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Axial Temperature Gradient Bending Stresses in Thin-Walled Tubes Adjoining a Heavy Section

F. G. Hammitt

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ABSTRACT

A study is conducted of the bending stresses induced in a thin-walled tube adjacent to a joint between the tube and a heavier section by an axial temperature gradient imposed across the assembly. A solution is shown utilizing basic relations taken from the existing literature for a tube of uniform wall thickness attached at either end to a flange of infinite stiffness. Both long and short tubes are considered.

The possibility of substantial reductions of stress through the use of a tapered wall thickness is discussed. Approximate methods are developed for the calculation of stress and wall thickness under conditions in which the wall thickness is varied in each instance to give: (1) constant radius of curvature (or arbitrary radii of curvature), (2) constant stress, and (3) arbitrary stress distribution.

The possibility of preventing plastic deformation for such structures under stringent operating conditions by the utilization of residual stresses is discussed.

NOMENCLATURE

D = Plate stiffness factor, $(\text{in.})^2(\text{lb})$.

E = Modulus of elasticity, lb/in.^2

μ = Poisson's ratio.

h = Tube wall thickness, in.

y = Dimension measured approximately normal to tube centerline (see Figs. 1 and 3), in.

x = Dimension measured approximately parallel to tube centerline (see Figs. 1 and 3), in.

R = Radius of curvature of a deflected beam, in.

$A, B, C, D, C_2, C_3, C_4, C_5, C_6$ = Constants of integration.

β = See nomenclature of Reference 1. Also constant used to define desired stress distribution. See equation (33).

ℓ = Tube length, in.

σ = Bending stress, lb/in.^2

Q = Heat flow, Btu/hr.

k = Thermal conductivity, $\frac{(\text{Btu})(\text{in.})}{(^\circ\text{F})(\text{in.}^2)(\text{hr})}$.

A = Area, in.^2

ΔT = Temperature difference between ends of tube, °F or °R.

d = Tube diameter, in.

r = Tube radius, in.

α = Temperature coefficient of expansion, $\frac{\text{inches}}{\text{inch-}^\circ\text{F}}$.

C_1 = Constant defined equation (13).

K_1 = Constant defined equation (16).

K_2 = Constant defined equation (21).

p = dy/dx.

e = Base of natural logarithms.

v = Shear, lb.

M = Moment, (in.)(lb).

INTRODUCTION

A major problem in the design of high-temperature equipment is the provision for an analysis of thermal stresses. This problem is of especial importance in heat-exchange equipment, certain types of turbomachinery (as turbines, compressors, and pumps handling high-temperature fluids), in nuclear reactors, furnaces, and numerous other equipment items.

The thermal-stress problems which must be considered involve (1) a radial temperature gradient across a tube wall as in a heat exchanger or across the disc of a rotating machine and (2) an axial temperature gradient which may be encountered in certain types of heat-exchange equipment, in the support casings of pumps and turbines, and in various other applications.

The present treatment is concerned with stresses due to the existence of an axial temperature gradient in a tube, which may be either a portion of a heat exchanger or of the casing or rotor of a turbomachine. In general, stresses of this sort will exist in combination with other types of stresses, as, for example, the direct stress due to pressure loading of a pipe. The total stress and deflection at a point are the result of a summation of their individual components. This paper considers only the components resulting from the thermal conditions.

GENERAL SITUATION

A thin-walled tube under a uniform axial temperature gradient, with unrestrained ends, is not thermally stressed, since it is free to assume the deflection curve dictated by the axial temperature distribution (it is assumed that the axial gradient is symmetrical about the centerline). Under these conditions the deflected tube will become a straight cone. However, if the tube is joined at the ends to members of increased cross section (i.e., flanges), the situation is basically altered. In general, the axial temperature gradient in the flanges will be less than that in the tube, or approximately zero, considering the larger cross-sectional area available to a constant heat flux. Consequently the angle of deflection assumed by the flange will be less than that of the tube. Since tube and flange are securely joined, the deflection curve and its slope must be continuous at this point, and the tube will be forced to accommodate itself approximately to the slope and deflection which would be assumed by the flange in the absence of the tube (due to the presumably much greater stiffness of the flange). Thus, as shown in Fig. 1, there will be bending stresses induced in the tube which are at a maximum at the joint and extend a considerable axial distance into the tube.

CONSTANT WALL THICKNESS CASE

If a tube of constant wall thickness is joined to a flange of relatively infinite stiffness, the tube being subjected to a uniform axial temperature gradient and the flange to zero axial gradient (assuming that a constant heat flux is conducted axially through the flange and tube), an analysis following existing literature[1] shows that the local bending of a thin-walled cylinder (i.e., "inward or outward crimping" of the end for example) may be compared to the case of a beam on an elastic foundation (i.e., the restoring force throughout the length of the beam is proportional to the deflection). This analogy is based on the fact that a thin longitudinal strip of the cylinder may be considered as supported by the neighboring strips (with the proper

1. S. Timoshenko, Strength of Materials (2nd ed.; New York: D. Van Nostrand Co., Inc., 1941) II, pp. 1 ff. and pp. 164 ff.

function of the angle applied)—Fig. 2. The stiffness of the strip is increased by the neighboring strips in the same manner as that of a plate and Timoshenko's factor

$$D = \frac{E\,h^3}{12\,(1 - \mu^2)} \tag{1}$$

used in the plate analysis must be used instead of EI.

The deflection curve for a semi-infinite, thin-walled tube, built into a flanged end, and under axial temperature gradient is shown in Fig. 1. It is assumed that the axis coincides with the deflected position of the tube wall for the case of unrestrained ends. Then for restrained ends, the tube-wall deflection at the end is zero (since the flange and tube are assumed at equal temperature at the joint), but the slope is equal to tan Θ where Θ is the angle of deflection of the unrestrained tube wall. This case may be solved by using Timoshenko's semi-infinite beam on an elastic foundation analysis, and substituting the plate stiffness factor D for EI. The basic differential equation is as follows:

$$D\,\frac{d^4y}{dx^4} = \frac{Ehy}{R^2}\ . \tag{2}$$

The deflection curve (and the bending stress) from this analysis is a damped harmonic curve. For most metallic structural materials including steel, (depending on the values of Poisson's ratio), the stress will be reduced to about 15% of the maximum value where the distance from the tube end is equal to about one-third of the square root of the product of tube diameter and wall thickness. However, the deflection is not reduced to this value until the square root of this product is equal to the distance from the beam end. Thus, if the tube is quite short, it will be necessary to consider both ends. This may be accomplished from Timoshenko's basic analysis, using the principle of superposition [since the differential equation (2) is linear, this procedure is justified]. It is possible to superimpose two infinite beams (the case of the single infinite beam loaded at one point is shown in detail in Timoshenko), one loaded at the point coinciding with one tube end, the other at the other tube end, with a moment and a shear (Fig. 3). Consider X positive to the left and the origin for each beam as shown. Then the region of interest (i.e., the region which coincides with the tube length) of beam No. 1 lies to the left of its origin and that of beam No. 2 to the right. The general solution for the differential equation for each beam separately is

$$y = e^{\beta x} (A \cos \beta x + B \sin \beta x) + e^{-\beta x} (C \cos \beta x + D \sin \beta x) \quad . \qquad (3)$$

Since for both beams the deflection at infinity must be zero, only half of the general solution is applicable for each beam (depending upon whether the region of interest is to the right or left of the origin). It is possible to transpose the solution for one beam into the coordinate system of the other and then add the solutions, giving the resultant deflection for the composite beam.

For beam No. 1,

$$y_1 = e^{\beta x_1} (C \cos \beta x_1 + D \sin \beta x_1) \quad . \qquad (4)$$

For beam No. 2,

$$y_2 = e^{\beta x_2} (A \cos \beta x_2 + B \sin \beta x_2) \quad . \qquad (5)$$

Set $x_2 = x_1 - \ell$, then

$$y = y_1 + y_2 = e^{\beta x_1} (C \cos \beta x_1 + D \sin \beta x_1) + e^{\beta(x_1-\ell)}[A \cos \beta(x_1-\ell) + B \sin \beta(x_1-\ell)] \quad . \qquad (6)$$

This expression contains four arbitrary constants and is subject to four boundary conditions; i.e., the deflection must be zero and the angle of deflection that of an unrestrained beam at each tube end. There are then four equations for the four constants. It is possible to calculate the moment from the second derivative equation and, hence, the stress at any point.

DESIGN METHODS FOR REDUCING STRESS

Where the axial temperature gradient is severe and the material is seriously weakened by elevated temperature, it may be found that the bending stresses, as calculated above, for the case of constant wall thickness, are too great for a successful design. Several possible approaches have been considered for the designing and calculating of structures consistent with reasonable stresses under the most stringent conditions. These methods involve a deviation from the concept of constant tube-wall thickness.

1. CONSTANT RADIUS OF CURVATURE AND POSSIBLE VARIATIONS

The bending stress in any beam of constant thickness is inversely proportional to the radius of curvature of the beam under deflection. Thus it would be desirable if this radius could be maximized at all points (if the resultant benefits were not sacrificed by too large an increase in wall thickness). Consider a tube built into a much heavier section (with consequent reduced axial temperature gradient and slope of deflection curve in the heavier section). The wall of such a tube will exhibit virtually zero slope of deflection curve at each end. As shown in Fig. 4, a maximum radius of curvature may be obtained at all points if the deflection curve is caused to follow a constant radius with an inflection point at midspan. To calculate this radius, it is first necessary to compute the change in tube radius Δr from the known temperatures at the tube ends and the properties of the material. The radius of curvature then may be easily found from geometric considerations (Fig. 4). If it is desired that the stress level be higher at one end of the tube than at the other, because of a difference in allowable stress due to different temperatures, it is possible in the same way to use a shorter radius at the high-stress end.

It is possible to design the tube-wall thickness in such a way that the natural deflection curve under temperature will follow the assumed constant radius curve. A rough estimate shows that substantial deviation from this temperature-controlled deflection curve is impossible because of the large restoring hoop stresses that must result. The necessary wall thickness may be calculated for each point from the consideration of continuity of heat flow. This may be accomplished graphically using the standard procedure of a heat-flow net. It may also be approximated by dividing the tube length into a number of segments and calculating the required mean thickness of each segment to pass the total heat flow, considering the temperature drop between segment medians as approximated from the assumed deflection curve—deflection being directly proportional to temperature. If desired, corrections for radiation and convection losses from the wall may be included. Remembering that the bending stress for a given moment is proportional to the thickness, the stress may be computed for any point. The relation between radius of curvature, wall thickness, and stress is

$$\sigma = \frac{Eh}{2R\,(1 - \mu^2)} \quad . \tag{7}$$

If the wall thicknesses are computed as suggested above, the actual deflection curve will not follow the assumed curve exactly since the bending stresses are in a direction to reduce the curvature. Thus the actual stresses will be less than those calculated from this method. The wall thickness resulting from calculations of this type will form a double taper with the minimum thickness at the center. Therefore, the stresses will be a maximum at the ends and minimum at the midpoint (where a weld might be advantageously located). Although the increased section at the ends is desirable, it may not be useful in some cases to utilize as thick a section as the above method would indicate because of the increase of stress with thickness.

If it is desired to change the stress distribution, the following procedure might be used. A new deflection curve can be assumed deviating in the desired direction. If this curve can be expressed as an equation, the radius of curvature can be calculated at any point from the second derivative. Otherwise, the curve can be plotted to enlarged vertical scale and the local radius of curvature found in any region from consideration of three points in the close vicinity and the application of the relations shown in Fig. 5. Once the deflection curve is assumed, the wall thicknesses can be computed as suggested for the constant radius case. Then it is possible to calculate the bending stresses by equation (7). To achieve the desired stress distribution in this way is a trial-and-error process. It is also possible to assume a wall-thickness distribution, compute deflection from temperature, and then check stresses by the method outlined above.

2. CONSTANT-STRESS DESIGN AND VARIATIONS

Consider again the case of a tube under axial temperature gradient built into relatively heavy end sections, where the temperature gradient is controlled by heat conduction axially through the wall.

For a given minimum and/or maximum wall thickness, that distribution of wall thickness which gives a constant stress throughout, or over a reasonable portion of the length, may be desired. On the other hand, it may be desirable in some cases to allow a higher stress at one end than the other because of temperature weakening effects on the material. Considering the first alternative, it will be noted from equation (7) that the stress is constant if h/R is constant. This condition leads to the differential equation and solution given below.

$$\sigma = \frac{Eh}{2R\ (1-\mu^2)} = \frac{Eh}{2\ (1-\mu^2)} \frac{d^2y}{dx^2} \ . \tag{8}$$

From the heat-flow equation,

$$Q = \frac{kA\ dT}{dx} \quad \text{and} \quad A = \pi dh \ . \tag{9}$$

Then

$$\frac{dx}{dT} = \frac{k\pi dh}{Q} \tag{10}$$

and

$$\frac{dy}{dT} = \alpha \frac{d}{2}, \quad \text{so that} \quad \frac{dx}{dy} = \frac{2}{\alpha d} \frac{k\pi d}{Q} h \tag{11}$$

or

$$h = \frac{dx}{dy} \bigg/ \frac{2k\pi}{\alpha Q} \ . \tag{12}$$

Let

$$C_1 = \frac{2k\pi}{\alpha Q}, \quad \text{so} \quad h = \frac{1}{C_1} \frac{dx}{dy} \ . \tag{13}$$

Substitute (13) into (8) to give

$$\frac{1}{C_1} \frac{d^2y}{dx^2} \frac{dx}{dy} \frac{E}{2\ (1-\mu^2)} = \sigma \ . \tag{14}$$

or

$$\frac{d^2y}{dx^2} - \frac{2\sigma\ C_1\ (1-\mu^2)}{E} \frac{dy}{dx} = 0 \ . \tag{15}$$

Let

$$K_1 = \frac{2\sigma\ (1-\mu^2)\ C_1}{E} \tag{16}$$

so that

$$\frac{d^2y}{dx^2} - k_1 \frac{dy}{dx} = 0 \ . \tag{17}$$

The solution to this second-order, linear, homogeneous, constant-coefficient, differential equation may be written

$$y = C_2 e^{K_1 x} + C_3 \quad . \tag{18}$$

Assume, for example, the following boundary conditions:

a) at $x = 0$, $y = 0$ b) at $x = \ell/2$, $y = \Delta r/2$.

The x-axis is considered to be parallel to the centerline of the tube and to pass through the tube wall at one end (Fig. 6). The second boundary condition implies that the deflection curve is to be symmetrical about the midspan point which is then a point of inflection. However, the equation here considered covers only half the length. For the total solution, a similar equation with the origin at the opposite end would be considered.

If these boundary conditions are applied, (18) becomes

$$y = \frac{\Delta r}{2(e^{K_1 \ell/2} - 1)} \left(e^{K_1 x} - 1\right) \quad . \tag{19}$$

The expression $\Delta r = \alpha \frac{d}{2} \Delta T$. Substitute this and equation (16) into equation (19); substituting also for C_1:

$$y = \frac{\alpha d \ \Delta T}{4\left(e^{\frac{2\sigma(1-\mu^2)\ k\pi\ell}{\alpha QE}} - 1\right)} \left[e^{\frac{4\sigma(1-\mu^2)\ k\pi x}{\alpha QE}} - 1\right] \quad . \tag{20}$$

Let

$$K_2 = \frac{2(1-\mu^2)\ k\pi}{\alpha E} \tag{21}$$

so that (20) becomes

$$y = \frac{\alpha d \ \Delta T}{4\left[e^{K_2 \frac{\sigma \ell}{Q}} - 1\right]} \left[e^{\frac{2K_2 \ \sigma x}{Q}} - 1\right] \quad . \tag{22}$$

Differentiating (22) we may write

$$\frac{dy}{dx} = \frac{\alpha d \ \Delta T \ K_2 \ \sigma}{2\left[e^{K_2 \frac{\sigma \ell}{Q}} - 1\right]} \ e^{\frac{2K_2 \ \sigma x}{Q}} \tag{23}$$

and then substituting (23) into (13)

$$h = \frac{\alpha Q}{2k\pi} \; \frac{2\left[e^{K_2 \frac{\sigma \ell}{Q}} - 1\right]}{\alpha d \; \Delta T \; K_2 \; \sigma} \; e^{\frac{-2K_2 \; \sigma x}{Q}} \qquad (24)$$

$$h = \frac{Q\left[e^{K_2 \frac{\sigma \ell}{Q}} - 1\right]}{k\pi d \; \Delta T \; K_2 \; \sigma} \; e^{\frac{-2K_2 \; \sigma x}{Q}} \; . \qquad (25)$$

Thus we have a solution showing an exponentially decreasing wall thickness up to the midpoint. Considering the other half of the tube length as a mirror image, the wall thickness will then increase in the same way.

To solve any particular case, consider equations (21) and (25). The steps in the solution are:

a. Evaluate K_2 from (21) from the material characteristics.

b. Select an allowable stress limit, σ, and evaluate h from (25) as a function of Q.

c. From physical conditions related either to the required thickness at the flange end for bolting, etc., or to the minimum allowable thickness at midspan governed by fabrication, pressure, etc., select h. Solve for Q.

d. If h at any point or Q are inacceptable for other reasons, re-examine σ and repeat.

In this way it will become apparent whether or not a constant stress solution is possible consistent with the other requirements.

VARYING STRESS

Rather than having a constant stress, it may be desirable that the stress vary in a predetermined manner along the tube axis. For example, it may be required that stress should increase as a function of distance from the high-temperature end (since allowable stress is an inverse function of the temperature). In that case, it is necessary to set equation (8) equal to f(x) rather than a constant σ. This leads to the following derivation:

(8) becomes
$$\frac{Eh}{2(1-\mu^2)} \frac{d^2y}{dx^2} = f(x) \tag{26}$$

and (15) may be written

$$\frac{d^2y}{dx^2} - \frac{2f(x)\ C_1\ (1-\mu^2)}{E} \frac{dy}{dx} = 0 \tag{27}$$

or

$$\frac{d^2y}{dx^2} - \frac{2K_2}{Q} f(x) \frac{dy}{dx} = 0 \ . \tag{28}$$

Call $dy/dx = p$ so that

$$\frac{dp}{dx} - \frac{2K_2}{Q} pf(x) = 0 \tag{29}$$

and

$$\frac{dp}{p} = \frac{2K_2}{Q} f(x)\ dx \ , \tag{30}$$

so

$$C_4\ e^{\frac{2K_2}{Q}\int_0^x f(x)\ dx} = \frac{dy}{dx} \tag{31}$$

or

$$y = C_4 \int e^{\frac{2K_2}{Q}\int_0^x f(x)\ dx}\ dx + C_5 \ . \tag{32}$$

If the boundary conditions used in the case of constant stress are assumed again, the condition that for $x = 0$, $y = 0$ gives $C_5 = 0$.

C_4 must be evaluated after integration. Depending on the assumption for the desired stress function, $f(x)$, this integration may or may not be graphical. With C_4 evaluated it is possible to substitute (32) into (13) to give the necessary expressions for h:

$$h = \frac{\alpha Q}{2k\pi\ C_4}\ e^{\frac{-2K_2}{Q}\int_0^x f(x)\ dx} \ . \tag{33}$$

If for example $f(x) = \sigma_o + \beta x$,

$$h = \frac{\alpha Q}{2k\pi \; C_4} \; e^{\frac{-2K_2}{Q} (\sigma_0 x + \beta \; x^2/x)} \quad ; \qquad (34)$$

(33) or (34) is analogous to (25) in the constant stress case. For a given solution it is necessary only to consider (21) and (33) and proceed in the manner previously outlined.

ALTERNATIVE APPROACHES

Whatever design method is used, it is always possible to use "locked-up" residual stresses to prevent excessive stresses (or plastic deformation) in the operating condition. In other words, if the component is prestressed to near yield in the direction opposite to that expected upon heating, bending stresses of approximately the sum of the yield stresses in the normal and elevated temperature conditions may be allowed without plastic deformation. This may be accomplished by proper application of a weld, by mechanical stressing, or simply by subjecting the component to conditions more severe than the normal operating condition and relying upon the resultant plastic deformation on this first cycle.

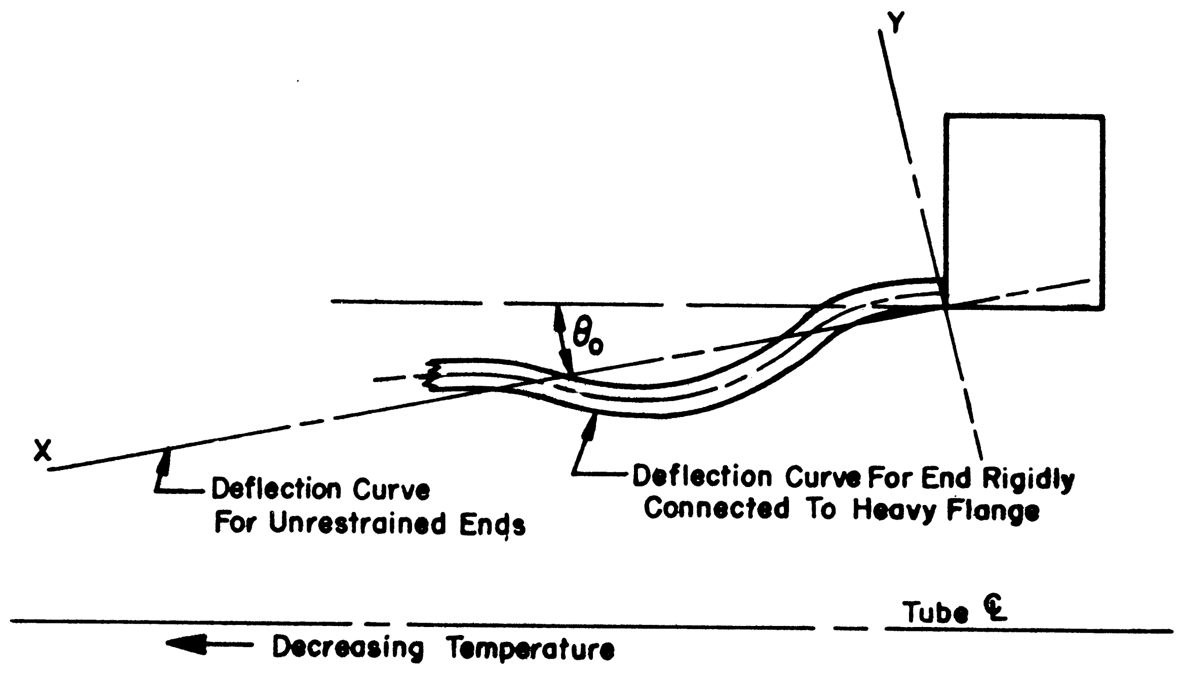


Fig. 1. Tube under uniform axial temperature gradient.

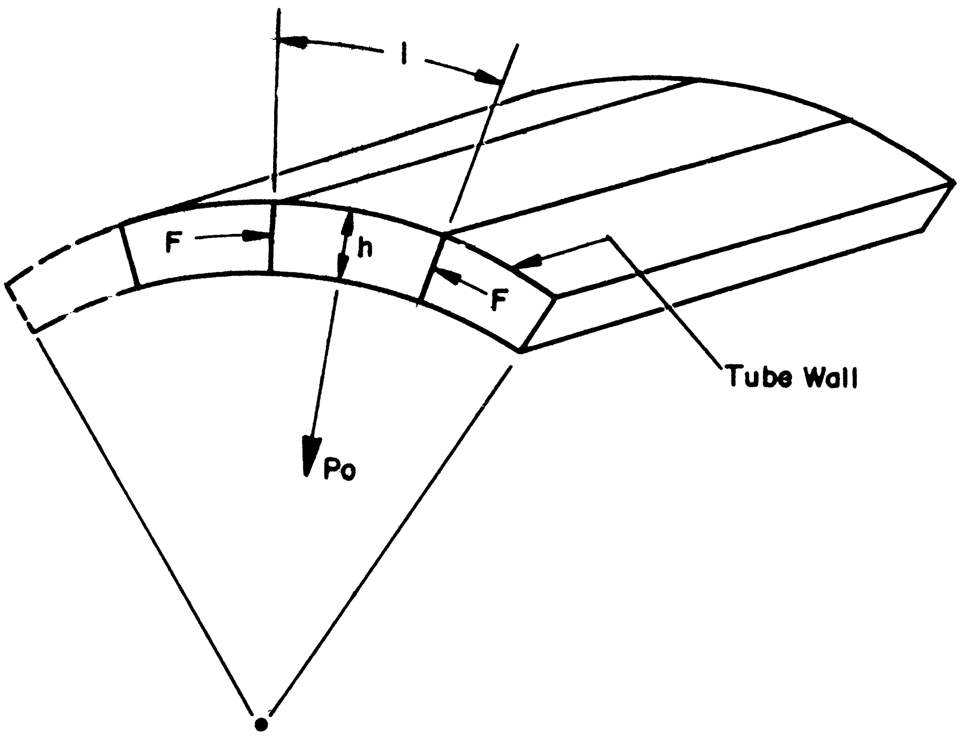


Fig. 2. Cylinder segment as beam on elastic foundation.

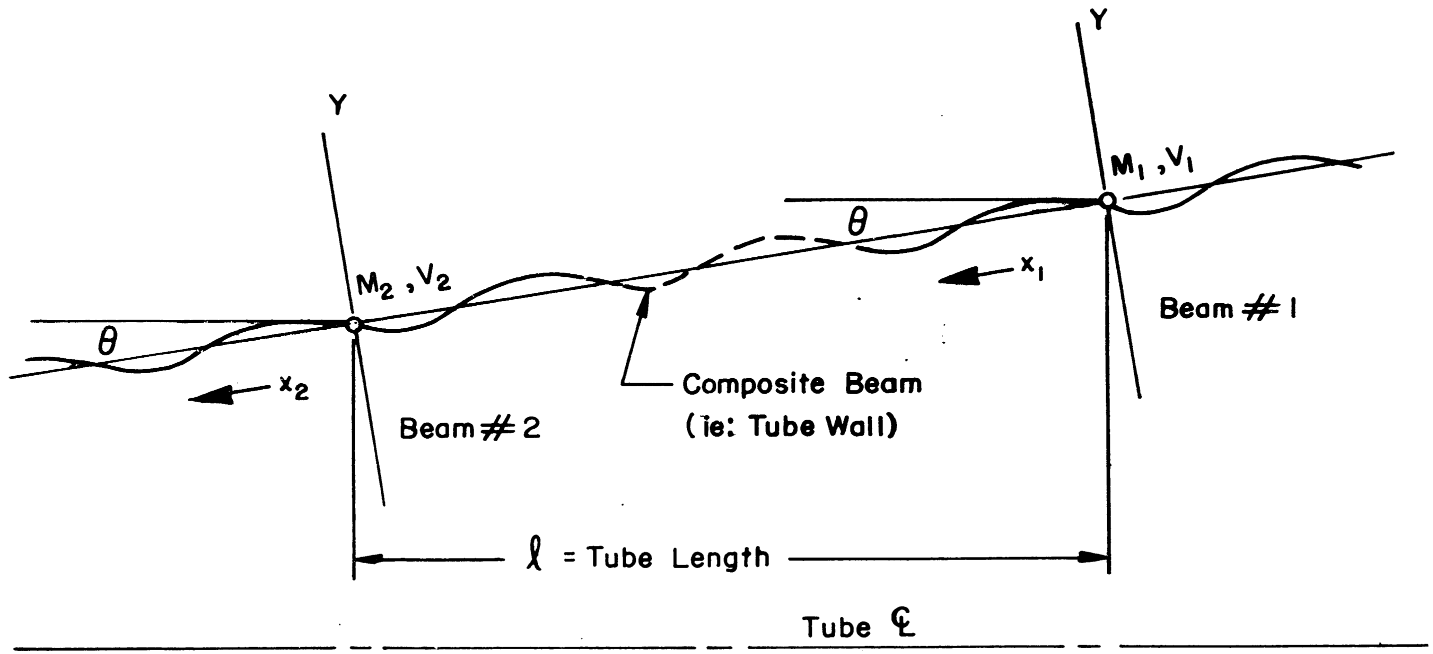
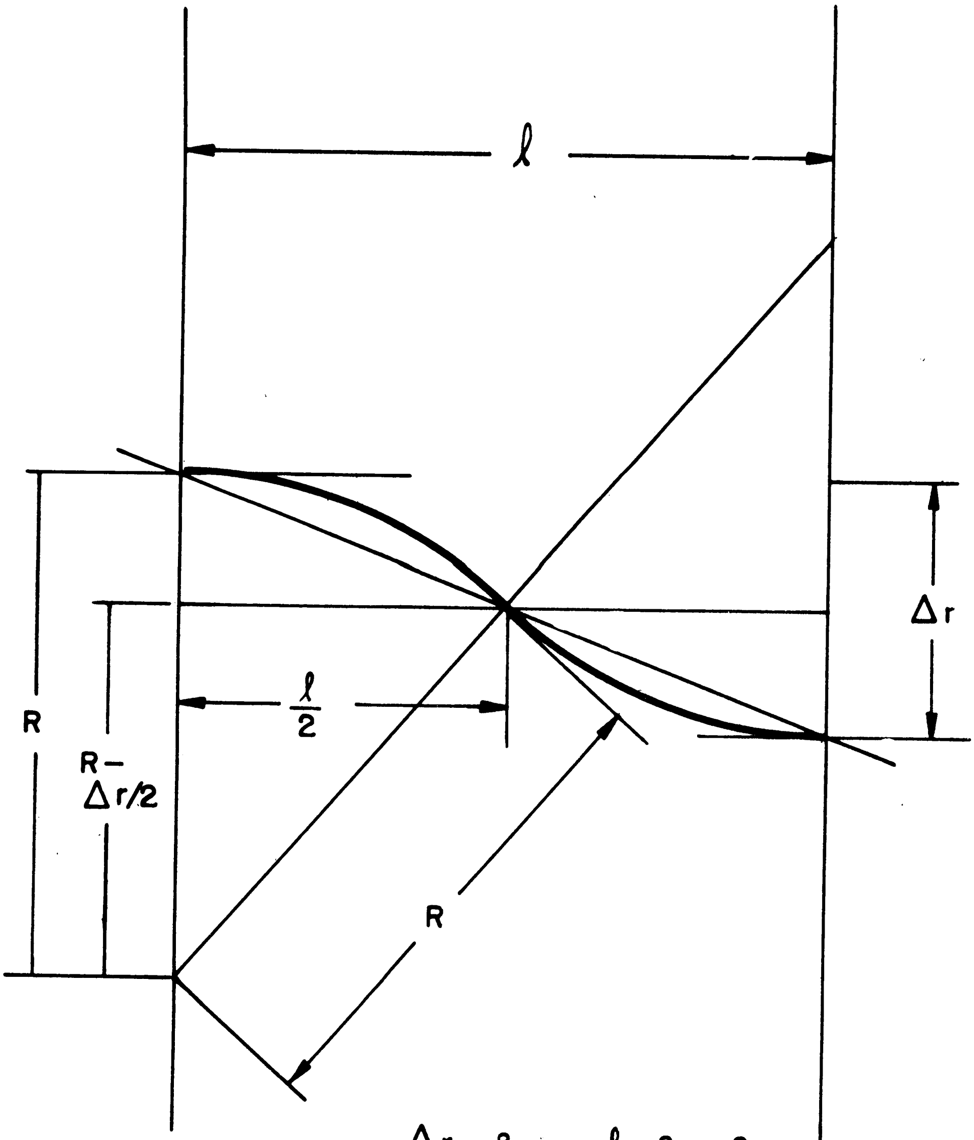


Fig. 3. Tube under axial temperature gradient as composite beam on elastic foundation.

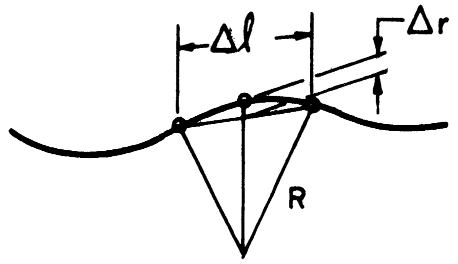


$$\left(R - \frac{\Delta r}{2}\right)^2 + \left(\frac{\ell}{2}\right)^2 = R^2$$

OR

$$R = \frac{(\Delta r)^2 + \ell^2}{4\Delta r} \cong \frac{\ell^2}{4\Delta r}$$

Fig. 4. Constant radius of curvature case.



$$R^2 = \left(\frac{\Delta \ell}{2}\right)^2 + (R - \Delta r)^2$$

OR

$$R \doteq \frac{(\Delta \ell)^2 + 4(\Delta r)^2}{8 \Delta r} \cong \frac{(\Delta \ell)^2}{8 \Delta r}$$

Fig. 5. Local radius of curvature.

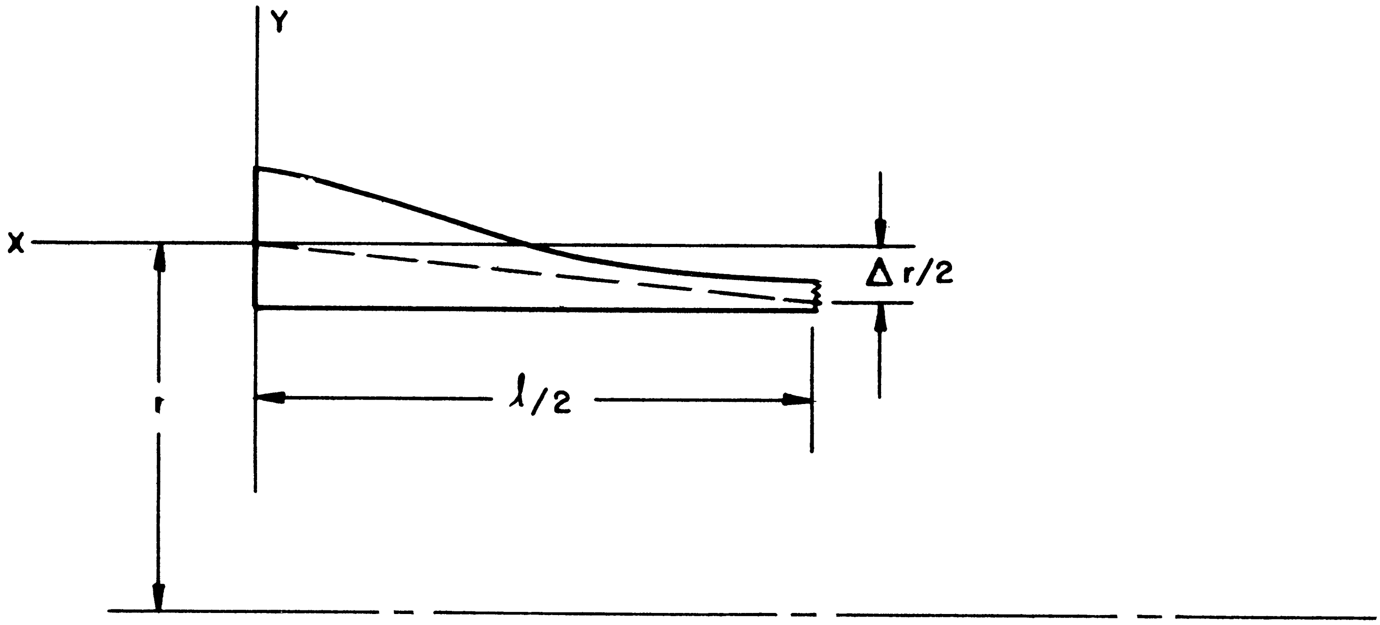


Fig. 6. Constant stress solution (schematic representation).