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THE EFFECT OF MODAL INTERACTION IN THE XENON INSTABILITY PROBLEM

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NOMENCLATURE

$a_i(t)$	the time-dependent part of the i'th term in the expansion of $\delta\phi(\underline{r},t)$
$b_i(t)$	same as above in the expansion of $\delta X(\underline{r},t)$
$c_i(t)$	same as above in the expansion of $\delta I(\underline{r},t)$
p	the resonance escape probability
q	the slowing down kernel
$\underline{r}$	space variable
t	time variable
x	space variable
C_n	the criticality factor of the n'th mode of the neutron flux distribution in the reactor
D	the thermal diffusion constant
F_{ij}	the modal coupling coefficient associated with the steady state neutron flux distribution
H	the thickness of the slab reactor
$I(\underline{r},t)$	the concentration of iodine-135
$I_o(\underline{r})$	the steady state iodine-135 distribution
$I_n(t)$	the reduced amplitude of the n'th mode in the expansion of $\delta I(\underline{r},t)$
$\delta I(\underline{r},t)$	the deviation of the iodine-135 distribution from the steady state
L	the thermal diffusion length
M	the migration length
$N_n(t)$	the reduced amplitude of the n'th mode in the expansion of $\delta\phi(\underline{r},t)$

P_{ij} the modal coupling coefficient associated with the steady state poison distribution

$Q(K_n)$ the fast leakage escape probability associated with the n'th mode

S_n the reduced subcriticality of the n'th mode of the neutron flux distribution

T the dimensionless time variable

$X(\underline{r},t)$ the concentration of xenon-135

$X_o(\underline{r})$ the steady state xenon-135 distribution

$X_n(t)$ the reduced amplitude of the n'th mode in the expansion of $\delta X(\underline{r},t)$

$\delta X(\underline{r},t)$ the deviation of the xenon-135 distribution from the steady state

γ_X the fission yield of xenon-135

γ_I the fission yield of iodine-135

γ $\gamma_X + \gamma_I$

δ_{ij} Kronecker's delta

ϵ the fast fission factor

λ_X the decay constant of xenon-135

λ_I the decay constant of iodine-135

λ $\lambda_X + \lambda_I$

ν the average number of neutrons produced upon fission

σ_X the microscopic thermal neutron absorption cross section of xenon-135

τ the neutron generation time in an infinite medium

$\phi(\underline{r},t)$	the thermal neutron flux
$\phi_o(\underline{r})$	the steady state thermal neutron flux distribution
$\phi_o(max)$	the maximum value of $\phi_o(\underline{r})$
$\delta\phi(\underline{r},t)$	the deviation of the thermal neutron flux distribution from the steady state
$\psi_n(\underline{r})$	the n'th characteristic function of the wave equation
$\psi_n(max)$	the maximum value of $\psi_n(\underline{r})$
Γ_n	the maximum effect of the poison on the criticality factor of the n'th mode
K_n	the n'th characteristic value of the wave equation
Σ_a	the thermal absorption cross section in the clean core
Σ_f	the thermal fission cross section
Φ	the reduced maximum steady state thermal neutron flux

INTRODUCTION

In the treatment of the space and time dependent reactor stability problem of neutron flux shape variations due to xenon-135, the method of harmonics is often used. Through this method, the finite set of partial differential equations describing the problem are replaced by an infinite set of ordinary differential equations which describe the time behavior of the various modes or harmonics of distribution. There is a finite subset of equations associated with each mode. It was pointed out by Kaplan (1) that particular sets of harmonics can be found such that each subset of these equations is independent of the other subsets, that is, such that the time behavior of each mode of distribution can be described independently of the behavior of the other modes. Often, however, it is desirable to choose a set of harmonics, such as the characteristic functions of the wave equation, which are simple, well known, and not dependent on the reactor power level. The use of such a set of harmonics will in general result in an interdependence of the subsets of ordinary differential equations mentioned above, such that the time behavior of each mode is influenced by the behavior of the other modes. It is convenient to attempt to predict the stability of the shape of the neutron flux distribution by examining the stability of each mode by itself, that is, when modal interaction is neglected. The results of such an attempt may not be very meaningful however, if it is possible that an unstable system is produced when several modes, each of which is by itself stable, interact. This work deals with the effect of modal interaction on stability. The characteristic functions of the wave equation are used in the modal expansion. The reactor model used is described in a subsequent section.

In this work, some of the terminology and notations of Weinberg and Wigner (2) are used. The term "criticality factor" and the symbol C are used in place of the more usual term "effective multiplication factor" and the symbol k_{eff}. In the modal expansion the first term will be called the first harmonic or fundamental mode, this in general should be distinguished from the steady state neutron flux distribution.

THE REACTOR MODEL

The reactor model used in this work is a bare, thermal reactor with stationary fuel, it is homogeneous except for the xenon poison. It is assumed that variation in the xenon-135 density is the only process through which a change in neutron flux level affects the properties of the core medium, that is, xenon poisoning is the only feedback effect. It is assumed that the xenon affects only the thermal absorption cross section. Linear theory is used: the treatment is restricted to small deviations from the steady state.

The attention is focused on the stablility of the shape of the neutron flux distribution, and therefore it is assumed that the fundamental mode of the flux distribution is held constant by a suitable control system which has negligible effect on the higher modes.

All numerical computations and results are given for a slab reactor with effective boundaries at $x = o$ and $x = H$, and in which no variations are allowed in the y and z directions in any of the variables.

Subject to the enumerated assumptions, the reactor system can be described to a good approximation by the following equations:

$$0 \approx \Sigma_a \tau \frac{\partial}{\partial t} \phi(\underline{r},t) \approx D\nabla^2 \phi(\underline{r},t) - \Sigma_a \phi(\underline{r},t) - \sigma_X X(\underline{r},t)\phi(\underline{r},t) + \nu\epsilon p \Sigma_f \int_{\text{all space}} \phi(\underline{r}',t) q(|\underline{r}-\underline{r}'|) d^3\underline{r}' \quad (1)$$

$$\frac{\partial}{\partial t} X(\underline{r},t) = \lambda_I I(\underline{r},t) + \gamma_X \epsilon \Sigma_f \phi(\underline{r},t) - \lambda_X X(\underline{r},t) - \sigma_X X(\underline{r},t)\phi(\underline{r},t) \quad (2)$$

$$\frac{\partial}{\partial t} I(\underline{r},t) = \gamma_I \epsilon \Sigma_f \phi(\underline{r},t) - \lambda_I I(\underline{r},t) \quad (3)$$

with the boundary conditions that the variables are zero at the effective boundaries.

The approximation that (1) is zero is reasonable for the higher modes of distribution, provided that the reactor dimensions are not more than a few hundred times the migration length.

THE MODAL EXPANSION

It is convenient to separate the variables into steady state and time dependent parts:

$$\phi(\underline{r},t) = \phi_0(\underline{r}) + \delta\phi(\underline{r},t) \tag{4}$$

$$X(\underline{r},t) = X_0(\underline{r}) + \delta X(\underline{r},t) \tag{5}$$

$$I(\underline{r},t) = I_0(\underline{r}) + \delta I(\underline{r},t) \tag{6}$$

Substituting these into the equations of motion, subtracting the steady state equations from the resulting ones, and then neglecting the terms involving the product $\delta\phi\delta X$, one obtains a set of equations describing the behavior of small deviations from the steady state. The time variables are now expanded in the following manner:

$$\delta\phi(\underline{r},t) = \sum_{i=1}^{\infty} a_i(t)\psi_i(\underline{r}) \tag{7}$$

$$\delta X(\underline{r},t) = \sum_{i=1}^{\infty} b_i(t)\psi_i(\underline{r}) \tag{8}$$

$$\delta I(\underline{r},t) = \sum_{i=1}^{\infty} c_i(t)\psi_i(\underline{r}) \tag{9}$$

where

$$\nabla^2\psi_i(\underline{r}) + K_i^2\psi_i(\underline{r}) = 0 \tag{10}$$

$$\int_{\text{reactor volume}} \psi_i(\underline{r})\psi_j(\underline{r})d^3\underline{r} = \delta_{ij} \tag{11}$$

and the ψ's satisfy the same boundary conditions as the variables. Multiplying the resulting equations by ψ_n and integrating them over the reactor volume, one obtains the desired infinite set of ordinary differential equations. These equations are cast into a convenient dimensionless form by defining the following quantities.

$$N_n(t) \equiv \frac{\psi_n(\max)}{\phi_o(\max)} a_n(t) \quad , \tag{12}$$

$$X_n(t) \equiv \frac{\lambda_X \psi_n(\max)}{\gamma \epsilon \Sigma_f \phi_o(\max)} b_n(t) \quad , \tag{13}$$

$$I_n(t) \equiv \frac{\lambda_I \psi_n(\max)}{\gamma_I \epsilon \Sigma_f \phi_o(\max)} c_n(t) \quad , \tag{14}$$

the reduced amplitudes of the n'th modes of the neutron flux, xenon, and iodine concentration deviations from steady state;

$$\Phi \equiv \frac{\sigma_X}{\lambda_X} \phi_o(\max) \quad , \tag{15}$$

the reduced maximum steady state flux,

$$F_{ij} \equiv \frac{\sigma_X}{\lambda_X} \int_{\text{reactor volume}} \phi_o(\underline{r})\psi_i(\underline{r})\psi_j(\underline{r})d^3\underline{r} \quad , \tag{16}$$

$$P_{ij} \equiv \frac{\sigma_X}{\gamma \epsilon \Sigma_f} \int_{\text{reactor volume}} X_o(\underline{r})\psi_i(\underline{r})\psi_j(\underline{r})d^3\underline{r} \quad , \tag{17}$$

the modal coupling coefficients associated with the steady state flux and poison distribution respectively;

$$\Gamma_n \equiv \frac{\gamma \epsilon \Sigma_f}{\Sigma_a(1+L^2K_n^2)} \quad , \tag{18}$$

the maximum effect of the steady state poison on the criticality factor of the n'th mode (so that $\Gamma_n P_{nn}$ is the actual effect on it);

$$C_n \equiv \frac{\nu \epsilon p \Sigma_f Q(K_n)}{\Sigma_a(1+L^2K_n^2)} - \Gamma_n P_{nn} \quad , \tag{19}$$

the criticality factor of the n'th mode;

$$S_n \equiv \frac{1-C_n}{\Gamma_n} , \tag{20}$$

the reduced subcriticality of the n'th mode, and finally

$$T \equiv \lambda_x t , \tag{21}$$

the dimensionless time variable.

In terms of the newly defined quantities, the equations of motion can be written in the following form.

$$0 \approx S_n N_n + F_{nn} X_n + \sum_{i \neq n} [P_{in} N_i + F_{in} X_i], \tag{22}$$

$$\frac{dX_n}{dT} = \frac{\gamma_I}{\gamma} I_n + (\frac{\gamma_x}{\gamma} - P_{nn}) N_n - (1+F_{nn}) X_n - \sum_{i \neq n} [P_{in} N_i + F_{in} X_i] , \tag{23}$$

$$\frac{dI_n}{dT} = \frac{\lambda_I}{\lambda_x} (N_n - I_n), \qquad n = 1, 2, 3, \ldots, \infty . \tag{24}$$

The influence of the i'th mode on the n'th one is represented by the expression within the brackets in Equations (22) and (23).

THE MODAL COUPLING COEFFICIENTS

The values of the modal coupling coefficients F_{ij} and P_{ij}, as indicated in (16) and (17), are influenced by the shapes of the steady state flux and poison distributions, the latter distribution being uniquely determined by the former. The shape of the flux distribution depends on the maximum flux level as well as on the reactor size. In the poisoned reactor the flux distribution is flatter than the fundamental mode of the expansion used here, but for the combinations of reactor sizes and flux levels considered in this work, it is not much flatter. For numerical computation of the coupling coefficients, therefore, the shape of the steady state flux distribution is approximated by the fundamental mode ψ_1. A flattening in ϕ_o, and therefore also in X_o, increases the values of F_{ij} and P_{ij} for $i=j$ (which might be called the self-coupling coefficients), and decreases the values of the coefficients for $i \neq j$, because the ψ's are orthogonal functions. The approximation mentioned above, therefore, results in a slight apparent strengthening of modal interaction. The values of some of the coupling coefficients are shown in Figure 1 as functions of the flux level. When the nonlinearities are neglected, as is done here, there is no interaction between an even and an odd harmonic. The absolute magnitudes of the coefficients fall off quite rapidly as $|i-j|$ increases, and therefore one may expect the strongest interaction to appear between the i'th and j'th modes when $j=i\pm2$.

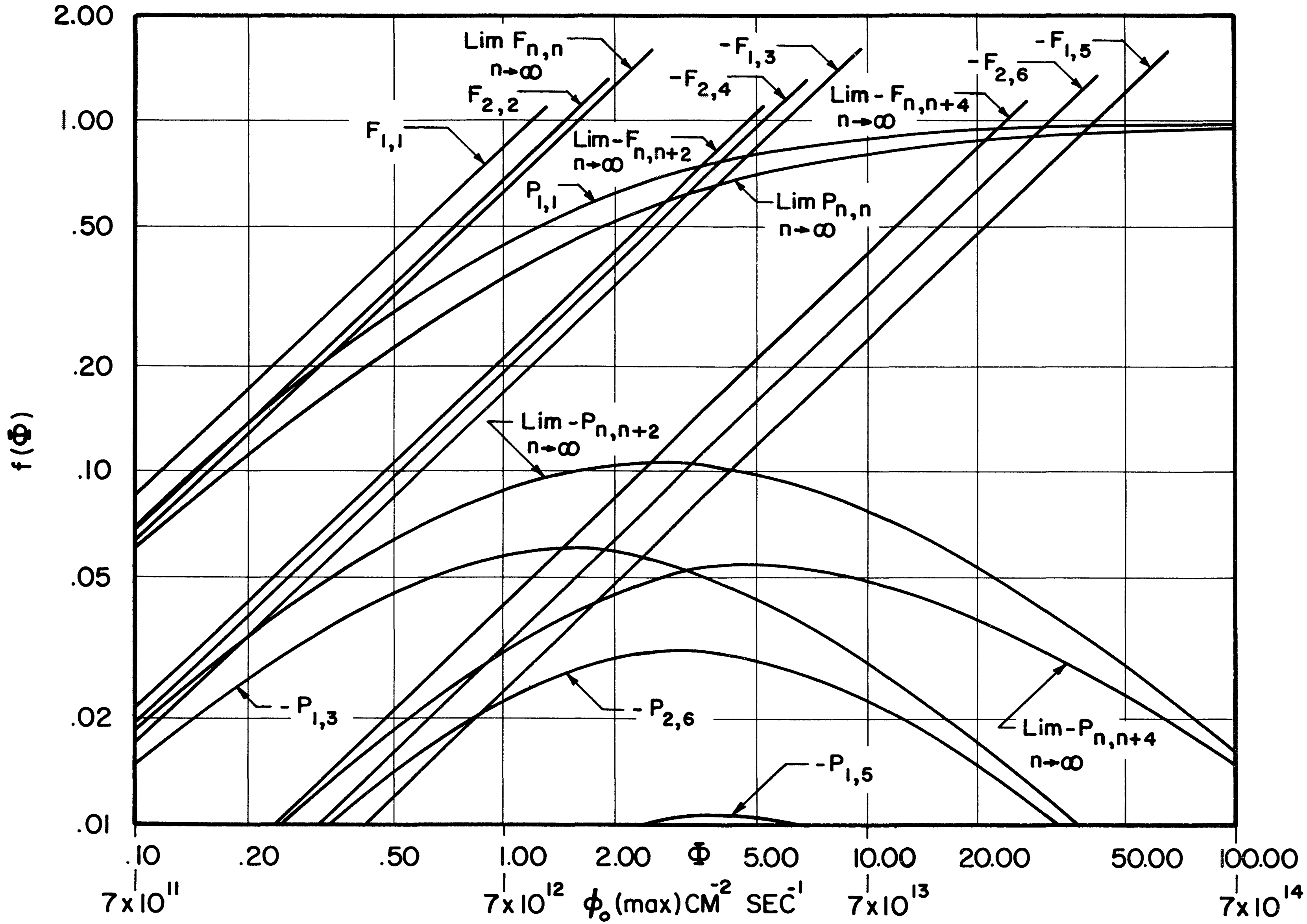

Figure 1. The Behavior of the Modal Coupling Coefficients F_{ij} and P_{ij} as Functions of Φ.

ANALYTICAL INVESTIGATION

An approximate stability criterion can be derived for the higher modes when modal interaction is neglected. Under this condition, the system represented by Equations (22), (23) and (24) is stable if

$$S_n > \frac{F_{nn}(P_{nn} - \gamma_x/\gamma)}{\lambda/\lambda_x + F_{nn}} , \quad n = 2, 3, \ldots, \infty . \tag{25}$$

This criterion predicts marginal stability for the n'th mode when S_n equals the right hand side of (25); this value will be referred to as the critical value of S_n as predicted by the approximate stability criterion. This critical value is shown as a function of flux level in Figure 2.

It is desirable to have a rough indication of what this criterion means in terms of reactor size rather than in terms of subcriticality. For $n \neq 1$, but not very large, and for the range of reactor sizes of interest here, one may write (3) as a rough approximation

$$S_n \approx \frac{M^2}{\Gamma_n} (K_n^2 - K_1^2) . \tag{26}$$

For a slab reactor with $\Gamma_n = .03$:

$$S_n \approx \frac{\pi^2}{.03} (n^2-1)\left(\frac{M}{H}\right)^2 . \tag{27}$$

Table I shows this approximate correspondence between reactor size and S_n. It should be noted here that the exact relationship between S_n and reactor size involves terms which are dependent on the power level.

The analytical investigation carried out was directed at answering the following question. If the n'th mode is predicted to be marginally stable by the approximate stability criterion given above, then will

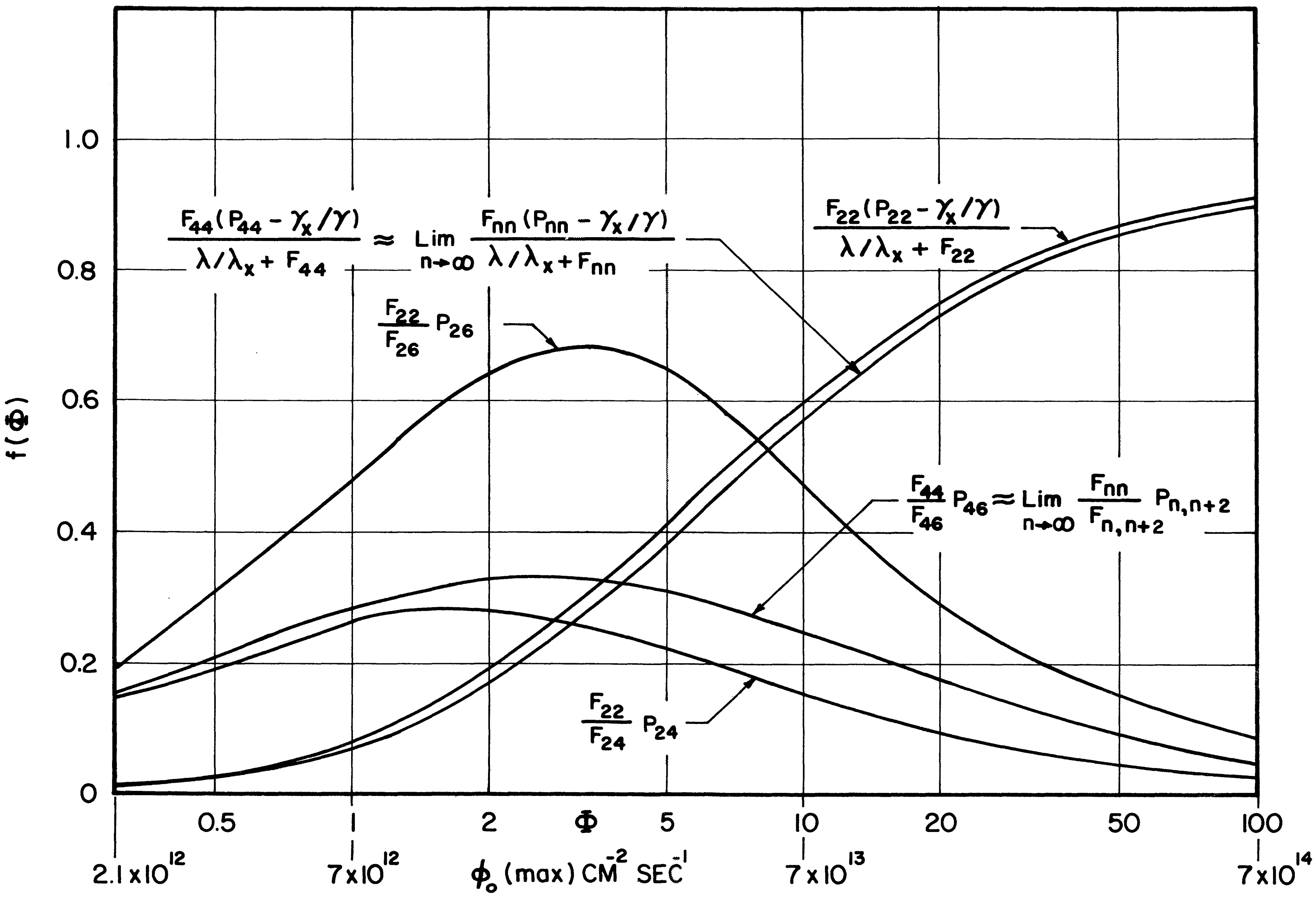

Figure 2. The Behavior of the Functions $\frac{F_{nn}(P_{nn} - \gamma_x/\gamma)}{\lambda/\lambda_x + F_{nn}}$ and $\frac{F_{nn}P_{nm}}{F_{nm}}$.

TABLE I

APPROXIMATE CORRESPONDENCE BETWEEN H/M AND S_n

H/M:	31	35	41	50	70	99	140
S_2:	1	.8	.6	.4	.2	.1	.05
S_4:	5	4	3	2	1	.5	.25

the combined system of the n'th and m'th modes be stable or unstable? In order to attempt to answer this question, the equations of motion (22), (23) and (24) of the n'th and m'th modes were combined under the assumption that $F_{ij} = P_{ij} = 0$ except when $i=n$ or $i=m$ and $j=n$ or $j=m$. The resulting fourth order differential equation, which is too long to exhibit here, was subjected to a rather tedious examination in the light of the Hurwitz-Routh stability criterion (4) under the condition that the n'th mode is predicted to be marginally stable by the approximate criterion (25). For the reactor model used here, this examination yielded the following results. For $S_n < F_{nn}P_{nm}/F_{nm}$ the combined system is unstable, irrespectively of the stability of the m'th mode. The behavior of the right hand side of this inequality as a function of flux level is shown in Figure 2. For $S_n > F_{nn}P_{nm}/F_{nm}$ the combined system is stable if the m'th mode is by itself sufficiently stable, otherwise it is unstable. Except for a very small region where $S_n \approx F_{nn}P_{nm}/F_{nm}$, in order that the m'th mode be by itself "sufficiently stable" in this sense, the requirement

$$S_m > \frac{F_{mm}(P_{mm} - \gamma_x/\gamma)}{\lambda/\lambda_x + F_{mm}} \qquad (28)$$

is necessary, but not sufficient. In other words, the value of S_m must be larger by some amount than that necessary for marginal stability as predicted by the approximate criterion (28). A necessary and sufficient condition simple enough to be useful was not found for this "sufficient stability" of the m'th mode. The numerical computations discussed in the next section, however, indicate that the value of S_m does not have to be significantly greater than that required by condition (28).

For very large values of S_m, the equation of motion of the combined system becomes independent of S_m and the effect of modal interaction approaches a nonzero limit. This fact is quite significant. According to Equation (22), an infinitely large S_m means that N_m, the m'th mode of the flux distribution is not allowed to vary. Nevertheless, it was found that interaction with the m'th mode has a nonzero effect on the stability of the n'th mode. The reason for this is that although the m'th mode of the flux distribution is not allowed to vary, the m'th mode of the xenon distribution will be forced into oscillation by variations in the n'th mode of the flux distribution. In other words, even if the oscillations in the flux distribution are purely in a single mode, the oscillations in the xenon distribution will involve all the other modes. This result agrees with those reported by Pearce (5). This means that even if the magnitude of the fundamental mode of the flux distribution is held constant by some control system, in examining the stability of the third mode, interaction between it and the fundamental must be taken into account.

NUMERICAL RESULTS

Since the least stable higher mode is the second harmonic, the numerical calculations were centered on it. An exploration, by means of analog computer simulation, of the behavior of the system consisting of the second, fourth, and sixth harmonics coupled together indicated that the effect of the sixth harmonic on the second is negligible unless the reactor size is in excess of a few hundred migration lengths. Such large reactor sizes are not considered here because they are beyond the range of validity of the approximations used. Therefore, in further numerical computations, which were done on a digital computer, the system consisting of only the second and fourth modes was considered. Table II shows the values of the constants which were used in the calculations.

For a given reactor composition the power level and the reactor size uniquely determine the values of S_2 and S_4, or alternately, the reactor power level and the value of S_2 determine the value of S_4. In the computations, however, the value of S_4 was varied in the full range $S_2 \leqslant S_4 \leqslant \infty$ (effectively). Although this led to some computations which may not be physically realizable, the results are quite interesting and do contribute to the understanding of the problem.

In the previous section, it was stated that in the region where $S_2 > F_{2,2}\ P_{2,4}/F_{2,4}$, interaction with the fourth mode will enhance stability provided the value of S_4 is sufficiently large. Figure 3 shows the results of the search for values of S_4 in this region such that the critical value of S_2 as predicted by the approximate stability

TABLE II

THE VALUES OF THE CONSTANTS USED IN THE NUMERICAL CALCULATIONS

$\sigma_X = 3 \times 10^{-18}\ cm^2$

$\lambda_X = 2.1 \times 10^{-5}\ sec^{-1}$

$\lambda_I = 2.9 \times 10^{-5}\ sec^{-1}$

$\gamma_X = .003$

$\gamma_I = .061$

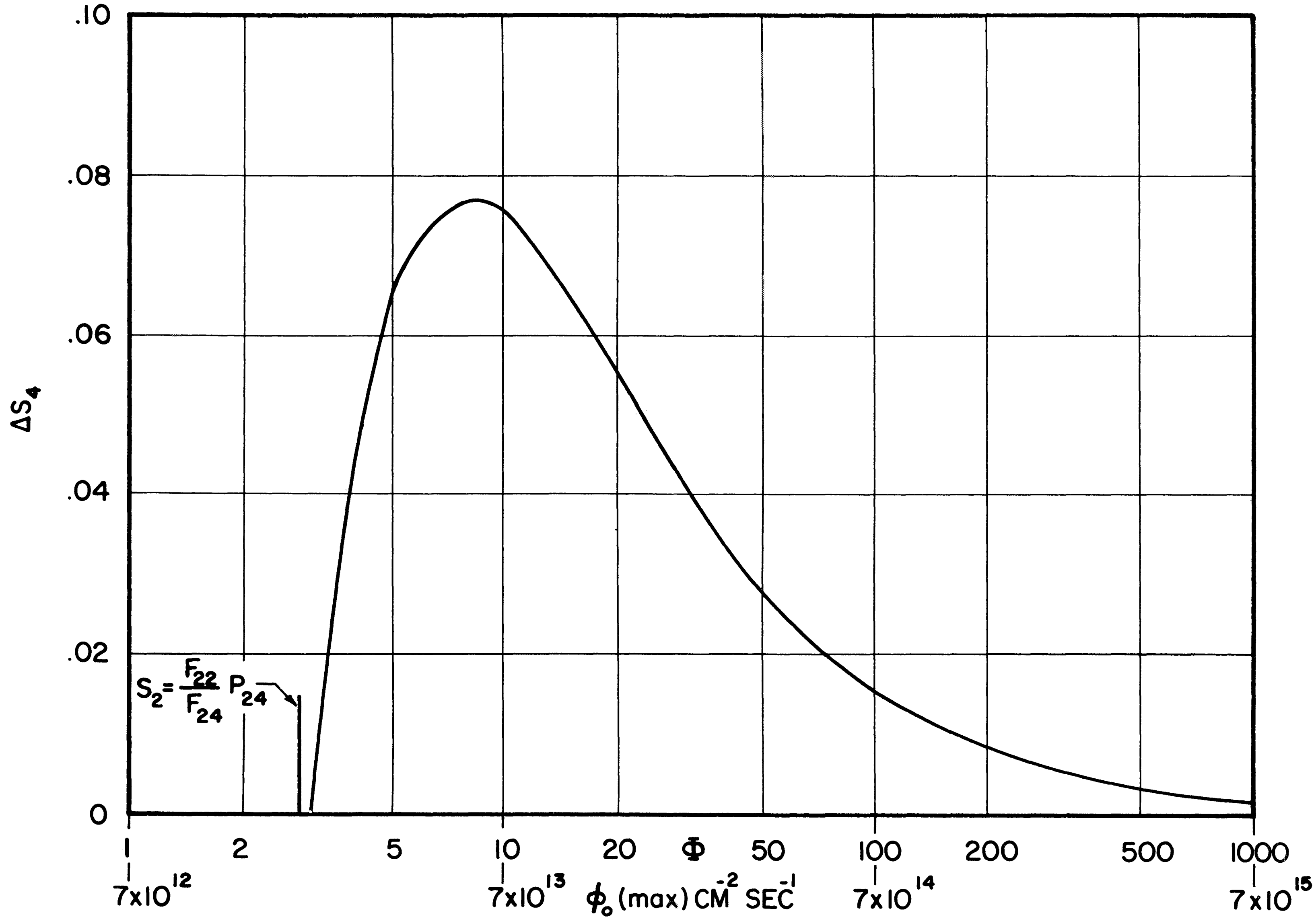

Figure 3. The Values of ΔS_4 for the Condition $\Delta S_2 = 0$ in the Region $S_2 > F_{2,2} P_{2,4}/F_{2,4}$.

criterion (25) is also the critical value for the combined system of the second and fourth modes. The plot shows ΔS_4: the difference between the values of S_4 found by this search and the critical values of S_4 as predicted by the approximate stability criterion. This difference is relatively small everywhere in this region. Table I shows that in this region the value of S_4 is considerably larger than the critical values found by the search described above. Therefore, in this region, if the second mode is predicted to be marginally stable by the approximate stability criterion, the combined system of the two modes considered here will be stable.

The remainder of the numerical calculations consisted of the search for values of S_2 for which the combined system is marginally stable. The flux level and the value of S_4 were varied over wide ranges. Figure 4 shows the computed values of ΔS_2: the difference between the values of S_2 found by this search and the critical values of S_2 as predicted by the approximate stability criterion.

The results indicate that if the reactor dimensions are not very large, less than about 60 times the migration length for the numerical values used here, second mode instability will set in at a high flux level: above 10^{13} neutrons/cm^2/sec. Interaction of the second mode with the fourth in this case enhances stability. The actual critical value of S_2 is percentagewise only slightly different from the critical value obtained when modal interaction is neglected.

If the reactor dimensions are large, 70 times the migration length or more, second mode instability sets in at flux levels below 10^{13} neutrons/cm^2/sec. Interaction with the fourth mode in this region

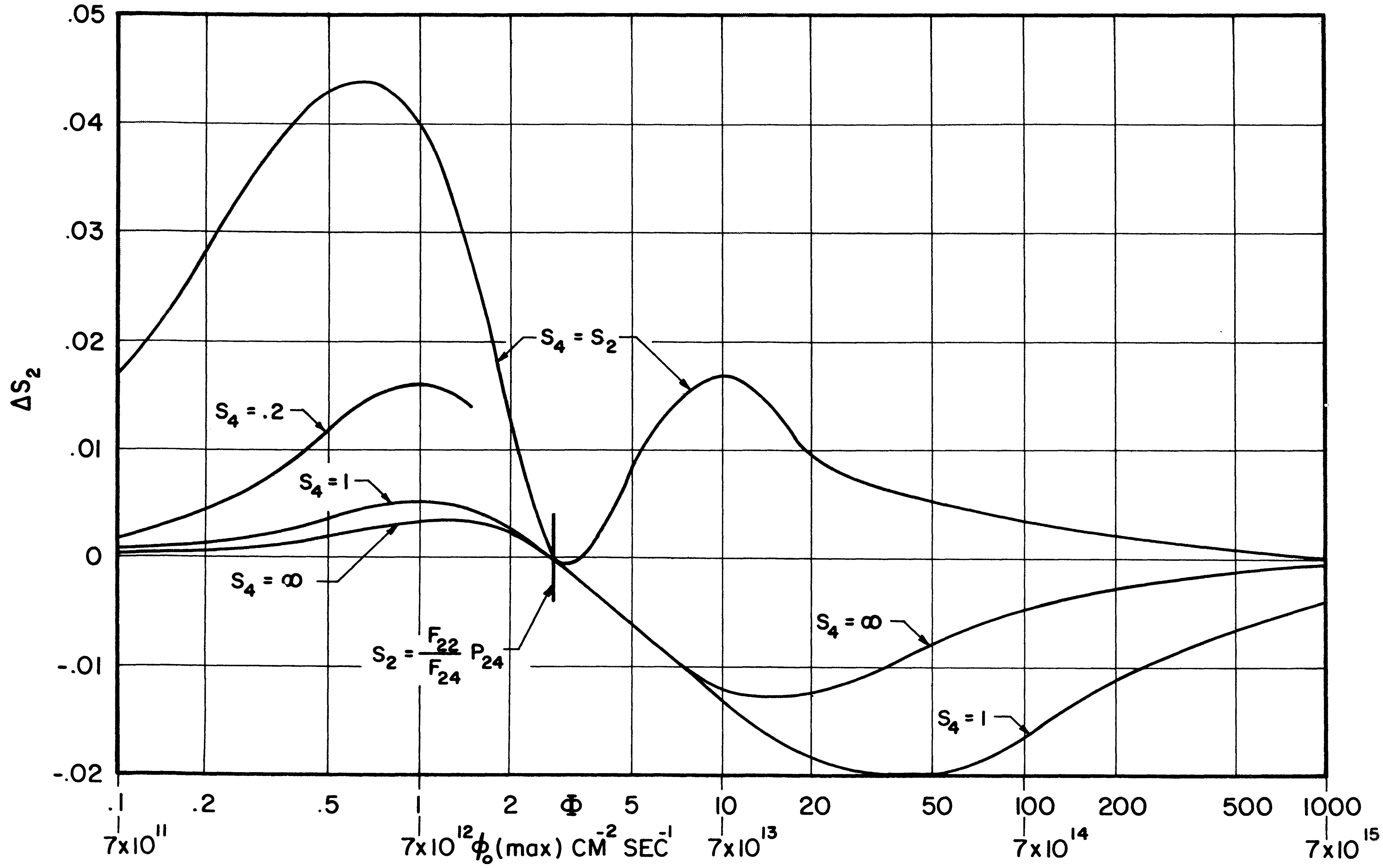

Figure 4. The Values of ΔS_2 as a Function of S_4 and Φ.

detracts from stability. As the reactor dimensions are made larger, the difference between the actual critical value of S_2 and that predicted when modal interaction is neglected becomes percentagewise quite large.

Thus the stability of a large reactor is seriously overestimated if modal interaction is neglected.

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