ROBUST TRACKING IN NONLINEAR SYSTEMS
AND ITS APPLICATION TO ROBOTICS

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October, 1984

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1This work was supported by the Air Force Office of Scientific Research, AF Systems Command, USAF under grant F 49620-82-C-0089
# TABLE OF CONTENTS

1. INTRODUCTION ............................................................................................................. 3

2. MAIN RESULT ................................................................................................................ 6

3. TRACKING CONTROL FOR A ROBOTIC MANIPULATOR ................................. 19

4. CONCLUSION ................................................................................................................ 27

5. REFERENCES ................................................................................................................ 29
ABSTRACT

A nonlinear feedback multivariable controller is used to implement multivariable tracking in a nonlinear system. The tracking error is measured by general function of system state and the input command. The controller is robust in the sense that the tracking error is ultimately bounded in the presence of modelling errors. Free parameters, which affect the form of the controller, allow flexibility in determining such factors as: the size of the ultimate bound, the rate of error decay, excursion of the control, conditions on the class of modelling errors, the level of system gain. Restrictive assumptions on the structure of model and the modelling errors are required. These assumptions hold for a robotic manipulator. This application is investigated at more length and it appears that the resulting control scheme may have advantages over others which have been proposed in the robotics literature.
1. INTRODUCTION

In this paper, we consider the nonlinear system:

\[ \dot{z}(t) = F(z(t), t) + G(z(t), t)u(t), \quad e(t) = E(z(t), t, y_d(t)), \quad (1.1) \]

where \( z(t) \in R^n \), \( u(t) \in R^m \), \( y_d(t) \in R^l \), \( e(t) \in R^s \), \( R^+ \triangleq [0, \infty) \), \( F: R^n \times R^+ \rightarrow R^n \), \( g: R^n \times R^+ \rightarrow R^m \times R^m \), and \( E: R^n \times R^+ \times R^l \rightarrow R^s \). The function \( y_d : R^+ \rightarrow R^l \) is a desired "output" for the system. Usually, the tracking error \( e \) is measured by \( E(z, t, y_d) = y_d - h(z) \), where \( y(t) = h(z(t)) \) is the output of the system. But in some applications (for instance, see [7, 20], \( E \) is more generally described as a function of \( z, t, y_d \). Roughly speaking, the problem of tracking is to control the system in (1.1) so that \( e \) is kept within a desirable tolerance. In practice, \( F, G, E \) may not be exactly known. Even when they are known exactly, they may be too complex to deal with easily. Then, they are replaced by \( \hat{F}, \hat{G}, \hat{E} \) (which are defined in the same sets as \( F, G, E \)) so that (1.1) is modelled by

\[ \dot{z}(t) = \hat{F}(z(t), t) + \hat{G}(z(t), t)u(t), \quad \hat{e}(t) = \hat{E}(z(t), t, y_d(t)). \quad (1.2) \]

The objective is to obtain a robust tracking controller, determined from the (simplified) model (1.2), such that the actual system (1.1) with the controller has acceptable tracking performance. Let us be more specific.

It is assumed that the controller has the form

\[ u(t) = K(z(t), t, Y'(t)), \quad (1.3) \]
where $Y' \triangleq (y_d, y_d^{(1)}, \ldots, y_d^{(r)})$, $r \leq n$ is an appropriate integer, and $y_d^{(j)}$ is the $j$th derivative of $y_d$. The resulting closed loop system is

$$
\dot{x}(t) = F(x(t), t) + G(x(t), t) K(z(t), t, Y'(t)), \quad \epsilon(t) = E(x(t), t, y_d(t)). \quad (1.4)
$$

The functions $y_d$ belong to $Y_d$, the class of $r$ times continuously differentiable functions from $R^+$ into $R^1$ satisfying $Y'(t) \in \Omega$, $t \in R^+$, where $\Omega$ is a specified subset of $R^{(r+1)}$. Normally, $K$ is chosen so that (1.2) with $u$ given by (1.3) gives $\dot{\epsilon}(t) \to 0$ as $t \to \infty$. The need for controllers involving the derivatives of $y_d$ becomes apparent when examples are considered. See [7] and Sections 2, 3.

The performance of the closed loop system (1.4) can be measured by a variety of definitions. The definitions considered here are adaptations, to tracking error, of definitions used in the literature of differential equations [10, 17] and robust regulators [1, 3, 19]. Let the Euclidean norm be denoted by $|\cdot|$. The tracking error is uniformly ultimately bounded with respect to $Y_d$, with bound $b$ if for every $d$, $t_o \in R^+$, there exists $\tau(d, t_o)$ such that $|x(t_o)| \leq d$ and $y_d \in Y_d$ imply $|\epsilon(t)| \leq b$, $t \geq t_o + \tau$. If $\tau$ is a function of $d$ only, the bound is said to be uniform with respect to $t$ and $Y_d$.

The controller $K$ is to be robust in the following sense. Given suitable conditions on the modelling errors,

$$
\Delta F \triangleq F - \hat{F}, \quad \Delta G \triangleq G - \hat{G}, \quad \Delta E \triangleq E - \hat{E}, \quad (1.5)
$$

the tracking error $\epsilon(t)$ for the closed-loop system (1.4) is to be uniformly ultimately bounded.
Robust design problems of a similar nature have been considered recently by many authors. See, for example, [1, 3, 8, 19]. In these papers, there is no input, output or error measure for the closed-loop system and the objective is to obtain ultimate bounds on the system state. Our approach to the tracking problem is to transform it into a problem where state bounds give tracking error bounds. Then we apply ideas chosen freely from the existing literature. The results obtained are not the most general possible. However, they do give robustness under quite weak hypotheses and avoid excessive complexity. The compromise between generality and complexity permits us to illuminate the design process and treat general examples in a relatively short space.

We now describe the main idea of the paper in greater detail. It is assumed that the model (1.2) has special structural characteristics so that it may be linearized by state feedback, precompensation of the inputs, and a transformation of state variables. Specifically, there exist functions

\[ \alpha : R^n \times R^+ \times R^{(r+1)} \to R^m, \beta : R^n \times R^+ \times R^{lr} \to R^m \times m, T : R^n \times R^+ \times R^{lr} \to R^n \]

such that

\[ \tilde{u} = \alpha(z, t, Y') + \beta(z, t, Y'^{-1})u, \]  
\[ \tilde{x} = T(z, t, Y'^{-1}) \]  

allow \( \hat{e} \) in (1.2) to be given by the linear system

\[ \dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t), \hat{e}(t) = C\tilde{x}(t), \]  

where \( A \in R^{n \times n}, B \in R^{n \times m}, C \in R^{r \times n} \). This step is used by Gilbert and Ha [7]
and, philosophically, is motivated by the work of Hunt, Meyer, Su and others, who consider similar transformations of \( \dot{z} = f(t) + g(z)u \) into \( \ddot{z} = A \ddot{z} + B \ddot{u} \). See [14] and the references indicated there. Once (1.8) is obtained, a robust saturation controller is designed using ideas which are similar to those used by Corless and Leitman [3] and Gutman [8]. Because our problem is more complex than theirs, the details are quite different.

The paper is organized as follows. Section 2 presents the main result (Theorem 2.1), its proof, and remarks concerning the various conditions and parameters used to determine \( K \). In Section 3, Theorem 2.1 is applied to the tracking control of a robotic manipulator, perhaps the most obvious and interesting example of a system where the special conditions required in Theorem 2.1 are met. The resulting controller which allows uncertainties and simplified modelling of manipulator dynamics, may be viewed as an improvement of the one presented in [7]. The use of simplified dynamics is important because it has the potential of significantly reducing computational complexity in the mechanization of the controller. The concluding section mentions further examples and certain limitations and advantages of our scheme.

2. MAIN RESULT

First, we introduce some general notation: \( I_p \in \mathbb{R}^{p \times p} \) is the identity matrix; the minimum and maximum of the real parts of the eigenvalues of \( A \in \mathbb{R}^{p \times p} \) are \( \sigma_m(A), \sigma_M(A) \), respectively; the matrix norm of \( A \in \mathbb{R}^{p \times p} \) is

\[
||A||_F \triangleq \max \{|AZ| : Z \in \mathbb{R}^p, |Z| = 1\} = (\sigma_M(A^T A))^{1/2}; f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^p \] is \( C^1 \)
if it is $k$ times continuously differentiable; the Jacobian matrices of $f \in C^1$ with respect to its first and second arguments at $(z_1, z_2) \in R^{n_1} \times R^{n_2}$ are denoted by

$$D_1f(z_1, z_2) \in R^{p \times n_1}, \quad D_2f(z_1, z_2) \in R^{p \times n_2},$$

respectively.

Next, we state assumptions on the problem data which are required for our main result. For

$$\widehat{\gamma}^r \triangleq (y^0, y^1, \ldots, y^r) \in R^{\ell(r+1)},$$

let

$$\widehat{\gamma}^j \triangleq (y^0, \ldots, y^i) \in R^{\ell(j+1)}, \quad \hat{\gamma}^j \triangleq (y^1, \ldots, y^{i+1}).$$

Let $\Omega^j = \{ \gamma^j : \gamma^r \in \Omega \}$ be the projection of $\Omega$ onto $R^{\ell(j+1)}$. In the assumptions, we can choose $\Omega \subset R^{\ell(r+1)}$, a $C^1$-matrix function $S : R^n \times R^+ \times R^r \rightarrow R^n \times n$ and $H \in R^{m \times n}$. The choice of $S$ and $H$ is free except for the constraints: (i) $S(z, t, \gamma^{r-1})$ is nonsingular, $z \in R^n$, $t \in R^+$, $\gamma^{r-1} \in \Omega^{-1}$, and (ii) the matrix $A_H \triangleq A + BH$ is stable and has simple structure ($A_H$ has linearly independent eigenvectors).

**Assumption A.1.** The functions $F$, $\hat{F}$, $G$, $\hat{G}$, $E$, $\hat{E}$ are $C^1$.

**Assumption A.2.** There exist an integer $r \geq 1$, $C^1$-mappings $\alpha$, $\beta$, $T$, and matrices $A$, $B$, $C$ (see Section 1) such that

$$B\beta(z, t, \gamma^{r-1}) = D_1T(z, t, \gamma^{r-1})\hat{G}(z, t), \quad (2.1)$$

$$AT(z, t, \gamma^{r-1}) + B\alpha(z, t, \gamma^r) = D_1T(z, t, \gamma^{r-1})\hat{F}(z, t) + D_2T(z, t, \gamma^{r-1}) + D_3T(z, t, \gamma^{r-1})\hat{\gamma}^{r-1}, \quad (2.2)$$

$$CT(z, t, \gamma^{r-1}) = \hat{E}(z, t, \gamma^r), \quad (2.3)$$

$$\beta(z, t, \gamma^{r-1}) \text{ is nonsingular,} \quad (2.4)$$
hold for all \( z \in R^n, t \in R^+, \overline{Y} \in \Omega \).

**Assumption A.3.** Over the class of uncertain \( \Delta F \) and \( \Delta G \), there are mappings \( \Delta F^*: R^n \times R^+ \times R^{r^*} \rightarrow R^n \) and \( \Delta G^*: R^n \times R^+ \times R^{r^*} \rightarrow R^n \times R^m \) satisfying

\[
D_1 T(z, t, \overline{Y}^{-1}) \Delta F(z, t) = B \Delta F^*(z, t, \overline{Y}^{-1}),
\]

(2.5)

\[
D_1 T(z, t, \overline{Y}^{-1}) \Delta G(z, t) = B \Delta G^*(z, t, \overline{Y}^{-1}) \beta(z, t, \overline{Y}^{-1}),
\]

(2.6)

for all \( z \in R^n, t \in R^+, \overline{Y}^{-1} \in \Omega^{-1} \).

**Assumption A.4.** Let

\[
\Phi(z, t, \overline{Y}^r) \triangleq [S(z, t, \overline{Y}^{-1})]^{-1} \{ \Delta G^*(z, t, \overline{Y}^{-1})(HT(z, t, \overline{Y}^{-1}) - a(z, t, \overline{Y}^r)) + \Delta F^*(z, t, \overline{Y}^{-1}) \}.
\]

(2.7)

There exists a \( C^r \) - function \( \phi: R^n \times R^+ \times R^{(r+1)} \rightarrow R^+ \) such that \( \phi(z, t, Y^r(t)) \) is locally lipschizian in \( R^n \times R^+ \) with respect to \( z \) and over the class of uncertain \( \Delta F, \Delta G \),

\[
|\Phi(z, t, \overline{Y}^r)| \leq \phi(z, t, \overline{Y}^r), \quad z \in R^n, t \in R^+, \overline{Y}^r \in \Omega.
\]

(2.8)

**Assumption A.5.** Define \( \Delta G^*_i: R^n \times R^+ \times R^{r^*} \rightarrow R^n \times R^m \) and \( \Gamma: R^n \times R^+ \times R^{r^*} \rightarrow R^n \times R^m \) by

\[
\Delta G^*_i(z, t, \overline{Y}^{-1}) \triangleq [S(z, t, \overline{Y}^{-1})]^{-1} \Delta G^*(z, t, \overline{Y}^{-1}) S(z, t, \overline{Y}^{-1}),
\]

(2.9)

\[
\Gamma(z, t, \overline{Y}^{-1}) \triangleq I_m + \frac{1}{2} \{ \Delta G^*_i(z, t, \overline{Y}^{-1}) + [\Delta G^*_i(z, t, \overline{Y}^{-1})]^T \}.
\]

(2.10)
There exists $\gamma > 0$ satisfying

$$\text{Min. } \{\sigma_m(\Gamma(x, t, \overline{Y}^{-1}) : x \in R^n, t \in R^+, \overline{Y}^{-1} \in \Omega^{-1}\} \geq \gamma. \quad (2.11)$$

**Assumption A.6.** There exists $\delta_E > 0$ such that

$$|\Delta E(x, t, \overline{Y}^z)| \leq \delta_E, \ z \in R^n, \ t \in R^+, \overline{Y}^z \in \Omega^z. \quad (2.12)$$

**Remark 2.1.** As can be seen by substituting (1.6), (1.7) into (1.8), assumption A.2 is equivalent to the existence of $\alpha, \beta, T$ which allows $\dot{e}$ to be given by (1.8). A limitation of our approach is the need to find such $\alpha, \beta, T$. In some cases, an appropriate choice for the model (1.2) may be necessary. In other cases, like the one of Section 3, the determination of $\alpha, \beta, T$ is evident.

**Remark 2.2.** The choice of $\alpha, \beta, T$ is not unique. For example, let $\overline{\alpha}, \overline{\beta}, \overline{T}, \overline{A}, \overline{B}, \overline{C}$ satisfy A.2 and consider the family of $\alpha, \beta, T$ generated by $\alpha = \Psi_2 \overline{\alpha} + \Psi_1 \overline{T}, \beta = \Psi_2 \overline{\beta}, T = \Psi_3 \overline{T}$, where $\Psi_1 \in R^n \times n, \Psi_2 \in R^n \times n, \Psi_3 \in R^n \times n$ are arbitrary, except for the nonsingularity of $\Psi_2$ and $\Psi_3$. Then, all members of this family allow A.2 to hold. Moreover, it is easy to verify that the corresponding $A, B, C$ allow A.3 - A.5 to be satisfied. In fact, if $H$ and $S$ are changed with $\Psi_1, \Psi_2, \Psi_3$ corresponding to the rules

$$H = (\Psi_2 \overline{H} + \Psi_1) \Psi_3^{-1}, \ S = \Psi_2 \overline{S}, \quad (2.13)$$

it follows that $\Phi$ and $\Delta G_s'$ are invariant with respect to $\Psi_1, \Psi_2, \Psi_3$. Thus, choosing different members of the family does not add to the flexibility in satisfying A.1 - A.6, already provided by $H$ and $S$. 
Remark 2.3. The conditions in (A.3) impose a structural restriction on the allowed class of $\Delta F, \Delta G$. They are similar to the "matching conditions" in [1, 8, 19].

Remark 2.4. Assumption A.5 may permit a fairly large modelling error $\Delta G$, since it only requires $\Gamma$ to be positive definite. If $\Delta G^* \neq 0$ and $\Delta G^* = 0$ are included in the class of modelling errors, $\Gamma$ is restricted to $[0, 1]$. The constraint on $\Delta G$ imposed by (2.11) is not appreciably weakened by taking $\gamma < .1$; for $\gamma > 0.9$, the constraint is quite severe. Usually, $\gamma \in [.3, .6]$. By choosing $\gamma_G = 1 - \gamma$, it can be seen that (2.11) can be replaced by the (stronger but simpler) condition:

$$||\Delta G_i'(z, t, \bar{Y}^{r-1})|| \leq \gamma_G < 1, \ z \in R^n, \ t \in R^+, \ \bar{Y}^{r-1} \in \Omega^{r-1}. \quad (2.11)'$$

Remark 2.5. The most natural choice for $S$ is $I_m$. But choice of $S \neq I_m$ allows different weights on the components of $\Phi, \Delta G^*$. This gives greater flexibility in imposing (2.8), (2.11), and (2.12), and may thus lead to stronger results.

Remark 2.6. By restricting the class of desired "output" functions $Y_d$, by making $\Omega$ smaller, the conditions (2.8), (2.11), (2.12) on the modelling errors are made less critical.

Now, we construct the controller in (1.3). First, define a mapping $K_1: \mathbb{R}^n \times R^+ \times R^{l(r+1)} \rightarrow \mathbb{R}^n$ by

$$K_1(z, t, Y^r) \triangleq [\beta(z, t, Y^{r-1})]^{-1}\{HT(z, t, Y^{r-1}) - \alpha(z, t, Y^r)\}. \quad (2.14)$$

Because $A_H$ has simple structure, there exists a nonsingular $P_H \in \mathbb{R}^n \times n$ [11] such that
\[ \dot{A}_H \triangleq P_H^{-1}A_HP_H = \text{diag} \Lambda_i, \quad (2.15) \]

where

\[ \Lambda_i \triangleq \begin{bmatrix} \sigma_i & -\omega_i \\ \omega_i & \sigma_i \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad i = 1, \ldots, n_c, \quad (2.16) \]

\[ |\lambda_i| \in \mathbb{R}^{1 \times 1}, \quad i = n_c + 1, \ldots, n - n_c, \]

and \( \sigma \pm j\omega_i, \quad i = 1, \ldots, n_c \) and \( \lambda_i, \quad i = n_c + 1, \ldots, n - n_c \) are, respectively, the complex and real characteristic roots of \( A_H \). For any such \( P_H \), define mappings \( W : \mathbb{R}^n \times \mathbb{R}^+ \times R^{l(r+1)} \to \mathbb{R}^m \) and \( \eta : \mathbb{R}^m \to \mathbb{R}^m \) by

\[ W(z, t, Y') \triangleq \gamma^{-1}\phi(z, t, Y')[S(z, t, Y'^{-1})]^T B^T (P_H^{-1})^T P_H^{-1} T(z, t, Y'^{-1}), \quad (2.17) \]

\[ \eta(\xi) \triangleq \begin{cases} \xi, & |\xi| \leq 1, \\ |\xi|^{-1}\xi, & |\xi| > 1. \end{cases} \quad (2.18) \]

Then, define a mapping \( K_2 : \mathbb{R}^n \times \mathbb{R}^+ \times R^{l(r+1)} \to \mathbb{R}^m \) by

\[ K_2(z, t, Y') \triangleq -\gamma^{-1}\phi(z, t, Y')[\beta(z, t, Y'^{-1})]^{-1} S(z, t, Y'^{-1}) \eta(W(z, t, Y')). \quad (2.19) \]

Finally, the desired controller is given by

\[ u = K(z, t, Y') \triangleq K_1(z, t, Y') + K_2(z, t, Y'). \quad (2.20) \]

\textbf{Remark 2.7.} The purpose of \( K_1 \) is to give good error response in the absence of modelling errors. In particular, letting \( u = K_1(z, t, Y') \) in (1.1) with
\[ \Delta F = 0, \Delta G = 0, \Delta E = 0 \] gives \( e = CZ \) where \( \dot{z} = A_H z \) and \( Z(t_o) = T(z(t_o), t_o, Y^{-1}(t_o)) \).

Hence \( e(t) \to 0 \) exponentially.

**Remark 2.8.** The controller \( K_2 \) in (2.19) is a kind of saturation function similar to the one proposed by Corless and Leitmann [3]. Its purpose is to give acceptable performance of (1.4) in the presence of the modelling errors.

**Remark 2.9.** Consider the family of \( \alpha, \beta, T \) generated in Remark 2.2 and let (2.13) hold. Then for every member of the family, it is possible to show \( P_H \) can be chosen so that \( K \) is unchanged by \( \Psi_1, \Psi_2, \Psi_3 \). Since both the Assumptions and \( K \) are unchanged when (2.13) holds, choosing different members of the family adds no generality to our subsequent results.

Since \( K \) is continuous on \( \mathbb{R}^n \times \mathbb{R}^+ \times \Omega \) and \( K(z, t, Y(t)) \) is locally lipschizian in \( \mathbb{R}^n \times \mathbb{R}^+ \) with respect to \( z \), we know from (1.4) and assumption A.1 that for any \( t_o \in \mathbb{R}^+ \), \( z(t_o) \in \mathbb{R}^n \), and \( y_d \in Y_d \), the system (1.4) has a solution defined on an open interval containing \( t_o \) [4, 9]. Since existence of the solution is needed for all \( t \geq t_o \), a final assumption is introduced.

**Assumption A.7.** For any \( t_o \in \mathbb{R}^+ \), \( z(t_o) \in \mathbb{R}^n \), and \( y_d \in Y_d \), the system (1.4) has a solution \( x : [t_o, \infty) \to \mathbb{R}^n \).

We can now state our main result.

Let

\[ \mu_H \triangleq -\sigma_M(A_H), \quad \xi_H \triangleq ||CP_H||. \] (2.21)
\[ \nu(z, t, Y^{-1}) \triangleq \| P_{H^{-1}} T(z, t, Y^{-1}) \|, \]  

(2.22)

\[ \delta_H \triangleq (\gamma \mu_N^{-1})^{1/2} \xi_H, \quad \delta_\nu \triangleq (\gamma \mu_H^{-1})^{1/2} \nu(z(t_0), t_0, Y^{-1}(t_0)). \]  

(2.23)

**Theorem 2.1.** Suppose that the assumptions A.1 - A.7 are satisfied. Then, the system (1.4) with the controller \( u \) in (2.20) has the following properties for any \( t_0 \in R^+ \), \( z(t_0) \in R^n \), and \( y_d \in Y_d \): (i) For all \( t \in [t_0, \infty) \),

\[ |e(t)| \leq \begin{cases} 
\delta_E + \delta_H, & \delta_\nu \leq 1, \\
\delta_E + \delta_H \{1 + (\delta_\nu^2 - 1) e^{-\gamma(t-t_0)}\}^{1/2}, & \delta_\nu > 1.
\end{cases} \]  

(2.24)

(ii) Suppose, in addition, that \( \Delta F \equiv 0, \Delta G \equiv 0 \), and \( \Delta E \equiv 0 \). Then,

\[ |e(t)| \leq \xi_H \nu(z(t_0), t_0, Y^{-1}(t_0)) e^{-\gamma(t-t_0)}, \quad t \in [t_0, \infty). \]  

(2.25)

**Proof.** Consider part (i). Let \( z : [t_0, \infty) \to R^n \) be the solution of the system (1.4) with \( K \) in (2.20) for any given \( t_0 \in R^+ \), \( z(t_0) \in R^n \), and \( y_d \in Y_d \). Let

\[ Z(t) \triangleq T(z(t), t, Y^{-1}(t)). \]  

Then, since \( T, z, Y^{-1} \) are all \( C^1 \), we can take the total time derivative of \( Z(t) \):

\[ \dot{Z} = D_1 T(z, t, Y^{-1})[F(z, t) + G(z, t)\{K_1(z, t, Y') + K_2(z, t, Y')\}] + D_2 T(z, t, Y^{-1}) + D_3 T(z, t, Y^{-1}) Y^{-1}. \]  

(2.26)

Then, substituting (2.5), (2.6), (2.7), (2.9), (2.14) into (2.29) and using the identities in (2.1), (2.2), we obtain
\[
\dot{Z} = A_H Z + B S(z, t, Y^{-1}) \Phi(z, t, Y') + \\
B S(z, t, Y^{-1}) \Delta G_s(z, t, Y^{-1}) |S(z, t, Y^{-1})|^{-1} \beta(z, t, Y^{-1}) K_2(z, t, Y') .
\] (2.27)

Let
\[
v(t) \triangleq P^{-1}_H Z(t), \quad V(v(t)) \triangleq \frac{1}{2} |v(t)|^2 . \quad (2.28)
\]

Taking the time derivative of \( V(v(t)) \) and substituting \( \dot{Z}, K_2, Z \) by (2.27), (2.19), (2.28), respectively gives
\[
\dot{V}(v) = v^T \hat{A}_H v + v^T P^{-1}_H B S(z, t, Y^{-1}) \Phi(z, t, Y') - \\
\gamma^{-2} |\phi(z, t, Y')|^2 v^T P^{-1}_H B S(z, t, Y^{-1}) \Gamma(z, t, Y^{-1}) |S(z, t, Y^{-1})|^T B^T P^{-1}_H v / \Theta(|W(z, t, Y')|) .
\] (2.29)

where \( \Theta : R^+ \rightarrow R^+ \) is defined as \( \Theta(a) \triangleq a \) if \( a > 1 \), \( \Theta(a) \triangleq 1 \) if \( a \leq 1 \). From (2.15), (2.16), and \( \mu_H = -\sigma_M(A_H) = -\sigma_M(\hat{A}_H) \), it can be verified that
\[
\xi^T \hat{A}_H \xi \leq -\mu_H |\xi|^2, \quad \xi \in R^n .
\] (2.30)

Then, this, (2.17), (2.29) and Assumptions A.4, A.5 imply
\[
\dot{V}(v) \leq -\mu_H |v|^2 + \gamma |W(z, t, Y')| - \gamma |W(z, t, Y')|^2 / \Theta(|W(z, t, Y')|) . \quad (2.31)
\]

We conclude
\[
\dot{V}(v(t)) \leq -2\mu_H V(v(t)) + \gamma, \quad t \in [t_s, \infty) . \quad (2.32)
\]

The following fact is an easy consequence of (2.32).

Robust Tracking
\[ |v(t)|^2 \leq \gamma \mu_H^{-1} + (|v(t_s)|^2 - \gamma \mu_H^{-1}) e^{-2\mu_H(t-t_s)}, \quad t \in [t_s, \infty). \] 

(2.33)

By (2.3), (2.12), and (2.28),

\[ |e| \leq ||CP_H|| |v| + \delta_E, \quad t \in [t_s, \infty), \] 

(2.34)

and property (i) follows immediately.

Next consider part (ii). If \( \Delta F \equiv 0 \) and \( \Delta G \equiv 0 \), (2.5), (2.6), and (2.7) imply \( \Phi \equiv 0 \). Thus, from (2.29) and (2.30), \( V(v) \leq -2\mu_H V(v) \) and therefore

\[ |v(t)| \leq |v(t_s)|e^{-\mu_H(t-t_s)}, \quad t \in [t_s, \infty). \] 

(2.35)

The inequality (2.34) with \( \delta_E = 0 \) completes the proof.

If \( \Omega \) is bounded, it follows from part (i) of the theorem that \( e \) is uniformly ultimately bounded with respect to \( Y_d \). In particular, \( b \) need only satisfy \( b > \delta_e \triangleq \delta_E + \delta_H \). Then,

\[ \tau \triangleq \begin{cases} (2\mu_H)^{-1} \ln \{ (|\gamma \mu_H^{-1} v(d, t_s)|^2 - 1) \delta_H^2 / [(b - \delta_E)^2 - \delta_H^2] \}, \\ 0, \quad \text{otherwise,} \end{cases} \] 

(2.36)

where \( v(d, t_s) \triangleq \sup \nu(x, t_s, Y^{-1}) \) for \( |x| \leq d \) and \( Y^{-1} \in \Omega^{-1} \). If \( T(x, t, \bar{Y}^{-1}) \) is independent of \( t \), \( v(d, t_s) = \bar{v}(d) \) and \( e \) is uniformly ultimately bounded with respect to \( t \) and \( Y_d \).
Part (ii) shows that the robust controller acts well when the modelling errors (1.5) are zero. It gives the same kind of exponential bound available when modelling errors are neglected in determination of $K$ (see Remark 2.7).

For good tracking performance, $\delta_e$ should be small and $\mu_H$ large. If the pair $A, B$ is controllable, $\mu_H$ may be chosen arbitrarily by selection of $H$. Clearly, $\delta_e$ cannot be smaller than $\delta_E$. Thus, $\Delta E$ must be small for small tracking error. In many applications, $\Delta E \equiv 0$ so that $\delta_e = \delta_H$. Although the remaining effects of $H, S$, and $P_H$ are quite complex, we can make a few general remarks.

Remark 2.10. Suppose that $S$ is given by $S \triangleq \zeta \bar{S}$, where $\zeta$ is a positive constant. Let $\Phi, \Delta G_s^*, \bar{W}, \bar{K}$ denote the functions resulting from (2.8), (2.9), (2.17), (2.20) when $\zeta = 1$. It follows that $\phi = \zeta^{-1} \Phi$, $\Delta G_s^* = \Delta \bar{G}_s^*$, $\gamma = \gamma$, and $W = \bar{W}$. Hence $K = \bar{K}$. Thus, scaling $S$ neither changes the controller $K$ nor affects the statement of Theorem 2.1. But for multivariable systems $(m \geq 2)$, $S$ can be used to affect changes in the control law (see (2.19)) and bounds on the modelling error (see Remark 2.5).

Remark 2.11. Let $\bar{P}_H$ satisfy (2.15). It is easy to show that the entire family of $P_H$ satisfying (2.15) is given by

$$P_H \triangleq \bar{P}_H \Theta, \quad \Theta \triangleq \text{diag } \Theta,$$

where

$$\Theta_i \triangleq \begin{cases} \Theta_i^0 \in R^{2 \times 2}, & i = 1, \ldots, n_c, \\ \Theta_i \in R^{1 \times 1}, & i = n_c + 1, \ldots, n - n_c, \end{cases}$$

Robust Tracking
and \( \theta_i \) is an arbitrary nonzero real number, and \( \Theta^*_i \) is an arbitrary orthogonal matrix. Since

\[
(P_H^{-1})^T P_H^{-1} = (\bar{P}_H^{-1})^T \Pi \bar{P}_H^{-1}, \quad \Pi \triangleq \text{diag} \, \Pi_i,
\]

where

\[
\Pi_{2i-1} = \Pi_{2i} = \theta_i^{-2}, \quad i = 1, \ldots, n_z,
\]

\[
\Pi_i = \theta_i^{-2}, \quad i = n_z + 1, \ldots, n - n_z.
\]

the \( \theta_i^2 \in R^+ \) parameterize, through (2.17) and (2.39), the family of controllers. As the \( \theta_i^2 \) are made smaller, the "controller gain" increases (consider \( K_2 \) in (2.19) when \( |W| \leq 1 \)) and the unsaturated region of control (\( |W| \leq 1 \)) becomes smaller. At the same time, \( \delta_H \) given by (2.23) decreases and may be made as small as desired through the use of sufficiently high gain. It is interesting to note that the \( \Theta^*_i \) have no effect on anything; neither \( K \) nor the error bounds (2.24) and (2.25) depend on them. This follows from (2.39), (2.40) and the orthogonality of the \( \Theta^*_i \), which implies \( \nu(x, t, Y^{-1}) \) and \( \xi_H \) are independent of the \( \Theta^*_i \).

**Remark 2.12.** Excessive decreases in the \( \theta_i^2 \) may lead to practical problems. Higher order dynamics neglected in the modelling process, together with the high gain corresponding to the small \( \theta_i^2 \), may lead to instabilities. Even without such modelling errors, there may be problems. Consider the simple situation where \( \phi \) and \( S \) are constant and \( \Delta F \equiv o, \Delta G \equiv o, \Delta E \equiv o \). If \( Z(t) = T(x(t), t, Y^{-1}(t)) \) is small so that \( \eta(W) = W \), then it can be shown from (2.27) that
\[ \dot{Z} = (A_H - \gamma^2 \phi^2 B S S^T B^T (P_H^{-1})^T P_H^{-1}) Z, \quad e = CZ. \quad (2.41) \]

Although the bound (2.25) must still hold, the linear dynamics are no longer determined by \( A_H \) and the error may be highly oscillatory. Because of (2.39), (2.40), the likelihood of this increases as the \( \theta_i^2 \) decreases.

**Remark 2.13.** The parameter \( \gamma \) appears in the expression for \( K_2 \) (as a coefficient \( \gamma^2 \), when \( |W| < 1 \), Assumption A.5, and \( \delta_H \) (as a coefficient \( \gamma^{1/2} \)). A reasonable compromise between "high gain" and the value of \( \delta_H \) or the constraints on \( \Delta G \) is achieved by \( \gamma \approx 0.5 \). See also Remark 2.4.

**Remark 2.14.** Since \( H \) appears almost everywhere, including the parameterization of \( P_H \), its full effects are difficult to judge. Because of (2.23) – (2.25), large values for \( \mu_H \) are favorable. But large \( \mu_H \) tends to require "large" \( H \). In turn, this tends to make \( \phi \) large, which again produces high gain.

**Remark 2.15.** High gain produced by large \( \gamma^{-1} \phi \) can be mitigated by increasing the \( \theta_i^2 \). But this affects \( \delta_H \) unfavorably (Remark 2.11).

**Remark 2.16.** By suitable choice of \( H, \phi, S \) and the \( \theta_i^2 \), the form of the controller, and the various bounds on modelling errors. Even more flexibility can be had by generalizing the form of the (Lyapunov) function \( V \), which appears in the proof of Theorem 2.1. The usual choice [1, 19] is \( V = \frac{1}{2} Z^T P Z \) where \( P \) is determined from \( Q \) by \( PA_H + A_H^T P = -Q \). This complicates considerably the statement of the results and assumptions and cannot improve the value for \( \mu_H \).
Finally, it is worth noting that Theorem 2.1 is still valid if A.1 - A.7 are replaced by A.1, A.2', A.3 - A.6 where A.2' is a strengthened version of A.2.

Assumption A.2'  Assumption A.2 holds, Ω is bounded, and T has a continuous inverse with respect to z. Specifically, there exists a continuous function $T^\dagger : R^n \times R^+ \times R^{r_l} \to R^n$ such that for all $(Z, t, \bar{V}^{r_l}) \in R^n \times R^+ \times \Omega^{r_l}$, the equation $Z = T(z, t, \bar{V}^{r_l})$ has a unique solution $z = T^\dagger(Z, t, \bar{V}^{r_l})$.

We sketch the proof that A.2' eliminates the need for A.7. By the existence of z on an interval $[t_*, t_f]$ and theorem 2.1, it is easy to verify that $|Z(t)| \leq ||P_H|| \nu(z(t_*), t_*, Y^{-1}(t_*)), t \in [t_*, t_f]$. From A.2', it then follows that there exists a $c_1 > 0$ such that $|z(t)| \leq c_1, t \in [t_*, t_f]$. This, (1.4), and A.1 then imply existence of $c_2 > 0$ such that $|\dot{z}(t)| \leq c_2, t \in [t_*, t_f]$. Using the bounds on $z(t)$ and $\dot{z}(t)$ and a continuation argument (see [4], p. 288) proves $z$ is defined on all of $[t_*, \infty)$.

3. TRACKING CONTROL FOR A ROBOTIC MANIPULATOR

As indicated in [7], a robotic manipulator may be described by

$$M(q)\ddot{q} + N(q, \dot{q}) + D(t) = u, \quad e = E_z(q, y_d).$$

(3.1)

where $q \in R^n$, $M : R^n \to R^n \times R^n$, $N : R^n \times R^n \to R^n$, $D : R^+ \to R^n$, $E_z \in R^n \times R^{r_l} \to R^n$. Here, $q$ is the vector of joint coordinates, $M$ is the generalized inertia matrix, $N$ is the vector of equivalent forces due to gravitational, centrifugal, Coriolis, viscous friction and actuator damping effects, and $D$ takes into account external disturbances.
In applications, $y_d$ may be a desired path for the joint coordinates $q$ or the position and orientation of the end-effector.

Usually, a manipulator is operated with various unknown loads. Thus, there are uncertainties in $M, N$. Even when $M, N$ are known exactly, they are very complex [21] and simplified models may produce significant reductions in the computations required for mechanization of the controller. Moreover, unmodelled external forces due to friction, actuator imperfections, etc. can be represented by $D(t)$. Since $E_o$ depends on a geometric relation between $q$ and $y_d$, it may be modelled quite precisely. Thus, letting $\dot{M}, \dot{N}, \dot{E}_o = E_o$ describe our model, we obtain the errors: $\Delta M = M - \dot{M}$, $\Delta N = N + D - \dot{N}$, and $\Delta E_o = E_o - \dot{E}_o \equiv 0$. The following conditions are imposed on the problem data. The sets $\Omega$ and $\Omega'$ are as described in Section 2. In the conditions, $\psi : R^m \to R^m \times m$ is a $C^1$ - matrix function, arbitrary except that $\Psi(q)$ is nonsingular, $q \in R^m$.

**Condition C.1.** $M, \dot{M}, N, \dot{N}, D \in C^1, E_o, y_d \in C^2$.

**Condition C.2.** $M(q), \dot{M}(q)$ are nonsingular, $q \in R^m$.

**Condition C.3.** $D_1 E_o(q, \tilde{Y})$ is nonsingular, $q \in R^m, \tilde{Y} \in \Omega'$.

**Condition C.4.** There exist $C^1$ - functions $\phi_M : R^m \to R^+, \phi_N : R^m \times R^m \times R^+ \to R^+$ such that

$$||[\Psi(q)]^{-1} [M(q)]^{-1} \Delta M(q)|| \leq \phi_M(q), \quad (3.2)$$

$$||[\Psi(q)]^{-1} [M(q)]^{-1} \Delta N(q, \dot{q}, t)|| \leq \phi_N(q, \dot{q}, t). \quad (3.3)$$

20 Robust Tracking
for all \( q \in \mathbb{R}^n \), \( \dot{q} \in \mathbb{R}^m \), and \( t \in \mathbb{R}^+ \).

**Condition C.5.** Define \( \Gamma_M : \mathbb{R}^n \to \mathbb{R}^{m \times m} \) by

\[
\Gamma_M(q) \triangleq I_m - \frac{1}{2} \left\{ ([\Psi(q)]^{-1} [M(q)]^{-1} \Delta M(q) \Psi(q)) + ([\Psi(q)]^{-1} [M(q)]^{-1} \Delta M(q) \Psi(q))^T \right\}. \tag{3.4}
\]

There exists \( r_s > 0 \) such that

\[
\text{Min.} \left\{ \sigma_m(\Gamma_M(q)) : q \in \mathbb{R}^n \right\} \geq r_s. \tag{3.5}
\]

**Remark 3.1.** Conditions C.4 and C.5 may be difficult to verify because they require knowledge of \( M(q) \), the inertial matrix for the actual manipulator. Suppose there exist a constant \( \hat{r}_s > 0 \) and a \( C^1 \) function \( \phi_N : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+ \) such that

\[
| | |[\Psi(q)]^{-1} [\dot{M}(q)]^{-1} \Delta M(q) \Psi(q)| | | \leq \hat{r}_s, \quad 0 < \hat{r}_s < 0.5, \tag{3.6}
\]

\[
| [\Psi(q)]^{-1} [\dot{M}(q)]^{-1} \Delta N(q, \dot{q}, t) | \leq \phi_N(q, \dot{q}, t). \tag{3.7}
\]

Then, it can be shown that Conditions C.4 and C.5 are satisfied by

\[
\phi_M(q) = (1 - \hat{r}_s)^{-1} \hat{r}_s | | |[\Psi(q)]^{-1}| | |, \quad \phi_N(q, \dot{q}, t) = (1 - \hat{r}_s)^{-1} \phi_N(q, \dot{q}, t),
\]

\[
r_s = (1 - \hat{r}_s)^{-1} (1 - 2\hat{r}_s). \tag{3.8}
\]

While (3.6), (3.7) are more restrictive than C.4, C.5, they are tested easily.

By introducing \( z = (x_1, x_2) = (q, \dot{q}) \), it is possible to reduce the manipulator and its model to systems of the form (1.1), (1.2).
For instance,

\[
\hat{F}(z, t) = \begin{bmatrix} x_2 \\ -[\hat{M}(x_1)]^{-1} \hat{N}(x_1, x_2) \end{bmatrix}, \quad \hat{G}(z, t) = \begin{bmatrix} 0 \\ [\hat{M}(x_1)]^{-1} \end{bmatrix},
\]

(3.9)

\[
\hat{E}(z, t, y_d) = E_e(x_1, y_d).
\]

We now show that if φ is chosen properly, Conditions C.1 - C.5 imply Assumptions A.1 - A.6. Assumption A.1 is obvious. Let \( Y^1 \triangleq (y_d, \dot{y}_d), Y^2 \triangleq (y_d, \ddot{y}_d, \dot{y}_d) \) and define the ith component of \( g(z, Y^2) \) by

\[
g_i(z, Y^2) \triangleq D_1^2 E_n(z_1, y_d)[x_2][z_2] + D_2 E_n(z_1, y_d)[\dot{y}_d] \\
+ 2D_1 D_2 E_n(z_1, y_d)[x_2][\dot{y}_d] + D_2^2 E_n(z_1, y_d)[\dot{y}_d][\dot{y}_d].
\]

(3.10)

Here, the second derivatives \( D_1^2 E_n \) and \( D_1 D_2 E_n \) are, respectively, quadratic and bilinear functions of the increments \( x_2 \) and \( \dot{y}_d \) (see [7]). Note that \( \dot{e} = D_1 E_e(x_1, y_d)[x_2] + g(z, Y^2) \). It is easily verified that A.2 is satisfied by \( r = 2 \) and

\[
g(z, Y^2) \triangleq g(z, Y^2) - D_1 E_e(x_1, y_d)[\hat{M}(x_1)]^{-1} \hat{N}(x_1, x_2),
\]

(3.11)

\[
\beta(x, t, Y^1) \triangleq D_1 E_e(x_1, y_d) [\hat{M}(x_1)]^{-1},
\]

(3.12)

\[
T(z, t, Y^1) \triangleq \begin{bmatrix} T_1(z, t, Y^1) \\ \vdots \\ T_m(z, t, Y^1) \end{bmatrix},
\]

(3.13)

where
\[ T_i(t, z, Y^1) \triangleq \begin{bmatrix} E_{i1}(z_1, y_d) \\ D_1E_i(z_1, y_d)z_2 + D_2E_i(z_1, y_d)y_d \end{bmatrix}, \quad i = 1, \ldots, m, \quad (3.14) \]

\[ A \triangleq \text{diag } A_i, \quad B \triangleq \text{diag } B_i, \quad C \triangleq \text{diag } C_i, \quad (3.15) \]

\[ A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = [1 \ 0], \quad i = 1, \ldots, m. \quad (3.16) \]

Taking

\[ \Delta G^*(z, t, Y^1) \triangleq -D_1E_i(z_1, y_d)[M(z_1)]^{-1}\Delta M(z_1)[D_1E_i(z_1, y_d)]^{-1}, \quad (3.17) \]

\[ \Delta F^*(z, t, Y^1) \triangleq D_1E_i(z_1, y_d)[M(z_1)]^{-1}\{-\Delta N(z_1, z_2, t) + \Delta M(z_1)[\dot{M}(z_1)]^{-1}\dot{N}(z_1, z_2)\} \quad (3.18) \]

verifies A.3. One choice of \( H \), which satisfies the requirement on \( A_H \), is

\[ H \triangleq \text{diag } H_i, \quad H_i \triangleq [-\sigma_i^2 + \omega_i^2 \ 2\sigma_i], \quad \sigma_i < \sigma, \ \omega_i > \sigma, \quad i = 1, \ldots, m. \quad (3.19) \]

For such \( H \), let

\[ U(z_1, z_2, Y^2) \triangleq [D_1E_i(z_1, y_d)]^{-1}\{ \begin{bmatrix} H_1T_i(z, t, Y^1) \\
\vdots \\
\vdots \\
H_mT_m(z, t, Y^1) \end{bmatrix} \begin{bmatrix} -g(z, Y^2) \end{bmatrix}. \quad (3.20) \]

Take
\[ S(z, t, Y^1) \triangleq D_1 E_z (z_1, y_d) \Psi(z_1). \] \hspace{1cm} (3.21)

Then, since

\[ \Phi(z, t, Y^2) = -[\Psi(z_1)]^{-1} [M(z_1)]^{-1} \{ \Delta M(z_1) \ U(x_1, x_2, Y^2) + \Delta N(x_1, x_2, t) \}, \] \hspace{1cm} (3.22)

\[ \Gamma(z, t, Y^1) = \Gamma_M(z_1), \] \hspace{1cm} (3.23)

Assumptions A.4 - A.6 are satisfied by \( \gamma = r_s, \ \delta_S = \sigma \), and

\[ \phi(z, t, Y^2) \triangleq \phi_{MN}(x_1, x_2, t, Y^2) \geq \phi_M(\phi(z_1) [U(x_1, x_2, Y^2)] + \phi_N(x_1, x_2, t). \] \hspace{1cm} (3.24)

Using the above notation in (2.16) yields the controller:

\[ u \triangleq K_1(q, \dot{q}, Y^2(t)) + K_2(q, \dot{q}, t, Y^2(t)), \] \hspace{1cm} (3.25)

where

\[ K_1(q, \dot{q}, Y^2(t)) \triangleq \hat{N}(q, \dot{q}) + \hat{M}(q) U(q, \dot{q}, Y^2(t)). \] \hspace{1cm} (3.26)

\[ K_2(q, \dot{q}, t, Y^2(t)) \triangleq -r_s^{-1} \phi_{MN}(q, \dot{q}, t, Y^2(t)) \hat{M}(q) \Psi(q) \eta(W(q, \dot{q}, t, Y^2(t)), \hspace{1cm} (3.27)

\[ W(q, \dot{q}, t, Y^2(t)) \triangleq r_s^{-1} \phi_{MN}(q, \dot{q}, t, Y^2(t)) [\Psi(q)]^T [D_1 E_z (q, y_d)]^T L(q, \dot{q}, t, Y^1(t)), \] \hspace{1cm} (3.28)
\[
L(q, \dot{q}, t, Y^i(t)) \triangleq \begin{bmatrix}
L_1(q, \dot{q}, t, Y^1(t)) \\
... \\
L_m(q, \dot{q}, t, Y^m(t))
\end{bmatrix}
\]

\[
L_i(q, \dot{q}, t, Y^i(t)) \triangleq \theta_i^2\omega_i^{-2} \{D_1E_{ei}(q, y_d)\dot{q} + D_2E_{eo}(q, y_d)\dot{y}_d - \sigma, E_{ei}(q, y_d)\}, \quad i = 1, \ldots, m.
\]

Here, we have chosen \( P_H \) according to Remark 2.11 with

\[
\bar{P}_H = \text{diag} \bar{P}_{Hi}, \quad \bar{P}_{Hi} = \begin{bmatrix}
1 & 0 \\
\sigma_i & -\omega_i
\end{bmatrix}, \quad i = 1, \ldots, m.
\]

(3.29)

Finally, because of the block diagonal structure of the various matrices, it is easy to show the key data in the statement of Theorem 2.1 are given by

\[
\mu_H = -\max \{\sigma_i, i = 1, \ldots, m\}, \quad \xi_H = \max \{|\theta_i|, i = 1, \ldots, m\}, \quad \gamma = r_s, \quad \delta_E = 0.
\]

(3.30)

The only thing lacking for Theorem 2.1 to hold is Assumption A.7. For \( E_s = y_d - q \) and \( \Omega \) bounded, A.2' is satisfied. Thus, A.7 is true automatically. For most manipulators, the situation \( E_s \not\equiv y_d - q \) leads to a more serious technical difficulty. Condition C.3 holds only for \( q \in Q \), where \( Q \) is an open subset of \( R^n \). For example, when \( E_s = y_d - h(q) \) and \( h(q) \) is the end effector position, it is known [21] that there are degenerate points where the Jacobians of \( h \) fails to exist. In such situations, we must interpret our results as being of a local character. If \( y_d \) is chosen appropriately, the local region may be quite large and the results predicted by Theorem will hold practically.
Remark 3.2. Because of the block diagonal structure of $C^1$ and $P_H$, (2.34) in the proof of Theorem 2.1 may be replaced by $|e_i| \leq \|C, P_H\| |v|, i = 1, ..., m$. This leads to bounds of the type (2.24) and (2.25) on each of the components of $e_i$. For instance, given any $b_i > \delta_H \triangleq (\gamma \mu_H^2) |\theta_i|$, there exists a $\tau_i \geq \delta$ such that $t \geq t_0 + \tau_i$ implies $|e_i(t)| \leq b_i$. Thus, by the choice of the $\theta_i^2$, the ultimate size of each component of the tracking error can be adjusted separately.

Remark 3.3. The choice of $H$ and $S$ has been specialized to simplify various expressions. For example, $H_i$ in (3.19) assigns only complex roots to $A_H$ and the specified form for $S$, (3.21), simplifies the inequalities in C.4 and C.4'. Other specializations and generalizations are treated easily.

Remark 3.4. The obvious choice for $\Psi$ is $I_m$, but $\Psi(q) \not\equiv I_m$ may allow larger uncertainty in $M$ (compare Remark 2.5). More importantly, $\Psi$ may be selected to influence the maximum excursions of $u$ produced by $K_2$ and hence give attention to actuator constraints. For example, suppose $\phi_{MN}(q, \dot{q}, t, Y^2)$ is a constant and $\Psi(q)$ is chosen so that $\dot{M}(q) \Psi(q) = Q$, a constant real matrix. Then by (2.18) and (3.27),

$$K_2(q, \dot{q}, t, Y^2) \in \{u : u = r, \phi_{MN} QW, \ |W| \leq 1\}$$

(3.31)

and the limits on $K_2$ are determined simply by $Q$. For instance, $Q = \text{diag} \ Q_i$ gives $|K_2| \leq r_s \phi_{MN} |Q_i|$. Of course, there is an interaction between $r_s$, $\phi_{MN}$, and the choice of $Q$. Using the same kind of reasoning that led to Remark 2.10, it is seen that there is no loss of generality if $Q$ is restricted so that $||Q|| = 1$. It is also worth noting that the choice $\Psi(q) = [\dot{M}(q)]^{-1} Q$ simplifies the verifications of (3.6), (3.7).
While the expressions and conditions of this section appear somewhat complicated, their special structure (compare with Section 2) gives considerable insight into the parameterization of the robust controller. This is especially true when joint coordinates are controlled and \( E_i = y_i - q \). Many of the functions are then simplified greatly: \( D_i E \equiv -I_m \), the components of \( T_i \) are \( y_i - x_{i1} \), and \( \dot{y}_i - x_{2i} \), \( U_i = -H_i T_i + \ddot{y}_i \), \( L_i = \theta_i^{-2} \omega_i^{-2} (\dot{\epsilon}_i - \sigma_i, \epsilon_i) \). If, in addition, \( M(q) \) and \( \Psi(q) \) are diagonal, \( K_2 \) becomes quite simple. Moreover, the issues raised in Remarks 2.12 – 2.15 are easily quantified and made specific.

4. CONCLUSION

An approach to robust tracking in nonlinear systems has been presented. Several assumptions are needed, the most demanding being A.2 and A.3. Assumption A.2 requires the equivalence of (1.2) and (1.8) under the transformation (1.8), (1.7). The general circumstances under which the equivalence holds have not been pursued here, but for specialized models and applications the equivalence is evident. Assumption A.3 requires that model uncertainties have special structure, similar to the matching conditions which appear in the theory of robust regulators [1, 8, 19]. While this structure appears in some applications, it may be more troublesome to satisfy than A.2, which can be influenced by the choice of the model (1.2). Prior literature on robust regulators [1] suggests that it is possible to relax A.3 by modifying \( K_2 \) and introducing additional, rather complex, assumptions.
The robotic manipulator, (3.1) is an interesting application where A.2 and A.3 are satisfied naturally. The resulting controller can be viewed as an extension of the proportional-derivative controller described in [7], which allows substantial model errors and simplifications in complexity of the controller. Although it has not been pursued in Section 3, more complex manipulator problems can be treated. For example, certain types of actuator dynamics can be incorporated, and it is possible to obtain a robust version of the proportional-integral-derivative controller in [7].

There have been a number of approaches to robust nonlinear tracking. Among them are [2, 13, 25]. Unfortunately, they limit severely the class of inputs and/or involve assumptions which are difficult to verify. Methods for attacking “robust” manipulator problem include adaptive control [5, 6, 12, 15, 18, 27], sliding and suction control [23, 24], and high gain nonlinear feedback [22]. The adaptive control methods lack a complete theory and do not attempt to account fully for nonlinear effects. In [22], a nonlinear feedback based on a singular perturbation techniques [20] is used to obtain a controller of the form (3.25) when \( E_x = y_d - q \). The function \( K_2 \) is different than ours. In order to assure ultimate boundedness of the tracking error, the gain of \( K_2 \) is required to be “sufficiently large”. There is little flexibility in meeting this requirement. There is no saturation function or way of interchanging accuracy for gain in the manner we have described. Thus, the method is more apt to lead to stability problems because of neglected actuator dynamics. In general form, our control law is most similar to the ones proposed in [23, 24]. However, the model assumptions in [23, 24] are more restrictive than ours and the supporting theory is not complete. The saturation function in [23, 24] is an each component of \( K_2 \). This may have practical advantages.
and is probably associated with the more restrictive assumptions about model structure.

5. REFERENCES


