THE UNIVERSITY OF MICHIGAN
COLLEGE OF ENGINEERING
Department of Civil Engineering

Technical Report
POST BUCKLING BEHAVIOR OF
TEE SHAPED ALUMINUM COLUMNS

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>ix</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>xiv</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>xxiv</td>
</tr>
<tr>
<td>CHAPTER I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1. Statement of the Problem</td>
<td>1</td>
</tr>
<tr>
<td>1.2. Review of the Prior Work</td>
<td>4</td>
</tr>
<tr>
<td>1.3. Initiation of the Research</td>
<td>11</td>
</tr>
<tr>
<td>CHAPTER II. BOUNDS OF BEHAVIOR OF A COLUMN</td>
<td>12</td>
</tr>
<tr>
<td>IN THE INELASTIC RANGE</td>
<td></td>
</tr>
<tr>
<td>General</td>
<td>12</td>
</tr>
<tr>
<td>2.1. Inelastic Buckling Gradient</td>
<td></td>
</tr>
<tr>
<td>i. Definition</td>
<td>13</td>
</tr>
<tr>
<td>ii. Evaluation</td>
<td>14</td>
</tr>
<tr>
<td>2.2. Tangent Modulus</td>
<td>19</td>
</tr>
<tr>
<td>2.3. Reduced Modulus</td>
<td></td>
</tr>
<tr>
<td>i. Introduction and Definition</td>
<td>20</td>
</tr>
<tr>
<td>ii. Evaluation</td>
<td>21</td>
</tr>
<tr>
<td>2.4. Critical Torsional Buckling</td>
<td>28</td>
</tr>
<tr>
<td>2.5. Local Buckling</td>
<td>30</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS (Continued)

CHAPTER III. THE EQUATIONS OF FLEXURAL-TORSIONAL BUCKLING .......................... 38

3.1. Selection of the Coordinates ............ 38
3.2. Transformation of Coordinates ........... 41
3.3. Boundary and Initial Conditions
     i. Boundary Conditions ............... 42
     ii. End Eccentricity ................... 44
     iii. Initial Crookedness of the Column 46
3.4. The Equations of Flexural-Torsional Buckling
     i. Moment-Curvature Relation ........ 47
     ii. External Moment and Force Components .................. 49
     iii. Internal Resisting Moment and Force Components .... 52
3.5. Equilibrium Conditions .................. 53
3.6. Warping Effect ......................... 54
3.7. Planar Bending .......................... 56
     Limiting Stage of Planar Bending .... 57

CHAPTER IV. TORSIONAL STIFFNESS .............. 61

4.1. Introduction .......................... 61
4.2. Assumptions .......................... 63
4.3. Displacement Equations .................. 63
4.4. Equilibrium Equations .................. 65
4.5. Boundary Conditions .................... 69
4.6. Difference Equations .................... 74
4.7. Polynomial Function ..................... 75
4.8. Approximate Differentiation ............. 76
4.9. Finite-Difference Approximation to the Parital Differential Equation of Equilibrium .... 76
TABLE OF CONTENTS (Continued)

4.10. Difference Equations for Boundary Conditions of T Section ........ 81
       Discontinuity .................. 83
       Points with Non-Equidistant Neighbors .................. 85
4.11. Convergence of the Iteration ........ 87
4.12. Shear Stress-Strain Relationship ...... 89
4.13. Evaluation of $G_s$ .................. 92
       Overrelaxation .................. 96
4.15. Technique ..................
       i. Choice of Mesh and Matrices .. 98
       ii. Evaluation of Matrices $[G_s]$ and $[\psi]$ ........ 101
       iii. Border Line of Discontinuity .. 101
4.16. Shear Stress and Torque ........ 102
4.17. Torsional Stiffness Coefficient ...... 103
4.18. Conclusions .................. 103

CHAPTER V. INCREMENTAL ANALYSIS OF "T" COLUMN BEHAVIOR ........ 106

5.1. Introduction .................. 106
5.2. Stress-Strain Relation ........ 107
5.3. Basic Relations used in the Incremental Procedure
       i. Division of the Column Length .. 111
       ii. Curvature-Deflection Relation .. 112
       iii. Equilibrium at the Nodal Points along the Column .... 121
       iv. Evaluation of the Stress at the Centroid of the Mesh .... 124
TABLE OF CONTENTS (Continued)

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>v. Planar Bending</td>
<td>127</td>
</tr>
<tr>
<td>5.4. Technique of the Incremental Analysis</td>
<td>129</td>
</tr>
<tr>
<td>5.5. Results</td>
<td>133</td>
</tr>
<tr>
<td>CHAPTER VI. TESTING</td>
<td>160</td>
</tr>
<tr>
<td>6.1. Introduction</td>
<td>160</td>
</tr>
<tr>
<td>6.2. Objectives of the Experiment</td>
<td>162</td>
</tr>
<tr>
<td>6.3. Material Properties</td>
<td>164</td>
</tr>
<tr>
<td>6.4. Column Dimensions</td>
<td>166</td>
</tr>
<tr>
<td>6.5. End Conditions</td>
<td>168</td>
</tr>
<tr>
<td>6.6. End Fixtures</td>
<td>169</td>
</tr>
<tr>
<td>6.7. Attachment of Strain Gages</td>
<td>172</td>
</tr>
<tr>
<td>6.9. Loading and Adjustments</td>
<td>178</td>
</tr>
<tr>
<td>6.10. Protective Measures</td>
<td>179</td>
</tr>
<tr>
<td>6.11. Summary and Results</td>
<td>180</td>
</tr>
<tr>
<td>CHAPTER VII. SUMMARY AND CONCLUSIONS</td>
<td>186</td>
</tr>
<tr>
<td>APPENDIX A. FLOW DIAGRAM AND RELATED PROGRAMS FOR COMPUTER</td>
<td>190</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>192</td>
</tr>
</tbody>
</table>
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. 1</td>
<td>Combinations Analyzed Incrementally by Digital Computer</td>
<td>135</td>
</tr>
<tr>
<td>5. 2</td>
<td>Ultimate Average Stress - Buckling towards or away from the flange (initial crookedness; $d = 1.5''$; stress-strain relation #1)</td>
<td>140</td>
</tr>
<tr>
<td>5. 3</td>
<td>(Ultimate Load/Critical Load) - Buckling towards or away from the flange (initial crookedness; $d = 1.5''$; stress-strain relation #1)</td>
<td>141</td>
</tr>
<tr>
<td>5. 4</td>
<td>Ultimate Average Stress - Buckling towards or away from the flange (initial crookedness; $d = 2.5''$, 3.''); stress-strain relation #1)</td>
<td>146</td>
</tr>
<tr>
<td>5. 5</td>
<td>(Ultimate Load/Critical Load) - Buckling towards or away from the flange (initial crookedness; $d = 2.5''$, 3.''); stress-strain relation #1)</td>
<td>146</td>
</tr>
<tr>
<td>5. 6</td>
<td>Ultimate Average Stress - Buckling towards or away from the flange - Comparison with the half-sine deflection shape assumption ($d = 1.5''$; stress-strain relations #1, #2, #3)</td>
<td>149</td>
</tr>
<tr>
<td>5. 7</td>
<td>(Ultimate Load/Critical Load) - Buckling towards or away from the flange - Comparison with the half-sine deflection shape assumption ($d = 1.5''$; stress-strain relations #1, #2, #3)</td>
<td>150</td>
</tr>
<tr>
<td>5. 8</td>
<td>Ultimate Average Stress - Buckling towards or away from the flange - Comparison of end eccentricity and initial crookedness ($d = 1.5''$; stress-strain relation #1)</td>
<td>156</td>
</tr>
<tr>
<td>Table</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>5.9</td>
<td>157</td>
<td></td>
</tr>
<tr>
<td>(Ultimate Load/Critical Load) - Buckling towards or away from the flange - Comparison of end eccentricity and initial crookedness ($d = 1.5&quot;$; stress-strain relation #1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.1</td>
<td>181</td>
<td></td>
</tr>
<tr>
<td>Table of Comparison of Experimental Results with Analytical Results</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>T-Section</td>
<td>2</td>
</tr>
<tr>
<td>1.2</td>
<td>Johnston's Strut Model</td>
<td>7</td>
</tr>
<tr>
<td>1.3</td>
<td>Augusti's Inelastic Strut Model</td>
<td>8</td>
</tr>
<tr>
<td>2.1</td>
<td>Strain Distribution at Initiation of Buckling</td>
<td>15</td>
</tr>
<tr>
<td>2.2</td>
<td>Stress and Strain Distribution at the Reduced Modulus Load</td>
<td>22</td>
</tr>
<tr>
<td>2.3</td>
<td>a. Stress Distribution - Z.S.C.A. in the web</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>b. Stress Distribution - Z.S.C.A. in the flange</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>c. Stress Distribution - Z.S.C.A. in the inner edge of the flange</td>
<td>26</td>
</tr>
<tr>
<td>2.4</td>
<td>Torsional Buckling</td>
<td>29</td>
</tr>
<tr>
<td>2.5</td>
<td>Coefficient k and the Edge Condition</td>
<td>31</td>
</tr>
<tr>
<td>2.6</td>
<td>Stress-Strain, Tangent Modulus Strain and Secant Shear Modulus Strain (#1)</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>Mathematical Fit and the Experiments</td>
<td></td>
</tr>
<tr>
<td>2.7</td>
<td>Tangent and Reduced Moduli in Buckling to the Left and to the Right vs.</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>Slenderness Ratio</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>Reduced Modulus in Buckling to the Left and to the Right vs. Tangent</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>Modulus</td>
<td></td>
</tr>
<tr>
<td>2.9</td>
<td>Tangent, Reduced, False Reduced Modulus Stresses and Ultimate Average Stress</td>
<td>36</td>
</tr>
<tr>
<td></td>
<td>vs. Slenderness Ratio</td>
<td></td>
</tr>
<tr>
<td>2.10</td>
<td>Critical Local Buckling for Flange and Web vs. Average Compressive Stress</td>
<td>37</td>
</tr>
<tr>
<td>3.1</td>
<td>a. &quot;Ideal&quot; Column and the Fixed Coordinates</td>
<td>39</td>
</tr>
<tr>
<td></td>
<td>b. Cross Section I-I</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>c. xyz and the $\xi\eta\zeta$ Coordinates</td>
<td>40</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>3.2 a.</td>
<td>Transformation in the Cross Section Parallel to the xy Plane</td>
<td>41</td>
</tr>
<tr>
<td>3.2 b.</td>
<td>Transformation in the Cross Section Parallel to the yz Plane</td>
<td>41</td>
</tr>
<tr>
<td>3.2 c.</td>
<td>Transformation in the Cross Section Parallel to the zx Plane</td>
<td>41</td>
</tr>
<tr>
<td>3.3 a.</td>
<td>Eccentric Loading</td>
<td>45</td>
</tr>
<tr>
<td>3.3 b, c.</td>
<td>Eccentricity of Loading at the Top and the Bottom Faces</td>
<td>45</td>
</tr>
<tr>
<td>3.4</td>
<td>Torque Component</td>
<td>51</td>
</tr>
<tr>
<td>3.5</td>
<td>Warping of the Cross Section</td>
<td>55</td>
</tr>
<tr>
<td>4.1</td>
<td>A Prismatic Bar under Torsion</td>
<td>64</td>
</tr>
<tr>
<td>4.2</td>
<td>Elastic, Tangent and Secant Moduli</td>
<td>66</td>
</tr>
<tr>
<td>4.3</td>
<td>Shear and Secant Shear Moduli</td>
<td>66</td>
</tr>
<tr>
<td>4.4</td>
<td>Cross Section and the Boundary</td>
<td>72</td>
</tr>
<tr>
<td>4.5</td>
<td>Elastic and Inelastic Region</td>
<td>73</td>
</tr>
<tr>
<td>4.6</td>
<td>Derivatives at 0, 1, and 2</td>
<td>76</td>
</tr>
<tr>
<td>4.7</td>
<td>Point &quot;o&quot; and the Neighboring Domains</td>
<td>77</td>
</tr>
<tr>
<td>4.8</td>
<td>Point (i, j) and the Neighboring Mesh</td>
<td>80</td>
</tr>
<tr>
<td>4.9</td>
<td>Boundary of the T Cross Section</td>
<td>82</td>
</tr>
<tr>
<td>4.10</td>
<td>Point on the Border Line of the Discontinuity</td>
<td>84</td>
</tr>
<tr>
<td>4.11</td>
<td>Point with Non-Equidistant Neighbors</td>
<td>85</td>
</tr>
<tr>
<td>4.12</td>
<td>Stepwise Approximation</td>
<td>87</td>
</tr>
<tr>
<td>4.13</td>
<td>Expansion of $\psi(x, y)$ in the Neighborhood of $(x, y)$</td>
<td>87</td>
</tr>
<tr>
<td>4.14</td>
<td>Total Strain vs. Stress</td>
<td>90</td>
</tr>
<tr>
<td>4.15</td>
<td>Plastic Strain vs. Stress</td>
<td>90</td>
</tr>
<tr>
<td>4.16</td>
<td>Shear Stress vs. Plastic Shear Strain</td>
<td>91</td>
</tr>
<tr>
<td>4.17, 4.18</td>
<td>Conversion of Tensile Stress-Strain Diagram to Shear Stress-Strain Diagram</td>
<td>92</td>
</tr>
<tr>
<td>4.19</td>
<td>Strain vs. Shear Secant Modulus</td>
<td>93</td>
</tr>
<tr>
<td>4.20</td>
<td>Choice of a Network</td>
<td>99</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>4.21</td>
<td>Junction of Flange and Web</td>
<td>105</td>
</tr>
<tr>
<td>5.1</td>
<td>Stress-Strain and Tangent Modulus-Strain Relation #2 (used in Batterman-Johnston column analysis)</td>
<td>109</td>
</tr>
<tr>
<td>5.2</td>
<td>Stress-Strain and Tangent Modulus-Strain Relation #3 (used in Johnston's strut model analysis)</td>
<td>110</td>
</tr>
<tr>
<td>5.3</td>
<td>Initial and Deflected Positions of the Column</td>
<td>111</td>
</tr>
<tr>
<td>5.4</td>
<td>Curvature Distribution along the Column</td>
<td>112</td>
</tr>
<tr>
<td>5.5</td>
<td>Concentrated Angle Changes</td>
<td>113</td>
</tr>
<tr>
<td>5.6</td>
<td>Average Shear and Deflection</td>
<td>121</td>
</tr>
<tr>
<td>5.7</td>
<td>Cross Section divided into Grid Networks</td>
<td>121</td>
</tr>
<tr>
<td>5.8</td>
<td>Strain Distribution at the $i^{th}$ Cross Section</td>
<td>126</td>
</tr>
<tr>
<td>5.9</td>
<td>Strain Regression - Stress Reversal</td>
<td>127</td>
</tr>
<tr>
<td>5.10</td>
<td>Eccentric Loading - Initial Crookedness</td>
<td>134</td>
</tr>
<tr>
<td>5.11</td>
<td>Dimensionless Load vs. Mid-length Deflection (initial crookedness; $L/r = 20$)</td>
<td>137</td>
</tr>
<tr>
<td>5.12</td>
<td>Dimensionless Load vs. Mid-length Deflection (initial crookedness; $L/r = 40$; half-sine curve deflection-shape comparison)</td>
<td>138</td>
</tr>
<tr>
<td>5.13</td>
<td>Dimensionless Load vs. Mid-length Deflection (initial crookedness; $L/r = 80$)</td>
<td>139</td>
</tr>
<tr>
<td>5.14</td>
<td>Dimensionless Load vs. Mid-length Deflections and Twist (initial crookedness; $d = 2.5''$; $L/r = 30$)</td>
<td>142</td>
</tr>
<tr>
<td>5.15</td>
<td>Dimensionless Load vs. Mid-length Deflections and Twist (initial crookedness; $d = 2.5''$; $L/r = 40$)</td>
<td>143</td>
</tr>
<tr>
<td>5.16</td>
<td>Dimensionless Load vs. Mid-length Deflections and Twist (initial crookedness; $d = 3.6''$; $L/r = 30$)</td>
<td>144</td>
</tr>
<tr>
<td>5.17</td>
<td>Dimensionless Load vs. Mid-length Deflections and Twist (initial crookedness; $d = 3.6''$; $L/r = 40$)</td>
<td>145</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>5.18</td>
<td>Dimensionless Load vs. Mid-length Deflection (initial crookedness; d = 1.5'; L/r = 40; stress-strain relation #2; half-sine curve deflection-shape comparison)</td>
<td>147</td>
</tr>
<tr>
<td>5.19</td>
<td>Dimensionless Load vs. Mid-length Deflection (initial crookedness; d = 1.5'; L/r = 40; stress-strain relation #3; half-sine curve deflection-shape comparison)</td>
<td>148</td>
</tr>
<tr>
<td>5.20</td>
<td>Ultimate Average Stress vs. Slenderness Ratio</td>
<td>151</td>
</tr>
<tr>
<td>5.21</td>
<td>(Ultimate Load/Critical Load) vs. Slenderness Ratio</td>
<td>152</td>
</tr>
<tr>
<td>5.22</td>
<td>Dimensionless Load vs. Mid-length Deflection (eccentric loading - initial crookedness comparison; L/r = 20)</td>
<td>153</td>
</tr>
<tr>
<td>5.23</td>
<td>Dimensionless Load vs. Mid-length Deflection (eccentric loading - initial crookedness comparison; L/r = 40)</td>
<td>154</td>
</tr>
<tr>
<td>5.24</td>
<td>Dimensionless Load vs. Mid-length Deflection (eccentric loading - initial crookedness comparison; L/r = 80)</td>
<td>155</td>
</tr>
<tr>
<td>5.25</td>
<td>Ultimate Average Stress vs. Slenderness Ratio (eccentric loading - initial crookedness comparison)</td>
<td>158</td>
</tr>
<tr>
<td>5.26</td>
<td>(Ultimate Load/Critical Load) vs. Slenderness Ratio (eccentric loading - initial crookedness comparison)</td>
<td>159</td>
</tr>
<tr>
<td>6.1</td>
<td>Column Supported Laterally</td>
<td>163</td>
</tr>
<tr>
<td>6.2</td>
<td>Specimens of the Material Properties Test</td>
<td>165</td>
</tr>
<tr>
<td>6.3</td>
<td>Tinius Olsen Mechanical Testing Machine</td>
<td>165</td>
</tr>
<tr>
<td>6.4</td>
<td>Cross Section Dimensions</td>
<td>167</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td>a.</td>
<td>169</td>
</tr>
<tr>
<td></td>
<td>End Fixture used by Bauschinger</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b.</td>
<td>169</td>
</tr>
<tr>
<td></td>
<td>End Fixture used by Von Karman</td>
<td></td>
</tr>
<tr>
<td>6.6</td>
<td>Different Parts of the End Fixture</td>
<td>171</td>
</tr>
<tr>
<td>6.7</td>
<td>End Fixture Movement</td>
<td>173</td>
</tr>
<tr>
<td>6.8</td>
<td>Displacement and Twist Measurement</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Attachments</td>
<td>173</td>
</tr>
<tr>
<td>6.9</td>
<td>Attached Pulleys and Chrome Blades</td>
<td>174</td>
</tr>
<tr>
<td>6.10</td>
<td>Translated and Twisted Mid-length Section</td>
<td>175</td>
</tr>
<tr>
<td>6.11</td>
<td>Reading Error</td>
<td>178</td>
</tr>
<tr>
<td>6.12</td>
<td>Column under Test</td>
<td>180</td>
</tr>
<tr>
<td>6.13</td>
<td>Average Stress vs. Mid-length Deflection Curves -</td>
<td>183</td>
</tr>
<tr>
<td></td>
<td>Test and Analytical Results (d = 1.5(^{\prime}); L/r = 30)</td>
<td></td>
</tr>
<tr>
<td>6.14</td>
<td>Average Stress vs. Mid-length Deflection Curves -</td>
<td>184</td>
</tr>
<tr>
<td></td>
<td>Test and Analytical Results (d = 1.5(^{\prime}); L/r = 40)</td>
<td></td>
</tr>
<tr>
<td>6.15</td>
<td>Average Stress vs. Mid-length Deflection Curves -</td>
<td>185</td>
</tr>
<tr>
<td></td>
<td>Test and Analytical Results (d = 2.5(^{\prime}); L/r = 30)</td>
<td></td>
</tr>
</tbody>
</table>

Appendix A

Computer Flow Diagram | 190
NOMENCLATURE

Abbreviations

(B. S.) \( \xi \), (B. S.) \( \eta \)  
Bending stiffness in the \( \xi \zeta \) and \( \eta \zeta \) planes

I. B. G.  
Inelastic buckling gradient (relation 1.1)

F. R. M. S. L. \( (\sigma_{r\ell})_f \)  
False reduced modulus stress in buckling to the left

F. R. M. S. R. \( (\sigma_{rr})_f \)  
False reduced modulus stress in buckling to the right

R. M. L. \( (E_{r\ell}) \)  
Reduced modulus in buckling to the left

R. M. R. \( (E_{rr}) \)  
Reduced modulus in buckling to the right

R. M. S.  
Reduced modulus stress

R. M. S. L. \( (\sigma_{r\ell}) \)  
Reduced modulus stress in buckling to the left

R. M. S. R. \( (\sigma_{rr}) \)  
Reduced modulus stress in buckling to the right

T. M. \( (E_t) \)  
Tangent modulus

T. M. S. \( (\sigma_t) \)  
Tangent modulus stress

Z. S. C. A.  
Zero stress change axis

i. b. g.  
\( (I. B. G.) (\frac{L}{P_t}) \) (relation 5.53)

r. m. s.  
\( (R. M. S.) / \text{critical stress} \)
A

Cross sectional area

\[ A \]

Matrix of the coefficients of the simultaneous linear equations (relation 4.70)

\[ A_l \]

Cross sectional area - left portion

\[ A_r \]

Cross sectional area - right portion

\[ A_1, A_2, A_3 \]

Values of the integrals of relation 3.32

\[ B \]

Ramberg-Osgood stress-strain relation constant (relation 4.65)

\[ \{ B \} \]

Vector, constants of the simultaneous linear equation (relation 4.70)

\[ C \]

Torsional stiffness coefficient

\[ [C] \]

Radius of curvature matrix (relation 5.36)

\[ C_a \]

Average stiffness coefficient (relation 4.83)

\[ C_w \]

Warping stiffness coefficient (relation 3.19)

\[ D \]

Dimension of a square

\[ D_l \]

Diameter of circle inscribed at juncture (Fig. 4.21)

\[ E \]

Elastic modulus

\[ E_{\text{eff}} \]

Effective elastic modulus

\[ E_r \]

Reduced modulus

\[ E_{rr} \]

Reduced modulus in buckling to the right

\[ E_{rl} \]

Reduced modulus in buckling to the left

\[ E_s \]

Secant modulus

\[ E_t \]

Tangent modulus

\[ [E_{tf}], [E_{tw}] \]

Matrix of the values of the tangent modulus at the flange and web networks
E_{tx}, E_{ty} 
E_{bx}, E_{by} 

Eccentricity of axial loading at the top and bottom faces, in the x and y directions (Fig. 3.3b)

F_{x(e)}, F_{x(i)} 
F_{y(e)}, F_{y(i)} 
F_{z(e)}, F_{z(i)} 

Components of the external and the internal forces along the x, y and z axes

F_{\xi (e)}, F_{\xi (i)} 
F_{\eta (e)}, F_{\eta (i)} 
F_{\zeta (e)}, F_{\zeta (i)} 

Components of the external and the internal forces along the \( \xi \), \( \eta \) and \( \zeta \) axes

G 
Shear modulus

G_E 
See relation 4.8

G_s 
Secant shear modulus

\begin{bmatrix} G_{s_f} \\ G_{s_w} \end{bmatrix} 
Secant shear modulus matrix for the flange and the web

I_p 
Polar moment of inertia

I_x, I_y 
Moment of inertia with respect to x and y axes

J 
Torsion constant (relation 2.39)

K 
Coefficient of effective slenderness ratio

\left( \frac{KL}{r} \right) r 
Effective reduced slenderness ratio

\left( \frac{KL}{r} \right)_{rf}, \left( \frac{KL}{r} \right)_{rr} 
Effective reduced slenderness ratios in buckling to the left or to the right (relation 2.37)

L 
Column length

M_r 
Moment at the reduced modulus load (relation 2.22)

M_x, M_y 
Moment about the x and y axes
\( M_z \quad \) Moment about the \( z \) axis (torsional moment)

\( M_{xe}, M_{xi}, M_{xd} \) External moment components due to eccentric loading, initial crookedness, and the lateral deflection, along the \( x \), \( y \) and \( z \) axes

\( M_y(e), M_y(i), M_y(d) \) External moment components along the \( x \), \( y \) and \( z \) axes

\( M_z(e), M_z(i) \) Normal vector

\( P, \{ P \} \) Axial load

\( P_{avg} \) Average of the normal forces calculated at the longitudinal nodal points (relation 5.47)

\( P_c \) Critical load

\( P_t \) Tangent modulus load

\( P_u \) Ultimate load

\( \{ Q \} \) Convergence optimizer (relation 5.46)

\( Q_1, Q_2 \) Components of \( Q \) (relation 5.50)

\( [R] \) Curvature-deflection transformation matrix (relations 5.16, 5.17)

\( \{ S \} \) Average shear vector (relation 5.10)

\( [SR] \) Matrix of the depth of strain regression (relation 5.37)

xvii
\{u\}, \{u_o\} \quad \text{Lateral deflection and initial crookedness in the x direction}

\{v\}, \{v_o\} \quad \text{Lateral deflection and initial crookedness in the y direction}

\{a\} \quad \text{Coefficient of the difference equation (relation 4.61)}

b \quad \text{Overall breadth of the flange (Fig. 1.1)}

[b] \quad \text{Transformation matrix (xyz and } \xi \eta \zeta \text{ coordinate systems) (relation 3.1)}

b_1, b_2, b_3, b_4 \quad \text{See Fig. 4.10}

[b] \quad \text{Matrices (relation 4.43)}

[c] \quad \text{Cycle of iteration for one cycle of increment}

d \quad \text{Overall depth of the section (Fig. 1.1)}

dA \quad \text{Element of area}

dn \quad \text{Element of length along the normal (Fig. 3.5)}

_d_n \quad \text{See relation 4.72}

ds \quad \text{Element of length along the center line (Fig. 3.5)}

e_{tx}, e_{ty} \quad \text{Eccentricity of loading in the x and the y directions per unit length at the top face (relation 3.4)}

e_x, e_y \quad \text{Symmetrical end eccentricity in the x and y directions}

f_l \quad \text{Fillet radius (Fig. 6.4)}

h \quad \text{Dimension of the mesh (Fig. 4.8)}

i, j, k, \ell, m, n, p, q \quad \text{Dummy variables}
k \quad \text{Edge condition coefficient (relation 2.42)}

m \quad \text{Number of the column segments (integer)}

m_f, m_w \quad \text{Number of flange and web network divisions in the x direction (integers) (Figs. 4.9, 5.7)}

n_f, n_w \quad \text{Number of flange and web network divisions in the y direction (integers) (Figs. 4.9, 5.7)}

n \quad \text{Cycle of increment}

[q_1] \quad \text{Transformation matrix (distributed load to concentrated load) (relation 5.8)}

[q_2] \quad \text{Transformation matrix (concentrated load to average shear) (relation 5.11)}

[q_3] \quad \text{Transformation matrix (average shear to moment) (relation 5.13)}

r, r_x, r_y \quad \text{Radius of gyration}

[r] \quad \frac{1}{L} [R]

r_{ex} \quad \text{Ratio of the eccentricity of loading in the x direction at the bottom to the same at the top}

r_{ey} \quad \text{Ratio of the eccentricity of loading in the y direction at the bottom to the same at the top}

t \quad \text{Thickness of flange (Fig. 1.1)}

\{u\}, \{u_o\} \quad \text{Lateral deflection and the initial crookedness in the x direction per unit length}

\{v\}, \{v_o\} \quad \text{Lateral deflection and the initial crookedness in the y direction per unit length}

u, v, w \quad \text{Displacement in the x, y and z directions (relations 4.4, 4.5)}

w_o \quad \text{Displacement of the point o in the z direction (relation 3.25)}

w \quad \text{Web thickness}
\{x\}, \{y\} \quad \text{Coordinates of the mesh}

\bar{x} \quad \text{Distance of the centroid of the cross section from the centroid of the flange (Fig. 3.3a)}

x, y, z \quad \text{Fixed coordinates (Fig. 3.1a)}

x_o \quad \text{Distance of the centroid from the outer surface (relation 1.1)}

x_{o}l', x_{o}r \quad \text{Distance of the centroid from the outer left or right surfaces (Fig. 1.1)}

x_l', x_r \quad \text{Distance of the Z.S.C.A. from the left face of the flange in buckling to the left or to the right (Fig. 2.2)}

z \quad \text{Distance from the top of the column}

\Delta_1', \Delta_2', \Delta_3', \Delta_4' \quad \text{Test-deflection readings (Fig. 6.10)}

\Delta A_f, \Delta A_w \quad \text{Area of the mesh of the flange and the web}

[\Delta A_f], [\Delta A_w] \quad \text{Area of the mesh of the flange and the web}

\Delta P, \left\langle \Delta P \right\rangle \quad \text{Increment of the axial load}

\Delta P_{avg} \quad \text{Increment of the average of the axial force evaluated at the longitudinal nodal sections}

\Delta P_l, \Delta P_r \quad \text{Increment of axial load of the left and the right portions}

\Delta_{ix}, \Delta_{iy} \quad \text{Initial crookedness at the mid-length of the column in the x and the y directions}

\Delta x, \Delta y \quad \text{Mid-length deflection in the x and the y directions}

\Delta \delta, [\Delta \delta] \quad \text{Increment of the lateral deflection}

\Delta \delta_o \quad \text{Increment of the lateral deflection at the mid-length}

\Delta \epsilon \quad \text{Increment of the strain}

\Delta \sigma \quad \text{Increment of the stress}
\( \Delta \phi, [-\Delta \phi] \)  
Increment of the curvature

\( \Delta \phi_0 \)  
Increment of the curvature at the mid-length

\( \alpha, \{\alpha\} \)  
Ratio of (B.S.) \( \eta \) to (B.S.) \( \xi \) (relations 3.37, 5.45)

\( \alpha_1 \)  
Coefficient of juncture correction to torsion constant (relation 4.83)

\( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \)  
See Fig. 4.11

\( \beta, \{\beta\} \)  
Rotation (twist) of the cross section about the \( \xi \) axis

\( \gamma \)  
Shear strain

\( \gamma_p \)  
Plastic shear strain

\( \gamma_{xy}, \gamma_{xz}, \gamma_{yz} \)  
Shear strain components

\( [\delta] \)  
Matrix, deflection per unit length in the x and the y directions

\( [\delta_0] \)  
Matrix, initial crookedness per unit length in the x and the y directions

\( \delta_1 \)  
See Figs. 6.10 and 6.11

\( \delta_{ix}, \delta_{iy} \)  
Initial crookedness at the mid-length per unit length in the x and the y directions

\( \delta_x, \delta_y \)  
Lateral deflection at the mid-length per unit length in the x and the y directions

\( \varepsilon \)  
Strain

\( \varepsilon_e \)  
Elastic strain

\( \{\varepsilon_0\} \)  
Strain at the origin of the \( \xi\eta\xi \) coordinate

\( [e_f(x, n_f, m_f)] \)  
Strain matrices for the flange and the web

\( [e_w(x, n_w, m_w)] \)

\( \varepsilon_p \)  
Plastic strain
\( \varepsilon_t \)  

Strain at the tangent modulus load

\( \varepsilon_{P.L.} \)  

Strain at the proportionality limit

\( \varepsilon_{xx}, \varepsilon_{xy}, \ldots \)  

Strain components

\( \eta \)  

Ratio of the tangent modulus to the elastic modulus (relation 2.26)

\( \{ \eta \}, \{ \xi \} \)  

Coordinates of the centroid of the mesh

\( \xi, \eta, \zeta \)  

Coordinate, fixed to the column cross section (Fig. 3.1b)

\( \theta \)  

Uniform twist per unit length

\( \{ \theta \} \)  

Concentrated angle change vector (relation 5.7)

\( \lambda \)  

Length of the column segment

\( \lambda \)  

See relation 4.71

\( \nu \)  

Poisson's ratio

\( \{ \xi \}, \{ \eta \} \)  

Coordinates of the centroid of the mesh

\( \bar{\xi}, \{ \bar{\xi} \} \)  

See relations 3.35, 5.44

\( \rho \)  

Distance of the element from the origin (Fig. 3.4)

\( [\rho_f^2], [\rho_w^2] \)  

Matrix of \( \rho^2 \) for the networks of the flange and the web

\( \sigma \)  

Stress

\( \sigma_c \)  

Critical stress

\( \sigma_{c_f} \)  

Critical local buckling stress

\( \sigma_{t} \)  

Critical torsional buckling stress

\( [\sigma_f(m \times n_f \times m_f)] \) \( [\sigma_w(m \times n_w \times m_w)] \)  

Stress matrices for the flange and the web
\[ \sigma_{\text{P. L.}} \] Stress at the proportionality limit

\[ \sigma_e \] Stress at the Euler load

\[ \sigma_t \] Tangent modulus stress

\[ \sigma_{xx}, \sigma_{yy}, \sigma_{zz} \] Components of \( \sigma \)

\[ \tau \] Shear stress vector

\[ \tau_{xy}, \tau_{yz}, \tau_{xz} \] Shear stress components

\[ \begin{bmatrix} \tau_{xxf} \\ \tau_{yxf} \end{bmatrix}, \begin{bmatrix} \tau_{yzf} \end{bmatrix} \] Matrices (shear stress components in the flange)

\[ \begin{bmatrix} \tau_{xyw} \end{bmatrix}, \begin{bmatrix} \tau_{yzw} \end{bmatrix} \] Matrices (shear stress components in the web)

\[ [-\phi] \] Curvature matrix (relation 5.5)

\[ \psi \] Warping function

\[ \begin{bmatrix} \psi_f \end{bmatrix}, \begin{bmatrix} \psi_w \end{bmatrix} \] Warping matrices for the flange and the web

\[ \omega \] Relaxation factor

\[ \omega_b \] Best value for \( \omega \)
ABSTRACT

The post buckling behavior of a tee shaped column of aluminum alloy is the subject under study in this thesis. Attention is focused on the cases where the buckling is initiated in the plane of symmetry. The effects of initial curvature and eccentric loading in the plane of symmetry and in the directions towards or away from the flange receive consideration. In the case where the buckling starts in the direction away from the flange, the flange of the tee section would be subjected to increasing compressive stress and there would be corresponding decreases in the value of the tangent modulus while increasing areas of the web would be subjected to strain regression at the elastic rate. Thus, progressively the ratio of the bending stiffness about the axis of symmetry to that of the bending in the plane of symmetry will decrease and in some cases, depending on the ratio of the bending stiffness about the principal axes in the unloaded stage, before the load reaches the potential maximum, biaxial bending accompanied with torsion would commence. On the contrary, if buckling commences in the plane of symmetry and towards the flange, theoretically, the column will not buckle out of its plane of symmetry and planar bending will govern the behavior of the column.

A complete evaluation of tee-section column behavior is needed to provide an accurate estimate of the effects of initial column imperfections, eccentricity of loading, the preferred modes of
buckling and to afford a confirmation of the qualitative predictions made in the preceding paragraph. To achieve the foregoing aims an algorithm has been developed for use on a digital computer to provide by an incremental process a complete simulation of tee-section column behavior.

Bounds of behavior of a column, specifically a tee column, have been discussed and the differing inelastic buckling gradients and the reduced moduli in buckling towards or away from the flange have been explained and illustrated graphically. The problem of torsion of a cross section subjected to a varying strain in the inelastic range with possible presence of strain regression region has been treated.

A series of tests has been conducted to confirm the theoretical predictions. To carry out the experiments, a procedure of measuring the lateral displacements and the twist of the hinged-end columns in combined bending and torsion has been developed.

The results of the incremental analyses are presented in the form of tables and graphs. These results include the ultimate load carrying capacity of tee columns of various slenderness ratio with various amounts of initial crookedness in buckling towards or away from the flange, comparison of initial crookedness and the eccentric loading, comparison of the assumed half-sine curve deflection shape and the more exact deflection shape, some selected graphs of the load-deflection history and finally the results of the experiments in the form of a table and some graphs.
A thorough survey of literature covering all aspects of the related studies was made and a complete bibliography of 107 references is provided.
CHAPTER I
INTRODUCTION

1.1. STATEMENT OF THE PROBLEM

The behavior of a centrally loaded column of mono-symmetric cross section, specifically a "T" section, of aluminum alloy, which is stronger about the axis of symmetry and which has no eccentricity of loading, imperfections or residual stress, is the subject under study in this thesis. The effects of initial curvature will receive consideration.

The critical or buckling load in the inelastic range for a concentrically loaded metal column with no initial curvature and hinged at both ends is predicted by the tangent modulus theory.\textsuperscript{(1)} In the limit as the increment above the tangent modulus load approaches zero, Shanley showed that bending would commence with no regression in compressive strain, but that for any finite amount of load above the tangent modulus load a strain regression would be initiated on the convex side and the strain at the concave side would start to increase.

By an incremental analysis, deflection of the column at a certain point, i.e., mid-length, and corresponding load values at different stages of deflection up to and beyond the maximum value of the load can be calculated and graphed. The slope of the load vs. deflection curve at the point of initiation of lateral deflection is called

-1-
the "Inelastic Buckling Gradient". If torsional buckling is absent at this point the value of Inelastic Buckling Gradient is given by

\[ \lim_{\Delta \delta \to 0} \frac{\Delta P}{\Delta \delta} = \frac{P_t}{r} \frac{x_o}{2} \]

(1.1)

where \( P_t \) is the tangent modulus load, \( r \) is the radius of gyration with respect to axis of bending and \( x_o \) is the distance between the centroid and the outer fiber of the convex side of the cross section measured on a line perpendicular to the buckling axis.

If buckling is to occur about the \( y \)-axis, the moment of inertia of the cross section about \( y \)-axis being less than that about \( x \)-axis, one may calculate two different values of Inelastic Buckling Gradient, depending on whether the column would buckle to the left or to the right (Fig. 1.1). In the case of no initial imperfection or eccentric loading, the most probable direction of buckling would perhaps be to the left, corresponding to the smaller I.B.G. (inelastic buckling gradient).

Fig. 1.1. T-Section

If initial imperfection in the "T" section were such that it would initiate buckling in one or the other direction or if it would be prevented from deflecting in one direction, the column would presumably buckle in the unsupported or initially curved direction. If buckling were initiated in the direction with the larger I.B.G., i.e., to the right,
the behavior of the column would be quite different from the first case. In the latter case the flange of the "T" section would be subjected to increasing compressive stress corresponding to decreasing value of tangent modulus and larger parts of the web would be subjected to strain regression at the elastic rate. Thus, progressively, the ratio of bending stiffness about the y-axis to that of the x-axis would increase and in some cases, depending on the ratio of the bending stiffness with respect to the x- and the y-axes in the unloaded stage before the load reaches the potential maximum, biaxial bending accompanied with torsion would commence. In other cases this happens in the unloading stage as the elastically behaving portion of the web increases.

The internal torsional resistance of the section could be presumed to be only due to St. Venant torsion (with the inelastic modification) and the warping torsional resistance can be neglected. If buckling originates in the direction of the smaller I.B.G., i.e., to the left, strain regression would start in the flange area and the web will be under increasing compressive strain with corresponding decrease in tangent modulus. In this case the ratio of bending stiffness about the x-axis relative to the y-axis will increase and more resistance to buckling out of the plane of the web will be introduced. Bending will then continue in the plane of the web as it originated.

In this thesis the inelastic buckling behavior of "T" sections of aluminum alloys will be studied quantitatively and compared with the qualitative predictions made in the preceding paragraphs. Also, quantitative study will be made on the effects of existence of initial
crookedness in the positive or negative directions of the x-axis in the plane of the web. The effects of eccentric loading in the plane of the web have also been studied and compared with the effect of initial imperfection. The study has been done for sections with different ratios of bending stiffness relative to the x- and the y-axes and with different material properties. Finally, a series of tests has been conducted to confirm the theoretically predicted behavior.

In order to evaluate the above buckling problem quantitatively, an algorithm has been developed for use on a high speed digital computer, that makes the manipulation of the incremental process possible. To carry out the experiment, a procedure of measuring the lateral displacements and the twist of the hinged-end columns in combined bending and torsion has been developed.

2.2. REVIEW OF THE PRIOR WORK

Musschenbroek\(^{(4)}\) built the first testing machine and proposed the first empirical column formula for the ultimate load carrying capacity of the columns of rectangular cross section which states that ultimate load per unit area is proportional to \( \left( \frac{D}{L} \right)^2 \), where D is the depth and L is the length of the column. This happened almost twenty-five years before Euler started to develop his column theory in 1744.

Euler's equation for critical load of a column is one of the most famous equations in strength of materials and was published in 1759. The translation of essential portions of this publication by
J. A. Vanden Broek (Am. J. Phys., Vol. 15, No. 4, July-August 1947, pp. 309-318) reveals Euler's ability to apply advanced mathematics to a practical problem. Euler's formula was in effect a generalization of Musschenbroek's special case for rectangular sections.

For many years Euler's formula was not generally applied to actual design, since tests of structures indicated that columns frequently failed below the Euler load. Euler's formula gives the average stress at which a slender (elastic) concentrically loaded and initially straight prismatic and homogeneous column will develop bifurcation of equilibrium.

Euler's formula gives a very high value for buckling load of short columns. In a period of over two hundred years between the introduction of Euler's formula and the presentation of Shanley's concept of column buckling, there have been a great many developments and modifications of the column theory. In 1889 Considère(5) conducted a series of 32 column tests and verified that for short columns the Euler formula gave too high a value for the buckling load. He suggested that if buckling occurs above the proportional limit the elastic modulus should be replaced in the Euler formula by an "$E_{\text{eff}}$" in order to evaluate the ultimate strength of the column. He also stated that the value of "$E_{\text{eff}}$" is between the modulus of elasticity ($E$) and the tangent modulus ($E_t$). During the same year Engesser, (6) independently of Considère, suggested that the column strength in the
inelastic range can be obtained by the substitution of the tangent modulus in place of elastic modulus in the Euler formula.

In 1895 Jasinski\(^{(7)}\) criticized the tangent modulus theory on the grounds that it did not account for the non-reversible characteristic of the stress-strain relation in the inelastic range. In 1910 Von Kármán\(^{(8)}\) evaluated the effective or "reduced modulus" for the rectangular and the idealized H-section columns. The reduced modulus is also called the "double modulus" as it is a function of both the elastic and the tangent modulus. The formula for the double modulus as derived by Von Kármán for the rectangular section is

\[
E_r = \frac{4E \varepsilon_t}{(\sqrt{E} + \sqrt{E_t})^2}
\]  

(1.2)

The double modulus theory was considered to be the correct theory of inelastic column behavior until 1946 when Shanley showed that it presented a paradox.\(^{(9)}\) In order to exceed the tangent modulus (Euler-Engesser formula), it is necessary that the effective modulus be greater than \(E_t\), i.e., some portion of the cross section must be subjected to a strain regression. This indicates that the column begins to bend before reaching the double modulus load. Shanley mathematically verified his conclusions\(^{(1)}\) which are as follows:\(^{(11)}\)

1. The Engesser load (tangent modulus load) represents the maximum load for which the column has only one equilibrium configuration. Up to this load the idealized column must remain straight; beyond this load it may bend.
2. The double-modulus load represents the upper limit for the load that can theoretically be reached as the column continues to bend with increasing load. To develop the double-modulus load would require infinite lateral deflection at a constant value of the tangent modulus which is impossible.

3. For a given material, the maximum load that can be developed will generally be only slightly greater than the Engesser load because of the rapid decrease in $E_t$ with increasing stress. Therefore, the Engesser load is considered as the practical upper limit for column strength.

Duberg and Wilder, III$^{(12)}$ applied Shanley's approach to an idealized H-section column with two point concentrations of area and with zero web area and they confirmed Shanley's concept of initiation of buckling at the tangent modulus load.

In 1950 Lin$^{(13)}$ studied slightly curved columns of rectangular cross section, including effects of strain reversal. In 1961 Johnston$^{(14)}$ analyzed the inelastic behavior of a concentrically loaded "model" consisting of two rigid end segments continuous with a short deformable central rectangular segment of aluminum alloy (Fig. 1.2).

Fig. 1.2. Johnston's Strut Model
In 1964 Augusti\(^{(15)}\) constructed a strut model made of two rigid limbs connected by a "central cell" and studied the buckling and collapse of structural elements in the inelastic range. The cell consisted of two straight filaments of vanishing thickness in the plane of the paper (Fig. 1.3).

![Diagram of strut model](image)

**Fig. 1.3. Augusti's Inelastic Strut Model**

In 1966 Batterman and Johnston\(^{(16)}\) presented the results of their study on column behavior. This study had a broad scope and included effects of the initial crookedness and residual stress on the ultimate load carrying capacity of H-shaped columns of steel and aluminum in bending about the weak and strong axes. Maximum load was determined for different values of the slenderness ratio, yield strength, residual stress and imperfection. In the above studies the effect of torsion was not considered. The study of combined torsion and flexure commenced in 1929 when Wagner\(^{(17)}\) presented the first theory of flexural-torsional buckling and also studied the effect of eccentric loading. He assumed that the center of the twist coincides with the shear center. Kappus\(^{(18)}\) later stated that in general Wagner's assumption was valid.
Ostenfeld\(^{(19)}\) presented a correct theory for flexural-torsional buckling of tee, angle and channel sections accompanied by experimental results. In 1934 Wagner and Pretschner\(^{(20)}\) presented some test results on angles and revised Wagner's earlier theory. In 1936 F. Bleich and H. Bleich derived the flexural-torsional buckling from energy considerations using variational principles;\(^{(21)}\) later they revised their theory.\(^{(22)}\) Lundquist and Fligg\(^{(23)}\) and Niles\(^{(24)}\) also studied and presented theories about torsional-flexural buckling. Goodier\(^{(25)}\) simplified Wagner's theory and extended it to the case of a column subjected to bending about both principal axes, but he neglected the effect of the moment amplification caused by the deflection of the column. Timoshenko\(^{(26, 27)}\) treated the torsional-flexural buckling of columns in the elastic region in 1945 and he also neglected the increase of the bending moment due to deflection of the column. Zickel\(^{(28)}\) considered the influence of initial deflections and twist on biaxial bending of wide flange shapes. He assumed the initial twist to be finite in magnitude. The resulting equations were nonlinear and no general solutions were given for these equations. Hill and Clark\(^{(29)}\) and Salvadori\(^{(30)}\) made extensive studies of eccentrically loaded columns. Thürlimann\(^{(31)}\) obtained an approximate solution for the case of small initial twist by modifying Zickel's equations. In 1966 Culver\(^{(32)}\) developed a mathematical model for the biaxially loaded column with initial imperfection.

In 1950 B. G. Neal\(^{(33)}\) presented the first solution for inelastic lateral torsional buckling of a rectangular beam. Galambos\(^{(34, 35)}\)
presented a solution for inelastic lateral torsional buckling of wide flange columns subjected to bending about the major principal axis as a PhD dissertation at the Lehigh University. In 1962 Birnstiel\textsuperscript{(36)} treated H-columns loaded into the inelastic range in biaxial bending by an incremental process and used a digital computer to evaluate the ultimate load carrying capacity of these columns under eccentric loading. He also recently completed a report on some experiments of such nature.\textsuperscript{(37)}

In 1964 Ringo\textsuperscript{(38)} in his PhD dissertation developed an equilibrium approach to the ultimate load on a biaxially loaded beam column. In 1966 Rossow, Barney and Lee\textsuperscript{(39)} introduced the effect of initial curvature into the moment-curvature relationship and solved the differential equation by double integration and then, using buckling criterion, found interaction curves and suggested semiempirical equations to approximate the interaction curves. In 1966 Lee, Fine and Hastreiter\textsuperscript{(40)} examined the problem of inelastic torsional buckling strength of axially loaded H-columns.

References (41) through (45) give a comprehensive review of the developments of the torsional, flexural or torsional-flexural buckling behavior of the columns.

Inelastic range studies have always been carried out with the assumption that the material yields at the proportionality limit, or the inelastic range of the behavior has been approximated by straight lines of stress-strain curve, except studies of column behavior by
Batterman and Johnston\(^{16}\) where they assume a close mathematical fit for the actual stress-strain relation. However, they neglect the effect of the torsion.

1.3. INITIATION OF THE RESEARCH

In 1961 the author assisted Dr. Johnston\(^{14}\) in his study of "Buckling Behavior above the Tangent Modulus Load" where, with the aid of a digital computer, he studied the behavior of structural aluminum alloy struts of various lengths. The buckling model used by Johnston in his study is shown on Fig. 1.2.

In Johnston's study of the "Inelastic Buckling Gradient",\(^{3}\) with which the author was closely associated, the most attention was given to the definition and the evaluation of the I.B.G. with particular reference to a tee section example. This study suggested an obvious need for a more thorough evaluation of the actual column behavior and obtaining a quantitative relation between column initial imperfection or eccentricity of loading and the preferred mode of buckling.

Later, the author assisted Batterman and Johnston\(^{16}\) in their study of "Behavior and Maximum Strength of Metal Columns". The experience gained in this task helped considerably in completion of this thesis.
CHAPTER II

BOUNDS OF BEHAVIOR OF A COLUMN IN THE INELASTIC RANGE

GENERAL

The bounds of behavior of an ideal column in the inelastic range will be discussed in this chapter. Included in these bounds is the "inelastic buckling gradient". Duberg and Wilder\(^{(2)}\) introduced the I.B.G. for the symmetrical cross section and also determined the complete load-deflection curve for an idealized H section column using the Ramberg-Osgood\(^{(46)}\) stress-strain relations. Later Johnston\(^{(3)}\) modified the I.B.G. for a monosymmetric section and also considered the presence of the residual stress. In particular, he noted the difference in the values of the I.B.G. for a T shaped aluminum column when it buckles in the plane of symmetry towards or away from the flange. Augusti\(^{(47)}\) in his discussion of Johnston's paper pointed out that the relation given for I.B.G. does not hold for the case where buckling takes place in the elastic range and the residual stress is not present. Augusti\(^{(15, 48)}\) also introduced another bound - the full yield line using his buckling model (Fig. 1.3). This is useful when the yielding stress can be explicitly defined. However, for a material which is continuously strain hardened the choice of a yield point is not easy. Earlier in 1920, Osgood\(^{(49)}\) noted the
difference in the reduced modulus loads for the different directions
of buckling of a "T" section and presented a graphical representation
of the variation of the reduced modulus in buckling towards or away
from the flange relative to the tangent modulus. In his study, Osgood
assumed that the area was concentrated along the center lines of the
web and the flange. When buckling is in the plane of symmetry, the
assumption of a line area for the flange would cause a considerable
error. In this study a detailed solution has been presented.

2.1. THE INELASTIC BUCKLING GRADIENT

i. Definition

The inelastic buckling gradient (I.B.G.)\(^{(3)}\) is defined as the
initial slope at the tangent modulus buckling load of the curve obtained
by plotting column load as ordinate and maximum lateral column
deflection as abscissa. Derivation of I.B.G. for a material with a
continuously strain hardening characteristic in the neighborhood of
the stress corresponding to the tangent modulus load is a very simple
process. The value of I.B.G. depends on the shape of the cross
section as well as the elastic and inelastic properties of the column.

In all the following discussions and computations, the
Ramberg-Osgood\(^{(46)}\) stress-strain curves can readily be adapted to
the procedure used in this study. However, it was desired to use
the actual stress-strain characteristic of the material and also
study the transition from elastic into inelastic region of behavior,
as Ramberg-Osgood curves do not have a truly straight line
representing the elastic range. The material properties used here are obtained by deriving a close mathematical fit for the stress-strain curves obtained from the experimental work (Chapter VI).

**ii. Evaluation**

The average critical stress of an initially straight and prismatic column of a homogeneous, isotropic and monotonically strain hardening material is given by

\[ \sigma_c = \sigma_t = \frac{2 \sigma_t}{(\frac{K L}{r})^2} \]  

(2.1)

where \( E_t \) = slope of the stress-strain curve at the critical stress 
\( r \) = radius of gyration of the cross section with respect to axis about which buckling is to occur 
\( K \) = the equivalent length factor, such that to make \( KL \) equal to the distance between two inflection points of the column deflection curve 
\( \sigma_t \) = tangent modulus stress

The tangent modulus load, \( P_t \), is

\[ P_t = \sigma_t A \]  

(2.2)

where \( A \) is the area of the cross section. If buckling occurs in the elastic range the average critical stress is obtained by substituting the tangent modulus in equation (2.1) by the elastic modulus

\[ \sigma_c = \sigma_e = \frac{2 \sigma_e}{(\frac{K L}{r})^2} \]  

(2.3)

where \( \sigma_e \) is the Euler's critical buckling stress.
For a perfect hinged end condition, the equivalent length factor $K = 1$ and the shape of the deflection curve at the pending of buckling, when the resistance to bending along the column and across the cross section is uniform, is that of a half sine curve. An increment of deflected shape can be written in the following form

$$\Delta \delta = \Delta \delta_0 \sin \frac{\pi z}{L}$$  \hspace{1cm} (2.4)

where $\Delta \delta$ is the increment of the lateral deflection at a point having a distance $z$ from the end of the column, and $\Delta \delta_0$ is the magnitude of the increment of the deflection at the mid-length.

Let us assume that the section of the column has only one axis of symmetry and the bending is to be initiated in the plane of symmetry of the column, i.e., the column is stronger in the direction of the other principal axis at this stage. Then, for an infinitesimal increase in the axial load above the buckling load, assuming deflection starts in the negative $x$-direction, the distribution of the stress at the mid-length section would be as shown in Fig. 2.1. The curvature of the column would be one.
at a cross section which has a distance $z$ from the end of the column will be

$$\Delta \phi = - \Delta \delta \frac{\pi^2}{L^2} \sin \frac{\pi z}{L} \quad (2.5)$$

At the mid-height, $z = \frac{L}{2}$, the curvature will be

$$\Delta \phi_o = \Delta \phi \bigg|_{z=\frac{L}{2}} = - \Delta \delta \frac{\pi^2}{L^2} \quad (2.6)$$

The strain increment of an element with a coordinate $(x, y)$ will be

$$\Delta \phi = \Delta \delta (x + \bar{d}) \frac{\pi^2}{L^2} \quad (2.7)$$

For a small increment of strain, the following relation exists between change of strain and stress

$$\Delta \sigma = \Delta \epsilon \ E_t \quad (2.8)$$

Substituting for $\Delta \epsilon$ from (2.7) we get

$$\Delta \sigma = \Delta \delta (x - \bar{d}) \ E_t \frac{\pi^2}{L^2} \quad (2.9)$$

Now the increment of the axial load can be obtained by integrating $\Delta \sigma$ over the area of the cross section

$$\Delta P = \int_A \Delta \sigma \ dA \quad (2.10)$$

or

$$\Delta P = \int_A \Delta \delta (x + \bar{d}) \ E_t \frac{\pi^2}{L^2} \ dA$$

or

$$\Delta P = \Delta \delta \ E_t \frac{\pi^2}{L^2} \int_A (x + \bar{d}) \ dA \quad (2.11)$$

If the centroid of the cross section is chosen as the origin and
x, y are the principal axes, then

\[ \int_{A} (x + \overline{d}) \, dA = \overline{d} \ A \]  \hspace{1cm} (2.12)

where \( \overline{d} \) is the distance of the y-axis from the farthest left fiber.

Replacing for the value of the integral in the equation (2.11) from (2.12)

\[ \Delta P = \Delta \delta \ \frac{\pi^2 E_t}{L^2} \ A \ \overline{d} \]  \hspace{1cm} (2.13)

according to (2.1) and (2.2), \( \frac{\pi^2 E_t}{L^2} \ A = \frac{P_t}{r^2} \), so that

\[ \frac{\Delta P}{\Delta \delta} = \frac{P_t}{r^2} \overline{d} \]  \hspace{1cm} (2.14)

As \( \Delta \delta \) tends to zero the left hand side of the equation (2.14) becomes the slope of the load vs. deflection curve at the point of initiation of bending

\[ \lim_{\Delta \delta \to 0} \left| \frac{\Delta P}{\Delta \delta} \right| = \frac{P_t}{r^2} (\overline{d}) \]  \hspace{1cm} (2.15)

Had we assumed that the deflection was initiated in the positive direction of the x-axis in the above equation, \( \overline{d} \) would have been \( (d - \overline{d}) \) and the I.B.G. would have been

\[ \lim_{\Delta \delta \to 0} \left| \frac{\Delta P}{\Delta \delta} \right| = \frac{P_t}{r^2} (d - \overline{d}) \]  \hspace{1cm} (2.16)

This is the Johnston's modification of Duberg and Wilder's\(^{(2)}\) derivation of inelastic buckling gradient for a symmetrical cross section.
In the equations (2.15) and (2.16), the value in parentheses is the distance of the centroid from the outer fiber of the convex side measured along a line parallel to the plane of bending.

It should be mentioned that if buckling begins in the elastic range the deflection $U$ will have the form of a half sine curve with an indeterminate amplitude "$U_o$" such that

$$U = U_o \sin \frac{\pi z}{L} \quad (2.17)$$

As there is strain regression on the convex side, relation (2.7) will not be true. Consequently, the equations (2.15) and (2.16) will not hold for the initiation of lateral buckling in the elastic range. It is important to note that in the elastic range the slope of the load vs. deflection curve at the point of incipient buckling is equal to zero, i.e.,

$$\lim_{\Delta \delta \to 0} \left| \frac{\Delta P}{\Delta \delta} \right| = 0 \quad (2.18)$$

Buckling then takes place with no increase in the axial load. The actual behavior of the column in this case can also be determined by the incremental procedure.

As the curvature of the load-deflection curve of the column is negative, the line tangent to the load-deflection curve at the tangent-modulus load will provide an upper bound for the inelastic column behavior.
2.2. TANGENT MODULUS

The tangent modulus stress and load can be obtained from equations (2.1) and (2.2) respectively. In the equation (2.1), $\sigma_t$ (the tangent modulus stress) is expressed implicitly as $E_t$ (tangent modulus) itself is a function of $\sigma_t$. Usually, as in the case of this problem, stress and the tangent modulus are expressed as functions of strain. Therefore, in order to evaluate the tangent modulus stress, for a specific equivalent slenderness ratio $\left(\frac{KL}{r}\right)$, one must follow some iterative process to pick up the right value for strain whose corresponding values of the stress and the tangent modulus would satisfy the equation (2.1).

Formula (2.1) can be solved for $\left(\frac{KL}{r}\right)$ and written in the following form

$$\frac{KL}{r} = \pi \sqrt{\frac{E_t}{\sigma_t}}$$  \hspace{1cm} (2.19)

If it is desired to have a graphical representation between the effective slenderness ratio and the tangent modulus stress, one may for numerous values of strain evaluate the corresponding stress and tangent modulus, and then by means of equation (2.19) determine the effective slenderness ratio, and thus avoid the iteration and reduce the length of the manipulation. Graphical representation of (2.19) along with other plots can be seen in Figures 2.6 through 2.10. A thorough explanation of these graphs will be given in the following pages. As the tangent modulus stress is the stress at which an ideal column with only infinitesimal imperfections from perfect
concentric loading starts to bend, the line \( P = \tau \sigma_0 A \) parallel to the
deflection axis is a lower bound for the ultimate load carrying
capacity of the column. Shanley states, "For a given material, the
maximum load that can be developed will generally be only slightly
greater than the Engesser (tangent modulus) load". The incre-
mental procedure here will provide a comparison between the tangent
modulus load and the ultimate load for some chosen "T" columns.

2.3. REDUCED MODULUS (DOUBLE MODULUS)

i. Introduction and Definition

The reduced or double modulus load can be defined as the
load at which the inelastic buckling gradient becomes zero, if a
column is prevented from buckling above the tangent modulus load.
Therefore, it is an upper bound for the ultimate load carrying
capacity of the column; in fact, the double modulus load can never
be attained in a real column. In derivation of the reduced modulus
\( E_r \) or reduced modulus load \( P_r \), the following assumptions are
made:

i. On the convex side of the bent column where the
strain regression is to occur, stress and strain
are related by the elastic modulus "E", i.e.,
unloading is elastic \( \Delta \sigma = E \Delta \varepsilon \); this is proven
experimentally.

ii. On the concave side of the bent column, the incre-
ments of the stress and the strain are related by
the tangent modulus \((\Delta \sigma = E \Delta \varepsilon)\) associated with
the compressive strain at which buckling starts.

One may make a gross mistake by using the tangent modulus
obtained at the tangent modulus load in the evaluation of the reduced
modulus. This means that the tangent modulus has been assumed to
remain constant beyond the tangent-modulus stress. The reduced
modulus stress, evaluated on such basis, is shown on Fig. 2.9
and may be termed "false reduced modulus" or "false reduced
modulus load". As it is seen in the graph, the error of such
assumption becomes larger and larger as one departs from the
linear portion of the stress-strain relation.

ii. Evaluation

The double modulus varies with the shape of the cross section.
Von Kármán \(^{(8)}\) derived the formula (1.2) for the reduced modulus of
a rectangular cross section. In the case of the "T" section, the
double modulus depends not only on the dimensions of the section
but also on the buckling direction. \(^{(3)}\)

Let us assume that buckling occurs in the positive x
direction and the curvature of the bending (in the plane of symmetry)
is \((-\phi)\). During bending, since there is no change in the column
load, the algebraic sum of the change in the load of the unloading
segment \((\Delta P_r)\) and the loading segment \((\Delta P_l)\) should be equal to
zero (Fig. 2.2)

\[
\int_A \Delta \sigma \, dA = \Delta P_l + \Delta P_r = 0 \tag{2.20}
\]
or
\[ \int_{A_f} (x_r - \bar{d} - x) \phi \, E_t \, dA_f + \int_{A_r} (x_r - \bar{d} - x) \phi \, E \, dA_r = 0 \quad (2.21) \]

During bending, the internal resisting moment \( M_r \) is equal to the integral of the moment of the elements of the load change \( \Delta P = \Delta \sigma \, dA \) about the bending axis. As bending takes place about the "zero strain change axis (Z.S.C.A.)", \( x_r \), the moment should be taken about this axis, i.e.,

\[ M_r = \int_A \Delta \sigma \left[ x - (x_r - \bar{d}) \right] \, dA \]

or

\[ M_r = \int_A \Delta \sigma (x - x_r) \, dA + \bar{d} \int_A \Delta \sigma \, dA \quad (2.22) \]

Strain distribution line after buckling

Uniform strain across the cross section of a column held straight prior to buckling

Uniform stress across the cross section of a column held straight prior to buckling

Stress distribution line after buckling

Fig. 2.2. Stress and Strain Distribution at the Reduced Modulus Load
According to (2.20), the second integral is equal to zero, then

$$M_r = \int_A \Delta \sigma (x - x_r) \, dA$$  \hspace{1cm} (2.23)

The right hand side of the equation (2.23) is the moment with respect to the $y$ principal axis. Thus, it is shown that the internal resisting moment with respect to the Z.S.C.A. and the $y$ principal axis are equal. Equation (2.22) can be written in the following form

$$M_r = \int_{A_f} \left[ x - (x_r - \overline{d}) \right]^2 \phi E_t \, dA_f + \int_{A_r} \left[ x - (x_r - \overline{d}) \right]^2 \phi E \, dA_r$$  \hspace{1cm} (2.24)

Now consider the equation relating the moment to the curvature

$$M_r = E_r I_{yy}$$  \hspace{1cm} (2.25)

by solving equation (2.21) for $x_r$ and evaluating $M_r$ from (2.24) one can obtain $E_r$ from (2.25) in which $I_{yy}$ is the moment of inertia with respect to the principal axis $y$. The reduced modulus obtained in this case, where buckling is to the right, will be named $E_{rr}$, and otherwise $E_{rf}$. In order to evaluate $E_{rf}$, $E$ and $E_t$ should be interchanged in the equations (2.21) and (2.24).

Now consider the evaluation of $E_{rr}$. Let

$$\eta = \frac{E_t}{E}$$  \hspace{1cm} (2.26)
We will have three cases:

1. The Z.S.C.A. is in the web (Fig. 2.3a)

\[
\Delta \sigma_1 = (x_r \phi E_t) \Delta P_1 = (x_r \phi E_t) \frac{t}{2}
\]

\[
\Delta \sigma_2 = (x_r - t) \phi E_t
\]

\[
\Delta P_{21} = (x_r - t) \phi E_t \frac{t}{2}
\]

\[
\Delta \sigma_3 = -(d - x_r) \phi E_t
\]

\[
\Delta P_{3} = -(d - x_r) \phi E_t \frac{(d - x_r)}{2}
\]

Line of uniform stress across the cross section of the column held straight prior to buckling

Fig. 2.3a. Stress Distribution - Z.S.C.A. in the web

then

\[
\Delta P_1 + \Delta P_2 + \Delta P_{22} + \Delta P_3 = 0
\]

or

\[
E \phi \left\{ \eta \left[ (2x_r - t) \frac{t}{2} + (x_r - t)^2 \frac{w}{2} \right] - (d - x_r) \frac{w}{2} \right\} = 0 \quad (2.27)
\]

Equation (2.27) is a quadratic equation in terms of \( x_r \) and can be simplified and solved for \( x_r \)

\[
x_r = \left\{ \left[ d - \eta \left( \frac{b}{w} - 1 \right) t \right] + \left[ (d - \eta \left( \frac{b}{w} - 1 \right) t) \left( d^2 + \left( \frac{b}{w} - 1 \right) t^2 \eta \right) (1 - \eta) \right] \right\} ^{\frac{1}{2}} / (1 - \eta)
\]

(2.28)
\[ M_r = \Delta P_1 \left( x_r - \frac{t}{3} \right) + \Delta P_2 \left( x_r - \frac{2t}{3} \right) + \Delta P_{22} \frac{2(x_r - t)}{3} + \Delta P_3 \left( d - x_r \right) \frac{2}{3} \]

or
\[ M_r = E\phi \left\{ \eta \left[ x_r bt \frac{(x_r - t)}{2} + bt (x_r - t) (x_r - \frac{2t}{3}) + (x_r - t)^3 \frac{w}{3} \right] \right. \\
+ \left. (d - x_r)^3 \frac{w}{3} \right\} \quad (2.29) \]

2. The Z.S.C.A. is in the flange (Fig. 2.3b)

\[ \Delta \sigma = x_r \phi E_t \phi \frac{b}{2} \]
\[ \Delta \sigma_2 = (t-x_r) \phi E \]
\[ \Delta \sigma_3 = (d-x_r) \phi E \]
\[ \Delta P_1 = x_r \phi E_t x \frac{b}{2} \]
\[ \Delta P_{21} = -(t-x_r) \phi E \frac{(t-x_r)}{2} \]
\[ \Delta P_{22} = -(t-x_r) \phi E (d-t) \frac{w}{2} \]
\[ \Delta P_3 = -(d-x_r) \phi E \]

Stress distribution line
after buckling

Fig. 2.3b. Stress Distribution - Z.S.C.A.
in the flange

\[ \Delta P = 0 \] will give
\[ x_r = \left\{ \left[ t + w \frac{(d-t)}{b} \right] + \left[ (t + w \frac{(d-t)}{b})^2 - \left( t^2 + w \frac{(d^2 - t^2)}{b} \right) (1 - \eta) \right] \right\} \frac{1}{(1 - \eta)} \]

\[ (2.30) \]
and

$$M_r = E \phi \left\{ \eta \left( \frac{x}{3} \right) + \left( t - x_r \right) \left[ \frac{(d-t)}{3} + (t - x_r) \right] \frac{w}{2} \right\}$$

(2.31)

3. The Z.S.C.A. coincides with inner surface of the flange (Fig. 2.3c).

This is the limiting point between cases (1) and (2).

$$\Delta \sigma_1 = t \phi \frac{E_t}{t} \quad \Delta P_1 = t^2 b \phi \frac{E_t}{2}$$

$$\Delta \sigma_2 = (d-t) \phi E \quad \Delta P_2 = -(d-t)^2 w \phi \frac{E_t}{2}$$

$$\Delta P = 0$$ gives:

$$\eta = \eta_1 = \frac{(d-t)^2 w}{t^2 b}$$

(2.32)

Fig. 2.3c. Stress Distribution - Z.S.C.A. at the inner edge of the flange

For values of $\eta > \eta_1$ The Z.S.C.A. will be at the flange

and

for values of $\eta < \eta_1$ The Z.S.C.A. will be at the web
Considering equations (2.24) or (2.29) or (2.31) we can write

$$M_r = E I_{rr} \phi$$  \hspace{1cm} (2.34)

where $I_{rr}$ is the weighted moment of inertia of the cross section with respect to the Z.S.C.A. with the area on the elastic and inelastic portions each having a weight equal to 1 and "η" respectively, where "η" is defined in (2.26). Then using equation (2.25)

$$E_{rr} = \frac{EI_{rr}}{l_{yy}}$$  \hspace{1cm} (2.35)

The preceding equations and discussion of $E_{rr}$ will hold for $E_{r\ell}$ by interchanging $x_r$ and $x_\ell$ (distance of the Z.S.C.A. from the left face of the flange, in buckling to the left) and also $E$ and $E_t$ (because the elastic and inelastic regions will interchange) in the equations (2.24) through (2.33).

The reduced modulus stress is given by the following relation

$$\sigma_r = \frac{2 E_r}{(KL/2)^2}$$  \hspace{1cm} (2.36)

which is the same as the expression for the tangent modulus stress in which $\sigma_t$ and $E_t$ are replaced by $\sigma_r$ and $E_r$. As for every value of tangent modulus we have two different reduced moduli, $E_{rr}$ and $E_{r\ell}$. We will have two distinct values for double modulus stress, one in buckling to the right, $\sigma_{rr}$, and the other in buckling to the left, $\sigma_{r\ell}$. Each is an upper bound for the ultimate load carrying capacity of the column in buckling towards the direction associated with its derivation. Equation (2.36) is an implicit function, as both
\( \sigma_r \) and \( E_r \) are functions of strain. (Both stress and the tangent modulus are expressed as functions of strain.) In order to evaluate the \( \sigma_r \) for a given \( \frac{KL}{r} \), one has to apply some iterative process.

For a chosen value of strain, stress and the tangent modulus and thereby the reduced modulus can be calculated. Having the reduced modulus, one can obtain the corresponding effective slenderness ratio from

\[
\left( \frac{KL}{r} \right)_r = \pi \sqrt{\frac{E_r}{\sigma}} \tag{2.37}
\]

This has been done for numerous values of strain, the graphs of \( \left( \frac{KL}{r} \right)_{rr} \) (or \( \left( \frac{KL}{r} \right)_{rf} \)) vs. \( \sigma \) and \( E_{rr} \) (or \( E_{rf} \)) are shown in Figures 2.7 and 2.8. Also the variation of \( \frac{E_r}{E} \) vs. \( \frac{E_t}{E} \) in buckling towards or away from the flange is shown in Fig. 2.9.

2.4. CRITICAL TORSIONAL BUCKLING

In order to have lateral buckling in the column, the dimensions of the column should be such that the torsional buckling will not occur prior to the lateral buckling.

In 1929 Wagner\(^{(50)}\) explained the phenomenon of the elastic torsional buckling in the following manner: Let the member be subjected to a twist and \( z \) be the axis of rotation passing through the center of twist of the cross section (Fig. 2.4). If rotation at \( z \) is \( \beta \) and at \( z + dz \) it is \( (\beta + \frac{d\beta}{dz} dz) \), then the rotation of a fiber
which has a distance $\rho$ from the center of twist will be $\rho \frac{d\beta}{dz}$.
Wagner suggested that the component of the normal stress ($\frac{P}{A}$), when displaced due to rotation, on the plane perpendicular to the rotation axis, will induce a torque, i.e.,

![Diagram](image)

**Fig. 2.4. Torsional Buckling**

\[
M_z = \int A \frac{P}{A} \rho \frac{d\beta}{dz} \rho \; dA = \frac{P}{A} I_p \frac{d\beta}{dz}
\]  

(2.38)

where $I_p$ = polar moment of inertia with respect to the center of twist.
If the torsion is uniform, $\frac{d\beta}{dz} = \theta$, and the warping rigidity $C_w$ is negligible, then at the equilibrium position

\[
M_z = C \theta
\]  

(2.39)

where

\[
C = \text{torsional stiffness (G J)}
\]

\[
G = \text{shear modulus}
\]

and for a T section

\[
J = \frac{1}{3}bt^3 + \frac{1}{3}w(d - t)^3 - 0.21t^4 - 0.105w^4 + \alpha_1D^4
\]

(For explanation refer to Fig. 4.21.) Combining equations (2.38) and (2.39), one may find the critical torsional buckling stress

\[
\sigma_{ct} = \frac{C}{I_p}
\]  

(2.40)
If the lateral buckling is to occur prior to the torsional buckling in the inelastic range, the above explanation and the derivation of critical torsional buckling stress will hold with one modification. Since in the inelastic range the shear stress and strain are related to one another by $G_5$ (secant shear modulus - equation (4.68)) instead of $G$ for the elastic range, then the torsional stiffness coefficient in the equation (2.40) will be

$$C = G_5 \ J$$

(2.41)

2.5. LOCAL BUCKLING

The local buckling phenomenon should be studied prior to the choice of the dimensions of the cross section. The web of some of the tee shape specimens chosen for the test buckled locally. In order to study the local buckling in the flange or the web of the tee column, one may treat it as a flat plate being compressed from opposite edges. At a critical stress the plate may buckle out of its plane prior to the commencement of the overall column buckling. Such buckling may occur either in the elastic or the inelastic range.

In 1891 Bryan\(^{51}\) presented a solution for the elastic buckling strength of a long rectangular plate simply supported along all edges and subjected to a longitudinal compressive load. Bleich\(^{22}\) suggested the following equation for the approximate local buckling stress (in the elastic or inelastic range) of a flat plate segment of a long column under uniform compressive load.\(^{55,56}\)
\[ \sigma_{c_t} = k \frac{\pi^2 E \sqrt{\eta}}{12(1 - \nu^2)(\frac{b}{t})^2} \]

where \[ \eta = \frac{E_t}{E} \]

\[ \nu = \text{Poisson's ratio} \]

\[ \frac{b}{t} = \text{width-thickness ratio} \]

and \( k \) depends on the longitudinal boundary conditions. For one edge fixed and the other free \( k = 1.277 \) (Fig. 2.5).

In recent years Stowell, (52) Haaijer (53) and Thürlimann (54) have studied the inelastic local buckling of steel plates in the strain-hardening range. However, the problem of this thesis is more complex than for a uniform compressive stress distribution, as the column cross section will be subjected to a varying strain. This condition results in a varying stress and differing tangent moduli for different elements. The presence of strain regression will make the problem still more complicated. For the practical purpose of this thesis equation (2.42) will be used together with the assumption that one edge of the plate is fixed and the other is free and that the maximum stress acting on the cross section of the flange or the web is acting uniformly across and along that segment of the column. The graph of uniform compressive
stress vs. the critical local buckling stress, based on equation (2.42),
for the different widths of the web (d = 1.5'', 2.5'', 3'') and the flange
(b = 4'') with the line $\sigma = \sigma_{c, \ell}$ (drawn to scale) is shown on Fig. 2.10.
Fig. 2.6. Stress-Strain, Tangent Modulus Strain and Secant Shear Modulus Strain (#1).

(Mathematical Fit and the Experiments)
Fig. 2.7. Tangent and Reduced Moduli in Buckling to the Left and to the Right vs. Slenderness Ratio
Fig. 2.8. Reduced Modulus in Buckling to the Left and to the Right vs. Tangent Modulus
Fig. 2.9. Tangent, Reduced, False Reduced Modulus Stresses and Ultimate Average Stress vs. Slenderness Ratio
Fig. 2.10. Critical Local Buckling for Flange and Web vs. Uniform Compressive Stress
CHAPTER III

THE EQUATIONS OF FLEXURAL-TORSIONAL BUCKLING

3.1. SELECTION OF THE COORDINATES

In order to best write the equations of flexural-torsional buckling of a column, it is essential to choose and define a proper coordinate system. In the case of a "T", the cross section is composed of two rectangular segments with a rather large width to thickness ratio. The following assumptions can be made which will result in considerable simplification of the method of solution of the problem with a negligible deviation from the exact solution:

i. Only St. Venant torsion is present and the contribution of warping restraint to the torsional resistance is negligible.

ii. At all stages of the deformation of the column, the center of twist remains at the point of intersection of the center lines of the web and the flange. At any cross section under consideration the movement of the twist center is determined by the manner of distribution of the normal stress, and also by the ratio of the width of the flange to the width of the web. However, the smallness of this movement justifies the above assumption.
In cases where it is desired to take the effect of the warping or the effect of change of location of the center of twist into consideration, one may adapt the procedures of this chapter with modifications which will be explained. For simplicity, the origin of the \(xyz\) coordinate system is chosen to coincide with the center of twist of the top face of the column. The stationary \(xyz\) Cartesian coordinate system is oriented with respect to the unloaded shape of the ideal column (Fig. 3.1a), such that \(x\)- and \(y\)-axes lie along the center line of the web and the flange respectively, and the positive direction of the \(z\)-axis is chosen from top to the bottom of the column.

The \(\xi\), \(\eta\) and \(\zeta\), mutually orthogonal coordinate axes, are so chosen that \(\xi\) and \(\eta\) axes coincide with the center-line axes of the flange and the web of the displaced cross section, respectively.
Thus, the $\xi\eta$ plane is taken to be fixed to the cross section under consideration. Therefore, its lateral motion and twist is the same as that of the column at that cross section. This makes the $\zeta$ axis remain tangential to the deflected longitudinal axis passing through the center of twist of the cross section of the column (Figs. 3.1b, c).
3.2. TRANSFORMATION OF COORDINATES

The x-, y- and z-axes are taken to be the initial reference axes and \( \xi, \eta \) and \( \zeta \) are the moving axes. In general, \( \xi\eta\zeta \) has a translation and a rotational motion relative to the xyz-axes. Neglecting the component of the translation of the \( \xi\eta\zeta \) axes in the z direction (due to shortening of the column) we can designate the components of the displacement of the center of twist (the origin of the \( \xi\eta\zeta \) axes) from its initial position in the x and y directions by \( U \) and \( V \) respectively (Figs. 3.2a, b, c).

Fig. 3.2a. Transformation in the Cross Section Parallel to the xy Plane

Fig. 3.2b. Transformation in the Cross Section Parallel to the yz Plane

Fig. 3.2c. Transformation in the Cross Section Parallel to the zx Plane
The rotational part of the motion can be designated by the angle of rotation \( \beta \), taken positive clockwise. The angle of rotation \( \beta \) is small (\( \beta \ll 1 \) radian) so we may approximate \( \sin \beta \) by \( \beta \) and \( \cos \beta \) by 1. Also we assume that \((U + U_o)^t\) and \((V + V_o)^t\) are small. Then we express the matrix \([b]\) in the following form

\[
[b] = \begin{bmatrix}
1 & \beta & -(U + U_o)^t \\
-\beta & 1 & -(V + V_o)^t \\
(U + U_o)^t & (V + V_o)^t & 1
\end{bmatrix}
\] (3.1)

where \( U \) and \( V \) are lateral deflections from the unloaded position in the \( x \) and \( y \) directions respectively and \( U_o \) and \( V_o \) are the initial crookedness. \( U, V, U_o \) and \( V_o \) all are functions of \( z \), the longitudinal axis.

\( \xi, \eta, \xi \) and \( x, y, z \) are related by \([b]\) in the following form

\[
\begin{bmatrix} \xi, \eta, \xi \end{bmatrix} = \begin{bmatrix} x, y, z \end{bmatrix} [b]^T
\] (3.2)

3.3. BOUNDARY AND INITIAL CONDITIONS

i. Boundary Conditions

In the case of the concentric loading the boundary conditions at the ends of the column \( z = 0 \) and \( z = L \) are

\[
U = V = 0 \quad \text{No lateral displacement.}
\]
\[
U'' = V'' = 0 \quad \text{No restraints against rotation about the x- and y-axes are imposed.}
\]
\[
\beta = 0 \quad \text{No restraint against twisting about the longitudinal axis is imposed, so it will}
\]
twist from its original position due to torque resulting from the component of the moment about the y-axis along the \( \zeta \) axis; but the position of the end section at \( z = 0 \) is assumed to be the reference position and therefore \( \beta \) is taken to be equal to zero. In the case of the symmetrical behavior with respect to the mid-height section of the column, i.e., symmetrical initial crookedness, the twist at the other end \( (z = L) \) will also be zero.

At the mid-height section, \( z = \frac{L}{2} \)

\[ U' = V' = 0 \quad \text{Slope of the lateral deflection curve is zero - symmetrical case.} \]

\[ \beta' = 0 \quad \text{Twist is maximum; torque is zero.} \]

In the case of eccentric loading the above conditions hold except when they are contrary to the following:

\begin{align*}
U'' \neq 0 \text{ at } z = 0, L & \quad \text{When end eccentricity exists in the } x \text{ direction.} \\
V'' \neq 0 \text{ at } z = 0, L & \quad \text{When end eccentricity exists in the } y \text{ direction.} \\
\beta \neq 0 \text{ at } z = L & \quad \text{When eccentricity at two ends and/or initial imperfection are unsymmetrical.}
\end{align*}
The clockwise twist is assumed to be positive. The twist of the columns results from a positive torque in the upper portion (from $z = 0$ to where $\beta' = 0$) of the column and a negative torque in the lower portion (where $\beta' = 0$ to $z = L$).

ii. End Eccentricity

In the case of eccentric loading the curvature of the deflected shape at any end section in the plane parallel to the line of eccentricity will be different from zero and its value will be determined by the magnitude of the eccentricity, the axial load and also by the moment-curvature properties of the cross section. In a general case, eccentricity with respect to both $x$- and $y$-axes and at both ends of the column can be considered. If eccentricity of loading exists with respect to the $y$-axis, then the biplanar bending accompanied by torsion will commence with the start of loading. The moments about the $x$- and $y$-axes resulting from the eccentric loading alone at a cross section which has a distance $z$ from the top can be written as (Figs. 3.3a, b, c)

$$M_{xe} = P \left[ E_{ty} + \left( \frac{E_{by} - E_{ty}}{L} \right) z \right]$$

$$M_{ye} = -P \left[ E_{tx} + \left( \frac{E_{bx} - E_{tx}}{L} \right) z \right]$$

(3.3)
Fig. 3.3a. Eccentric Loading

where $E_{tx}$ and $E_{ty}$ are the eccentricity in the x and y directions at the top of the column and $E_{bx}$ and $E_{by}$ are the same values at the bottom section of the column. Let us assume
\[ e_{tx} = \frac{E_{tx}}{L} \]

\[ e_{ty} = \frac{E_{ty}}{L} \]

and

\[ r_{ex} = \frac{E_{bx}}{E_{tx}} \]

\[ r_{ey} = \frac{E_{by}}{E_{ty}} \]  \hspace{1cm} (3.4)

Then the moment due to eccentricity will be

\[ M_{xe} = P \frac{e_{ty}}{L} \left[ 1 + (r_{ey} - 1) \frac{z}{L} \right] \]

\[ M_{ye} = -P \frac{e_{tx}}{L} \left[ 1 + (r_{ex} - 1) \frac{z}{L} \right] \]  \hspace{1cm} (3.5)

iii. Initial Crookedness of the Column

Initial imperfection in the direction of one or both principal axes can be taken into consideration in this analysis. Initial imperfection can be either expressed in the form of a function or a vector which gives the value of the initial crookedness at the nodal points along the length of the column. These values can be obtained by measuring the actual initial crookedness at the required sections.

A sinusoidal or parabolic shape of crookedness can be expressed in the following forms:

sinusoidal shape:

\[ u_o(z) = \Delta_{ix} \sin \left( \frac{\pi z}{L} \right) \]

\[ u_o(z) = \delta_{ix} \sin \frac{\pi z}{L} \]

or

\[ v_o(z) = \Delta_{iy} \sin \left( \frac{\pi z}{L} \right) \]

\[ v_o(z) = \delta_{iy} \sin \frac{\pi z}{L} \]  \hspace{1cm} (3.6)
parabolic shape:

\[
U_o(z) = \Delta_{ix} \frac{z(L - z)}{4L^2} \quad u_o(z) = \delta_{ix} \frac{z}{4L} \left(1 - \frac{z}{L}\right)
\]

\[
V_o(z) = \Delta_{iy} \frac{z(L - z)}{4L^2} \quad v_o(z) = \delta_{iy} \frac{z}{4L} \left(1 - \frac{z}{L}\right)
\]

or

\[
\Delta_{ix} \text{ and } \Delta_{iy} \text{ are the magnitude of initial crookedness at the mid-height section and}
\]

\[
\delta_{ix} = \frac{\Delta_{ix}}{L}
\]

\[
\delta_{iy} = \frac{\Delta_{iy}}{L}
\]

\[
u_o = \frac{U_o}{L}
\]

\[
v_o = \frac{V_o}{L}
\]

\[
\text{The moment due to initial crookedness alone at a section which has a distance } z \text{ from the top section will be}
\]

\[
M_{x1}(z) = -P v_o(z) L
\]

\[
M_{y1}(z) = P u_o(z) L
\]

3.4. THE EQUATIONS OF FLEXURAL-TORSIONAL BUCKLING

i. Moment Curvature Relation

Positive moment or torque is represented by a vector pointing in the positive direction of the \(xyz\) or \(\xi\eta\zeta\) coordinate systems with directions expressed in accordance with the right-hand rule at the top face. In the elastic range of the behavior of the column, the
following relations hold between the moment and the flexure at all
the sections along the length of the column \(^{(57)}\)

\[
\begin{align*}
M_x &= -EI_x V''(z) \\
M_y &= EI_y U''(z)
\end{align*}
\] (3.10)

Beyond the proportionality limit equations (3.10) no longer hold and
the introduction of a functional relationship between the moment and
the curvature is extremely difficult, if not impossible.

In planar bending the moment curvature relation will be a
function of the cross sectional shape, the average stress, the depth
of the strain regression, the direction of buckling, and the stress-
strain relation. In biplanar bending the difficulties in expressing
the moment-curvature relation will increase because the moment-
curvature relation in one principal plane will be an implicit function
of the same relation in the other principal plane. Interaction curves
can be prepared for any stress-strain relation and cross section
to give the relation between the moment and the curvature to be used
in manual calculation. However, in feeding these interaction curves
to the computer memory either they must be expressed as algebraic
functions (or a group of algebraic functions), or be given in the form
of a matrix.

In the first case one has to develop these functions for any
specific stress-strain relationship and the cross sectional shape.
In general, these functions cannot precisely represent the inter-
action curves, but some acceptable approximation can be made.
If the interaction curves are prepared in the form of a matrix, the number of the elements of the matrix grows larger as one tends to reduce the error of the solution. For an acceptable error the elements of the matrix will occupy a considerable number of locations of the computer memory and therefore make the other parts of the computer manipulation extremely difficult. An alternative solution is to choose the number of the elements of the matrix as few as possible and then use the interpolation procedure to obtain a closer answer. However, this process requires extra computation in addition to the preparation of interaction curves and the related matrix and the error of computation will be significant.

For the above mentioned reasons and because in incremental column analysis, the parameters of the moment-curvature relation, such as average normal strain, depth of strain regression, axial load and lateral deflection in x and y directions also have to be determined on an iterative basis, no attempt has been made to develop a separate and direct moment-curvature relation for the inelastic range of the column behavior. However, during every increment of the axial load and the lateral deflection of the column the moment and curvature relation has been evaluated. A detailed description of the iterative procedure is given in Chapter V.

ii. External Moment and Force Components

Moment Components:

The components of the moment due to deflection caused by the concentric load $P$ can be written as
\[ M_{xd}(z) = -PL \nu(z) \]
\[ M_{yd}(z) = -PL \left[ \frac{X}{L} - u(z) \right] \]

The moment components due to the load eccentricity, initial imperfection and the lateral deflection from the unloaded position will be

\[ M_{x(e)} = M_{xd} + M_{xi} + M_{xe} \]
\[ M_{y(e)} = M_{yd} + M_{yi} + M_{ye} \] \hspace{1cm} (3.12)

\[ M_{z(e)} = 0 \]

or

\[ M_{x(e)} = -PL \left[ \nu(z) + \nu_o(z) - e_{ty} (1 + (r_{ey} - 1) \frac{z}{L}) \right] \]
\[ M_{y(e)} = -PL \left[ \frac{X}{L} - u(z) - u_o(z) + e_{tx} (1 + (r_{ex} - 1) \frac{z}{L}) \right] \] \hspace{1cm} (3.13)

\[ M_{z(e)} = 0 \]

Using the transformation relation (3.2) we can write

\[ M_{\xi(e)} = M_{x(e)} + \beta M_{y(e)} \]
\[ M_{\eta(e)} = -\beta M_{x(e)} + M_{y(e)} \] \hspace{1cm} (3.14)

\[ M_{\xi1(e)} = M_{x(e)} \left[ U(z) + U_o(z) \right] + M_{y(e)} \left[ V(z) + V_o(z) \right] \]

Torque applied externally on the cross section under consideration will have another component besides \( M_{\xi1(e)} \). This component results from the torque produced by the components of the normal stress on the plane of the cross section (refer to equation (2.38)). Hence,

\[ M_{\xi2(e)} = \frac{d\beta}{dz} \int_A \sigma \rho^2 dA \]
where \( \rho \) is the distance of the element from the center of the twist and \( \sigma \) is the normal stress acting on that element (Fig. 3.4). Therefore, torque exerted can be written as

\[
M_{\xi}(e) = M_x [U(z) + U_o(z)]' + M_y [V(z) + V_o(z)]' - \frac{d\beta}{dz} \int_A \sigma \rho^2 \, dA
\]  
(3.15)

**Force Components:**

The only force assumed to be acting is the axial load \( P \) so

\[
F_x(e) = 0 \\
F_y(e) = 0 \\
F_z(e) = P
\]  
(3.16)

Using the transformation relations we get

\[
F_{\xi}(e) = -P [U(z) + U_o(z)]' \\
F_{\eta}(e) = -P [V(z) + V_o(z)]' \\
F_{\zeta}(e) = P
\]  
(3.17)
iii. Internal Resisting Moment and Force Components

Moment Components:

It is assumed that:

1. the effect of the shear stress in lateral displacement is negligible.

2. the slope of the deflected curve is small \((U', V' << 1)\);

therefore, the components of the axial force \(P\) on the \(\xi\) and \(\eta\) axes can be taken equal to zero

\[
F_{\xi}(e) = F_{\eta}(e) = 0 \quad (3.18)
\]

3. uniaxial stress-strain relation will not change due to the shear stress (Chapter IV).

4. strain at any element of the cross section is proportional to its distance from the center of curvature.

Having made the above assumptions we can write

\[
M_{\xi(i)} = -\int_A \sigma(\xi, \eta) \eta \ dA \\
M_{\eta(i)} = \int_A \sigma(\xi, \eta) \xi \ dA 
\]  

(3.19)

and the resisting torque \(^{(58)}\)

\[
M_{\xi(i)} = C\beta' - C_w \beta'''
\]

where \(C\) is the torsional stiffness and is discussed in Chapter IV.

The torque resisted by bending resulting from non-uniform warping of adjacent cross sections \((C_w \beta''')\), where \(C_w\) is called warping rigidity, is assumed to be negligible in this case; therefore,

\[
M_{\xi(i)} = C\beta' \quad (3.20)
\]
Force Components:

\[ F_{\xi(i)} = 0 \]
\[ F_{\eta(i)} = 0 \]
\[ F_{\xi(i)} = -\int_A \sigma(\xi, \eta) \, dA \]  \hspace{1cm} (3.21)

3.5. EQUILIBRIUM CONDITIONS

The equilibrium conditions can be written as

\[ F_{\xi(e)} + F_{\xi(i)} = 0 \]
\[ F_{\eta(e)} + F_{\eta(i)} = 0 \]  \hspace{1cm} (3.22)
\[ F_{\xi(e)} + F_{\xi(i)} = 0 \]

and

\[ M_{\xi(e)} + M_{\xi(i)} = 0 \]
\[ M_{\eta(e)} + M_{\eta(i)} = 0 \]  \hspace{1cm} (3.23)
\[ M_{\zeta(e)} + M_{\zeta(i)} = 0 \]

substituting for the values of \( F \)'s and \( M \)'s in the above equations
\begin{align*}
P - \int_A \sigma(\xi, \eta) \, dA &= 0 \\
PL \left\{ \begin{array}{l}
\left[ v(z) + v_0(z) \right] - e_{ty} \left[ 1 + (r_{ey} - 1) \frac{z}{L} \right] \\
+ \beta \left[ \frac{x}{L} - u(z) - u_0(z) \right] + e_{tx} \left[ 1 + (r_{ex} - 1) \frac{z}{L} \right] \end{array} \right\} + \int_A \sigma(\xi, \eta) \, dA = 0 \\
PL \left\{ \begin{array}{l}
\beta \left[ v(z) + v_0(z) \right] - e_{ty} \left[ 1 + (r_{ey} - 1) \frac{z}{L} \right] \\
- \left[ \frac{x}{L} - u(z) - u_0(z) \right] + e_{tx} \left[ 1 + (r_{ex} - 1) \frac{z}{L} \right] \end{array} \right\} + \int_A \sigma(\xi, \eta) \, dA = 0 \\
PL^2 \left\{ \begin{array}{l}
\left[ v(z) + v_0(z) \right] - e_{ty} \left[ 1 + (r_{ey} - 1) \frac{z}{L} \right] \left[ u(z) + u_0(z) \right]' \\
+ \left[ \frac{x}{L} - u(z) - u_0(z) \right] + e_{tx} \left[ 1 + (r_{ex} - 1) \frac{z}{L} \right] \left[ v(z) + v_0(z) \right]' \end{array} \right\} \\
+ \beta' \int_A \sigma(\xi, \eta) \rho^2 \, dA - C\beta' = 0 \quad (3.24)
\end{align*}

Equilibrium equations (3.24) should be satisfied for all the cross sections along the column. The above equilibrium equations are very general and take into account the biplanar bending accompanied by the twist with or without initial crookedness and end eccentricity of loading.

3.6. WARPING EFFECT

In case consideration of warping deformation is needed for a more refined solution, the elastic warping displacement at any point of a cross section, \(^{*}\) having relatively thin walls, can be calculated from\(^{36, 59}\)

\(^{*}\)Does not apply near the longitudinal edges but this has a negligible effect for thin walled sections.
\[ w = w_o - \beta' \int_0^s r_s \, ds - \int_0^n r_n \, dn \] (3.25)

where \( w_o \) denotes the displacement in the z direction of the point from which \( s \) is measured (Fig. 3.5).

The integral \( \int_0^s r_s \, ds \) represents twice the area swept by the radius \( r \) as we move along the middle line of the cross section from the origin of \( s \), while the integral \( \int_0^n r_n \, dn \) represents twice the area swept by the same radius as we move from the center line to the element along the normal line. The swept area should be taken positive if the radius \( r \) is rotating in the positive direction, i.e., from \( \xi \) to \( \eta \). The contribution of the second integral is negligible for thin-walled cross sections.

In the inelastic region equation (3.25) cannot be used and a rigorous solution is needed to evaluate the warping. Warping of cross sections subjected to combined elastic and inelastic torsion is discussed in Chapter IV. However, in Chapter IV it is assumed that the center of twist coincides with the intersection of the center lines of the flange and the web. In order to modify this assumption
one should, by using the shear stress values obtained from rigorous solution, determine the new center of twist. If the new center of twist did not (with an acceptable approximation) coincide with the assumed one, a new center of twist should be chosen and the evaluation of warping displacement, strain, shear stress, and the center of twist should be repeated until an acceptable solution is obtained. After the instantaneous center of twist is located, the term "$p$" in equation (3.15) should be redefined as the distance between the element and the instantaneous center of twist. Also, the change of strain due to the warping displacement should be taken into consideration.

3.7. PLANAR BENDING

Equilibrium equations (3.24) can be simplified for the case of planar bending. Cases studied in this thesis exclude the initial imperfection or eccentricity of loading in the plane parallel to the flange. Also, it is assumed that the column is initially stronger in the flange direction than in the web direction. Previously it was explained that in case buckling starts in the plane of symmetry of the "T" section towards the flange, theoretically, planar bending will govern the behavior of the column. But when buckling initiates in the direction away from the flange (to the right) up to the stage where bending stiffness of the column in the plane of the symmetry is greater than the bending stiffness in the plane parallel to the flange,
the biplanar bending will not commence. Therefore, it is advanta-
geous to express the simplified equilibrium conditions for the planar
bending in the plane of symmetry. In this case

\[ \beta = \beta' = v = v_o = v' = v'_o = e_{ty} = r_{ey} = 0 \]

and

\[ \sigma(\xi, \eta) \equiv \sigma(\xi) \]  

(3.26)

Hence, the equilibrium equations will reduce to two equations as
follows

\[
P - \int_{-\frac{t}{2}}^{\frac{d-\frac{t}{2}}{2}} t(\xi) \sigma(\xi) \, d\xi = 0
\]

(3.27)

\[
PL \left[ \frac{\bar{x}}{L} - u(z) - u_o(z) \right] + e_{tx} \left[ 1 + (r_{ex} - 1) \frac{z}{L} \right] - \int_{-\frac{t}{2}}^{\frac{d-\frac{t}{2}}{2}} t(\xi) \sigma(\xi) \, d\xi = 0
\]

(3.27)

\[ t(\xi) \] is the dimension of the cross section in \( \eta \) direction

\[
t(\xi) = b \quad -\frac{t}{2} < \xi < \frac{t}{2}
\]

(3.28)

\[
t(\xi) = w \quad \frac{t}{2} < \xi < d-\frac{t}{2}
\]

Limiting Stage of Planar Bending

When the non-linear distribution of the stress exists on the
cross section, the bending axis will no longer coincide with the
principal axis parallel to the flange, or we may say that the "principal
axis" about which the bending stiffness of the cross section (weighted
moment of inertia) is to be minimum, will move parallel to its
original position. If the bending axis is a line \( \xi = \bar{\xi} \) then the bending
stiffness in the plane of symmetry can be expressed as
\[(B.S.)_{\xi} = \int_{-\frac{t}{2}}^{\frac{d}{2}} \frac{d}{2} t(\xi) (\xi - \bar{\xi})^2 E_t(\xi) \, d\xi \quad (3.29)\]

\[\bar{\xi} \text{ has to be evaluated such that it would make } (B.S.)_{\xi} \text{ a minimum, i.e.,}\]

\[\frac{d}{d\xi} (B.S.)_{\xi} = 0 \quad (3.30)\]

\[\frac{d}{d\xi} \int_{-\frac{t}{2}}^{\frac{d}{2}} t(\xi) (\xi - \bar{\xi})^2 E_t(\xi) \, d\xi = 0\]

t(\xi) in the above equation is sectionally continuous (3.28).

If the stress and its derivative with respect to the strain in the inelastic range is a continuous function of strain, then \(E(\xi)\) will also be sectionally continuous, separated by the border line of the elastic and inelastic zones. The integral equation (3.30) can be divided to the sectionally continuous domains and integrated and finally written in the following form

\[\frac{d}{d\xi} \left[ A_1 \frac{\xi^2}{\xi} + A_2 \bar{\xi} + A_3 \right] = 0 \quad (3.31)\]

where

\[A_1 = \int_{-\frac{t}{2}}^{\frac{d}{2}} t(\xi) E(\xi) \, d\xi\]

\[A_2 = -2 \int_{-\frac{t}{2}}^{\frac{d}{2}} \xi \cdot t(\xi) E(\xi) \, d\xi \quad (3.32)\]

\[A_3 = \int_{-\frac{t}{2}}^{\frac{d}{2}} t(\xi) \cdot \xi^2 E(\xi) \, d\xi\]
Equation (3.31) is a continuous function with respect to $\xi$; therefore, according to Leibnitz's rule

$$
\int_{-\frac{t}{2}}^{d-\frac{t}{2}} \frac{d}{d\xi} \left[ t(\xi) \ E_t(\xi) \ (\xi - \xi_0)^2 \right] d\xi = 0 \quad (3.33)
$$

or

$$
\int_{-\frac{t}{2}}^{d-\frac{t}{2}} \left[ t(\xi) \ E_t(\xi) \ (\xi - \xi_0) \right] d\xi = 0 \quad (3.34)
$$

Solving (3.33) for $\xi$ we get

$$
\xi = \frac{\int_{-\frac{t}{2}}^{d-\frac{t}{2}} t(\xi) \ E_t(\xi) \ d\xi}{\int_{-\frac{t}{2}}^{d-\frac{t}{2}} t(\xi) \ E_t(\xi) \ d\xi} \quad (3.35)
$$

Now the bending stiffness in the plane of symmetry can be found by substituting the value of $\xi$ from (3.35) in (3.29). While bending is taking place in the plane of symmetry, axis of symmetry remains as the axis about which the bending stiffness is maximum. Bending stiffness in bending out of the plane of symmetry can be expressed in the following form

$$
(B.S.)_\eta = \int_{\xi_1}^{\xi_2} \int_{\eta_1}^{\eta_2} E_t(\xi) \ \eta^2 \ d\eta \ d\xi
$$

or
\[
\text{(B.S.)}_\eta = \frac{b^3}{3} \int_{-\frac{t}{2}}^{\frac{t}{2}} E_t(\xi) \, d\xi + \frac{w^3}{3} \int_{-\frac{d}{2}}^{\frac{d}{2}} E_t(\xi) \, d\xi \quad (3.36)
\]

Let \( \alpha \) be the ratio of the bending stiffness in the plane parallel to the flange \((\text{B.S.})_\eta\) to the bending stiffness in the plane of symmetry \((\text{B.S.})_\xi\), i.e.,

\[
\alpha = \frac{(\text{B.S.})_\eta}{(\text{B.S.})_\xi} \quad (3.37)
\]

As long as the value of \( \alpha \) for all the cross sections along the column remains greater than unity, biplanar bending will not commence.
CHAPTER IV

TORSIONAL STIFFNESS

4.1. INTRODUCTION

At the stage where the elastic limit is surpassed and a linear relation between stress and strain no longer exists, the presence of bending moment in the cross section will subject different elements of the cross section to a varying strain, and therefore their inelastic properties (i.e., tangent modulus, secant shear modulus, Poisson's ratio) will vary from one point to another. At some stage of loading when strain regression is present, an abrupt change in the material properties of the neighboring elements of the border line of the elastic (regression) and inelastic zones of the cross section exists. Evaluation of the torsional stiffness coefficient of a cross section under above condition is a complex problem. The behavior of a member in the above condition is similar to that of a composite member made of numerous elements with different "elastic" properties.

In inelastic column analysis, when torsion commences, one encounters such problems as the relation between the amount of torque applied and the rotation produced or the shear stress distribution.
The precise solution of this problem is very complex, involving three dimensional analysis, as torsional moment as well as flexural moment varies along the column. In practice a variety of geometrically complicated composite sections are being used. Composite sections are used in reinforced concrete and aircraft, as well as some very specialized applications. Some problems on the torsion of compound prismatic bars have been treated analytically by Muskhelishvili,\(^{61}\) Cowan,\(^{62}\) Mitra,\(^{63}\) Gorgidze,\(^{64}\) Sherman,\(^{65}\) Craven,\(^{66}\) Takeyama,\(^{67}\) and Suharevski.\(^{68}\) These problems generally dealt with cross-sectional shapes that could easily be conformally mapped. The torsion problem of a member with constant elastic properties across the cross section and with a nonclassical geometry of cross section may be solved by relaxation methods, as used by Christopherson,\(^{69}\) Southwell,\(^{70}\) Shaw,\(^{71,72}\) Allen,\(^{73}\) Dobie,\(^{74}\) and others. Ely and Zienkiewicz\(^{75}\) applied relaxation method to the solution of torsion of some composite bars of special cross sections, made of two different materials \((G_1, G_2)\). In this chapter (starting from compatibility conditions and using the relaxation method) a procedure will be introduced to solve the torsion problem of composite members with application to the torsion of a "T" shaped member in column analysis. The iteration process is carried out by means of a MAD program written for the 7090 digital computer.
4.2. ASSUMPTIONS

To reduce the complexities and apply the solution to the problem of torsion of a "T" shaped column, certain assumptions have been made.

The structural shape has a uniform cross section and there is no restraint against longitudinal displacement of the elements of the cross section.

The torsional moment is assumed constant along any one segment of the member. (By segment is meant an arbitrary increment of column length chosen for the numerical solution.) As we increase the number of segments in column analysis, this assumption tends to become more accurate.

With the above assumptions only St. Venant torsion will be present and thus the magnitude of the rotation, $\beta$, along the member within the length of the segment of the column will be a linear function of the length of the segment.

4.3. DISPLACEMENT EQUATIONS

Assume a prismatic bar of length $L$ fixed at one end to the xy plane and with origin of the coordinate system fixed at the center of twist of the bar with the z-axis coinciding with the longitudinal axis of the bar and torque $M_z$ acting counterclockwise on the front face (Fig. 4.1).
If $\theta$ is assumed to be the rotation per unit length (uniform torsion)
then
\[ \beta = \theta z \quad (4.1) \]
and point $A(x, y, z)$ after application of torque will displace to
$A'(x+u, y+v, z+w)$, then
\[ u = OA[\cos(\alpha + \beta) - \cos(\alpha)] = OA[\cos\alpha \cos\beta - \sin\alpha \sin\beta - \cos\alpha] \quad (4.2) \]
\[ v = OA[\sin(\alpha + \beta) - \sin(\alpha)] = OA[\sin\alpha \cos\beta + \cos\alpha \sin\beta - \sin\alpha] \]
assuming that $\beta \ll 1$ then $\sin\beta \approx \beta$ and $\cos\beta \approx 1$, and $\overline{OA}\sin\alpha = y$, $\overline{OA}\cos\alpha = x$.
\[ u = -y\beta \quad (4.3) \]
\[ v = x\beta \]
or

\[ u = -\theta yz \]  \hspace{1cm} (4.4)  \\
\[ v = \theta xz \]

The displacement in the z direction, \( w \), is assumed to be independent of \( z \) and only a function of \( x \) and \( y \) and therefore can be written

\[ w = \theta \psi(x, y) \]  \hspace{1cm} (4.5)

\( \psi \) which defines the warping of the end surface is called the warping function.

The above assumptions are used in St. Venant's semi-inverse method to reduce the governing equilibrium equations to one differential equation. Thus, the objective will be to find the function \( \psi(x, y) \) so that the equilibrium equations and the boundary conditions will be satisfied.

4.4. EQUILIBRIUM EQUATIONS

If in general terms we assume that the displacement vector is \( u \) (i.e., \( u_1 = u, u_2 = v, u_3 = w \)) then strain vector in Cartesian coordinates will be

\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{k,i} \delta_{ik,j}) \]  \hspace{1cm} (4.6)

\( (i = 1, 2, 3; j = 1, 2, 3; k = 1, 2, 3) \)

neglecting the nonlinear term \( u_{k,i} \delta_{ik,j} \) equation (4.6) can be written as

\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]  \hspace{1cm} (4.7)
which is a linear approximation and is known as Lagrange's equation of compatibility of strains. Assuming that there exists a linear relation between the strain and stress components in the Cartesian coordinates and in the neighborhood of the state of the present stress or strain, stress vector $\tau$ can be written

$$\tau_{ij} = (G\varepsilon)_{ij} \epsilon_{ij} \quad (4.8)$$

where $(G\varepsilon)_{ij}$ is equal to $E_s$ when $i = j$ and is equal to $2G_s$ when $i \neq j$. The secant elastic modulus $(E_s)$ and the secant shear modulus $(G_s)$ are defined in the following relations $^{(11)}$ (Figs. 4.2 and 4.3)

![Fig. 4.2. Elastic, Tangent and Secant Moduli](image)

![Fig. 4.3. Shear and Secant Shear Moduli](image)

$$E_s = \frac{\sigma}{\epsilon} \quad (4.9)$$

$$G_s = \frac{\tau}{\gamma}$$

where $\sigma = $ axial stress

$\epsilon = $ total strain, including inelastic strain

$\tau = $ shear stress in pure shear

$\gamma = $ shear strain in pure shear
and $\sigma$ and $\epsilon$ are evaluated in the uniaxial stress. The equilibrium equations can be written in the following form

$$\tau_{ij} + F_i = 0$$  \hfill (4.10)

Using equations (4.7), (4.8) and (4.10) along with displacement equations (4.4) and (4.5) strain components in xyz coordinates will be

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = 0$$

$$\gamma_{xy} \text{ (or } 2\epsilon_{xy} \text{ )} = \gamma_{yx} \text{ (or } 2\epsilon_{yx} \text{ )} = 0$$  \hfill (4.11)

$$\gamma_{yz} \text{ (or } 2\epsilon_{yz} \text{ )} = \gamma_{zy} = \theta \left( \frac{\partial \psi}{\partial y} + x \right)$$

$$\gamma_{zx} \text{ (or } 2\epsilon_{zx} \text{ )} = \gamma_{xz} = \theta \left( \frac{\partial \psi}{\partial x} - y \right)$$

and stress components will be

$$\sigma_x \text{ (or } \tau_{xx} \text{ )} = \sigma_y \text{ (or } \tau_{yy} \text{ )} = \sigma_z \text{ (or } \tau_{zz} \text{ )} = 0$$

$$\tau_{xy} = \tau_{yx} = 0$$  \hfill (4.12)

$$\tau_{yz} = \tau_{zy} = G_s(x, y) \theta \left( \frac{\partial \psi}{\partial y} + x \right)$$

$$\tau_{zx} = \tau_{xz} = G_s(x, y) \theta \left( \frac{\partial \psi}{\partial x} - y \right)$$

Equations (4.12) show that the secant shear modulus ($G_s$) is assumed to be a function of $x$ and $y$ only and independent of $z$. As we reduce the length of the element the correctness of this assumption will improve. Using the equilibrium equations it will be shown that the assumption $G_s$ being independent of $z$ is compatible with the assumed displacement vector. Equilibrium equation (4.10) can be written
\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0
\]
\[
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0
\tag{4.13}
\]
\[
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0
\]

There is no external force acting on the element; therefore,

\[
F_x = F_y = F_z = 0
\]

Using stress relations (4.12) the first two equations of (4.13) are satisfied.

Had we assumed \( G_s \) to be a function of \( z \) then the first two equations of (4.13) should have been written as

\[
0 + 0 + \frac{\partial G_s(x, y, z)}{\partial z} \theta \left( \frac{\partial \psi}{\partial x} - y \right) + 0 = 0
\tag{4.14}
\]
\[
0 + 0 + \frac{\partial G_s(x, y, z)}{\partial z} \theta \left( \frac{\partial \psi}{\partial y} + x \right) + 0 = 0
\]

as

\[
\left( \frac{\partial \psi}{\partial x} - y \right) = 0
\]

and

\[
\left( \frac{\partial \psi}{\partial y} + x \right) = 0
\]

give a trivial solution of \( \tau_{yz} = \tau_{zy} = \tau_{xz} = \tau_{zx} = 0 \). Therefore, \( \frac{\partial G_s(x, y, z)}{\partial z} \) should be taken equal to zero (\( G_s = G_s(x, y) \)).

The third and governing equilibrium equation will be
\[ \frac{\partial G_s(x, y)}{\partial y} \theta (\frac{\partial \psi}{\partial y} + x) + G_s(x, y) \theta \frac{\partial^2 \psi}{\partial y^2} \]

\[ + \frac{\partial G_s(x, y)}{\partial x} \theta (\frac{\partial \psi}{\partial x} - y) + G_s(x, y) \theta \frac{\partial^2 \psi}{\partial x^2} = 0 \]

or

\[ G_s(x, y)[\nabla^2 \psi(x, y)] + \frac{\partial G_s(x, y)}{\partial x} (\frac{\partial \psi}{\partial x} - y) + \frac{\partial G_s(x, y)}{\partial y} (\frac{\partial \psi}{\partial y} + x) = 0 \] (4.15)

In the case of constant \( G_s \) across the cross section

\[ \frac{\partial G_s(x, y)}{\partial x} = \frac{\partial G_s(x, y)}{\partial y} = 0 \]

and equation (4.15) will reduce to the Laplace equation \( (\nabla^2 \psi = 0) \).

4.5. BOUNDARY CONDITIONS

The surface of the bar is free of external forces \((F_x = F_y = F_z = 0)\); therefore, the normal component of the stress at the boundary should be equal to zero. Unit vector \( \mathbf{N} \) normal to the boundary can be written as

\[ \mathbf{N} = \frac{dy}{ds} \mathbf{i} - \frac{dx}{ds} \mathbf{j} \] (4.16)

and the stress vector

\[ \mathbf{\tau} = \tau_{xz} \mathbf{i} + \tau_{yz} \mathbf{j} \] (4.17)

to satisfy the above condition

\[ \mathbf{N} \cdot \mathbf{\tau} = 0 \] (4.18)

or

\[ \tau_{xz} \frac{dy}{ds} - \tau_{yz} \frac{dx}{ds} = 0 \] (4.19)
Substituting for \( \tau_{xy} \) and \( \tau_{yz} \) from (4.12)

\[
\theta G_s(x, y) \left[ (\psi_x - y) \frac{dy}{ds} - (\psi_y + x) \frac{dx}{ds} \right] = 0
\]
as \( G_s(x, y) \neq 0 \)

\[
(\psi_x - y) \frac{dy}{ds} - (\psi_y + x) \frac{dx}{ds} = 0 \text{ on } S \quad (4.20)
\]
equation (4.20) should be satisfied on the lateral surface of the element.

On the two boundary surfaces at the end of the element defined by planes of \( z = 0 \) and \( z = L \), stresses given by (4.12) should produce a torque equivalent to the one applied externally and moreover, the resultant force should be equivalent to zero. The first condition requires that

\[
M_z = \int_A \left( x \tau_{yz} - y \tau_{xz} \right) \, dA \quad (4.21)
\]
or

\[
M_z = \theta \int_A \int_A G_s(x, y) \left( x^2 + y^2 + x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial y} \right) \, dx \, dy \quad (4.22)
\]

Equation (4.22) gives the relation between the applied torque (\( M_z \)) and the rotation per unit length, \( \theta \). Even if we could express the warping function \( \psi \) as an explicit function of \( x \) and \( y \), the analytical integration of the above equation would not be a simple problem, as in general, \( G_s \) is not a continuous function of \( x \) and \( y \). As we shall see, the warping function \( \psi \) will also be discontinuous.
A third condition demands that

\[ \int_A \tau_{xz} \, dA = 0 \]

and

\[ \int_A \tau_{zy} \, dA = 0 \]  \hspace{1cm} (4.23)

Substituting for \( \tau_{zy} \) and \( \tau_{zx} \) from (4.12)

\[ \int \int_A e G_s(x, y) \left( \frac{\partial \psi}{\partial x} - y \right) \, dx \, dy = 0 \]  \hspace{1cm} (4.24)

\[ \int \int_A G_s(x, y) \left( \frac{\partial \psi}{\partial y} + x \right) \, dx \, dy = 0 \]

As \( e \) is constant, it can be taken out of the integral sign and canceled; therefore, we should prove that

\[ \int \int_A G_s(x, y) \left[ (\frac{\partial \psi}{\partial y} + x) \right] \, dx \, dy = 0 \]  \hspace{1cm} (4.25)

and similarly the second equation can be proved. Multiplying both sides of equation (4.15) by "y" and integrating over the end section boundary \( A \) and adding it to equation (4.25),

\[ \int \int_A G_s(x, y) (\psi_y + x) \, dx \, dy + \int \int_A y \left[ G_s(x, y) \nabla^2 \psi + G_s(x, y) (\psi_{xx} - y) + \right. \]

\[ + G_s(x, y) (\psi_{yy} + x) \left. \right] \, dx \, dy = 0 + \int \int_A (y \psi) \, dx \, dy = 0 \]

Left hand side can be written as

\[ \int \int_A \left[ y \left( G_s(x, y), x \left( \psi_y + x \right) + y \left( G_s(x, y) \psi_{xx} + G_s(x, y) (\psi_{yy} + x) + \ldots \right) \right. \right. \]

\[ y G_s(x, y), y \left( \psi_y + x \right) + y G_s(x, y) \psi_{yy} \left. \right] \, dx \, dy = 0 \]
or
\[
\int \int_A \left[ \frac{\partial}{\partial x} \left[ y G_s(x, y) \right] + \frac{\partial}{\partial y} \left[ y G_s(x, y) \right] \right] \, dx \, dy + \\
\int \int_A \left[ y G_s(x, y) \right] \, dx \, dy = 0
\]

The above equation can be written as \(^{(60)}\) (Fig. 4.4)
\[
\oint_S \left[ y G_s(x, y) \right] \, dy = -\oint_S \left[ y G_s(x, y) \right] \, dx = 0
\]
or
\[
\oint_S \left[ y G_s(x, y) \right] \frac{dy}{ds} - \left[ y G_s(x, y) \right] \frac{dx}{ds} \, ds = 0
\]
or
\[
\oint_S y G_s(x, y) \left[ (\psi_x - y) \frac{dy}{ds} - (\psi_y + x) \frac{dx}{ds} \right] \, ds = 0
\]
or
\[
\oint_S f \, ds = 0 \quad (4.26)
\]

where
\[
f = f(x, y) = y G_s(x, y) \left[ (\psi_x - y) \frac{dy}{ds} - (\psi_y + x) \frac{dx}{ds} \right] \quad (4.27)
\]

The term inside the bracket is the same as equation (4.20). If the lateral boundary condition is satisfied, this bracket will be equal to zero and consequently equation (4.25) will be satisfied.

Fig. 4.4. Cross Section and the Boundary
When discontinuity is present in the functions $\psi$ and $G_s$, as in the case of the existence of strain regression, the boundary can be divided into two or more regions and the above procedure followed and integration taken in different regions separately. Finally, it will be concluded that the integration (4.26) should be taken over all boundaries of divided regions, for example, as in the case of the Fig. 4.5

$$\int_{S_1} (f)_1 ds + \int_{S_2} (f)_2 ds = 0 \quad (4.28)$$

$S_1$ and $S_2$ can be divided into two parts

$$S_1 = S'_1 + S''_1 \quad \quad S_2 = S'_2 = S''_2$$

then equation (4.27) can be written as

$$\int_{S'_1} (f)_1 ds + \int_{S''_1} (f)_1 ds + \int_{S'_2} (f)_2 ds + \int_{S''_2} (f)_2 ds = 0$$

$S'_1$ and $S'_2$ form the outer boundary of the surface (A) and (4.2) is satisfied on the boundary

$$\int_{S'_1} (f) ds + \int_{S'_2} (f) ds = \oint_S (f) ds = 0$$
Now conditions (4.26) will become
\[ \int_{S''_1} (f)_1 \, ds + \int_{S''_2} (f)_2 \, ds = 0 \]
as \( S''_1 = -S''_2 = S'' \), then
\[ \int_{S''} (f)_1 \, ds - \int_{S''} (f)_2 \, ds = 0 \quad (4.29) \]

The subscript on (4.26) means equation (4.26) written for that region.

Equation (4.29) indicates that the components of the shear stress at the separating boundary when approaching from region (1) or (2) should be the same.

\[ (\tau_{xz})_1 = (\tau_{xz})_2 \quad (4.30) \]
\[ (\tau_{yz})_1 = (\tau_{yz})_2 \]
or, in other words, if equations (4.30) together with equations (4.20) are satisfied the boundary conditions (4.23) will also be satisfied.

4.6. DIFFERENCE EQUATIONS

In order to solve the problem of inelastic torsion one should find a warping function \( \psi(x,y) \) that satisfies both the differential equation of equilibrium (4.15) and the boundary condition (4.20) with the supplemental relation (4.30).

Except for a few cases for which analytical solutions may be possible, solutions of the differential equations of equilibrium with complex boundary conditions are not presently available. Numerical
methods must be used to solve such a problem. The differential
equations of equilibrium are replaced by "finite difference equations".
The relaxation procedure for solution of finite difference equations
was initiated by Christopherson (69) and Southwell (70). Shaw (71, 72)
has also applied the relaxation method to some torsion problems.

In this analysis a modification of the relaxation method will
be used to evaluate the warping function and consequently the torque-
rotation relations. Prior to forming the finite difference equation
for the differential equation (4.15) the following should be reviewed:

4.7. POLYNOMIAL FUNCTION

Suppose that y, a function of the independent variable x, is
required to have the values \( y_0, y_1, y_2, \ldots, y_n \) when x has the
values 0, h, 2h, \ldots, nh. It can be proved that the function

\[ y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \]  

(4.31)

which is a polynomial in x of degree n, and in which the \((n + 1)\)
quantities of a are constants, satisfies the imposed requirements,
and all the differentials of (4.31) are continuous and have finite
values in the considered range (x being finite). The \( n^{th} \) differential
\( \frac{d^n y}{dx^n} \) has the constant value of \( na_n \) and the higher differentials
are zero.
4.8. APPROXIMATE DIFFERENTIATION

In this procedure a range of three points will be applied in calculation of the differentials.

Therefore, the following relations will be given without proof

(Fig. 4.6)

\[
\begin{align*}
(n = 2, \text{ given: } y_0, y_1, y_2, h) & \\
\frac{h}{0} \frac{dy}{dx} &= \frac{1}{2} (-3y_0 + 4y_1 - y_2) \\
\frac{h}{1} \frac{dy}{dx} &= \frac{1}{2} (-y_0 + y_2) \\
\frac{h}{2} \frac{dy}{dx} &= \frac{1}{2} (y_0 - 4y_1 + 3y_2) \\
h^2 \frac{d^2y}{dx^2} &= (y_0 - 2y_1 + y_2) 
\end{align*}
\]

(4.32)

4.9. FINITE-DIFFERENCE APPROXIMATION TO THE PARTIAL DIFFERENTIAL EQUATION OF EQUILIBRIUM

After a range of three points and a network of squares of side "h" has been considered, we shall formulate the finite-difference equation for the differential equation (4.15).
Let us assume that functions \( G_s(x, y) \) and \( \psi(x, y) \) and the
first derivative of \( G_s(x, y) \) and the
first and second derivatives of
\( \psi(x, y) \) with respect to \( x \) and \( y \) are
continuous in the neighborhood of
point "o" (Fig. 4.7) within a domain
of radius "h". In order to treat the
differential equation (4.15) in a
manner more suitable for this
discussion it can be written in
the following form

\[
\frac{\partial}{\partial x} \left[ G_s(x, y) (\psi_x - y) \right] + \frac{\partial}{\partial y} \left[ G_s(x, y) (\psi_y + x) \right] = 0
\]

or

\[
\frac{\partial}{\partial x} \left[ G_s(x, y) \psi_x \right] + \frac{\partial}{\partial y} \left[ G_s(x, y) \psi_y \right] = \frac{\partial}{\partial x} \left[ y G_s(x, y) \right] - \frac{\partial}{\partial y} \left[ x G_s(x, y) \right]
\]

or

\[
\frac{\partial}{\partial x} \left[ G_s(x, y) \psi_x \right] + \frac{\partial}{\partial y} \left[ G_s(x, y) \psi_y \right] = \left[ y G_s, x - x G_s, y \right]
\]

or

\[
\frac{\partial}{\partial x} \left[ G_s(x, y) \psi_x \right] + \frac{\partial}{\partial y} \left[ G_s(x, y) \psi_y \right] = g(x, y)
\]

where

\[
g(x, y) = y \frac{\partial G_s(x, y)}{\partial x} - x \frac{\partial G_s(x, y)}{\partial y}
\]
Now consider point $o$ of the mesh with four surrounding points $N$, $E$, $S$, $W$ and four mid-points $n$, $e$, $s$, $w$ (Fig. 4.7), then using equations (4.32), $g(x, y)$ can be expressed for point $o$ as follows

$$
g(x, y)_o = \frac{1}{h} \left[ y(o) \left[ G_s(e) - G_s(w) \right] - x(o) \left[ G_s(s) - G_s(n) \right] \right] = b(o) \tag{4.35}$$

having the values of $G_s$ at the nodal points the right hand side of the equation (4.33) can be evaluated for the mesh. These values are constant for a chosen mesh and will tend to a more accurate value as mesh size becomes smaller. To evaluate the left side of (4.33)

$$\begin{align*}
\left( \frac{\partial}{\partial x} \left( G_s \psi, x \right) \right)_o &= \frac{\psi(E) - \psi(o)}{h} \\
\left( \frac{\partial}{\partial x} \left( G_s \psi, x \right) \right)_w &= \frac{\psi(o) - \psi(W)}{h} \\
\left( \frac{\partial}{\partial y} \left( G_s \psi, y \right) \right)_n &= \frac{\psi(o) - \psi(N)}{h} \\
\left( \frac{\partial}{\partial y} \left( G_s \psi, y \right) \right)_s &= \frac{\psi(S) - \psi(o)}{h}
\end{align*} \tag{4.36}$$

and

$$\frac{\left( G_s \psi, x \right)_e - \left( G_s \psi, x \right)_w}{h} = \frac{1}{h^2} \left[ G_s(e) \left[ \psi(E) - \psi(o) \right] - G_s(w) \left[ \psi(o) - \psi(W) \right] \right]$$

$$= \frac{1}{h^2} \left[ G_s(e) \psi(E) + G_s(w) \psi(W) - \left[ G_s(e) + G_s(w) \right] \psi(o) \right]$$

similarly

$$\frac{\left( \frac{\partial}{\partial x} \left( G_s \psi, y \right) \right)_o}{h} = \frac{1}{h^2} \left[ G_s(s) \psi(S) + G_s(n) \psi(N) - \left[ G_s(s) + G_s(n) \right] \psi(o) \right] \tag{4.37}$$
and thus the finite-difference equation for the differential equation of the equilibrium can be written as

$$\frac{1}{h^2} \left[ G_s(n) \psi(N) + G_s(e) \psi(E) + G_s(s) \psi(S) + G_s(w) \psi(W) - \left[ G_s(n) + G_s(e) + G_s(s) + G_s(w) \right] \psi(o) \right] = b(o)$$  (4.38)

where $b(o)$ is expressed in (4.35) and has a constant value at every nodal point. Solving (4.37) for $\psi(o)$ we will have

$$\psi(o) = \left[ G_s(n) \psi(N) + G_s(e) \psi(E) + G_s(s) \psi(S) + G_s(w) \psi(W) - h^2 b(o) \right] / \left[ G_s(n) + G_s(e) + G_s(s) + G_s(w) \right]$$

or

$$\psi(o) = \left[ G_s(n) \psi(N) + G_s(e) \psi(E) + G_s(s) \psi(S) + G_s(w) \psi(W) \right] / c(o) + d(o)$$  (4.39)

where $d(o) = -h^2 b(o)/c(o)$

$$c(o) = G_s(n) + G_s(e) + G_s(s) + G_s(w)$$  (4.40)

Now we write the difference equation for the point $(i, j)$ ($i^{th}$ row and $j^{th}$ column) of the mesh (Fig. 4.8)
Fig. 4.8. Point \((i, j)\) and the Neighboring Mesh

where

\[ i' = 2i - 1 \]

\[ j' = 2j - 1 \]  \hspace{1cm} (4.41)
\[
\psi(i, j) = \left[ G_s(i'-1, j') \psi(i-1, j) + G_s(i'+1, j') \psi(i+1, j) + G_s(i', j'-1) \psi(i, j-1) + G_s(i', j'+1) \psi(i, j+1) \right] / c(i, j) \\
- \hbar^2 b(i, j) / c(i, j) 
\]

where

\[
c(i, j) = G_s(i', j'-1) + G_s(i', j'+1) + G_s(i'+1, j') + G_s(i'-1, j')
\]

and

\[
b(i, j) = \frac{1}{\hbar} \left[ y(i) [G_s(i', j'+1) - G_s(i', j'-1)] - x(j) [G_s(i'+1, j') - G_s(i'-1, j')] \right]
\]

If \( G_s \) is constant, equation (4.42) will reduce to the finite difference equation of the Laplacian operator \( \nabla^2 \)

\[
G_s, x = G_s, y = 0
\]

\[
G_s(i', j'-1) = G_s(i', j'+1) = G_s(i'+1, j') = G_s(i'-1, j') = c(i, j) / r
\]

\[
b(i, j) = 0
\]

and equation (4.42) will become

\[
\psi(i, j) = \left[ \psi(i, j+1) + \psi(i, j-1) + \psi(i+1, j) + \psi(i-1, j) \right] / 4 
\]

4.10. DIFFERENCE EQUATIONS FOR BOUNDARY CONDITIONS OF T SECTION

Choosing the xy coordinate system as shown below with the origin in the centroid of the flange (assumed to be the center of the twist) and x, y-axis parallel to the sides of the cross section, boundary condition (4.20) will be simplified. On the sides parallel to the x-axis, (Fig. 4.9), \( dy = 0 \), \( dS = dx \) and equation (4.20) becomes
\[(\psi_y + x) = 0 \quad (4.45)\]

On the sides parallel to the \(y\)-axis, \(dx = 0\), \(dy = dS\) and equation (4.20) becomes
\[(\psi_x - y) = 0 \quad (4.46)\]

Fig. 4.9. Boundary of the \(T\) Cross Section

Applying equations (4.32) to the above boundary conditions we can write

For sides parallel to \(x\)
- on the negative side of \(y\)
  \[\frac{1}{2h} \left[ -3\psi(1,j) + 4\psi(2,j) - \psi(3,j) \right] + x(j) = 0 \quad (4.47)\]
- on the positive side of \(y\)
  \[\frac{1}{2h} \left[ -3\psi(n,j) + 4\psi(n-1,j) - \psi(n-2,j) \right] - x(j) = 0 \]

For sides parallel to \(y\)
- on the left side
  \[\frac{1}{2h} \left[ -3\psi(i,1) + 4\psi(i,2) - \psi(i,3) \right] - y(i) = 0 \quad (4.48)\]
- on the right side
  \[\frac{1}{2h} \left[ -3\psi(i,m) + 4\psi(i,m-1) - \psi(i,m-2) \right] + y(i) = 0 \]
In order to treat the boundary nodal points in the process of iteration, the values of function $\psi$ for the boundary points should be given in terms of the value of the $\psi$ of the neighboring points; therefore, equations (4.47) and (4.48) should be solved for the boundary nodal points

**Negative side of $y$ parallel to $x$**

$$\psi(1, j) = \left[ 4\psi(2, j) - \psi(3, j) + 2hx(j) \right] / 3$$

**Positive side of $y$ parallel to $x$**

$$\psi(n, j) = \left[ 4\psi(n-1, j) - \psi(n-2, j) - 2hx(j) \right] / 3$$

**Left side parallel to $y$**

$$\psi(i, 1) = \left[ 4\psi(i, 2) - \psi(i, 3) - 2hy(i) \right] / 3$$

**Right side parallel to $y$**

$$\psi(i, m) = \left[ 4\psi(i, m-1) - \psi(i, m-2) + 2hy(i) \right] / 3$$

**Discontinuity**

Equations (4.30) should be satisfied along the line of discontinuity. The basic assumption that has been made is that all elements of the cross section go through the same amount of unit rotation $\theta$ about the center of twist (centroid of the flange).

Considering this assumption and using stress relations (4.12) for the elements along the discontinuity line, the following should hold

$$\left[ G_s(x, y) (\psi_y + x) \right]_1 = \left[ G_s(x, y) (\psi_y + x) \right]_2$$

$$\left[ G_s(x, y) (\psi_x - y) \right]_1 = \left[ G_s(x, y) (\psi_x - y) \right]_2$$

(4.51)
\([ \ ]_1 \) and \([ \ ]_2 \) mean the term evaluated in the regions (1) and (2) respectively. In terms of the difference equation, assuming that the nodal point is on the border line of the two regions, equations (4.51) can be written as follows (Fig. 4.10)

\[
2h[G_s(i, m) - G_s(i, 1)]x(m) = G_s(i, 1)[-3\psi(i, 1) + 4\psi(i, 2) - \psi(i, 3)]
+ G_s(i, m)[-3\psi(i, m) + 4\psi(i, m-1) - \psi(i, m-2)]
\]

\[
2h[G_s(i, m) - G_s(i, 1)]y(i) = G_s(i, 1)[-3\psi(i, 1) + 4\psi(i-1, 1) - \psi(i-2, 1)]
+ G_s(i, m)[-3\psi(i, m) + 4\psi(i+1, m) - \psi(i+2, m)]
\]

(4.52)

The above equations can be solved for \(\psi(i, 1)\) and \(\psi(i, m)\) so that in the iteration process either one can be calculated in terms of the other.
\[
\psi(i, 1) = \left\{ \frac{[G_s(i, m)/G_s(i, 1) + 1] \cdot 2hx(m) + 4\psi(i, 2) - \psi(i, 3)}{3} + \left[ \frac{G_s(i, m)}{G_s(i, 1)} \right] \cdot [-3\psi(i, m) + 4\psi(i, m-1) - \psi(i, m-2)] \right\} / 3
\]

\[
\psi(i, m) = \left\{ \frac{[-1 + G_s(i, 1)/G_s(i, m)] \cdot 2hy(i) + 4\psi(i+1, m) - \psi(i+2, m)}{3} + \left[ \frac{G_s(i, 1)}{G_s(i, m)} \right] \cdot [-3\psi(i, 1) + 4\psi(i-1, 1) - \psi(i-2, 1)] \right\} / 3
\]

Other forms of discontinuity can be treated in the same manner.

**Points with Non-Equidistant Neighbors**

In many cases a point near a boundary or the border line of two regions will not be surrounded by equidistant neighboring nodal points, or one may wish to make a change of net interval in a special region. Special formulas must be provided for such cases. Starting from the polynomial function (4.31) for a general case such as the one on the Fig. 4.11, equations (4.32) can be written as following

![Diagram](image-url)

**Fig. 4.11.** Point with Non-Equidistant Neighbors
\[
\begin{align*}
\left( \frac{\partial \psi}{\partial x} \right)_0 &= \frac{2\psi(2) + (\alpha_2^2 - \alpha_1^2)\psi(0) - \alpha_2^2\psi(1)}{\alpha_1\alpha_2(\alpha_1 + \alpha_2)h} \\
&= \frac{\alpha_1}{\alpha_2(\alpha_1 + \alpha_2)h} \psi(2) + \frac{\alpha_2 - \alpha_1}{\alpha_1\alpha_2h} \psi(1) - \frac{\alpha_2^2\psi(0)}{\alpha_1(\alpha_1 + \alpha_2)h} \\
\left( \frac{\partial \psi}{\partial x} \right)_1 &= \frac{-\alpha_1^2\psi(2) + (\alpha_1 + \alpha_2)^2\psi(0) - \alpha_2(2\alpha_1 + \alpha_2)\psi(1)}{\alpha_1\alpha_2(\alpha_1 + \alpha_2)h} \\
&= \frac{-\alpha_1}{\alpha_2(\alpha_1 + \alpha_2)h} \psi(2) + \frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2h} \psi(0) - \frac{2\alpha_1 + \alpha_2}{\alpha_1(\alpha_1 + \alpha_2)h} \psi(1) \\
\left( \frac{\partial \psi}{\partial x} \right)_2 &= \frac{\alpha_1(2\alpha_2 + \alpha_1)\psi(2) - (\alpha_1 + \alpha_2)^2\psi(0) - \alpha_2^2\psi(1)}{\alpha_1\alpha_2(\alpha_1 + \alpha_2)h} \\
&= \frac{2\alpha_2 + \alpha_1}{\alpha_1(\alpha_1 + \alpha_2)h} \psi(2) - \frac{(\alpha_1 + \alpha_2)}{\alpha_1\alpha_2h} \psi(0) + \frac{\alpha_2}{\alpha_2(\alpha_1 + \alpha_2)h} \psi(1) \\
\left( \frac{\partial^2 \psi}{\partial x^2} \right)_{201} &= 2 \frac{\alpha_1\psi(2) - (\alpha_1 + \alpha_2)\psi(0) + \alpha_2\psi(1)}{\alpha_1\alpha_2(\alpha_1 + \alpha_2)h^2} \\
&= \frac{2}{h^2} \left[ \frac{1}{\alpha_1(\alpha_1 + \alpha_2)} \psi(2) - \frac{1}{\alpha_1\alpha_2} \psi(0) + \frac{1}{\alpha_1(\alpha_1 + \alpha_2)} \psi(1) \right]
\end{align*}
\]

By changing the \(\psi(1), \psi(2), \alpha_1, \alpha_2\) respectively to the \(\psi(3), \psi(4), \alpha_3, \alpha_4\), the above relations will give the derivatives of \(\psi\) with respect to \(y\). For \(\alpha_1 = \alpha_2 = 1\) relations (4.54) reduce to those expressed
in (4.32). Using the above relations, governing difference equations and the boundary conditions for points with non-equidistant neighbors can be obtained; however, in this study to simplify the manipulation, the straight lines of discontinuity have been approximated by a step-wise line (Fig. 4.12).

**Fig. 4.12. Stepwise Approximation**

4.11. CONVERGENCE OF THE ITERATION

Relations (4.32) and (4.54) can be obtained by applying Taylor's theorem and expanding the function $\psi(x, y)$ at the neighborhood of $0$ (Fig. 4.13), as done by Young.

**Fig. 4.13. Expansion of $\psi(x, y)$ in the Neighborhood of $(x, y)$**
\[ \psi(2) = \psi(0) + h(\psi, x_0) + \frac{h^2}{2} (\psi, xx)_0 + \frac{h^3}{3} (\psi, xxx)_0 + \frac{h^4}{4} (\psi, xxxx)_0, 2 \]  

where

\[ (\psi, xxxx)_{0, 2} = \psi, xxxx (x_0 + \xi, y_0) \quad 0 < \xi < h \]  

(4.56)

similarly

\[ \psi(1) = \psi(0) - h(\psi, x_0) + \frac{h^2}{2} (\psi, xx)_0 - \frac{h^3}{3} (\psi, xxx)_0 + \frac{h^4}{4} (\psi, xxxx)_0, 2 \]  

(4.57)

Combining (4.55) and (4.57)

\[ (\psi, xx)_0 = \frac{\psi(2) + \psi(1) - 2\psi(0)}{h^2} - \frac{h^4}{4} [(\psi, xxxx)_0, 2 + (\psi, xxxx)_0, 1] \]  

(4.58)

\[ (\psi, x)_0 = \frac{\psi(2) - \psi(1)}{2h} - \frac{h^2}{12} [(\psi, xxx)_0, 2 + (\psi, xxxx)_0, 1] \]

similarly the derivative with respect to \( y \) can be expressed

\[ (\psi, yy)_0 = \frac{\psi(4) + \psi(3) - 2\psi(0)}{h^2} - \frac{h^4}{4} [(\psi, yyyy)_0, 4 + (\psi, yyyy)_0, 3] \]  

(4.59)

\[ (\psi, y)_0 = \frac{\psi(4) - \psi(3)}{2h} - \frac{h^2}{12} [(\psi, yyyy)_0, 4 + (\psi, yyyy)_0, 3] \]

Neglecting the higher order terms and substituting (4.58) and (4.59)

in (4.15), for each point \((x, y)\) in the domain, we get
\[
\left[ G_s(x, y) + \frac{h}{Z} G_s(x, y),_x \right] \psi(x+h, y) + \left[ G_s(x, y) - \frac{h}{Z} G_s(x, y),_x \right] \psi(x-h, y) \\
+ \left[ G_s(x, y) + \frac{h}{Z} G_s(x, y),_y \right] \psi(x, y+h) + \left[ G_s(x, y) - \frac{h}{Z} G_s(x, y),_y \right] \psi(x, y-h) \\
- 4G_s(x, y) \psi(x, y) = h^2 \left[ y G_s(x, y),_x - x G_s(x, y),_y \right] 
\]

(4.60)

(4.60) can be written in the following form

\[
a_1 \psi(x-h, y) + a_2 \psi(x+h, y) + a_3 \psi(x, y-h) + a_4 \psi(x, y+h) - a_0 \psi(x, y) = t(x, y)
\]

(4.61)

where \(a_i (i=0, 1, 2, 3, 4)\) are functions of \((x, y)\).

we observe that

\[
a_0 = a_1 + a_2 + a_3 + a_4
\]

If \(h\) is chosen such that

\[
h < \min \left\{ \frac{2 G_s(x, y)}{G_s(x, y),_x}, \frac{2 G_s(x, y)}{G_s(x, y),_y} \right\}
\]

(4.62)

the minimum being evaluated in the domain of the operation, then all the \(a_i\) will be positive and convergence of the iteration process will be guaranteed.*

4.12. SHEAR-STRESS-STRAIN RELATIONSHIP

In order to derive the shear stress relations (4.12) from shear strain relations (4.11) the values of \(G_s\) used in equation

\[
\tau = G_s \gamma
\]

(4.63)

*For further details refer to reference (78)
should be defined and evaluated for the inelastic range. Suppose the relation between stress and strain is given by the Fig. 4.14. Total strain can be written as the summation of two strains, $\epsilon_e$, elastic strain, and $\epsilon_p$, plastic strain:

$$\epsilon = \epsilon_e + \epsilon_p \quad (4.64)$$

or

$$\epsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)^n \quad (4.65)$$

Now we plot the stress against the plastic strain. The relation between the plastic shear stress $(\tau)_p$ and strain $(\gamma)_p$ can be formed by dividing the stress ordinate of the Fig. 4.15 by two.
The reason for this is: for pure compression along the $z$-axis

$$\sigma_y = \sigma_x = 0 \text{ and } \sigma_z = -\sigma,$$

then the principal shear stresses will be

$$\tau_{xy} = \frac{\sigma_x - \sigma_y}{2} = 0$$

$$\tau_{xz} = \frac{\sigma_x - \sigma_z}{2} = 0$$

$$\tau_{yz} = \frac{\sigma_y - \sigma_z}{2} = \frac{\sigma}{2} \tag{4.66}$$

Since $\tau_{xz}$ and $\tau_{yz}$ are equal in pure compression the corresponding inelastic strains $\varepsilon_{xz}$ and $\varepsilon_{yz}$ must also be equal, each being half of the total inelastic strain $\varepsilon_P$. Therefore, to convert Fig. 4.14 to a principal shear stress vs. the corresponding strain, stress ordinate of Fig. 4.15 should be divided by two (Fig. 4.16). To form the complete $\tau$ vs. $\gamma$ diagram from $\sigma$ vs. $\varepsilon$ diagram simply for every value of plastic strain $\varepsilon_{pl}$ should be calculated and then corresponding point on ($\tau$ vs. $\gamma$) diagram will be

$$\left(\tau_1 = \frac{\sigma_1}{2} \text{ vs. } (\gamma_e + \varepsilon_{pl}) \right)$$

(Figs. 4.17 and 4.18).

Fig. 4.16. Shear Stress vs. Plastic Shear Strain
The relation between modulus of elasticity $E$ and shear modulus $G$ is given by the formula

$$G = \frac{E}{2(1 + \nu)}$$  \hspace{1cm} (4.67)

in which $\nu$ is the Poisson's ratio.

4.13. EVALUATION OF $G_s$

For a given strain $\epsilon_1$, the stress $\sigma_1$ can be calculated graphically or mathematically. In case stress is given by an algebraic formula in terms of strain,

$$\tau_1 = \frac{\sigma_1}{2}$$
and plastic strain

$$\varepsilon_{p_1} = \varepsilon_1 - \frac{\sigma_1}{E}$$

shear strain

$$\gamma = \frac{\tau_1}{G} + \varepsilon_{p_1}$$

$$G_{s_1} = \frac{\tau_1}{\gamma_1} = \frac{\frac{\sigma_1}{2G} + \varepsilon_1}{\frac{\sigma_1}{E} + \varepsilon_1} = \frac{1}{2}$$

or

$$G_{s_1} = \frac{E}{\frac{\varepsilon_1 E}{\sigma_1}} = \frac{E}{2(\nu + \frac{\varepsilon_1}{\sigma_1})}$$

or in general

$$G_s = \frac{E}{2(\nu + \frac{\varepsilon}{\sigma})} \quad (4.68)$$

It can be seen from equation (4.68) that in the inelastic range the value of $G_s$ varies as a function of $\varepsilon$ (Fig. 4.19).

In the elastic range, relation $\sigma = \varepsilon E$ reduces the equation (4.68) to the relation (4.67) between $G$ and $E$.

Fig. 4.19. Strain vs. Shear Secant Modulus
It is obvious that in the case of combined compression and torsion, the value corresponding to the combined stress according to one of the failure criteria, such as VonMises criterion, should be considered; but in this problem the amount of torque and rotation and consequently the shear stress due to torsion are very small compared to the compressive stress. It is assumed that $G_s$ is constant in the neighborhood of the considered strain. Moreover, the tensile (or compressive) stress-strain curve is used directly to compute $\tau = f(\gamma)$.

4.14. METHOD OF ITERATION

In any numerical solution of a differential equation, the continuum of points on the boundary and the interior region should be replaced by a discrete set of points. The value of the function is then considered only at these points rather than at every point of the governing region.

Liebmann(79) suggested an iterative procedure in the year 1918 and demonstrated the solution for the Laplace's equation. He divided the region into a square net and assigned approximate values to the function at the interior nodal points and known values at the nodal points of the boundary. Then traversing the region in an orderly manner repeatedly, and replacing the value of the function at each interior nodal point by the mean of the values of the four neighboring points, i.e.,
\[ \psi(i,j) = \left[ \psi(i,j-1) + \psi(i,j+1) + \psi(i+1,j) + \psi(i-1,j) \right] / 4 \]

(a special case of equation (4.42))

and using the newly calculated values immediately, he obtained a solution for the Laplace's equation.

Richardson\(^{(80)}\) had suggested a similar method, earlier (1910), in which he did not replace the old values by the newly calculated ones until the whole cycle was completed. This method converges somewhat slower than the one suggested by Liebmann. In general, for each interior point \((\psi_{ij})_i\) and each boundary point \((\psi_{pq})_b\) we can form an equation, and hence will have a system of linear equations for the nodal points. The most popular method for solving the simultaneous linear equation is Gaussian elimination.

A dense system of \(N\) equations for \(N\) unknowns can be solved with an order of \(\frac{N^3}{3}\) multiplications, plus other arithmetic operations.

Elimination methods which are called direct methods are rarely applied to solution of elliptic difference equations because of the large storage space required to store matrix \(a_{klij}\) in the relation

\[ a_{klij} \psi_{ij} = b_{kl} \quad (4.69) \]

(repeat of the subscripts means summation on these subscripts)

For partial differential equations, \(N\) is usually chosen very large such that we can only store some multiple of \(N\) but not \(N^2\) elements.

However, it is obvious that, disregarding the round-off errors, the elimination method will give the exact solution in a finite
number of steps. On the other hand, the indirect or iteration methods, both in operation and in storage, take full advantage of the numerous zero elements in the matrix \([a]\). Moreover, in contrast to the elimination method, iteration methods are self-correcting in the process of iteration and, therefore, minimize the round-off error.

The algorithm for an iterative method is very simple, but normally an exact solution can be achieved only with a very high number of iteration cycles.

The algorithm for the direct method is rather complicated. The above methods have been treated extensively by Householder \(^{(81)}\) (1953) and Bodewig \(^{(82)}\) (1956). If we write the two-dimensional matrix \([[\psi_{ij}]]\) in the form of a vector \([\psi]\) then equation (4.69) can be written as follows

\[
[A] [\psi] = [B] \tag{4.70}
\]

considering the difference equation (4.42) matrices \([A]\) and \([B]\) can be formed. \([A]\) is a matrix with no zero diagonal element and has diagonal dominance and also is not reducible. Then the method of successive displacements, sometimes called relaxation method, will converge in solving this problem. \(^{(83)}\) This method was introduced by Seidel (1874), but is known as Gauss-Seidel method.

Liebmann applied this method to the elliptic difference equations (1918). \(^{(79)}\)

**Overrelaxation**

Overrelaxation is the change of any element of \([\psi]\) by \(\omega\) times the exact amount of the change necessary to make that
element satisfy the equation written for it ($\omega > 1$). Young\(^{84}\) and Frankel\(^{85}\) introduced this procedure and Friedman,\(^{86}\) Householder,\(^{87}\) Arms, Gates and Zondek,\(^{88}\) Keller,\(^{89}\) and others have studied and applied this theorem. They suggest and prove that overrelaxation speeds the process of iteration considerably. $\omega$ is called overrelaxation factor and the best value is\(^{90}\)

$$\omega_b = 1 + \frac{\lambda}{(1 + \sqrt{1 - \lambda})^2}$$  \hspace{1cm} (4.71)

where $\lambda$ is the limit value of $d_n / d_{n-1}$ and

$$d_n = \max [\psi(i,j)]^n - [\psi(i,j)]^{n-1}$$  \hspace{1cm} (4.72)

$n$ is the number of the iteration cycle. The theory indicates that it is much better to overestimate than to underestimate $\omega$ to achieve a better convergence rate.

The best method of choosing $\omega$ is to perform a number of iterations with the iteration factor equal to 1, and estimate the value of $\lambda$ and thereby $\omega_b$.

For solving the problem of torsional stiffness in the inelastic range, i.e., the difference equation (4.42), the indirect method with the advantages of overrelaxation procedure, explained above, has been used and the difference equation (4.42) has been written in the following form
\[
[\psi(i,j)]^{n+1} = \omega \left[ G_s(i'-1,j')[\psi(i-1,j)]^n + G_s(i'+1,j')[\psi(i+1,j)]^n \\
+ G_s(i',j'-1)[\psi(i,j-1)]^n + G_s(i',j'+1)[\psi(i,j+1)]^n / c(i,j) \\
- h^2 b(i,j)/c(i,j) \right] + (1 - \omega) [\psi(i,j)]^n
\] (4.73)

4.15. TECHNIQUE

i. Choice of Mesh and Matrices

In order to use all the locations assigned to the variable \(\psi(i,j)\) in carrying out the manipulation of the problem with the electrical digital computer, two different variable names have been assigned to the flange and the web, \([\psi_f]\) and \([\psi_w]\), respectively. \([\psi_f]\) has a dimension of \((n_f, m_f + 1)\) and the dimension of \([\psi_w]\) is \((n_w, m_w + 1)\) (Fig. 4.20). In order to preserve the continuity of \([\psi_f]\) and \([\psi_w]\) in the intersection the following relation is kept between them

\[
\psi_w(i,0) = \psi_f(n+i, m_f - 1) \quad i=1,2,\ldots,n_w
\]

\[
\psi_w(i,1) = \psi_f(n+i, m_f) \quad i=2,3,\ldots,(n_w - 1)
\] (4.74)

\[
\psi_w(i,2) = \psi_f(n+i, m_f + 1) \quad i=1,2,\ldots,n_w
\]
Fig. 4.20. Choice of a Network

The reason for excluding nodal points \((1, 1)\) and \((n_w, 1)\) of the web and the corresponding nodal points of the flange in continuity relations is the indeterminate nature of the values of \(\psi\) at these corner points.
when approaching from \( x \) or \( y \) directions towards these points.

Theoretically, the torsional stress at these corners is infinite. In this solution two extremely close points are assigned to the corner points, and therefore, an abrupt discontinuity exists between the values of \( \psi \) at these two points. In evaluation of torsional stiffness these corner points have been ignored.

For the same reason, two matrices have been designated for the values of secant shear modulus for the flange and the web,

\[
\begin{bmatrix}
G_s \cdot (2n_f + 1, 2m_f + 1)
\end{bmatrix}
\text{ and }
\begin{bmatrix}
G_s \cdot (2n_w + 1, 2m_w + 1)
\end{bmatrix},
\]

respectively.

For simplicity in this analysis the dimensions of the \( T \) section have been chosen equal to some even multiples of the mesh size \( h \), except for the length of the stem of the web, which can be taken any odd or even multiple of \( h \). By doing so, existence of the non-equidistant points due to the geometry of the cross section have been avoided. Moreover, the center of the twist will coincide with a nodal point.

The dimensions of the cross section will be

\[
t = (m_f - 1)h \\
b = (n_f - 1)h \\
w = (n_w - 1)h \\
d = (m_w - 1)h + t
\]

\( (m_f, n_f \text{ and } n_w \text{ are odd integers}) \)
ii. Evaluation of Matrices $[G_s]$ and $[\psi]$

In order to form the secant shear modulus matrices one has to evaluate the strain at that stage of bending and at the previous stage of bending for every nodal point, and determine whether the regression of strain has commenced at that point or not. In the former case the properties of the element, represented by that point, is chosen to be elastic, i.e., the secant shear modulus equal to $G$, and in the latter case with the strain at hand, inelastic secant shear modulus can be evaluated according to the previous discussion. Strain at the nodal points for two successive stages can be evaluated as follows

$$
(\epsilon)^1 = (\epsilon_o)^1 - x(\phi_x)^1 - y(\phi_y)^1 \\
(\epsilon)^2 = (\epsilon_o)^2 - x(\phi_x)^2 - y(\phi_y)^2 
$$

(4.76)

In the above equations $x$ and $y$ are the coordinates of the respective nodal point and $\epsilon_o$ is the strain of the origin of the $xy$-axes (center of twist) and $(-\phi_x)$ and $(-\phi_y)$ are the curvature in the $x$ and $y$ directions respectively. Superscripts 1 and 2 indicate the stage at which those values are given.

iii. Border Line of Discontinuity

Equating $(\epsilon)^1$ to $(\epsilon)^2$ the separating line of elastic and inelastic regions can be expressed

$$
\left[(\phi_x)^2 - (\phi_x)^1 \right] x - \left[(\phi_y)^2 - (\phi_y)^1 \right] y = \left[(\epsilon_o)^2 - (\epsilon_o)^1 \right] 
$$

(4.77)
This line will intersect the x- and y-axes at A and B respectively.

\[
\begin{align*}
A: & \quad x = \frac{(\epsilon_o^2 - \epsilon_o^1)}{(\phi_x^2 - \phi_x^1)} \\
& \quad y = 0 \\
B: & \quad x = 0 \\
& \quad y = \frac{(\epsilon_o^2 - \epsilon_o^1)}{(\phi_y^2 - \phi_y^1)} \\
\end{align*}
\]

(4.78)

In case of the existence of the line of discontinuity, stepwise approximation of the border lines in the flange and the web can be made. Then some arbitrary values must be assigned to the elements of matrices \( [\psi_f] \) and \( [\psi_w] \) and the iteration to the extent of acceptable approximation carried out.

4.16. SHEAR STRESS AND TORQUE

After determining the values of the warping function, components of the shear stress can be found from the following relations

\[
\begin{align*}
[\tau_{xz}(i,j)]_f = \theta G_{s_f} (i',j') \left[ \frac{\psi_f(i,j+1) - \psi_f(i,j-1)}{2h} - y_f(i) \right] \\
& \quad \text{and} \\
[\tau_{yz}(i,j)]_f = \theta G_{s_f} (i',j') \left[ \frac{\psi_f(i+1,j) - \psi_f(i-1,j)}{2h} + x_f(j) \right] \\
\end{align*}
\]

(4.79)

By changing the subscript \( f \) to \( w \) shear stress components for the web can be expressed in terms of \( [\psi_w] \), \( [G_{s_w}] \), \( \{x_w\} \) and \( \{y_w\} \). Then torque \( M_z \) can be evaluated using equation (4.22)
\[ M_z = \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} \left[ x_f(j) \tau_{yzf}(i, j) - y_f(i) \tau_{xzf}(i, j) \right] \Delta A_f(i, j) \]

\[ + \sum_{k=1}^{n_w} \sum_{\ell=1}^{m_w} \left[ x_w(\ell) \tau_{yzw}(k, \ell) - y_w(k) \tau_{xzw}(k, \ell) \right] \Delta A_w(k, \ell) \quad (4.80) \]

where

\[ \Delta A_f(i, j) = \Delta A_w(k, \ell) = \frac{h^2}{2^{\frac{2}{2}}} \] for the interior points

\[ = \frac{h^2}{2} \] for the boundary points

\[ = \frac{h^2}{4} \] for the corner points

\[ = \frac{3h^2}{4} \] for the three corner point

(line separating flange and web counted on boundary line)

4.17. TORSIONAL STIFFNESS COEFFICIENT

Thus, having evaluated \( M_z \) in terms of \( \theta \), stiffness coefficient can be evaluated from the relation

\[ C = \frac{M_z}{\theta} \quad (4.81) \]

4.18. CONCLUSIONS

The objective of this study was to determine an approximate acceptable value for the torsional stiffness coefficient under the conditions explained in this chapter, and to use this value in
the torsional behavior of the column under study. Going through this rigorous calculation for every iteration cycle of the deflection of the column is very much time consuming and rather impossible.

A study was made for a T section of the following dimensions

\[
\begin{align*}
t &= 1 \text{ inch} \\
b &= 6 \text{ inches} \\
w &= 1 \text{ inch} \\
d &= 7 \text{ inches}
\end{align*}
\]

and twelve different cases of strain distribution and the stiffness coefficient for each case was evaluated, and then for the same cases the following computation was followed

\[
C_a = \frac{\sum_{i=1}^{n_f} \sum_{j=1}^{m_f} G_{s_f}(i, j) \Delta A_f(i, j)}{\sum_{k=1}^{n_f} \sum_{\ell=1}^{m_f} \Delta A_f(i, j)} \left(\frac{1}{3}bt^3 - 0.21t^4\right)
\]

\[
+ \frac{\sum_{k=1}^{n_w} \sum_{\ell=1}^{m_w} G_{s_w}(k, \ell) \Delta A_w(k, \ell)}{\sum_{k=1}^{n_w} \sum_{\ell=1}^{m_w} \Delta A_w(k, \ell)} \left[\frac{1}{3}(d - t)w^3 - 0.105w^4\right]
\]

\[
+ \frac{\sum_{i=1}^{n_f} \sum_{j=1}^{m_f} G_{s_f}(i, j) \Delta A_f(i, j) + \sum_{k=1}^{n_w} \sum_{\ell=1}^{m_w} G_{s_w}(k, \ell) \Delta A_w(k, \ell)}{\sum_{i=1}^{n_f} \sum_{j=1}^{m_f} \Delta A_f(i, j) + \sum_{k=1}^{n_w} \sum_{\ell=1}^{m_w} \Delta A_w(k, \ell)} \alpha_1 D_1^4
\]
For this example \( h \) was taken 0.1 inch and \( \alpha_1 = 0.105 \) (reference (91), figure 7) and \( D_1 = 1.25 \) inch (Fig. 4.21).

**Fig. 4.21. Junction of Flange and Web**

The range of variation for these twelve cases was found to be

\[
0.92 C_a < C < 1.54 C_a \quad (4.84)
\]

(4.83) indicates that assumption of \( C_a \) as the stiffness coefficient is more a conservative assumption and does not deviate "too much" from the exact solution. To formulate a closer solution more cases have to be considered and solved.

In the study of the column problem, equation (4.83) has been used as an approximate value for the torsional stiffness of the "T" section.
CHAPTER V
INCREMENTAL ANALYSIS OF "T" COLUMN BEHAVIOR

5.1. INTRODUCTION

The procedure presented herein is the one used to determine the load deflection history and thereby the ultimate load carrying capacity of the column. This procedure is based on Newmark's numerical method\(^{(92)}\) and uses Batterman and Johnston's method of expressing strains in terms of the radius of the curvature.\(^{(16)}\)

In order to evaluate the integrals in the equilibrium, torsional stiffness, and the bending stiffness equations over the area of the cross section of the column, the cross section was divided into a grid network in a manner similar to Birnstiel's method.\(^{(36)}\) Integrals were taken over each mesh area (assuming that the value of the integrant is constant over the mesh area and is equal to the same value for its centroid), and then summed over the cross sectional area. The effect of initial crookedness in the plane of symmetry and in either direction of the x-axis has been taken into consideration. Eccentric loading in the plane of symmetry has been studied and compared with the initial crookedness. Different ratios of bending stiffness with respect to two principal axes at the unloaded state have been studied. Comparison has been made between the assumed
sinusoidal deflected shape and the more exact deflected shape. Also, different stress-strain relations have been used to find the load-deflection history of the column. The results of the above studies will be illustrated graphically.

The steps followed in this analysis and also the MAD$^{(93)}$ language computer programs, written for the IBM 7090 to carry out the numerical manipulation, will be explained in this chapter.

5.2. STRESS-STRAIN RELATION

For the purpose of comparison of the analytical and the experimental results it was necessary to use experimentally determined elastic and inelastic properties of the material in the incremental analysis. This necessitated that the stress-strain relation be prepared in a manner that could easily be fed into the computer memory. The ideal case was to find a mathematical fit, as close as possible, to the experimental results.

The best fit curve consisted of three parts, beginning with a straight line portion and ending with a segment of a hyperbola, connected together with a portion of a parabola of third degree. These three segments were chosen such that the slopes of the two segments at the point of their intersection were the same. The method of least squares$^{(94)}$ was used to choose the separating points of the above three segments and the numerical calculations were done by the digital computer. The graphical representation of the algebraic fit for the stress-strain relation along with the
elastic (tangent) modulus and the secant shear modulus, evaluated on the basis of discussion in Chapter IV, is shown on Fig. 2.6.

The equations of the best fit curve are:

Straight line segment \( \varepsilon \leq 0.0035 \)

\[ \sigma = 10290 \times \varepsilon \]

\[ E = \frac{\sigma}{\varepsilon} = 10290 \]

Parabola segment \( 0.0035 \leq \varepsilon \leq 0.0050 \)

\[ \sigma = 4.54444 \times 10^8 \varepsilon^3 - 8.83111 \times 10^6 \varepsilon^2 + 5.54069 \times 10^4 \varepsilon - 69.2125 \]

\[ E_t = \frac{d\sigma}{d\varepsilon} = 13.63332 \times 10^8 \varepsilon^2 - 17.66222 \times 10^6 \varepsilon + 5.54069 \times 10^4 \quad \text{(5.1)} \]

Hyperbola segment \( \varepsilon \geq 0.0050 \)

\[ \sigma = \left[ -1.77152 \times 10^{-3} / (\varepsilon - 3.7743 \times 10^{-3}) \right] + 45.2953 \]

\[ E_t = \frac{d\sigma}{d\varepsilon} = \left[ -1.77152 \times 10^{-3} \right] / (\varepsilon - 3.7743 \times 10^{-3})^2 \]

Also two alternative stress-strain relations are used in the numerical evaluations. The stress-strain curve used by Batterman and Johnston\(^{(16)}\) is called stress-strain curve #2 (Fig. 5.1). The stress-strain curve used by Johnston\(^{(14)}\) in the strut model study is called stress-strain curve #3 (Fig. 5.2).
Fig. 5.1. Stress-Strain and Tangent-Modulus Strain Relation #2
(used in Batterman-Johnston column analysis)
Fig. 5.2. Stress-Strain and Tangent Modulus-Strain Relation #3
5.3. BASIC RELATIONS USED IN THE INCREMENTAL PROCEDURE

i. Division of the Column Length

The column is divided into \( m \) equal segments of length \( \lambda \)

\[
\lambda = \frac{L}{m} \tag{5.2}
\]

The nodal points along the column are numbered consecutively starting
with 0 from the top (left) and ending with \( m \) at the bottom (right).

In this chapter the longitudinal view of the column will be
designated by an axis passing through the centroid of the flange (twist
center) of the initially straight column (Fig. 5.3).

![Initial and Deflected Positions of the Column](image)

Fig. 5.3. Initial and Deflected Positions of the Column

If it is desired to have the mid-height section as a nodal point, \( m \)
should be chosen an even number. All behavior characteristics of
every nodal point, such as deflection, curvature, depth of strain
regression, and others which will be explained, can be distinguished by the values of the subscripts of every variable, i.e., \([-\phi(\ell q)]_n\) represents the curvature at the \(\ell^{th}\) nodal point \((\ell = 0, 1, \ldots, m)\) along the \(q^{th}\) axis \((q = 1\) is x-axis, \(q = 2\) is y-axis), after \(n\) cycle of incrementing the fixed increment. The fixed increment in this analysis is chosen to be the increment of the deflection of the mid-height of the column in the x direction, i.e.,

\[
[\Delta \delta (\frac{m}{2}, 1)]_1 = [\Delta \delta (\frac{m}{2}, 1)]_2 = \ldots = [\Delta \delta (\frac{m}{2}, 1)]_n
\]

(5.3)

ii. Curvature-Deflection Relation

Consider the vector \(\{U\}\) with \((m + 1)\) elements. The element \(U_\ell\) of the above vector represents the deflection of the column in a longitudinal plane of the column under consideration. If \(\{-\phi\}\) is the curvature vector of the deflection \(\{U\}\) (Fig. 5.4), then, using the Newmark's procedure, one may find the matrix \([R]\) such that

\[
\{U\} = [R]\{\phi\}
\]

(5.4)

![Curvature Distribution along the Column](Fig. 5.4)
where \( \{ U \} \) and \( \{ \phi \} \) are

\[
\begin{pmatrix}
U_0 \\
U_1 \\
U_2 \\
\vdots \\
U_{m-1} \\
U_m
\end{pmatrix}
= \begin{pmatrix}
\phi_0 \\
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_{m-1} \\
\phi_m
\end{pmatrix}
\quad (5.5)
\]

Matrix \([ R ]\) is the product of three matrices \([ q_1 ]\), \([ q_2 ]\) and \([ q_3 ]\).
Matrix \([ q_1 ]\) converts the distributed values of the curvature, \(\{-\phi\}\), into the concentrated angle changes, \(\{ \theta \}\), according to the following relation (Fig. 5.5)

\[
\{ \theta \} = [q_1] \{ \phi \}
\quad (5.6)
\]

Fig. 5.5. Concentrated Angle Changes
where

\[
\{\theta\} = \begin{bmatrix}
\theta_0 \\
\theta_1 \\
\theta_2 \\
\vdots \\
\theta_{m-1} \\
\theta_m
\end{bmatrix}
\]  

(5.7)

and

\[
[q_{1(m+1,m+1)}] = \frac{L}{I2m} \begin{bmatrix}
3.5 & 3. & -5 & 0 & 0 & . & 0 & 0 & 0 \\
1 & 10 & 1 & 0 & 0 & . & 0 & 0 & 0 \\
0 & 1 & 10 & 1 & 0 & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & . & 0 & 1 & 10 & 1 & 0 \\
0 & 0 & 0 & . & 0 & 0 & 1 & 10 & 1 \\
0 & 0 & 0 & . & 0 & 0 & -.5 & 3. & 3.5
\end{bmatrix}
\]  

(5.8)

The moment vector in the conjugate beam loaded with the angle changes will be equivalent to the displacement vector \(\{U\}\). Matrix \([q_2]\) relates the concentrated angle change (load), i.e., \(\{\theta\}\), to the average slope of the deflection curve (shear in the conjugate beam), \(\{S\}\), in the following equation \(^{(95)}\)

\[
\{S\} = [q_2] \{\theta\}
\]  

(5.9)
where

\[
\{S\} = \begin{bmatrix}
S_1 \\
S_2 \\
\vdots \\
S_{m-2} \\
S_{m-1}
\end{bmatrix}
\tag{5.10}
\]

and

\[
[q_{2(m-1,m+1)}] = \frac{1}{m} \begin{bmatrix}
0 & m-1 & m-2 & m-3 & \ldots & 2 & 1 & 0 \\
0 & -1 & m-2 & m-3 & \ldots & 2 & 1 & 0 \\
0 & -1 & -2 & m-3 & \ldots & 2 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & -1 & -2 & \ldots & \ldots & -(m-2) & 1 & 0 \\
0 & -1 & -2 & \ldots & \ldots & -(m-2) & -(m-1) & 0
\end{bmatrix}
\tag{5.11}
\]

Matrix \([q_3]\) transforms the average slope (shear) into the deflection (moment in the conjugate beam) in the following form (Fig. 5.6)

\[
\{U\} = [q_3] \{S\}
\tag{5.12}
\]
Fig. 5.6. Average Shear and Deflection

where

\[
[q_{3(m+1,m-1)}] = \frac{L}{m}
\]

\[
\begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
1 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & 1 & 0 \\
1 & 1 & \ldots & 1 & 1 \\
\end{bmatrix}
\]  

(5.13)

Combining (5.6), (5.9) and (5.12) we get

\[
\left\{ U \right\} = [q_3] [q_2] [q_1] \left\{ \phi \right\}
\]

(5.14)

Matrix \([R]\) can be found by comparing (5.4) and (5.14)

\[
[R] = [q_3] [q_2] [q_1]
\]

(5.15)

One can find \([R]\) by replacing for \([q_1]\), \([q_2]\) and \([q_3]\) and performing the operation
\[ R_{ij} = 0 \quad \text{for } i = 0, m \]

\[ = \frac{(m - i) L^2}{12 m^3} \quad \text{for } j = 0 \]

(except for \( i = 0, m \))

\[ = \frac{i L^2}{12 m^3} \quad \text{for } j = m \]

(except for \( i = 0, m \))

\[ = \frac{12i (m - i) - m}{12 m^3} L^2 \quad \text{for } i = j \]

\[ = \frac{i (m - i) L^2}{m^3} \quad \text{for } i > j \]

\[ = \frac{i (m - j) L^2}{m^3} \quad \text{for } i < j \]

The above relations show that, except for the boundary elements, the matrix \([R]\) is a symmetrical matrix

\[ R_{ij} = R_{ji} \]

Matrix \([R]\) relates the curvature to the deflection as well as it relates the increment of the curvature to the increment of the deflection. This follows from distributive property of the matrices, as we can write

\[ \{ \phi \}_{n+1} = \{ \phi \}_n + \{ \Delta \phi \}_{n+1} \quad (5.18) \]

Using the relation (5.4)

\[ \{ U \}_{n+1} = [R] \{ \phi \}_{n+1} = [R] \{ \phi \}_n + [R] \{ \Delta \phi \}_{n+1} = \{ U \}_n + \{ \Delta U \}_{n+1} \quad (5.19) \]

where

\[ \{ \Delta U \}_{n+1} = [R] \{ \Delta \phi \}_{n+1} \quad (5.20) \]
Here, in every cycle of increment the algebraic increments of the curvature and the deflection are considered and related to one another. Because in the general case we shall have to evaluate the deflection in two directions, $\delta_{\ell 1}$ and $\delta_{\ell 2}$ will represent the deflection per unit length of the column at $\ell^{th}$ nodal point along the column in the x and y directions respectively. $\Delta \delta_{\ell 1}$ and $\Delta \delta_{\ell 2}$ will represent the increments of the above values. Using the matrix $[R]$ we can write

$$\Delta \delta_{\ell q} = \sum_{k=0}^{m} r_{\ell k} \Delta \phi_{k q} \quad (q = 1, 2; \ell = 0, 1, \ldots, m) \quad (5.21)$$

where $\Delta \phi_{k q}$ is the increment of the curvature of deflection in the $q^{th}$ ($q = 1, 2$) direction at the $\ell^{th}$ nodal point along the column, and

$$r_{\ell k} = R_{\ell k} / L \quad (5.22)$$

Relation (5.15) (or (5.17)) shows that elements of the first and the last rows of the matrix $[R]$ are zero, then

$$\text{det} [R] = 0 \quad (5.23)$$

and therefore $[R]$ is a singular matrix and its inversion is not possible. This means that in the general case for which the curvatures of the end sections are non-zero, with the ends prevented from deflecting, one cannot obtain the curvature vector $\{-\phi\}$ from the displacement vector $\{U\}$. In the case of concentric loading of the pinned-end column the end section curvatures are zero and therefore the first and the last elements of the deflection and curvature vectors and also the first and the last rows and columns of the matrix $[R]$ can be dropped so as to eliminate the singularity of the matrix $[R]$. 
In this incremental procedure the vector \( \{ \Delta \phi \} \) was assumed, then, using \([R]\), the increment of the displacement vector, \( \{ \Delta U \} \), was evaluated by (5.20) and thus the need for the inversion of \([R]\) was avoided. The advantage was taken of the matrix properties in the process of modification of the assumed \( \{ \Delta \phi \} \) for a better value and normalization of the \( \{ \Delta U \} \), i.e., both the modified \( \{ \Delta \phi \} \) and the corresponding \( \{ \Delta U \} \) were multiplied by a scalar value to make the deflection at the mid-height section equal to the assigned value, as

\[
\{ \Delta U \} = [R] \{ \Delta \phi \}
\]

\[
a \{ \Delta U \} = a [R] \{ \Delta \phi \}
\]

\[
\{ a \Delta U \} = [R] \{ a \Delta \phi \}
\]

(5.24)

where \( a \) is a scalar factor. Part of the incremental analysis is devoted to the assumption of a sinusoidal curve for the deflected shape of the column. In this case we can write

\[
U(z) = U_o \sin \frac{\pi z}{L}
\]

(5.25)

where \( U_o \) is the mid-height deflection, then

\[
[- \phi(z)] = U(z) = -\frac{L^2}{\pi} U_o \]

and

\[
\frac{L^2}{\pi} \varepsilon_{\frac{L}{2}} \phi_{\frac{L}{2}} = \frac{L^2}{\pi} \phi_{\frac{L}{2}}
\]

(5.26)

Thus the matrix \([R]\) will have only one element, i.e.,

\[
[R] = \left[ \frac{L^2}{\pi} \right]
\]

(5.27)
iii. Equilibrium at the Nodal Points along the Column

In order to evaluate the values of the integrals in equations (3.24) or (3.27) for planar bending, the distribution of stress over the area of the cross section must be known. In general, the stress cannot be expressed in the form of a single function in terms of $\xi$ and $\eta$. Thus, the integration cannot be carried out algebraically. Numerical evaluation of the integrals required the division of the area of the cross section into a grid network (Fig. 5.7). Then the values of the integrant at the centroid of every mesh and thereby the values of the integral over its area were determined and summed over the cross sectional area.

Two different variable names were chosen to express the coordinates of the mesh centroids to differentiate between a mesh of the flange and of the web.  

Fig. 5.7. Cross Section Divided into Grid Networks
Coordinates of the point \((i, j)\) of the flange are

\[
\begin{align*}
\xi_f &= \xi_{fj} = \frac{t}{2} + (j - \frac{1}{2}) \frac{t}{m_f} \\
\eta_f &= \eta_{fi} = -\frac{b}{2} + (i - \frac{1}{2}) \frac{b}{n_f}
\end{align*}
\]  

(5.28)

Coordinates of the point \((k, p)\) of the web are

\[
\begin{align*}
\xi_w &= \xi_{wp} = \frac{t}{2} + (p - \frac{1}{2}) \frac{d - t}{m_w} \\
\eta_w &= \eta_{wk} = -\frac{w}{2} + (k - \frac{1}{2}) \frac{w}{n_w}
\end{align*}
\]  

(5.29)

Now the integrals of the equations (3.24) at the \(i^{th}\) nodal point along the column can be written in a summation form

\[
\int_A \sigma(\xi, \eta) \, dA = \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} \sigma_{fij} \Delta A_{fij} + \sum_{k=1}^{n_w} \sum_{p=1}^{m_w} \sigma_{wkp} \Delta A_{wkp}
\]

\[
\int_A \sigma(\xi, \eta) \, dA = \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} \sigma_{fij} \xi_{fj} \eta_{fi} \Delta A_{fij} + \sum_{k=1}^{n_w} \sum_{p=1}^{m_w} \sigma_{wkp} \eta_{wk} \Delta A_{wkp}
\]

\[
\int_A \sigma(\xi, \eta) \, dA = \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} \sigma_{fij} \xi_{fj} \Delta A_{fij} + \sum_{k=1}^{n_w} \sum_{p=1}^{m_w} \sigma_{wkp} \xi_{wp} \Delta A_{wkp}
\]

\[
\int_A \sigma(\xi, \eta) \, dA = \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} \sigma_{fij} (\xi_{fj}^2 + \eta_{fi}^2) \Delta A_{fij} + \sum_{k=1}^{n_w} \sum_{p=1}^{m_w} \sigma_{wkp} (\xi_{wp}^2 + \eta_{wk}^2) \Delta A_{wkp}
\]  

(5.30)
where
\[
\Delta A_{fi} = \frac{bt}{n_f m_f} = \Delta A_f
\]
\[
\Delta A_{wkp} = \frac{(d - t)w}{n_w m_w} = \Delta A_w
\]

(5.31)

The area of the mesh is constant for the flange or for the web.

Therefore, \( \Delta A_{fi} \) and \( \Delta A_{wkp} \) can be factored out of the summation.

Using the above relations for \( \ell \)th nodal point along the column the equilibrium conditions (3.24) can be written in the following form

\[
P_{\ell} \left[ \Delta A_{f} \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} \sigma_{\ell fij} + \Delta A_{w} \sum_{k=1}^{n_w} \sum_{p=1}^{m_w} \sigma_{\ell wp} \right] = 0
\]

(5.31a)

\[
P_{\ell} L \left\{ \left( \delta_{\ell 2} + \delta_{o\ell 2} \right) - e_{t2} \left[ 1 + (r_{e2} - 1) \frac{L}{m} \right] \right\} + \beta_{\ell} \left\{ \left( \frac{\bar{X}}{L} - \delta_{\ell 1} - \delta_{o\ell 1} \right) + e_{t1} \left[ 1 + (r_{e1} - 1) \frac{L}{m} \right] \right\}
\]

\[
+ \Delta A_{f} \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} \sigma_{\ell fij} \eta_{fi} + \Delta A_{w} \sum_{k=1}^{n_w} \sum_{p=1}^{m_w} \sigma_{\ell wp} \eta_{wk} = 0
\]

(5.31b)

\[
P_{\ell} L \left\{ \beta_{\ell} \left( \delta_{\ell 2} + \delta_{o\ell 2} \right) - e_{t2} \left[ 1 + (r_{e2} - 1) \frac{L}{m} \right] \right\} - \left( \frac{\bar{X}}{L} - \delta_{\ell 1} - \delta_{o\ell 1} \right) + e_{t1} \left[ 1 + (r_{e1} - 1) \frac{L}{m} \right] \right\}
\]

\[
+ \Delta A_{f} \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} \sigma_{\ell fij} \xi_{fi} + \Delta A_{w} \sum_{k=1}^{n_w} \sum_{p=1}^{m_w} \sigma_{\ell wp} \xi_{wp} = 0
\]

(5.31c)
\[ P_\ell L^2 \left\{ \left[ (\delta_{t2} + \delta_{o12}) - e_{t2} \left[ 1 + (r_{e2} - 1) \frac{\ell}{m} \right] \right] (\delta_{t1} + \delta_{o11}) + \left[ \frac{x}{L} - \delta_{t1} - \delta_{o11} \right] + e_{t1} \left[ 1 + (r_{e1} - 1) \frac{\ell}{m} \right] \right\} (\delta_{t2} + \delta_{o12}) \]

\[ + \beta_\ell \left\{ \Delta A_f \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} \sigma_{fij} \rho_{fij}^2 + \Delta A_w \sum_{k=1}^{n_w} \sum_{p=1}^{m_w} \sigma_{wkp} \rho_{wkp}^2 \right\} \]

\[ - \left\{ \frac{1}{3} \frac{b_t^3 - 0.21t^4}{n_f m_f} \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} G_{s_{fij}} \right\} \]

\[ + \frac{1}{3} \frac{(d-t)^3 - 0.105w^4}{n_w m_w} \sum_{k=1}^{n_w} \sum_{p=1}^{m_w} G_{s_{wkp}} + \frac{\alpha_1 D_1^4}{n_m m_f + n_m m_w} \left\{ \sum_{i=1}^{n_f} \sum_{j=1}^{m_f} G_{s_{fij}} + \sum_{k=1}^{n_w} \sum_{p=1}^{m_w} G_{s_{wkp}} \right\} \right\} = 0 \quad (5.32d) \]

where

\[ \rho_{fij}^2 = \xi_{fij}^2 + \eta_{fij}^2 \]

\[ \rho_{wkp}^2 = \xi_{wkp}^2 + \eta_{wkp}^2 \quad (5.33) \]

The last term, in brackets, of the equation (5.32d) is the torsional stiffness of cross section of the column at the \( \ell^{\text{th}} \) nodal point evaluated according to the equation (4.83).

iv. Evaluation of the Stress at the Centroid of the Mesh

**Stress is given as a function of strain; therefore, the strain at the centroid of the mesh should be known so that the stress at that**
point could be evaluated. Having made the assumption that the plane sections remain plane, in order to express the strain across any cross section along the column, it is necessary to know the strain at one point of the cross section, i.e., at the origin of the $\xi \eta \zeta$ axes, and the curvature of the deflected shape of the column in two non-parallel longitudinal planes at the same cross section. In general, in order to express the orientation of the strain distribution plane relative to the unstrained position, it is required to know the strain at three points which are not on a straight line. Let $\varepsilon_{\ell}$ be the strain at the centroid of the flange at the $\ell$th cross section. Strain $\varepsilon$ at point $(\xi, \eta)$ of this cross section will be

$$\varepsilon_{\ell}(\xi, \eta) = \varepsilon_{\ell} - (\phi_{\ell1} + \beta_{\ell} \phi_{\ell2}) \xi - (-\beta_{\ell} \phi_{\ell1} + \phi_{\ell2}) \eta$$ (5.34)

For the mesh of the flange and the web

$$\varepsilon_{lij} = \varepsilon_{\ell} - (\phi_{\ell1} + \beta_{\ell} \phi_{\ell2}) \xi_{lij} - (-\beta_{\ell} \phi_{\ell1} + \phi_{\ell2}) \eta_{lij}$$

$$\varepsilon_{wp} = \varepsilon_{\ell} - (\phi_{\ell1} + \beta_{\ell} \phi_{\ell2}) \xi_{wp} - (-\beta_{\ell} \phi_{\ell1} + \phi_{\ell2}) \eta_{\ell k}$$ (5.35)

where $(-\phi_{\ell1})$ and $(-\phi_{\ell2})$ are the curvature of the deflected column in the xz and yz planes at the $\ell$th section, respectively. The strain at the origin is expressed in the following form

$$\varepsilon_{\ell} = \phi_{\ell1} C_{\ell1} = \phi_{\ell2} C_{\ell2} \quad (\ell = 1, 2, \ldots, m)$$ (5.36)

The magnitude of $C_{\ell q}$ is the same as the corresponding curvature of the deflected shape. Figure 5.8 gives the strain distribution at the $\ell$th cross section along the column and at the two consecutive incremental steps, $n$ and $(n + 1)$. 
Fig. 5.8. Strain Distribution at the $\ell$th Cross Section

The depth of strain regression SR at the $(n + 1)^{th}$ incremental step, $\ell$th section and on the $q$th axis will be

$$
[SR_{\ell q}]_{n+1} = \frac{[\phi_{\ell q} C_{\ell q}]_{n+1} - [\phi_{\ell q} C_{\ell q}]_{n}}{[\Delta \phi_{\ell q}]_{n+1}} \quad (q=1, 2; \ell=1, 2, \ldots, m)
$$

(5.37)

Stress at the Strain Regression Region

Having evaluated the strain at the centroid of the mesh, if strain regression has occurred, i.e.,

$$
(\epsilon_{\ell ij})_{n+1} < (\epsilon_{\ell ij})_{n}
$$

(5.38)
then the stress will be
\[
(\sigma_{ij})_{n+1} = (\sigma_{ij})_{\text{max}} - E \left[ (\varepsilon_{ij})_{\text{max}} - (\varepsilon_{ij})_{n+1} \right]
\]  
(5.39)
where \((\varepsilon_{ij})_{\text{max}}\) and \((\sigma_{ij})_{\text{max}}\)
are the maximum strain
recorded at the mesh \((l, j)\).
If strain reversal is present
(the fiber is in tension)
(Fig. 5.9), the stress will
correspond to the effective
strain \((\varepsilon_{\text{eff}})\), in the stress-strain diagram, where

\[
(\varepsilon_{\text{eff}}) = (\varepsilon_{ij})_{n+1} - \left[ (\varepsilon_{ij})_{\text{max}} - \frac{(\sigma_{ij})_{\text{max}}}{E} \right]
\]  
(5.40)

v. Planar Bending

In the stage where the biplanar bending accompanied with
torsion has not yet commenced, one should take advantage of the
simplifications which can be made in the preceding relations. In
planar bending (bending in the plane of symmetry)
\[
\phi_{l2} = \Delta \phi_{l2} = U_{l2} = \Delta U_{l2} = \beta_{l} = 0 \quad (l=1,2,\ldots,m)
\]
then

\[ \varepsilon_{f \ell 1j} = \varepsilon_{f \ell 2j} = \ldots = \varepsilon_{f \ell n_f j} \text{ and } \varepsilon_{w \ell 1p} = \varepsilon_{w \ell 2p} = \ldots = \varepsilon_{w \ell n_w p} \]

and

\[ \sigma_{f \ell 1j} = \sigma_{f \ell 2j} = \ldots = \sigma_{f \ell n_f j} \text{ and } \sigma_{w \ell 1p} = \sigma_{w \ell 2p} = \ldots = \sigma_{w \ell n_w p} \]  \hspace{1cm} (5.41)

The equilibrium equations will reduce to

\[ P_{f \ell} - n_f \Delta A_f \sum_{j=1}^{m_f} \sigma_{f \ell 1j} + n_w \Delta A_w \sum_{p=1}^{m_w} \sigma_{w \ell 1p} = 0 \]  \hspace{1cm} (5.42a)

\[ -P_{f \ell} L \left( \frac{x}{L} - \delta_{f \ell 1} - \delta_{o \ell 1} \right) + e_{t1} \left[ 1 + (r_{e1} - 1) \frac{f}{m} \right] \]

\[ + n_f \Delta A_f \sum_{j=1}^{m_f} \sigma_{f \ell 1j} \xi_{fj} + n_w \Delta A_w \sum_{p=1}^{m_w} \sigma_{w \ell 1p} \xi_{wp} = 0 \]  \hspace{1cm} (5.42b)

and the strain at the centroid of the mesh will be

\[ \varepsilon_{f \ell ij} = \varepsilon_{o \ell} - \phi_{f \ell 1} \xi_{fj} \quad (i=1, 2, \ldots, n_f) \]  \hspace{1cm} (5.43)

\[ \varepsilon_{w \ell kp} = \varepsilon_{o \ell} - \phi_{f \ell 1} \xi_{wp} \quad (k=1, 2, \ldots, n_w) \]

For the evaluation of the limiting point of the planar bending

equation (3.35) can be written in the following form

\[ \overline{\xi_{\ell}} = \frac{n_f \Delta A_f \sum_{j=1}^{m_f} E_{t f \ell j} \xi_{fj} + n_w \Delta A_w \sum_{p=1}^{m_w} E_{t w \ell p} \xi_{wp}} {\frac{m_f}{m} + \frac{m_w}{m}} \]  \hspace{1cm} (5.44)
where
\[ E_{t_{\text{fl}j}} = \text{the tangent modulus at the point } \xi_{ij} \text{ of the flange} \]
at the \( l^{\text{th}} \) section of the column
\[ E_{t_{\text{wp}l}} = \text{the tangent modulus at the point } \xi_{wp} \text{ of the web} \]
at the \( l^{\text{th}} \) section of the column

The ratio of the stiffness \( \alpha \) at the \( l^{\text{th}} \) cross section of the column will be

\[
\alpha_l = \frac{\frac{b_t}{3m_f} \sum_{j=1}^{m_f} E_{t_{\text{fl}j}} + \frac{w}{3m_w} \sum_{p=1}^{m_w} E_{t_{\text{wp}l}}}{\frac{bt}{m_f} \sum_{j=1}^{m_f} (\xi_{ij} - \xi_l)^2 E_{t_{\text{fl}j}} + \frac{w(d - t)}{m_w} \sum_{p=1}^{m_w} (\xi_{wp} - \xi_l)^2 E_{t_{\text{wp}l}}} \tag{5.45}
\]

5.4. TECHNIQUE OF THE INCREMENTAL ANALYSIS

The technique of the incremental analysis may be summarized as follows:

Given:

i. the dimensions of the cross section \( b, d, t, w \)

ii. the length of the column, \( L \), or the slenderness ratio, \( \frac{L}{r} \)

iii. the number of divisions of the grid, \( n_f, m_f, n_w, m_w \)

iv. the number of the segments the column is to be divided into, \( m \)

v. the increment of the deflection \( \Delta x \) (at \( z = \frac{L}{2} \), positive or negative)
vi. initial crookedness $\delta_{ix}$, $\delta_{iy}$; eccentricity of loading, $e_x$, $e_y$ (per unit length)

vii. relation between strain and stress, strain and tangent modulus, and strain and secant shear modulus

viii. acceptable error in the evaluation of the axial load in the longitudinal sections, satisfying the moment equilibrium equations and the calculation of the twist at the longitudinal sections.

Desired:

Equilibrium position after each increment of the deflection together with all the related characteristics such as deflection and curvature in the x and y directions, twist along the column, axial load, moment about the x- and y-axes, stress distribution across the column and at the longitudinal nodal points, and the border line of strain regression.

The following are some important points to be noted in the process of convergence to an equilibrium position after each increment:

i. The estimated change of curvature vector in the $n^{th}$ increment $(-\Delta \phi)_n$ was revised for a better estimate according to the following relation

$$
\left[ (\Delta \phi_{\ell})_n \right]_{(c+\frac{1}{2})} = \left( \Delta \phi_{\ell} \right)_n \frac{(\Delta P_{avg})_n + Q}{(\Delta P_{\ell})_n + Q} 
$$

(5.46)

where $c$ = the number of the cycle of iteration
\[(\Delta P_i)_n = (P_i)_n - (P_{avg})_{n-1}\]

\[(\Delta P_{avg})_n = (P_{avg})_n - (P_{avg})_{n-1}\]  \hspace{1cm} (5.47)

\[(P_{avg})_n = \frac{1}{m+1} \sum_{\ell=0}^{m} (P_i)_n\]

Having revised \[\{(\Delta \phi)_n\}_c \text{ to } \{(\Delta \phi)_n\}_{c+\frac{1}{2}}\], using matrix \([R]\) corresponding increment of deflection \[\{(\Delta U)_n\}_{c+\frac{1}{2}}\] can be obtained. Normalization of the curvature and the deflection vectors may be done according to (5.24),

\[\{(\Delta \phi)_n\}_{c+1} = a \{(\Delta \phi)_n\}_{c+\frac{1}{2}}\]  \hspace{1cm} (5.48)

\[\{(\Delta U)_n\}_{c+1} = a \{(\Delta U)_n\}_{c+\frac{1}{2}}\]

where

\[a = \frac{(\Delta x)_{inc}}{[(\Delta U)_{inc}(\frac{m}{Z})]_{n+\frac{1}{2}}}\]  \hspace{1cm} (5.49)

and \((\Delta x)_{inc}\) = mid-length deflection increment

The term \(Q\) in the equation (5.46) is used to optimize the convergence rate. \(^{(16)}\) The value of \(Q\) should be chosen such that the numerator and the denominator of the fraction in the equation (5.46) would always be positive. In the unloading stage where \((\Delta P_{avg})\)
and (or) $(\Delta P_f)$ are negative, the fraction may become
negative and consequently the revised values in the
cycles of iteration will oscillate between a negative
and a positive value without achieving convergence.
For the same reason $Q$ was assumed to be the sum
of two values
\[ Q = Q_1 + Q_2 \tag{5.50} \]
where
\[ Q_1 = \text{largest value of } \{\Delta P\} \]
\[ Q_2 = \text{a positive value which should be} \]
reduced if the convergence rate is
slow and visa versa
In the loading stage where $(\Delta P_{avg})$ is positive if the
magnitude of $(\Delta P_f)$ is smaller than the $(\Delta P_{avg})$
a greater curvature change and otherwise a smaller
curvature change should be assumed. In the unloading
stage where $(\Delta P_{avg})$ is negative, the curvature
revision scheme should be inverse of the above
procedure. The fraction in the equation (5.37) follows
the above scheme.

ii. The change of curvature vector $\{-\Delta \phi\}$ should be
controlled such that if its magnitude becomes smaller
than a specific value the direction (sign) of that value
will be changed. This will prevent the underflow of
the values of the change of curvature vector $\{-\Delta \phi\}$. 
The change of direction of the increment of the curvature occurs in the unloading stage and at the sections close to the end of the column in the case of eccentric loading.

The steps followed to obtain the desired results are summarized in the computer program flow chart in Appendix A.

5.5. RESULTS

The cross sectional dimensions of the tee columns which were analyzed incrementally by means of the 7090 digital computer are as follows (Fig. 5.7)

\[ b = 4. \text{ inch} \]
\[ t = .279 \text{ inch} \]
\[ w = .218 \text{ inch} \]
\[ d = 1.5, 2.5, 3. \text{ inch} \]

Other section properties were

<table>
<thead>
<tr>
<th>( d )</th>
<th>( d = 1.5 \text{ inch} )</th>
<th>( d = 2.5 \text{ inch} )</th>
<th>( d = 3. \text{ inch} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area, ( \text{inch}^2 )</td>
<td>1.382</td>
<td>1.600</td>
<td>1.709</td>
</tr>
<tr>
<td>( I_x, \text{inch}^4 )</td>
<td>1.489</td>
<td>1.490</td>
<td>1.490</td>
</tr>
<tr>
<td>( I_y, \text{inch}^4 )</td>
<td>.161</td>
<td>.734</td>
<td>1.245</td>
</tr>
<tr>
<td>( r_x, \text{inch} )</td>
<td>1.038</td>
<td>.965</td>
<td>.934</td>
</tr>
<tr>
<td>( r_y, \text{inch} )</td>
<td>.342</td>
<td>.677</td>
<td>.853</td>
</tr>
<tr>
<td>( \alpha = \frac{I_x}{I_y} )</td>
<td>9.248</td>
<td>2.030</td>
<td>1.197</td>
</tr>
<tr>
<td>( \overline{d}, \text{inch} )</td>
<td>.283</td>
<td>.518</td>
<td>.660</td>
</tr>
</tbody>
</table>
Load-deflection history of columns of various combinations of slenderness ratio, initial crookedness, eccentrically or concentrically loaded (Fig. 5.10) (with three different stress-strain relations) was determined. Combinations which were studied are summarized in Table 5.1.

Fig. 5.10. Eccentric Loading-Initial Crookedness
### Table 5.1

**Combinations Analyzed Incrementally by Digital Computer**

<table>
<thead>
<tr>
<th>Stress-Strain Relations</th>
<th>d</th>
<th>KL/r</th>
<th>Initial Crookedness*</th>
<th>Eccentric Loading**</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>1.5&quot;</td>
<td>10</td>
<td>*</td>
<td>**</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>40</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
<td></td>
<td></td>
</tr>
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<td></td>
<td></td>
<td>80</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>120</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>160</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>200</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#2 and #3</td>
<td>2.5&quot; and 3.&quot;</td>
<td>30</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>40</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.5&quot;</td>
<td>40</td>
<td>*</td>
<td></td>
</tr>
</tbody>
</table>

*concentric loading - initial crookedness assumed to be a half-sine curve (equation 3.6); \( \delta_{iy} = 0, \delta_{ix} = (\pm 0.0, \pm 0.0001, \pm 0.0005, \pm 0.001, \pm 0.002, \pm 0.004) \).

**no initial crookedness - \( e_{tx} = e_{x}, r_{ex} = 1, e_{ty} = 0 \) (equation 3.4, Fig. 5.10), \( e_{x} = (\pm 0.001, \pm 0.002) \).
In the following pages the summary of the results of the incremental analysis will be shown graphically and tabulated. Some selected graphs of the load-deflection history will also be presented. For all the graphs the stress-strain relation used is #1 and $d = 1.5''$ unless otherwise is indicated.

The lines indicated by (i. b. g.) in the graphs are the tangent lines to the dimensionless load-deflection curves, i.e.,

\[
(i. \ b. \ g.) = \lim_{\Delta \delta \to 0} \left| \frac{\Delta P/P_t}{\Delta \delta/L} \right| = \lim_{\Delta \delta \to 0} \left| \frac{\Delta P}{\Delta \delta} \right| = \frac{L}{P_t} (I. \ B. \ G. ) (\frac{L}{P_t})
\]

(5.53)
Fig. 5.11. Dimensionless Load vs. Mid-length Deflection (initial crookedness; $L/r = 20$)
Fig. 5.19. Dimensionless Load vs. Mid-length Deflection
(initial crookedness; d = 1.5"; L/r = 40; stress-strain relation #3;
half-sine curve deflection-shape comparison)
Fig. 5.13. Dimensionless Load vs. Mid-length Deflection (initial crookedness; \(L/r = 80\))
Table 5.2
Ultimate Average Stress
Buckling towards or away from the flange
(initial crookedness; d = 1.5"; stress-strain relation #1)

<table>
<thead>
<tr>
<th>KL/f</th>
<th>Buckling to the Left</th>
<th>Buckling to the Right</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Initial Crookedness, δ_IX</td>
<td>Critical Stress, σ_c</td>
</tr>
<tr>
<td></td>
<td>(-LC)</td>
<td>(-0.0040)</td>
</tr>
<tr>
<td>10</td>
<td>40.914</td>
<td>42.409</td>
</tr>
<tr>
<td>20</td>
<td>35.762</td>
<td>35.870</td>
</tr>
<tr>
<td>30</td>
<td>30.505</td>
<td>35.690</td>
</tr>
<tr>
<td>40</td>
<td>24.883</td>
<td>29.623</td>
</tr>
<tr>
<td>50</td>
<td>20.020</td>
<td>24.158</td>
</tr>
<tr>
<td>100</td>
<td>7.409</td>
<td>8.433</td>
</tr>
<tr>
<td>200</td>
<td>2.163</td>
<td>2.322</td>
</tr>
</tbody>
</table>
MID-LENGTH DEFLECTION AND TWIST PER UNIT LENGTH; $\delta_x, \delta_y$ INCH/INCH; $\beta/L$ RAD/INCH

Fig. 5.14. Dimensionless Load vs. Mid-length Deflections and Twist
(initial crookedness; $d = 2.5''$; $L/r = 30$)
MID-LENGTH DEFLECTION AND TWIST PER UNIT LENGTH; $\delta_x$, $\delta_y$ INCH/INCH; $\beta/L$ RAD/INCH

Fig. 5.15. Dimensionless Load vs. Mid-length Deflections and Twist
(initial crookedness; $d = 2.5''$; $L/r = 40$)
MID-LENGTH DEFLECTION AND TWIST PER UNIT LENGTH; $\delta_x$, $\delta_y$ INCH/INCH; $\beta/L$ RAD/INCH

Fig. 5.16. Dimensionless Load vs. Mid-length Deflections and Twist
(initial crookedness; $d = 3.11''$; $L/r = 30$)
MID-LENGTH DEFLECTION AND TWIST PER UNIT LENGTH; $\delta_x$, $\delta_y$ INCH/INCH, $\beta/L$ RAD/INCH

Fig. 5.17. Dimensionless Load vs. Mid-length Deflections and Twist (initial crookedness; $d = 3''$; $L/r = 40$)
### Table 5.4
Ultimate Average Stress
Buckling towards or away from the flange
(initial crookedness; d = 2.5", 3"; stress-strain relation #1)

<table>
<thead>
<tr>
<th>d</th>
<th>KL/F</th>
<th>Initial Crookedness, δ₁ₓ</th>
<th>Reduced Modulus Stress, f₁ₓ</th>
<th>Reduced Stress, σ₁ₓ</th>
<th>Critical Stress, f₁</th>
<th>I.B.G. Stress, f₁ chained</th>
<th>Reduced Modulus Load/P₁</th>
<th>Reduced Load/P₁</th>
<th>Initial Crookedness, δ₁ₓ</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5&quot;</td>
<td>30</td>
<td>31.746</td>
<td>49.764</td>
<td>42.870</td>
<td>974.9</td>
<td>42.472</td>
<td>3732.6</td>
<td>44.120</td>
<td>89.162</td>
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<td></td>
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<td>35.688</td>
<td>44.132</td>
<td>42.021</td>
<td>811.1</td>
<td>42.200</td>
<td>41.782</td>
<td>40.920</td>
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<td>40.720</td>
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<td>4718.2</td>
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<td></td>
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<td>41.782</td>
<td>40.920</td>
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</tr>
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<td>43.551</td>
<td>40.720</td>
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<td>40.247</td>
<td>4718.2</td>
<td>43.300</td>
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<td>36.543</td>
<td>41.497</td>
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<td>44.120</td>
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<td>49.764</td>
<td>42.870</td>
<td>974.9</td>
<td>42.472</td>
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<td>41.782</td>
<td>40.920</td>
<td>39.180</td>
</tr>
</tbody>
</table>

### Table 5.5
(Ultimate Load/Critical Load)
Buckling towards or away from the flange
(initial crookedness; d = 2.5", 3"; stress-strain relation #1)

<table>
<thead>
<tr>
<th>d</th>
<th>KL/F</th>
<th>Initial Crookedness, δ₁ₓ</th>
<th>Reduced Modulus Stress, f₁ₓ</th>
<th>Reduced Stress, σ₁ₓ</th>
<th>Critical Stress, f₁</th>
<th>I.B.G. Stress, f₁ chained</th>
<th>Reduced Modulus Load/P₁</th>
<th>Reduced Load/P₁</th>
<th>Initial Crookedness, δ₁ₓ</th>
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</thead>
<tbody>
<tr>
<td>2.5&quot;</td>
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<td>1.009</td>
<td>22.953</td>
<td>87.884</td>
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<td>2.099</td>
<td>1.015</td>
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<td>.840</td>
<td>.934</td>
<td>.976</td>
<td>1.001</td>
<td>1.172</td>
<td>1.009</td>
<td>22.953</td>
<td>1.008</td>
</tr>
<tr>
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<td>40</td>
<td>.656</td>
<td>.849</td>
<td>.934</td>
<td>.976</td>
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<td>1.172</td>
<td>1.009</td>
<td>.994</td>
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<td>.934</td>
<td>.976</td>
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<td>1.172</td>
<td>1.009</td>
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<td>.977</td>
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<td>1.172</td>
<td>1.009</td>
<td>23.219</td>
<td>1.015</td>
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<td></td>
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<td>.903</td>
<td>.977</td>
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<td>1.009</td>
<td>23.219</td>
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<td>.857</td>
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<td>.964</td>
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<td>.964</td>
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<td>1.082</td>
<td>30.958</td>
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<td>1.001</td>
<td>1.172</td>
<td>1.009</td>
<td>23.219</td>
<td>.970</td>
</tr>
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<td>1.172</td>
<td>1.009</td>
<td>23.219</td>
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<td>.777</td>
<td>.857</td>
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<td>.964</td>
<td>1.002</td>
<td>1.082</td>
<td>30.958</td>
<td>.972</td>
</tr>
</tbody>
</table>
Fig. 5.18. Dimensionless Load vs. Mid-length Deflection
(initial crookedness; d = 1.5"; L/r = 40; stress-strain relation #2;
half-sine curve deflection-shape comparison)
Fig. 5.12. Dimensionless Load vs. Mid-length Deflection
(initial crookedness; L/r = 40;
half-sine curve deflection-shape comparison)
### Table 5.6

**Ultimate Average Stress**

Comparison with the half-sine deflection shape assumption 
(d = 1.5"; stress-strain relations #1, #2, #3)

<table>
<thead>
<tr>
<th>Stress-Strain</th>
<th>KL/l</th>
<th>Deflection Shape</th>
<th>Buckling to the Left</th>
<th>Buckling to the Right</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Initial Crookedness, δ_{1x}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.0040</td>
<td>-0.0020</td>
<td>-0.0010</td>
<td>-0.0005</td>
</tr>
<tr>
<td>#1 30 ⊙</td>
<td>30.305</td>
<td>34.690</td>
<td>37.54</td>
<td>39.278</td>
</tr>
<tr>
<td>#2 20 ⊙</td>
<td>29.603</td>
<td>34.139</td>
<td>37.176</td>
<td>39.028</td>
</tr>
<tr>
<td>#3 40 **</td>
<td>24.883</td>
<td>29.623</td>
<td>33.290</td>
<td>35.754</td>
</tr>
</tbody>
</table>

*Deflection shape is assumed to be a half sine curve

**Column divided into six equal segments
Table 5.7

(Ultimate Load/Critical Load)

Buckling towards or away from the flange
Comparison with the half-sine deflection shape assumption
(d = 1.5"; stress-strain relations #1, #2, #3)

<table>
<thead>
<tr>
<th>Stress-Strain Relation</th>
<th>Deflection Shape</th>
<th>Buckling to the Left</th>
<th>Buckling to the Right</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Initial Crookedness, $\delta_{ix}$</td>
<td>False Reduced Load/Pl</td>
<td>Reduced Load/Pl</td>
</tr>
<tr>
<td></td>
<td>.0040</td>
<td>.0020</td>
<td>.0010</td>
</tr>
<tr>
<td>** 30</td>
<td>0.714</td>
<td>0.817</td>
<td>0.884</td>
</tr>
<tr>
<td>* 40</td>
<td>0.697</td>
<td>0.804</td>
<td>0.875</td>
</tr>
<tr>
<td>** 40</td>
<td>0.618</td>
<td>0.736</td>
<td>0.827</td>
</tr>
<tr>
<td>* 40</td>
<td>0.601</td>
<td>0.720</td>
<td>0.813</td>
</tr>
<tr>
<td>** 40</td>
<td>0.642</td>
<td>0.755</td>
<td>0.839</td>
</tr>
<tr>
<td>* 40</td>
<td>0.622</td>
<td>0.737</td>
<td>0.826</td>
</tr>
<tr>
<td>** 40</td>
<td>0.616</td>
<td>0.726</td>
<td>0.811</td>
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<tr>
<td>* 40</td>
<td>0.597</td>
<td>0.709</td>
<td>0.798</td>
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</table>

* Deflection shape is assumed to be a half sine curve
** Column divided into six equal segments
Fig. 5.20. Ultimate Average Stress, \( \frac{P_y}{A} \), in KSI
Fig. 5.22. Dimensionless Load vs. Mid-length Deflection (eccentric loading - initial crookedness comparison; L/r = 20)
Fig. 5.23. Dimensionless Load vs. Mid-length Deflection (eccentric loading - initial crookedness comparison; L/r = 40)
MID-LENGTH DEFLECTION PER UNIT LENGTH OF THE COLUMN, $\delta_x$, INCH/INCH

Fig. 5.24. Dimensionless Load vs. Mid-length Deflection (eccentric loading - initial crookedness comparison; $L/r = 80$)
Table 5.8

Ultimate Average Stress

Buckling towards or away from the flange
Comparison of end eccentricity and initial crookedness
(d = 1.5"; stress-strain relation #1)

<table>
<thead>
<tr>
<th>KL/r</th>
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<th>Buckling to the Right</th>
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</thead>
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<tr>
<td></td>
<td>Δix = -.002</td>
<td>eₓ = +.002</td>
<td>Δix = -.001</td>
<td>eₓ = +.001</td>
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<tr>
<td>10</td>
<td>42.409</td>
<td>42.181</td>
<td>43.233</td>
<td>43.031</td>
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<tr>
<td>20</td>
<td>38.870</td>
<td>38.415</td>
<td>40.748</td>
<td>40.435</td>
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<td>34.690</td>
<td>33.944</td>
<td>37.548</td>
<td>37.034</td>
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<td>40</td>
<td>29.623</td>
<td>28.660</td>
<td>33.290</td>
<td>32.495</td>
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<td>50</td>
<td>24.158</td>
<td>23.218</td>
<td>27.668</td>
<td>26.809</td>
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</table>
Table 5.9

(Ultimate Load/Critical Load)
Buckling towards or away from the flange
Comparison of end eccentricity and initial crookedness
(d = 1.5"; stress-strain relation #1)

<table>
<thead>
<tr>
<th>KL/r</th>
<th>Crookedness $\delta_{ix} = - .002$</th>
<th>Eccentricity $e_x = + .002$</th>
<th>Crookedness $\delta_{ix} = - .001$</th>
<th>Eccentricity $e_x = + .001$</th>
<th>Eccentricity $e_x = - .001$</th>
<th>Crookedness $\delta_{ix} = + .001$</th>
<th>Eccentricity $e_x = - .002$</th>
<th>Crookedness $\delta_{ix} = + .002$</th>
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<td>1.003</td>
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<td>.872</td>
<td>.998</td>
<td>1.001</td>
<td>.978</td>
<td>.984</td>
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<td>.996</td>
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<td>.779</td>
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Fig. 5.25. Ultimate Average Stress vs. Slenderness Ratio
(eccentric loading - initial crookedness comparison)
Fig. 5.26. (Ultimate Load/Critical Load) vs. Slenderness Ratio (eccentric loading - initial crookedness comparison)
CHAPTER VI

TESTING

6.1. INTRODUCTION

Previous Experiments

In a historical note on column construction, Hoff\(^{(41, 42)}\) points out that the columns which were hewn out of solid rocks and erected almost five thousand years ago in Egypt, and twenty-five hundred years ago in Persopolis still stand, while the scientific and mathematical approach to the study of column problem started with Euler's formula in 1744. Although Musschenbroek built the first testing machine about 25 years prior to that date, his experimental research on columns did not receive attention until it was discovered that Euler's formula failed to give an answer to the column buckling in the inelastic range. In 1889 Considère conducted a series of 32 column tests that were important in relation to the tangent modulus theory.

Of large number of column tests which have been carried out, a few will be cited, as follows:

In the 1920's several series of tests on rolled built-up columns were conducted at the University of Wisconsin.\(^{(97)}\) In these experiments the effects of crookedness, eccentricity and imperfections of rolling were studied and the results were compared with the "ideal" column analysis. In these tests planar bending about the strong axis
was considered. These tests led to the development of design specification formulas in railing engineering practice.

Massonnet and Campus\(^{(98)}\) performed 95 tests on columns in Belgium and developed some design formulas for columns under oblique compression, and made a comparison with European column specifications. In these tests they studied symmetric or unsymmetric eccentrical loadings.

In 1956 Baker, Horne and Heyman\(^{(99)}\) published results of the tests performed on small models of three-dimensional frameworks in which the columns were subjected to biaxial bending in Cambridge.

In 1958 Mason, Fisher and Winter\(^{(100)}\) performed tests on 24 welded hat-shaped sections, loaded eccentrically and bent about the weak axis parallel to the flange.

Estuar and Tall\(^{(101)}\) in 1964 reported on a test procedure on pinned end columns. In January 1966, Lay, Aglietti and Galambos\(^{(102)}\) suggested some testing techniques for restrained beam-columns. In May 1966 Yarimci, Yaura and Lu\(^{(103)}\) published a report on techniques for testing structures permitted to sway. The above mentioned techniques were considerably helpful in developing the testing procedure for the measurement of the lateral displacements and the twist of the column explained in this chapter.

Recently Birnstiel, Leu, Tesoro and Tomasetti\(^{(37)}\) concluded a report on the experiments of the H-columns under biaxial bending, in order to compare the results with the values determined from the
interaction formula published in the "Column Research Council Guide to Design Criteria for Metal Compression Members," and found the formula generally very conservative. In this experiment 16 H-columns of high strength low-alloy V65 steel were used. Twelve of these experiments were loaded with substantially alike eccentricities at both ends with respect to both axes of the column and not alike eccentricity of the two ends on the rest of the experiments. In the process of the experiment, load vs. deflection in two directions and the rotation were recorded. As in the general case of the behavior of the column, biaxial bending combined with torsion will deduce.

6.2. OBJECTIVE OF THE EXPERIMENT

In order to study the applicability of the analytical results and compare the results obtained from incremental analysis with those obtained experimentally, it was necessary to perform a series of tests on the behavior of "T" shaped aluminum alloy columns. However, the author experienced a great deal of technical difficulties, and some of the experiments in the early stages yielded surprising and unexpected results. However, the experiences gained and the guidance obtained cannot be overlooked.

In the first series of tests it was attempted to support the column at the mid-height at either the flange or the web side and in the direction parallel to the web and normal to the column axis in order to study the load vs. deflection history of the column after
initiation of buckling in the positive
or negative x direction (Fig. 6.1).
Surprisingly, considerably
higher ultimate load carrying
capacity than the one anticipated
by theory was obtained in the
case where the column was
prevented from deflection
towards the negative direction
of the x-axis. The reasons
for this are threefold:

![Diagram](image)

**Fig. 6.1. Column Supported Laterally**

1. The existence of the mid-height support changed the
effective length and the mode of behavior of the column.

2. The existence of the eccentricity of loading at the ends,
no matter how small, in the positive direction of x was
producing horizontal force in the lateral support and this,
in turn, was changing the mode of deflection.

3. The existence of even negligible initial imperfection or
crookedness in the plane parallel to the web and towards
the negative direction of x had the same effect as end
eccentricity.
However, in this series of tests, because of the above reasons and also over-estimation of the critical local buckling load, the failure occurred in local buckling of the flange. To eliminate the local buckling, the widths of the original test specimens were reduced. Also, the scheme of the experiment was changed, lateral support was omitted, and by intentionally giving end eccentricity along the x-axis and in one direction or the other the column was forced to deflect in the arbitrarily chosen direction.

6.3. MATERIAL PROPERTIES

The material properties of the columns used in the experiments were determined by compression tests on two specimens of the complete rolled aluminum alloy H-section which was then split longitudinally to form the T sections used as column specimens. The length of the first specimen was chosen to be 12 inches but occurrence of local buckling at 45.7 ksi determined the last point obtained for the curve of the stress-strain relation. The second specimen was cut shorter to an 8 inch length and was braced to delay the local buckling, but it buckled locally a step sooner than the first specimen as the rods which were used to brace the specimen might have exerted lateral pressure and changed the mode of local buckling towards outside (Fig. 6.2). However, the range of the stress-strain relation which was obtained was far more than enough for use in relating the test results to analytical studies, as that limit of stress was beyond that of the loading stage of the chosen column tests.
M-M "Precision Strain Gages" (made by MicroMeasurements, Inc., Romulus, Michigan) were used as electrical strain gages and the TINIUS OLSEN 400000 pounds Mechanical Testing Machine in the Civil Engineering Department Laboratory of The University of Michigan (Fig. 6.3) was used as a loading device.

The rate of machine head movement was chosen to be about 0.025 inch per minute in the elastic range and 0.05 inch in the inelastic region. The graph of stress-strain for the above tests and the approximated algebraic fit for them are given in Fig. 2.6. The average Poisson's ratio was 0.325.
6.4. COLUMN DIMENSIONS

Columns are compression members of considerable length compared to their cross sectional dimensions. Tee column test specimens of various lengths were prepared by cutting and milling from a 6 x 4 6061-T6 aluminum alloy H-section. Before cutting to length, the web of the H-section was cut along the line parallel to longitudinal axis. The end cut surfaces were milled to give a smooth surface with a rather precise right angle to the adjoint surfaces. Very negligible effect, if any, of the relief of the residual stress was observed as the specimens did not tend to become crooked or change their initial form after cutting. The cross sectional dimensions of each specimen were measured at a few points along the length by means of micrometer and vernier calipers with accuracy of 0.001 inch. The result of the measurements of the web thickness is somewhat differed from the one given in the Alcoa Structural Handbook. (104)
The dimensions of the cross section were (Fig. 6.4)

\[ f_1 = 0.230 \text{ inch (0.250 in the handbook)} \]
\[ t = 0.279 \text{ inch} \]
\[ w = 0.218 \text{ inch (0.230 in the handbook)} \]
\[ b = 4. \text{ inch} \]
\[ d = 2.5 \text{ inch (first series)}; 1.5 \text{ inch (second series)} \]
\[ L = \text{length of the specimens, chosen such that the slenderness ratio would be multiples of 10} \]

Fig. 6.4. Cross Section Dimensions

Other section properties were calculated as follows:

For H section used as stress-strain test specimen:

Area = 3.464

For T section:

\[ d = 1.5 \text{ inch} \] \hspace{2cm} \[ d = 2.5 \text{ inch} \]

Area = 1.404 inch\(^2\) \hspace{2cm} Area = 1.622 square inch

\[ I_x = 1.489 \text{ inch}^4 \] \hspace{2cm} \[ I_x = 1.490 \text{ inch}^4 \]
\[ I_y = 0.161 \text{ inch}^4 \] \hspace{2cm} \[ I_y = 0.734 \text{ inch}^4 \]
\[ r_x = 1.038 \text{ inch} \] \hspace{2cm} \[ r_x = 0.965 \text{ inch} \]
\[ r_y = 0.342 \text{ inch} \] \hspace{2cm} \[ r_y = 0.677 \text{ inch} \]
\[ \overline{d} = 0.284 \text{ inch} \] \hspace{2cm} \[ \overline{d} = 0.519 \text{ inch} \]
6.5. END CONDITION

Although a pinned-end column does not exist in actual structures and a column with absolutely no restraints at the end is a highly idealized member, study of its behavior is essential to the column design. In practice the end condition of columns lies between the two limiting points of full restraint and no restraint at all. Providing full end restraint in testing is perhaps even more difficult than furnishing restraint-free end condition. In the case of a column with full end restraint, especially short columns, where ultimate load is achieved in the inelastic range of the stress-strain relation due to St. Venant's effect, in the distribution of stress at the sections close to the end, a non-uniform flow will exist and the effectiveness of the end restraint will gradually reduce. Moreover, the presence of mixed problems will prevent us from studying the sole problem of overall buckling of the column.

Establishment of a complete restraint-free end condition is also difficult. However, the end resistance to rotation can be minimized to a negligible extent by means of techniques such as were used in these experiments. In the case of a pinned-end column the critical and governing state of stress will exist in the mid-height section with no end effects. For these reasons and because the basic problem of overall buckling was under study, the pinned-end condition was selected for the experiments and the analytical study. Of course, consideration of effective length for different end conditions, the effect of residual stress, transverse
load, symmetrical and unsymmetrical eccentric loading, and planar or biplanar bending for varied slenderness ratios and cross sections with different ratios of moment of inertia in the x and y directions is also a feature of this study.

6.6. END FIXTURES

A few variations of end fixtures have been used in testing of the pinned-end columns. In 1887 Bauschinger used the conical end fixtures\(^{105}\) (Fig. 6.5a) which allowed free rotation of column ends and also ensured concentric loading. Later Von Kármán, in his column tests, used a knife-edged bar bearing on a plane surface as an end fixture\(^{27}\) (Fig. 6.5b).

![Fig. 6.5a. End Fixture used by Bauschinger](image)

![Fig. 6.5b. End Fixture used by Von Kármán](image)
In this experiment a segment of a sphere with attachments, shown in Fig. 6.6, was utilized to provide a hinge action for each end of the column. Each end fixture consisted of four parts:

1. A segment of sphere of radius $2 \frac{1}{4}$" and height 2" (part #1), made of hardened crucible steel. The maximum load carrying capacity of the sphere was calculated from the equation\(^{(106)}\)

$$P_{\text{max}} = C \ d^2 \quad (6.3)$$

in which \(d\) is in inches and \(P\) is in kips. For \(d = 4.5\"\) and \(C = 5 \text{ kips/inch}^2\)

$$P_{\text{max}} \approx 100 \text{ kips} \quad (6.4)$$

which provides a safety factor of almost 1.5 for this experiment in the highest loading case.

2. A circular plate of radius $5 \frac{1}{2}$" and thickness $\frac{5}{16}$" made of hardened steel of sufficient shear resistance, set between the flat surface of the sphere and the end of the column to give smooth distribution of load and prevent local yielding (part #2).

3. A circular ring, to attach parts (#1) and (#2), and together with part (#4), prevent the column from slipping and help set the column in the desired position, as the three small screws could move the segment of the sphere with respect to the rest of the set-up some fraction of an inch (part #3). This part is also made of hardened steel.
4. A circular lead plate (part #4) of radius $5\frac{1}{2}$" poured with the cross section of the column cut on it such that the centroid of the cut segment is at the desired position with respect to the center of the plate which coincides with both centers of the parts (#1) and (#2).

5. A circular hardened steel plate (part #5) of radius 7" and 1" thickness set between the part (#1) and the testing machine to distribute the load at the top and bottom of the machine on a larger area than the surface of contact of the sphere and thus prevent the local yielding of the headings of the machine.

Details of the end fixtures are shown in Fig. 6.6.

![Fig. 6.6. Different Parts of the End Fixture](image)

Part (#6) in Fig. 6.6 is a device which was used to determine the centroid of the cross section.
6.7. ATTACHMENT OF STRAIN GAGES

In the tests for obtaining the material properties, M-M Precision Strain Gages were used in order to have rather accurate strain readings in the inelastic range. For the column tests SR-4, A-8, strain gages were used. Strain gages were carefully attached. The surface of the contact was cleaned with a range of medium to soft emery paper. Then the surface was cleaned with carbon tetra-chloride and finally with acetone. Duco cement, specially made for strain gage attachment, was used as adhesive and moderate pressure applied until the cement dried enough so that the edges of the gage would not draw away from the specimen. Protective measures were taken to secure the gages. Enough time was given for the cement to harden before testing the specimen.

The purpose of attaching the strain gages to the column was mainly to check the concentricity, or in some cases the correctness of intended eccentricity, in the initial steps of loading.

6.8. MEASUREMENT OF THE LATERAL DISPLACEMENTS AND THE TWIST

As the spherical end fixtures are free to rotate and displace their contact points with respect to fixed surfaces of the headings of the testing machine (Fig. 6.7), measurement of the displacements and the twist of a point along the column relative to the end points requires special attention and technique.
The method used in this experiment to measure the mid-height displacement and the twist, relative to the end section, is as follows:

Near each end of the column four pulleys were attached, two on two ends of the flange, one at the end of the web and the last one on the face of the flange. The centers of these pulleys had a certain measured distance \( \frac{3''}{4} \) from the end of the column (Fig. 6.8). The pulleys were adjusted such that the piano strings passing over them were exactly at the center of the cross section.

Fig. 6.7. End Fixture Movement

Fig. 6.8. Displacement and Twist Measurement Attachments
At the mid-height section 4 ferro type chrome plates (blades), with a scale divided to (1/100") and each with their axis perpendicular to the respective piano string, were attached to the column with epoxy glue (Fig. 6.9). The grooves of the pulleys were very narrow in order to prevent the string from sliding from its original position. The wires were fastened at the lower end and by means of some weight attached to the other end of the wire, after passing over the upper pulley, the wires were kept taut. The pulleys were lubricated to eliminate the friction between the wire and the surface of the groove. At the first step (400 lbs. of load to keep the column in position) and at every step of reading the position of wire on the chrome plate scale was recorded (such that the wire and its image on the chrome plate and the reading edge were on a straight line). A magnifying glass was used to read with a better accuracy. The mid-height deflection and twist can be obtained from the above readings by the aid of the following explanation:
Let us assume that the centroid of the flange is the center of the twist. xyz coordinate is fixed to the end of the column with its origin (o) at the center of twist, and x- and y-axes respectively, the axes of symmetry for the web and the flange (Fig. 6.10). $\xi\eta\zeta$ coordinate is fixed to the mid section of the column with its origin (o') at the center of twist and, $\xi$ and $\eta$ axes respectively, the axes of symmetry for the web and the flange at the mid-height section. When there are no displacements or twist, and no initial imperfection, xyz will coincide with $\xi\eta\zeta$. At some deformed stage, the mid-height section will have, in general, lateral deflection (U, V) and the twist ($\beta$).

Fig. 6.10. Translated and Twisted Mid-length Section
Readings on chrome plate scales (minus the original readings) will give the values of $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$, which is the distance between the reflection of the initial position of the string on the plate and the present one. (The distance $\Delta$ is smaller than the distance between the wire and the chrome plate and the chrome plate will not interfere with the reading for small twist.) The following gives the relation between the readings and the displacements and the twist

$$\Delta_1 \cos \beta = V + b_1 \sin \beta$$

$$-\Delta_2 \cos \beta = U - b_2 \sin \beta$$

$$-\Delta_3 \cos \beta = V - b_3 \sin \beta$$

$$\Delta_4 \cos \beta = U + b_4 \sin \beta$$

(6.5)

Solving (6.5) for the twist

$$\tan \beta = \frac{\Delta_1 + \Delta_3}{b_1 + b_3} = \frac{\Delta_2 + \Delta_4}{b_2 + b_4}$$

(6.6)

and the lateral displacements

$$U = \frac{1}{2} \left[ (\Delta_4 - \Delta_2) \cos \beta + (b_4 - b_2) \sin \beta \right]$$

$$V = \frac{1}{2} \left[ (\Delta_1 - \Delta_3) \cos \beta + (b_1 - b_3) \sin \beta \right]$$

(6.7)
Assuming $\beta$ to be very small, then

$$\tan \frac{\beta}{\beta} = \sin \frac{\beta}{\beta} = \beta$$ and $\cos \frac{\beta}{\beta} = 1$

$$\beta = \frac{\Delta_1 + \Delta_3}{b_1 + b_3} = \frac{\Delta_2 + \Delta_4}{b_2 + b_4} \quad \text{(radian)}$$

$$U = \frac{1}{2} \left[ (\Delta_4 - \Delta_2) + \frac{b_4 - b_2}{b_4 + b_2} (\Delta_4 + \Delta_2) \right] \quad (6.8)$$

$$V = \frac{1}{2} \left[ (\Delta_1 - \Delta_3) + \frac{b_1 - b_3}{b_1 + b_3} (\Delta_1 + \Delta_3) \right]$$

In this experiment

$$b_2 = b_4 = \frac{b}{2}$$

$$b_1 = (d - \frac{t}{2})$$

$$b_3 = \frac{t}{2}$$

and

$$\beta = \frac{\Delta_2 + \Delta_4}{b} = \frac{\Delta_1 + \Delta_3}{d}$$

$$U = \frac{1}{2} \left[ (\Delta_4 - \Delta_2) \right] \quad (6.9)$$

$$V = \frac{1}{2} \left[ (\Delta_1 - \Delta_3) + \left( \frac{d - t}{d} \right) (\Delta_1 + \Delta_3) \right]$$

$\Delta$'s are positive if the wire moves to the right (increasing the figures on the chrome plate scale) and negative otherwise. Accuracy of the data can be checked by the first equation of (6.9).

In reading the $\Delta$'s a slight error is encountered which can be corrected. Instead of $\Delta_1 \ldots$ in the above equations the value of
\[ \Delta' = \Delta_1 - \delta_1 \sin \beta \]

where \( \delta_1 \) is the distance between the wire and the plate. As both \( \beta \) and \( \delta_1 \) are small, and as \( \delta_1 \) decreases with increasing \( \beta \), the above error may be neglected, as it is done in this experiment. Thus, the lateral deflections and the twist relative to the location of the center of the pulleys were determined. By a simple parabolic-interpolation, the deflections and the twist with respect to the end section were calculated. Mechanical deflection gages were also used in the first few steps to verify the other readings. They were also used in the cases not involving the twist, i.e., \( d = 1.5'' \).

6.9. LOADING AND ADJUSTMENT

Loading the specimen concentrically, or with a certain eccentricity, was the objective. To achieve this goal, the centroid of the cross section used in the experiment cut to \( \frac{1}{4}'' \) length was determined by means of the device shown on Fig. 6.4 (#6). Part #4 was poured of lead such that its center would coincide or would have the desired eccentricity with respect to the centroid of the cross section and to
hold the column in the end fixture. By loading in the elastic range and checking the strains and deflections, by means of three small screws of part #3, the necessary adjustments were made to set the column in the desired position. This was a very long and delicate job and required some amount of patience. The rate of head movement of the testing machine was chosen to be 0.025 inch per minute. Except for some scattered reading in two or three experiments, the author was unable to take the desired data for the unloading part of the experiment, as load dropped suddenly in all cases - in some cases because of local buckling, in others because the unloading time was extremely small.

Paris\(^{(107)}\) in his study of plastic dynamic buckling of columns took also the behavior of the testing machine into consideration. He noted that the machine was set to a slow rate of head movement and left on throughout the test. As the unloading time was extremely small, therefore, little additional axial deformation could be attributed to the turning of the screws of the testing machine. Paris stated that comparatively large additional deformation noted in the unloading stage was due to the release of strains within the testing machine itself.

6.10. PROTECTIVE MEASURES

Although part #3 (the ring) of the end fixture would somewhat prevent the column from sliding and skipping out, in the case of extensive end rotation, the danger of end fixture and the column
being thrown out exists. In this experiment protective measures were taken, and protective plates were set around the testing machine (Fig. 6.3) and the top end fixtures were loosely attached to the machine by means of wire. Wire was also wrapped around the poles of the testing machine to prevent the parts from flying out (Fig. 6.12).

6.11. SUMMARY AND RESULTS

The main objective of conducting the experiments was to verify the analytical solution to the problem of asymmetrical behavior of the "T" shaped column of aluminum alloy. Also, the material properties were to be determined in order to be used in analytical solution. The material properties (the stress-strain relation, the Poisson's ratio) obtained from the experiment and its mathematical fit, the elastic (tangent) and shear (secant shear) moduli evaluated from the mathematical fit and according to discussion on torsional stiffness (equation (4.68)) are shown on Fig. 2.6. Fourteen columns
were tested, eight with \( d = 2.5'' \) and the rest with \( d = 1.5'' \). The ultimate load carrying capacity of the tested columns and the corresponding values obtained by incremental procedure (computer results) are tabulated below:

### Table 6.1
Table of Comparison of Experimental Results with Analytical Results

<table>
<thead>
<tr>
<th>( d )</th>
<th>K/L/r</th>
<th>Eccentricity</th>
<th>Experimental</th>
<th>Anal - Exp.</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5''</td>
<td>30</td>
<td>+.002</td>
<td>33.944</td>
<td>34.585</td>
<td>-1.88%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+0</td>
<td>42.511</td>
<td>42.870</td>
<td>-0.84%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-.002</td>
<td>43.542</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>+.002</td>
<td>28.660</td>
<td>29.362</td>
<td>+2.01%</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>+.0</td>
<td>40.290</td>
<td>40.512</td>
<td>-0.55%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-.002</td>
<td>42.047</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>2.5''</td>
<td>30</td>
<td>+.002</td>
<td>35.006</td>
<td>36.120</td>
<td>-3.18%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+0</td>
<td>42.521</td>
<td>41.800</td>
<td>+1.70%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-.002</td>
<td>43.128</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>+.002</td>
<td>40.140</td>
<td>38.950</td>
<td>+2.96%</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td>+.002</td>
<td>23.851</td>
<td>24.520</td>
<td>-2.80%</td>
</tr>
<tr>
<td>60</td>
<td></td>
<td>-.002</td>
<td>23.338</td>
<td>24.084</td>
<td>+3.20%</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>+.00</td>
<td>43.664</td>
<td>44.800</td>
<td>-2.60%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+0.002</td>
<td>43.664</td>
<td>42.600</td>
<td>+2.44%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.002</td>
<td>43.883</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>+0.002</td>
<td>40.310</td>
<td>43.150</td>
<td>-7.05%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.002</td>
<td>41.682</td>
<td>*</td>
<td></td>
</tr>
</tbody>
</table>

*Buckling did not follow this direction

*No torsion or local buckling

*Web buckled locally after the ultimate load

*Failure due to local buckling of web

*Web buckled locally prior to the ultimate load

*Web buckled locally after the ultimate load

*No torsion was induced

*Failure due to local buckling in web

*Failure due to local buckling in web

*Considerable difference between the experiment and the analytical result due to the lateral support. Failure due to local buckling in web
Table 6.1 also gives the percentage difference between the computed and the tested values with respect to the computed values.

It can be noticed that test results are lower than the computed values in the case of negative eccentricity (buckling towards the web) and higher than the computed values otherwise. Also, it is noticeable that in the cases of concentric loading the columns deflected to the flange side (negative x direction) and as it is seen from Figs. 6.13 and 6.14 deflection started below their respective tangent modulus load and test ultimate loads were higher than the computed values. Although some discrepancies between the test and analytical results were observed (Figs. 6.13, 6.14, 6.15), the tests as well as the incremental procedure show that in buckling behavior of a tee column a considerable difference exists in the ultimate load carrying capacity of the T shaped aluminum alloy column when it buckles in the plane of symmetry towards or away from the flange.
Fig. 6.13. Average Stress vs. Mid-length Deflection Curves: Test and Analytical Results (d = 1.5", L/r = 30)
Fig. 6.14. Average Stress vs. Mid-length Deflection Curves Test and Analytical Results (d = 1.5"; L/r = 40)
MID-LENGTH DEFLECTION AND TWIST PER UNIT LENGTH; $\delta_x$, $\delta_y$ INCH/INCH, $\beta/L$ RAD/INCH

Fig. 6.15. Average Stress vs. Mid-length Deflection Curves
Test and Analytical Results ($d = 2.5''$; $L/r = 30$)
CHAPTER VII
SUMMARY AND CONCLUSIONS

The post buckling behavior of a tee shaped aluminum column has been studied. Attention has been focused on the cases where the buckling commences in the plane of symmetry. The effects of initial curvature and eccentricity of loading in the web or the flange directions have been considered. The problems discussed and the conclusions derived can be summarized as follows:

1. Bounds of behavior of a column, specifically a T column, have been discussed in Chapter II. The differing inelastic buckling gradients and the reduced moduli for buckling towards or away from the flange have been explained and illustrated graphically.

2. Equations of torsional flexural buckling and the choice of the coordinates have been discussed in Chapter III. The initial crookedness and the eccentricity of loading, in a general form, have been introduced in these equations. The limiting point up to which the column will not buckle out of its plane of symmetry has been determined and simplifications of the equilibrium equations for the state of planar bending have been made.
3. In Chapter IV the problem of torsion of a cross section subjected to a varying strain in the inelastic range with possible presence of a strain regression region has been treated. This is the problem encountered in the evaluation of the torsional stiffness of the column cross section in the inelastic range of biplanar bending. At the conclusion of this chapter the relation (4.83) (which has been used in the column analysis) has been suggested to approximate the torsional stiffness of the column cross section under the above conditions.

4. An incremental procedure has been presented in Chapter V for the purpose of obtaining a quantitative evaluation of the ultimate load carrying capacity of the tee shaped columns and a comparison is made with the qualitative predictions that bound the behavior of the column. The 7090 digital computer was utilized to carry out the manipulation. Combinations studied were summarized in Table 5.1. A selected number of load-deflection history and the tables of the results were presented in Chapter V.

The following conclusions may be drawn from the incremental analysis:

i. For \( d = 1.5'' \ (\alpha = 9.248) \), except for \( L/r = 10 \), the column did not buckle out of its plane of symmetry in the loading stage. For \( L/r = 10 \), biplanar bending accompanied with torsion
started almost at the ultimate load. In Fig. 5.11, dashed lines show the case where the column was prevented from deflection in the y direction.

ii. For $d = 2.5''$ ($\alpha = 2.030$) and $d = 3.0''$ ($\alpha = 1.197$) the dimensionless graphs of the load vs. the components of the deflection in the x and the y directions as well as the twist are shown on Figs. 5.14 through 5.17. In these figures the behavior of the column is indicated where prevented from buckling out of the plane of symmetry. From these figures it can be seen that as $\alpha = \frac{I_x}{I_y}$ becomes smaller (d becomes larger) the difference between the ultimate average stress when buckling accompanies torsion and the ultimate average stress, in buckling away from the flange (to the right), when the column is prevented from buckling in the y direction becomes larger.

iii. A noticeable difference exists between the ultimate load carrying capacity of the T column in buckling towards or away from the flange. As $\alpha$ tends towards unity this difference becomes smaller. The ultimate load for
different values of slenderness ratio and initial crookedness in buckling towards or away from the flange is given on Figs. 5. 20 and 5. 21.

iv. Figures 5. 12, 5. 18 and 5. 19 and also Tables 5. 6 and 5. 7 show the difference between the ultimate load evaluated on the basis of a half-sine curve deflection shape assumption and a more exact deflection shape. It can be noticed that as initial crookedness becomes larger the difference between the ultimate load evaluated on the above two bases becomes larger.

v. Figures 5. 22 through 5. 26 and also Tables 5. 8 and 5. 9 present a comparison between the initial crookedness and the eccentric loading.

vi. Three different stress-strain relations have been used in the incremental analysis (Figs. 5. 18 and 5. 19).

5. A series of tests was conducted to confirm the analytical results. A procedure of deflection and twist measurement for the pinned-end columns has been developed and presented in Chapter III. Table 6. 1 compares the analytical and the experimental results.
RELATED PROGRAMS

CALALF. Calculates the vector \( \{a\} \) according to relation 5.45 and determines if biplanar bending has commenced.

CALBET. Calculates \( \{\beta'\} \), then \( \{\beta\} \) using relation 5.32d, when end section is deleted \( (\phi = 0) \), uses FINEPS. to find the strain at that section.

CALMAP. When planar bending:
- Calculates axial load (relation 5.42a).
- Also evaluates the moment equilibrium equation (5.42b).
When biplanar bending:
- Calculates axial load (relation 5.32a).
- Also evaluates the moment equilibrium equation (relation 5.32b when \( q = 2 \) and relation 5.32c when \( q = 1 \)).

ET. Calculates tangent modulus for a given strain (relation 5.1, Fig. 2.6).

FINEPS. Finds strain for a given stress (from relation 5.1).

FINTAN. Finds the stress and the strain at the tangent modulus load for a given slenderness ratio.

GS. Finds the secant shear modulus for a given strain (relation 4.68).

RMATIX. Generates the matrix \([R]\).

SIGMA. Finds the stress for a given strain (relation 5.1).
REFERENCES


31. Thurlimann, B., "Deformations of, and Stresses in, Initially Twisted and Eccentrically Loaded Columns of Thin-Walled Open Cross Section," Report No. 3 to the Column Research Council and the Rhode Island Department of Public Works, Graduate Division of Applied Mathematics, Brown University, June 1953.


