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Technical Report

ANALYTIC SOLUTION FOR ATMOSPHERIC DENSITY FROM SATELLITE MEASUREMENTS OF STELLAR REFRACTION

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I. BACKGROUND

The deduction of atmospheric density from satellite measurements of stellar refraction has been described in two previous University of Michigan Technical Reports.[1,2] One gap in the theory existed at the time of printing, namely, an exact expression for retrieval of the density function when the refraction is known. In both reports this inversion was approximated by an integration along a straight, rather than refracted, ray path. For this approximate case, an analytic solution could be obtained. In Reference 2, doubt was expressed that an analytic solution could be obtained in the exact case, and numerical methods were begun.

However, an exact analytic solution has now been demonstrated. The appropriate equation for this type of solution was used by Bateman[3] in 1909 while investigating the propagation of seismic waves. That the seismic and refraction cases are quite analogous is evident since both types of rays are brachistochrones and traverse spherically-stratified media. Spherical stratification of the atmosphere is assumed throughout; and that this assumption leads to negligible error was proved in Reference 2.

II. DEMONSTRATION OF SOLUTION

If μ is the index of refraction at radius r from the earth's center and z is the obtuse angle formed by the ray and the earth's radius at any point, the refraction R measured at the satellite is given by

$$R = 2\mu_o \, r_o \sin z_o \int_1^{\mu_o} \frac{d\mu}{\mu[\mu^2 r^2 - \mu_o^2 r_o^2 \sin^2 z_o]^{1/2}}$$

The subscript o refers to the vertex, or point of symmetry, of the ray and subscript s will refer to the satellite position. Evidently $z_o = \pi/2$, so

$$R = 2r_o\mu_o \int_1^{\mu_o} \frac{d\mu}{\mu[\mu^2 r^2 - \mu_o^2 r_o^2]^{1/2}} \qquad (1)$$

Since μ is a single-valued function of r, we may write $r(\mu)$ and $R = R(\mu_o r_o)$. Thus:

$$R(\mu_o r_o) = 2\mu_o r_o \int_{\mu_o r_o}^{r_{max}} \frac{\frac{d \log \mu}{d(\mu r)} d(\mu r)}{[\mu^2 r^2(\mu) - \mu_o^2 r_o^2]^{1/2}} \tag{2}$$

where $r(\mu)$ is to be found. The notation r_{max} refers to any r so large that $\mu(r) = 1$—in particular, r_s, which cannot be exceeded in the physical sense. However, r_{max} may be taken to be ∞, if desired. For comparison with Bateman, we adopt the substitutions

$$\eta = \mu r ; \quad \alpha = \mu_o r_o ; \quad \beta = r_{max}$$

Thus:

$$R(\alpha) = 2\alpha \int_{\alpha}^{\beta} \frac{\frac{d\log\mu}{dn} dn}{(\eta^2 - \alpha^2)^{1/2}} \tag{3}$$

Now substitute:

$$R(\alpha) = R(s) ; \quad \frac{d\log\mu}{dn} = 2t\phi(t) ; \quad \alpha^2 = \frac{1}{s} ; \quad \beta^2 = \frac{1}{a} ; \quad \eta^2 = \frac{1}{t}$$

Since $dn/dt = -\eta^3/2$, if $\eta = \alpha$, $t = 1/\alpha^2 = s$, and if $\eta = \beta$, $t = 1/\beta^2 = a$, we have

$$F(s) = \frac{2}{\sqrt{s}} \int_s^a - \frac{\eta^3}{2} \frac{2t\phi(t)dt}{\left(\frac{1}{t} - \frac{1}{s}\right)^{1/2}} = 2\int_a^s \frac{\phi(t)dt}{(s - t)^{1/2}} \tag{4}$$

which is an integral equation solved by Abel. The solution is

$$\phi(t) = \frac{1}{2\pi} \frac{d}{dt} \int_a^t \frac{F(s)ds}{\sqrt{t - s}} \tag{5}$$

provided: (1) $F(s)$ is continuous in the closed interval $(a \leq t \leq b)$;

(2) $F(a) = 0$; and

(3) $\int_a^x \frac{F(s)ds}{\sqrt{x - s}}$ have a continuous derivative throughout the same closed interval.

The first condition is obviously met by the nature of $F(s) = R(\alpha) = R(\mu_o r_o)$, refraction during a scan being continuous throughout. The refraction angle is 0 at the upper limit, where t is smallest. To check the third condition note that

$$\frac{d}{dx}\int_a^x \frac{F(s)ds}{\sqrt{x-s}} = k\phi(t)$$

$\phi(t)$ is proportional to the derivative of atmospheric density and obviously continuous over the interval.

Resubstituting in Eq. (5), we have:

$$\frac{1}{\mu}\frac{d\mu}{d(\mu r)} = \frac{-1}{\pi}\mu r \frac{d}{d(\mu r)}\int_{\mu r}^{r_{max}} \frac{\mu r\ R(\mu_o r_o)d(\mu_o r_o)}{\mu_o^2 r_o^2\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}} \tag{6}$$

Bateman[3] points out the following simplification:

$$\mu r \frac{d}{d(\mu r)}\int_{\mu r}^{r_{max}} \frac{\mu r R(\mu_o r_o)d(\mu_o r_o)}{\mu_o^2 r_o^2\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}} = \frac{d}{d(\mu r)}\int_{\mu r}^{r_{max}} \frac{\mu^2 r^2 R(\mu_o r_o)d(\mu_o r_o)}{\mu_o^2 r_o^2\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}} -$$

$$- \int_{\mu r}^{r_{max}} \frac{\mu r R(\mu_o r_o)d(\mu_o r_o)}{\mu_o^2 r_o^2\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}} = \frac{d}{d(\mu r)}\int_{\mu r}^{r_{max}} \frac{R(\mu_o r_o)d(\mu_o r_o)}{\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}} -$$

$$- \left[\frac{d}{d(\mu r)}\int_{\mu r}^{r_{max}} \sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}\right] \frac{R(\mu_o r_o)d(\mu_o r_o)}{\mu_o^2 r_o^2} - \int_{\mu r}^{r_{max}} \frac{\mu r R(\mu_o r_o)d(\mu_o r_o)}{\mu_o^2 r_o^2\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}}$$

$$= \frac{d}{d(\mu r)}\int_{\mu r}^{r_{max}} \frac{R(\mu_o r_o)d(\mu_o r_o)}{\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}} \tag{7}$$

Thus Eq. (6) becomes

$$\frac{1}{\mu}\frac{d\mu}{d(\mu r)} = \frac{-1}{\pi}\frac{d}{d(\mu r)}\int_{\mu r}^{r_{max}} \frac{R(\mu_o r_o)d(\mu_o r_o)}{\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}} \tag{8}$$

and

$$\log \mu(r_{max}) - \log \mu(\mu r) = -\frac{1}{\pi}\left[0 - \int_{\mu r}^{r_{max}} \frac{R(\mu_o r_o)d(\mu_o r_o)}{\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}}\right] \qquad (9)$$

$$\log \mu = -\frac{1}{\pi}\int_{\mu r}^{r_{max}} \frac{R(\mu_o r_o)d(\mu_o r_o)}{\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}} \qquad (10)$$

or

$$\mu = \exp\left\{-\frac{1}{\pi}\int_{\mu r}^{r_{max}} \frac{R(\mu_o r_o)d(\mu_o r_o)}{\sqrt{\mu_o^2 r_o^2 - \mu^2 r^2}}\right\} \qquad (11)$$

which has the form

$$\mu = \psi(\mu r)$$

At any point, $r = \mu r/\mu$ so that $r(\mu)$ is given explicitly, which was required.

It should be noted that determination of the density function is based on knowledge of the refraction angle as a function of $\mu_o r_o$, which is the ray path constant. That is,

$$\mu_o r_o = \mu_o r_o \sin z_o = \mu r \sin z = \mu_s r_s \sin z_s$$

and since $\mu_s = 1$

$$r_s \sin z_s = r_o \mu_o$$

Measurement of refraction angle R is tantamount to measurement of z_s, and satellite tracking yields r_s. Thus telemetry of R and knowledge of satellite position is equivalent to the measurement of $R(\mu_o r_o)$.

Although it seems extremely fortunate that the analytic expression requires knowledge of the precise refraction function, $R(\mu_o r_o)$, that is available experimentally, this result is not altogether surprising. The ray path constant being fundamental to the general concept of brachistochrones, there is considerable a priori likelihood that it would appear in the inversion equation and would be readily susceptible to measurement as well.

III. IMPLICATIONS OF SOLUTION

The demonstration of an analytic solution for index of refraction (i.e., density) as a function of height is of fundamental importance to the satellite refraction technique. First, it proves the existence of a unique inverse function. Second, it allows a simpler, much quicker, evaluation of the function, virtually assuring that the data reduction can be completed in real time. Third, it allows immediate programming of the data reduction process, and in turn a sophisticated error analysis, including simulation of satellite data and density-function retrievals. Fourth, it focusses attention on equipment requirements, which become the only major area in which feasibility has not been established. Variation of parameters in the simulated data will assist in specifying the instruments.

REFERENCES

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3. Bateman, H., The solution of the integral equation connecting the velocity of propagation of an earthquake-wave in the interior of the earth with the times which the disturbance takes to travel to the different stations on the earth's surface, Phil. Mag. S.6, 19, 576-587, 1910.