \theta_{\eta}^*} + \ldots \right] \left[ 1 - e^{\sum_{i=1}^{n} \theta_{\eta_{i}}} \right]^{-2}
\]

\[
= \frac{2 \varepsilon}{\delta_{0}^*} \left[ \theta_{\eta}^* + e^{\theta_{\eta}^*} + e^{2 \theta_{\eta}^*} + \ldots \right] \left[ 1 - e^{\delta_{0}^*} + e^{2 \delta_{0}^*} + \ldots \right] = \delta_{0}^* + \epsilon \delta_{1}^* + \epsilon^2 \delta_{2}^* + \ldots
\]
where:

\[ g_0^* = \frac{5}{\delta_0^*} q_0^* \]
\[ g_1^* = \frac{5}{\delta_0^*} q_1^* - \frac{3}{\delta_0^*} \delta_1^* \]
\[ g_2^* = \frac{5}{\delta_0^*} q_2^* - \frac{3}{\delta_0^*} \delta_2^* \]

\[ \delta_1^* \delta_2^* \left( q_1^* - \frac{3}{\delta_0^*} q_0^* \right) \]
APPENDIX XXIV

CALCULATION FOR EXPRESSING THE VELOCITY, WALL SHEAR STRESS, AND NUSSELT NUMBER EXPLICITLY IN TERMS OF THE SOLUTIONS TO THE ORDINARY DIFFERENTIAL EQUATION

Using $\psi_0^* = 2a \xi f_0$ and $\delta_0^* = 2a \xi$ in the Eqs. (140) we get:

\[ u_0^* = X_e f_0' \]
\[ u_1^* = \frac{X_e}{2a \xi} [\psi_1^* - f_0' \delta_1^*] \]
\[ u_2^* = \frac{X_e}{2a \xi} [\psi_2^* - f_0' \delta_2^* + \frac{\delta_1^*}{2a \xi} (f_0' \delta_1^*-\psi_1^*)] \]

Since:

\[ \psi_1^* = A e 2a \sum_{n=1}^{\infty} f_{1n} \xi^n e^{i \tau} \]
\[ \delta_1^* = A e 2a \sum_{n=1}^{\infty} a_{1n} \xi^n e^{i \tau} \]

we set:

\[ u_1^* = A e \sum_{n=0}^{\infty} g_{1n} \xi^n e^{i \tau} \]

and we have:

\[ \sum_{n=0}^{\infty} g_{1n} \xi^n = X_e \sum_{n=1}^{\infty} (f_{1n} - f_0 a_{1n}) \xi^{n-1} \]
\[ = X_e \sum_{n=0}^{\infty} (f_{1(n+1)} - f_0 a_{1(n+1)}) \xi^n \]

and equating coefficients:

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\[ g_{1n} = X_e (f'_{1(n+1)} - f'_0 a_{1(n+1)}) \text{; for } n = 0, 2, 4, 6, \ldots \]

since \( f_{1n} = a_{1n} = 0 \) for \( n = 0, 2, 4, 6, \ldots \)

Since:

\[
\psi_2^* = \mathcal{G} e^{2a} \sum_{n=1}^{\infty} (g^{2n} e^{2i\tau} + \tilde{g}^{2n}) \xi^n
\]

\[
\delta_2^* = \mathcal{G} e^{2a} \sum_{n=1}^{\infty} (a^{2n} e^{2i\tau} + \tilde{a}^{2n}) \xi^n
\]

we set:

\[
u_2^* = \mathcal{G} e^{2a} \sum_{n=0}^{\infty} (g^{2n} e^{2i\tau} + \tilde{g}^{2n}) \xi^n
\]

and we have, using (111) and equating the coefficients of \( e^{2i\tau} \):

\[
\sum_{n=0}^{\infty} g_{2n} \xi^n = X_e \sum_{n=1}^{\infty} (f'_{2n} - f'_0 a_{2n}) \xi^n
\]

\[
- \frac{X_e}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n} (f'_{1m} - f'_0 a_{1m}) \xi^{n+m-2}
\]

\[
= X_e \sum_{n=0}^{\infty} (f'_{2(n+1)} - f'_0 a_{2(n+1)}) \xi^n - \frac{X_e}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n} (f'_{1m} - f'_0 a_{1m}) \xi^{n+m-2}
\]

and equating coefficients of like powers of \( \xi \), we get:

\[ g_{2n} = X_e (f'_{2(n+1)} - f'_0 a_{2(n+1)} - h_n) \text{; } n = 0, 2, 4, 6, \ldots \]

\[ h_0 = \frac{1}{2} a_{11} (f'_{11} - f'_0 a_{11}) \]

\[ h_2 = \frac{1}{2} [a_{13} (f'_{11} - f'_0 a_{11}) + a_{11} (f'_{13} - f'_0 a_{13})] \]
\[
\begin{align*}
= \frac{1}{2} [a_{13} f_{11}' + a_{11} f_{13}' - 2 a_{11} a_{13} f_0'] \\

h_4 = \frac{1}{2} [a_{15} (f_{11}' - f_0 a_{11}) + a_{13} (f_{13}' - f_0 a_{13}) \\
+ a_{11} (f_{15}' - f_0 a_{15})] \\
= \frac{1}{2} [a_{15} f_{11}' + a_{13} f_{13}' + a_{11} f_{15}' - f_0' (2 a_{11} a_{15} + a_{13} a_{13})]
\end{align*}
\]

Since:
\[f_{1n}, a_{1n}, f_{2n}, a_{2n} = 0; \text{for } n = 0, 2, 4, \ldots\]

and similarly for the time independent part:
\[\tilde{f}_{2n} = X_e (\tilde{f}_{2(n+1)}' - f_0 a_{2(n+1)} - \tilde{a}_n); \text{ for } n = 0, 4, 8, \ldots\]
\[\tilde{a}_0 = \frac{1}{2} a_{11} (f_{11}' - f_0 a_{11})\]
\[\tilde{h}_4 = \frac{1}{2} [\tilde{a}_{15} f_{11} + \tilde{a}_{13} f_{13} + \tilde{a}_{11} f_{15}' - f_0' (2 \tilde{a}_{11} a_{15} + \tilde{a}_{13} a_{13})]\]

Since:
\[\tilde{f}_{2n}, \tilde{a}_{2n} = 0; \text{ for } n \neq 1, 5, 9, \ldots\]

Collecting the results, we get:
\[u_0^* = X_e f_0'; \quad u_1^* = \sum_{n=0}^{\infty} g_{1n} e^{i \tau}; \quad u_2^* = \sum_{n=0}^{\infty} (g_{2n} e^{2i \tau} \tilde{a}_n) e^{i \tau}\]
\[g_{1n} = X_e (f_{1(n+1)}' - f_0 a_{1(n+1)}) \quad \text{for } n = 0, 2, 4, \ldots\]
\[g_{2n} = X_e (f_{2(n+1)}' - f_0 a_{2(n+1)}) \quad \text{for } n = 0, 2, 4, \ldots\]
\[h_0 = \frac{1}{2} a_{11} (f_{11}' - f_0 a_{11})\]
\[h_2 = \frac{1}{2} [a_{13} f_{11} + a_{11} f_{13} - 2 a_{11} a_{13} f_0']\]
\[h_4 = \frac{1}{2} [a_{15} f_{11}' + a_{13} f_{13}' + a_{11} f_{15}' - f_0' (2 a_{11} a_{15} + a_{13} a_{13})]\]

..................................................
\[ \tilde{\eta}_n = x_0 (\tilde{f}_0'_{2(n+1)} - f_0' \tilde{\eta}_n - \tilde{\eta}_n) \quad n = 0, 1, \ldots \]

\[ \tilde{h}_0 = \frac{1}{2} \tilde{a}_{11} (f_{11}' - f_0 \tilde{a}_{11}) \]

\[ \tilde{h}_4 = \frac{1}{2} \left( \tilde{a}_{11} f_{11}' + \tilde{a}_{13} f_{13}' - \tilde{a}_{11} f_{15}' - f_0 (\tilde{a}_{11} a_{15} + \tilde{a}_{13} a_{13}) \right) \]

Using the relations:

\[ \psi_0^* = 2a \tau_0 \quad \text{and} \quad \delta_0^* = 2a \xi \]

in Eqs. (141), we get:

\[ \tau_0^* = \frac{1}{a} f_0'' \]

\[ \tau_1^* = \frac{1}{2a^2 \xi} \left[ \psi_1 \eta - 2 f_0'' \delta_1 \right] \]

\[ \tau_2^* = \frac{1}{2a^2 \xi} \left[ \psi_2 \eta - 2 f_0'' \delta_2^* - \frac{\delta_1}{2a^2 \xi} (2 \psi_1 \eta - 3 f_0'' \delta_1^*) \right] \]

Setting:

\[ \tau_1^* = \frac{1}{a} \sum_{n=0}^{\infty} e_{1n}^* e_{1n} \]

We get from using the expansion (103) and (104) in the second of the above equations: \( e_{1n} = f_{1n}'(n+1) - 2f_0'' a_1(n+1) \): \( n = 0, 1, 2, \ldots \) since \( a_{1n}, f_{1n} \equiv 0 \) for \( n \), \( 0, 2, 4, \ldots \) and setting:

\[ \tau_2^* = \frac{1}{a} \sum_{n=0}^{\infty} (e_{2n} e^{2i\tau} + \tilde{e}_{2n}) \]

We get by using the expansions (112) and (113), equating first time dependent and time independent parts and then the coefficients of powers of \( \xi \).
\[ e_{2n} = \bar{f}_{2}^{n}(n+1)-2\bar{f}_{0}^{n}a_{2}(n+1)-r_{n}; \quad n=0, 2, 4, \ldots \]

\[ r_{0} = \dfrac{1}{2}a_{11}(2\bar{f}_{11}^{n}-3\bar{f}_{0}^{n}a_{11}) \]

\[ r_{2} = \dfrac{1}{2}[2a_{13}\bar{f}_{11}^{n}+2a_{11}\bar{f}_{13}^{n}-6a_{11}a_{13}\bar{f}_{0}^{n}] \]

\[ r_{4} = \dfrac{1}{2}[2a_{15}\bar{f}_{11}^{n}+2a_{13}\bar{f}_{15}^{n}+2a_{11}\bar{f}_{15}^{n}-3\bar{f}_{0}^{n}(2a_{11}a_{15}+a_{13}a_{15})] \]

\[ \tilde{e}_{2n} = \bar{f}_{2}^{n}(n+1)-2\bar{f}_{0}^{n}\tilde{a}_{2}(n+1)-\tilde{r}_{n}; \quad n=0, 4, \ldots \]

\[ \tilde{r}_{0} = \dfrac{1}{2}\tilde{a}_{11}(2\bar{f}_{11}^{n}-3\bar{f}_{0}^{n}a_{11}) \]

\[ \tilde{r}_{4} = \dfrac{1}{2}[2\tilde{a}_{15}\bar{f}_{11}^{n}+2a_{13}\bar{f}_{15}^{n}+2\tilde{a}_{11}\bar{f}_{15}^{n}-3\bar{f}_{0}^{n}(2\tilde{a}_{11}a_{15}+\tilde{a}_{13}a_{13})] \]

since

\[ a_{1n}, \bar{f}_{1n}, a_{2n}, f_{2n} \equiv 0; \text{ for } n=0, 2, 4, 6, \ldots \]

and

\[ \tilde{a}_{2n}, \bar{f}_{2n} \equiv 0 \quad \text{ for } n\neq 1, 5, 9, \ldots \]

And we have also used Eqs. (111).

\[ \tau_{0}^{*} = \dfrac{1}{a}\bar{f}_{0}^{n}; \quad \tau_{1}^{*} = \sum_{n=0}^{\infty}e_{1n}^{n}e^{i\tau}; \quad \tau_{2}^{*} = \sum_{n=0}^{\infty}(e_{2n}^{n}e^{2i\tau}+\tilde{e}_{2n})^{n} \]

\[ e_{1n} = \bar{f}_{1}^{n}(n+1)-2\bar{f}_{0}^{n}a_{1}(n+1) \quad n=0, 2, 4, \ldots \]

\[ e_{2n} = \bar{f}_{2}^{n}(n+1)-2\bar{f}_{0}^{n}a_{2}(n+1)-r_{n} \quad n=0, 2, 4, \ldots \]

\[ r_{0} = \dfrac{1}{2}a_{11}(2\bar{f}_{11}^{n}-3\bar{f}_{0}^{n}a_{11}) \]

\[ r_{2} = \dfrac{1}{2}[2a_{13}\bar{f}_{11}^{n}+2a_{11}\bar{f}_{13}^{n}-6a_{11}a_{13}\bar{f}_{0}^{n}] \]

\[ r_{4} = \dfrac{1}{2}[2a_{15}\bar{f}_{11}^{n}+2a_{13}\bar{f}_{15}^{n}+2a_{11}\bar{f}_{15}^{n}-3\bar{f}_{0}^{n}(2a_{11}a_{15}+a_{13}a_{15})] \]

\[ \tilde{e}_{2n} = \bar{f}_{2}^{n}(n+1)-2\bar{f}_{0}^{n}\tilde{a}_{2}(n+1)-\tilde{r}_{n} \quad n=0, 4, \ldots \]
\[ r_0 = \frac{1}{2} a_{11} (2f_{11}'' - 3f_{0}''a_{11}) \]
\[ r_4 = \frac{1}{2} [2a_{15}f_{11}'' + 2a_{13}f_{13}'' + 2a_{11}f_{15}'' - 3f_{0}''(2a_{11}a_{15} + a_{13}a_{13})] \]

Using the relations (91) and (99) in Eqs. (142) gives:

\[ q_{0}^* = \frac{1}{2a} F'_{0} \]
\[ q_{1}^* = \frac{1}{2a} \Phi_{1n}^{*} - \frac{1}{4a^2 \xi} \delta_{1}^{*} F'_{0} \]
\[ q_{2}^* = \frac{1}{2a} \Phi_{2n}^{*} - \frac{1}{4a^2 \xi} \delta_{2}^{*} F'_{0} - \frac{1}{4a^2 \xi} \delta_{1}^{*}(\Phi_{1n}^{*} - \frac{\delta_{1}^{*}}{2a} F'_{0}) \]

Since:

\[ \delta_{1}^{*} = \left( e^{2a} \sum_{n=1}^{\infty} a_{1n} \xi^{n} e^{i \tau} \right) \]
\[ \Phi_{1n}^{*} = \left( e^{2a} \sum_{n=1}^{\infty} F_{1n} \xi^{n} e^{i \tau} \right) \]

we set:

\[ q_{1}^* = \left( e^{2a} \frac{1}{2a} \sum_{n=0}^{\infty} G_{1n} \xi^{n} e^{i \tau} \right) \]

and we have

\[ \sum_{n=0}^{\infty} G_{1n} \xi^{n} = \sum_{n=0}^{\infty} F_{1n}' \xi^{n} - F_{0}' \sum_{n=1}^{\infty} a_{1n} \xi^{n-1} \]

And equating coefficients of \( \xi \):

\[ G_{1n} = F_{1n}' - a_{1(n+1)}F_{0}'; \quad n=0, 2, 4, \ldots \]

since \( a_{1n} = 0 \) for \( n=0, 2, 4, \ldots \); \( F_{1n} \equiv 0 \) for \( n=1, 3, 5, \ldots \)

Since:
\[ \delta_{2n}^* = \chi e^{2a} \sum_{n=1}^{\infty} (a_{2n}e^{2i\tau} + \tilde{a}_{2n}) \xi^n \]

\[ \Phi_{2n}^* = \chi e^{2a} \sum_{n=0}^{\infty} [F_{2n} e^{2i\tau} + \tilde{F}_{2n}] \xi^n \]

We set:

\[ \eta_{2n}^* = \chi e^{2a} \frac{1}{2a} \sum_{n=0}^{\infty} [G_{2n} e^{2i\tau} + \tilde{G}_{2n}] \xi^n \]

and we have, using (111) for the coefficients of \( e^{2i\tau} \):

\[ \sum_{n=0}^{\infty} G_{2n} \xi^n = \sum_{n=0}^{\infty} F_{2n} \xi^n - F_0^1 \sum_{n=1}^{\infty} a_{2n} \xi^{n-1} \]

\[ - \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_{1n} F_{1m}^1 \xi^{n+m-1} \]

\[ + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{1n} a_{1m} \xi^{n+m-2} \]

And equating the coefficients of like powers of \( \xi \) we get:

\[ G_{2n} = F_{2n}^1 - a_{2(n+1)} F_0^1 - R_n \quad n=0,2,4,\ldots \]

\[ R_0 = \frac{1}{2} [a_{11} F_{10} - a_{11} a_{11} F_0^1] \]

\[ R_2 = \frac{1}{2} [a_{13} F_{10} + a_{11} F_{12} - 2F_0^1 a_{11} a_{13}] \]

\[ R_4 = \frac{1}{2} [a_{15} F_{10} + a_{13} F_{12} + a_{11} F_{14} - F_0^1 (2a_{11} a_{15} + a_{13} a_{13})] \]

Since \( a_{1n}, a_{2n} = 0 \) for \( n=0,2,4,\ldots \) and \( F_{1n}, F_{2n} = 0 \) for \( n=1,3,5,\ldots \)
And similarly for the time independent parts:

\[ \tilde{G}_n = \tilde{F}_n - \tilde{a}_n(n+1) \tilde{F}_0 - \tilde{R}_n \quad n=0,4,\ldots \]

\[ R_0 = \frac{1}{2} [\overline{a}_{11} F_{10} - \overline{a}_{11} a_{11} F_{0}'] \]

\[ \tilde{R}_4 = \frac{1}{2} [a_{15} F_{10} + a_{13} F_{12} + a_{11} F_{14} - F_{0}'(2a_{11}a_{15}+a_{13}a_{13})] \]

Since:

\[ \tilde{F}_n = 0 \quad \text{for} \quad n \neq 0,4,8,\ldots \]

\[ \tilde{a}_n = 0 \quad \text{for} \quad n \neq 1,5,9,\ldots \]

Collecting the results:

\[ q_0^* = \frac{1}{2a} \tilde{F}_0' \]

\[ q_1^* = \mathcal{R} e^{-\frac{1}{2a} \tau} \sum_{n=0}^{\infty} G_{1n}^* e^{n \tau} \]

\[ q_2^* = \mathcal{R} e^{-\frac{1}{2a} \tau} \sum_{n=0}^{\infty} [G_{2n} e^{n \tau} + \tilde{G}_n^*]^n \]

\[ G_{1n} = F_{1n} - a_1(n+1) F_{0}' \quad n=0,2,4,\ldots \]

\[ G_{2n} = F_{2n} - a_2(n+1) F_{0}' - R_n \quad n=0,2,4,\ldots \]

\[ R_0 = \frac{1}{2} [a_{11} F_{10} - a_{11} a_{11} F_{0}'] \]

\[ R_2 = \frac{1}{2} [a_{13} F_{10} + a_{11} F_{12} - 2F_{0}'a_{11} a_{13}] \]

\[ R_4 = \frac{1}{2} [a_{15} F_{10} + a_{13} F_{12} + a_{11} F_{14} - F_{0}'(2a_{11}a_{15}+a_{13}a_{13})] \]

\[ \text{.................................} \]

\[ \text{.................................} \]
\[ \tilde{G}_{2n} = \tilde{F}_{2n} - \tilde{a}_{2(n+1)}F'_0 - \tilde{R}_n \quad n=0,4,\ldots \]

\[ \tilde{R}_0 = \frac{1}{2} [\bar{a}_{11}F_{10} - \bar{a}_{11}a_{11}F'_0] \]

\[ \tilde{R}_4 = \frac{1}{2} [\bar{a}_{13}F_{10} + \bar{a}_{13}F_{12} + \bar{a}_{11}F_{14} - F'_0(2\bar{a}_{11}a_{15} + \bar{a}_{13}a_{13})] \]
APPENDIX XXV

COMPUTER PROGRAMS INVOLVING INTEGRATION OF DIFFERENTIAL EQUATIONS

All the computer programs which involve the integration of differential equations by the Runge-Kutta method are listed in this appendix along with the nomenclature which is necessary for their understanding and for connecting them with the equations derived above. Each program is divided into blocks whose functions are indicated by titles.

NOTATION FOR COMPUTER PROGRAMS INVOLVING THE INTEGRATION OF DIFFERENTIAL EQUATIONS

\[ Y(1) \sim \text{calculated value of solution to differential equation} \]
\[ Y(2) \sim \text{first derivative of } Y(1) \]
\[ Y(3) \sim \text{second derivative of } Y(1) \]
\[ Y(4) \sim \text{third derivative of } Y(1) \]
\[ F(1) \left( U(1) \right) \sim \begin{cases} (U(1), U(2) \text{ are used for } F(1) \text{ and } F(2) \text{ for cylinder problem with } 0.1 \leq E \leq 0.1) \text{ Terms of the differential equations which have been rewritten as a series of first order equations of the form:} \\ Y(2) = F(1) \\ Y(3) = F(2) \\ Y(4) = F(3) \end{cases} \]

(c.f. discussion in Chapter III)
CORRESPONDENCE BETWEEN NOMENCLATURE USED IN COMPUTER PROGRAM FOR CYLINDER PROBLEM WITH $E^2 > .01$ AND NOMENCLATURE USED IN TEXT

\[ A(0) \sim a_0 \]
\[ A(k) \sim b_k; k = 2,4,6,\ldots \]
\[ B(k) \sim B_k \]
\[ C(k) \sim A_k \]
\[ \varepsilon \sim \varepsilon^* \]
\[ E^2 \sim E^2 \]
\[ Fr \sim \frac{\tau_w \sqrt{\frac{2R_0 U_\infty}{\nu}}}{\frac{1}{2} \rho U_\infty^2} \]
\[ H(k,\iota) \sim H_k(\eta); K, k = 3,5,7,\ldots \]
\[ H(k,\iota) \sim G_k(\eta); K, k = 2,4,6,\ldots \]
\[ Nu(j) \sim q_j \]
\[ O \sim N_u/\sqrt{\frac{2R_0 U_\infty}{\nu}} \]
\[ P(\iota) \sim \eta \]
\[ Pr \sim P_r \]
\[ Re \sim U_\infty R_0/\nu \]
\[ Tau(j) \sim \tau_j \]
\[ U(\iota) \sim f_1(\eta) \]
\[ U'(\iota) \sim f'_1(\eta) \]
\[ UDP(\iota) \sim f''_1(\eta) \]
\[ UTP(\iota) \sim f'''_1(\eta) \]
\[ V(k,\iota) \sim f_k(\eta) \]
\[ Vp(k,\iota) \sim f'_k(\eta) \]
\[ VDP(k,\iota) \sim f''_k(\eta) \]

K, k = 3,5,7,\ldots
\[ V(K, I) \sim \sim F_k(\eta) \]
\[ VP(K, I) \sim \sim F'_k(\eta) \]
\[ VDP(K, I) \sim \sim \hat{F}_k(\eta) \]

\[ W(J, I) \sim u_J(\eta) \]

\[ X \sim \eta \]

\[ X E \sim X_e \]

\[ X I \sim \xi \]

\[ k, K \quad 0, 2, 4, 6, \ldots \]
CYLINDER PROBLEM WITH $E^2$ SQUARED GREATER THAN 0.01

SET UP

---

DIMENSION Y(3), F(3), D(3)
EXECUTE SETCO((14), Y(1), F(1), D(1), STEP)
INTEGER J,K,M,N, I, T, EPS, TS, MAX, MVAX, NM
VECTOR VALUES FM=81H, $S_4$, $S_4$, $S_4$, $S_4$
VECTOR VALUES FN=81H, $4F12.6$, $4F12.6$, $4F12.6$
DIMENSION P(150), U(150), V(150), W(150), B(11), C(11)

---

DIMENSION A(I,11), TAU(I), N(I)
DIMENSION A(I,150), R(I,150)
DIMENSION V(I,150), VS
DIMENSION VP(I,150), PS
DIMENSION VD(I,150), DS
DIMENSION VD(I,150), TW
DIMENSION H(I,150), HS
VECTOR VALUES V=2.150, 150
VECTOR VALUES PS=2.150, 150
VECTOR VALUES DS=2.150, 150
VECTOR VALUES TM=2.150, 150
VECTOR VALUES HS=2.150, 150
READ AND PRINT DATA
T2=1.00
EPS=2.12*2 RE
YDP=4.0/(ES.P, 0.50)
TS=1
T=1

SW16
WHENEVER T.E.1, EPS=2
WHENEVER T.G.1, EPS=1

---

CALC. OF CORRECTED VALUES OF CONSTANT PARAMETER FROM ERRORS IN BOUNDARY COND.'S

---

TA=AC
CA=-C-01
CY=-C-01
H(J)=1.0
R(I)=1.0
P=1

SW2
C1=CF1*(P1(M)/P1(M-1)-1)
WHENEVER Q1.G.87.20, Q2=G7.20
WHENEVER Q1.L.(87.20), Q1=G7.20
Q2=CA*(2.15540/EXP.+Q1)-EXP.(-Q1)1+1
CADD
P=P+1
TA=TA+DA
AN=A(1)
AP=TA
CALC. OF CORR. VALUES OF INITIAL COND'S FROM PREVIOUS ERRORS IN BOUNDARY COND'S

\[ \text{SW1} \]
- \( R_{21(I)} = 1.5 \)
- \( R_{21(I+1)} = 0.0 \)
- \( n = 1 \)

\[ \text{SW4} \]
- \( Q_{2} = C(F2 + R2(N+1)/R2(N-I)-1) \)
- WHENEVER \( U_{2} = U_{2} \neq B7.20 \)
- WHENEVER \( Q_{2} < 2.0 \)
- \( DX = DT \times ([U_{2} - 20]/[\exp(Q_{2}) - \exp(Q_{2} - 1)]) \)
- \( DX = DX \)
- \( n = n + 1 \)
- \( Y_{X} = X + Y_{2} \)
- WHENEVER \( T.S.E = 1 \)
- \( Y_{(I)} = Y_{X} \)
- \( Y_{(2)} = 0 \)
- \( Y_{(1)} = 0 \)
- \( GR \) WHENEVER \( T.S.E = 0 \)
- \( Y_{(2)} = Y_{X} \)
- \( Y_{(1)} = \text{EPS} - 1 \)
- END OF CONDITIONAL

\[ \text{SW3} \]
- \( j = 0 \)
- \( A = 0 \)

\[ \text{START} \]
- \( i = 1 \)
- \( f(1) = X \)
- WHENEVER \( T.E.1 \)
- \( u(1) = y(1) \)
- \( u(1) = y(1) \)
- \( u(1) = y(3) \)
- \( u(1) = f(3) \)
- \( u(1) = u(1) \)
- \( v(t, i) = y(t) \)
- \( v(t, i) = y(t) \)
- \( v(t, i) = y(t) \)
- WHENEVER \( T.S.E = 1, V.D.P(t, i) = y(3) \)
- \( V.D.P(t, i) = y(3) \)
- WHENEVER \( T.S.E = 0, V.D.P(t, i) = f(3) \)
- \( V.D.P(t, i) = f(3) \)
- END OF CONDITIONAL
- \( V.D.P(t, i) = f(3) \)

\[ \text{SW5} \]
- \( f(1) = y(t) \)
- \( f(2) = y(t) \)
- WHENEVER \( T.E.1, f(3) = A(1) \times y(1) \times y(3) - (y(2) - P.2) \times 1 \)
- \( x = y(t) \times y(t) \)
- \( x = y(t) \times y(t) \)
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- \( x = y(t) \times y(t) \)
- \( x = y(t) \times y(t) \)
- \( x = y(t) \times y(t) \)
- END OF CONDITIONAL

\[ \text{CALC} \]
- \( x = \text{MKPLC} \)
- WHENEVER \( x = 1 \), TRANSFER TO SW5
- WHENEVER \( T.E.S = 1 \), TRANSFER TO SW6

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- 091
- 092
TRANSFER TO START

---

SW6

WHENEVER TS.E.1, TRANSFER TO SW7

WHENEVER TS.E.0, TRANSFER TO SW8

---

CALC. OF ERROR IN CONT. BOUNDARY COND.

SW7

WHENEVER ABS(Y(1)-(1.0+EPS/(T+1.0))).GT.CTOL AND N.L.NMA

R2(N)=Y(1)-(EPS /(T+1.0))

TRANSFER TO SW4

---

CALC. OF ERROR IN MOM. BOUNDARY COND.

OTHERWISE

WHENEVER N.GE.NMAX, PRINT RESULTS R2(1)...R2(N)

A(T+1)=AP

C(T+1)=A0.0*(A.P.2)

MBC = A2*Y(2) + C(T+1)*(-(Z2).P.((T+1)/2))+(2.*T/(T+1))

1

1

END OF CONDITIONAL

WHENEVER ABS(Y(3) + MBC*ES+0.5).GT.MTOL AND M.GE.NMAX

R1(N)=Y(1)*MGE*ES+0.5

TRANSFER TO SW2

OTHERWISE

WHENEVER M.GE.NMAX, PRINT RESULTS R1(1)...R1(M)

---

STORAGE OF SOL'S AT LAST GRID PT.

WHENEVER T.E.1

U(I+1)=Y(I)

UP(I+1)=Y(I)

UPM(I+1)=Y(I)

UPF(I+1)=Y(I)

OTHERWISE

V(T,I)=Y(I)

VPT(I,I)=Y(I)

VCP(I,I)=Y(I)

END OF CONDITIONAL

END OF CONDITIONAL

---

CALC. OF TERMS TO BE USED IN HIGHER ORDER MOM. EQU'S

THROUGH SW14 FOR J=1,J.G.(GRSIZ+1)

WHENEVER T.E.1, H(3,J)=0

WHENEVER T.E.2, H(5,J)=((V(P3,J)+P2)-(V3,J)*VDP3,J)

WHENEVER T.E.3, H(7,J)=((V(P3,J)+P2)-(V3,J)*VDP3,J)

WHENEVER T.E.4, H(7,J)=((V(P3,J)+P2)-(V3,J)*VDP3,J)

SW14

1

1

END OF CONDITIONAL

END OF CONDITIONAL

---

CALC. OF TERMS TO BE USED IN HIGHER ORDER MOM. EQU'S

THROUGH SW14 FOR J=1,J.G.(GRSIZ+1)

WHENEVER T.E.1, H(3,J)=0

WHENEVER T.E.2, H(5,J)=((V(P3,J)+P2)-(V3,J)*VDP3,J)

WHENEVER T.E.3, H(7,J)=((V(P3,J)+P2)-(V3,J)*VDP3,J)

WHENEVER T.E.4, H(7,J)=((V(P3,J)+P2)-(V3,J)*VDP3,J)

---

CALC. OF TERMS TO BE USED IN HIGHER ORDER MOM. EQU'S

THROUGH SW14 FOR J=1,J.G.(GRSIZ+1)

WHENEVER T.E.1, H(3,J)=0

WHENEVER T.E.2, H(5,J)=((V(P3,J)+P2)-(V3,J)*VDP3,J)

WHENEVER T.E.3, H(7,J)=((V(P3,J)+P2)-(V3,J)*VDP3,J)

WHENEVER T.E.4, H(7,J)=((V(P3,J)+P2)-(V3,J)*VDP3,J)
1  J1-1.015*V(I,J)+V(P2,J)-2.01V(I,J)*V(V2,J)+15.*A(2)*((4.*
2  U(I,J)+V(V4,J)-U(J)+V(P4,J)-10.*V(V3,J)+V(I,J)*V(P2
3  J1)+15.*A(4)*12.01U(I,J)+V(V2,J)-30.*13.*A(2)*V
4  (5,J)-2.01A(4)+V(V3,J)+V(P0,J)
5  WHENEVER T.E.2, B6=0
6  WHENEVER T.E.2, B8=6.01(VI2,GRSIZ+1)+A(2)-1.+A(2)*V(0,GRS
7  I2(1))
8  WHENEVER T.E.4, B6=15.01(VI4,GRSIZ+1)*A(2)+V(V2,GRSIZ+1)*A(4
9  1.0-V(4,GRSIZ+1)-2.01V(V2,GRSIZ+1)*A(2)-V(0,GRSIZ+1)*A(4)
2  SIZ11*V(0,GRSIZ+1)*A(2))
5  WHENEVER T.E.3, NU2=V(0,G11)1/A
6  WHENEVER T.E.2, NU2=V(V2+1)=A(2)*V(P1,11)/A
5  WHENEVER T.E.4, NU2=V(4+1)=0.01A(2)*V(V2+1)+6.01A(2)*P+2
6  1)1)1.0*V(P0,11)1/A
5  WHENEVER T.E.6, NU6=(VP1+1)-15.*A(2)*V(4+1)-15.*6.*(A(2)*P
6  1)1=A(4)+V(V2+1)+19.01(A(2)*P)+15.*A(2)+A(4)+16.01V(V01))
7  Z+A

--------------------

TEST LOC. TO DET. WHICH ENERGY EQ. IS BEING SOLVED AND
WHEN ENERGY PROB. IS COMPL.

-----

I+1
5  WHENEVER ETOL.L.0.001, ETOL=ETOL*(10,0,P.(T/2))
6  YE=YP
7  EPS1=1
8  WHENEVER T.L,TMAX, TRANSFER TO SW1

-----

PRINT STATEMENTS AND MISCELLANEOUS

-----

PRINT FORMAT FM1
PRINT FORMAT FM2

THROUGH SW0, FOR K=1,1,K,G.(GRSIZ+1)
PRINT FORMAT FM3, PK(I),I(K),VS3,K),VI5,K),VI7,K)
PRINT FORMAT FM1
PRINT FORMAT FM2

THROUGH SW4, FOR K=1,1,K,G.(GRSIZ+1)
PRINT FORMAT FM2, PK(I),I(K),W3,K),K,S3,K),W5,K),W7,K)
PRINT FORMAT FM1
PRINT FORMAT FM2

THROUGH SW4, FOR K=1,1,K,G.(GRSIZ+1)
PRINT FORMAT FM2, PK(I),UP1,K),VP3,K),VP5,K),VK(T,K)
PRINT FORMAT FM1
PRINT FORMAT FM2

THROUGH SW1, FOR K=1,1,K,G.(GRSIZ+1)
PRINT FORMAT FM2, PK(I),UP(K),VP3,K),VP5(K),VP7(K)
PRINT FORMAT FM1
PRINT FORMAT FM2

THROUGH SW1, FOR K=1,1,K,G.(GRSIZ+1)
PRINT FORMAT FM2, PK(I),I(K),VDP3,K),VDP5,K),VDP7(K)
PRINT FORMAT FM1
PRINT FORMAT FM2

THROUGH SW1, FOR K=1,1,K,G.(GRSIZ+1)
PRINT FORMAT FM2, PK(I),V0(K),W3(K),W5(K),W6(K)
PRINT FORMAT FM1
PRINT FORMAT FM2

PRINT FORMAT FM4, A(2),V4,4,4
PRINT FORMAT FM4, TAU(I1),TAU(U3),TAU(S1),TAU(U7)
PRINT FORMAT FM1
PRINT FORMAT FM2

THROUGH SW4, FOR K=1,001,1,1,K,G1.5
CE=A1(*AU2(T1/2.1)*X1.P.2)+A4(4.2)+X1.P.4*(A61.201.1)+1
1 (X1.P.6.1)
CE=CE(ES3.5)

TAU(I1)+X1-TAU(U3)+X1.P.3+TAU(S1)+TAU(U7)+1.201.1*(X1.P.51-ITA
1 U(T)/100.+X1.P.7)
The procedure used in the program for calculating drop trajectories is to use the Runge-Kutta method to integrate the equations of motion for a drop starting at particular values of $y_d^*$ until the drop almost reaches the cylinder or until it entirely misses the cylinder. If the drop is going to hit the cylinder the calculated values of the solution are extended by linear interpolation to the required values at the cylinder. These values are then used to calculate the quantities of Eqs. (17*A.C.) which are needed to describe the conditions in the liquid film.

At each step of the integration the current value of the absolute value of the vector difference between the drop velocity and the gas velocity at the position of the drop is used to calculate an instantaneous Reynolds number based on the drop diameter. The drag coefficient corresponding to this Reynolds number is obtained by interpolation from the tabulated values of Table II, which are put into the program as "data" and were obtained from Figure 28 which is a smoothing of the values obtained from Reference 28. This value of the drag coefficient is then used in the equations of motion to calculate the new position of the drop.
Figure 28. Values of drag coefficients for spheres used in calculating drop trajectories.
<table>
<thead>
<tr>
<th>Re</th>
<th>C_D</th>
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<tbody>
<tr>
<td>0.2</td>
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<tr>
<td>0.4</td>
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<tr>
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<td>44.5</td>
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<td>36.0</td>
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<td>14.4</td>
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<td>4</td>
<td>8.4</td>
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<td>6</td>
<td>6.2</td>
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<td>8</td>
<td>5.1</td>
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<td>10</td>
<td>4.4</td>
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<td>20</td>
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<td>40</td>
<td>1.88</td>
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<td>600</td>
<td>0.53</td>
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<td>800</td>
<td>0.49</td>
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<tr>
<td>1000</td>
<td>0.48</td>
</tr>
<tr>
<td>2000</td>
<td>0.40</td>
</tr>
</tbody>
</table>
CORRESPONDENCE BETWEEN NOMENCLATURE USED IN COMPUTER PROGRAM FOR CALCULATING DROP TRAJECTORIES AND NOTATION USED IN TEXT

CD1 ~ CD

H ~ ΔV*

MDOT ~ J

MF(N) ~ J (tabulated value)

MTOT ~ \( \dot{\mathcal{M}} \)

P(I) ~ t*

PH ~ \( \varphi_d \)

PHI(K) ~ \( \varphi_d \) (tabulated table)

R(I) ~ \( \left| \frac{x^*}{r^*_d} \right| \)

REL ~ 2raΔV*/\( \nu_g \) (drop Reynolds number)

RINF ~ 2RoU*/\( \nu_g \)

T ~ t*

V1 ~ \( \frac{V^*_{x_0}}{x_0} \)

V2 ~ \( \frac{V^*_{y_0}}{y_0} \)

VPHI ~ \( \frac{V^*}{\varphi} \)

VXF ~ \( \frac{V^*_{x_0f}}{x_{0f}} \)

VYF ~ \( \frac{V^*_{y_0f}}{y_{0f}} \)

X1 ~ \( \frac{x^*}{x_d} \)

X2 ~ \( \frac{y^*}{y_d} \)

XBAR ~ \( \bar{x} \)

XF ~ \( \frac{x^*}{df} \)
\[ \text{XF}(k) \sim x^*_{df} \quad \text{(stored value)} \]
\[ \text{YF} \sim y^*_{df} \]
\[ \text{YF}(k) \sim y^*_{df} \quad \text{(stored value)} \]
\[ \text{YINF}(k) \sim y^*_{d\infty} \]
$COMPILE MAD, PRINT OBJECT

MAD (01 MAY 1965 VERSION) PROGRAM LISTING .........

....CALC. OF DROP TRAJECTORIES

SET UP

VECTOR VALUES FM1=$1H/1H-#$
VECTOR VALUES FM2=$1H,$4,4H1H-#$
VECTOR VALUES FM3=$1H,$4,4F12.6-#$
DIMENSION Y(4),F(4),D(4)
EXECUTE SETRKO,(4,Y(1),F(1),0,T,STEP)
DIMENSION IMAGE(10000)
EXECUTE PLOT2,(IMAGE,0,,-.5,2.0,89350,0.,)
DIMENSION RED(1000),CD(1000),P1(1000),X1(1000),X2(1000),Y1(1000),Y2(1
000),MF(1000),XP(1000),YP(1000),PHI(1000),VPHI(1000),VINF(10)
2 000),R(1000)
INTEGER JMAX,l,IMAX,K,L,KX,MY,N
INTEGER SW
INTEGER II,KI,IP,KP
READ AND PRINT DATA
K=0
SW=0
KI=0
ALPH=3.0,ROH/RR*8.)

INITIAL CONDITIONS

SW1
K=K+1
WHENEVER KI,E.5, KI=0
KI=KI+1
Y(1)=XINF
Y(2)=INCY*(K-1,)
Y(3)=1.0
Y(4)=0.0
I=0
T=0

INTEGRATION OF DIFFERENTIAL EQUATIONS

II=0
I=I+1
II=II+1
P[I]=T
X(I)=Y(1)
X2(I)=Y2(2)
V1(I)=Y(3)
V2(I)=Y(4)
R(I)=Y(1)*P2+Y(2)*P2.*PO.5
WHENEVER KI,E,KP .AND. II,E.IP
EXECUTE PLOT3,(I05,Y(I),Y(2),L1)
II=0
OTHERWISE
CONTINUE
END OF CONDITIONAL

F(1)=Y(3)
F(2)=Y(4)
F(3)=1.0+((Y(1)+P.2-Y(2)+P.2)/(Y(1)+P.2+Y(2)+P.2))
F(4)=2.0+Y(1)+Y(2)/(Y(1)+P.2+Y(2)+P.2)
H=H(1)+H(2)+H(3)+H(4)
REL=INF+4.0*ABS(H)+R
WHENEVER REL<GE.1.0
CD1=TA.B(REL1;RE.D(1),CD1);1,1,4,XMAX,T5)
OTHERWISE
CD1=24.*((1.+T3./0.)*REL1/REL)
END OF CONDITIONAL
F(3)=ALPH*CD1*H1*H
F(4)=ALPH*CD1*H2*H
S=RKDEQ(0)
WHENEVER S.E.1.0, TRANSFER TO SW5
WHENEVER (Y(1)+P.2+Y(2).P.2).LE.1.0 OR Y(1).GE.0., TRANSFER TO SW6
TRANSFER TO START

STORAGE OF SOL'S AT LAST GRID PT.

SW6
P(I+1)=T
X1(I+1)=Y(1)
X2(I+1)=Y(2)
V1(I+1)=Y(3)
V2(I+1)=Y(4)
R(I+1)=Y(1)+P.2+Y(2).P.0.S

CALC. OF TANG. VEL. AND TOTAL MASS FLOW

WHENEVER R(I+1).LE.1.0
R(I+1)=X1(I)+P.2+X2(I)+P.0.5
A=(X1(I)+X2(I)+X1(I)+X2(I-1))/(X1(I)-X1(I-1))
B=(X2(I)+X1(I)+X2(I)-X1(I))/(X1(I)-X1(I-1))
C=(B.P.0.5-A.P.0.5)
X=(C.P.0.5)+A
Y=1.0
VX=VX+VX
VV=VY+VY
TAPH=VY/ABS(VX)
PH=ATAN(TAPH)
VPH(K)=PH
VPH(K)=VY+PH+VY+COS(PH)
VY=ICY*(K-1)
VF(K)=XK
VY(K)=Y

CALC. OF LAST DROP TO HIT CYLINDER

WHENEVER SW.E.0
SW=1
KE=K
END OF CONDITIONAL

DET. OF LAST DROP IN PATH OF CYLINDER

WHENEVER X2(I).LE.1.0, TRANSFER TO SW5

PLOT STATEMENT
PRINT FORMAT FM1  #085
PRINT COMMENT $  #086
1 DMQP TRAJECTORIES  #087
EXECUTE PLOT4.O$LABEL
CALC OF MASS FLUX

MF(KE-2)=0.  #088
MF(KE-1)=0.  #089
THROUGH SWB, FOR N=1,1,1,1,4, (KE-2)  #090
MF(N)=NCG/PHI(N+1)-PHI(N)  #091

PRINT STATEMENTS

PRINT RESULTS KE
KE=KE+1  #092
PRINT FORMAT FM1  #093
PRINT FORMAT FM2
THROUGH SW9, FOR M=1,1,XBAR,G,PHI(KE-2)  #095
XBAR=PHI*M  #096
MDOT=TAB,XBAR,PHI(1),MF(1),1,1,4,KE-2,T61  #097
VPHI=TAB,XBAR,PHI(1),VPHI(1),1,1,4,KE-2,T71  #098
MTOT=TAB,XBAR,PHI(1),YINF(1),1,1,4,KE-2,T71  #099
PUNCH FORMAT FM3, XBAR,MTOT,MDOT,VPHI  #100
PRINT FORMAT FM3, XBAR,MTOT,MDOT,VPHI  #101

END OF PROGRAM  #102

SW9
CORRESPONDENCE BETWEEN NOMENCLATURE USED IN COMPUTER PROGRAM FOR CYLINDER PROBLEM WITH $E \leq 0.1$ AND NOMENCLATURE USED IN TEXT

\begin{align*}
\text{ALP}(2*J) & \sim \alpha_{2j} \\
\text{BC.(I,J)} & \sim \left( \begin{array}{c} I \\ J \end{array} \right) \\
\text{Beta}(2*J) & \sim \beta_{2j} \\
\text{DEL(I)} & \sim \tilde{v}(x) \text{ (tabulated value)} \\
\text{DELT} & \sim \tilde{e} \\
E & \sim \epsilon \\
F.(N) & \sim N ! \\
G & \sim G_k \\
NU & \sim \frac{N_u}{\sqrt{\frac{2R_0U_\infty}{\nu}}} \\
P(I) & \sim \nu \\
P(D(J)) & \sim \tilde{x} \text{ (tabulated value)} \\
P(R) & \sim P_r \\
RINF & \sim 2R_0U_\infty/\nu_g \\
RR & \sim r_d/R_0 \\
V(2*K,I) & \sim F_{2k}(\eta) \\
VP(2*K,I) & \sim F'_{2k}(\eta) \\
X & \sim \eta \\
XI & \sim \tilde{x}
\end{align*}
SOLUTION TO ENERGY EQUATION FOR SMALL 'E'

SET UP

INTEGER K,HMAX,1,J,N,NMAX,JMAX,1,J,IM,IN
INTEGER T,TMAX
INTEGER M
DIMENSION ALP(50),BET(50),R(2001),P(150),NO(75)
DIMENSION DLE(2001),PD(200)
DIMENSION Y(J),U(J),DI(3)
EXECUTE SETRAD(J),Y(J),U(J),0,X,STEP
VECTOR VALUES FM=1H/1H=5
VECTOR VALUES FMZ=1H ,54,82,1H=5
VECTOR VALUES FM3=1H ,54,5.3,54,10F13.6=5
VECTOR VALUES FM4=1H ,54,13.2F20.6=5
VECTOR VALUES FM5=1H ,54,10F13.6=5
VECTOR VALUES FM6=1H ,54,5.3,512.6=5
VECTOR VALUES FM7=1H ,54,5.3,13F=5
DIMENSION V(2000,VS)
DIMENSION VP(2000,VS)
VECTOR VALUES V5=2,150,150
VECTOR VALUES V5=2,150,150
READ AND PRINT DATA

DEFINITION OF F.

INTERNAL FUNCTION (J)
ENTRY TO F.
FACT=1
WHENEVER J,J,0, FUNCTION RETURN FACT
THROUGH LOOP, FOR II=1,1,II,G,1J
FACT=FACTII
FUNCTION RETURN FACT
END OF FUNCTION

DEFINITION OF B.C.

INTERNAL FUNCTION BC,(1,1,1,1)=F,(1,1,1,1)/F,(1,1,1,1)

INPUT FROM MOM. PROGRAM

T=0
SW13
T=T+1
READ FORMAT FM7,6,M
THROUGH SW10, FOR J=1,1,J,G,M
SW10
READ FORMAT FM6, PD(J),DE(J)
THROUGH SW1, FOR J=1,1,J,G,JMAX
SW1
READ FORMAT FM4, NO(J),ALP(2+J),BET(2+J)
CALC. OF INITIAL COND'S FROM ERROR IN B.C.
OTHERWISE
PRINT RESULTS N
V(2*K+1)=Y(1)
VP(2*K+1)=Y(2)
END OF CONDITIONAL

K=K+1

CALC., AND PRINT-OUT OF NUSSELT NO.

PRINT RESULTS E,RR,RRINF,PR
THROUGH SW12, FOR J=1,1:(J+1)*INXI1.G,XIMAX
X1=J*INXI1

DELX=MAX(X1,F1PD1,DELX1,1.1,4,M,T6)
NUM=.

THROUGH SW11, FOR I=3,1:3.G,KMAX

NU=NU+VPZP1(1)*X1FZ21*FZ21*

PRINT STATEMENTS

PRINT FORMAT FM1
PRINT FORMAT FM2

THROUGH SW9, FOR I=1,1:G.(G5S21)
PRINT FORMAT FM3, P(1),V(0,1),V(2,1),V(4,1),V(6,1),V(8,1)
PRINT FORMAT FM1
PRINT FORMAT FM2
PRINT FORMAT FM3, VP(0,1),VP(2,1),VP(4,1),VP(6,1),VP(8,1)
WHenever T.E.TMAX, TRANSFER TO SW13
END OF PROGRAM
Note: that all quantities which occur in program for the oscillating plate are either pure imaginary or real. The equations are written in this program so that the $i = \sqrt{-1}$ does not appear. It is to be understood that all quantities calculated in this program which correspond to pure imaginary numbers should be multiplied by $i$ to get the quantities appearing in the text.

CORRESPONDENCE BETWEEN NOMENCIATURE USED IN COMPUTER PROGRAM FOR OSCILLATING PLATE WITH NOMENCIATURE IN TEXT

\begin{align*}
A(0) & \sim a \\
A(j) & \sim a_{1j} \\
A2(j) & \sim a_{2j} \\
A2H(j) & \sim \tilde{a}_{2j} \\
AD1 & \sim |\tilde{\delta}_1| \\
AF1 & \sim |\tilde{\tau}_1| \\
AN1 & \sim |\tilde{q}_1| \\
ARD1 & \sim \Phi_{\tilde{\delta}_1} \\
ARF1 & \sim \Phi_{\tilde{\tau}_1} \\
ARN1 & \sim \Phi_{\tilde{q}_1} \\
\text{DELTA.}(I,J) & \sim \delta_{ij} \\
E1(j) & \sim e_{1j} \\
E2(j) & \sim e_{2j} \\
\tilde{E}(j) & \sim \tilde{e}_{2j} \\
G1(j) & \sim G_{1j}
\end{align*}
G2(J) \sim G_{2j}

GH(J) \sim G_{2j}

H(I) nonhomogeneous part of differential equations for momentum problem, terms appearing in nonhomogeneous part of differential equations for energy problem.

\nu \sim (\nu_{\text{avg.}} - \nu_{\text{steady}}) / \sqrt{\frac{xU_\infty}{v}}

\eta

PB1 \sim (\varepsilon/x)N \frac{U_0x}{v}

PB2 \sim \delta_0 N \frac{U_0x}{v}

PB3 \sim \tau_0 N \frac{U_0x}{v} / \left( \frac{1}{2} \rho U_\infty^2 \right)

\pi/2

R \sim R_j, r_j; j = 0, 2, 4, \ldots \text{ depending on which equation is being solved}

RH \sim \tilde{R}_j, \tilde{r}_j; j = 0, 2, 4, \ldots \text{ depending on which equation is being solved}

RHS nonhomogeneous part of differential equation for energy problem

SG \sim u^*_0

SG1 \sim \xi_{10}

SG12 \sim \xi_{12}

SG14 \sim \xi_{14}

SG2 \sim \xi_{20}

SG22 \sim \xi_{22}

SG24 \sim \xi_{24}

SGH \sim \tilde{\xi}_{20}
$$\text{SGH}^4 \sim \tilde{\zeta}_{24}$$

$$U(I) \sim f_0(\eta)$$

$$\text{UP}(I) \sim f'_0(\eta)$$

$$\text{UDP}(I) \sim f''_0(\eta)$$

$$V(J,I) \sim \begin{cases} f_{1,j}(\eta) \\ f_{2,j}(\eta) \\ f_{3,j}(\eta) \end{cases} \quad j,j = 1,3,5,\ldots$$

$$V(J,I) \sim \begin{cases} f_{1,j}(\eta) \\ f_{2,j}(\eta) \\ f_{3,j}(\eta) \end{cases} \quad j,j = 0,2,4,\ldots$$

$$W(J,I) \sim \begin{cases} f_{2,j}(\eta) \\ f_{3,j}(\eta) \end{cases} \quad j,j = 1,3,5,\ldots$$

$$W(J,I) \sim \begin{cases} f_{2,j}(\eta) \\ f_{3,j}(\eta) \end{cases} \quad j,j = 0,2,4,\ldots$$

$$X \sim \eta$$

$$X_e \sim X_e$$

$$Z(I) \sim f_0(\eta)$$

$$ZP(I) \sim f'_0(\eta)$$

$$ZDP(I) \sim f''_0(\eta)$$
PROGRAM FOR OSC. FLAT PLATE

SET UP

DIMENSION Y(3),F(3),I(3)
EXECUTE SETRKO,(3,Y(1),F(1),D,X,STEP)
INTEGER J,K,N,F,N,T,CPY,GRSIZ,TAU,MP,NP,I,T1,T2,hFE,ASE,TMAX
I,MMAX,NMAX,TS
INTEGER DELTA.

1,73
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READ AND PRINT DATA
I2=1.00
T1=1
TAU0=0.044

INDEX FOR MGM.EQU'S
*:045
WHENEVER TAU.LE.T1
*:046
T2=TAU-1
*:047
EPS2=EPS(TAU)
*:048
CR WHENEVER TAU.LE.T2
*:049
T2*(TAU-T1)-1
*:050
EPS=EPS(TAU-T1)
*:051
CR WHENEVER TAU.LE.T3
*:052
T4=4*(TAU-T2)-3
*:053
EPS2=EPS(TAU-T2)
*:054
END OF CONDITIONAL

CALC. OF CORRECTED VALUES OF INITIAL COND.'S FROM PREVIOUS ERRORS IN BOUNDARY COND.'S
*:055
DA=0.001
*:056
DY=0.001
*:057
R1(0)=1.5
*:058
R1([1]=1.0
*:059
P=1
*:060
Q1=[Q1(M-1)]
*:061
WHENEVER Q1.GT.87.20
*:062
Q8=Q8+0.01
*:063
C8=CAH(Q1,2.35047/EXP(Q1-EXP(-Q1))+11)
*:064
DA=DB
*:065
P=0
*:066
TA=TA+DA
*:067
ANX=ANX
*:068
AP=AP+TA
*:069

CALC. OF CORR. VALUES OF INITIAL COND.'S FROM PREVIOUS ERRORS IN BOUNDARY COND.'S
*:070
Y=X+Y
*:071
R2(0)=1.0
*:072
R2([1]=1.0
*:073
N=1
*:074
WHENEVER Q2.GT.87.20
*:075
Q8=Q8+0.01
*:076
U=X+Y
*:077
U=X+Y
*:078
S=N+1
*:079
Y=X+Y
*:080
WHENEVER IS.E.1
*:081
Y([1])=Y
*:082
Y([2])=0
*:083
Y([1])=0
*:084
Y([2])=Y
*:085
Y([2])=Y
*:086
Y=[CELTAU+TAU)
*:087
END OF CONDITIONAL
INTEGRATION OF DIFF. EQ'S

SW3.
I=0
X=0

START
I=I+1

P[I]=X
WHENEVER TAU.E.0 AND TS.E.1
U[I]=Y[I]
UP[I]=Y2
UDP[I]=Y3
CR WHENEVER TAU.E.0 AND TS.E.0
Z[I]=Y[I]
ZP[I]=Y2
ZDP[I]=F2
CR WHENEVER TAU.LE.T1
VT[I]=Y[I]
W[I]=Y2
WHENEVER TS.E.1, WDP[T[I]=Y3]
WHENEVER TS.E.0, WDP[T[I]=F2
CR WHENEVER TAU.LE.T2
WHENEVER TS.E.1, WDP[T[I]=Y3]
WHENEVER TS.E.0, WDP[T[I]=F2
CR WHENEVER TAU.LE.T3
WHENEVER TS.E.1, WDP[T[I]=Y3]
WHENEVER TS.E.0, WDP[T[I]=F2
END OF CONDITIONAL
WHENEVER TS.E.1, UTP[I]=F3

WHENEVER TS.E.1
F[I]=Y2
F2=I+1
WHENEVER TAU.E.0
F[I]=2.*X-AP.*P.*.2.*Y.*Y1
LIMEWISE
F[I]=2.*X-AP.*P.*.2.*Y1+U[I]Y3-(-1)*UP[I]Y2+U*UDP[I]Y1+1
1(T1)+UP[I]P.2+U[I]U*UDP[I]AP-103
END OF CONDITIONAL
WHENEVER TAU.E.0
F[I]=Y2
WHENEVER TAU.E.1
RHS=0
CR WHENEVER TAU.LE.T1
WHENEVER TAU.LE.T2
WHENEVER TAU.LE.T3
END OF CONDITIONAL
WHENEVER TAU.LE.T1
WHENEVER TAU.LE.T2
WHENEVER TAU.LE.T3
END OF CONDITIONAL
SWRAEQ..[1]
WHENEVER 1.E.GRSLZ, TRANSFER TO SW6
WHENEVER 1.E.GRSLZ, TRANSFER TO SW6
END OF CONDITIONAL

TAU=TAU+1
WHENEVER TAU.E.E1, TRANSFER TO SW6

SET UP OF ENERGY PROB.

T=0
TAU=0
T5=0
MU=2.0/(A.P.2)*PR*XE
YY=YP
TRANSFER TO SW1

CALC. OF TERMS FOR ENERGY BOUNDARY CCNC.

WHENEVER TAU.E.E1

E.RC=G

G.R. WHENEVER TAU.E.E1, T1

E.RC=(Z(T)+1)*A(T)+0.5*DELTA.(C.T)-2.0*EPS+(1-DELTA.(C.T))*XE*

1 A(T)=A(T)+1

G.R. WHENEVER TAU.E.E1, T1

E.RC=(X(T)+1)*2.0*(A(2)+C.2)*AI(I)-0.5*(AI(I)+P.Z)-(AI(I)-0.25)

1 X(0)=I

G.R. WHENEVER TAU.E.E1, T1+2

E.RC=(Z(I)+1)*4.0*(A(2)+2.5)*AI(I)+2.0*AI(I)*AI(I)+XE*(4.0*AZ(I)-2.0*AI(I)+I)

1 A(I)+P.Z)(AI(I)+2.0-0.25)*V(2.1)+0.5*(4.0*AZ(I)-2.0*AI(I))*V(I)

G.R. WHENEVER TAU.E.E1, T2+1

E.RC=(Z(I)+1)*4.0*(A(2)+2.5)*AI(I)+2.0*AI(I)*AI(I)+XE*(4.0*AZ(I)-2.0*AI(I))*V(I)

G.R. WHENEVER TAU.E.E1, T3

E.RC=(Z(I)+1)*4.0*(A(2)+2.5)*AI(I)+2.0*AI(I)*AI(I)+XE*(4.0*AZ(I)-2.0*AI(I))*V(I)

G.R. WHENEVER TAU.E.E1, T3

E.RC=(Z(I)+1)*4.0*(A(2)+2.5)*AI(I)+2.0*AI(I)*AI(I)+XE*(4.0*AZ(I)-2.0*AI(I))*V(I)
1 \*V(5)+i+1) = 244
1+WHENEVER TAU.E.(T+2)
16 \*Z(1)*1+1) = 245
1*WHENEVER TAU.E(5)-0.25*A(5)+3*A(1)+5*B+e(1)*A(3)
1+V(1)+5.25%V(1)+1+1) = 246
2(4*A(1)+2*A(1)+1)*V(1)+3+1+1) = 246
END OF CONDITIONAL

CALC. OF ERROR IN ENERGY B.C.

WHENEVER \*ABS(V(1)+V(1))=EBC,G,EPL AND H.L.MAX
R2(1)=1+Y(1)+MU*V(1)+*EBC

TRANSFER TO SW4

END OF CONDITIONAL

STORAGE OF SOLS AT LAST GRID PT.

OTHERWISE

WHENEVER TAU.F.0

V(1)-V(1) = 251
Z(1)+1-1) = 252
ZP(1)+1-1) = 253
ZCP(1)+1-1) = 254

END OF CONDITIONAL

END OF CONDITIONAL

CALC. OF TERMS TO BE USED IN HIGHER ORDER ENERGY EQUS.

THROUGH SW5, F5G J=1,2,J=5*GRS(I)+1

WHENEVER TAU.E.I, H(J)=5*ZP(J)+A(I)\*V(1)+4+\*V(J)+C.J=5*VP(C.J)+V(V(J)+1)+C.J=5*VP(C.J)+V(V(J)+1)

1 \*I+1) = 273
1+WHENEVER TAU.E.(T+1)\*H(J)=5*ZP(J)+A(I)
1+EP(*)\*A(I)+A(I)\*A(I)+C.5*V(1)+2*V(1)+2*V(1)+V(1)+1+1
2-1=V(1)+V(1)+V(1)+V(1)+V(1)+1+1
2=V(1)+V(1)+V(1)+V(1)+V(1)+1+1
3=A(I)+C.5*V(1)+3*STEP+J*V(1)+1+1

WHENEVER TAU.E.(T+1)\*H(J)=5*ZP(J)+A(I)\*V(1)+4+STEP+J*A(I)

1 \*A(I)+C.5*V(1)+3*STEP+J*V(1)+1+1
2=A(I)+C.5*V(1)+3*STEP+J*V(1)+1+1
3=A(I)+C.5*V(1)+3*STEP+J*V(1)+1+1
4=V(1)+V(1)+V(1)+V(1)+V(1)+1+1
5=A(I)+C.5*V(1)+3*STEP+J*V(1)+1+1

TEP*J*VP(C.J)

WHENEVER TAU.E.T2, H(J)=5*ZP(J)+A(I)\*V(1)+4+\*V(J)+C.J=5*VP(C.J)+V(V(J)+1)

1 \*V(J)+1) = 277

SW15

WHENEVER TAU.E.(T+1), H(J)=5*ZP(J)+A(I)\*V(1)+4+\*V(J)+C.J=5*VP(C.J)+V(V(J)+1)

1 \*J+1) = 277
1+WHENEVER TAU.E.(T+1), H(J)=5*ZP(J)+A(I)\*V(1)+4+\*V(J)+C.J=5*VP(C.J)+V(V(J)+1)
WHenever TAU.E.T1, R = 0.5 * A1(1) * G1(C)
WHenever TAU.E.T1*1, R = 0.5 * A1(1) * G1(2) + A3(1) * G1(C)
WHenever TAU.E.T1*2, R = 0.5 * A1(1) * G1(3) + A1(1) * G1(4) - A3(1) * G1(2)
WHenever TAU.E.T2, R = 0.5 * A1(1) * G1(I0)
WHenever TAU.E.T2*1, R = 0.5 * A1(5) * G1(1) + A1(1) * G1(14) + A3(1) * G1(12)

TEST LOC. TO DET. WHICH ENERGY EQ. IS BEING SOLVED AND
WHEN ENERGY PROB. IS COMPL.

TAU = TAU + 1
Y = Y
WHenever TAU.E.T1
T = T - 1
WHenever TAU.E.T1
T = T - 2
WHenever TAU.E.T2
T = T - 2
WHenever TAU.E.T3
T = T - 2
WHenever TAU.E.T3
T = T - 4
WHenever TAU.E.T3
END OF CONDITIONAL
WHenever TAU.E.T3, TRANSFER TO SW1

PRINT STATEMENTS AND MISS. CALC'S

PRINT FORMAT FM1
PRINT FORMAT FM2
PRINT FORMAT FM3
PRINT FORMAT FM4
PRINT FORMAT FM5
PRINT FORMAT FM6
PRINT FORMAT FM7
PRINT FORMAT FM8
PRINT FORMAT FM9
PRINT FORMAT FM10

S10

WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)

PRINT FORMAT F11
PRINT FORMAT F12
PRINT FORMAT F13
PRINT FORMAT F14
PRINT FORMAT F15
PRINT FORMAT F16
PRINT FORMAT F17
PRINT FORMAT F18
PRINT FORMAT F19
PRINT FORMAT F20

S11

WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)

PRINT FORMAT F21
PRINT FORMAT F22
PRINT FORMAT F23
PRINT FORMAT F24
PRINT FORMAT F25
PRINT FORMAT F26
PRINT FORMAT F27
PRINT FORMAT F28
PRINT FORMAT F29
PRINT FORMAT F30

S12

WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)

PRINT FORMAT F31
PRINT FORMAT F32
PRINT FORMAT F33
PRINT FORMAT F34
PRINT FORMAT F35
PRINT FORMAT F36
PRINT FORMAT F37
PRINT FORMAT F38
PRINT FORMAT F39
PRINT FORMAT F40

S13

WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)
WHenever SW1, FOR K = 1, L, K.G.(GSI=1)

PRINT FORMAT F41
PRINT FORMAT F42
PRINT FORMAT F43
PRINT FORMAT F44
PRINT FORMAT F45
PRINT FORMAT F46
PRINT FORMAT F47
PRINT FORMAT F48
PRINT FORMAT F49
PRINT FORMAT F50
PRINT FORMAT FM2
#1=1.5708
THROUGH SW30, FOR XI=1,..,11 X1=0.5
1+(P.2)*(X1.P.4)*(X1.P.5)/G1.0
AF1=(ABS(E1.0)))*(1.+C*5*(E1.2)*P.2+2.*E1.4*E1.0)/E1.0
1)*P.2)*(X1.P.4)*XE/A
AN1=(ABS(G1.0)))*(1.+C*5*(G1.2)*P.2+2.*G1.4*G1.0)/G1.0
1)*P.2)*(X1.P.4)*XE/A
AM1=PI-ATAN1(A11)*A5*(XI.P.4)/(A11*(XI.P.2))
AF1=PI-ATAN1(E1.0)*E1.4*(XI.P.4)/(E1.2)*(XI.P.2.1)
AN1=PI-ATAN1(G1.0)*G1.4*(XI.P.4)/(G1.2)*(XI.P.2.1)

WHENEVER CAT.LF,CATM, TRANSFER TO SW31
END OF PROGRAM
APPENDIX XXVI

COMPUTER PROGRAM FOR CALCULATING FILM THICKNESS, LOCAL SKIN FRICTION, AND THE TERMS OF THE EXPANSIONS (52)

This program calculates the liquid film thickness, local skin friction, and the terms of the expansions (52) from the numerical values of $\hat{h}$, $J$, and $V_\phi$ obtained from the program for drop trajectories.

Once $\hat{\delta}$ is calculated, the product $\hat{h} \hat{\delta}$ and $J \hat{\delta}$ are formed, and are tabulated. The expansions (52) are assumed to contain a finite number of terms and a system of linear equations is obtained by writing these expansions at various values of $\xi$ which we denote by $\xi_i$.

Thus we have:

$$\hat{h} \hat{\delta}(\xi_i) = \sum_{n=0}^{m} \frac{\alpha_{2n}}{(2n)!} \xi_i^{2n}; \quad i = 0, 1, 2, \ldots, m$$

This system of equations is written in matrix form and solved for the $\alpha_{2n}$'s by inverting the matrix. Thus if we denote the matrix $A$ by:

$$A = \begin{bmatrix} \xi_i^{2n} \\ \xi_i \\ \frac{1}{(2n)!} \end{bmatrix}$$

the vector $\tilde{c}$ by
\[
\alpha_0 = \begin{bmatrix}
\dot{\alpha}(\xi_0) \\
\dot{\alpha}(\xi_1) \\
\vdots \\
\dot{\alpha}(\xi_n)
\end{bmatrix}
\]

and the vector \( \tilde{\alpha} \) by:

\[
\tilde{\alpha} = \begin{bmatrix}
\alpha_0 \\
\alpha_2 \\
\vdots \\
\alpha_{2m}
\end{bmatrix}
\]

We have:

\[
\tilde{c} = A \tilde{\alpha}
\]

And:

\[
\tilde{\alpha} = A^{-1} \tilde{c}
\]

CORRESPONDENCE BETWEEN NOTATION FOR COMPUTER PROGRAM FOR CALCULATING FILM THICKNESS, LOCAL SKIN FRICTION, AND TERMS OF THE EXPANSION (52), AND NOTATIONS USED IN THE TEXT

\( AG(2k-1) \sim \gamma_{2k-1} \)

\( DBL(M) \sim \tilde{\delta} \)

\( DR \sim \rho_g/\rho \)

\( E \sim E \)

\( F.(N) \sim N! \)
\[
\text{FF}(M) \sim \tau^* \sqrt{\frac{2R_0 U_\infty}{\nu}} \frac{1}{\frac{1}{2} \rho U_\infty^2}
\]

\[
\text{GAM} \sim \gamma
\]

\[
\text{MDOT} \sim \mathcal{M}
\]

\[
P(M) \sim \tilde{x}
\]

\[
R \sim \tau_{\text{eff}}
\]

\[
\text{RINF} \sim 2R_0 U_\infty / v_g
\]

\[
\text{RR} \sim \frac{r_d}{R_0}
\]

\[
\text{VR} \sim \sqrt{\frac{v_g}{\nu}}
\]

\[
X \sim \tilde{x}
\]

\[
Z(x) \sim Z(x)
\]
SET UP

INTEGER I
INTEGER TMAX
INTEGER IMAX, SM
INTEGER M,K, N, LP
DIMENSION Q(8), D(8), AG(20)
DIMENSION C(100), D(100)
DIMENSION FF(200), A(200), R(200), ALP(200), RET(200), CV(200), P(200)
DIMENSION XBAR(500), MDOT(500), J(500), VPHI(500)
DIMENSION DEL(200)
DIMENSION C(14000, CS)
DIMENSION C(4000, CT)
VECTOR VALUES CS=2.1, 60
VECTOR VALUES CT=2.1, 60
VECTOR VALUES FM1=1H1/IH-#
VECTOR VALUES FM2=1H1 , S4, S3/IH-#
VECTOR VALUES FM3=1H1 , S4, S3,F53,F12.6*#
VECTOR VALUES FM4=1H1 , S4, S3, F20.6*#
VECTOR VALUES FM5=1H1 , S4, S3, F12.6*#
VECTOR VALUES FM6=1H1 , S4, S3, F12.6*#
VECTOR VALUES FM7=1H1 , S4, S3, F12.6*#
READ AND PRINT DATA

SW9
THROUGH SW9, FOR K=1,1,K,G, KMAX
READ FORMAT FMS, XBAR(K), MDOT(K), J(K), VPHI(K)

SW4
PRINT RESULTS AG(2*K-1)

INTERNAL FUNCTION F(I,J)=GAMMA(I)*J

DEFINITION OF Z.

INTERNAL FUNCTION (IX)
ENTRY TO Z.
THROUGH SW9, FOR K=2,1,K,G,6

FUNCTION RETURN SZ
END OF FUNCTION

CALC. OF DELTA AND RELATED QUANTITIES

T=0
E=0

SW10
T=T+1
E=E+0.02
KI=DR/V
K2=4.*KI*VR
M=0

SW1
M=M+1
X=M*INCX
N=IP*M
P(M)=X
Q=(4./3.)*K1*SN*(2*XK)
R=K2*LI)*J(N)*VPHI(N)
S=MDOT(N)
WHENEVER (.ABS.(1-(27./2.))*(Q.P.2)*S/(R.P.3))=1.0
M=M-1
TRANSFER TO SW15
OTHERWISE

CONTINUE

END OF CONDITIONAL
GAM=ARCCOS((-1-(27./2.))*(Q.P.2)*S/(R.P.3)))
DEL(M)=(R/3.Q)*((2.*COS.(GAM/3.))-1)

CALC. OF LOCAL SKIN FRICTION

FF(M)=2.*E=R+3.*E+Q+DEL(M)

CALC. OF ALPHA SUB. 2J AND BETA SUB. 2J

C(M)=DEL(M)*J(N)
D(M)=DEL(M)*MDOT(N)
WHENEVER X.L.XMAX, TRANSFER TO SW1
THROUGH SW3, FOR K=1,1,K.G.IMAX
WHENEVER K.E.1
X=P(I)
C(I)=C(M)
D(I)=D(M)

OR WHENEVER K.E.IMAX
X=P(M)
C(I)=C(M)
D(I)=D(M)

OTHERWISE

X=X*(P(M)-P(I))/((MAX-1.)
A(I)=TAR.X.P(I),C(I),11,1,4,M,T6)
B(I)=TAR.X.P(I),C(I),11,1,4,M,T6)
END OF CONDITIONAL
THROUGH SW3, FOR I=1,1,I.G.IMAX
C(I,J)=X.P.(2*J-2)

SW3
C2(K,J)=X.P.(2*J-1)
S1=SLC.(I.MAX,60,C1(I,J),ALP(I)+A(1),CV.D)
PRINT RESULTS S1
S2=SLC.(I.MAX,60,C2(1,J),BET(I)+B(1),CV.D)
PRINT RESULTS S2

PRINT STATEMENTS
PRINT RESULTS RINF
PRINT RESULTS RR
PRINT RESULTS E,M
PUNCH FORMAT FM7, E,M
PRINT FORMAT FM1
PRINT FORMAT FM2
THROUGH SW7, FOR K=1, K.G.M
PUNCH FORMAT FM6, P(K), DEL(K)
PRINT FORMAT FM3, P(K), DEL(K), FFI(K)
PRINT FORMAT FM1
PRINT FORMAT FM2
THROUGH SW8, FOR K=1, 1, K.G.IMAX
ALP(K)=ALP(K)*F.(2.*K-2.)
BET(K)=BET(K)*F.(2.*K-1.)
PUNCH FORMAT FM4, K, ALP(K), BET(K)
PRINT FORMAT FM4, K, ALP(K), BET(K)
WHENEVER T.LE.TMAX, TRANSFER TO SW1G
END
END OF PROGRAM
APPENDIX XXVII

REDUCTION OF EXPERIMENTAL DATA FROM REFERENCE (1)

From Reference 1 we shall consider the data corresponding to the SC-6 nozzle with an air velocity of 110 ft/sec and a water flow rate of 0.083 lb/sec, given in Table II-B, page 48. The reported value of $2R_0\frac{U_\infty}{\nu_g}$ was $8 \times 10^4$.

The measured total value of flow using a drop collecting device and performing a numerical integration was 0.061 lb/sec. The actual total flow obtained from their rotameters was 0.083 lb/sec. The measured value of the flow impinging on the cylinder using the drop collecting device was 0.002 lb/sec. We therefore take as a corrected value for the total flow impinging on the cylinder 0.002 x (0.083/0.061) or 0.00272 lb/sec. On page 24, Reference 1, is reported that for an air velocity of 110 ft/sec the drop velocity was 85 ft/sec. From Reference 23, Table E-2, $\nu = 0.045 \text{ ft}^2/\text{hr}$ at the drop temperature of $57^\circ$. The diameter of their cylinder was 1.5 in.

Hence:

$$\frac{2R_0U_\infty}{\nu} = \frac{1.5}{12} \times \frac{3600}{.046} \times 85 = 8.30 \times 10^5$$

Since the cylinder is 2 in. in length, the mean liquid flux impinging on the cylinder

$$= \frac{\text{actual flow impinging on the cylinder}}{\text{area of cylinder}}$$

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\[ = \frac{0.00272}{1.5 \times 2} \times 144 = 0.13 \text{ lb/sec ft}^2 \]

Hence the effective mass of liquid per unit volume in the free stream is:

\[ \frac{0.13}{85} = 1.52 \times 10^{-3} \text{ lb/ft}^3 \]

and the volume fraction of liquid in the free stream, \( x_e \), is

\[ x_e = \frac{1.52}{62.4} \times 10^{-3} = 2.44 \times 10^{-5} \]

Hence:

\[ \sqrt{\frac{2R_e U_\infty}{\nu} N} = \sqrt{8.30 \times 10^5} = 910 \]

\[ E = x_e \sqrt{\frac{2R_e U_\infty}{\nu}} = 0.022 \]

From Reference 23, Table E-2, at a wall temperature of 78°, \( k = 0.35 \text{ Btu/hr-ft°F} \).

Hence:

\[ \frac{2R_o}{k \sqrt{\frac{2R_o U_\infty}{\nu}}} = \frac{1.5}{12 \times 35 \times 910} = 3.94 \]

Table II-B (from Reference 1)

<table>
<thead>
<tr>
<th>Position</th>
<th>( h_{\text{average}} ) Btu/ft(^2)-hr-°F</th>
<th>( \sqrt{\frac{2R_o U_\infty}{\nu}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>451</td>
<td>0.178</td>
</tr>
<tr>
<td>30°</td>
<td>372</td>
<td>0.147</td>
</tr>
<tr>
<td>60°</td>
<td>270</td>
<td>0.1065</td>
</tr>
<tr>
<td>330°</td>
<td>448</td>
<td>0.168</td>
</tr>
</tbody>
</table>