T H E   U N I V E R S I T Y   O F   M I C H I G A N

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Technical Note

THE NUMBER OF TRANSITIVITY SETS OF BOOLEAN FUNCTIONS

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SUMMARY

Three groups are defined as transformation groups on the class of Boolean functions. The transitivity classes are counted using the famous combinatorial theorem of Pólya.[4] In particular, a concise algorithm is found for counting the classes under the group of complementations and permutations thus simplifying a result of Slepian.[6]

I. INTRODUCTION

There is a well known connection between switching theory and Boolean functions.[6] Since the Boolean algebra of functions is a free algebra on n generators, it has 2^n atoms and contains 2^{2^n} functions. Many of the techniques of switching theory require enumeration of functions. To a mathematician, these problems of switching theory are trivial, since if n is finite, 2^{2^n} is finite and all of the functions may be enumerated. On the other hand, for $n \geqq 9$, 2^{2^n} is larger than the number of electrons and protons in the universe and enumeration is impractical. A technique developed to help cut down the number of functions to be enumerated is to define a group as operating on the class of Boolean functions. If the group is intransitive on Boolean functions, one has only to enumerate equivalence classes (also called orbits or transitivity classes) rather than functions.

We will consider three groups operating on Boolean functions and in each case we shall count the number of equivalence classes of functions with k atoms in their (unique) normal form expansion by the use of the Hauptsatz of Pólya.[4]

II. PÓLYA'S THEOREM

In his classic study of trees, chemical isomers, and their relatives, Pólya[4] proved a theorem which has subsequently become the most famous and most important single result of combinatorial analysis. Pólya's theorem has achieved this distinction because of its generality, and simplicity. We shall briefly review this result; using the general formulation of DeBruijn.[2] Our discussion involves more generality than is needed for our results; the importance of the theorem justifies the additional generality.

Let F be the class of all functions from a finite set D to a finite set R. Suppose D has s elements and that $\mathfrak{G}$ is a permutation group of degree s and order g acting on D. Two functions f_1, $f_2 \in F$ are called equivalent if there exists a permutation $\alpha \in \mathfrak{G}$ such that $f_1(d) = f_2(\alpha(d))$ for all $d \in D$ ($\alpha(d)$ denotes the image of d under the permutation α). We represent R as the union of r disjoint subsets, i.e., $R = \bigcup_{i=1}^{r} R_i$ and $R_i \cap R_j = \emptyset$ if $i \neq j$.

Let $k_1,\ldots,k_s$ be any partition of s. Pólya's theorem tells us the number of equivalence classes of functions from D to R such that for k_i values of $d \in D$, the image $f(d) \in R_i$ for $i = 1,\ldots,r$.

Let ψ_i be the number of elements in R_i and define the figure counting series as

$$\psi(x) = \sum_{i=0}^{\infty} \psi_i x_i$$

Let $P(x_1,\ldots,x_s)$ be the generating function for the number of classes of functions with the property that for k_i values of $d \in D$, the image $f(d)$ is in R_i e.g., the desired number is the coefficient of ${x_1}^{k_1}\ldots{x_s}^{k_s}$ in $P(x_1,\ldots,x_s)$. The polynomial $P(x_1,\ldots,x_s)$ is sometimes called the configuration counting series.

Before stating Pólya's theorem, we must develop the concept of the cycle index polynomial of $\mathfrak{G}$ (Zyklenzeiger), denoted by $Z_{\mathfrak{G}}$. Let $f_1,\ldots,f_s$ be s indeterminates, and let $g_{j_1,j_2,\ldots,j_s}$ be the number of permutations of $\mathfrak{G}$

having j_k cycles of length k for k = 1, 2,...,s. Naturally

$$\sum_{i=1}^{s} i j_i = s \qquad (1)$$

Then we define

$$Z_{\mathfrak{G}} = \frac{1}{g} \sum_{(j)} g_{j_1, j_2, \ldots, j_s} f_1^{j_1} f_2^{j_2} \ldots f_s^{j_s}$$

where the sum is taken over all partitions of s which satisfy (1). Let $Z_{\mathfrak{G}}(g(x), g(x^2), \ldots, g(x^s))$ denote the cycle index polynomial with f_i replaced by $g(x^i)$ (i = 1,...,s) for any arbitrary function g(x).

Now we can finally state Pólya's theorem which reduces the problem of determining the number of equivalence classes to the determination of the figure counting series and the cycle index polynomial.

Theorem 2.1 (Pólya). The configuration counting series is obtained by substituting the figure counting series into the cycle index polynomial of $\mathfrak{G}$. Symbolically

$$P(x_1, \ldots, x_s) = Z_{\mathfrak{G}}(\psi(x^1), \psi(x^2), \ldots, \psi(x^s))$$

For the problems to be considered here, Pólya's theorem takes an even simpler form. First of all, we deal only with generating functions of one variable. When one considers equivalence of Boolean functions, we have $D = \{0, 1\}^n$, $s = 2^n$, and $R = \{0, 1\}$. Thus

$$\psi(x) = 1 + x$$

The only non-trivial calculations that need to be made will be the construction of the cycle index for each group. We can then compute P(x); the coefficient of x^k in $P(x)$ will be the number of classes of functions having k atoms in their normal form expansions.

III. THE GROUP Σ_2^n

The first group to be considered is the direct sum of n copies of the cyclic group of order 2, denoted by Σ_2^n. The order of this group is 2^n and the elements are n-tuples of zeroes and ones. The group operation is written $\oplus$. Now we define this group as operating on the class of Boolean functions following Ashenhurst[1] who first studied this group.

Definition 3.1. *Let* $i \in \Sigma_2^n$ *and let* $f(x_1,\dots,x_n)$ *be a Boolean function of n variables. We define*

$$if \;=\; (i_1,\dots,i_n)\, f(x_1,\dots,x_n) \;=\; f(x_1^{i_1},\dots,x_n^{i_n})$$

where

$$x_j^{i_j} = \begin{cases} x_j & \text{if } i_j = 0 \\ \overline{x}_j & \text{if } i_j = 1 \end{cases} \qquad \textit{for } j = 1,\dots,n$$

One defines two functions as equivalent under Σ_2^n if there is an operation of the group which maps one function into the other. Intuitively the functions are equivalent if one can be obtained from the other by complementing some of the variables.

Example: The two Boolean functions $x_1\overline{x}_2+\overline{x}_1x_2$ and $\overline{x}_1\overline{x}_2+x_1x_2$ are equivalent under Σ_2^2.

The effect of a permutation $i \in \Sigma_2^n$ is to permute the n-tuples of zeroes and ones for which $f(x_1,\dots,x_n)$ is one i.e., the atoms of the normal form expansion of $f(x_1,\dots,x_n)$. It will be convenient to have a notation for the atoms and for the permutations of atoms. n-tuples of zeroes and ones are associated both with the atoms and with the decimal equivalent of the binary number. The correspondence is made clearer in the following table for three variables.

Atom	Binary Representation	Decimal Symbol
$\bar{x}_1\bar{x}_2\bar{x}_3$	000	0
$\bar{x}_1\bar{x}_2 x_3$	001	1
$\bar{x}_1 x_2\bar{x}_3$	010	2
$\bar{x}_1 x_2 x_3$	011	3
$x_1\bar{x}_2\bar{x}_3$	100	4
$x_1\bar{x}_2 x_3$	101	5
$x_1 x_2\bar{x}_3$	110	6
$x_1 x_2 x_3$	111	7

If $i, j \in \sum_2^n$, then let ρ_i, ρ_j denote the corresponding permutations of the atoms, and let ρ_{ij} denote the permutation corresponding to $i \oplus j$. It is easy to prove that $\rho_{ij} = \rho_i\rho_j$.* Thus the mapping from $i \in \sum_2^n$ into the permutation group on the atoms denoted by Γ_{2^n} is a homomorphism. The mapping is easily seen to be one-to-one but properly into. Γ_{2^n} is easily seen to be the automorphism group of the Boolean algebra of functions.

*If $f = \sum_k A_k$ is the normal form expansion of f, then $\rho_i\rho_j f = \sum_k A_{i \oplus j \oplus k}$. Since $\rho_{ij} f = \sum_k A_{i \oplus j \oplus k}$, we have $\rho_{ij} = \rho_i\rho_j$. $i \oplus j$ denotes the digit-wise modulo two sum of i and j.

Example: Let us apply the permutation i = (0, 1, 1) to all the functions of three variables. We apply the permutation to the atoms and write the result using the conventional cyclic notation for permutations.

$$(03)(12)(47)(56)$$

To count the equivalence classes we must determine the cycle structure of the permutations in our representation of $\sum_2^n$.

Theorem 3.2. <u>Every</u> $i \in \sum_2^n$ <u>different from the identity induces a permutation of the atoms which has</u> 2^{n-1} <u>transpositions</u>.

Proof. Let A_j and A_k be two atoms and let $i \in \sum_2^n$ be different from the identity. Suppose $i(A_j) = A_k$, then $A_j = i(A_k)$ because i is of order 2. Thus the permutation of atoms associated with i, has disjoint transpositions for its cycle structure. It cannot have less than 2^{n-1} transpositions because no atom is invariant under any i.

Theorem 3.3. (Ashenhurst) $Z_{\sum_2^n} = \frac{1}{2^n}(f_1^{2^n} + (2^n - 1)f_2^{2^{n-1}})$.*

Proof. The first term corresponds to the identity element while the second term corresponds to all other elements of the group.

*This is a special case of a more general result. If one takes $\mathfrak{h}$ to be the regular representation of the group of order p^k and type (p,...,p), then it can be shown that $Z_{\mathfrak{h}} = \frac{1}{p^k}(f_1^{p^k} + (p^k-1)f_p^{p^{k-1}})$ for primes $p \geqslant 2$ and $k > 1$.

Corollary 3.4. <u>The number of equivalence classes of functions having s atoms in their (unique) normal form expansion under $\sum_2^n$ is</u>

$$\frac{1}{2^n}\binom{2^n}{s} \qquad \underline{\text{if } s \equiv 1 \bmod 2}$$

and

$$\frac{1}{2^n}\left(\binom{2^n}{s} + (2^n - 1)\binom{2^{n-1}}{s/2}\right) \qquad \underline{\text{if } s \equiv 0 \bmod 2}$$

Proof. By Pólya's theorem, we get

$$Z_{\sum_2^n}(1 + x) = \frac{1}{2^n}\left((1 + x)^{2^n} + (2^n - 1)(1 + x^2)^{2^{n-1}}\right)$$

$$= \frac{1}{2^n}\left(\sum_{k=0}^{2^n}\binom{2^n}{k} x^k + (2^n - 1)\sum_{j=0}^{2^{n-1}}\binom{2^{n-1}}{j} x^{2j}\right)$$

If $s \equiv 1 \bmod 2$, the coefficient of x^s is $\frac{1}{2^n}\binom{2^n}{s}$.

If $s \equiv 0 \bmod 2$, the coefficient of x^s is

$$\frac{1}{2^n}\left(\binom{2^n}{s} + (2^n - 1)\binom{2^{n-1}}{s/2}\right)$$

Corollary 3.5. <u>The total number of equivalence classes of Boolean functions under $\sum_2^n$ is</u>

$$\frac{1}{2^n}\left(2^{2^n} + (2^n - 1)2^{2^{n-1}}\right)$$

Proof. By Pólya's theorem, take $f_i = 1+1^i = 2$ in $Z_{\sum_2^n}$.

Some typical calculations for this group have been made and the results are tabulated below. We use T_n for the total number of equivalence classes and N_n^k for the number of classes of functions having k atoms in their normal form expansions. Note that $N_n^k = N_n^{2^n-k}$ which is a consequence of the law of duality for Boolean functions.

k \ n	N_n^k				
	1	2	3	4	5
0	1	1	1	1	1
1	1	1	1	1	1
2	1	3	7	15	31
3		1	7	35	155
4		1	14	140	1,240
5			7	273	6,293
6			7	553	28,861
7			1	715	105,183
8			1	870	330,460
9				715	876,525
10				553	2,020,239
11				273	4,032,015
12				140	7,063,784
13				35	10,855,425
14				15	14,743,445
15				1	17,678,835
16				1	18,796,230
T_n	3	7	46	4,336	134,281,216

IV. THE GROUP $\mathcal{S}_n$

Now we define the symmetric group on n letters as a group of operators on the Boolean functions.

Definition 4.1. For any $\sigma \in \mathcal{S}_n$ and any $f(x_1,\ldots,x_n) \in F$, we define

$$\sigma f(x_1,\ldots,x_n) = f(x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$

Again we induce a permutation on the atoms exactly as before. This mapping is again a one-to-one homomorphism from $\mathcal{S}_n$ into $\mathcal{S}_{2^n}$.

The cycle index for $\mathcal{S}_n$ as a group on n letters is well known, but we need a representation of $\mathcal{S}_n$, as a permutation group on 2^n objects.* The following theorem allows us to use the cycle index of $\mathcal{S}_n$ as a group on n letters to get the cycle index of the group on the atoms.

Theorem 4.2. Permutations of the same cycle structure in $\mathcal{S}_n$ induce permutations of the same cycle structure in $\mathcal{S}_{2^n}$.

Proof. Let h be the homomorphism from $\mathcal{S}_n$ into $\mathcal{S}_{2^n}$. Since all permutations of the same cycle structure are conjugates, it is sufficient to show that conjugates of σ in $\mathcal{S}_n$ correspond to conjugates of $h(\sigma)$ in $\mathcal{S}_{2^n}$. This is trivial since

$$h(\rho^{-1}\sigma\rho) = h(\rho^{-1})h(\sigma)h(\rho) = (h(\rho))^{-1}h(\sigma)h(\rho)$$

because h is a homomorphism.

We must devise an algorithm to pass from the symmetric group on n letters to a representation of the group on 2^n letters. To accomplish this, we calculate the effect of a cycle of length k on the atoms of the Boolean algebra.

Definition 4.3.

$$e(1) = 2$$

$$e(k) = \frac{2^k - \sum_{\substack{d|k \\ d<k}} d\, e(d)}{k}$$

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*Two groups may be isomorphic but not permutationally equivalent. Permutation groups $\mathfrak{A}$ and $\mathfrak{B}$ on object sets X and Y are called permutationally equivalent if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic as abstract groups and there is a one-to-one correspondence h: $X \longleftrightarrow Y$ such that if γ is the abstract isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, then for every $x \in X$, $\alpha \in \mathfrak{A}$, we have $h(\alpha x) = (\gamma \alpha)h(x)$.

Theorem 4.4. *A cycle of length k in* $\mathfrak{S}_n$ *as a group on n letters induces a permutation of the atoms whose cycle structure is given by*

$$\prod_{d|k} f_d^{\,e(d)}$$

where the f_d *are the indeterminates of the cycle index of* $\mathfrak{S}_n$ *on* 2^n *letters.*

Proof. Write the numbers from 0 to 2^k-1 in binary notation and in natural order. The effect of a permutation of length k can be obtained by removing the left-hand column and writing it as the right-hand column. This has the effect of doubling each number modulo 2^k-1. The exponent of f_d is independent of n and has the same value every time it occurs. Note that if k is prime, then we get $f_1^2 f_k^{\frac{2^k-2}{k}}$. The exponent of f_k is an integer; this fact is a consequence of Fermat's theorem.

Before writing down the explicit formula for the cycle index of $\mathfrak{S}_n$ as a permutation group on the 2^n atoms, we construct a multiplication of indeterminates which will facilitate computation.

Definition 4.5. *Let* $a_1^{i_1}\ldots a_r^{i_r}$ *and* $b_1^{j_1}\ldots b_s^{j_s}$ *be two products of indeterminates; the letters* a_k *and* b_k *are not necessarily distinct. The product of these terms (written x) is given by*

$$\prod_{p,q}(a_p^{i_p} \times b_q^{j_q})$$

where

$$a_p^{i_p} \times b_q^{j_q} = f_{\langle p,q\rangle}^{\,i_p j_q (p,q)}$$

where $\langle p,q \rangle$ denotes the least common multiple of p and q and (p,q) is the greatest common divisor of p and q. Of course, x is an associative and commutative operation.

Example: We compute the cycle structure of the permutation which corresponds to the product of a cycle of length 2 and a cycle of length 3. This is denoted by t_2t_3.

$$t_2t_3 \longrightarrow (f_1^2f_2) \times (f_1^2f_3^2) = (f_1^2f_1^2)(f_1^2f_3^2)(f_1^2f_2)(f_2f_3^2)$$

$$= f_1^4f_3^4f_2^2f_6^2 = f_1^4f_2^2f_3^4f_6^2$$

The symbol $\mathop{\times}\limits_{i=1}^{n} a_i$ means $a_1 \times \ldots \times a_n$. The reason that $f_p \times f_q = f_{<p,q>}^{(p,q)}$ is that we are changing the degree of our representation from p+q to pq. The subscript will be $<p,q>$ as Slepian[6] showed and the exponent occurs because $(p,q)\ <p,q> = pq$, the number of objects being permuted.

Theorem 4.6. <u>The cycle index for $\mathfrak{M}_n$ as a permutation group on the 2^n atoms of the Boolean algebra of functions is</u>

$$Z_{\mathfrak{M}_n} = \frac{1}{n!} \sum_{(j)} \frac{n!}{j_1!1^{j_1}j_2!2^{j_2}\ldots j_n!n^{j_n}} \mathop{\times}\limits_{i=1}^{n} \left(\prod_{d|i} f_d^{e(d)} \right)^{j_i}$$

<u>where the sum is over all partitions of n such that</u>

$$\sum_{i=1}^{n} ij_i = n$$

<u>and $\mathop{\times}\limits_{i=1}^{n} y^{j_i} = y^{j_1} \times y^{j_2} \times \ldots \times y^{j_n}$ where $y^{j_i} = \overbrace{y \times y \times \ldots \times y}^{j_i}$</u>

Proof. The coefficient in the sum is the number of permutations of the n letters with j_i cycles of length $i (i = 1,\ldots,n)$. To see this, choose any permutation with the required cycle structure. If all n! permutations are applied to the letters in the cycles, then the resulting permutations are not distinct for just two reasons, (1) the relative position of the cycles is irrelevant and (2) all cycles with the same elements in the same order are the same. The number of duplications for the first reason $j_1!\; j_2! \ldots j_n!$. The number of duplications for the second reason is $1^{j_1}\, 2^{j_2} \ldots n^{j_n}$ so that the desired number is

$$\frac{n!}{\prod_{i=1}^{n} j_i!\; i^{j_i}} .$$

Corollary 4.7. *The number of equivalence classes under* $\mathcal{J}^n_n$ *of functions with k atoms in their normal form expansion is the coefficient of* x^k *in*

$$P(x) = Z_{\mathcal{J}^n_n}(1 + x).$$

Corollary 4.8. *The total number of equivalence classes of functions under* $\mathcal{J}^n_n$ *is*

$$P(1) = Z_{\mathcal{J}^n_n}(2).$$

Example: We perform the calculations for n = 2.

$$Z_{\mathcal{J}^n_2} = \frac{1}{2}\,(f_1^4 + f_1^2 f_2)$$

$$P(x) = 1 + 3x + 4x^2 + 3x^3 + x^4 .$$

The twelve equivalence classes are listed below.

[0]	[1]	[xy]	$[\overline{xy}]$
[x+y]	$[\overline{x}+\overline{y}]$	$[x \oplus y]$	$[x \equiv y]$
[x,y]	$[\overline{x},\overline{y}]$	$[\overline{x}+y, x+\overline{y}]$	$[x\overline{y},\overline{x}y]$

The results of some computations for modest values of n are shown below. The number of variables is denoted by n and N_n^k is the number of classes of functions of n variables having k-atoms. Note that $N_n^k = N_n^{2^n-k}$; again T_n denotes the total number of classes.

k \ n	N_n^k 1	2	3	4	5
0	1	1	1	1	1
1	2	3	4	5	6
2	1	4	9	17	28
3		3	16	52	134
4		1	20	136	625
5			16	284	2,674
6			9	477	10,195
7			4	655	34,230
8			1	730	100,577
9				655	258,092
10				477	579,208
11				284	1,140,090
12				136	1,974,438
13				52	3,016,994
14				17	4,077,077
15				5	4,881,092
16				1	5,182,326
T_n	4	12	80	3,984	37,333,248

V. THE GROUP $\mathcal{G}_n$

The next group to be considered is the group which allows both complementation and permutations of the variables. This group, denoted by $\mathcal{G}_n$ has order $n!2^n$; a general element of the group is of the form $i\sigma$ where $i \in \Sigma_2^n$ and $\sigma \in \mathcal{S}_n$. $\mathcal{G}_n$ is defined as a transformation group on the Boolean functions as follows:

$$i\sigma f(x_1,\ldots,x_n) = f(x_{\sigma(1)}^{i_1},\ldots,x_{\sigma(n)}^{i_n})$$

$\mathcal{G}_n$ is also the symmetry group of the n-cube and the dual of the n-cube, the hyperoctahedron.* Pólya (5, footnote 7) mentions that $\mathcal{G}_n$ is the wreath product (Kranzgruppen) of $\mathcal{S}_n$ and Σ_2 denoted by $\mathcal{S}_n[\Sigma_2]$, however this gives $\mathcal{G}_n$ a representation of degree 2n. The 2n objects may be taken as the faces of the n-cube or the vertices of the hyperoctahedron inscribed in the hypercube.

Our problem requires that $\mathcal{G}_n$ be given as a permutation group of degree 2^n. Harary[3] has constructed an operation called exponentiation of groups for precisely this reason; we review Harary's defintion briefly.

Let $\mathcal{A}$ and $\mathcal{B}$ be permutation groups of degrees a and b operating on object sets X and Y; let the orders of $\mathcal{A}$ and $\mathcal{B}$ be m and n respectively. The exponentiation group $\mathcal{B}^{\mathcal{A}}$ has Y^X, the class of all functions from X to Y, as its object set. The elements of $\mathcal{B}^{\mathcal{A}}$ are constructed by permuting the domain X using a permutation in $\mathcal{A}$ and then permuting the image objects for every domain object by elements of $\mathcal{B}$. The properties of this group are given below along with the properties of $\mathcal{A}[\mathcal{B}]$. The explicit construction of $\mathcal{A}[\mathcal{B}]$ may be found in Pólya's paper.[4]

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*This problem may be interpreted as counting the number of distinct ways that the vertices of the n-cube may be painted with two colors. We mean that two paintings of the n-cube are distinct in case one cannot be transformed into the other by a rotation or reflection of the n-cube.

			Wreath Product	Exponentiation
Group	$\mathfrak{A}$	$\mathfrak{B}$		
Object Set	X	Y	XxY	Y^X
Degree	a	b	ab	b^a
Order	m	n	mn^a	mn^a

Thus $\mathfrak{G}_n = \Sigma_2^{\mathfrak{S}_n}$. Note that $\Sigma_2^{\mathfrak{S}_n}$ is isomorphic to $\mathfrak{S}_n[\Sigma_2]$, but the groups are not permutationally equivalent since they have different degrees.

The important problem of determining $Z_{\mathfrak{B}^{\mathfrak{A}}}$ in terms of $Z_{\mathfrak{A}}$ and $Z_{\mathfrak{B}}$ is still unsolved. We present a method of computing the cycle index of the exponentiation group in this special case. Unfortunately the technique cannot be generalized to arbitrary exponentiations. Slepian[6] has already counted the number of equivalence classes under $\mathfrak{G}_n$, but his argument was unnecessarily complicated. His method required counting the conjugate classes of the hyperoctahedral group; our result depends on the knowledge of the cycle index of $\mathfrak{S}_n$.

The derivation of the cycle index rests on the following argument. Since $\mathfrak{G}_n = \Sigma_2^n \mathfrak{S}_n$,* every element of $\mathfrak{G}_n$ is of the form $i\sigma$ where $i \in \Sigma_2^n$

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*By the product $\mathfrak{A}\mathfrak{B}$ of two groups $\mathfrak{G}$ and $\mathfrak{B}$ which are subgroups of a larger group $\mathfrak{G}$ we mean the group whose domain is $\{ab \mid a \in \mathfrak{A}, b \in \mathfrak{B}\}$. This group is defined when one of $\mathfrak{A}$ or $\mathfrak{B}$ is normal in $\mathfrak{G}$. In our case Σ_2^n is normal in $\mathfrak{G}_n$. Thus $\mathfrak{G}_n$ is the least group containing Σ_2^n and $\mathfrak{S}_n$.

and $\sigma \in \mathcal{J}_n$. Since every element of Σ_2^n except the identity has the same cycle structure, we need only compute the effect of the permutation of $\mathcal{G}_n$ pre-multiplied by any permutation consisting of 2^{n-1} transpositions because we already know the structure of $\mathcal{J}_n$ as a group on the 2^n atoms.

Definition 5.1. <u>The function g(d) is defined as follows</u>:

$$g(2) = 1$$

$$g(2k) = \frac{2^k - \sum\limits_{\substack{d \nmid k \\ d | 2k \\ d < 2k}} dg(d)}{2k}$$

Theorem 5.2. <u>If t_k is an indeterminate of the cycle index of $\mathcal{J}_n$, then the following correspondence indicates the cycle structure induced on the atoms in $\mathcal{G}_n$.</u>

$$t_k \rightarrow \frac{1}{2}\left(\prod_{d|k} f_d^{\,e(d)} + \prod_{\substack{d \nmid k \\ d | 2k}} f_d^{\,g(d)} \right)$$

Proof. The first term describes the elements of $\mathcal{G}_n$ when $i = (0,\ldots,0)$; this subgroup is just $\mathcal{J}_n$. The second term describes the cycle structure obtained by pre-multiplying the permutations of $\mathcal{J}_n$ by any element Σ_2^n, say $(0\ 1)(2\ 3)\ldots(2^n-2,\ 2^n-1)$.

The multiplication of indeterminates is exactly the same as for $\mathcal{J}_n$ and for exactly the same reasons.

Theorem 5.3. *The cycle index of* $\mathcal{G}_n = \Sigma_2^{\mathcal{S}_n}$ *is given by*

$$Z_{\mathcal{G}_n} = \frac{1}{n!2^n} \sum_{(j)} \frac{n!}{\prod_{i=1}^{n} j_i!\, i^{j_i}} \prod_{i=1}^{n} \left(\prod_{d|i} f_d^{e(d)} + \prod_{\substack{d \nmid i \\ d|2i}} f_d^{g(d)} \right)^{j_i}$$

where the sum is over all partitions of n such that

$$\sum_{i=1}^{n} i j_i = n.$$

The reason that our calculation of the cycle index of $\mathcal{G}_n$ is so simple is that the cycle index of Σ_2^n has only two terms.

Corollary 5.4. *The number of equivalence classes under* $\mathcal{G}_n$ *of functions with k atoms in their normal form expansions is the coefficient of* x^k *in*

$$P(x) = Z_{\mathcal{G}_n}(1 + x)$$

Corollary 5.5. *The total number of equivalence classes of functions under* $\mathcal{G}_n$ *is*

$$P(1) = Z_{\mathcal{G}_n}(2)$$

Example. The calculations for n = 2 yield

$$Z_{\mathcal{G}_2} = \frac{1}{8}\left((f_1^2 + f_2)^2 + f_1^2 f_2 + f_4\right)$$

$$= \frac{1}{8}\left(f_1^4 + 3f_1^2 f_2 + f_2^2 + f_4\right)$$

$$P(x) = 1 + x + 2x^2 + x^3 + x^4$$

The classes are [0], [1], $[\overline{x}\overline{y}, \overline{x}y, x\overline{y}, xy]$

$$[x,\overline{x},y,\overline{y}],[x \oplus y, x \equiv y], [\overline{x}+\overline{y},\overline{x}+y,x+\overline{y},x+y]$$

Some results are tabulated below.

k \ n	1	2	3	4	5
	N_n^k				
0	1	1	1	1	1
1	1	1	1	1	1
2	1	2	3	4	5
3		1	3	6	10
4		1	6	19	47
5			3	27	131
6			3	50	472
7			1	56	1,326
8			1	74	3,779
9				56	9,013
10				50	19,963
11				27	38,073
12				19	65,664
13				6	98,804
14				4	133,576
15				1	158,658
16				1	169,112
T_n	3	6	22	402	1,228,158

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