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Technical Report

THE RANGE OF THE YIELD CONDITION IN STABLE, IDEALLY PLASTIC SOLIDS

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SUMMARY

The property of stability in an ideally plastic solid is employed to define bounding yield conditions which remain valid when test results are available for only one stress state, such as simple tension or simple shear. For material in which yield is insensitive to changes in mean stress, the bounding yield conditions are shown to be the maximum shear-stress criterion of Tresca and a new criterion designated the maximum reduced stress criterion. Use of both criteria enables the best possible bounds to be obtained for the carrying capacity of a structure in the absence of further information. Parallel results are obtained for material in which yield strength is a linear function of the mean stress, and some examples are given.

I. INTRODUCTION

The powerful variational principles of the theory of ideally plastic solids have been employed mainly in the development of approximate solutions to well-defined problems in which the yield properties of the material are supposed to be known exactly. The carrying capacity may then be bounded from above and below by means of the theorems of limit analysis, developed by Gvozdev[1] and by Drucker, Prager, and Greenberg,[2] or by the use of escribed and inscribed yield conditions.[3] On the other hand, the question of finding approximate solutions when information about the yield condition is incomplete does not seem to have been investigated, despite its technological importance.

In this paper, the problem of finding upper and lower bounds to the yield load of a body is discussed for the case when the yield strength of the material is known exactly for only one or two stress states. An isotropic material in which yield is independent of the mean stress is considered first, and then the analysis is extended to material in which the yield criterion is linearly dependent on the mean stress. The former model is widely used in the analysis of metals, while interest in the latter springs from recent attempts to apply the theory of plasticity to soils.[4,5]

II. MATERIAL INSENSITIVE TO MEAN STRESS

A. BOUNDING YIELD SURFACES

Consider an ideally plastic, isotropic material in which the yield criterion is independent of the mean stress. In view of the property of isotropy, the orientation in space of the principal stress directions is unimportant and the state of stress may be represented completely by the principal stress components σ_1, σ_2, σ_3. It is then possible to represent the yield criterion by means of a surface in principal stress space having components σ_1, σ_2, σ_3 as co-ordinates. In the case of yield criteria independent of the mean stress, this surface will be formed by generators at right angles to the planes $\sigma_1 + \sigma_2 + \sigma_3 = \text{const.}$, i.e., parallel to the axis making equal angles with the co-ordinate directions, and any intersection of the surface with a plane $\sigma_1 + \sigma_2 + \sigma_3 = \text{const.}$ will be typical.

Two cases of intersections of this type are shown in Fig. 1. Consider first the case where the results of tensile and compressive tests are available (Fig. 1a). In the case of metals, tensile and compressive yield strengths are often nearly equal, suggesting the additional assumption, which will be made here, that reversal of the sign of a stress does not alter the stress magnitude at yield. Test points are represented by the six small circles in Fig. 1a, because any one of the three principal stresses can be taken as the nonzero component. Alternatively, considering a case where the result of a shear test (e.g., simple torsion) is available, the six points indicated by the small circles in Fig. 1b will then have been determined experimentally.

We now consider the question: What are the largest and smallest yield surfaces that can be drawn through the test data in the two cases? When finding the smallest surfaces, it is sufficient to note the requirement of convexity for any yield surface associated with a stable plastic material,[6] in order to arrive at the two hexagons shown. The largest surfaces are found by noting in addition that the cross sections shown must have 30° symmetry, so that reversal of all three stress components does not influence yield.

The hexagons marked 1 in Fig. 1 will be recognized as representing the familiar maximum shear stress criterion of Tresca:

$$\max.\ (|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) = \sigma_0 = 2\tau_0 \qquad (1)$$

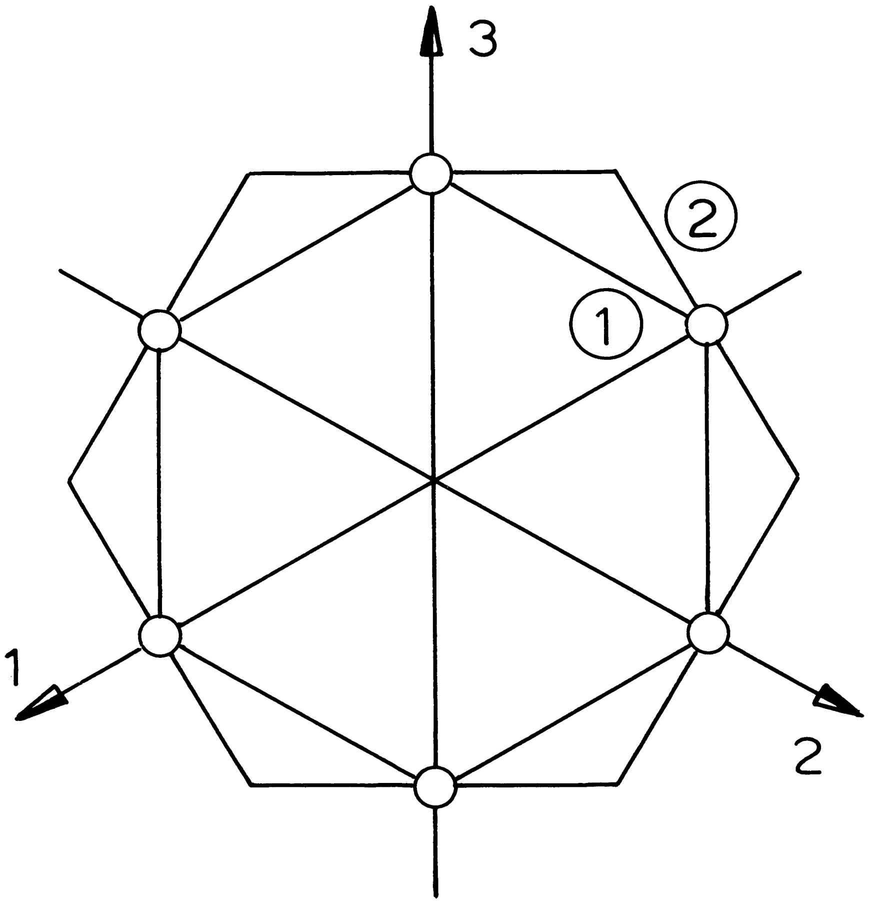

Fig. 1a. Material insensitive to mean stress. Bounding yield criteria when tensile test results are available.

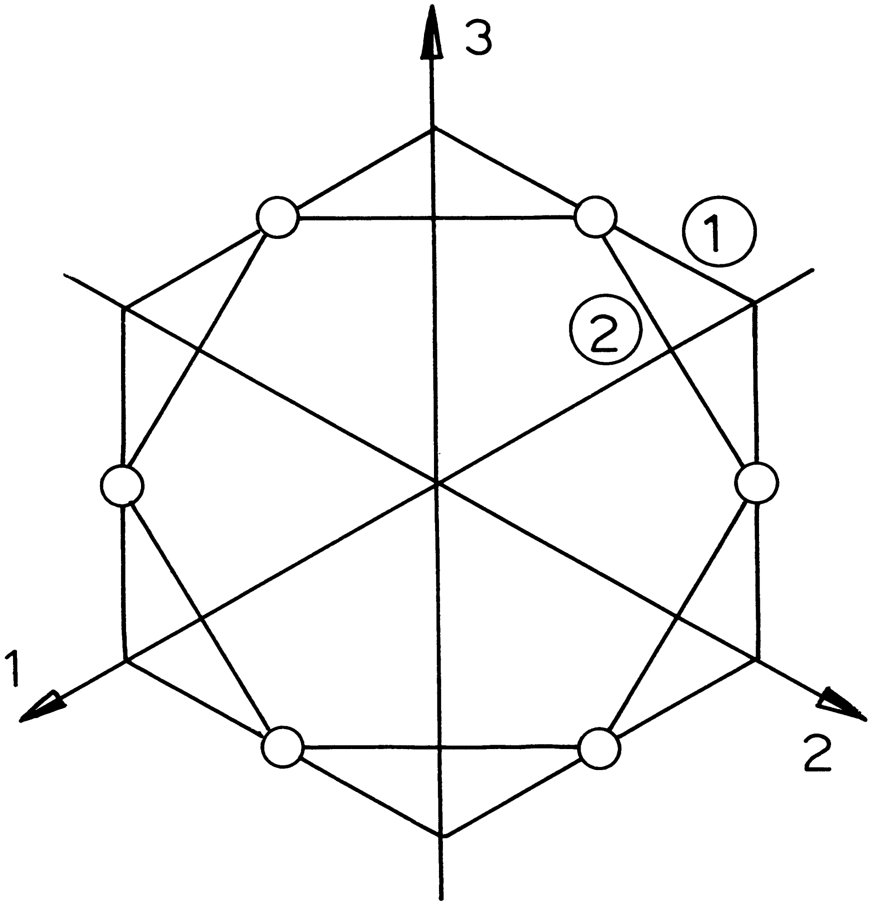

Fig. 1b. Material insensitive to mean stress. Bounding yield criteria when simple shear-test results are available.

where σ_o is the yield stress in simple tension and τ_o the yield stress in simple shear. The hexagons marked 2 in Fig. 1 represent criteria in which a restriction is placed on the value of the maximum reduced stress (which in Fig. 1 is measured parallel to the stress axes):

$$\max.\ (|\sigma_1 - \sigma|, |\sigma_2 - \sigma|, |\sigma_3 - \sigma|) = \tau_o = \frac{2}{3}\sigma_o \qquad (2)$$

where σ_o, τ_o are defined as for Eq. (1) and $\sigma = (\sigma_1 + \sigma_2 + \sigma_3)/3$.

The hexagons shown in Fig. 1 and represented analytically by Eqs. (1) and (2) owe their significance to the property that all possible yield conditions must lie between them. A lower bound obtained by the use of the inscribed hexagon will be a lower bound for all stable materials irrespective of the details of the yield condition, and an upper bound obtained by the use of the escribed hexagon will be similarly an upper bound in all cases.

Hill[7] has suggested Eq. (2) as a suitable linear approximation to v.Mises' yield surface when the stress state is close to a uniaxial compression. Ivlev[8] introduced the hexagons shown in Fig. 1a and designated them as bounding hexagons, but did not make this property dependent on the availability of tensile test data. Neither author anticipates the particular use of the criteria which is suggested here.

When determining upper bounds, the computations are greatly simplified if a compact expression can be found for the rate of energy dissipation in terms of the strain rates. The theory of the generalized plastic potential[9] indicates that the associated strain-rate vectors coincide with the outward drawn normal to the yield surface when corresponding axes are superimposed. Thus

$$\epsilon_{ij} = \sum_{p=1}^{n} \lambda_p \frac{\partial f_p}{\partial \sigma_{ij}} \qquad (3)$$

where λ_p are positive constants and the f_p are regular functions, the equations of various parts of the yield surface being $f_p(\sigma_{ij}) = 0$. It is then a simple matter to show that the rate of energy dissipation is

$$D = |\epsilon|_{max.}\sigma_o \qquad (4)$$

in the case of the maximum shear-stress criterion, and

$$D = |\gamma|_{max.}\tau_o \tag{5}$$

in the case of the maximum reduced stress criterion.

Still further restrictions on the range of the yield criterion can be made if the results of more than one type of test are available. If, for example, results for both tension and torsion tests were known and these results were consistent with the maximum shear-stress criterion, an examination of Fig. 1 reveals that the criterion must hold exactly for all other stress states. On the other hand, if these tests were consistent with the maximum reduced stress criterion, that criterion would be the only possible one. In both cases the range of possible criteria is reduced to a band of zero width.

In other cases, it is evident that additional results from only a very few combined stress tests would be sufficient to confine the yield surface for a stable material to an extremely narrow range.

As might be expected, application of the yield criteria developed above leads to bounds on the carrying capacity which in some cases are much closer than can be obtained by a simple factoring process based on inscribed and escribed yield surfaces of the same shape. If tensile test values are available, an upper bound based on the Tresca yield criterion must employ the outer hexagon shown in Fig. 2 while an upper bound based on the maximum reduced stress criterion can employ the inner hexagon. The latter will always be at least as favorable because, as a consequence of the lower bound theorem of limit analysis,[2] increasing the yield strength of a body in any zone can never reduce the collapse load. An example follows.

B. A PLANE-STRESS EXAMPLE

Consider a circular plate of radius R and thickness $t \ll R$, simply supported around the edge and subjected to a lateral pressure p uniformly distributed over the top surface. Supposing the yield stress of the material in tension to be given, it is desired to find the closest upper and lower bounds to the carrying capacity. For the lower bound, a solution using the Tresca yield criterion will be appropriate. This has been given by Hopkins and Prager.[10] The corresponding solution for the reduced stress criterion will now be developed.

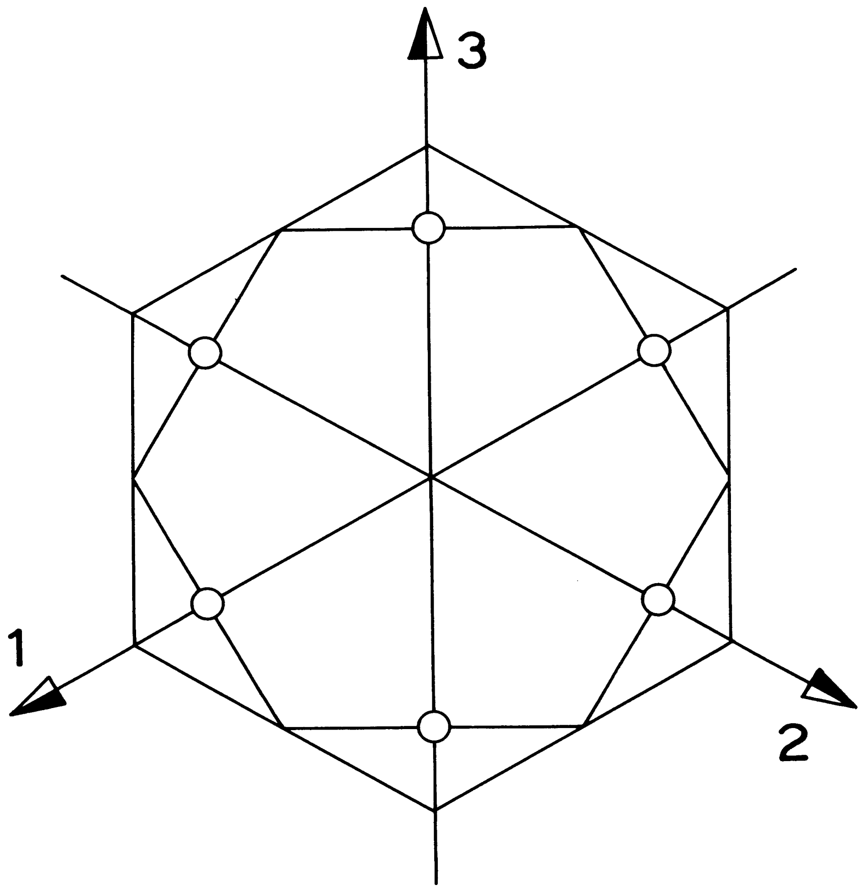

Fig. 2. Alternative escribed yield criteria when tensile test results are available.

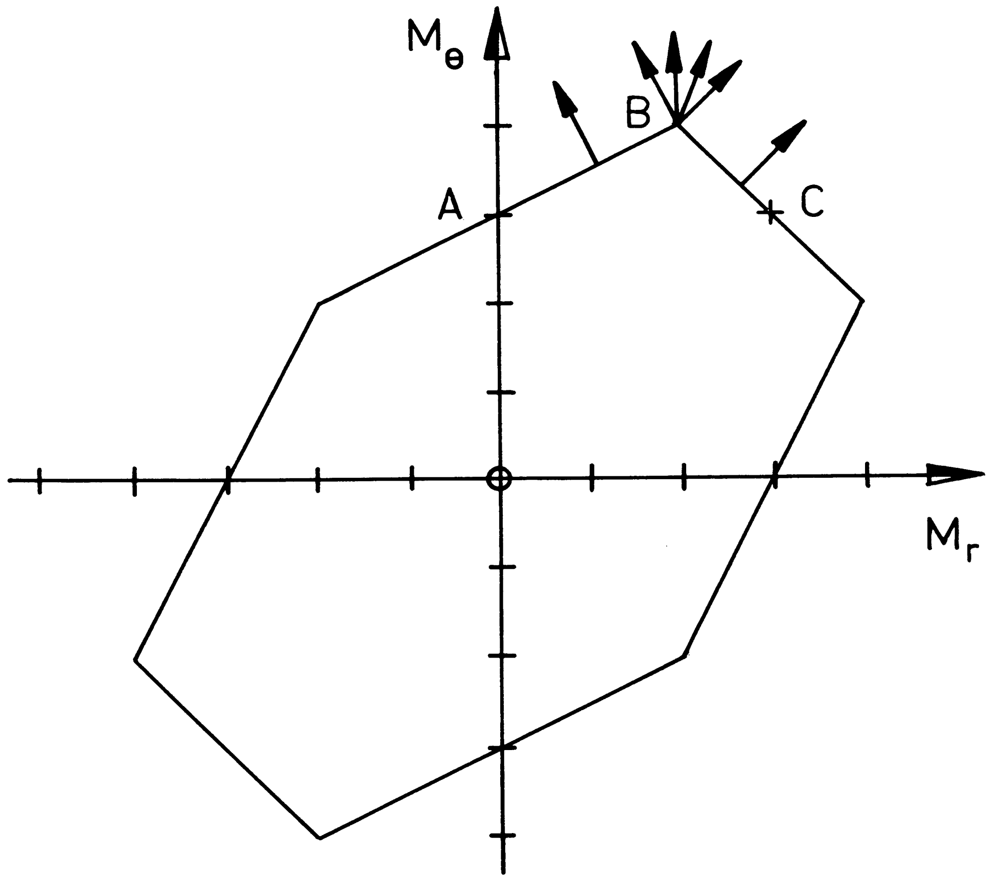

Fig. 3. The reduced stress criterion for plate bending (plane stress).

In the analysis, the usual assumptions of plate theory will be adopted. If the effects of shear are neglected, and if the lateral pressure is small compared with the stresses in the plane of the plate, then the state of the plate will be approximately one of plane stress, the nonzero principal stresses lying in the plane of the plate. Adopting the maximum reduced stress criterion, the yield curve drawn in terms of the radial moment per unit length M_r and the circumferential moment per unit length M_θ will be as shown in Fig. 3. Since M_r is zero at the outer edge and equal to M_θ at the center (by symmetry), the state points are expected to lie in the range ABC in Fig. 3. The equation of AB is

$$M_\theta - \frac{1}{2} M_r = M_o \qquad (6)$$

and that of BC is

$$\frac{1}{2} M_\theta + \frac{1}{2} M_r = M_o \qquad (7)$$

Eliminating the shear force from the radial equilibrium equation by means of the vertical equilibrium equation, we obtain

$$\frac{d}{dr}(rM_r) - M_\theta + \frac{1}{2} pr^2 = 0 \qquad (8)$$

where the signs have been assigned so that positive bending moments tend to induce sag of the plate in the same direction as the action of the pressure p.

Substituting Eqs. (6) and (7) in (8) and integrating the resultant first-order ordinary differential equations,

$$M_r = 2M_o - \frac{1}{5} pr^2 + c_1 r^{-\frac{1}{2}} \qquad (9a)$$

$$M_\theta = 2M_o - \frac{1}{10} pr^2 + c_1 r^{-\frac{1}{2}} \qquad (9b)$$

on AB and

$$M_r = M_o - \frac{1}{8} pr^2 + c_2 r^{-2} \qquad (10a)$$

$$M_\theta = M_o + \frac{1}{8} pr^2 - c_2 r^{-2} \qquad (10b)$$

on BC.

Substituting the boundary conditions $M_r = 0$, $M_\theta = M_0$ at $r = R$ and $M_r = M_\theta = M_0$ at $r = 0$ and equating the resultant expressions for M_r at $r = b = \beta R$, the following transcendental equation for β is obtained

$$6\beta^{2.5} - 15\beta^2 + 4 = 0 \qquad (11)$$

The root in the range $1 \geq \beta \geq 0$ is

$$\beta = 0.6242$$

which on substitution in Eqs. (9) and (10) leads to

$$p = 6.852 \frac{M_0}{R^2} \qquad (12)$$

The above pressure is statically admissible and so is a lower bound. It can be shown to be the actual collapse load by associating a velocity field.

The velocities are determined to within an arbitrary constant through the position of the state point in Fig. 3. According to the theory of plasticity for generalized stresses,[11] the generalized strains corresponding to M_r, M_θ will form a vector coinciding with the outwards drawn normal to the yield criterion, when corresponding axes are superimposed. The generalized strain rates corresponding to the moments are the curvature rates

$$\kappa_r = -\frac{d^2w}{dr^2} \quad ; \quad \kappa_\theta = -\frac{1}{r}\frac{dw}{dr} \qquad (13)$$

where w is the vertical velocity. The normal to the line described by Eq. (6) leads to the curvature components

$$\kappa_\theta = -\lambda \quad : \quad \kappa_r = +\frac{1}{2}\lambda$$

so $\kappa_\theta + 2\kappa_r = 0$ and hence

$$\frac{d^2w}{dr^2} + \frac{1}{2r}\frac{dw}{dr} = 0$$

On integration, this becomes

$$w = c_3 r^{\frac{1}{2}} + c_4 \tag{14}$$

and in a similar fashion the normal to Eq. (7) leads to

$$w = c_5 r^2 + c_6 \tag{15}$$

Setting the deflection zero at the supports and noting that there must be continuity of deflection and slope at $r = b$, we obtain

$$w = 2.45(1 - \rho^{\frac{1}{2}})w_o \qquad 1 \geqslant \rho \geqslant \beta \tag{16a}$$

$$w = (1 - 1.240\,\rho^2)w_o \qquad \beta \geqslant \rho \geqslant 0 \tag{16b}$$

where w_o is the deflection rate at the center. A velocity distribution has been associated with the statically admissible load, Eq. (12), and the solution is complete.

We are now in a position to compare the estimates of collapse pressure obtained using the two criteria. In the case of Tresca's yield criterion, the collapse pressure given by Hopkins and Prager[10] is

$$p = \frac{6M_o}{R^2} \tag{17}$$

and this will be the best lower bound. Comparing the expressions (12) and (17),

$$6.852 \geqslant \frac{pR^2}{M_o} \geqslant 6 \tag{18}$$

and the pressure is bounded to within 14.2%.

In contrast, if the Tresca yield criterion were used to obtain the upper bound, the outer hexagon shown in Fig. 2 would be employed and

$$8 \geq \frac{pR^2}{M_o} \geq 6 \qquad (19)$$

The pressure is bounded to within 33.3%.

III. MATERIAL SENSITIVE TO MEAN STRESS

A. BOUNDING YIELD SURFACES

Consider an ideally plastic, isotropic material in which the stress level at yield is a linear function of the mean stress. This constitutes a first step in the generalization of the analysis in Section II. There is evidence that it may be appropriate for certain soils, and also possibly for some concretes and cast metals.

Intersections of the yield surfaces with the planes $\sigma_1 + \sigma_2 + \sigma_3 =$ const. are shown in Fig. 4. Figure 4a shows the case where the results of tensile and compressive tests (represented by the circles) are available and Fig. 4b the case where shear tests are available. The limiting lines are obtained by arguments parallel to those used in Section II.

When expressing the bounding criteria mathematically, it is convenient to introduce the cohesion c and the angle of internal friction ϕ in the fashion in which they are commonly employed in statements of Coulomb's yield criterion. Any line which cuts the octahedral axis can then be defined in terms of c and ϕ values appropriate to the generalized Coulomb yield criterion which contains the line. The algebra is straightforward and will not be given here: the resulting expressions for the various bounding criteria are summarized in Table I. When tensile and compressive tests are available, these are not necessarily consistent with the same generalized Coulomb yield surface (see, for instance, the test results for sand quoted in Ref. 5), so values c, ϕ corresponding to the compressive test results and c', ϕ' corresponding to the tensile test results are introduced. The ratio $\alpha = \phi/\phi'$ is limited by the requirement of convexity to the range

$$2 - \sin\phi \geqslant \alpha \geqslant \frac{(1 - \sin\phi)}{2} \qquad (20)$$

the extreme values representing the attainment of triangular cross sections to the yield surfaces A and B. When $\alpha = 1$, surface A reduces to the Coulomb yield surface (see Table I).

The flow equations in the last column of Table I are obtained by substituting in Eq. (3) the analytic expressions for the yield surfaces given in the first column.

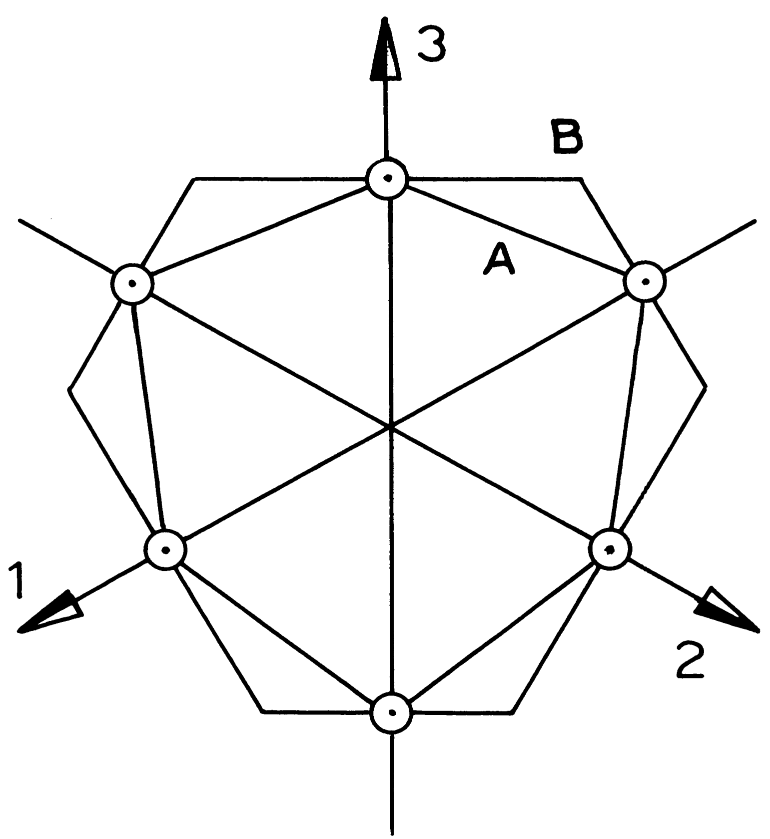

Fig. 4a. Material sensitive to mean stress. Bounding yield criteria when tensile test results are available.

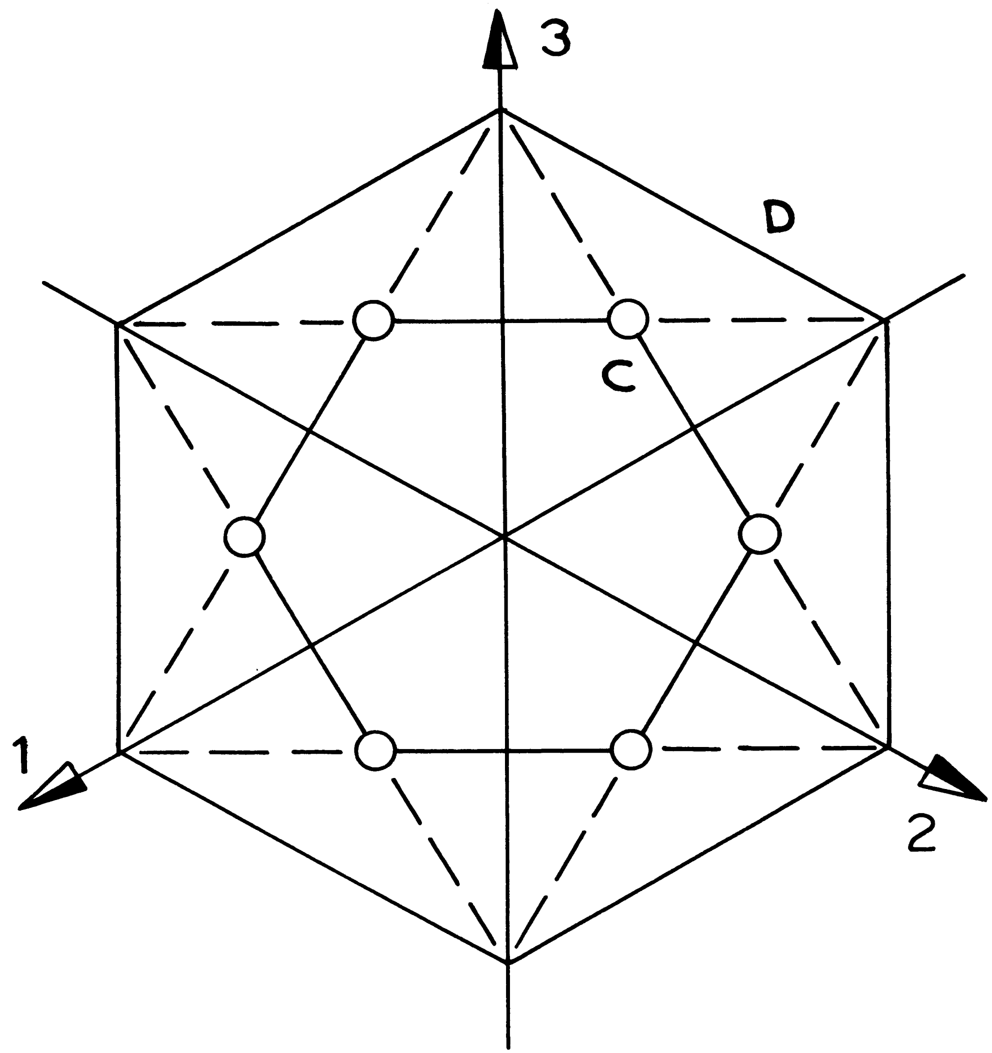

Fig. 4b. Material sensitive to mean stress. Bounding yield criteria when simple shear-test results are available.

TABLE I

EXPRESSIONS FOR THE YIELD CRITERIA ILLUSTRATED IN FIG. 4

Surface	Yield Criteria	Strain-Rate Equations
A	$(\alpha+\sin\phi)\sigma_1+(1-\alpha)\sigma_2-(1-\sin\phi)\sigma_3 = 2c\cos\phi$ where $2-\sin\phi \geqslant \alpha \geqslant (1-\sin\phi)/2$ $(\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3;\ \alpha = \sin\phi/\sin\phi')$	$(1-\sin\phi)(\epsilon_1+\epsilon_2)+(1+\sin\phi)\epsilon_3 = 0$ $(\sigma_1 = \sigma_2 > \sigma_3)$ $(1-\sin\phi')\epsilon_1+(1+\sin\phi')(\epsilon_2+\epsilon_3) = 0$ $(\sigma_1 > \sigma_2 = \sigma_3)$ $\epsilon_1:\epsilon_2:\epsilon_3 = \alpha+\sin\phi:1-\alpha:\sin\phi-1$ $(\sigma_1 > \sigma_2 > \sigma_3)$
	$\alpha = 2-\sin\phi$ $\max.(\sigma_1-\sigma,\sigma_2-\sigma,\sigma_3-\sigma) = 2(c\cos\phi-\sigma\sin\phi)/(3-\sin\phi)$ or $\sigma_1 = (\sigma_2+\sigma_3)(1-\sin\phi)/2+c\cos\phi$ $(\sigma_1 \geqslant \sigma_2,\sigma_3)$	$(1-\sin\phi)(\epsilon_1+\epsilon_2)+(1+\sin\phi)\epsilon_3 = 0$ $(\sigma_1 = \sigma_2 > \sigma_3)$ $\epsilon_1:\epsilon_2:\epsilon_3 = 2/(1-\sin\phi):-1:-1$ $(\sigma_1 > \sigma_2 > \sigma_3)$
	$\alpha = (1-\sin\phi)/2$ $\min.(\sigma_1-\sigma,\sigma_2-\sigma,\sigma_3-\sigma) = 4(\sigma\sin\phi-c\cos\phi)/(3-\sin\phi)$	$(1-3\sin\phi)\epsilon_1+(1+\sin\phi)(\epsilon_2+\epsilon_3) = 0$ $(\sigma_1 > \sigma_2 = \sigma_3)$ $\epsilon_1:\epsilon_2:\epsilon_3 = 1:1:-2(1-\sin\phi)(1+\sin\phi)$ $(\sigma_1 > \sigma_2 > \sigma_3)$
	$\alpha = 1$ $\sigma_1-\sigma_3+(\sigma_1+\sigma_3)\sin\phi = 2c\cos\phi$ $(\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3)$	$\epsilon_1:\epsilon_2:\epsilon_3 = (1+\sin\phi)(1-\sin\phi):0:-1$

TABLE I (Concluded)

Surface	Yield Criteria	Strain-Rate Equations
B	$(1+\sin\phi)(\sigma_1+\sigma_2)/2-(1-\sin\phi)\sigma_3 = 2c\cos\phi$ $(\sigma_2 \geqslant \sigma_2^c = [(2\alpha-1+\sin\phi)\sigma_1+(2-\alpha-\sin\phi)\sigma_3]/(1+\alpha))$ $(1-\sin\phi')(\sigma_1+\sigma_2)/2-(1+\sin\phi')\sigma_3 = -2c\cos\phi$ $(\sigma_2 < \sigma_2^c)$ $(\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3; \alpha = \sin\phi/\sin\phi')$	$(3-\sin\phi)(1-\sin\phi')\epsilon_1+$ $[4\cos\phi\cos\phi'-(1+\sin\phi)(1-\sin\phi')]\epsilon_2+$ $(1+\sin\phi)(3+\sin\phi')\epsilon_3 = 0$ $(\sigma_2 = \sigma_2^c)$ $\epsilon_1:\epsilon_2:\epsilon_3 = 1:1:-2(1-\sin\phi)/(1+\sin\phi)$ $(\sigma_2 \geqslant \sigma_2^c)$ $\epsilon_1:\epsilon_2:\epsilon_3 = 1:1:-(1+\sin\phi')/2(1-\sin\phi')$ $(\sigma_2 \leqslant \sigma_2^c)$
C	$\max.(\lvert\sigma_1-\sigma\rvert, \lvert\sigma_2-\sigma\rvert, \lvert\sigma_3-\sigma\rvert) = c\cos\phi-\sigma\sin\phi$ or $(2-\sin\phi)\sigma_1-(1+\sin\phi)(\sigma_2+\sigma_3)+3c\cos\phi = 0$ $(\sigma_2 \geqslant (\sigma_1+\sigma_3)/2)$	$(1+\sin\phi)\epsilon_1+\epsilon_2+(1-\sin\phi)\epsilon_3 = 0$
D	$\max.(\lvert\sigma_1-\sigma_2\rvert, \lvert\sigma_2-\sigma_3\rvert, \lvert\sigma_3-\sigma_1\rvert) = 3(c\cos\phi-\sigma\sin\phi)$ or $(1+\sin\phi)\sigma_1+\sigma_2\sin\phi-(1-\sin\phi)\sigma_3 = 3c\cos\phi$ $(\sigma_1 > \sigma_2 > \sigma_3)$	$(1-\sin\phi)(\epsilon_1+\epsilon_2)+(1+2\sin\phi)\epsilon_3 = 0$ $(\sigma_1 = \sigma_2 > \sigma_3)$ $(1-2\sin\phi)\epsilon_1+(1+\sin\phi)(\epsilon_2+\epsilon_3) = 0$ $(\sigma_1 > \sigma_2 = \sigma_3)$ $\epsilon_1:\epsilon_2:\epsilon_3 = 1+\sin\phi:\sin\phi:\sin\phi-1$ $(\sigma_1 > \sigma_2 > \sigma_3)$

At the point $\sigma_1 = \sigma_2 = \sigma_3 = c \cot \phi$, which is the apex for all the yield surfaces, the rate of energy dissipation is

$$D = \sigma_{ij} \epsilon_{ij} = c \cot \phi \, (\epsilon_1 + \epsilon_2 + \epsilon_3) \qquad (21)$$

Equation (21) also applies to all other points on the surfaces. The apex can be considered as belonging to every side and on any one side the projection of stresses in the direction normal to the side is always the same. In the case of surface C, the rate of energy dissipation can be written in the form

$$D = |\gamma|_{max.} \, c \cos \phi \qquad (22)$$

where $|\gamma|_{max.}$ is the value of the largest shear strain rate. Equally simple expressions do not appear to exist for the other surfaces and Eq. (21) has to be used.

B. PLANE STRAIN

In the case of plane strain, all the bounding criteria introduced above can be expressed in terms of equivalent Coulomb criteria, with a suitable choice of the constants. A solution which has employed the Coulomb criterion can then be readily adapted to apply to any of the other criteria by a simple factoring process.

In plane strain, one principal strain rate is to be set equal to zero. The ordering $\epsilon_1 \geqslant \epsilon_2 \geqslant \epsilon_3$ is already established as a consequence of the ordering $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ adopted in Table I, and it remains to determine which strain rate is to be zero. The dilatation is everywhere positive, so $\epsilon_1 \neq 0$. If $\epsilon_3 = 0$, then ϵ_1 and ϵ_2 are both greater than zero, but by the flow equations in Table I they are both of opposite sign in every case, which is a contradiction; hence $\epsilon_2 = 0$.

For surface A, Fig. 4, substitution of $\epsilon_2 = 0$ in the flow equations, Table I, corresponding to the two types of corner, gives

$$(1 - \sin \phi)\, \epsilon_1 + (1 + \sin \phi)\, \epsilon_3 = 0 \qquad (23a)$$

when $\sigma_1 = \sigma_2 > \sigma_3$ and

$$(1 - \sin \phi')\epsilon_1 + (1 + \sin \phi')\epsilon_3 = 0 \tag{23b}$$

when $\sigma_1 > \sigma_2 = \sigma_3$. Examination of the plane stress cross section (not shown) reveals that Eq. (23a) defines a vector lying between normals to adjacent flats providing $\phi \geqslant \phi'$ and Eq. (23b) defines a similar vector providing $\phi \leqslant \phi'$. Substituting the stress conditions associated with (23a) and (23b) in the equation for the yield surface given in Table I reduces the latter to

$$(1 + \sin \phi)\sigma_1 - (1 - \sin \phi)\sigma_3 = 2c \cos \phi \tag{24a}$$

when $\phi \geqslant \phi'$ and

$$(1 + \sin \phi')\sigma_1 - (1 - \sin \phi')\sigma_3 = 2c' \cos \phi' \tag{24b}$$

when $\phi \leqslant \phi'$. Thus, for plane strain, surface A reduces to Coulomb's criterion, the constants c, ϕ being used when $\phi \geqslant \phi'$ and c', ϕ' when $\phi \leqslant \phi'$.

For surface B, Fig. 4, there is only one type of corner. Substituting the corresponding stress $\sigma_2 = \sigma_2^c$ in the yield criterion, Table I, we obtain after some reduction an equivalent Coulomb yield criterion in which the effective values ϕ^* and c^* are defined by

$$\sin \phi^* = \frac{2(\sin \phi + \sin \phi')}{3 + \sin \phi - \sin \phi' + \sin \phi \sin \phi'} \tag{25a}$$

$$\frac{c^*}{c} = 2\left(1 + \frac{\sin \phi'}{\sin \phi}\right) \left[\frac{1 - \sin \phi}{(3 + \sin \phi')(3 - \sin \phi)(1 - \sin \phi')}\right]^{\frac{1}{2}} \tag{25b}$$

For surface C, Fig. 4, the yield criterion in Table I reduces immediately to the Coulomb criterion when the value $\sigma_2 = (\sigma_1 + \sigma_3)/2$, which holds at a corner, is substituted. This is as expected, since this value of σ_2 represents the state of simple shear for which the experimental data were available in the first place.

For surface D, Fig. 4, substitution of $\epsilon_2 = 0$ in the flow equations,

Table I, corresponding to the two types of corner, gives

$$(1 - \sin \phi)\,\epsilon_1 + (1 + 2 \sin \phi)\,\epsilon_3 = 0 \qquad (26a)$$

when $\sigma_1 = \sigma_2 > \sigma_3$ and

$$(1 - 2 \sin \phi)\,\epsilon_1 + (1 + \sin \phi)\,\epsilon_3 = 0 \qquad (26b)$$

when $\sigma_1 > \sigma_2 = \sigma_3$. Examination of the plane-stress cross section of the yield surface shows that the first equation is inadmissible because it is not the normal to a supporting plane of the yield surface. Substituting $\sigma_2 = \sigma_3$ in the yield criterion, Table I, an equivalent Coulomb yield criterion is obtained in which the effective values ϕ^{**} and c^{**} are defined by

$$\sin \phi^{**} = \frac{3 \sin \phi}{2 - \sin \phi} \qquad (27a)$$

$$\frac{c^{**}}{c} = \frac{3}{2}\left[\frac{1 - 2 \sin \phi}{1 - \sin \phi}\right]^{\frac{1}{2}} \qquad (27b)$$

As an alternative approach, the same effective values of c and ϕ for substitution in an equivalent Coulomb yield criterion can be found by comparing the corresponding flow equations given in Table I.

All problems of plane strain have now been shown to reduce to equivalent problems formulated in terms of the Coulomb yield criterion. In the case of materials insensitive to mean stress, the parallel result is well known. It must be recalled, however, that in the presence of dilatation the material is no longer in a state of simple shear and the argument used in the latter case is not applicable.

C. PLANE-STRAIN EXAMPLE

Consider the indentation of a half space by a long, rigid punch in the particular case when the tensile and compressive strengths have been measured and have been found to be consistent with the same constants in Coulomb's law, so that $\alpha = 1$. As stated by Prandtl,[12] the indentation pressure is

$$p = c \cot \phi (e^{\pi \tan^2 \phi} \tan^2(\frac{\pi}{4} + \frac{\phi}{2}) - 1) \qquad (28)$$

Prandtl's solution is complete in the sense than an incipient velocity field can be associated with the stresses in the deformable zone and the material can be shown to be necessarily rigid in the remainder of the half space (see Shield[13,14]).

In plane strain, surface A, Fig. 4, reduces to the Coulomb criterion with the original constants c and ϕ, while surface B requires the new constants defined by Eq. (25) after setting $\phi' = \phi$. Surface C reduces to the Coulomb criterion with the original constants and surface D requires the new constants defined in Eqs. (27).

Equation (28) can be applied equally well using the modified values ϕ^* and c^*, or ϕ^{**} and c^{**} because the resultant stresses and velocities are consistent with the various yield surfaces of Fig. 4, in the plane-strain case. The resulting upper and lower bounds are shown in Fig. 5 for various ϕ, the range between the upper and lower curves representing the maximum variation in pressure which can occur with any convex, conical yield surface.

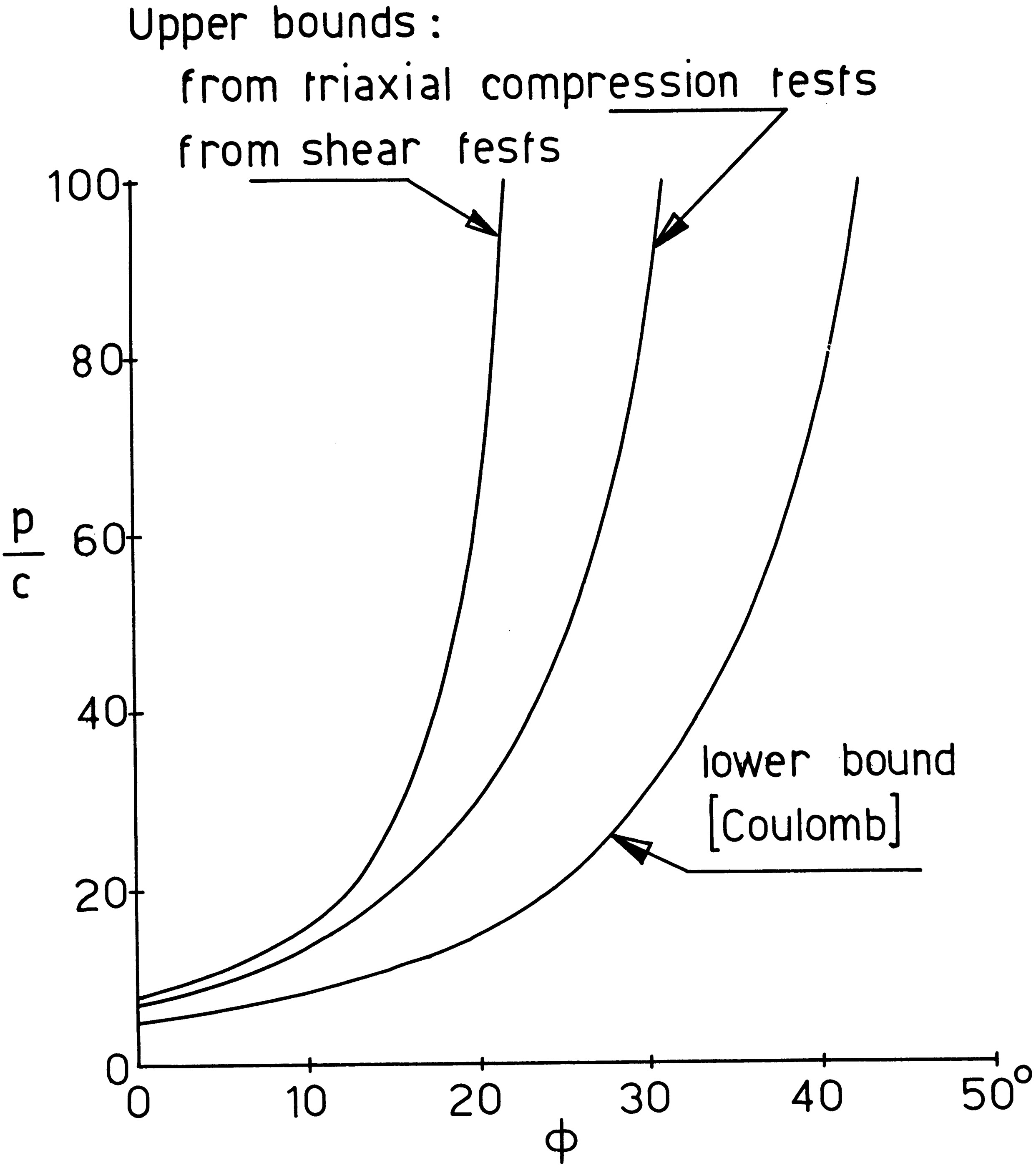

Fig. 5. Punch indentation (plane strain) for a material sensitive to mean stress.

IV. DISCUSSION

Certain yield criteria have been established as intrinsically suitable for engineering analysis because they lead to upper and lower bounds, as the case may be, without any reliance on a detailed knowledge of the actual yield criterion. All the criteria are piecewise linear, with the attendant simplifications of the analytical work which are well known.

It is perhaps worth emphasizing that results from a few tests may serve to define the yield criterion exactly or nearly so if the results fall close to one of the bounding criteria. If, for example, a metal develops yield strengths in tension and torsion consistent with Tresca's criterion, no further testing is necessary to identify this as the correct criterion providing there is sufficient other evidence to establish the metal as an ideally plastic material. A similar remark applies for the Coulomb criterion in the case of a material for which yield is a linear function of the main stress.

REFERENCES

1. A. A. Gvozdev, Proc. Conf. Plastic Deformations, 1936 (Academy of Sciences, U.S.S.R., 1938), 19; English trans: Int. J. Mech. Sci., 1 (1960), 322.

2. D. C. Drucker, W. Prager, and H. J. Greenberg, Q. App. Math., 9 (1952), 381.

3. R. Hill, Phil. Mag., 42 (1951), 868.

4. D. C. Drucker, R. E. Gibson, and D. J. Henkel, Proc. Am. Soc. Civil Eng., 81 (1955), 791.

5. R. M. Haythornthwaite, Proc. Am. Soc. Civil Eng., 86SM5 (1960), 35.

6. D. C. Drucker, Proc. 1st. U. S. Nat. Cong. App. Mech. (Am. Soc. Mech. Eng., 1951), 487.

7. R. Hill, Phil. Mag., 41 (1950), 733.

8. D. D. Ivlev, Prikl. Mat. Mekh. (Academy of Sciences, U.S.S.R.), 22 (1958), 850; English trans.: Q. App. Math. Mech.

9. W. T. Koiter, Q. App. Math., 11 (1953), 350.

10. H. G. Hopkins and W. Prager, J. Mech. Phys. Solids, 2 (1953), 1.

11. W. Prager, Proc. 8th. Int. Cong. Appl Mech., 1952 (Istanbul, 1955), 2, 65.

12. L. Prandtl, Nachr. Koeniglichen Gesell. Wiss. Goettinger, Math. Phys. Kl. (1920), 74.

13. R. T. Shield, Q. App. Math., 11 (1953), 61.

14. R. T. Shield, J. App. Mech., 21 (1954), 193.

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