IMPROVED TRUNCATION FOR SPRT WITH NORMAL OBSERVATIONS

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Abstract:

A desirable truncation of the SPRT would produce actual error probabilities less than or equal to desired error probabilities. We present such a truncation, as a function of desired error probabilities and distribution parameters, when the observations are IID Normal. This truncation is shown to be superior to others in the literature, particularly to Johnson's working rule.

Key words: SPRT, truncation point, discrimination factor, error probabilities.

Introduction:

Suppose we wish to test H_0: $W = w_0$ against H_1: $W = w_1$, by sequentially observing independent random variables X_1, X_2, ..., X_n with density $f_i(x_i/w_j)$, j=0, 1. Wald (7) proposed the SPRT, truncated at point m, as follows:

Define the log-likelihood ratio at time i

$$z_i = \ln \left\{ \frac{f_i(x_i/w_1)}{f_i(x_i/w_0)} \right\},$$

and the log-likelihood ratio up to time n

$$\underline{Z}_n = \sum_{i=1}^{n} z_i .$$

Then,

reject H_0 if $\underline{Z}_n \geq b$ for n = 1,2,...,m

accept H_0 if $\underline{Z}_n \leq a$ for n = 1,2,...,m

and take one more observation if

$$a < \underline{Z}_n < b \quad \text{for} \quad n = 1,2,...,m-1.$$

If the experiment has not stopped at or before m observations then

reject H_0 if $b > \underline{Z}_m > 0$

and accept H_0 if $a < \underline{Z}_m < 0$ . . . . . . (1)

Then stopping bounds a and b are given by the approximation

$$a \cong \ln \left\{ \frac{\beta_d}{1-\alpha_d} \right\} \quad \text{and} \quad b \cong \ln \left\{ \frac{1-\beta_d}{\alpha_d} \right\} \qquad (2)$$

where α_d and β_d are the desired error probabilities of type I and type II respectively.

It is our objective to choose (in advance) the smallest truncation point m such that the actual error probabilities α_a and β_a, achieved by the test, are less than or equal to the desired values α_d and β_d. In addition, we restrict ourselves to using Wald's bounds (relation (2)), since they are easily computable and well understood. We thus do not consider non-constant bound techniques such as those in (1,2).

Wald (7) suggested setting m "large enough" such that the effect of truncation on the actual error probabilities would be minimal. (This tends to produce a conservative test, as we shall see). In this spirit, Ghosh (4) and Johnson (6) give a working rule for choosing a non-integer truncation point $m_j{}^*$ when the observations are independent and identically distributed Normal random variables: set $m_j{}^* = 3 \, \mathrm{Sup}_W \, E[N/W]$, where $E[N/W]$ is the expected number of observations for the <u>untruncated</u> SPRT when W is the true state of nature. The integer point $m_j{}^{**}$ is then obtained by rounding up $m_j{}^*$ to the next higher integer. It will be shown later that the $m_j{}^{**}$ thus chosen tends to be conservative. Aroian and Robison (3) showed that for small m, actual error probabilities can be computed to any desired degree of accuracy by using numerical integration. Their method, however, becomes tedious for large m as discussed by Golhar (5). We present here a more efficient method, which produces (for IID Normal random variables, and for the symmetric case, i.e., when $\alpha_d = \beta_d$) a simple relationship between m* (the non-integer truncation point), $\alpha_d = \beta_d$ and the distribution parameters. We also show that the value of m* obtained through such a relationship is superior to the one obtained by Johnson's working rule, in the sense that the resulting truncated SPRT gives a smaller

E(N) for any desired error probability.

<u>Computing the non-integer truncation point m^*</u>:

To find the operating characteristic function $L_j = L(w_j)$, we need the probability density of $\underset{\sim}{Z}_n$ given $\bigcap_{k=1}^{n-1} a<\underset{\sim}{Z}_k<b$. Let $P_j(z,n) = \text{Prob.}\left\{(\underset{\sim}{Z}_n \leq z) \bigcap_{k=1}^{n-1} (a < Z_k < b) \middle| W=w_j\right\}$ and let $p_j(z,n)$ be the derivative of $P_j(z,n)$ with respect to z. Then successive convolutions are required to calculate $p_j(z,n)$ namely

$$p_j(z,n) = f_1(z/w_j)$$

$$p_j(z,n) = \int_a^b p_j(u,n-1) \quad f(z-u/w_j)\ du \qquad \text{for } n > 1$$

Using these relationships we can calculate the operating characteristic function

$$L_j(m) = \sum_{n=1}^{m-1} \left[\int_{-\infty}^{a} p_j(z,n)\ d_z\right] + \int_{-\infty}^{0} p_j(z,m)\ d_z \quad \ldots(3)$$

The following procedure is used to obtain m^* when $f_i(X_i/w_j)$ is Normal with mean w_j (j=0,1) and variance σ^2 :

i) Given $\alpha \equiv \alpha_d = \beta_d$, compute Wald's stopping bounds a and b from equation (2).

ii) For a given value of discrimination factor $d = (w_0-w_1)/\sigma$ use relationship (3) to compute $\alpha_a(m)$ $(=\beta_a(m))$ = $L_1(m)$ for different values of m.

iii) Find m^* by interpolation so that $\alpha_a(m) = \alpha_d$.

<u>Results</u>:

Values of α were varied between .01 and .2, and values of d between .5 and 2. For each α, figure 1 shows ln(m*) vs. ln(d). An essentially linear relationship between ln(m*) and

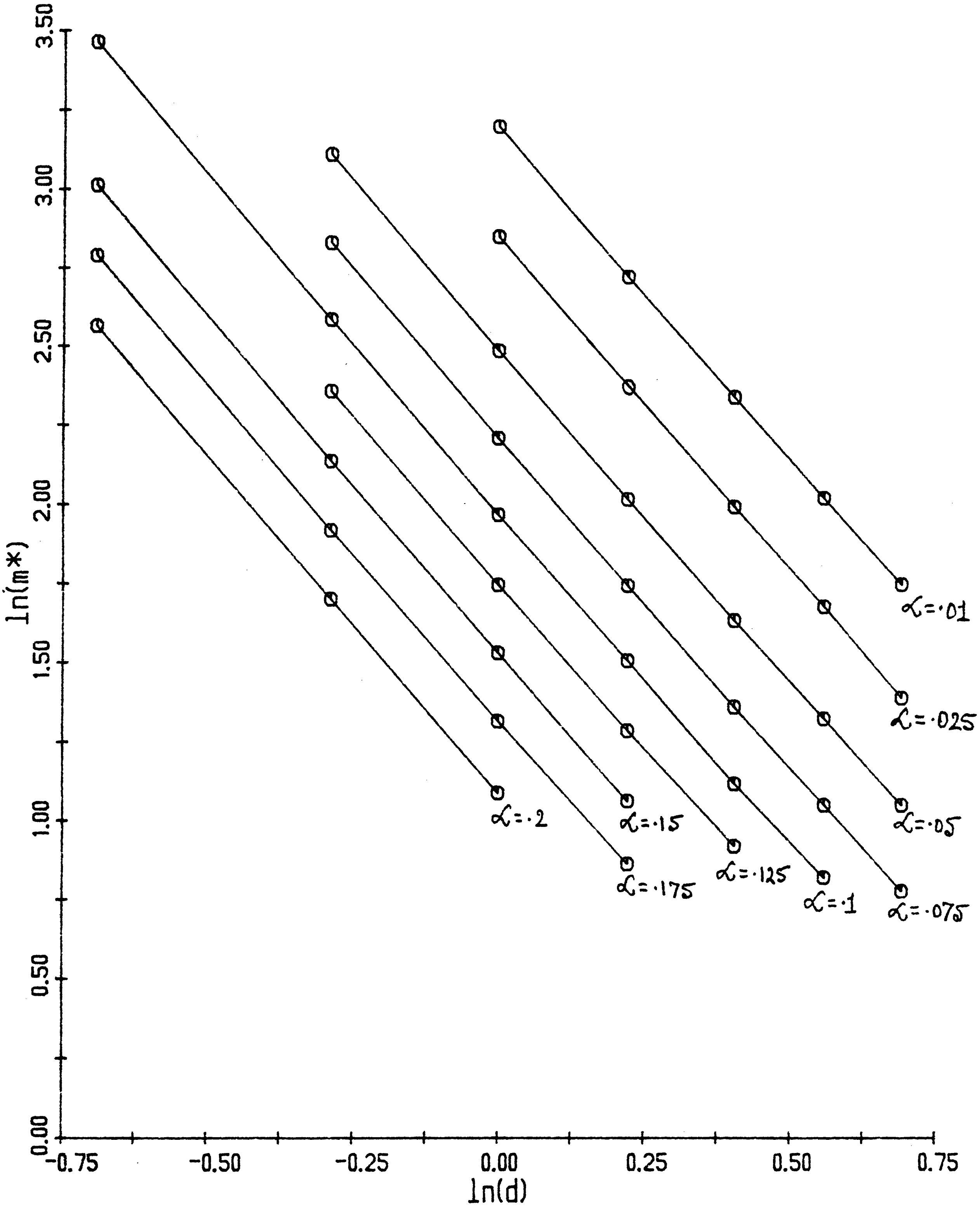

Figure 1 - The relationship between $\ln(m^*)$ and $\ln(d)$ for IID Normal random variables.

$\ln(d)$ is seen. Using a least squares procedure, a constant slope of -2.09 is obtained for all fixed α between .01 and .2. This suggests that m* and d have the following simple relationship:

$$\ln(m^*) \cong \ln[k(\alpha)] - 2.09 \ln(d), \qquad (4)$$

$$\text{or,} \quad m^* \cong \frac{k(\alpha)}{d^{2.09}}$$

where $\ln[k(\alpha)]$ is the intercept at d=1, and depends upon the value of α.

To obtain $k(\alpha)$, m* was plotted against for d=1 as shown in figure 2. The curve is well fit by the equation:

$$k(\alpha) \cong -79 + 72 \, (\alpha)^{-.079} \qquad (5)$$

Using equation (5), relationship (4) can be written:

$$\ln(m^*) \cong \ln[-79 + 72 \, (\alpha)^{-.079}] - 2.09 \ln(d) \qquad (6)$$

Thus there is a useful and simple linear relationship (6) between m*, α and the discrimination factor d. We can obtain the smallest integer m** by rounding up the solution to (6).

To show that the truncated SPRT using m** thus chosen gives a smaller E(N) than the one obtained by using Johnson's working rule it suffices to demonstrate that $m^* < m_j^*$ for different d and α. The solid lines in figure 3 show (linear) relationships between ln(m*) vs. ln(d) for α = .01, .1, and .2. The dotted lines correspond to $\ln(m_j^*)$ vs. ln(d) for the same values of α. As can be seen, m_j^* is 15% to 50% higher than m*. Thus, Johnson's rule gives a very conservative truncation point, giving a value of E(N/W) greater than that obtained by using m*.

Conclusion:

We have established a simple relationship between a

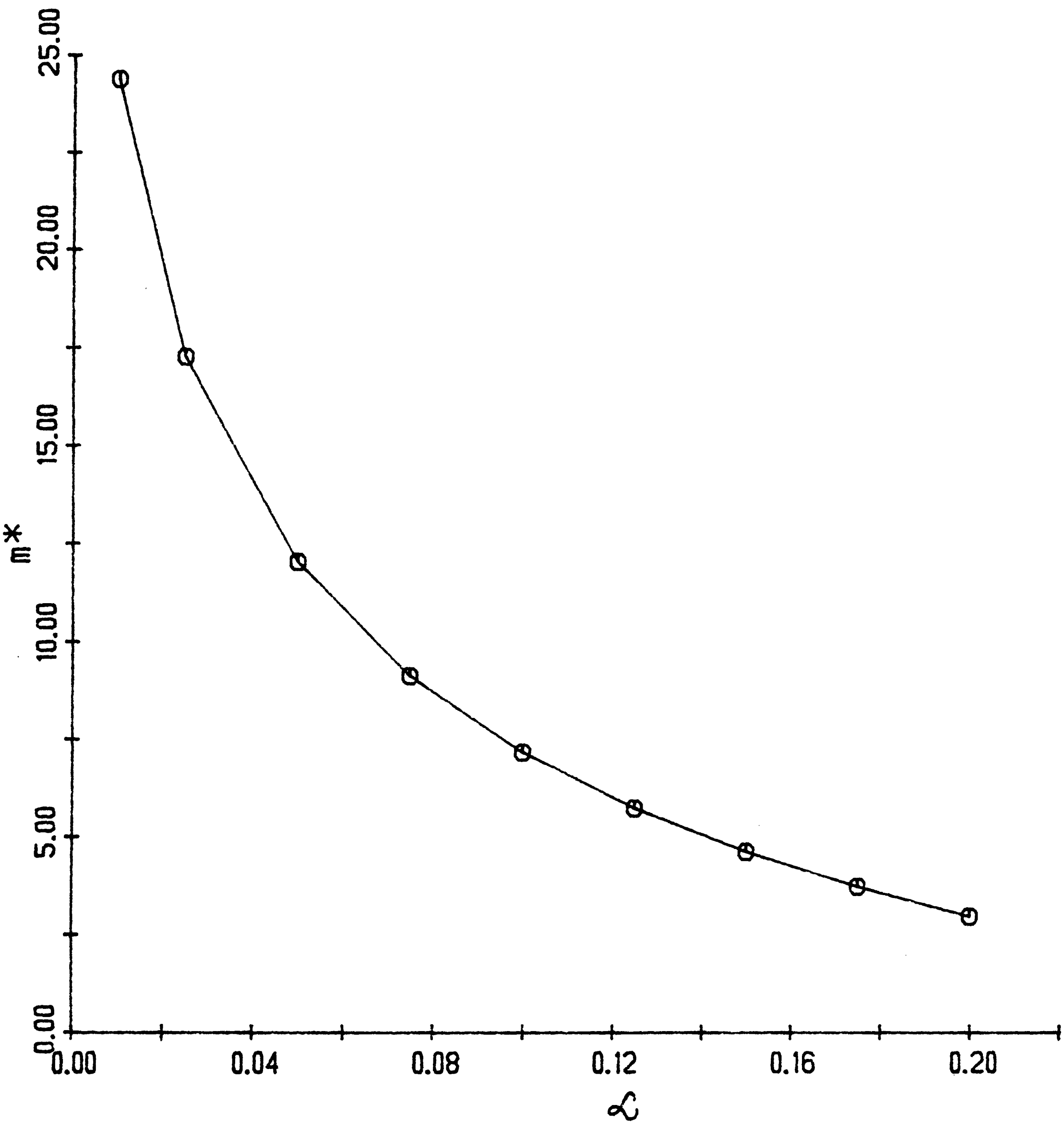

Figure 2 - The relationship between m and α when d=1 for IID Normal random variables.

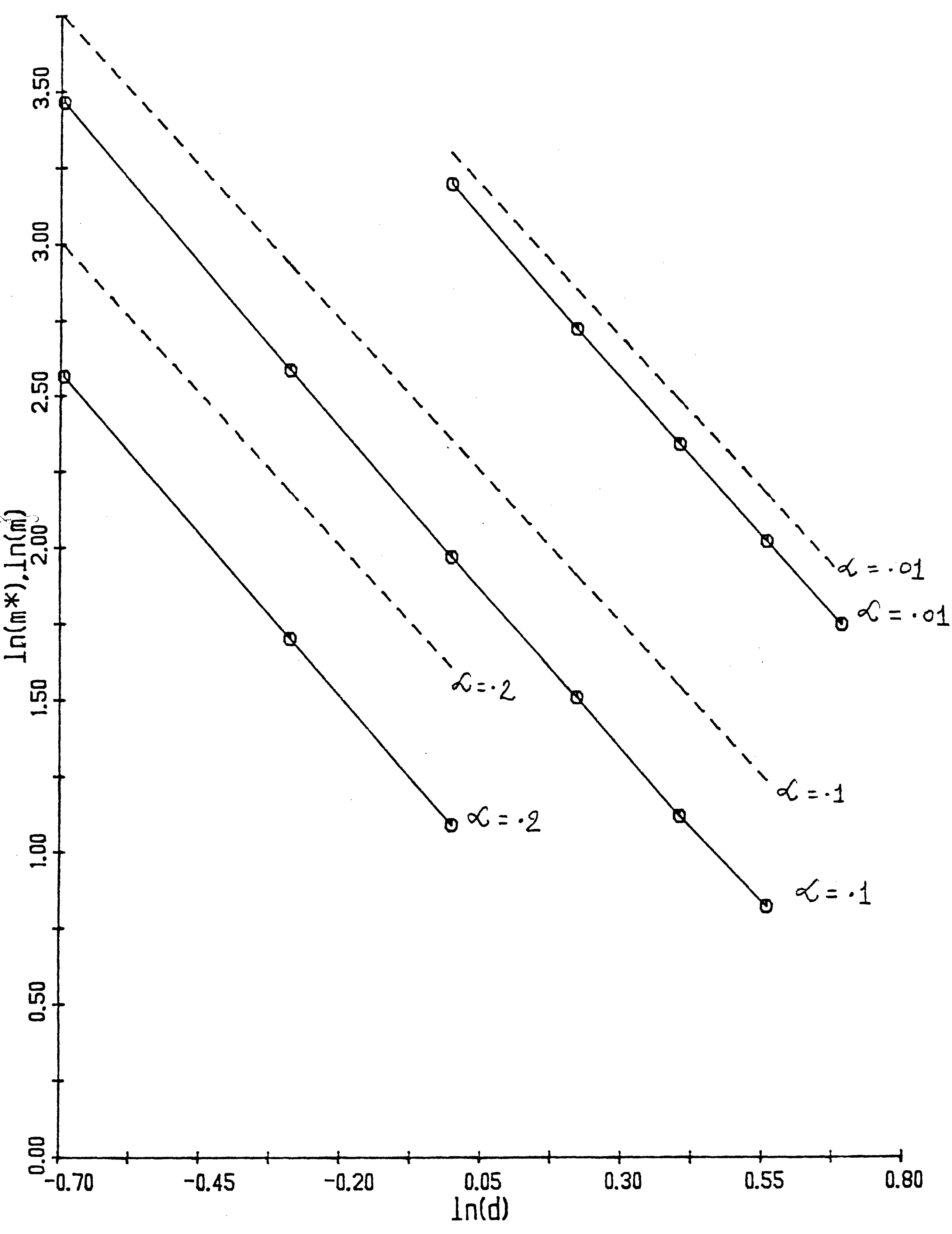

Figure 3 - A comparison of m^* (solid lines) with Johnson's approximate m_j^* (dotted lines) for IID Normal random variables.

useful truncation point m**, α and d for tests using Wald's bounds. We have also shown that the SPRT using this truncation point gives a smaller E(N/W) than using a truncation point obtained from Johnson's working rule, yet still gives actual error probabilities within desired limits.

In this paper we only analyzed the symmetric case ($\alpha_d = \beta_d$), and Normal observations. Work is in progress on the non-symmetric case, and when the observations are exponentially distributed.

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