THE UNIVERSITY OF MICHIGAN
INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

MODIFIED BOUNDARY LAYER TYPE SOLUTION FOR FREE CONVECTION FLOW IN VERTICAL CLOSED TUBE WITH ARBITRARILY DISTRIBUTED INTERNAL HEAT SOURCE AND WALL TEMPERATURE

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1.0 INTRODUCTION

The problem of natural convection heat and mass transfer within closed vessels wherein the heat is generated internal to the fluid and removed through coolant passages surrounding or within the fluid vessel has assumed increasing importance of late in connection with homogeneous nuclear reactors. The problem of natural convection flow without internal heat sources in a tubular vessel with a single closed end (opposite end connected to an infinite reservoir) which is also of moderate length to diameter ratio has been considered in some detail by Lighthill[1] for the case with constant wall temperature. The case of the similarity regime (defined later) in a vertical tubular vessel, closed at one end and connected with an infinite reservoir at the other as in the Lighthill case, but with a linear, axially-varying wall temperature, is examined by Ostrach.[2]

It is the purpose of this paper to present a method of solution for the case of a vertical tubular vessel with both ends closed in which heat is generated in an arbitrary axial distribution of heat source strength and removed through walls which are under arbitrary temperature distribution. This case has been examined because of the application to homogeneous nuclear reactors. Since the heat source term in a power reactor is of an order of magnitude higher than conventional sources, it has been assumed that a modified boundary layer solution may be applied. It was pointed out by Lighthill[1] that the internal flow might be expected to fall within one of the following regimes, depending upon a parameter which is the product of the Rayleigh Number and radius to length ratio for the tube:

1. Similarity Regime. If the above parameter is small, i.e., Rayleigh Number small or tube slender so that the temperature and velocity profiles are fully developed and their shapes (not magnitudes) do not vary with axial position. In other words, the boundary layer completely fills the tube.

2. Intermediate Regime. The boundary layer fills a substantial portion of the tube radius but the "fully developed" regime is not attained.

3. Boundary Layer Regime. If the above parameter is very large, i.e., large Rayleigh Number or short tube, the boundary layer does not have sufficient axial extent to grow into the central portion of the tube and thus occupies only a negligible portion of the radius. In the extreme case, the flow is completely analogous to a flat plate in an infinite fluid. However, the type of solution may be extended somewhat into the "intermediate regime" by utilizing partially arbitrary assumptions of temperature and velocity profile for the boundary layer and the central core.

It is this latter procedure which is extended in this paper to the case with both ends closed, internal heat source, arbitrary axial wall temperature distribution. As stated by Lighthill[1] the applicability of this type of solution depends upon the product of Rayleigh Number and radius to length ratio being large. In terms of the present analysis, some of the following factors must apply in varying degree if the modified boundary layer assumptions are to be used.

1. Absolute tube dimensions large, or
2. Heat source strong, or

3. Length to radius ratio of tube small, or

4. Thermal diffusivity and kinematic viscosity of fluid small.

If the application is to be homogeneous nuclear power reactors, the heat source may be assumed strong, but the passage dimensions and length to radius ratio may be small. Also, if the fluid is a liquid metal, the thermal diffusivity may be relatively very large. However, as will be seen later, the assumptions which are made appear to limit the type of solution to fluids with a Prandtl Number of the order of unity. It has been assumed for the purposes of this paper that the heat source term for aqueous homogeneous power reactors is sufficiently large so that the modified boundary layer type solution herein described is of interest. The analysis presented applies only for laminar flow. It is realized that turbulence may well exist for many applicable cases. However, the laminar analysis should be of interest in at least indicating the proper trends.

Although the present analysis appears limited to aqueous fluids, the trends which are indicated should apply also to some extent to liquid metals.

It is hoped that this paper may be followed by others in the near future reporting the results of machine calculations for a variety of applicable boundary conditions using the method of solution herein presented, and also experimental data to be derived from a program presently underway.

2.0 ANALYSIS OF PROBLEM

2.1 Lighthill Approach

As previously mentioned, the Lighthill analysis considered a vertical tube, closed at one end and open at the other to an infinite fluid reservoir. The tube walls were maintained at a constant temperature, different from the reservoir temperature. According to the analysis, if the product of Rayleigh Number and radius to length ratio is sufficiently large, the flow is of the boundary layer type. If it is somewhat smaller, it is necessary to consider the effect of the return velocity in the central portion of the tube. Under these conditions, the return flow in the core will be at constant temperature since the influence of thermal conduction between the core and the boundary layer is small.

To analyze this situation, it is assumed that

1. The boundary layer approximations apply.
2. Fluid inertial forces are small compared to bouyancy and shear. As Lighthill shows, this condition results if the Prandtl Number is large. (Thus, direct application to liquid metals appears unlikely.)
3. The radial extent of the temperature and velocity boundary layers is the same.

Further, a velocity and temperature profile as shown in Figure 1 are assumed. These are caused to satisfy the physical boundary conditions at the wall, the centerline, and $r = \beta$ for the quantities and their first derivatives. Integral equations for the conservation of mass, momentum, and energy are written for each radial disc (Figure 2); i.e., these quantities are not satisfied point by point but on an integrated basis.

The assumed velocity and temperature profiles, in non-dimensional form are as below. The meanings of the symbols are given in Nomenclature. The parameters β, γ, and δ are functions of the axial position, x. However, the non-dimensional core temperature, t_1*, is constant with x.

$$u = \begin{cases} -\gamma \text{ for } 0 < \beta < r \\ -\gamma\left[1 - \left(\frac{r - \beta}{1 - \beta}\right)^2 \left\{1 + \delta(r - 1)\right\}\right] \text{ for } \beta < r < 1 \end{cases} \tag{1}$$

$$t = \begin{cases} t_1 \text{ for } 0 < r < \beta \\ t_1\left[1 - \left(\frac{r - \beta}{1 - \beta}\right)^2\right] \text{ for } \beta < r < 1 \end{cases} \tag{2}$$

2.2 Modification for Volumetric Heat Source and Arbitrary Axial Wall Temperature Distribution

The procedure to include the effect of the volume heat source and the arbitrary wall temperature distribution is similar to that of Lighthill except that the energy equation is modified to include the heat source term, and the assumed velocity and temperature profiles modified to allow a variation of core temperature in the axial direction. Under these conditions, the assumed profiles are:

$$u = \begin{cases} -\gamma \text{ for } 0 < \beta < r \\ -\gamma\left[1 - \left(\frac{r - \beta}{1 - \beta}\right)^2 \left\{1 + \delta(r - 1)\right\}\right] \text{ for } \beta < r < 1 \end{cases} \tag{1'}$$

*Non-dimensional temperature, t, is actually the product of the Rayleigh Number based on radius and the radius to length ratio. See Nomenclature.

which is unchanged from the Lighthill relation (1) and

$$t = \begin{cases} t(x) & \text{for } 0 < \beta < r \\ t(x)\left[1 - \left(\frac{r - \beta}{1 - \beta}\right)^2\right] & \text{for } \beta < r < 1 \end{cases} \qquad (2')$$

The maximum fluid temperature will be attained at the top of the vessel in the core. In the non-dimensional quantities, this will be $t'_o + t_{w_o}$, defined in terms of the temperature difference between fluid at center-line and wall at top plus the temperature difference between the top and bottom of the wall. (See Figure 2 for illustration.)

As given by Lighthill, the integrated forms of the equations expressing conservation of mass and momentum are:

$$\int_0^1 r\ u\ dr = 0 \qquad (3)$$

$$\int_0^1 r\ t\ dr + \frac{1}{2}(t)_{r=0} + \left(\frac{\partial u}{\partial r}\right)_{r=1} = 0 \qquad (4)$$

The integrated form of the energy relation, broadened over Lighthill's case to include the heat source term, is:

$$\frac{\partial}{\partial x}\int_0^1 r\ u\ t\ dr = \left(\frac{\partial t}{\partial r}\right)_{r=1} - \frac{q_v}{2} \qquad (5)$$

where q_v is a non-dimensional heat source term which is proportional to the product of the local volumetric heat source and a grouping of the physical fluid quantities. The precise definition is given in the Nomenclature.

The assumed velocity and temperature profiles, (1') and (2'), are substituted into (3) and (4) as in the Lighthill analysis. Equation (3) yields δ as $\delta(\beta)$ exactly as in the Lighthill case. Equation (4), using the relation for δ from (3), yields γ as $\gamma(t, \beta)$. The expression is the same as that of Lighthill except that t_1 of Lighthill becomes $t(x)$. Thus,

$$\gamma = t(x) \frac{(3 + \beta)(3 + 2\beta)(1 - \beta)^3}{36(3 + 4\beta + 3\beta^2)} = t(x)G(\beta) \tag{6}$$

The relations (1'), (2'), and (6) are substituted into the energy relation (5), following the Lighthill procedure. The right side of the resultant relation will differ from Lighthill's case because of the presence of the heat source term. The left side, however, will also differ. The term in question is

$$\frac{\partial}{\partial x} \int_0^1 r\, u\, t\, dr$$

In the Lighthill analysis, t was a function of β multiplied by a constant, t_1. Now, however, the multiplier temperature is a function of x.

As shown in the Appendix

$$\frac{\partial}{\partial x} \int_0^1 r\, u\, t\, dr = 2 \frac{d}{dx}[t^2(x)F(\beta)] \tag{7}$$

where $F(\beta)$ is as evaluated by Lighthill and is given in the Appendix. Then the whole energy relation, integrated in an axial direction, is:

$$\int_{x_{N-1}}^{x_N} d[t^2(x)F(\beta)] = \int_{x_{N-1}}^{x_N} \frac{t(x)}{1-\beta}\, dx - \frac{q_v}{4} \int_{x_{N-1}}^{x_N} dx \tag{8}$$

$$\underbrace{[t^2(x)F(\beta)]_{x_{N-1}}^{x_N}}_{\text{Convection}} = \underbrace{\int_{x_{N-1}}^{x_N} \frac{t(x)}{1-\beta}\, dx}_{\text{Wall Conduction}} - \underbrace{\frac{q_v(x_N - x_{N-1})}{4}}_{\text{Heat Source}} \tag{9}$$

Equation (9) is essentially the energy relation for the radial disc shown in Figure 2. However, the relations for conservation of mass and momentum have also been used in the derivation. There are two independent variables in (9), t and x. Consequently, it is necessary to formulate an additional relationship in order to allow a solution.

The addition relation which has been used is that between axial position and the fluid temperature along the vessel centerline. Consider the control volume sketched in Figure 3. It is assumed that fluid transfer from the core to the boundary layer has only negligible effect on energy conservation for the core. It is also assumed that thermal conduction to or from the control volume is negligible. Order of magnitude calculations for those cases to which this type of modified boundary layer solution may be applied show that these assumptions are justified. Conservation of energy for the control volume gives:

$$Q_v[\Delta x \pi (\beta R)^2] = C_v \rho \pi (\beta R)^2 \Delta x \left(- \frac{dT_c}{dx}\right) U_c \tag{10}$$

so that

$$\frac{dT_c}{dx} = - \frac{Q_v}{\rho C_v U_c} \tag{11}$$

or in the non-dimensional quantities (see Appendix)

$$\frac{dt(x)}{dx} = \frac{q_v}{\gamma} \tag{12}$$

Thus,

$$[t(x)]_{x_{N-1}}^{x_N} = \frac{q_v}{\gamma} (x_N - x_{N-1}) \tag{13}$$

The basic set of equations to be solved are then (9) and (13).

2.3 Difference Equations for Arbitrary Wall Temperature and Heat Source Axial Distribution

Rather than attempting to find analytic solutions for various specialized conditions of the boundary value problem it was considered more useful and practical to utilize an approximate numerical procedure and maintain a completely arbitrary specification of the axial distribution of wall temperature and heat source. The numerical procedure was designed for programming into a high-speed digital computer.

It is evident that the derivation for Equations (9) and (13) has considered a constant wall temperature where the t variable is related to the temperature difference between the fluid at any axial and radial position and the tube wall at that axial position. A change in variables is now made so that $t = t' + t_w$, where t' is related to the temperature difference between wall and fluid at any axial position as was t, and t_w is related to the temperature difference between the wall at any axial position and the wall at the bottom. This latter location represents the minimum temperature in the system.

As previously mentioned, Equations (9) and (13) are written for any of a series of radial discs (Figure 2) which together comprise the entire vessel. If a sufficient number of such discs are employed, it is permissible to consider the wall temperature constant for each disc. Then, Equation (9), which is basically the energy balance for a disc, does not involve the axial wall temperature gradient and is the same whether variable or constant wall temperature is considered. In other words, the t_w portion of t makes no contribution.

On the other hand, Equation (13), expressing the fluid axial temperature distribution along the centerline, must consider the fluid temperature as such, including both portions which make up the total, rather than just the temperature difference between the fluid and wall at a given axial position. The heat source term may be considered as a mean value for each disc, q_{v_N}. Equation (9) is unchanged except that t' is used for t, and (13) becomes (15) below.

$$[t'^2(x)F(\beta)]_{x_{N-1}}^{x_N} = \int_{x_{N-1}}^{x_N} \frac{t(x)}{1-\beta}\, dx - \frac{q_{v_N}(x_N - x_{N-1})}{4} \qquad (14)$$

$$[t'(x) + t_w(x)]_{x_{N-1}}^{x_N} = \frac{\overline{q}_v(x_N - x_{N-1})}{\overline{\gamma}} \qquad (15)$$

As is shown in detail in the Appendix, these may be reduced to the approximate difference equations listed below.

$$\frac{t_N'^2F_N - t_{N-1}'^2F_{N-1}}{x_N - x_{N-1}} - \frac{t_N' + t_{N-1}'}{2-\beta_N-\beta_{N-1}} + \frac{q_{v_N}}{4} = \Delta \qquad (16)$$

$$(t' + t_w)_N = (t' + t_w)_{N-1} \sqrt{1 - \frac{4q_{v_N}(x_N - x_{N-1})}{(t' + t_w)^2_{N-1}(G_N + G_{N-1})}} \quad (17)$$

For a consistent solution, $\Delta \to 0$.

2.4 Boundary Conditions

The boundary conditions of zero slip on the walls, walls impervious to the fluid, wall temperature equal to the fluid temperature adjacent to the wall, heat flux through the wall equal to the conductive heat flux normal to the wall within the fluid and insulated end sections* are all implicit in the basic equations used and the assumed temperature and velocity profiles. The remaining conditions to be applied are concerned with the end effects. Physically, these may be stated in various ways. For example, if the tube as an entirety is considered, then the overall heat convection term in Equation (9) must vanish, and the total heat source equal the total wall conduction. From another viewpoint, the mixed mean boundary layer effluent temperature at the bottom must equal the mixed mean temperature of the ascending core at that point. Also at the top, the mixed mean temperature of the core effluent must equal that of the boundary layer.

The latter statement can be satisfied with the assumed temperature profile only if

1. there is no radial temperature gradient at the tube ends, or
2. the boundary layer thickness at the ends is zero.

In this last case, assuming non-infinite velocities, the core and boundary layer flows at the ends must be zero. (The postulated

*It is possible to consider heat flow through the tube ends if desired by simply selecting a suitable q_v for each extreme discs.

over-simplified flow model is not capable of an examination of the detailed end conditions.) Since there is a temperature difference between the fluid at the tube centerline and the wall at the top, it is necessary that the boundary layer thickness at the top be zero. This is physically necessary in any case from consideration of the basic boundary layer flow mechanism.

Examination of Equation (9) and the form of $F(\beta)$ given in the Appendix, shows that when β is 1.0 (i.e., zero boundary layer thickness), F is zero, and hence t^2F is zero. Since it is necessary that the overall convective term vanish for the tube, t^2F at the bottom must be also zero. This is accomplished if either the boundary layer thickness or the radial temperature gradient becomes zero at the bottom. As the calculations show, the specification of either of these conditions carries with it the other. In this manner, all the boundary conditions are satisfied.

2.5 Calculating Procedure

The calculating procedure is a double iteration designed to satisfy the condition of zero boundary layer thickness at either end. Physically, if the wall temperature distribution and heat source are specified, the solution to the problem must be unique (assuming given physical properties of the fluid). Assuming that these quantities are specified, each radial disc, starting at the top, is considered separately. An initial estimate of t'_o at the top is made. It is then possible to compute, for an estimated β, the value of t' at the bottom of the first disc from Equation (17). These values for t' and β at either end of the first disc are substituted into Equation (16). If they are consistent, $\Delta = 0$. If not, an improved estimate for β is made. In this way, the calculation proceeds to the bottom of the tube. If $\beta \neq 0$ at the bottom, it is necessary to repeat with an improved estimate for the initial t'_o.

This procedure has been programmed for an IBM-650 high-speed digital computer and various preliminary points calculated. These preliminary results are given in the next section. It is anticipated that more comprehensive analytical results as well as the results of an experimental program which is presently in progress may be presented in the future.

3.0 RESULTS OF ANALYSIS

The generalized characteristic of the flow described by this analysis is a boundary layer descending along the cooled tube walls and a core of fluid ascending. To this extent, the flow regime is similar to that described by Lighthill in Reference 1. The primary difference lies in the fact that the fluid temperature along the tube centerline is not constant as in the Lighthill case, but even for constant wall temperature there is a strong temperature gradient* along the tube centerline, with the temperature increasing toward the top.

As previously stated, physically, if the wall temperature distribution, the heat source strength and distribution, and fluid physical properties are specified, then the entire problem is specified. An examination of the definitions for $\bar{q}_V$ (i.e., mean q_V) and $t_o' + t_{w_o}$ shows that they are algebraically related through the Nusselt Number based on tube radius and the maximum internal temperature difference as shown by Equation (18). The algebraic manipulations are shown in the Appendix.

$$Nu_a = \frac{\bar{q}_V}{2(t_o' + t_{w_o})} \tag{18}$$

It is possible then to plot the Nusselt Number as a function of either $\bar{q}_V$ or $t_o' + t_w$ with the wall temperature distribution and/or heat source distribution as a parameter. Curves of these sorts where the Lighthill results (Reference 1) are compared with the results for constant wall

*Even so, the radial gradient adjacent to the wall is considerably greater. However, the centerline gradient extends for the length of the tube while the radial gradient only extends across the boundary layer. Thus, the temperature differences associated with the centerline gradient are in many cases much greater.

temperature with internal heat generation and uniform heat source distribution are presented in Figures 4 and 5. Figure 6 is a plot of temperature against heat source, again comparing the Lighthill results with the results of the present analysis. It should be mentioned that the Lighthill case corresponds physically to the application of all the heat through a radial disc of differential height at the bottom of the tube, and $\bar{q}_V$ must be interpreted in this manner. Figure 7 shows the variation of boundary layer thickness with axial position for various uniformly-distributed heat source strengths and constant wall temperature.

It is noted that the Nusselt Number for the case of internal heat generation, constant wall temperature, uniform heat source distribution in a closed tube in the modified boundary layer regime with laminar flow is considerably smaller than that for a tube under similar conditions but open at one end to an infinite reservoir. From another viewpoint, this latter case can be stated as that of a closed tube with internal heat generation wherein the heat is supplied along the bottom end plate rather than according to a uniform distribution. From a rather naive viewpoint, the situation may be explained physically by the fact that in the latter case the maximum temperature difference (upon which the Nusselt Number has been based) exists across the boundary layer for the entire length of the tube, whereas for the case of uniform heat source distribution, the maximum temperature difference across the boundary layer is applied only at the top (it decreases to zero at the bottom).

It is anticipated that as the digital computer calculations proceed further detailed results for the above cases and for various wall temperatures and heat source distributions may be presented along with experimental results from a program presently in progress.

4.0 NOMENCLATURE

T	Temperature
U, u	Dimensional and non-dimensional velocity in axial direction
X, x	Dimensional and non-dimensional coordinate in axial direction
R, r	Dimensional and non-dimensional coordinate in radial direction
ℓ, a	Length and radius of tube
ρ	Density
c_V	Specific heat
k	Thermal conductivity
κ	Thermal diffusivity, $\kappa = k/\rho c_V$
ν	Kinematic viscosity
A_s	Area of tube wall
t	Non-dimensional temperature = $\frac{\alpha g a^4 \Delta T}{\nu \kappa \ell}$
t'	Non-dimensional temperature difference between wall and fluid at any given axial position. Subscript 0 applies to top of tube at centerline. Subscript c applies to centerline (or core) in general.
t_w	Non-dimensional temperature difference between top and bottom of wall
$t_o' + t_{w_o}$	Maximum non-dimensional temperature in the system = $Ra \cdot \frac{a}{\ell}$

Ra_a Rayleigh Number based on radius and maximum temperature difference = $\frac{\alpha g a^3 (T_{wall_{min}} - T_{fluid_{max}})}{\nu \kappa}$

q_v Non-dimensional volumetric heat source = $\frac{Q_v a^6 \alpha g}{\rho \nu \kappa^2 \ell c_v}$

Nu_a Nusselt Number based on radius = $\frac{ha}{k}$

g Acceleration of gravity

α Coefficient of volumetric expansion

β $1 - \beta$ is the non-dimensional boundary layer thickness and βR is the radius at which the boundary layer is terminated. See Figure 1 and 3.

γ, γ', $F(\beta)$, $G(\beta)$ Non-dimensional functions defined in text

TEMPERATURE AND VELOCITIES PROFILES

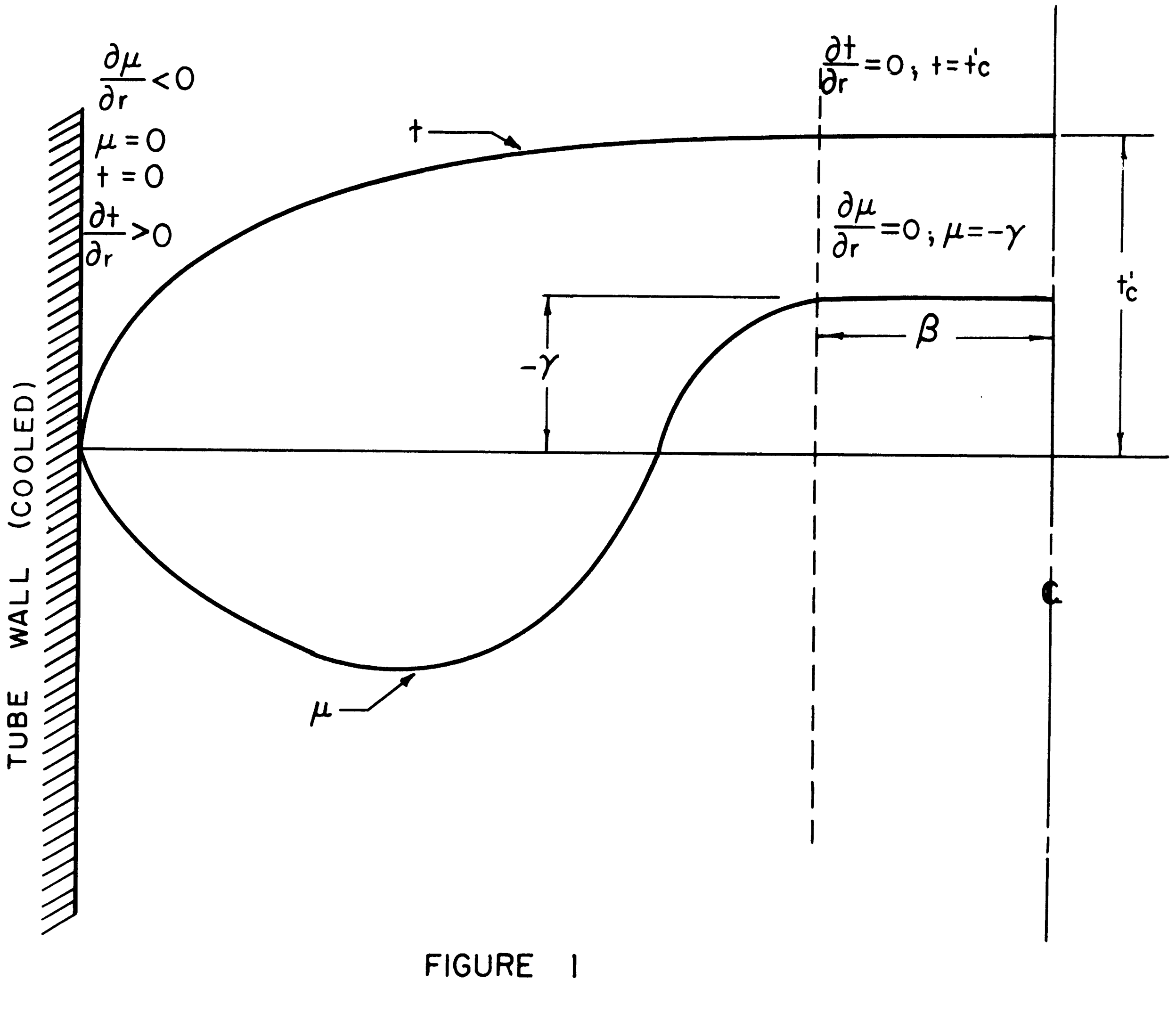
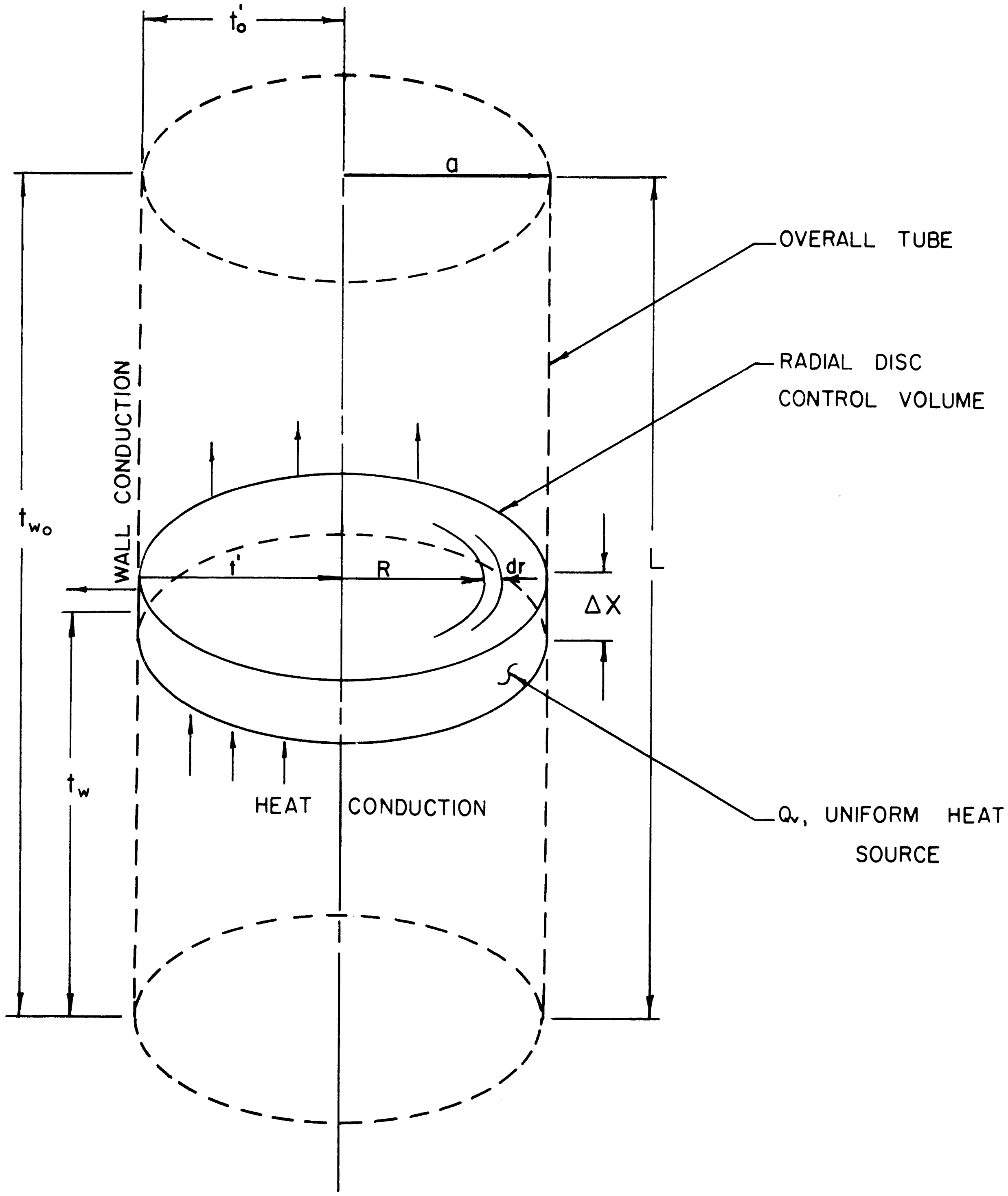


FIGURE 1



t'o
a
OVERALL TUBE
RADIAL DISC
CONTROL VOLUME
WALL CONDUCTION
t_wo
t'
R
dr
ΔX
L
t_w
HEAT CONDUCTION
Qv, UNIFORM HEAT
SOURCE

FIGURE 2

CORE TEMPERATURE GRADIENT

CONTROL VOLUME

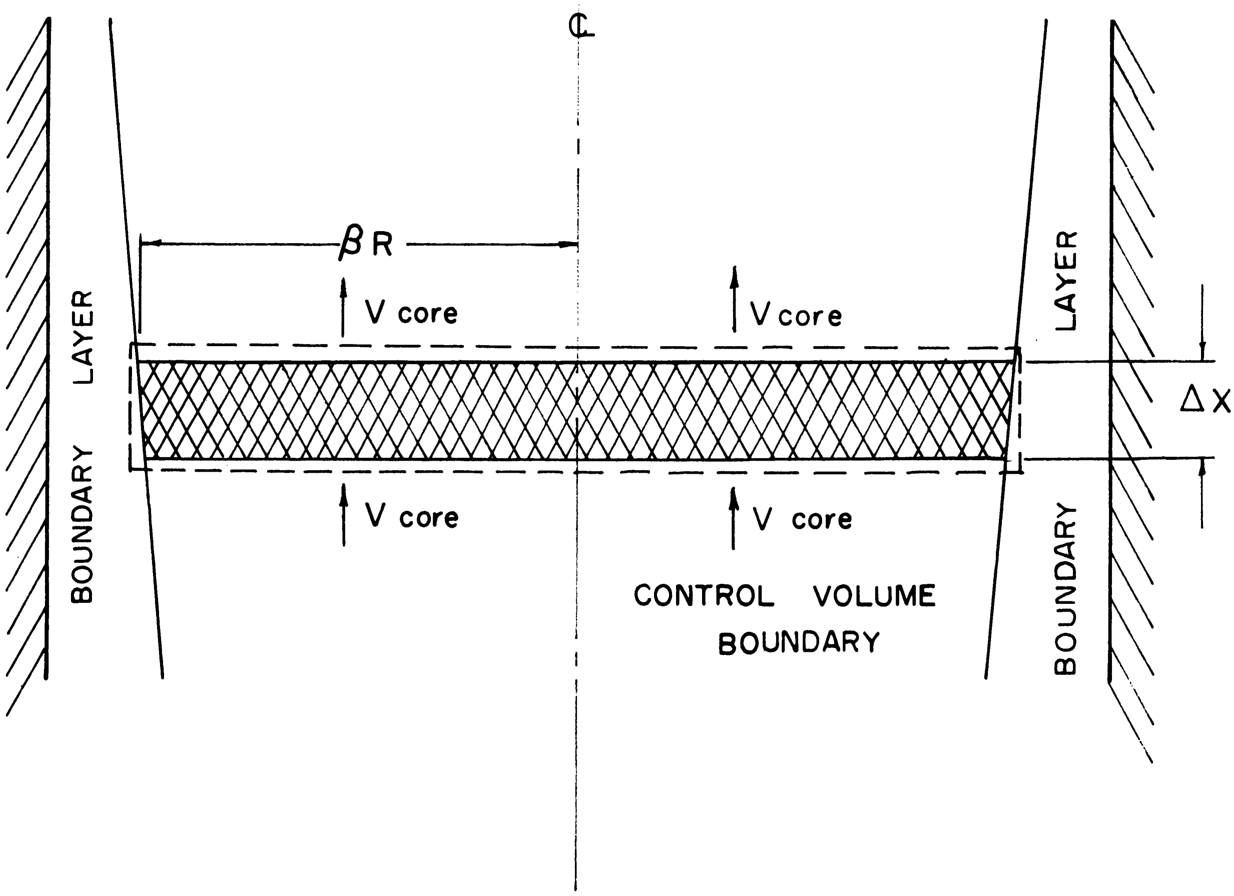


FIGURE 3

NUSSELT NUMBER
vs
NON - DIMENSIONAL HEAT SOURCE

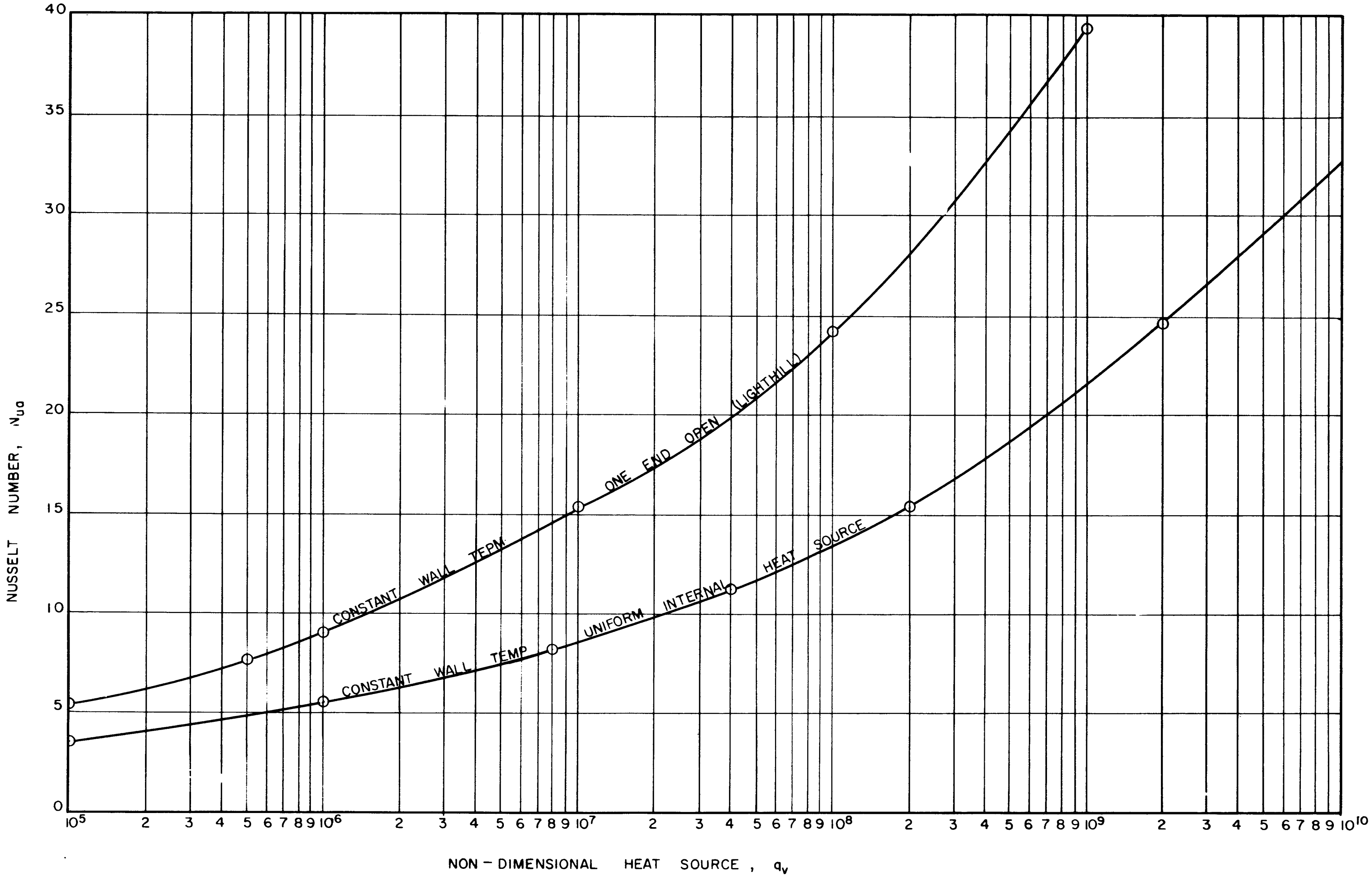


FIGURE 4

NUSSELT NUMBE

vs

NON – DIMENSIONAL TEMPERATURE DIFFERENCE

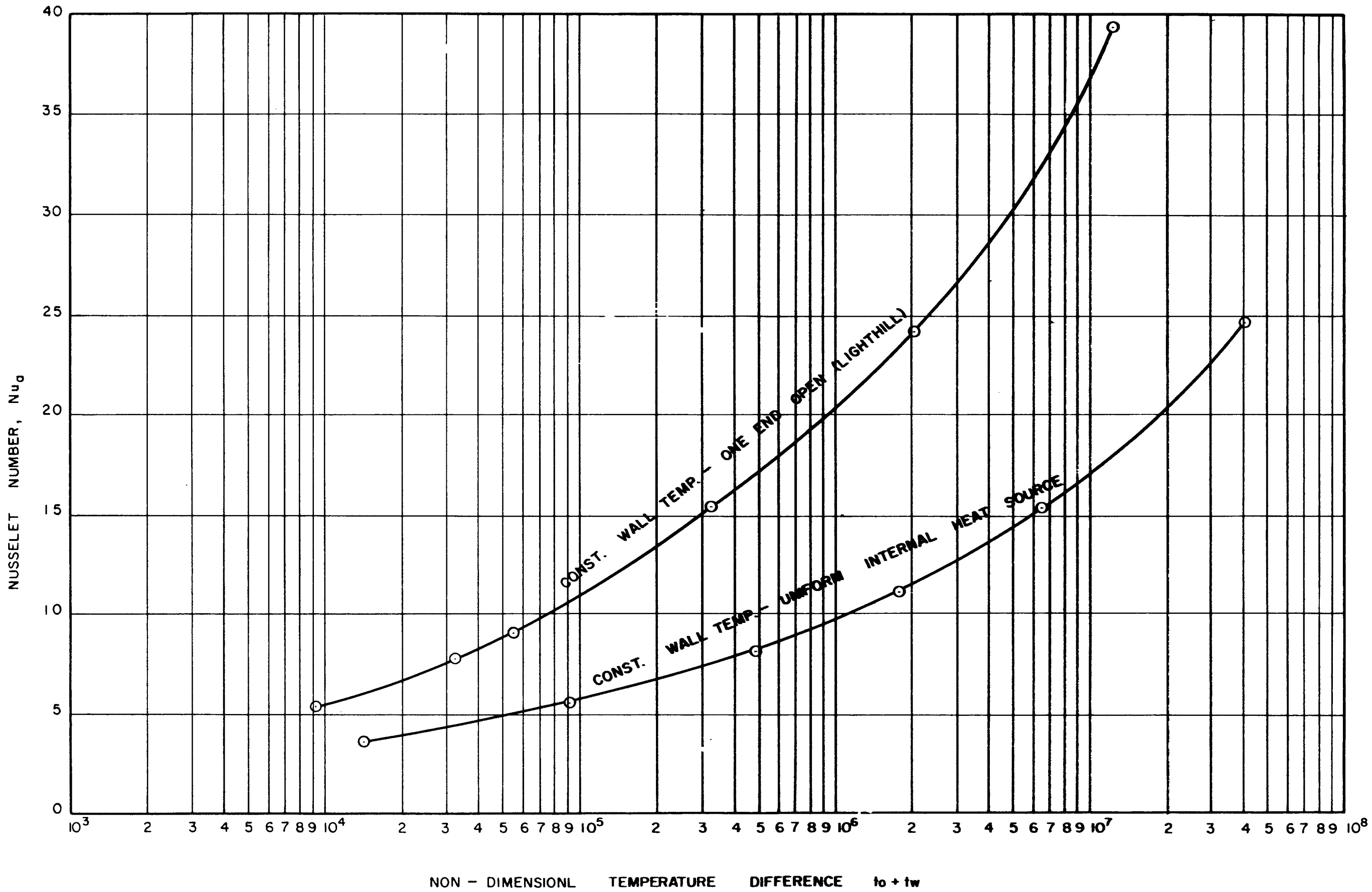


FIGURE 5

NON- DIMENESIONAL HEAT SOURCE
vs
NON - DIMENSIONAL TEMPERATURE DIFFERENCE

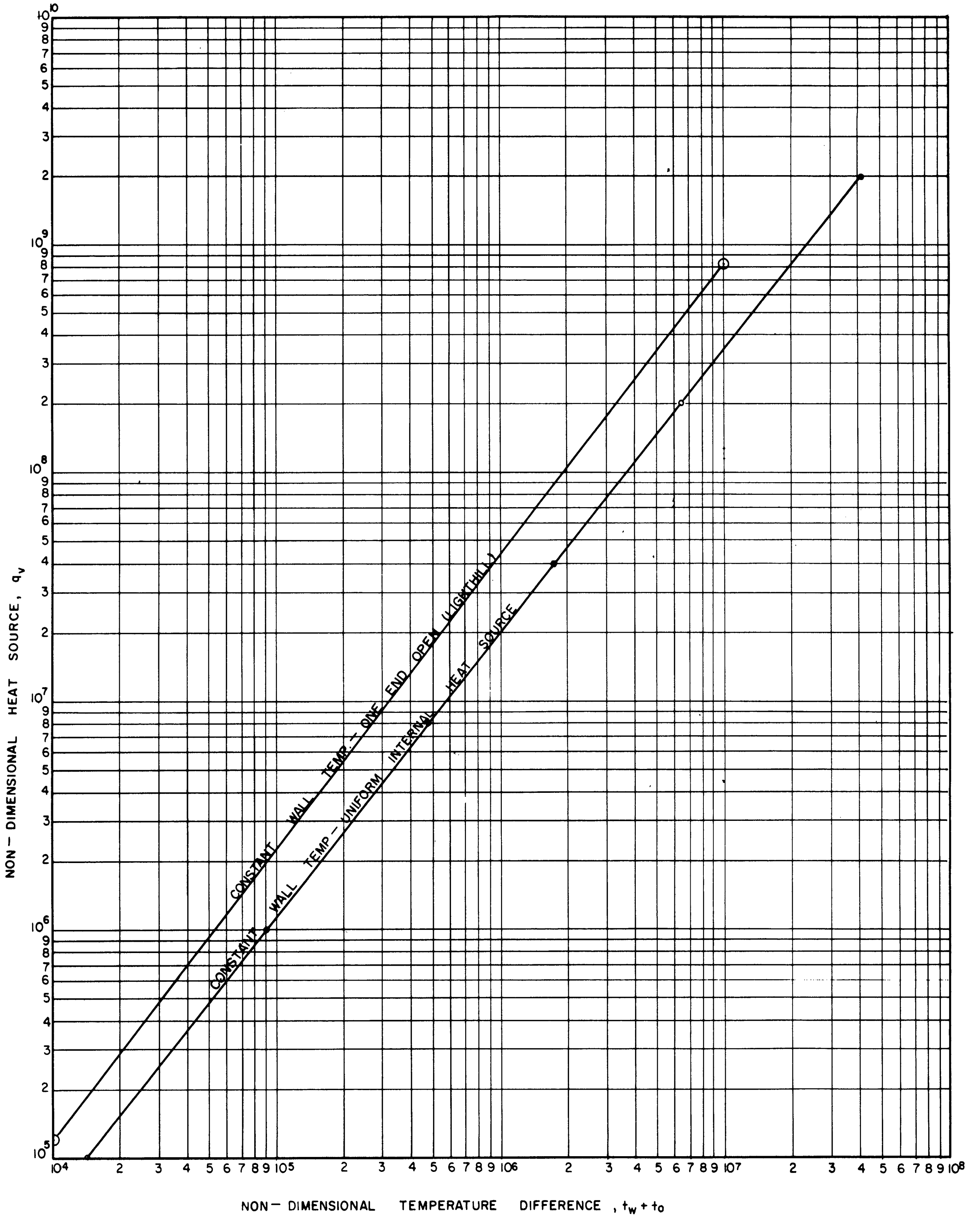


FIGURE 6

NON – DIMENSIONAL CORE THICKNESS
vs
NON – DIMENSIONAL AXIAL POSITION

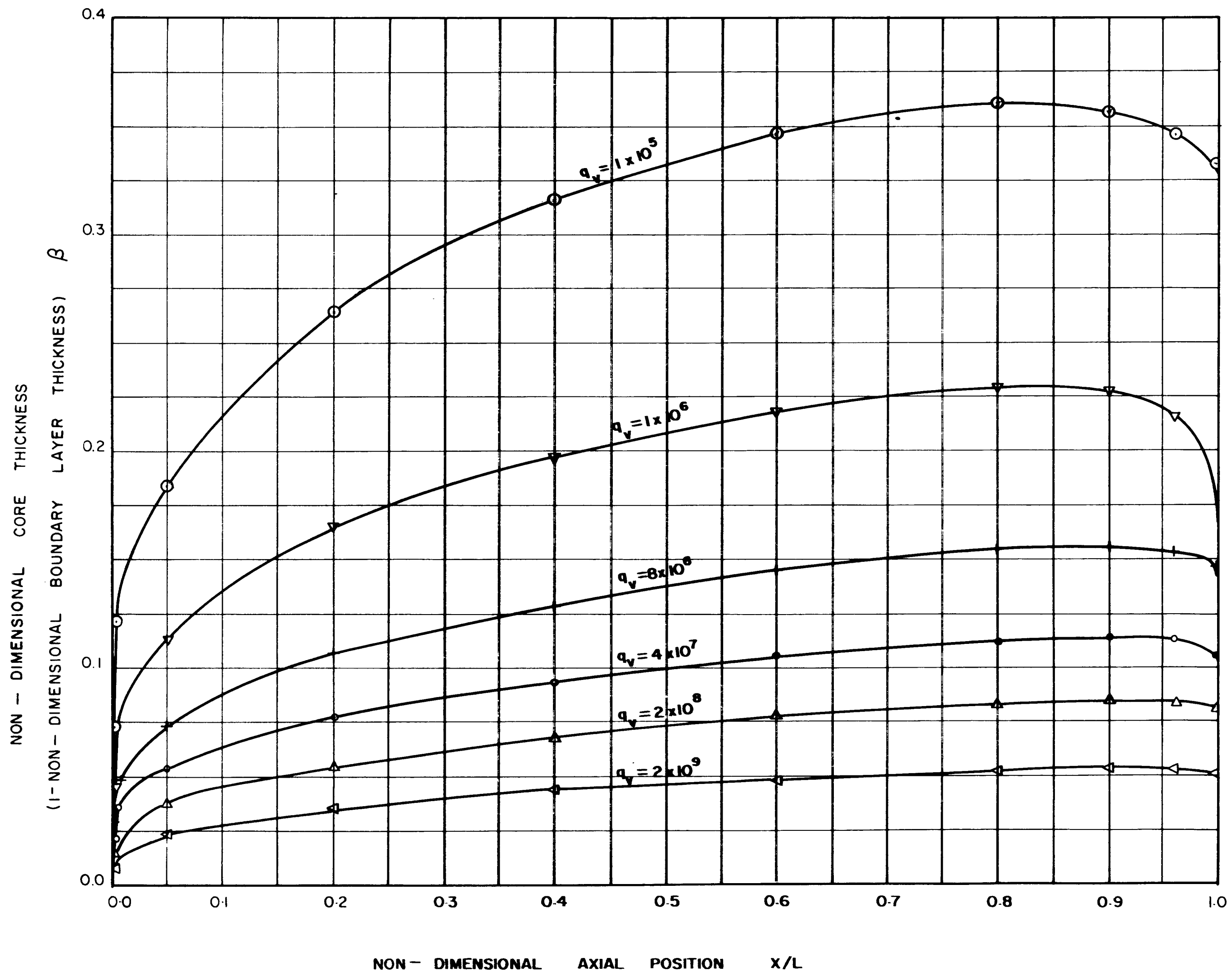


FIGURE 7

APPENDIX

1-A Derivation of Energy Equation

Consider a disc as shown in Figure 2 and write an energy balance:

$$\rho c_v\left[\frac{\partial}{\partial x}\int_0^a UT2\pi R dR\right]\Delta x = k\left(\frac{\partial T}{\partial R}\right)_{R=a}\Delta x 2\pi a + Q_v \pi a^2 \Delta x \qquad (1\text{-a})$$

or

$$\frac{\partial}{\partial x}\int_0^a 2UTRdR = 2\frac{k}{\rho c_v}\, a\left(\frac{\partial T}{\partial R}\right)_{R=a} + a^2\frac{Q_v}{\rho c_v} \qquad (2\text{-a})$$

and

$$\kappa = \frac{k}{\rho c_v}$$

Make the substitutions as below to obtain non-dimensional relations.

$$X = x\ell$$

$$R = ra$$

$$T = T_{wall_{min}} - \frac{\nu\kappa\ell}{\alpha g a^4}\, t$$

$$U = \frac{\kappa\ell}{a^2}\, u$$

$$\frac{1}{\ell}\frac{\partial}{\partial x}\int_0^1 \frac{2\kappa\ell}{a^2}\, u\left(T_{wall_{min}} - \frac{\nu\kappa\ell}{\alpha g a^4}\, t\right)(ar)d(ar) = \frac{2\kappa a}{a}\frac{\partial}{\partial r}\left(T_{wall_{min}} - \frac{\gamma\kappa\ell}{\alpha g a^4}\, t\right)_{r=1} + \frac{a^2 Q_v}{\rho c_v}$$

$$(3\text{-a})$$

$$\frac{\partial}{\partial x}\int_0^1 2u\, t\, r\, dr = 2\left(\frac{\partial t}{\partial r}\right)_{r=1} - \frac{a^2}{\rho c_v}\frac{\alpha g a^4}{\kappa^2 \nu\ell}\, Q_v \qquad (4\text{-a})$$

$$\frac{\partial}{\partial x}\int_0^1 u\, t\, r\, dr = \left(\frac{\partial t}{\partial r}\right)_{r=1} - \frac{q_v}{2} \qquad (5\text{-a})$$

$$\left(\frac{\partial t}{\partial r}\right)_{r=1} = -t(x)\left\{2\left(\frac{r-\beta}{1-\beta}\right)\left(\frac{1}{1-\beta}\right)\right\}_{r=1} = -\frac{2t(x)}{1-\beta} \qquad (6\text{-a})$$

utilizing the assumed temperature profile(2').

$$\frac{\partial}{\partial x}\int_0^1 r\,u\,t\,dr = \frac{\partial}{\partial x}\left[\int_0^\beta r\,u\,t\,dr + \int_\beta^1 r\,u\,t\,dr\right] = \frac{\partial}{\partial x}\left[\int_0^\beta - \gamma t(x) r\,dr - \gamma t(x)\int_1^\beta\left[1-\left(\frac{r-\beta}{1-\beta}\right)^2\right]\right.$$

$$\left.\left[1 - \left(\frac{r-\beta}{1-\beta}\right)^2\left\{1 + \delta(r-1)\right\}\; r\,dr\right]\right] \tag{7-a}$$

utilizing the assumed velocity profile (1'), and the assumed temperature profile (2').

Following the procedure of Reference 1, we should get

$$\frac{\partial}{\partial x}\int_0^1 r\,u\,t\,dr = \underbrace{\left[-\frac{2t(x)}{1-\beta} - \frac{q_v}{2}\right]}_{\text{from (5-a) and (6-a)}}$$

and also

$$= \underbrace{2\frac{d}{dx}\left[t^2(x)F(\beta)\right]}_{\text{from (7-a)}} \tag{8-a}$$

where

$$F(\beta) = \frac{(1-\beta)^3(3+\beta)(45 + 132\beta + 181\beta^2 + 62\beta^3)}{30240(3 + 4\beta + 3\beta^2)}$$

Integrating both sides of (8-a), and considering the disc of Figure 2, we get

$$[t^2(x)F(\beta)]_{x_2}^{x_1} = -\int_{x_2}^{x_1}\frac{t(x)}{1-\beta}\,dx - \frac{q_v(x_1 - x_2)}{4} \tag{9-a}$$

2-A Derivation of Centerline Temperature Gradient

In dimensional quantities, it was shown in the text (Equation 11):

$$\frac{dT_c}{dx} = -\frac{Q_v}{\rho c_v U_c} \tag{11}$$

Making the non-dimensionalizing substitutions of Section 2-A

$$\frac{\partial}{\partial(x\ell)}\left[T_{wall_\beta} - \frac{\nu\kappa\ell}{\alpha g a^4}t\right] = -\frac{Q_v}{\rho c_v \frac{\kappa\ell}{a^2}\gamma} \qquad (10\text{-a})$$

$$\frac{\partial t}{\partial x} = \frac{Q_v a^6 \alpha g}{\rho\nu\kappa^2\ell c_v\gamma} = \frac{q_v}{\gamma} \qquad (11\text{-a})$$

3-A Derivation of Difference Equations

Equation (9) may be written in an approximate form as

$$\frac{t_N'^2 F_N - t_{N-1}'^2 F_{N-1}}{x_N - x_{N-1}} - \left(\frac{t_N' + t_{N-1}'}{2}\right)\Bigg/\left(1 - \frac{\beta_N + \beta_{N-1}}{2}\right) + \frac{q_{v_N}}{4} = \Delta \qquad (16)$$

For a consistent solution,

$$\Delta \rightarrow 0$$

Equation (13) combined with (6) gives

$$\bar{G}(\beta)\bar{t}(x)\Delta t(x) = q_{v_N}\Delta x$$

where

$$\bar{t} = t_{N-1} - \frac{\Delta t}{2}$$

or

$$\bar{t}\Delta t = \frac{q_{v_N}\Delta x}{\bar{G}}$$

and

$$\Delta t\left(t_{N-1} - \frac{\Delta t}{2}\right) - \frac{q_{v_N}\Delta x}{\bar{G}} = 0$$

or

$$\Delta t = t_{N-1} \pm \sqrt{t_{N-1}^2 - \frac{2q_{v_N}\Delta x}{\bar{G}}}$$

where only the negative sign has physical significance.

so

$$(\Delta t)_{N,N-1} = t_{N-1}\left(1 - \sqrt{1 - \frac{2q_{v_N}\Delta x}{\bar{G}t_{N-1}^2}}\right)$$

and

$$\Delta t = t_{N-1} - t_N;$$

$$\bar{G} = \frac{G_N + G_{N-1}}{2}$$

so that

$$t_N = t_{N-1}\sqrt{1 - \frac{4q_v\Delta x}{t_{N-1}^2[G_N + G_{N-1}]}}$$

or, as explained in the text,

$$(t' + t_w)_N = (t' + t_w)_{N-1}\sqrt{1 - \frac{4q_v(x_N - x_{N-1})}{(t' + t_w)_{N-1}^2[G_N + G_{N-1}]}} \qquad (17)$$

4-A <u>Relations Between Nusselt Number, q_v, $t_o' + t_{w_o}$</u>

$$Nu_a = \frac{ha}{k} = \frac{Qa}{A_s\Delta T_{max}k} = \frac{\bar{Q}_v\pi a^3\ell}{2\pi\ell a\Delta T_{max}k} = \frac{\bar{Q}_v a^2}{2\Delta T_{max}k} \qquad (12\text{-a})$$

where

Q is total heat input rate

$\bar{Q}_v$ is mean volumetric heat input rate

By definition:

$$\frac{q_v}{2(t_o' + t_{w_o})} = \frac{\dfrac{Q_v a^6\alpha g}{\nu\kappa^2 c_v\rho\ell}}{\dfrac{\alpha g a^4\Delta T_{max}}{\nu\kappa\ell}} = \frac{\bar{Q}_v a^2}{2\Delta T_{max}k} = Nu_a$$

REFERENCES

1. Lighthill, M. J., "Theoretical Considerations on Free Convection in Tubes," Quarterly Journal of Mechanics and Applied Mathematics, Vol. 6, 1953, pp. 398-439.

2. Ostrach, Simon and Thornton, P. R., "On the Stagnation of Natural-Convection Flows in Closed-End Tubes," NACA, Lewis Flight Propulsion Laboratory, ASME Paper No. 57-SA-2.