Technical Report No. 123

ERRATA

Page 10, Fig. 3: "Take Action A_1" & "Take Action A_2" should be interchanged.

Page 17, Line 8: $z_\gamma = .343$

Line 10: $0 \leq z \leq .343$

Line 11: $.343 < z < .657$, defer; expected loss is $G_1(z)$.

Pages 23 & 24, Figure 8 & figure 9: The graphs should be labeled, as in Figure 7, in descending order; n = 1, n = 2,....

Page 30, In the block diagram the block $z(I) \leftarrow z(I) + .01$ should be replaced by $C \leftarrow C + .01$

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DEFERRED DECISION THEORY

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PREFACE

Before his untimely death, in October of 1960, Professor Goode had completed the following work in rough form. Credit should be given to Blackwell and Girschick (Ref. 11) for the mathematical formulation of the problem. Because of Prof. Goode's experience in the use of high speed computer devices he realized that the calculations implied by the mathematics were feasible with today's high speed computers. He also had the insight to see how these calculations could be applied to present-day decision problems, e.g., radar.

The work was done under US Army Signal Corps Contract No. DA-36-039 sc-78283 at the Cooley Electronics Laboratory. Mr. T. G. Birdsall, Mrs. P. Elliot, Mr. R. A. Roberts, and Mr. W. Evans have made it possible to present Professor Goode's work. Without funds provided by the Office of Naval Research under Contract No. Nonr-1224(36) this report could not have been completed.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ABSTRACT

The problem examined in this paper is the case of the simple two-alternative decision in which it is allowable to defer decision until further observations have been made. Given the costs of making errors and the cost of deferring a decision, the problem is to determine a course of action which will minimize the expected losses. An iterative algorithm is presented for calculation of decision points. The existence of decision points is proved by proving convergence for the algorithm. Results for Gaussian distributions run on the IBM 704 are included. For the parameters examined it is shown that the decision points converge within four deferrals.

DEFERRED DECISION THEORY

1. INTRODUCTION

Some of the earliest notions concerning decision making between two alternatives on the basis of observations of a stochastic variable were introduced by LaPlace (Ref. 1). In his view, for a variable representing an outcome which is either a success or a failure, the quantity to be calculated is

$$p = \frac{r + 1}{n + 1}, \tag{1}$$

where *n* is the number of trials and *r* the number of successful outcomes. If *p*, now conventionally considered the probability of success, is large, an action compatible with a successful outcome is indicated. If *p* is small, the converse action is indicated.

Subsequently, Bayes (Ref. 2) constructed a model of decision which, for the two-alternative case, sets up a choice to be made between two hypotheses. The choice depends on values of a stochastic variable to be observed. It is assumed that the *a priori* probability of each hypothesis' being true is known. The distribution of the observed-variable values when either hypothesis holds is also known. More explicitly, let the hypotheses be H_1 and H_2, their associated *a priori* probabilities z and 1-z, respectively, and let the distribution of *y*, the observed variable, be $p_1(y)$ when H_1 is true and $p_2(y)$ when H_2 is true. Then

$$P_y(H_1) = \frac{zp_1(y)}{z\, p_1(y) + (1-z)p_2(y)} \tag{2}$$

is Bayes' formula, where $P_y(H_1)$ denotes the probability that if y was observed H_1 is true.

In the context of decision making, this formula incorporates the following parameters:

1. The difference in value between alternatives if these can be expressed quantitatively.
2. Prior knowledge about the hypotheses.
3. The distribution of outcomes under each hypothesis.

Another factor, important in making decisions, which is not accounted for by the formula is the set of losses and gains associated with the several possible combinations of hypotheses and actions taken, as, for example, an action taken compatible with H_2 when H_1 is true.

Because of widespread misunderstanding about the meaning and use of the a priori probabilities, Bayes' formula fell into disuse. Its place was taken by a considerable amount of confused thinking, represented, for example, by the view that the population mean is distributed about the sample mean instead of the converse.

Into this state of affairs, Fisher (Ref. 3) injected precise statements concerning inferences to be drawn from observations. He defined the null hypothesis as one of interest whose validity was to be tested. He rejected the null hypothesis when sets of observations occurred which would be improbable if the null hypothesis were true. The level of improbability at which the null was to be rejected was labeled the level of significance. However, the alternative hypothesis was not in evidence.

Neyman and Pearson (Ref. 4) reintroduced the alternative hypothesis. They stated, reasonably, that if one alternative is rejected, another is accepted. (This might be a complex of alternatives all considered at once.) This, in turn, led to the recognition of two types of error: accepting H_2 when H_1 is true (Type I), and accepting H_1 when H_2 is true (Type II). Because the probability of each type of error could be calculated if the decision policy relative to values observed were stated, it was possible to make, in some sense, an optimum decision. Neyman and Pearson suggested holding α, the Type I error probability, constant and so choosing the decision mechanism that β, the Type II error, would be minimized.

For example, consider the situation represented in Fig. 1, where $p_1(y)$ and $p_2(y)$ are both normal with different means and a single observation is made. If one decides to accept H_2 whenever y lies in the interval $y_1 \leq y \leq y_2$ then,

$$\alpha = \int_{y_1}^{y_2} p_1(y)\, dy ,$$

and

$$1-\beta = \int_{y_1}^{y_2} p_2(y)\, dy .$$

These areas are indicated for two different decision intervals in Fig. 1. When the observation falls in a region α, which is fixed in area, H_2 will be accepted. Otherwise H_1 will be accepted. The only restriction on the choice of α is its fixed area. Minimizing β is equivalent to maximizing $1-\beta$. It is intuitively clear that the α region in the right tail of $p_1(y)$ maximizes $1-\beta$ in $p_2(y)$. It is also rigorously correct. Thus, Neyman and Pearson had restored the alternatives in the two-alternative decision but made no use of prior knowledge.

Wald (Ref. 5) took a major step in his treatment of statistical decision functions. He introduced three new characteristics of the

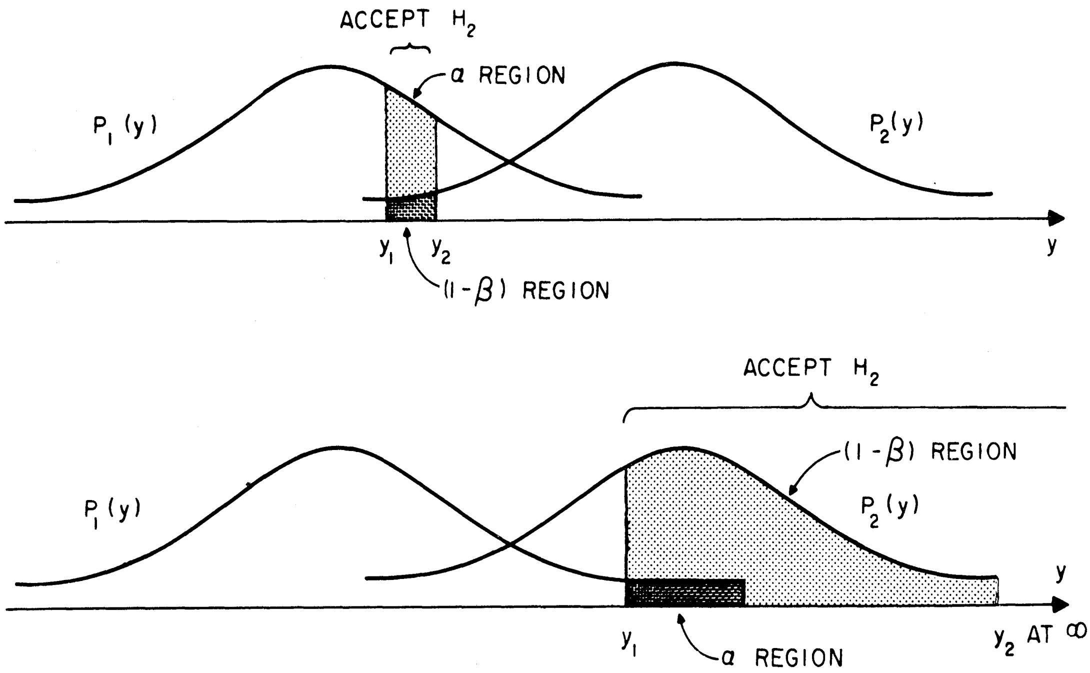

Fig. 1. Two choices of the α region and the corresponding $1-\beta$ regions for two normal distributions.

decision and unified the entire treatment. The three new characteristics are:

a) The introduction of the costs of errors as determinants of the sizes of α and β. It remains true that there are two types of error, but this fact is less important than the losses due to each.

b) The inclusion of the alternative of putting off decisions until it pays to make a decision. Up to this point, the decision was assumed to be made once and for all after observations were taken. Wald introduced the clearly desirable notion of waiting until making a decision is worthwhile.

c) The possibility that an opponent was determining the situation was introduced. This led to the minimax criterion which Von Neumann (Ref. 6) had introduced earlier, and the result was shown to be equivalent to the game solution for zero-sum two-person games. Wald used the expected value as a criterion for action.

In the present paper, we are concerned with the case of a two-alternative decision in which it is allowable to defer decision (deferred decision case) until further observations have been made. The problem is to determine the course of actions (which will minimize expected losses). The major objectives of the paper are twofold: to set forth the solution of this problem as simply as possible so as to exhibit its implications for practical decision making and to provide a means of determining numerical answers for special cases. To our knowledge this has not previously been done.

We begin with a short review of the two-alternative decision under the approach used in present-day statistical decision theory. Then the problem is restated in deferred-decision terms. An iterative algorithm is produced for the calculation of decision points if, in fact, they exist. The existence of decision points is proved by proving convergence for the algorithm.

Since the formulas are difficult to handle analytically, a computer program has been written. The flow diagram for the program and a summary of the results for Gaussian distributions and some specific numerical costs are given. Finally, some conclusions are drawn regarding the usefulness of the results.

2. STATISTICAL DECISION - TWO-ALTERNATIVE

Suppose we have two possible alternatives, H_1 and H_2, which we know from prior knowledge have a probability of materializing of $\underline{z}$ and $1-\underline{z}$, respectively. To make decisions, we observe a variable $\underline{y}$ whose probability distributions, if H_i is true, are $p_i(y)$ with $i = 1,2$. We also know the costs of making each of the possible errors: the cost is ω_{12} for taking action A_2 consistent with H_2 when in fact H_1 is true. The cost is ω_{21} for taking action A_1 consistent with H_1 when in fact H_2 is true. The gains from making correct decisions have been normalized at zero with no loss of generality. Our problem is to choose to take A_1 or A_2 on the basis of an observed $\underline{y}$ in such a fashion that the expected loss is minimized.

Our choices may, of course, be made under several different criteria. We will deal with the criterion of minimizing the expected loss as being mathematically tractable, of frequent occurrence, and reasonable in many cases.

After the observation of a value of $\underline{y}$, the state of our knowledge of the truth of H_1 will be given by Eq. 2. It states the newly calculated probability that H_1 is true, $P_y(H_1)$.

The probability (from our view) that H_1 is true is $P_y(H_1)$, and the expected loss for taking A_2 is $P_y(H_1)\omega_{12}$. The expected loss for taking action A_1 is $P_y(H_2)\omega_{21} = [1 - P_y(H_1)]\omega_{21}$. These lines are plotted in Fig. 2. The double-ruled portions of these curves are the smaller loss parts, and if we are forced to be on one of these curves, we should

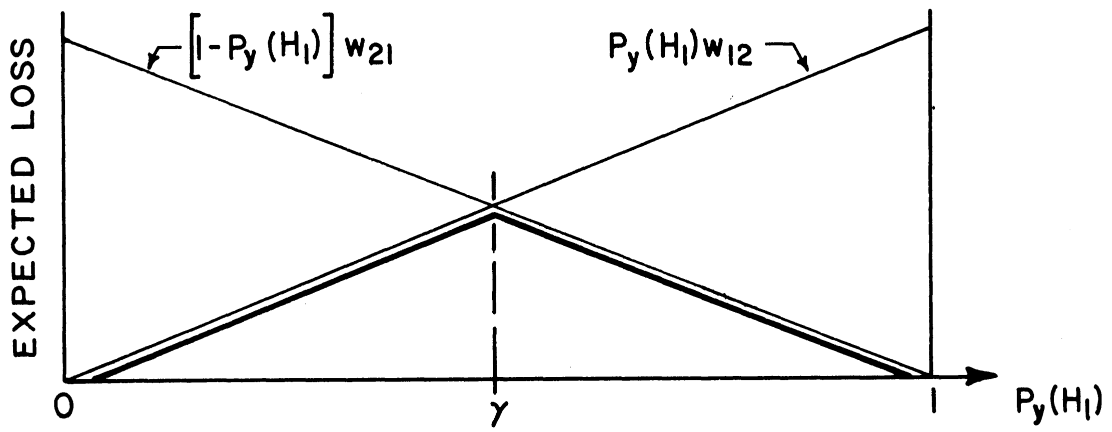

Fig. 2. Plot of expected loss for deciding on the A_1 and A_2 as a function of the probability that event H_1 occurred given observation $\underline{y}$.

try to be on the double-ruled portions. This can be accomplished by taking A_2 when $P_y(H_1) < \gamma$ and A_1 otherwise. γ is given by

$$\gamma = \frac{\omega_{21}}{\omega_{12} + \omega_{21}} \quad . \tag{3}$$

Some notes should be made about this result and its relation to other terminology. Frequently, likelihood ratio, $\ell(y) \triangleq p_2(y)/p_1(y)$, is of interest. $P_y(H_1)$ is monotonic decreasing with $\ell(y)$ as shown by substitution into Eq. 2:

$$P_y(H_1) = \frac{z}{z + (1-z)\ell(y)} \quad , \tag{4}$$

or

$$\ell(y) = \frac{z[1 - P_y(H_1)]}{(1-z)\ P_y(H_1)} \quad . \tag{5}$$

$\ell(y)$ goes from ∞ to 0 as $P_y(H_1)$ varies from 0 to 1. The decision point in $\ell(y)$ is

$$\ell(\gamma) = \frac{z}{1 - z} \frac{\omega_{12}}{\omega_{21}} \qquad (6)$$

i.e., take A_2 when $\ell(y) > \ell(\gamma)$.

Further, one frequently deals with four values: the two costs already stated plus the gains associated with taking A_2 when in fact H_2 is true; and similarly for A_1. Let these gains by ω_{22} and ω_{11}. Then if one calculates expected value taking these into account, ω_{12}/ω_{21} is replaced by $\frac{\omega_{12} + \omega_{11}}{\omega_{21} + \omega_{22}}$. Since the gains are negative costs, choosing ω_{11} and ω_{22} as zero is merely the choice of new origins from which to measure costs. We still retain freedom to choose a unit of measure for cost statements. We reserve its use until later.

Finally, we note that $\underline{y}$ can be representative of a larger number of observations than just one. In case several observations are made, $p_1(y)$ and $p_2(y)$ must be calculable. For example, for independence and a single distribution, $p_1(y) = p_1(y_1)p_1(y_2) \ldots p_1(y_k)$, where $\underline{k}$ observations are made. It is emphasized that the state of our knowledge before seeing $\underline{y}$ is $\underline{z}$ and $1-\underline{z}$, whereas after seeing $\underline{y}$, it is $P_y(H_1)$ and $P_y(H_2)$. We make the decision after seeing $\underline{y}$. If we had to make it before seeing $\underline{y}$, the same cut-off point, $P_y(H_1) < \gamma$, would hold but $P_y(H_1)$ would be replaced by $\underline{z}$.

3. DEFERRED DECISION PROBLEM

We now examine the following situation: things are as in the two-alternative case, but we are told after arriving at the state of knowledge $P_y(H_1)$ that we do not have to make a decision. We can hold off for another observation or as many as we like until a total of $\underline{n}$ have been taken. The penalty for taking observations is the cost of an observation, which we take as 1 (the unit of loss measure). Whether we should delay decision will depend on whether the expected loss for taking another observation is greater than for making the decision now. But, of course, the expected value of taking another observation will depend on the expected value of taking another one beyond that, etc., until we have exhausted our right to take observations. In fact, the expected value of taking one more will depend on how many we may be allowed, the actual value being the result of a complicated nesting process.

3.1 General Theory

To start the process, suppose no observations may be taken. That is, a decision must be made at once. If $T(z)$ is the expected loss for a terminal decision, then the expected loss $E_o(z_o)$ is,

$$E_o(z_o) = T(z_o) = z_o\,\omega_{12} \quad , \qquad 0 \leq z_o < \gamma_o$$
$$= (1-z_o)\,\omega_{21}, \qquad \gamma_o \leq z_o < 1 \qquad (7)$$

where:

$$\gamma_o = \frac{\omega_{21}}{\omega_{21} + \omega_{12}} \;, \text{ and } \omega_{12}, \omega_{21} > 0 \quad .$$

The above follows from the fact that the probability of H_1 being true is z_o and the cost of taking action A_2 is ω_{12} so that the probability of a loss when action A_2 is taken is z_o and the expected loss is $z_o\,\omega_{12}$. Similarly, the probability of H_2 being true is $(1-z_o)$, the cost of taking action A_1 is in this case ω_{21}, and the expected loss is $(1-z_o)\,\omega_{21}$. A plot of these two expressions as a function of z_o is shown in Fig. 3. Actions should be taken as indicated in Fig. 3.

Suppose now the option is offered of one further observation.

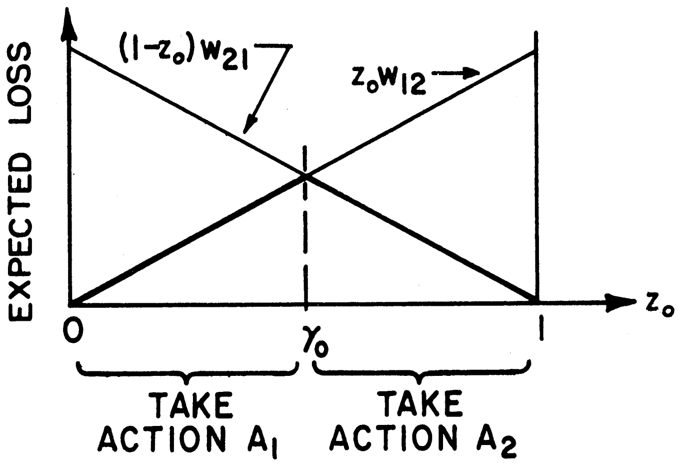

Fig. 3. Expected losses making up $E_0(z)$, the terminal decision curve.

The expected loss curve for each value of z_1 in this case must be calculated. For fixed z_1, the probability that the value actually observed is y is given by

$$p(y) = z_1 \, p_1(y) + (1-z_1) \, p_2(y) \, . \tag{8}$$

If y is observed, the probability that H_1 is true can be calculated from

$$P_y(H_1) = z_1 \, \frac{p_1(y)}{p(y)} \, . \tag{9}$$

The observer will then be in a state such that no more observations are permitted and the probability of H_1 is $P_y(H_1)$. The expected loss for any given observation y is $E_0[P_y(H_1)]$. For all possible y, the expected loss is $\sum_{\text{all } y} p(y) \, E_0[P_y(H_1)]$. Thus the expected cost of deferring a decision is,

$$G_1(z) = \sum_{\text{all } y} p(y) \, E_0[P_y(H_1)] + 1 \, . \tag{10}$$

Conjoining the expected values of not taking an observation at all, [T(z)], and taking the one observation, $[G_1(z)]$, the curves appear as in Fig. 4.

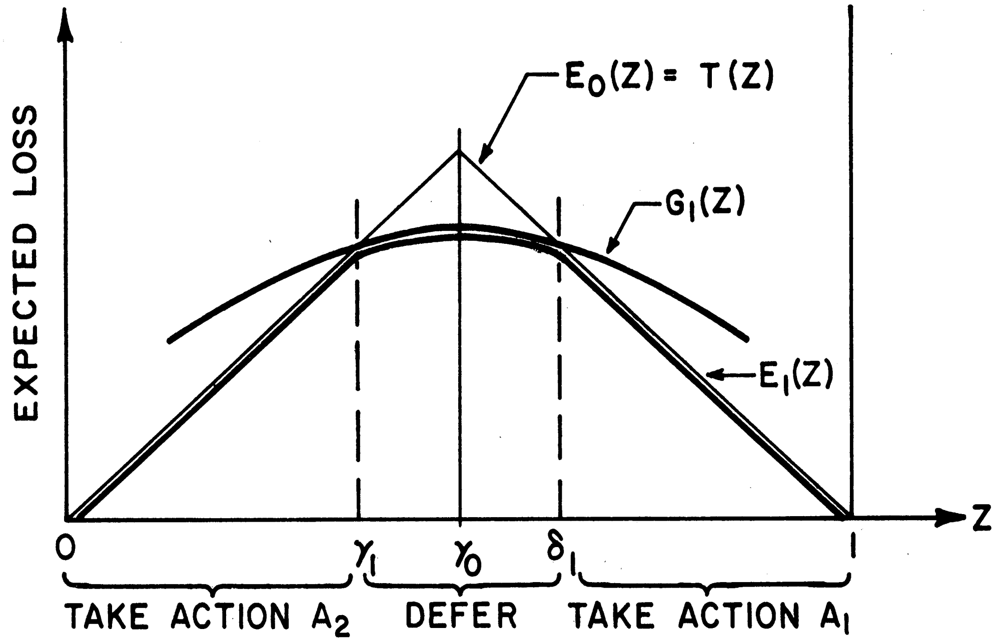

Fig. 4. $E_1(z)$, $G_1(z)$, $E_o(z)$ as functions of $\underline{z}$.

Alternatives should be chosen as shown in Fig. 4. The optimized expected loss curve is

$$E_1(z) \quad = \quad \min\ [T(z),\ G_1(z)] \ . \qquad (11)$$

Suppose now two observations may be taken and the observer must decide whether to stop and decide now, or take the first of the two observations. Again, the expected value of each move must be calculated. In case a decision is made at once, T(z) holds. In the case of making another observation, suppose $\underline{y}$ is observed. The probability of this occurring for a given $\underline{z}_2$ is as before,

$$p(y) \quad = \quad z_2 p_1(y) + (1-z_2)p_2(y) \ . \qquad (12)$$

The probability of H_1 being true is given by

$$P_y(H_1) = \frac{z_2 p_1(y)}{p(y)} . \qquad (13)$$

At this point, the observer is in the position of having the probability of H_1 being true equal to $P_y(H_1)$ and an option of taking one more observation for which the expected loss is $E_1(z)$. For this observation y, the expected loss is $E_1[P_y(H_1)]$. Averaged over all y the expected loss is, $\sum_{\text{all } y} p(y)\, E_1[P_y(H_1)]$. Adding the cost of the observation, the expected loss for taking an observation if two more are permitted is

$$G_2(z) = \sum_{\text{all } y} p(y)\, E_1[P_y(H_1)] + 1 . \qquad (14)$$

To compare the expected losses for taking and not taking an observation, Fig. 5 shows the plot of $G_2(z)$, $E_2(z)$, $E_1(z)$, and $T(z)$.

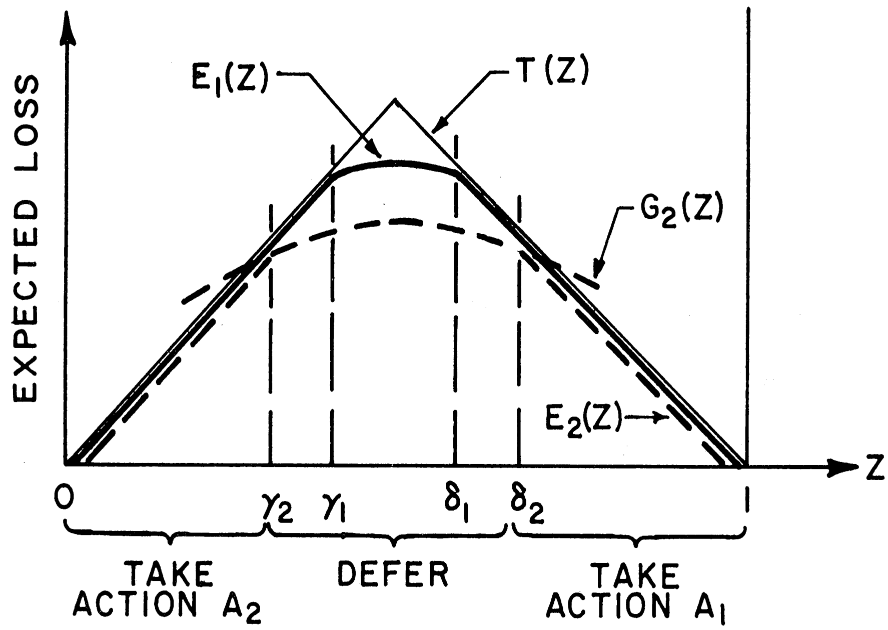

Fig. 5. Plot of $G_2(z)$, $E_2(z)$, $E_1(z)$, and $T(z)$.

The optimized expected loss curve for the a priori probability z and the option of taking as many as two more observations is given by

$$E_2(z) = \min\,[T(z),\, G_2(z)]\ . \qquad (15)$$

The process generalizes to the set of steps:

1) $p(y) = z_n\, p_1(y) + (1-z_n)\, p_2(y)$; probability that the value y will be observed with the option of taking up to n more observations.

2) $P_y(H_1) = z_n \dfrac{p_1(y)}{p(y)}$; probability of H_1 being true, given the observation y and the a priori probability z_n.

3) $G_n(z) = \sum_{\text{all } y} p(y)\, E_{n-1}[P_y(H_1)] + 1$; expected loss for taking an observation with n possible observations remaining.

4) $E_n(z) = \min\,[T(z),\, G_n(z)]$; optimized expected loss with n possible observation remaining and where T(z) is expected loss of making a terminal decision.

In the case of no observations permitted, the set of decision points are the same, $\gamma_o = \delta_o$, and in general, the decision points are a pair γ_n, δ_n, the intersection of T(z) and $G_n(z)$.

3.2 An Example: The Normal Case

Consider the case where the logarithm of the likelihood ratio is normally distributed under either hypothesis. (It follows that it will be similarly distributed under the other hypothesis.)

Without loss of generality we can consider the observation to be real valued with normal density functions with unit variance, and means zero and d'. Let the cost of a Type I error be ω_{12}, and that of a Type II error be ω_{21}, and let the cost of deferring for one observation be 1.00. If one is allowed to defer and take at most n more observations, should one terminate and take action A_1, or take action A_2, or defer and take an observation? This is the decision problem at "Stage n."

<u>Solution, n = 0</u>

Specifically

$$p_1(y) = \frac{1}{\sqrt{2\pi}} e^{-1/2\, y^2}$$

and

$$p_2(y) = \frac{1}{\sqrt{2\pi}} e^{-1/2(y-d')^2}$$

T(z) is the expected loss for a terminal decision. Let z_o be the given *a priori* probability of H_1. We have no observations left to take, and we must make a decision. This is the simple two-alternative probability case and as previously discussed in this paper we say H_1 when $0 \leq z_o \leq \gamma_o$ and we say H_2 when $\gamma_o \leq z_o \leq 1$ where γ_o is given by $\gamma_o = \dfrac{\omega_{21}}{\omega_{12} + \omega_{21}}$. Thus our expected loss for a terminal decision is,

$$\begin{aligned} T(z_o) &= z_o\, \omega_{12} , && 0 \leq z_o \leq \gamma_o \\ &= (1-z_o)\, \omega_{21} , && \gamma_o \leq z_o \leq 1 \end{aligned} \qquad (16)$$

<u>n = 1</u>: We have the possibility of taking one more observation. We wish to know if we should accept H_1 or H_2, or defer and take our one observation at which time we then will apply the results of the "0" state. We are given z_1, the *a priori* probability of H_1. From Bayes' Theorem we are able to calculate the *a posteriori* probability z_o on the condition we have an observation y,

$$z_o = P_y(H_1) = z_1 \frac{p_1(y)}{p(y)}$$

But,

$$p(y) = z_1 \, p_1(y) + (1-z_1) \, p_2(y)$$

Therefore,

$$z_o = P_y(H_1) = \frac{z_1 \, e^{-y^2/2}}{z_1 \, e^{-y^2/2} + (1-z_1) \, e^{-\left(\frac{y-d'}{2}\right)^2}}$$

$$= \frac{z_1}{z_1 + (1-z_1) \, e^{yd'} \, e^{\frac{-d'^2}{2}}} \tag{17}$$

$$G_1(z) = \int_{-\infty}^{f(\gamma)} p(y)(1 - z_o)\omega_{21} \, dy + \int_{f(\gamma)}^{+\infty} p(y) \cdot (z_o)\omega_{12} \, dy + 1 \tag{18}$$

Substituting the expressions for $p(y)$ and z_o from above we have,

$$G_1(z) = \omega_{21} \int_{-\infty}^{f(\gamma)} (1-z_1) \, p_2(y)dy + \omega_{12} \int_{f(\gamma)}^{+\infty} z_1 p_1(y) \, dy + 1 \tag{19}$$

where:

$$f(\gamma) = \text{value of } y \text{ for which } P_y(H_1) = \gamma$$

$$G_1(z) = \frac{\omega_{21}(1-z_1)}{\sqrt{2\pi}} \int_{-\infty}^{f(\gamma)} e^{-1/2(y-d')^2} \, dy$$

$$+ \left(\frac{\omega_{12} z_1}{\sqrt{2\pi}}\right) \int_{f(\gamma)}^{+\infty} e^{-1/2 \, y^2} \, dy + 1 \tag{20}$$

$$= \frac{\omega_{21}(1-z_1)}{\sqrt{2\pi}} \int_{-\infty}^{f(\gamma)-d'} e^{-t^2/2} \, dt + \frac{z_1 \omega_{12}}{\sqrt{2\pi}} \int_{f(\gamma)}^{+\infty} e^{-t^2/2} dt + 1$$

We now wish to find an expression for $f(\gamma)$.

$$P_y(H_1) = \frac{z_1}{z_1 + (1-z_1)\, e^{-d^2/2}\, e^{+yd}}$$

For $P_y(H_1) = \gamma$ what is the corresponding value for y?

$$y = f(\gamma) \Longleftrightarrow (1-z_1)\, e^{-d'^2/2}\, e^{yd'} = \frac{z_1(1-\gamma)}{\gamma}$$

Explicitly

$$\frac{1}{d'} \left\{ \frac{d'^2}{2} + \ell n \frac{z_1(1-\gamma)}{\gamma(1-z_1)} \right\} = f(\gamma)$$

Therefore,

$$G_1(z) = \frac{(1-z)\,\omega_{21}}{\sqrt{2\pi}} \int_{-\infty}^{L_1} e^{-t^2/2}\, dt + \frac{z\,\omega_{12}}{\sqrt{2\pi}} \int_{L_2}^{+\infty} e^{-t^2/2}\, dt + 1$$

$$L_1 = \frac{1}{d'}\left\{ \ell n \frac{z(1-\gamma)}{\gamma(1-z)} - d'/2 \right\}$$

$$L_2 = \frac{1}{d'}\left\{ \ell n \frac{z(1-\gamma)}{\gamma(1-z)} + d'/2 \right\}$$

To find the decision points, set $T(z) = G_1(z)$ and solve for z. Specifically, for the lower decision point, z_γ, solve,

$$z_\gamma \omega_{12} = G_1(z_\gamma) \qquad \text{for } z_\gamma \,. \tag{21}$$

And for the upper decision point, z_δ, solve,

$$\omega_{21}(1 - z_\delta) = G_1(z_\delta) \qquad \text{for } z_\delta \,. \tag{22}$$

Equations 21 and 22 for the upper and lower decision points cannot be solved in generality. To illustrate further let $\omega_{12} = \omega_{21} = w$. $G_1(z)$ then becomes,

$$G(z) = (1-z)\, w\Phi\left\{\frac{1}{d'}\, \ell n\left(\frac{z}{1-z}\right) - \frac{d'}{2}\right\} + zw\,\Phi\left\{-\frac{1}{d'}\, \ell n\frac{z}{1-z} - \frac{d'}{2}\right\} + 1. \quad (23)$$

where:

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}\, dt}{\sqrt{2\pi}}$$

To obtain the upper and lower decision points we must solve,

$$z_\gamma w = (1-z_\gamma)\, w\Phi\left\{\frac{1}{d'}\, \ell n\left(\frac{z}{1-z}\right) - \frac{d'}{2}\right\} + z_\gamma w\, \Phi\left\{-\frac{1}{d'}\, \ell n\, \frac{z_\gamma}{1-z_\gamma} - \frac{d'}{2}\right\} + 1 \quad (24)$$

and,

$$w(1-z_\delta) = (1-z_\delta)\, w\Phi\left\{\frac{1}{d'}\, \ell n\, \frac{z_\delta}{1-z_\delta} - \frac{d'}{2}\right\} + z_\delta w\, \Phi\left\{-\frac{1}{d'}\, \ell n\, \frac{z_\delta}{1-z_\delta} - \frac{d'}{2}\right\} + 1 \quad (25)$$

of course, by symmetry, $z_\delta = 1-z_\gamma$. This may be used, or saved for a check. Equations 24 and 25 were solved graphically for $w = 15$ and $d' = 1$. The lower decision point, $z_\gamma = .342$, and the upper decision point, $z_\delta = .657$. Thus, our decision criteria is as follows:

For $0 \leq z \leq .342$, choose H_2; expected loss is $15z$.

For $.342 \leq z \leq .657$, defer; expected loss is $G_1(z) + 1$

$$= 15(1-z)\ \Phi\left\{\ell n\, \frac{z}{1-z} - \frac{1}{2}\right\} + 15z\ \Phi\left\{-\ell n\, \frac{z}{1-z} - \frac{1}{2}\right\} + 1$$

For $.657 \leq z \leq 1$, choose H_1; expected loss is $15(1-z)$.

$n = 2$: left for reader.

4. CONVERGENCE

To prove that the process converges, we shall show that all $E_n(z)$ are bounded below by 0, and for each z is a monotonic nonincreasing function of n. Thus there is a limiting function $E(z)$, and corresponding limiting decision points γ and δ. Further, the limit is nondegenerate, i.e., $\gamma > 0$ and $\delta < 1$ and there exists a z such that $E(z) \neq 0$. Both the lower bound and monotoniety follow inductively.

Note that T(z) is nonnegative, since $\omega_{ij} \geq 0$ and $T(z) = \min [z\,\omega_{12}, (1-z)\,\omega_{12}]$, both nonnegative quantities. Since $E_o(z)$ is equal to T(z), $E_o(z)$ is nonnegative. Now if some $E_k(z)$ is nonnegative, any average of $E_k(z)$ values is nonnegative. Thus $G_{k+1}(z)$, which is an average of E_k values, plus the cost of observation (which is 1) is bounded below by 1. Finally, $E_{k+1}(z)$ is simply the lesser of $G_{k+1}(z)$ and T(z), and so it is nonnegative and the induction is complete. This establishes the lower bound for $E_n(z)$.

It also follows that since $G_n(z) \geq 1$ for all n and z, that for very small z, and for very large z, a terminal decision is always appropriate. Specifically,

$$\left.\begin{array}{l} 0 \leq z \leq \dfrac{1}{\omega_{12}} \\ \text{or} \\ 1 - \dfrac{1}{\omega_{21}} \leq z \leq 1 \end{array}\right\} \Rightarrow T(z) < 1 \Longrightarrow E_n(z) = T(z)$$

This establishes the nondegenerate bounds on the decision points

$$\gamma_n \geq \frac{1}{\omega_{12}} \qquad \delta_n \leq 1 - \frac{1}{\omega_{21}}$$

and confirms the contention that at least for each n, $E_n(z)$ is not identically zero, since $E_n\left(\frac{1}{\omega_{12}}\right) = 1$ for all n.

The monotone decreasing nature of $E_n(z)$ is also established by induction. By definition, since $E_o(z) = T(z)$

$$E_1(z) = \min\left[E_o(z), G_1(z)\right] \leq E_o(z) .$$

Suppose

$$E_{n-1}(z) \leq E_{n-2}(z), \text{ i.e., } E_{n-1}(z) - E_{n-2}(z) \leq 0$$

$$G_n(z) - G_{n-1}(z) = \int p(y)\, E_{n-1}\left[\frac{z}{z+(1-z)\ell(y)}\right] dy - \int p(y)\, E_{n-2}\left[\frac{z}{z+(1-z)\ell(y)}\right] dy$$

$$= \int p(y) \left\{ E_{n-1}\left[\frac{z}{z+(1-z)\ell(y)}\right] - E_{n-2}\left[\frac{z}{z+(1-2)\ell(y)}\right]\right\} dy \leq 0$$

since $p(y)$ is nonnegative and $E_{n-1} - E_{n-2}$ is nonpositive at every y. Since $G_n(z) \leq G_{n-1}(z)$, $E_n(z) \leq E_{n-1}(z)$ and the induction is complete.

From the viewpoint of the subsequent computer work, we should pull out one comment from the above proof. If at some pair of successive stages, the maximum discrepancy of $E_n(z)$ and $E_{n-1}(z)$ is ϵ, then $G_{n+1}(z)$ will be within ϵ of $G_n(z)$, and this will enforce a corresponding bound in the maximum discrepancy between $E_{n+1}(z)$ and $E_n(z)$. Thus the amount of reduction of $E_n(z)$ "gained" each iteration is monotone decreasing.

5. COMPUTER ANALYSIS

The expressions of Eqs. 13, 14, 15, define the process by which we may iteratively determine the E_n under either a truncated or nontruncated process. Since hand calculation is prohibitive, we resorted to the use of the University of Michigan's IBM 704. The solution for normal distributions of mean zero and d''s of 2, 1, and .5 is given here; but changes required for other parameters or other distributions or changing costs, will be obvious. The binary cards in machine language for this program can be made available. The flow diagram of Fig. 16 was used to set up the computer program. A more complete flow diagram is included in Appendix I. The parameter d', roughly speaking, is a measure of the discriminability content of an observation.

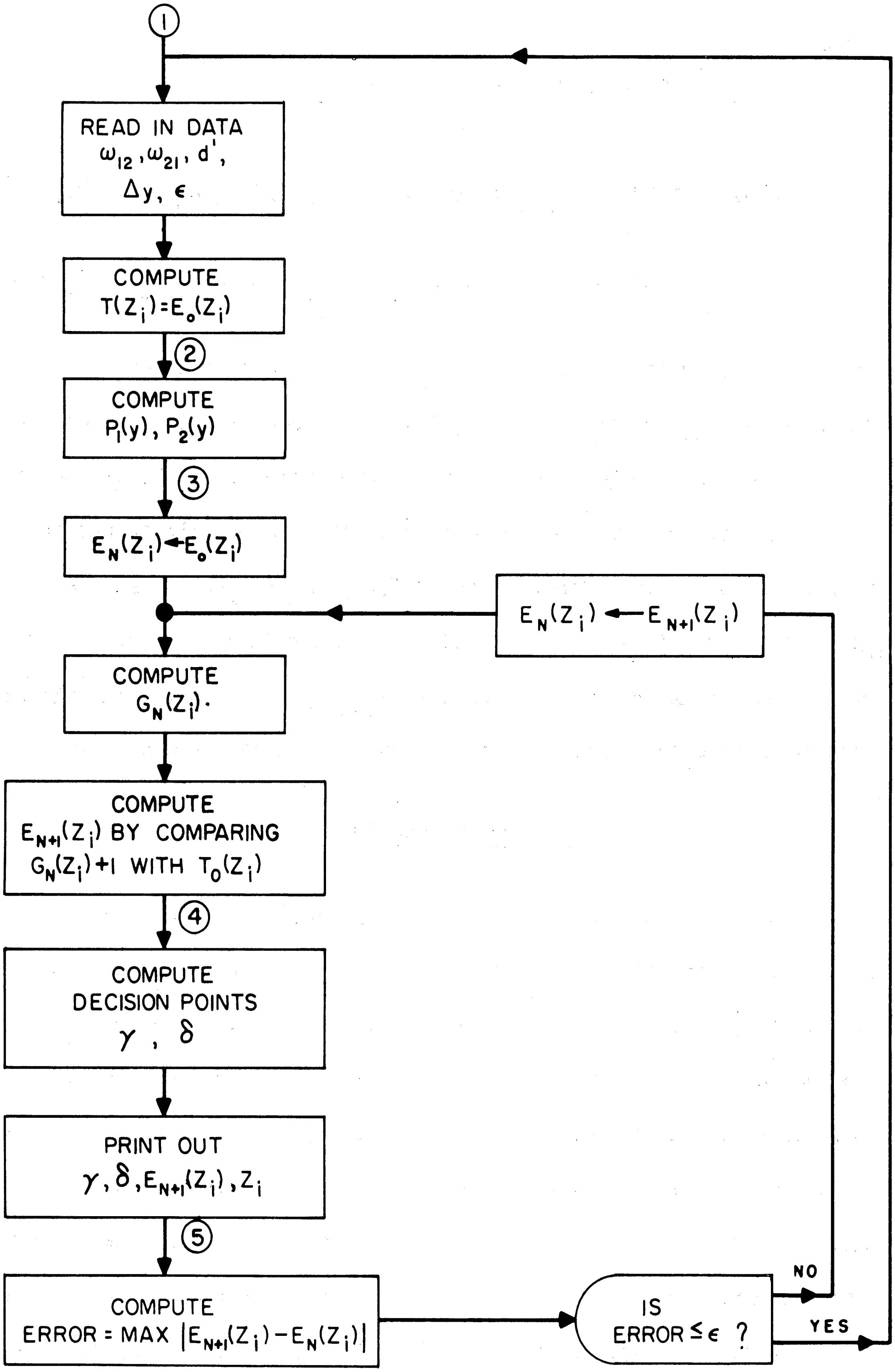

Fig. 6. Computer block diagram.

6. CONCLUSIONS AND APPLICATIONS

Professor Goode did not write any conclusions before his unexpected death and so the only conclusions the writer will draw from the data is that which Professor Goode commented on after he had glanced at his computer data. The reader is as able as the writer to draw other conclusions from the data.

Professor Goode was interested in determining how quickly the intersection points γ, δ of the "terminal loss curve" and the "further observation loss curve" converge to definite limits (see Figs. 7, 8, 9). He was interested in this because he wanted to be able to take the costs of errors, the cost of deferring a decision, and the a priori probability of H_1 and from these parameters find the "optimum points" α, β and minimum expected number of deferrals in terms of the costs and a priori prob. of H_1. The "optimum points" α and β are defined as the coordinates on the power curve, or the ROC.* In other words, Professor Goode wanted to take the costs involved in deferring a decision, compare this against the costs in making a terminal decision and from this find the point of operation on the ROC. This is the inverse of the problem solved by Wald in his formulation of sequential analysis. Wald first sets his point on the ROC, i.e., picks α and β, and then finds a sequential test that minimizes the expected number of deferrals to meet his operating point on the ROC.

The table of Fig. 10 shows how quickly the γ and δ converge to definite limits. This then is the conclusion drawn by Professor Goode from his computer data. The γ and δ points converge very rapidly

*The power curve (Ref. 7) is a plot of Type I error vs. Type II error, for all values of y. It has been shown that for "optimum" performance in making decisions one should operate on the upper boundary of the ROC.

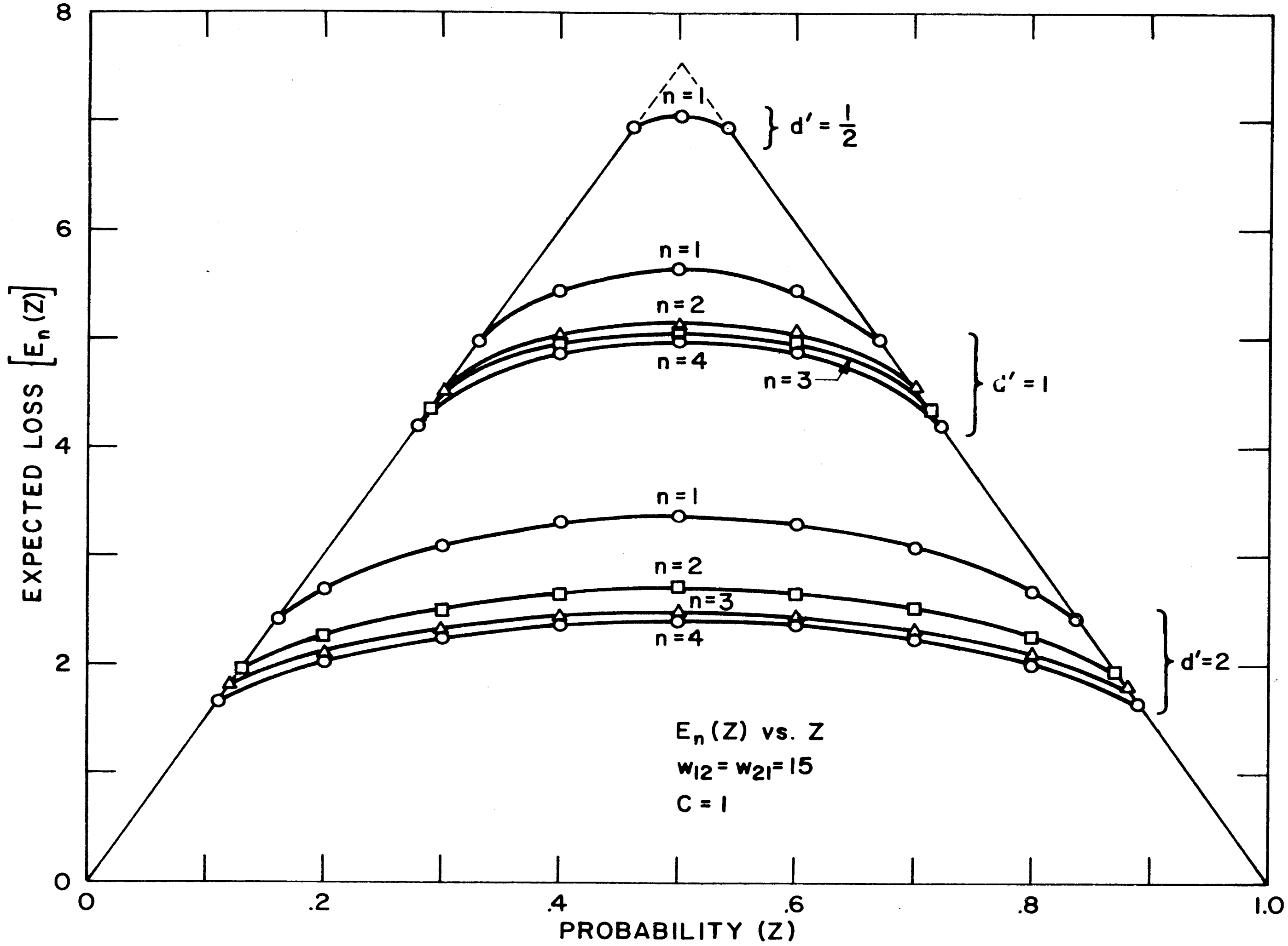

Fig. 7. Expected loss curves for the normal case.

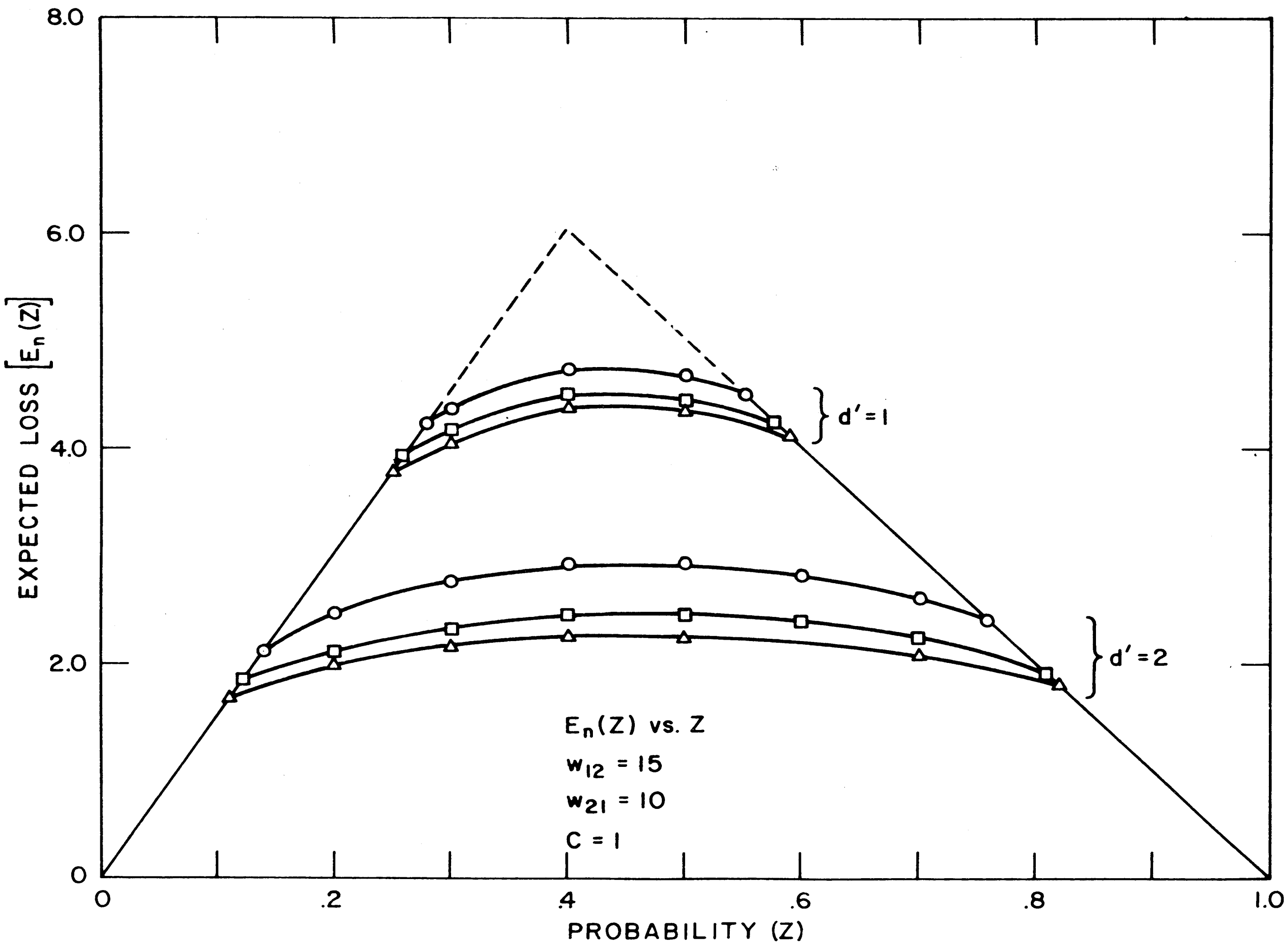

Fig. 8. Expected loss curves for the normal case.

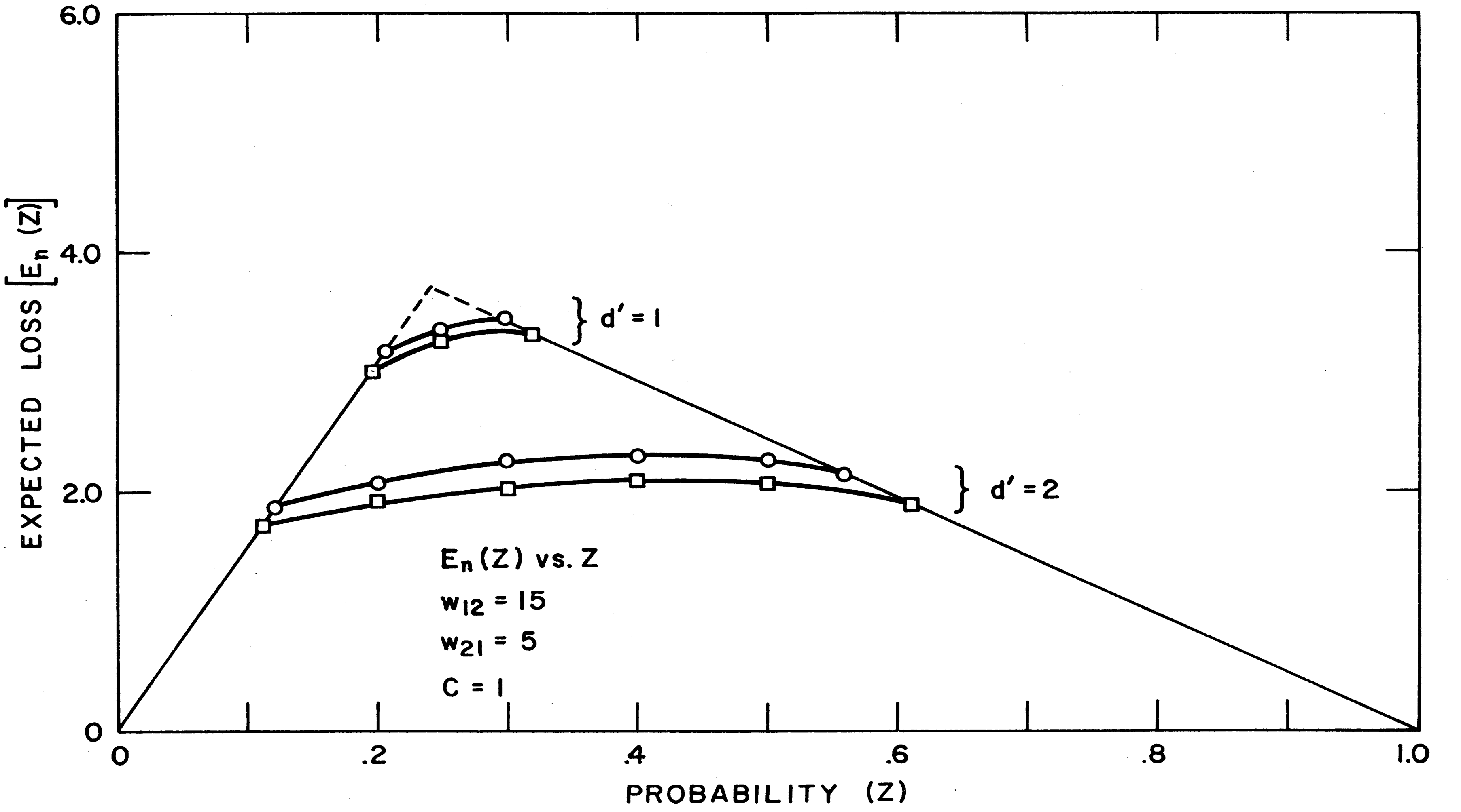

Fig. 9. Expected loss curves for the normal case.

d'	ω_{12}	ω_{21}	n = 1		n = 2		n = 3		n = 4	
			γ	δ	γ	δ	γ	δ	γ	δ
2	15	15	.16	.84	.13	.87	.12	.88	.11	.89
2	15	10	.14	.76	.12	.81	.11	.82		
2	15	5	.12	.56	.11	.61	.11	.62		
2	15	1	(.062)							
1	15	15	.33	.67	.30	.70	.29	.71	.28	.72
1	15	10	.28	.55	.26	.58	.25	.59		
1	15	5	.21	.32	.20	.32	.20	.33		
.5	15	15	.46	.54	.45	.55				
.5	15	10	.38	.42						
.5	15	5	(.25)							
.5	15	1	(0.62)							

γ = lower decision point δ = upper decision point

( ) = value of Gamma, no observation required.

Fig. 10. Table of decision points for the normal case.

to definite limits for the costs and d''s examined.

The rest of the paper that follows is Professor Goode's work.

Examination of the relationship between the "terminal loss curve" and the "further observation loss curve" yields an interesting insight into the motive of penalties and gains for acting in one or another nonoptimum fashion. Figure 11 is a plot of the difference between a typical pair of such loss curves. Between γ and δ the curve represents the added loss incurred when z is between these limits and a decision is made without taking advantage of the possibility of making more observations.

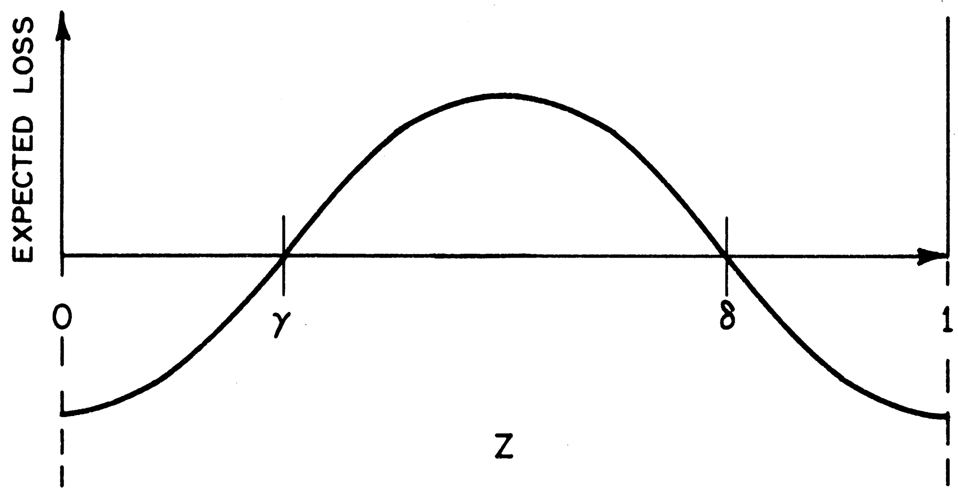

Fig. 11. Difference between "terminal loss curve" and "further observation loss curve."

On the other hand, for $z < \gamma$ and $z > \delta$, the dip in the curve represents the added loss for delaying decision by taking more observations. While the "further observation loss curve" is represented here generally, it must of course correspond to some specific allowable number of further observations. However, for the numerical values of the costs chosen in the computed examples (where the cost of a loss is large relative to the cost of an observation), the greatest change occurs between the "terminal loss curve" and the "one-further-observation-allowed loss curve." After that, relatively smaller effects are obtained from two or more observations allowed. In the case of costs approximately equal to the cost of observation, the penalty for taking observation holds over the entire range of z.

In summary, when a priori knowledge says that the probability of one or the other hypothesis being correct is high, cost for delaying is incurred. When a priori knowledge is uncertain, it pays to gather information if the cost of observations are not great relative to the possible losses incurred with decision.

Applications

The developments in decision-making techniques have implications for many practical areas. One such area, for example, is in the search radar. At the outset in using a search radar, a display was provided to the human operator and with little instruction concerning the decision to be made the operator was told to report all "targets" detected. But the elements of a decision mechanism are clearly recognizable in the process. "Target" is one hypothesis and the alternative is "Noise." The human was not instructed concerning the choice of a cutoff point and widely varying procedures were used (and still are) from waiting for many scans before decision to a relatively few, from reporting varying degrees of intensity blips to only reporting very intense blips. To some extent in warfare the *a priori* probability was occasionally introduced by putting the operator on the "qui vive" when enemy raids were expected.

With the need for automatic detection, some attention had to be given to the process of deciding between target and noise. In one radar, the pulse repetition frequency was such that about ten hits could be expected on a target in a single scan. In the mechanization of detection in the radar the cutoff point was set at four pulse returns above an arbitrary threshold level. This choice was made on an intuitive basis.

As the art developed, some statistical technique crept in. The fact that the set might be saturated with targets, many false, led to the consideration of methods for implementing a change in the threshold above which a return was called a target. This threshold was so manipulated that the "false alarm rate," i.e., the Type I error, α, was kept constant. Thus records were kept of the number of targets turning out false (discovered by the fact that the next return did not materialize) and the threshold reset so as to keep the fraction constant.

While this implementation was not introduced to improve decision making, it began to use elements associated with the anatomy of decision. More recently the α and β errors have been introduced into the radar and a sequential test employed (Ref. 8), storage being provided for information on the successive returns from a given set of pulses. To date, to this author's knowledge, no attempt has been made to introduce the *a priori* probability of a target occurring into the setting of threshold and no place has been provided for the choice of α and β based on costs of errors. The implication of deferred decision is that information should be stored scan-to-scan and that the *a posteriori* probability of a target should be computed. When the value of the *a posteriori* probability is less than γ, the information should be discarded. When the *a posteriori* probability is more than δ, a target should be recorded and the information discarded (unless a decision does not need to be made, in which case the *a posteriori* probability should be recorded). Of course, the γ and δ to be used depends on the values of costs and *a priori* probabilities. These would need to be precomputed and stored in the radar for use in setting the thresholds.

APPENDIX

Figures 12 through 16 give a complete flow diagram for the computer program used in obtaining the data presented in this paper. The general block diagram of Fig. 6 has been broken into five detailed blocks, as indicated in Fig. 6.

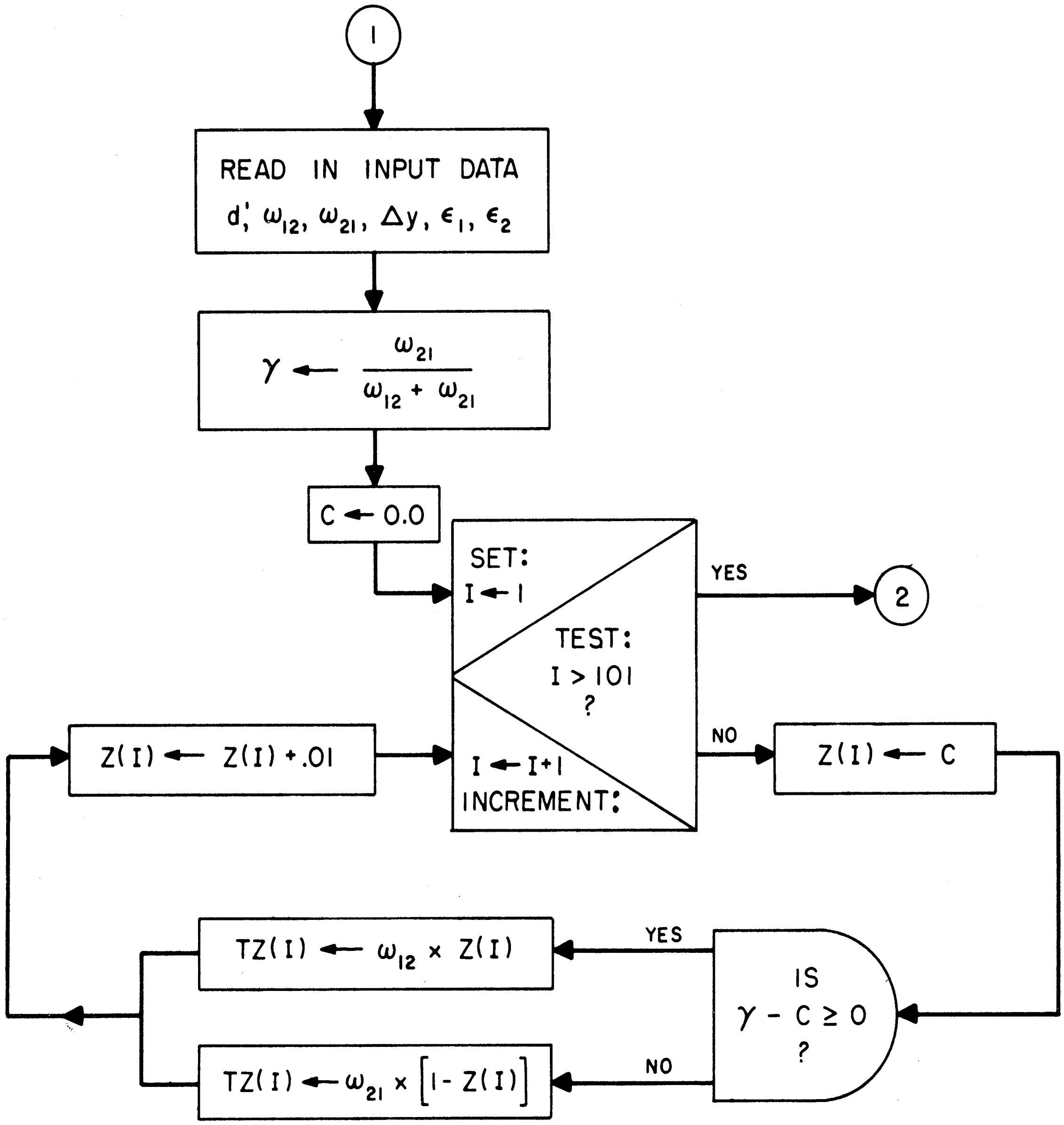

Fig. 12. Computation of T(z).

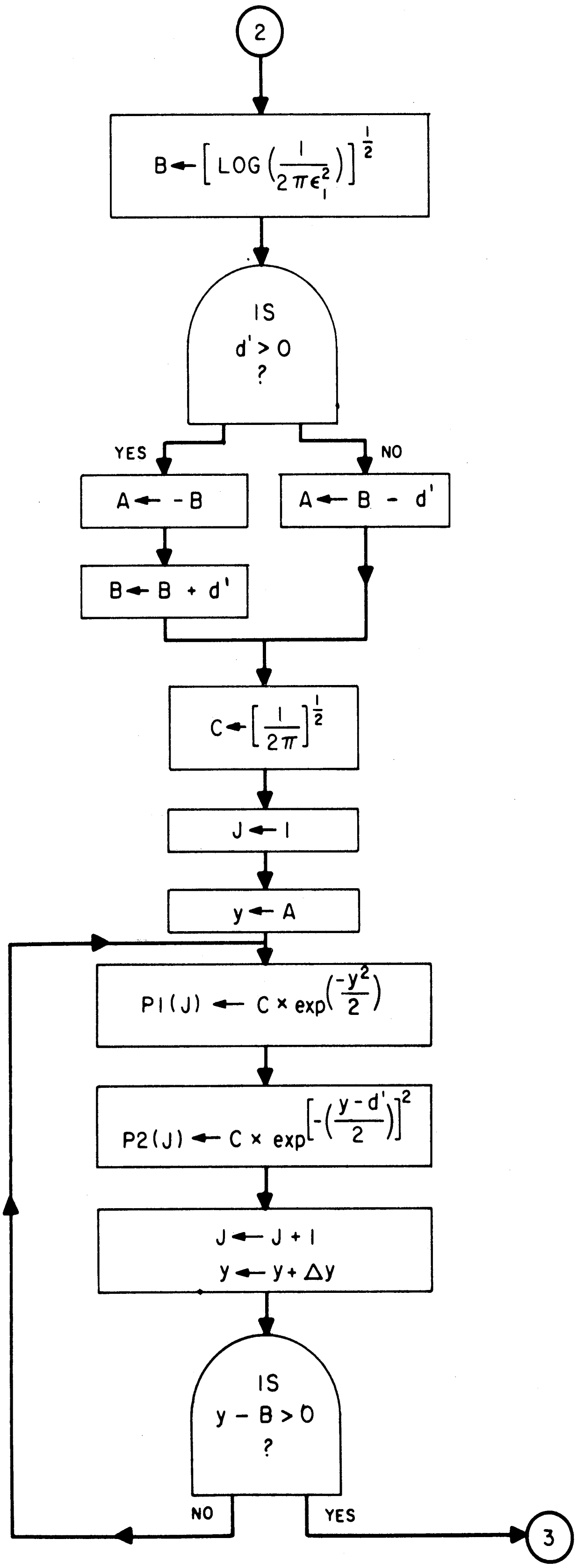

Fig. 13. Computation of $p_1(y)$ and $p_2(y)$.

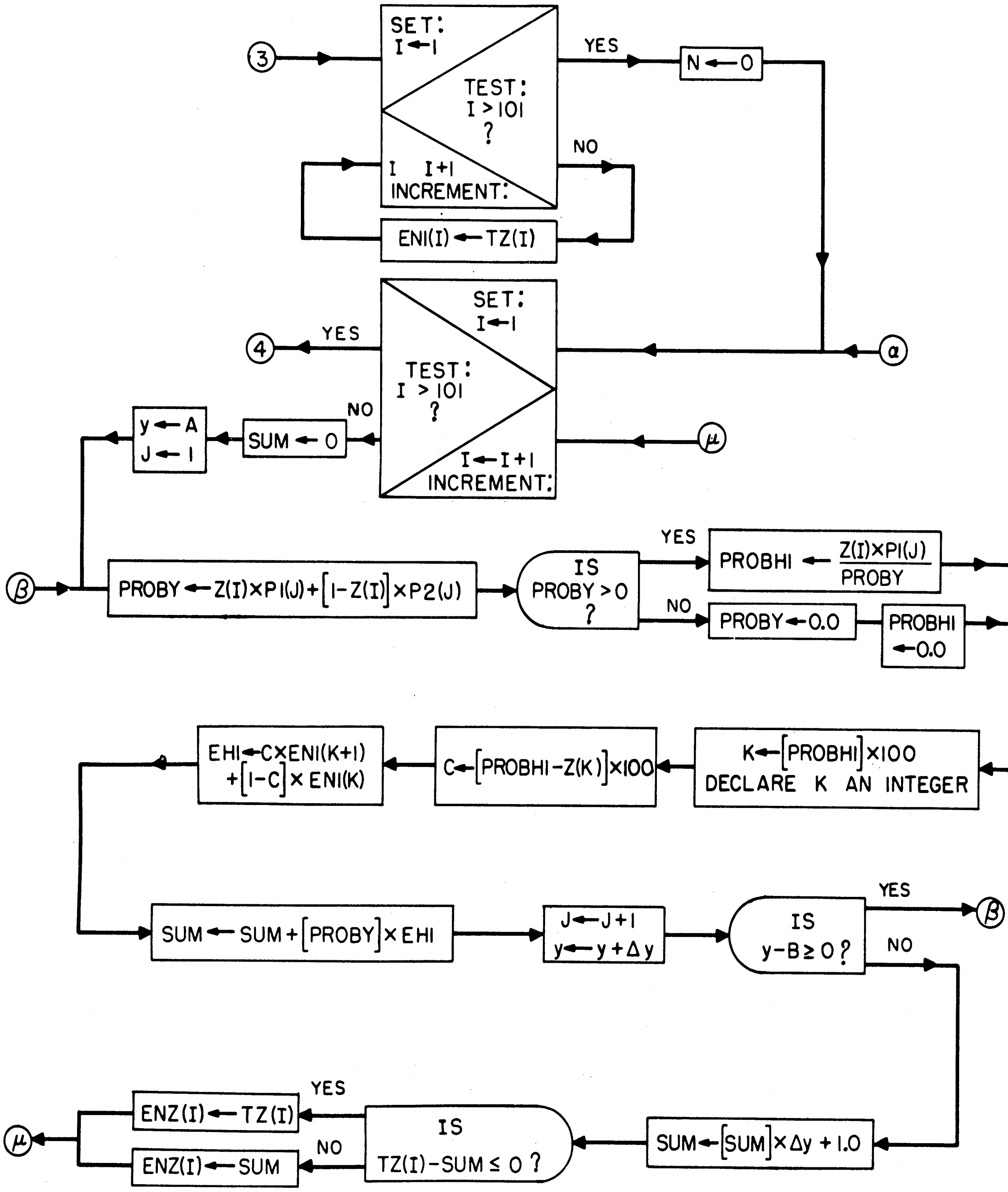

Fig. 14. Computation of $G_n(Z_i)$ and $E_n(Z_i)$.

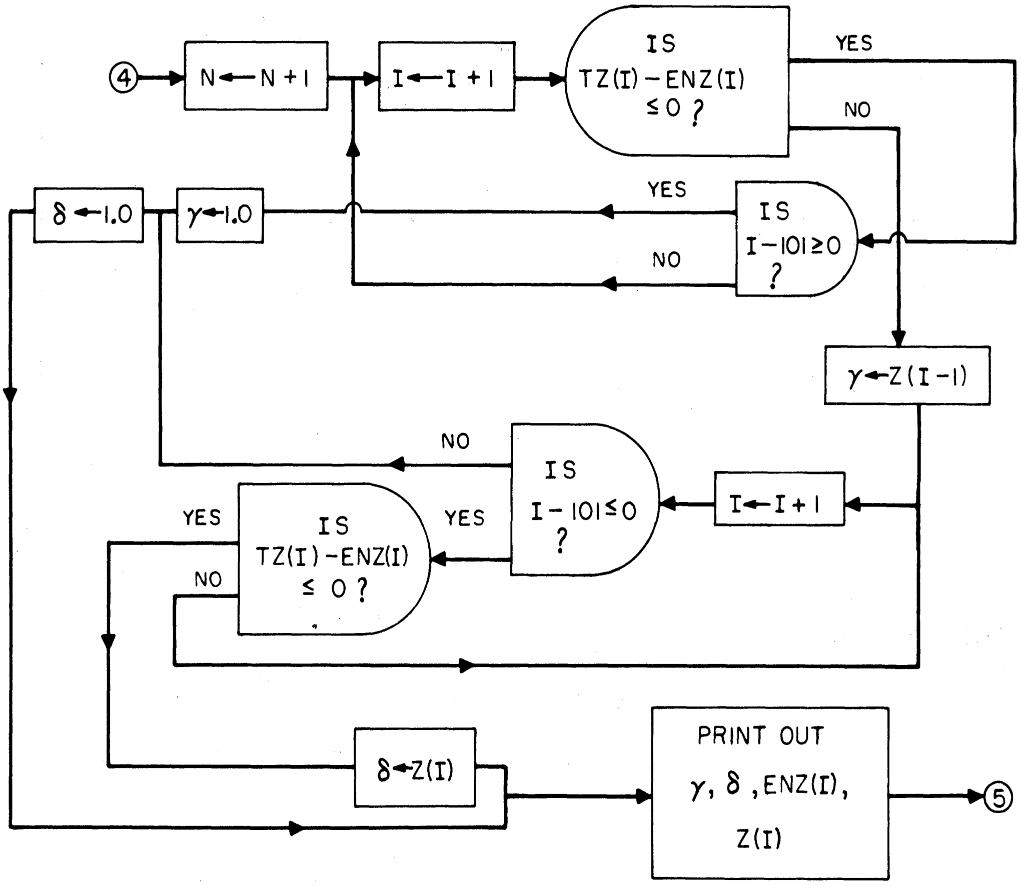

Fig. 15. Computation of decision points, γ and δ.

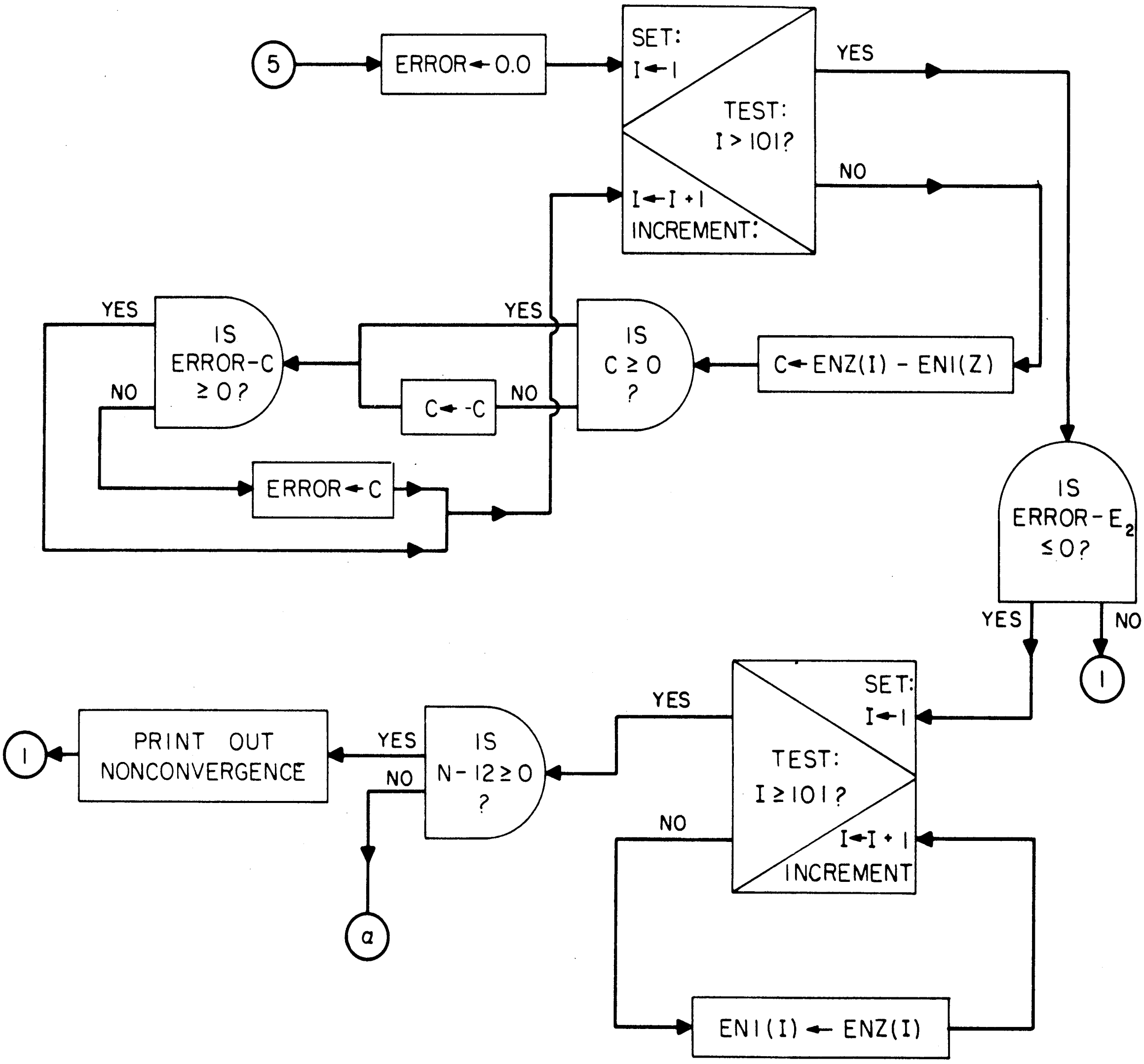

Fig. 16. Computation for convergence.

REFERENCES

1. P. E. LaPlace, Theorie Analytique des Probabilities (3rd ed.; Paris, 1820).

2. T. Bayes, "An Essay Towards Solving a Problem in the Doctrine of Chance," Phil. Trans. Royal Soc. of London, Vol. 53, pp. 370-418, 1763.

3. R. A. Fisher, Statistical Methods for Research Workers (8th ed.; Edinburgh: Oliver and Boyd Ltd., 1941).

4. ______, Design of Experiments (2nd ed.; Edinburgh: Oliver and Boyd Ltd., 1937).

5. J. Neyman and E. S. Pearson, "On the Use and Interpretation of Certain Test Criteria for Purposes of Statistical Inference," Biometrica, 1928.

6. ______, "On the Problem of the Most Efficient Tests of Statistical Hypotheses," Phil. Trans. Royal Soc. of London, 1933.

7. A. Wald, Sequential Analysis (New York: John Wiley and Sons, 1947).

8. J. Von Neumann, "Zur Theorie der Gesellschaftsspiele," Mathematics Annalem, Vol. 100, 1928.

9. W. W. Peterson, T. G. Birdsall, "The Theory of Signal Detectability," Cooley Electronics Laboratory Technical Report No. 13, The University of Michigan, June 1953.

10. G. W. Preston, "The Search Efficiency of the Probability Ratio Sequential Search Radar," General Atronics Corp., IRE International Convention Record, March 1960.

11. D. Blackwell, M. A. Girshick, Theory of Games and Statistical Decisions (New York: John Wiley and Sons, 1954).

    The mathematical formulation of the deferred decision problem is given in Section 10.2 of Blackwell and Girshick. The following is quoted from Section 10.2,

    > "It is to be observed that, while the averaging process is laborious from a computational point of view, the fact that the determination of the stopping regions and the Bayes risk involves nothing more complicated than taking expectations is of theoretical interest. Also this method can be considered as an iterative procedure for obtaining the Bayes risk and the stopping regions of the non-truncated sequential procedure."

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