# NEAR CAUSTIC SURFACE FIELD

by

R. F. Goodrich

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## TABLE OF CONTENTS

## ABSTRACT

At high frequency the acoustic surface fields for plane wave illumination of a prolate spheroid are found for near symmetric illumination. The surface caustic, in contrast to the symmetric case, is found to be diffuse. The values of the surface fields are used to find the modification of the creeping wave contribution to back scattering.

# I

# INTRODUCTION

We consider the use of the creeping wave formalism (Keller, 1953; Fock, 1946) for the prediction of surface fields induced by a plane wave on smooth convex scatterers. The application is straightforward away from regions of the surface in which the creeping waves converge due to the geometry of the body. Here the local geometry is taken to be cylindrical. If the creeping waves converge to a point, we speak of this point as a caustic and the creeping wave formalism can be modified in order to predict the fields in the region of a caustic. However, if the creeping waves do not quite converge to a point and yet cannot be treated in the cylinder approximation of the bodies in question by cylinders, a more subtle modification of the formalism is required. We shall consider this problem and designate the near-caustic region as a diffuse caustic.

We attack the problem of describing the field in the region of the diffuse caustic through a canonical problem. We will study the surface fields on a prolate spheroid for a plane wave which is near axially incident and then attempt a generalization to other shapes. In analyzing the prolate spheroid we shall extend slightly the work of Kazarinoff and Ritt (1959) using the asymptotic analysis of Langer (1935, 1949). The generalization of these results to general convex bodies will then follow from a consideration of the creeping wave paths, the geodesics, on a prolate spheroid.

# II

# SCALAR DIFFRACTION BY A PROLATE SPHEROID

## 2.1 The Integral Representation

We suppose a scalar plane wave be incident on a prolate spheroid and study the solutions of the Helmholtz equation satisfying either the Neumann or Dirichlet boundary conditions. We use the prolate spheroidal coordinates $(\xi, \eta, \phi)$ where

$$\begin{cases} x = c\sqrt{\xi^2 - 1}\,\sqrt{1-\eta^2}\,\cos\phi\;, \\ y = c\sqrt{\xi^2 - 1}\,\sqrt{1-\eta^2}\,\sin\phi\;, \\ z = c\,\xi\,\eta\;, \end{cases} \tag{2.1}$$

where c is the semifocal distance. The equation of the spheroid is $\xi = \xi_o$ so the major and minor axes are given by

$$\begin{cases} a = c\,\xi_o\;, \\ b = c\sqrt{\xi_o^2 - 1}\;. \end{cases} \tag{2.2}$$

The Helmholtz equation

$$(\nabla^2 + k^2)\psi = 0 \tag{2.3}$$

becomes, in the prolate spheroidal coordinates,

$$\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial\Psi}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\Psi}{\partial\eta}\right]+\left[\frac{1}{\xi^2-1}+\frac{1}{1-\eta^2}\right]\frac{\partial^2\Psi}{\partial\phi^2}+\gamma^2\left[(\xi^2-1)+(1-\eta^2)\right]\Psi=0\,, \tag{2.4}$$

where $\gamma = kc$. We require $\Psi$ to be of the form

$$\Psi = e^{i\vec{k}\cdot\vec{r}} + \Psi_s \tag{2.5}$$

where $\Psi_s$ satisfies the radiation condition and $\Psi$ is to satisfy either of the boundary conditions

$$\Psi(\xi_o, \eta, \phi) = 0\,,$$

or

$$\left.\frac{\partial\Psi}{\partial\xi}(\xi, \eta, \phi)\right|_{\xi_o} = 0\,, \tag{2.6}$$

and when $e^{i\vec{k}\cdot\vec{r}}$ is the incident wave with

$$\vec{k} = k(\sin\theta_o,\ 0,\ \cos\theta_o)\ . \tag{2.7}$$

We make the expansion

$$\Psi = \sum_{-\infty}^{\infty} \Psi_m e^{im\phi} \tag{2.8}$$

and consider the component equations

$$\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial\psi}{\partial\xi}\right]-\frac{m^2\psi}{\xi^2-1}+\gamma^2(\xi^2-1)\psi+\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\psi}{\partial\eta}\right]-\frac{m^2}{1-\eta^2}\psi+\gamma^2(1-\eta^2)\psi=0\ . \qquad (2.9)$$

With $\lambda$ denoting a separation constant, (2.9) is seen to be equivalent to the separated equations

$$\left[\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}-\frac{m^2}{\xi^2-1}+\gamma^2(\xi^2-1)-\lambda\right]\ u=0 \qquad (2.10a)$$

$$\left[\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}-\frac{m^2}{1-\eta^2}+\gamma^2(1-\eta^2)+\lambda\right]\ \omega=0 \qquad (2.10b)$$

From the solutions of (2.10a) which are asymptotically in $\xi$ of the form

$$y_2(\xi,\lambda)\sim\frac{e^{-i\gamma\xi}}{\xi}$$

and (2.11)

$$y_1(\xi,\lambda)\sim\frac{e^{i\gamma\xi}}{\xi}$$

we form either

$$\phi_2^D(\xi,\lambda)=y_1(\xi,\lambda)\,y_2(\xi_o,\lambda)-y_2(\xi,\lambda)\,y_1(\xi_o,\lambda)$$

or (2.12a)

$$\phi_2^N(\xi,\lambda)=y_1(\xi,\lambda)\,y_2'(\xi_o,\lambda)-y_2(\xi,\lambda)\,y_1'(\xi_o,\lambda)$$

and

$$\phi_1(\xi,\lambda)=y_1(\xi,\lambda) \qquad (2.13)$$

With the solutions $\phi_1$ and $\phi_2$ we then form the resolvent Green's function

$$G = \frac{\phi_2(\xi<)\phi_1(\xi>)}{(\xi^2-1)W(\phi_1,\phi_2)} \qquad (2.14)$$

where $W(\phi_1,\phi_2)$ is the Wronskian of the solutions $\phi_1$ and $\phi_2$ . This becomes for the Dirichlet and Neuman boundary conditions

$$G^D = \frac{1}{2i\gamma\phi_1(\xi_o,\lambda)}\ \phi_2^D(\xi<,\lambda)\phi_1(\xi>,\lambda) \qquad (2.15a)$$

$$G^N = \frac{1}{2i\gamma\phi_1'(\xi_o,\lambda)}\ \phi_2^N(\xi<,\lambda)\phi_1(\xi>,\lambda) \qquad (2.15b)$$

Similarly, we take the solutions of (2.10b) which are regular at $\eta=1, \psi_1(\eta,-\lambda)$ and at $\eta=1$, $\psi_1(\eta,-\lambda)$, where

$$\psi_{-1}(\eta,-\lambda) = \psi_1(-\eta,-\lambda) \quad , \qquad (2.16)$$

and form the resolvent Green's function

$$\tilde{G} = \frac{1}{(1-\eta^2)W(\psi_{-1},\psi_1)}\ \psi_{-1}(\eta<,-\lambda)\psi_1(\eta>,-\lambda) \qquad (2.17)$$

The solution $\psi_m$ can then be represented by the plane wave limit of

$$\psi_m = \frac{1}{2\pi i}\int_\Gamma \tilde{G}(\eta,\eta',-\lambda)\,G(\xi,\xi',\lambda)\,d\lambda \qquad (2.18)$$

## 2.2 The Asymptotic Analysis

We wish to evaluate (2.18) for $\gamma$ large. We will parallel the work of Kazarinoff and Ritt (1959) making a turning point analysis of (2.6a) (Langer, 1949) and a Stokes phenomenon analysis of (2.6b) (Langer, 1935).

On puting

$$u = \frac{1}{\sqrt{\xi^2 - 1}} v \quad , \tag{2.19}$$

Eq. (2.6a) becomes

$$v'' + \left[\gamma^2 - \frac{\lambda}{\xi^2 - 1} + \frac{1 - m^2}{(\xi^2 - 1)^2}\right] v = 0 \tag{2.20}$$

We define $\xi_r$ by

$$\lambda = (\xi_r^2 - 1)(\gamma^2 + \frac{1 - m^2}{(\xi_r^2 - 1)^2}) \quad , \tag{2.21}$$

and rewrite (2.20) ,

$$v'' + \gamma^2 \frac{\xi^2 - \xi_r^2}{\xi^2 - 1} \left\{ 1 + \frac{m^2 - 1}{\gamma^2} \frac{1}{(\xi^2 - 1)(\xi_r^2 - 1)} \right\} v = 0 \tag{2.22}$$

Now for $m = 0$ or for $m \ll \gamma$ the asymptotic forms of two solutions of (2.22) for $\gamma \longrightarrow \infty$ with $\xi > 1$ are, after Kazarinoff and Ritt

$$v^{(j)} = \gamma^{1/6} e^{\mp i\gamma f(\xi_r)} V^j(\xi) \qquad (j = 1, 2) \tag{2.23}$$

$$V^j(\xi) = \sqrt{\frac{\pi}{2}}\; e^{\pm 5\pi i/12}\, \Psi(\xi)\, \xi^{1/3} H^{(j)}_{1/3}(\xi)$$

$$\zeta = \gamma \int_{\xi_r}^{\xi} u(\tau)\, d\tau$$

$$u^2(\xi) = \frac{\xi^2 - \xi_r^2}{\xi^2 - 1}$$

$$\Psi = \left(\frac{\zeta}{\gamma}\right)^{1/6} \cdot u^{-1/2}(\xi)$$

$$f = -\int_0^1 \sqrt{\frac{\xi_r^2 - t^2}{1 - t^2}}\, dt$$

As m grows large, each solution retains the same form with

$$\gamma \longrightarrow \gamma_m = \gamma \sqrt{1 + \frac{m^2 - 1}{\gamma^2(\xi_r^2 - 1)}} \tag{2.24}$$

provided $\xi$ is near $\xi_r$. We note that for $\xi$ increasing without bound, the asymptotic solutions have leading terms independent of m.

Since we shall wish to evaluate (2.18) as a residue series (where this is possible) we look for the roots of

$$\phi_1(\xi_o, \lambda) = 0$$

or (2.25)

$$\phi_2'(\xi_o, \lambda) = 0 \quad .$$

The zeros of these solutions occur at the zeros of

$$H^{(1)}_{1/3}(h_r) = 0 \quad ,$$

or (2.26)

$$H^{(1)\,'}_{1/3}(h_r') = 0 \quad .$$

From the definition of $\zeta$ these are the values of $\xi_r$ such that

$$h_r \text{ or } h_r' = \gamma_m \int_{\xi_1}^{\xi_o} \sqrt{\frac{\xi^2 - \xi_r^2}{\xi^2 - 1}} \, d\xi \quad . \tag{2.27}$$

Moreover, if $\xi_r$ is near to $\xi_o$

$$h_r \text{ or } h_r' \cong \gamma_m \frac{2}{3} \left(\frac{2\xi_o}{\xi_o^2 - 1}\right)^{1/2} (\xi_o - \xi_r)^{3/2} \quad . \tag{2.28}$$

We now rewrite (2.23) using the Airy integral

$$\omega(t) = \sqrt{\frac{\pi}{3}} \, e^{2\pi i/3} (-t)^{1/2} H^{(1)}_{1/3}\left(\frac{2}{3}(-t)^{3/2}\right) \tag{2.29}$$

hence, with $\zeta = \frac{2}{3}(-t)^{3/2}$

$$v^{(1)}(\xi) = e^{-\pi i/4} e^{-i\gamma f} (-t)^{1/4} \left(\frac{\xi^2 - 1}{\xi^2 - \xi_r^2}\right)^{1/4} \omega(t) \quad . \tag{2.30}$$

We now approximate $v^{(1)}(\xi_o)$ and $v^{(1)'}(\xi_o)$ with $\xi_r$ assumed to be near $\xi_o$. From (2.28) we find

$$\xi_r - \xi_o = \frac{1}{\gamma_m}\left(\frac{\gamma_m(\xi_o^2 - 1)}{2\xi_o}\right)^{1/3} t_{r_1} \tag{2.31}$$

where $t_r$ is a root of

$$\omega(t) = 0 \,, \quad \text{(Dirichlet problem)}$$

or

$$\omega'(t) = 0 \quad \text{(Neumann problem)} \quad .$$

Substituting in (2.30)

$$v^{(1)}(\xi_o) \cong e^{-\pi i/4} e^{-i\gamma f(\xi_1)} \left(\frac{\gamma_m(\xi_o^2 - 1)}{2\xi_o}\right)^{1/6} \omega(t) \tag{2.32}$$

where we have made the approximation

$$\xi_r + \xi_o \cong 2\xi_o \quad .$$

On noting that

$$\left.\frac{\partial}{\partial \xi}\right|_{\xi_o} = \gamma_m \left(\frac{\gamma_m(\xi_o^2 - 1)}{2\xi_o}\right)^{-1/3} \frac{\partial}{\partial t}$$

we find

$$v^{(1)'}(\xi_o) \cong e^{-\pi i/4}\, e^{-i\gamma f(\xi_1)}\, \gamma_m \left(\frac{\gamma_m(\xi_o^2-1)}{2\xi_o}\right)^{-1/6} \omega'(t) \tag{2.33}$$

For the case $\gamma >> m$, we put

$$\gamma_m \cong \gamma ,$$

and define

$$M = \left(\frac{\gamma(\xi_o^2-1)}{2\xi_o}\right)^{1/3} ,$$

$$= \left(\frac{kb^2}{2a}\right)^{1/3} . \tag{2.34}$$

Hence (2.32) and (2.33) become

$$v^{(1)}(\xi_o) = e^{-i\pi/4}\, e^{-i\gamma f(\xi_1)}\, M^{1/2}\, \omega(t) \tag{2.35}$$

$$v^{(1)'}(\xi_o) = e^{-i\pi/4}\, e^{-i\gamma f(\xi_1)}\, M^{-1/2}\, \gamma \omega^1(t) . \tag{2.36}$$

The angular Eq. (2.10b) is transformed by letting

$$\omega = \frac{1}{\sqrt{1-\eta^2}}\, z \tag{2.37}$$

and becomes

$$z'' + \left\{ \gamma^2 + \frac{\lambda}{1-\eta^2} + \frac{1-m^2}{(1-\eta^2)^2} \right\} z = 0 . \tag{2.38}$$

As before, we put

$$\lambda = \gamma^2(\xi_r^2 - 1)\left(1 + \frac{1-m^2}{\gamma^2(\xi_r^2-1)^2}\right) \qquad (2.39)$$

so that (2.26) becomes

$$v'' + \frac{\gamma^2}{1-\eta^2}\left\{\xi_r^2 - \eta^2 + \frac{1-m^2}{\gamma^2(\xi_r^2-1)}\right\}v + \frac{1-m^2}{(1-\eta^2)^2}\,v = 0 \qquad (2.40)$$

or, by defining

$$\eta_r^2 = \xi_r^2 + \frac{1-m^2}{\gamma^2(\xi_r^2-1)} \quad ,$$

$$v'' + \frac{\gamma^2}{1-\eta^2}(\eta_r^2 - \eta^2)v + \frac{1-m^2}{(1-\eta^2)^2}\,v = 0 \quad . \qquad (2.41)$$

Using the Stokes phenomenon analysis of Langer (1935) we may approximate the solution of (2.41) that is regular at $\eta = 1$, as follows:

$$v \cong \sqrt{\frac{\sigma}{P\gamma}}\, J_m(\sigma) \qquad (2.42)$$

where $\sigma = \gamma \int_{\eta}^{1} P(t)\,dt$ ,

$$P^2(\eta) = \frac{\eta_r^2 - \eta^2}{1-\eta^2} \quad .$$

The solutions of (2.10b) are

$$\psi_1(\eta,-\lambda) \cong \sqrt{\frac{\sigma}{\gamma}}\left[(1-\eta^2)(\eta_r^2-\eta^2)\right]^{-1/4} J_m(\sigma) \tag{2.43a}$$

$$\psi_1(\eta, -\lambda) = \psi_1(-\eta,-\lambda) \tag{2.43b}$$

and

$$(1-\eta^2)W(\psi_{-1}, \psi_1) \cong \frac{2}{\pi}\cos\big(2\sigma(o)\big)(-)^m \quad . \tag{2.44}$$

The angular Green's function is from (2.44) and (2.17)

$$\tilde{G}(\eta, \eta', -\lambda) = \frac{\psi_{-1}(\eta<, -\lambda)\psi_1(\eta>, -\lambda)}{\frac{2}{\pi}\cos\big(2\sigma(o)\big)(-)^m} \quad . \tag{2.45}$$

We now form the integral representations for the surface fields using Eqs. (2.45) and (2.35) or (2.36). In the case of the Dirichlet boundary condition the normal derivative of the field on the surface is given by

$$\left.\frac{\partial\psi^D}{\partial n}\right|_{\xi_o} = \frac{e^{-\pi i/4}}{M\pi}\sqrt{\frac{\gamma\xi_o}{2}}\,\frac{1}{\gamma}\left(\frac{1}{(1-\eta^2)(1-\eta_o^2)}\right)^{1/4}\sum_m e^{im(\phi+\pi)}$$
$$\cdot\int_{\Gamma'} dt\,\frac{e^{-i\sigma(o)}}{2\cos\big(2\sigma(o)\big)}\left(\frac{1}{(\eta_r^2-\eta^2)(\eta_r^2-\eta_o^2)}\right)^{1/4}\frac{J_m\big(\sigma(\eta)\big)J_m\big(\sigma(\eta_o)\big)}{\omega(t)}\sqrt{\sigma(\eta)\sigma(\eta_o)} \tag{2.46}$$

where we have noted that $\gamma f(\xi_r) = -\sigma(o)$ and that $d\lambda = 2\gamma\xi_o M dt$.

If we take the incident direction to be near axial so that $\theta_o$ is small, we can truncate the m series at some $m=m_o$ so that $\gamma\xi_o$ can be chosen so large that $\gamma\xi_o >> m_o$. In this case we put

$$\eta_r = \xi_r >> 1$$

and, except in oscillatory functions, put

$$\xi_r \cong \xi_o \quad .$$

Equation (2.43) then becomes

$$\left.\frac{\partial\psi^D}{\partial n}\right|_{\xi_o} \cong \frac{e^{-\pi i/4}}{M\pi}\frac{1}{\sqrt{2}}\left(\frac{\xi_o^2}{\xi_o^2-\eta^2}\right)^{1/4}\int_{\Gamma'} dt\,\frac{e^{-i\sigma(o)}}{2\cos(2\sigma(o))}\cdot\sqrt{\sigma(\eta)\sigma(\eta_o)}\left(\frac{1}{(1-\eta^2)(1-\eta_o^2)}\right)^{1/4}\cdot$$

$$\cdot\frac{1}{\omega(t)}\sum e^{im(\phi+\pi)}J_m\left(\sigma(\eta)\right)J_m\left(\sigma(\eta_o)\right) \quad . \qquad (2.47)$$

We note that the summation in (2.47) can be done explicitly to give

$$\left.\frac{\partial\psi^D}{\partial n}\right|_{\xi_o} = \frac{e^{-\pi i/4}}{M\pi}\frac{1}{\sqrt{2}}\left(\frac{\xi_o^2}{\xi_o^2-\eta^2}\right)^{1/4}\cdot\frac{1}{(1-\eta^2)^{1/4}}\int_{\Gamma'} dt\,\frac{e^{-i\sigma(o)}}{2\cos(2\sigma(o))}\sqrt{\sigma(\eta)}\cdot$$

$$\cdot\frac{1}{\omega(t)}J_o\left(\sqrt{\sigma^2(\eta)+\sigma^2(\eta_o)+2\sigma(\eta)\sigma(\eta_o)\cos\phi}\right) , \qquad (2.48)$$

(Magnus and Oberhettinger, 1949).

With a similar argument the surface field satisfying the Neumann boundary conditions can be written as

$$\psi^N(\xi_o) = -\frac{e^{-\pi i/4}}{\pi} \frac{1}{\sqrt{2}} \left(\frac{\xi_o^2}{\xi_o^2-\eta^2}\right)^{1/4} \frac{1}{(1-\eta^2)^{1/4}} \int_{\Gamma'} dt \frac{e^{-i\sigma(o)}}{2\cos(2\sigma(o))} \sqrt{\sigma(\eta)}$$

$$\frac{1}{\omega'(t)} J_o\left(\sum\right) \tag{2.49}$$

where we have put

$$\sum^2 = \sigma^2(\eta)+\sigma'^2(\eta_o)+2\sigma(\eta)\sigma(\eta_o)\cos\phi \tag{2.50}$$

From the definition of $\sigma(\eta)$ we write

$$\sigma(\eta)=\sigma(o)-\gamma(\eta)=\gamma\int_0^1\sqrt{\frac{\xi_r^2-x^2}{1-x^2}}dx-\gamma\int_0^\eta\sqrt{\frac{\xi_r^2-x^2}{1-x^2}}\,dx \tag{2.51}$$

and expand about $\xi_r=\xi_o$

$$\gamma\int_0^\eta\sqrt{\frac{\xi_r^2-x^2}{1-x^2}}\,dx \cong ka\int_0^\eta\sqrt{\frac{1-\epsilon^2x^2}{1-x^2}}\,dx+M\int_0^\eta\frac{dx}{\sqrt{(1-\epsilon^2x^2)(1-x^2)}}t\,. \tag{2.52}$$

The first term on the right of (2.52) is the wave number k multiplied by the path length measured from $\eta = 0$ to $\eta$ along a curve of constant $\phi$ ; the second term is the reduced distance given by Fock (1946) multiplied by $t_r$. Rewriting

$$\check{\sigma}(\eta) = kS(\eta) + \xi(\eta)\, t_r \qquad (2.53)$$

where

$$S(\eta) = a\int_0^{\eta} \sqrt{\frac{1-\epsilon^2 x^2}{1-x^2}}\, dx \qquad (2.54)$$

$$\xi(\eta) = \int_0^{S(\eta)} \left(\frac{kR(s)}{2}\right)^{1/3} \frac{ds}{R(s)} \qquad (2.55)$$

$$= M\int_0^{\eta} \frac{dx}{\sqrt{(1-\epsilon^2 x^2)(1-x^2)}}$$

with R(s) the radius of curvature at $S(\eta)$ along a curve of constant $\phi$.

Since the path of integration $\Gamma'$ is such that Imt > 0 we have from (2.52) that $\mathrm{Im}\,\sigma(o) > 0$ and we make the convergent expansion

$$\frac{e^{-i\sigma(o)}}{2\cos\left(2\sigma(o)\right)} = e^{i\sigma(o)} \sum_{\ell=0}^{\infty} (-)^{\ell}\, e^{4i\ell\sigma(o)} \quad . \qquad (2.56)$$

We now consider the forms (2.48) and (2.49) for $\eta$ near enough zero so that $|\sigma(\eta)| >> 1$ . In this case, we may use the asymptotic form of the Bessel function and approximate $\sum$ by

$$\sum \cong \sigma(\eta) + \sigma(\eta_o)\cos\phi$$

$$= \sigma(o) - \check{\sigma}(\eta) + \sigma(\eta_o)\cos\phi \qquad (2.57)$$

in oscillatory functions and

$$\sum \cong \sigma\,(\eta) \qquad (2.58)$$

in non-oscillatory functions. On substituting the asymptotic form of the Bessel function and the expansion (2.56) in (2.48) we have

$$\left.\frac{\partial\psi^D}{\partial n}\right|_{\xi_o} = \frac{1}{2M\pi^{3/2}}\left(\frac{\xi_o^2}{\xi_o^2-\eta^2}\right)^{1/4}\frac{1}{(1-\eta^2)^{1/4}}\sum_{\ell=0}^{\infty}(-)^{\ell}\int_{\Gamma}\frac{dt}{\omega(t)}\;\cdot$$

$$\cdot\; e^{4i\ell\sigma(o)}\left\{e^{i\left(\check{\sigma}(\eta)-\sigma(\eta_o)\cos\phi\right)} - i\,e^{i\left(2\sigma(o)-\check{\sigma}(\eta)+\sigma(\eta_o)\cos\phi\right)}\right\}. \qquad (2.59)$$

Using the forms (2.51) and (2.53) and noting that for $\eta_o$ near to one

$$\sigma(\eta_o) \cong kb\,\theta_o + \frac{a}{b}\,Mt\,\theta_o \qquad (2.60)$$

and rewriting

$$\sigma(o) = k\frac{L}{4} + \frac{\Xi}{4}\,t,$$

where

$$\frac{L}{4} = a\int_0^1 \sqrt{\frac{1-\epsilon^2x^2}{1-x^2}}\,dx$$

is one quarter of the distance around, the spheroid for $\phi$ = constant mod $\pi$ and

$$\Xi = M \int_0^1 \frac{dx}{\sqrt{(1-\epsilon^2 x^2)(1-x^2)}}$$

is the corresponding reduced distance, we have

$$\left.\frac{\partial \psi^D}{\partial n}\right|_{\xi_o} = \frac{1}{2M\pi^{3/2}} \left(\frac{\xi_o^2}{\xi_o^2 - \eta^2}\right)^{1/4} \frac{1}{(1-\eta^2)^{1/4}} \sum_{\ell=0}^{\infty} (-)^{\ell} \int_{\Gamma} \frac{dt}{\omega(t)} e^{i\ell kL + 4i\ell \Xi t} \cdot$$

$$\cdot \left\{ e^{i\left[kS(\eta) - kb\,\dot{\theta}_o \cos\phi\right]} e^{i\left[\zeta(\eta) - \frac{a}{b} M\theta_o \sin\phi\right] t} - i e^{i\left[k\frac{L}{2} - kS(\eta) + kb\,\theta_o \cos\phi\right.} \right.$$

$$\cdot\, e^{i\left[\frac{\Xi}{4} - \zeta(\eta) + \frac{a}{b} M\theta_o \sin\phi\right] t} \qquad (2.61)$$

If we now evaluate (2.61) on the shadow boundary and include only the first term in the back scattered direction we have an expression of the form

$$I_B(\eta_1, \phi_1) = \frac{1}{2M\pi^{3/2}} \left(\frac{\xi_o^2}{\xi_o^2 - \eta_1^2}\right)^{1/4} \frac{1}{(1-\eta_1^2)^{1/4}} (-i) e^{i\left[k\frac{L}{2} - kS(\eta_1) + kb\theta_o \cos\phi_1\right]} \cdot$$

$$\cdot \int_{\Gamma} \frac{dt}{\omega(t)} e^{i\left[\Xi/4 - \zeta(\eta_1) + \frac{a}{b} M\theta_o \sin\phi_1\right] t} \qquad (2.62)$$

where $(\eta_1, \phi_1)$ are the coordinates of the shadow boundary and

$$\eta_1 \cong \frac{a}{b}\theta_o \cos\phi_1 \quad . \tag{2.63}$$

Since $\eta_1$ is small we make the approximation

$$\begin{aligned} S(\eta_1) &\cong a\eta_1 \\ &= \frac{a^2}{b}\theta_o \cos\phi_1 \end{aligned} \tag{2.64}$$

We will neglect the terms of order $\theta_o$ in the t-integration and include terms of this order only in the phase. This gives the approximation

$$I_B(\eta_1, \phi_1) \cong I_B(o)\, e^{ik\frac{c^2}{b}\theta_o \cos\phi_1} \tag{2.65}$$

where $I_B(o)$ is the value taken on the shadow boundary for symmetric illumination.

The back scattered field will be proportional to the integral of (2.65) over the azimuthal angle $\phi_1$ from 0 to $2\pi$, that is

$$\psi_B(\theta_o) \sim I_B(o) \int_0^{2\pi} e^{ik\frac{c^2}{b}\theta_o \cos\phi_1}\, d\phi_1 \quad . \tag{2.66}$$

But the zeroth order Bessel function has the integral representation

$$J_o(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \cos t} \, dt$$

so that the leading creeping wave term in the back scattered field is given

$$\psi_B(\theta_o) \cong \psi_B(o) J_o(k \frac{c^2}{b} \theta_o) \quad .$$

There is an analogous expression for the Neumann boundary condition.

We have then , that for near axial illumination of the spheroid that the leading creeping wave term in the back scattered field is given by the field arising from symmetric illumination multiplied by the shape factor $J_o(k \frac{c^2}{b} \theta_o)$. We note that this factor takes the value of one for either symmetric illumination, $\theta_o = 0$, or for the sphere limit, $c = 0$ .

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13. ABSTRACT

At high frequency the acoustic surface fields for plane wave illumination of a prolate spheroid are found for near symmetric illumination. The surface caustic, in contrast to the symmetric case, is found to be diffuse. The values of the surface fields are used to find the modification of the creeping wave contribution to back scattering.

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| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
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