THE UNIVERSITY OF MICHIGAN
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Technical Report

# THE TILTING PUNCH

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ORA Project 05894

under contract with:

DEPARTMENT OF THE ARMY
ORDNANCE TANK-AUTOMOTIVE COMMAND
DETROIT ORDNANCE DISTRICT
CONTRACT NO. DA-20-018-AMC-0980T
DETROIT, MICHIGAN

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

July 1965

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SUMMARY

The solution of Prandtl[1] for plastic indentation of a half-space by a rigid punch in plane strain is extended to cases where the load is applied eccentrically. Complete solutions are obtained for all possible combinations of applied load. The solution is also given for the obtuse angle truncated wedge loaded at the end, of which the former problem is a special case.

# 1. NOTATION

| | |
|---|---|
| $M$ | applied moment |
| $W$ | applied load |
| $c, \varphi$ | material constants for a Coulomb body |
| $k$ | shear stress at yield |
| $m, w$ | dimensionless variables |
| $p$ | indentation pressure |
| $s$ | width of punch |
| $\alpha$ | width ratio |
| $\beta$ | wedge semi-angle |
| $\delta$ | displacement |
| $\theta$ | rotation |
| $\psi$ | $= s\theta/8$ |

## 2. INTRODUCTION

The extensive literature on the theory of punch indentation seems devoid of any reference to tilting modes. The latter are to be expected when the load is applied eccentrically, and they have been observed in tests on sands[2-4] and clays.[3] An estimate of the failure load can be found easily[3] by introducing an eccentrically placed stress field of the type used by Prandtl,[1] with the zone of surface pressure extending over part of the width of the punch, the remainder being unloaded. The solution is incomplete in the sense this term[5] is used in plasticity theory, unless the load can be shown to be statically admissible and unless at least one kinematically admissible mode of deformation can be associated with the stress field. It is the purpose of this note to supply these ingredients for the plane strain indentation of a level surface and also for the more general case of an obtuse angle truncated wedge loaded at the end. It will also be shown that a series of tilting mechanisms are possible even when the loading is concentric. The latter point will be dealt with first.

## 3. CONCENTRIC LOADING

Three velocity fields have been proposed for plane strain indentation of a half-space by a smooth rigid punch,[1,6,7] but none accommodates rotation of the punch. A broad class of velocity fields accommodating rotation can be defined by noting the restriction placed on the velocities by the stress field, which requires only that the sign of the plastic power be everywhere positive. An example of a possible velocity field is shown in Figure 1, where the extent of the deforming zone is shown in (a), and the corresponding hodograph in (b). The material has been assumed to be one in which the yield point stress is independent of the hydrostatic component of stress, so that the stress deformation characteristics form an orthogonal net and there is no dilation during plastic flow. The instantaneous center for motion of the punch has been placed in an arbitrary position outside a zone defined by vertical lines drawn through the corners of the punch B and $B_1$, so that the indenting surface of the punch does not lift at any point. The slip between punch and surface, bb' at B, is defined at once from the requirement that the velocity in ABC near AB must be normal to AB, in view of the presence of rigid material below AC. The slip at other points on the punch surface is not uniquely determined but must meet certain conditions. There is a one-to-one correspondence between points on $BB_1$ in (a) and points on $bb_1$ in (b). Progression from B to $B_1$ is accompanied by progression from b to $b_1$. In a progression from B to $B_1$, the change in the velocity component in direction EF cannot be positive and the change in direction $EF_1$ cannot be negative, in order that no element of material should act as an

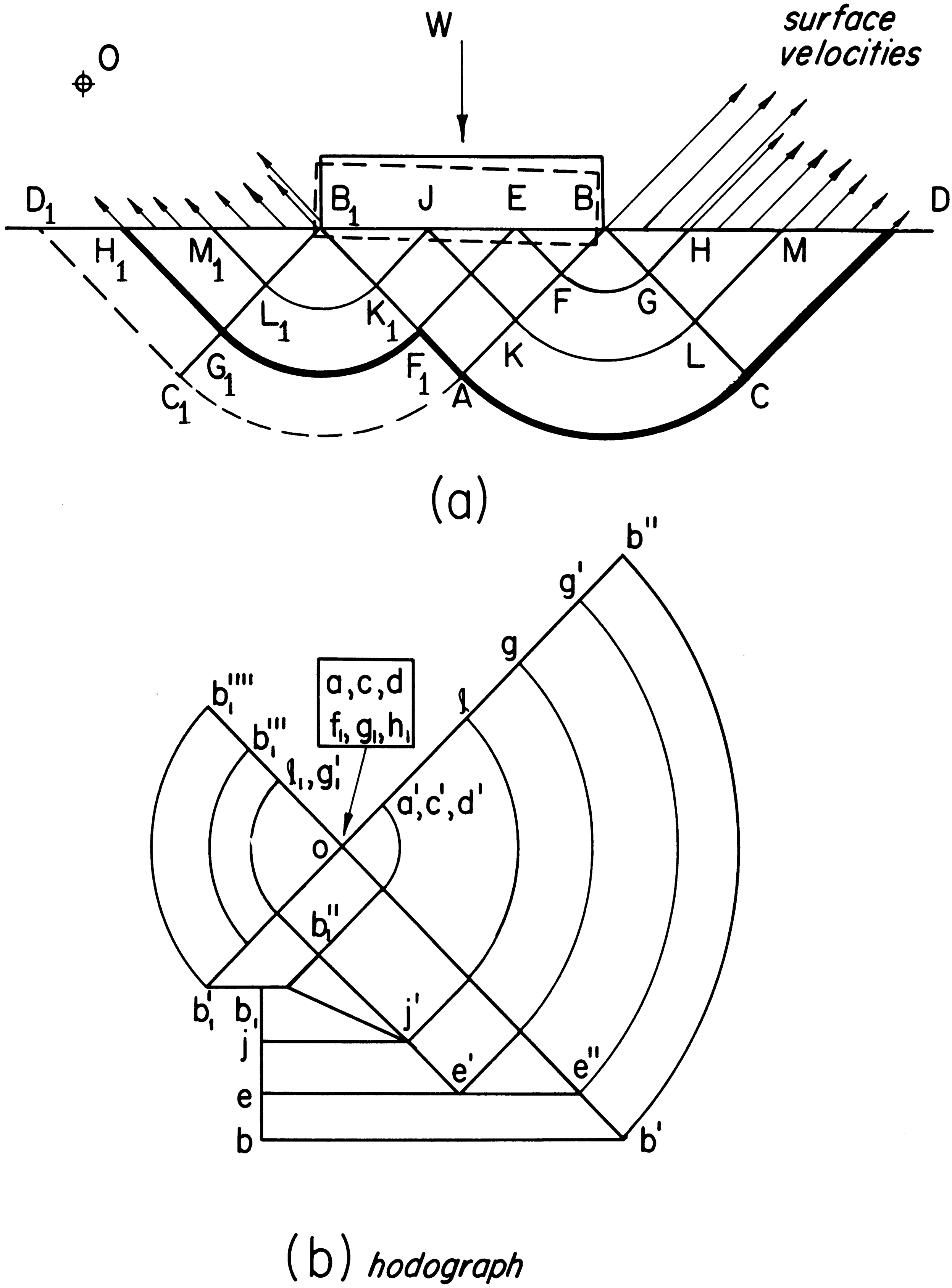


Figure 1. A possible deformation mode for the tilting punch with concentric load.

engine during the motion. It follows that the change in velocity for the indented material on line $BB_1$ for a progression from B to $B_1$ must have a negative projection on ob' and a positive projection on $ob_1'$. In terms of the hodograph, Figure 1b, vectors for increments of velocity associated with progression from B to $B_1$ must lie within an arc between the directions b'o and b'b. In the figure, the path $b'e''e'j'b_1''b_1'$ has been chosen arbitrarily to meet this condition and to illustrate various types of discontinuity that can occur. It is interesting to note that a velocity discontinuity oa' on ACD must be accompanied by a discontinuity $b_1''' \; b_1''''$ in surface velocity at $B_1$. The point $b_1''$ could be chosen on ob' and in that case, zone $AC_1D_1B_1$ is rigid and motion is confined to zone $ACDBB_1$. On the other hand, no choice of surface slip velocities can eliminate deformation from any part of the latter zone. The surface velocities for points on BD cannot be the same, but must decrease from B to D, with possible discontinuities, as at H and D in the example shown. On $B_1D_1$, the velocity can be the same at various positions, as between $M_1$ and $H_1$, and in particular zero, as between $H_1$ and $D_1$ in the example. The non-tilting punch can be considered as a limiting case of the tilting punch when the instantaneous center is removed an indefinite distance either to the right or to the left of the punch. The velocity field (Figure 1) then coincides with that given by Spencer[7] as the limiting (quasi-static) field for dynamic indentation without tilting. On the other hand, the velocity fields proposed by Prandtl[1] and Hill[6] are inadmissible in the presence of even the smallest amount of tilting, because it becomes impossible to ensure that every element of material in the deforming zone dissipates work.

## 4. ECCENTRIC LOADING

A case in which the instantaneous center of motion of the punch lies between the vertical edges of the punch is shown in Figure 2. Clockwise motion about the center O is assumed, so that points on the indenting surface to the left of $B_1$ (Figure 2a), will rise and points to the right will fall. It appears reasonable to introduce a Prandtl stress field $ACDD_1C_1$ which is associated with contact pressure over $BB_1$ and which leaves the rest of the punch surface unloaded.

To prove this choice of stress field correct, it is necessary to establish an associated velocity field and also to show the existence of an extension of the stress field which is statically admissible throughout the half-space.[8] A velocity field can be found quite simply by adapting the field discussed for the case of concentric loading (Figure 1). Due to the presence of the punch surface to the left of $B_1$ (Figure 2a), the vertical component of velocity just to the left of $B_1$ must tend to zero, and this condition is met by the field shown (Figures 2a and 2c).

A statically admissible extension of the Prandtl stress field was first given by J. F. W. Bishop.[8] An alternative suggested by Shield[9] will be repeated here because it is simpler and deserves more recognition. The field (Figure 2a) is obtained by continuing the stresses of the radial zone ABC outwards and introducing a discontinuity AE chosen so that the principal stresses just to the left of AE are horizontal and vertical. The path of the discontinuity is constructed starting at A (Figure 2a) using a succession of constructions of the type illustrated in the stress plane, (Figure 2b) which shows the discontinuity

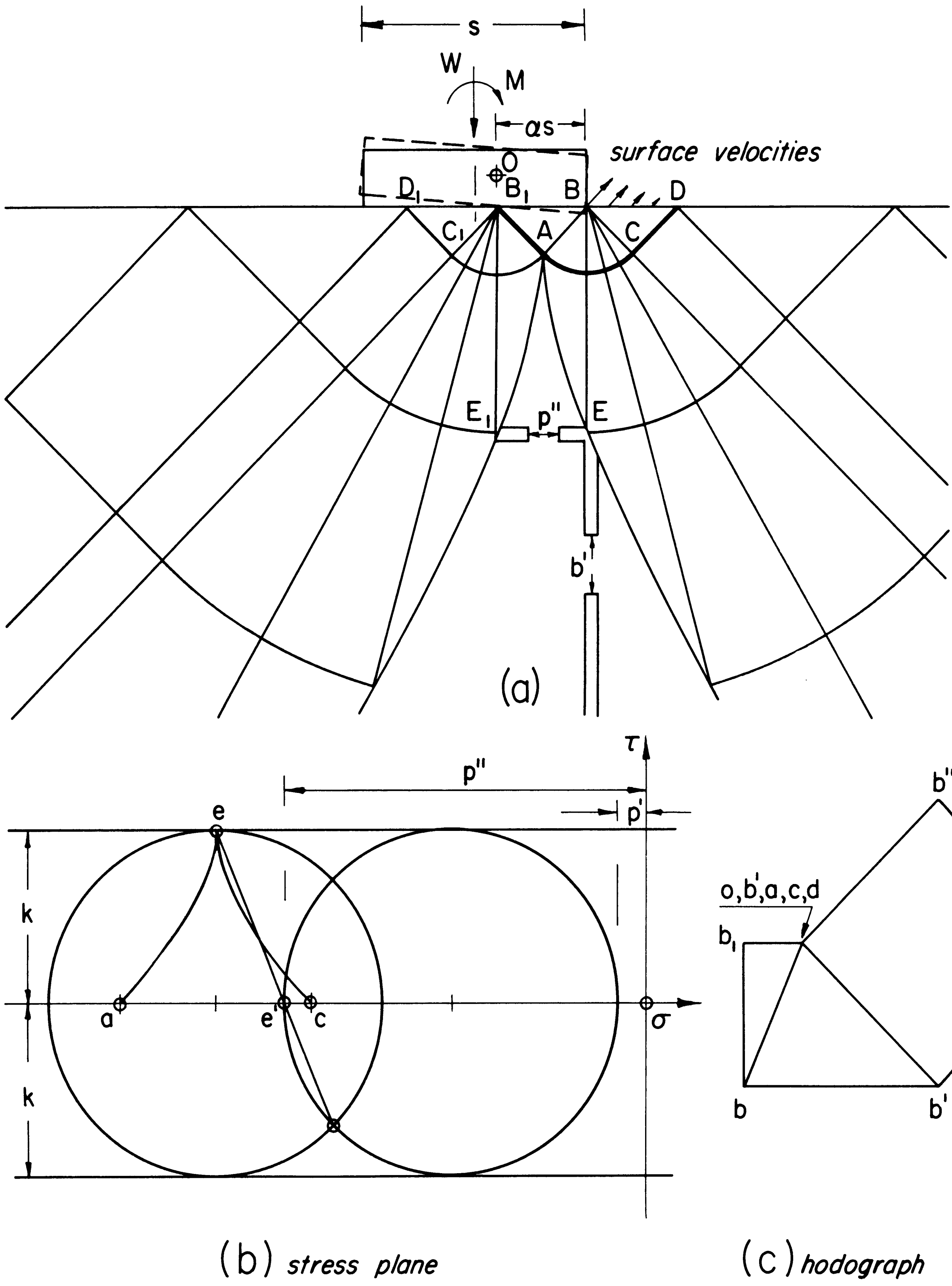


Figure 2. Deformation mode for the punch with an eccentric load. One possible statically admissible extension of the stress field is shown in (a).

for point E (Figure 2a). To the left of AE, the horizontal and vertical stresses on AE are assumed to be maintained in columns of material downwards and to the left. The pressure $p_2$ decreases monotonically as the depth increases, so the material in the zone $EAE_1$ is below yield.

The cycloidal trace of poles[10] for stresses in zone ABC is shown in Figure 2b and it is confirmed that the velocities shown in Figure 2c are associated with absorbtion of work by every element of material.

The deformable zone can be restricted to $ACDBB_1$ in two steps. Zone $EAE_1$ is below yield and hence rigid, so it will be rigid in all solutions.[5] Inextensibility of the slip lines ACD and $AC_1D_1$ then ensures that the entire zone below $DCAC_1D_1$ is necessarily rigid in this (and hence in all) solutions. In establishing the velocity field, it has been noted that the velocity at $B_1$ just to the left of $AB_1$ must be zero to avoid interference with the punch at $B_1$; however, in view of the sense of the shear stress in zone $B_1C_1D_1$, the velocity in the direction $AB_1$ cannot increase across any element along the line $B_1C_1$. As a monotonic function zero at both $B_1$ and $C_1$, the velocity must be zero everwhere on $B_1C_1$ and so the zone $AB_1C_1D_1$ must be rigid. The deformable zone is now confined to $ACDBB_1$.

Contact with the punch must be maintained along $BB_1$ in the assumed stress solution. Also, there is no velocity across $AB_1$ in view of the rigidity of $AB_1C_1$; hence velocities in $ABB_1$ and thus in the rest of the deforming zone $ACDB_1$ are completely determined by the motion of the punch. The deforming region must reach out to D and there is no alternative possible.

In the case of concentric loading discussed in Section 2, an extended stress field of the type shown in Figure 2 can be constructed below $ACDD_1C_1$ of Figure 1 and hence the material below that zone can be shown to be necessarily rigid. The deformable zone comprises the whole of $ACDD_1C_1$.

COLLAPSE LOADS

The Prandtl indentation pressure[1] is

$$p_1 = (2 + \pi)k \tag{1}$$

where k is the yield stress in simple shear. This pressure acts across the interface $BB_1$, and equilibrating it with an applied force W and a moment M applied to the center of the punch, we obtain

$$\left.\begin{aligned} W &= \alpha s p_1 \\ M &= (1-\alpha)sW/2 \end{aligned}\right\} \tag{2}$$

where s is the width of the punch and $\alpha s$ is the length of $BB_1$. The yield curve in the M, W plane for the range $M \geq 0$ is found by eliminating $\alpha$ between equations (2). The analysis is repeated for $M \leq 0$. It is convenient to introduce the dimensionless variables

$$\left.\begin{aligned} m &= 8M/p_1 s^2 \\ w &= W/p_1 s \end{aligned}\right\} \tag{3}$$

and the results can then be summarized by the single expression

$$|m| = 4w(1 - w) \tag{4}$$

which defines the yield curve shown in Figure 3.

The rate of energy dissipation is $W\dot{\delta} + M\dot{\Theta}$, where $\dot{\delta}$ is the displacement rate of W and $\dot{\Theta}$ the rotation rate of M. It can be rewritten

$$\dot{D} = p_1 s(w\dot{\delta} + m\dot{\psi}) \tag{5}$$

where $\dot{\psi} = s\Theta/8$. Hence[11] w and m can be regarded as generalized stresses and $\dot{\delta}$ and $\dot{\psi}$ as corresponding generalized strains for the purposes of plasticity theory. It follows that the yield surface shown in Figure 3 is necessarily convex, and if a corresponding generalized strain vector is drawn for any point on the surface, it will lie in the direction of the outwards drawn normal to the yield surface.

With this interpretation, the various features of Figure 3 can be readily identified. Point A corresponds to concentric loading, and the presence of a corner in the yield surface at that point permits a fan of normals to various supporting planes; each corresponding to a different position for the instantaneous center O (Figure 1) as it ranges over all possible locations outside the punch. These many modes of deformation, with various ratios of sinking and tilting are all associated with the same indentation load. Point B corresponds to lifting of the punch with no indentation, and hence zero load. The arrows shown at that point in Figure 3 correspond to tilting about one or other of the corners. The curves between A and B represent the loads associated with the tilting mode described in this section.

SOILS

The analysis can be repeated for the indentation of material, the yield-point stress of which is sensitive to the level of mean stress. Such materials

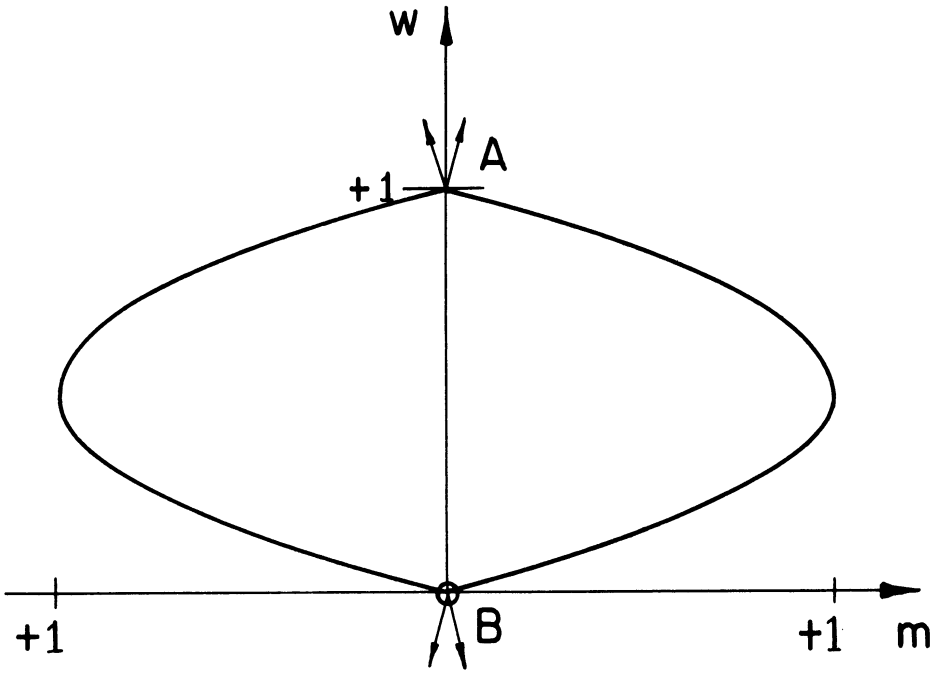


Figure 3. Yield curve for the tilting punch drawn in terms of the non-dimensional variables $m = 8M/ps^2$; $w = W/ps$.

are commonly used as models for granular media and other soils. In order to illustrate the differences in procedure, a material of this type will be used in the following discussion of the truncated wedge.

## 5. TRUNCATED WEDGE

The analysis for indentation of a half space can be adapted to the general case of the truncated wedge without introducing new difficulties. The material of the wedge will be assumed to obey the Coulomb yield criterion[12] in which it is postulated that deformation can occur when, on any plane in the material, the shear stress reaches the value $c - \sigma \tan \varphi$, where $\sigma$ is the normal stress across the plane and $c, \varphi$ are material constants. This model is often used to represent granular media. In terms of the stress plane (Figure 4b), yield will occur when the largest Mohr circle associated with the stress state touches the sloping line defined by the dimension c and the angle $\varphi$. If the body is ideally plastic, yield will be accompanied by dilation[13] such that relative motions will not be parallel to the directions of slip, but will subtend an angle $\varphi$ with them.

Details of the construction for a wedge with semi-angle $45^{\circ}$ are shown in Figure 4, where the same notations have been used as in Figure 2 to facilitate comparisons. The pressure across $BB_1$ is[14]

$$p_2 = c.\cot\varphi\,[e^{2\beta \tan \varphi} \tan^2(\pi/4 + \varphi/2) - 1] \qquad (6)$$

where $\beta$ is the wedge semi-angle. The non-dimensional yield curve shown in Figure 3 applies to this case if the pressure defined in equation (6) is substituted in equation (3) for the Prandtl indentation pressure $p_1 = (2 + \pi)k$.

For material insensitive to mean stress, the pressure across $BB_1$ is

$$p_3 = (2 + 2\beta)k \qquad (7)$$

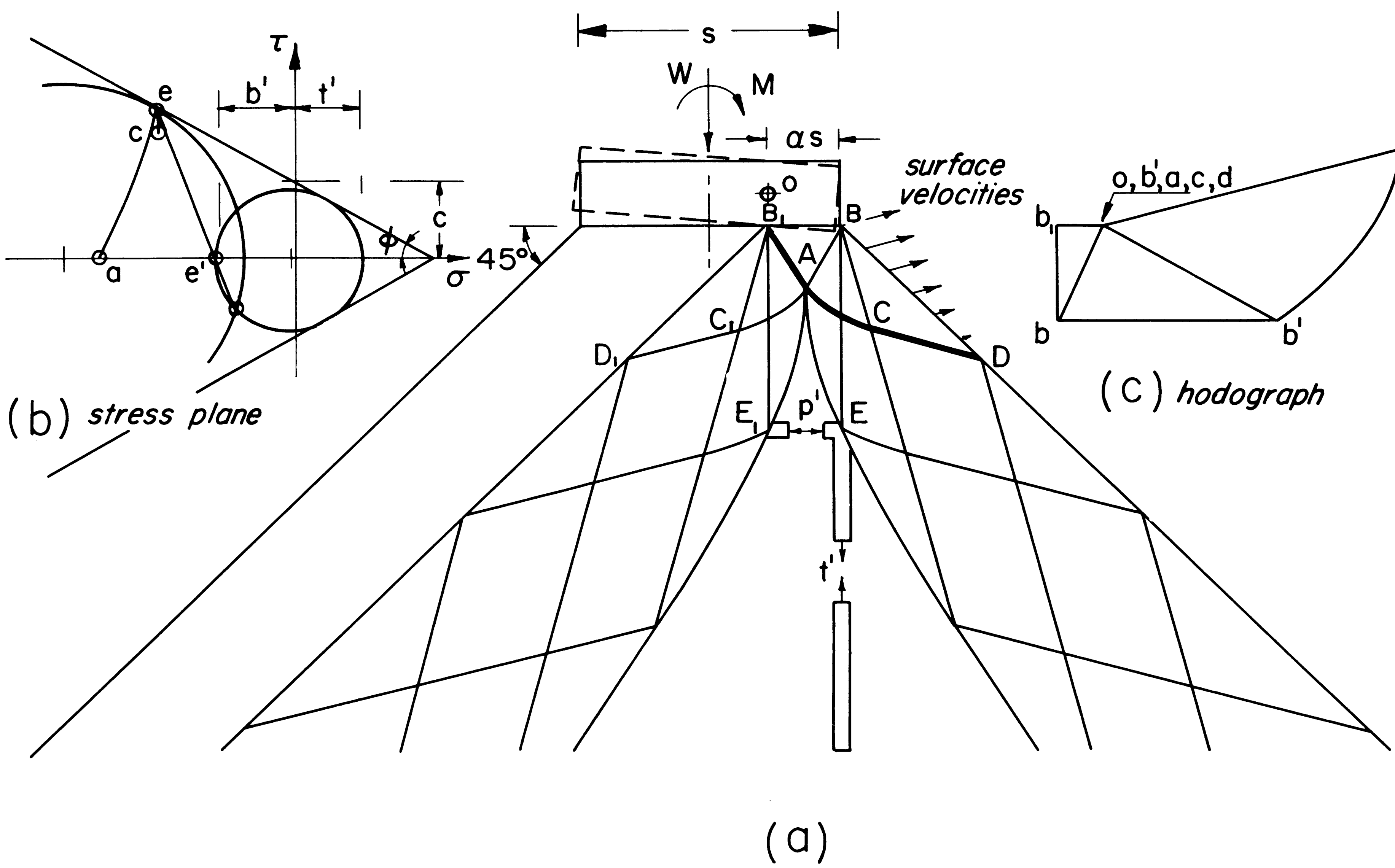


Figure 4. Deformation mode for eccentric loading of a punch bearing on the end of a truncated wedge. One possible statically admissible extension of the stress field is shown in (a).

Again, the yield curve shown in Figure 3 applies if $p_3$ is substituted for $p_1$ in equation (3).

The solutions are valid for $\pi/2 \geq \beta \geq 0$. If $\beta < 0$, the wedge is reduced in size at points remote from the punch and a local failure would occur there. If $\beta > \pi/2$, the extension of the stress field $AC_1D_1B_1$ cannot be constructed due to the presence of the punch interface within the required zone.

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