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Technical Report

HOMOMORPHISMS OF GRAPHS

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RESEARCH PROGRESS REPORT

Title: "Homomorphisms of Graphs," S. Hedetniemi, University of Michigan Technical Report

Background: The Logic of Computers Group of the Communication Sciences Program of The University of Michigan is investigating the application of logic and mathematics to the design of computing automata. The use of graph-theoretical techniques in the study of automata forms a part of this investigation.

Condensed Report Contents:

The works of Hartmanis and Stearns, Krohn and Rhodes, Yoeli and Ginzburg, and Zeiger amply demonstrate the usefulness of homomorphisms in studying decompositions of finite automata. Yoeli and Ginzburg's approach is slightly different from the others in that it is more concerned with aspects of the state transition graphs of finite automata. In their paper, "On Homomorphic Images of Transition Graphs," they give a complete characterization of the class of homomorphisms of the graphs which correspond to input-free automata.

This paper was motivated by an interest in extending these results of Yoeli and Ginzburg in the direction of a characterization of the class of homomorphisms of graphs which correspond to arbitrary finite automata. It is a review and a classification of most of the published definitions and results on mappings of graphs which have been called homomorphisms. The paper contains, in addition, several new results and several new definitions of homomorphisms of graphs.

For Further Information: The complete report is available in the major Navy technical libraries and can be obtained from the Defense Documentation Center. A few copies are available for distribution by the author.

PREFACE

The works of Hartmanis and Stearns [10], Krohn and Rhodes [12], Yoeli and Ginzburg [22], and Zeiger [24] amply demonstrate the usefulness of homomorphisms in studying decompositions of finite automata. Yoeli and Ginzburg's approach is slightly different from the others in that it is more concerned with aspects of the state transition graphs of finite automata. In their paper, "On Homomorphic Images of Transition Graphs" [22], they give a complete characterization of the class of homomorphisms of the graphs which correspond to input-free automata.

This paper was motivated by an interest in extending these results of Yoeli and Ginzburg in the direction of a characterization of the class of homomorphisms of graphs which correspond to arbitrary finite automata. It is a review and a classification of most of the published definitions and results on mappings of graphs which have been called homomorphisms. The paper contains, in addition, several new results and several new definitions of homomorphisms of graphs, which are found primarily in Sections 1.3 — 1.6, 1.8, and 1.9.

The mappings of graphs which are defined in this paper are divided into three classes; Homomorphisms, Contractions, and Relational Homomorphisms. The distinction between the first two classes, while not very important, is a very natural one to make, and can best be understood after the class of Homomorphisms has been defined and studied in Section 1. It is for this reason that the discussion of this distinction is postponed until Section 2, where Contractions are defined and examined.

It is likely that there are definitions and results on mappings of graphs which have recently been published but which are not referenced here. The author would welcome any information concerning such omissions.

For the reader not well versed in the terminology of graph theory, terms which are undefined in the text are defined in the Appendix.

Proofs of statements which are quite easily constructed have, in many instances, been omitted, it is hoped without any loss of content or continuity.

The author is especially indebted to Professor Frank Harary, of the Mathematics Department, and to Yehoshafat Give'on, of the Logic of

Computers Group, for their many critical comments, suggestions, and sharp reading of the manuscript, which greatly improved the organization and presentation of definitions, results, and proofs.

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NOTATION

Of a graph $G = \langle V,\rho \rangle$:

V or $V(G)$ denotes the set of points of G;

ρ denotes the adjacency relation of G, $\rho \subseteq V\times V$;

$\rho(a,b)$ or (a,b) means there is a line from point a to point b in G;

$\bar{\rho}(a,b)$ means there is no line from point a to point b in G;

$$\rho(a) = \{b : (a,b)\epsilon\rho\};$$

$$\rho^{-1}(b) = \{a : (a,b)\epsilon\rho\};$$

$$n(a) = \rho(a)\cup\rho^{-1}(a);$$

$\chi(G)$ denotes the chromatic number of G.

K_n denotes the complete graph on the n points, k_1, k_2, ..., k_n.

P denotes a cycle; P_n, a cycle of length n with points p_1, p_2, ..., p_n.

C denotes a chain; C_n, a chain of length n with points c_1, c_2, ..., c_{n+1}.

1. HOMOMORPHISMS

1.1. General Definitions.

A graph $G = \langle V, \rho \rangle$ is a relational system whose set is the finite set V of points of G and whose set of relations consists of one binary relation $\rho \subseteq V \times V$, the adjacency relation. A graph $G' = \langle V', \rho' \rangle$ is a subgraph of G, written $G' \subseteq G$, iff $V' \subseteq V$ and $\rho' \subseteq \rho$; G' is a full subgraph of G iff for every $a,b \in V'$, $(a,b) \in \rho \Longrightarrow (a,b) \in \rho'$.[1] Let ϕ be a function from V into V', and define the extension $\phi : V \times V \longrightarrow V' \times V'$, by

$$(a,b)\phi = (a\phi, b\phi).$$

If ϕ satisfies the condition

(1) $\rho\phi \subseteq \rho'$, or equivalently, $(a,b) \in \rho \Longrightarrow (a\phi, b\phi) \in \rho'$,

then ϕ is a homomorphism of G into G'. If ϕ also satisfies the condition

(2) $(a\phi, b\phi) \in \rho' \Longrightarrow (\exists c)(\exists d)[c\phi = a\phi \wedge d\phi = b\phi \wedge (c,d) \in \rho]$,

then ϕ is a full homomorphism of G into G'. From the definition of homomorphism it follows that the product of homomorphisms is itself a homomorphism, i.e., if G, G', and G'' are graphs and $\phi : G \longrightarrow G'$ and

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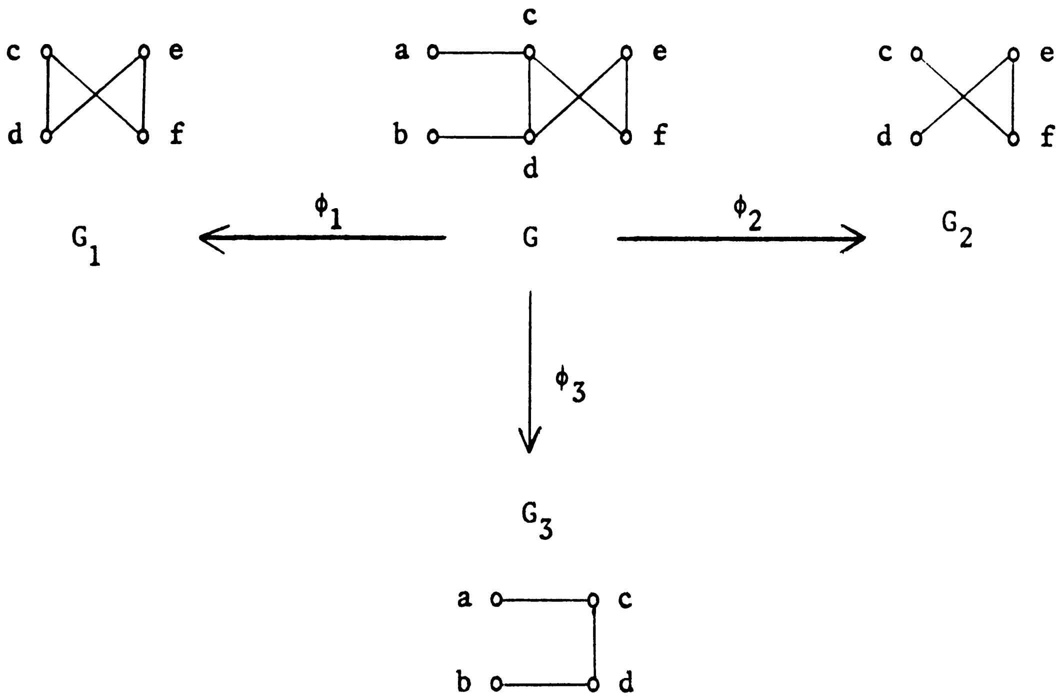
[1] Subgraphs and full subgraphs correspond in Ore's terminology to subgraphs and section graphs, and in Berge's terminology to partial subgraphs and subgraphs, respectively.

$\psi : G' \longrightarrow G''$ are homomorphisms, then the product mapping $\phi\psi$, for which

$a\phi\psi = (a\phi)\psi$ and $(a,b)\phi\psi = ((a,b)\phi)\psi = (a\phi,b\phi)\psi = ((a\phi)\psi,(b\phi)\psi) = (a\phi\psi,b\phi\psi)$, is a homomorphism of G into G''. It follows also from (1) that if $G\phi = \langle V\phi,\rho\phi\rangle$ is the <u>image</u> of G under the homomorphism ϕ then $G\phi$ is a subgraph of G', and if ϕ is a full homomorphism then $G\phi$ is a full subgraph of G'. If, in addition, ϕ maps V <u>onto</u> V' then ϕ is said to be a (full) homomorphism of G onto G'. Note that if ϕ is a full homomorphism of G onto G' then $\rho\phi = \rho'$.

In the case $G' \subseteq G$ and ϕ is a (full) homomorphism of G into G', ϕ is said to be a (full) <u>endomorphism</u> of G; when $G\phi = G' = G$, ϕ is an <u>automorphism</u> of G. A full homomorphism ϕ of a graph G onto a graph G' is an <u>isomorphism</u> when ϕ is 1-1 from V onto V', in which case G and G' are said to be <u>isomorphic</u>, written $G \simeq G'$.

Figure 1 illustrates some of these definitions. The homomorphisms ϕ_1 and ϕ_3 are full endomorphisms of G onto the full subgraphs G_1 and G_3, respectively. The endomorphism ϕ_2 is not full, since G_2 is not a full subgraph of G. Note that G_2 and G_3 are isomorphic. Note also that



ϕ_1:	ϕ_2:	ϕ_3:
$a,d \longrightarrow d$	$a \longrightarrow d$	$a \longrightarrow a$
$b,c \longrightarrow c$	$b \longrightarrow c$	$b \longrightarrow b$
$e \longrightarrow e$	$c,e \longrightarrow e$	$c,e \longrightarrow c$
$f \longrightarrow f$	$d,f \longrightarrow f$	$d,f \longrightarrow d$

FIG. 1.

$G\phi_1\phi_1 = G\phi_1$ and $G\phi_3\phi_3 = G\phi_3$, but $G\phi_2\phi_2 \neq G\phi_2$.

1.2. Properties Preserved under Homomorphisms.

H. J. Keisler has written a paper [11] which is concerned essentially with a logical formulation of the properties of a relational system which are preserved under two classes of homomorphisms, which he calls "strong homomorphisms"[1] and "retracts". Using our terminology for graphs, these "strong homomorphisms" are full homomorphisms for which the function is onto, and "retracts" are endomorphisms which act as the identity mapping on the image, i.e., $G\phi = G'$ is a subgraph of G and $b\phi = b$ for every $b \varepsilon V'$. It follows from this definition that "retracts" are those endomorphisms which are idempotent, i.e., for which $G\phi^2 = G\phi$. Two theorems are stated in the paper, one for each class of mappings, to the effect that a property of a relational system is preserved if and only if it is logically equivalent to a sentence of a particular form in the first order predicate calculus with identity. Although these results obviously can be applied to homomorphisms of graphs, they have not been as yet.

1.3. Complete Homomorphisms.

An ordinary graph is a graph whose adjacency relation is symmetric and irreflexive; stated in other words, a graph is ordinary if it is undirected and has no loops or multi-lines. In this section, and in Sections 1.4, 1.5,

[1] Later in the paper, Section 1.9, we will define a different class of mappings and call them strong homomorphisms.

and 1.6, we will consider only full homomorphisms of ordinary graphs onto ordinary graphs; these mappings ϕ must satisfy the condition

(3) $a\phi = b\phi \Longrightarrow (a,b)\notin\rho$,

and we will call them <u>ordinary</u> homomorphisms.[1] We will, however, omit the word "ordinary" when referring to graphs and homomorphisms in these four sections.

A homomorphism of a graph G onto a graph G' is called <u>complete</u> iff $G' = K_n$, for some n. A complete homomorphism of a graph G onto K_n is said to be of <u>order</u> n. A <u>coloring</u> of a graph G is an assignment of colors to the points of G such that no two adjacent points are assigned the same color. More formally, an <u>n-coloring</u> of a graph $G = \langle V,\rho\rangle$ is a function η from V onto N, where N is the set of natural numbers (colors) $\{1, 2, \ldots, n\}$, satisfying the condition that $(a,b)\varepsilon\rho \Longrightarrow a\eta \neq b\eta$. An n-coloring η is <u>complete</u> iff for every i,j, $i \neq j$, $(\exists a)(\exists b)[a\eta = i \wedge b\eta = j \wedge (a,b)\varepsilon\rho]$. The <u>chromatic number</u> of a graph G is the smallest number n such that G has an n-coloring.

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[1]Full homomorphisms satisfying this condition are called <u>independent</u> by Ore, cf. Section 2.2.

Theorem 1.3.1. If $\eta : V \longrightarrow N$ *is a (complete) n-coloring of graph* $G = \langle V, \rho \rangle$, *then* η *determines a (complete) homomorphism* ϕ_η *of* G *onto* K_n, *and conversely.*

Proof. *If* $a\eta = i$, *let* $a\phi_\eta = k_i$. ||

The next eight corollaries follow more or less directly from this theorem; the proofs of these corollaries are quite simple and are omitted.

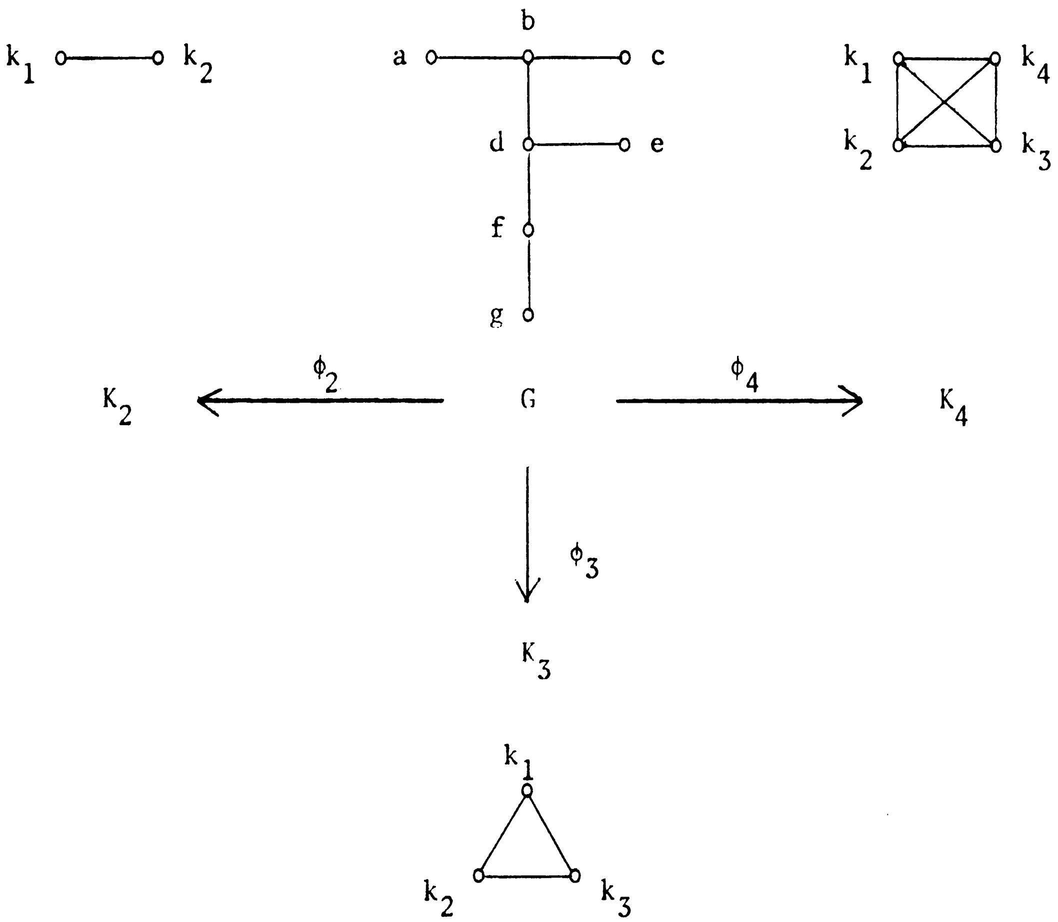
Corollary 1.3.2. (Ore) If the chromatic number of a graph G is n, then G has a complete homomorphism of order n.

While this corollary defines the order of a particular complete homomorphism that every graph must have, a given graph may have several other complete homomorphisms of different orders. Figure 2 illustrates this possibility; the graph G has complete homomorphisms of orders 2, 3, and 4.

Corollary 1.3.3. (Prins) The smallest number s for which a given graph G has a complete homomorphism[1] *of order s is the chromatic number of G.*

Corollary 1.3.4. If ϕ *is a homomorphism of a graph G, then* $\chi(G) \leq \chi(G\phi)$.

[1] In a manuscript, "Complete Contractions of Graphs," dated February 1963, Prins refers to these mappings as complete contractions.



ϕ_2: $a, c, d, g \longrightarrow k_1$; $b, e, f \longrightarrow k_2$

ϕ_3: $a, c, d \longrightarrow k_1$; $b, e, g \longrightarrow k_2$; $f \longrightarrow k_3$

ϕ_4: $a, e, g \longrightarrow k_1$; $b \longrightarrow k_2$; $d \longrightarrow k_3$; $c, f \longrightarrow k_4$

FIG. 2.

Corollary 1.3.5. If G is any graph and S is any independent set of points of G, then $\chi(G) \leq 1 + \chi(G-S)$.

Corollary 1.3.6. (Dirac)[1] *If G is a critical graph and S is*

[1] c.f. Dirac [4].

any independent set of points of G, then $\chi(G) = 1 + \chi(G-S)$.

Corollary 1.3.7. If G is any graph having n points and point independence number α_0, *then* $\chi(G) \leq 1 + n - \alpha_0$.

It is interesting to note in passing the relationship of this result to the result stated in Berge [2, p. 37] and Ore [16, p. 225] which asserts that

$$n/\alpha_0 \leq \chi(G).$$

A graph G is <u>simple</u> iff every homomorphic image of G is isomorphic to G.

Corollary 1.3.8. A graph G is simple[1] *iff* $G = K_n$, *for some n.*

A homomorphism ε of a graph G is <u>elementary</u> iff for two points $a,b \in V(G)$, $a\varepsilon = b\varepsilon$ and ε is 1-1 on $V(G) - \{a,b\}$.

Lemma 1.3.9. Every ordinary homomorphism of a graph G onto a graph G' can be expressed as a product of elementary homomorphisms.

As an immediate consequence of Corollary 1.3.5. we have

Corollary 1.3.10. The chromatic number of an elementary homo-

[1]Simple with respect to ordinary homomorphisms.

morphic image of G is at most one greater than the chromatic number of G.

And because of *Corollary 1.3.6* we have

Corollary 1.3.11. Every elementary homomorphic image of a critical graph G has the same chromatic number as G.

Theorem 1.3.12. (Prins) If the chromatic number of a graph G is s, and G has a complete homomorphism of order t, then for all n, $s \leq n < t$, G has a complete homomorphism of order n.

Proof. The theorem follows directly from *Lemma 1.3.9* and *Corollary 1.3.10.* Consider any homomorphism ϕ which maps G onto K_t. By *Lemma 1.3.9,* ϕ can be expressed as a product $\varepsilon_1\varepsilon_2 \ldots \varepsilon_m$ of elementary homomorphisms. Let $G_1 = G\varepsilon_1$, $G_2 = G_1\varepsilon_2$, ..., $G_m = G_{m-1}\varepsilon_m = K_t$. By *Corollary 1.3.10* we know that the chromatic number of G_i is at most one greater than the chromatic number of G_{i-1}. It follows therefore that if $\chi(G) = s$ and $s \leq n < t$, then there exists at least one G_i whose chromatic number is n. By *Corollary 1.3.2,* this G_i has a complete homomorphism ϕ_i onto K_n. Hence G has a complete homomorphism $\varepsilon_1\varepsilon_2 \ldots \varepsilon_i\phi_i$ onto K_n. ||

Prins, in considering complete homomorphisms, made an interesting distinction between two types. <u>Type-1</u> complete homomorphisms of a graph G are obtained in the following manner. Take any maximal independent set of points V_1 in V(G). Remove the points in V_1 from G and take a second maximal independent set of points V_2 in $V(G-V_1)$. Iterate this process until V(G) has been completely depleted, and has been partitioned into sets $V_1, V_2, \ldots, V_m$. This partition defines a complete homomorphism ϕ of G onto K_m; $a\phi = k_i$ iff $a \epsilon V_i$. If a complete homomorphism is not of type-1 it is said to be of <u>type-2</u>.

Prins has shown that the homomorphisms stated in *Corollary 1.3.2* and *Theorem 1.3.12* can be required to be of type-1, i.e., if the chromatic number of G is s and if G has a type-1 complete homomorphism of order t, then for every n, $s \leq n < t$, G has a type-1 complete homomorphism of order n. Prins has attempted to characterize the type-2 complete homomorphisms, but has not yet succeeded.

While *Corollary 1.3.3* establishes the minimum order s, it remains an open question to determine a decent description of the maximum order t of all complete homomorphisms of a given graph G. The following result, which extends *Corollary 1.3.7*, establishes a bound for this number t.

Theorem 1.3.13. Let G be a graph having n points and point independence number α_0, and let t be the largest order of all complete homomorphisms of G. Then $\chi(G) \leq t \leq n + 1 - \alpha_0$.

Proof. Let ϕ be a complete homomorphism of G onto K_t, where t is maximum, and consider the partition of V(G) determined by the sets $\phi^{-1}(k_i)$, i = 1, 2, ..., t. Let V_1 be any maximal independent set of G containing α_0 points, and consider where among the sets $\phi^{-1}(k_i)$ the points in V_1 lie. Three possibilities exists; $\phi^{-1}(k_i)$ contains no points of V_1, $\phi^{-1}(k_i)$ contains some points of V_1 and some points of $V-V_1$, or $\phi^{-1}(k_i)$ contains only points of V_1. Note, however, that at most one set $\phi^{-1}(k_i)$ can contain only members of V_1. It follows therefore that we can pick representatives from at least t-1 of these sets which are not members of V_1, and therefore the total number of points in G minus these t-1 representatives leaves at least the α_0 points of V_1 remaining. Thus

$$n - (t-1) \geq \alpha_0$$

$$n + 1 - \alpha_0 \geq t\ .$$

Corollary 1.3.14. $t \leq \beta_0(G) + 1$, *where* $\beta_0(G)$ *is the* *__point__* *__cover__* *number of G.*

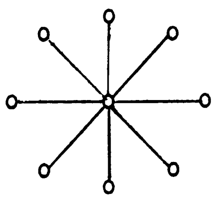
Proof. This follows since $n = \alpha_0(G) + \beta_0(G)$.[1] ||

Perhaps the most natural bound for the largest order t of all complete homomorphisms of a given graph G, having q lines, is the largest integer r such that

$$\binom{r}{2} \leq q.$$

As one might expect, however, there are cases in which each of the two bounds, r and $n+1-\alpha_0$, gives a better estimate than the other, as the examples in Figure 3 illustrate.

G_1:



G_2:

FIG. 3.

In G_1, $n = 9$, $q = 8$, and $\alpha_0 = 8$, hence $r = 4$, $n+1-\alpha_0 = 2$, while $t = 2$.

[1] cf. Gallai [7] .

If G_2, $n = 10$, $q = 9$, and $\alpha_o = 5$, hence $r = 4$, $n+1-\alpha_o = 6$, while $t = 4$.

1.4. Type-n Homomorphisms.

An <u>n-basis</u> of a graph $G = \langle V,\rho\rangle$ is a subset of points, $V_n \subseteq V$, satisfying

$$\text{(i)} \quad a,b\varepsilon V_n \Longrightarrow d(a,b) > n;$$

$$\text{(ii)} \quad c\not\varepsilon V_n \Longrightarrow d(c,V_n) \leq n.$$

A Prins' type-1 complete homomorphism (cf. Section 1.3) can be described in terms of an iterated subtraction from a graph of maximal independent sets of points. Since from the definition of an n-basis it follows that maximal independent sets of points are 1-bases, the suggestion naturally arises of considering complete homomorphisms of <u>type-n</u>, mappings which can be described in terms of an iterated subtraction from a graph of n-bases.

Even though the process of obtaining a series of n-bases from a given graph does not always define a complete homomorphism, the fact that some complete homomorphisms which are not of Prins' type-1 can be expressed as complete homomorphisms of type-n seems to justify the following definition.

A full homomorphism ϕ of a graph $G = \langle V,\rho\rangle$ onto a graph $G' = \langle V',\rho'\rangle$ is of <u>type-n</u> iff there exists an ordering of the sets $\phi^{-1}(a')$, for $a' \varepsilon V'$, say $V_1, V_2, \ldots, V_k$, such that

(i) V_1 is an n-basis of G, and

(ii) for $i \geq 2$, V_i is an n-basis of $G - \bigcup_{j=1}^{i-1} V_j$,

and for no m, $0 < m < n$, is ϕ of type-m.

Note that not every complete homomorphism is of type-n, for some n. An example of this is the complete homomorphism given in Figure 4; neither of the sets {a,d}, {c,e} or {b,f} is an n-basis of G, for any n.

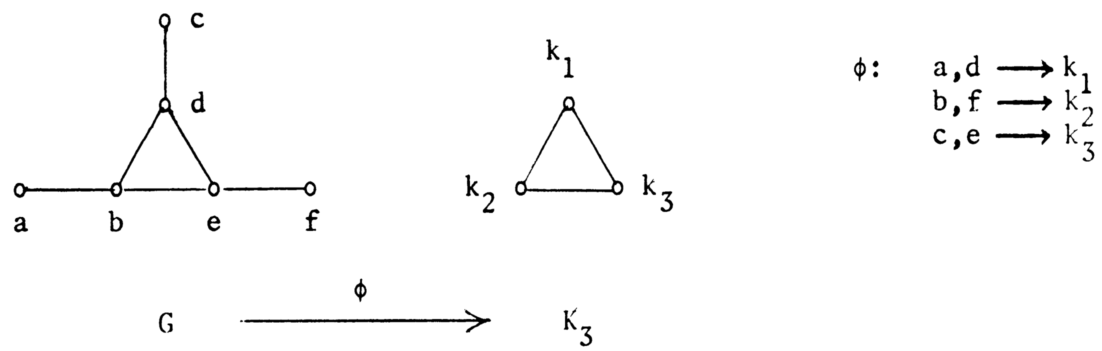
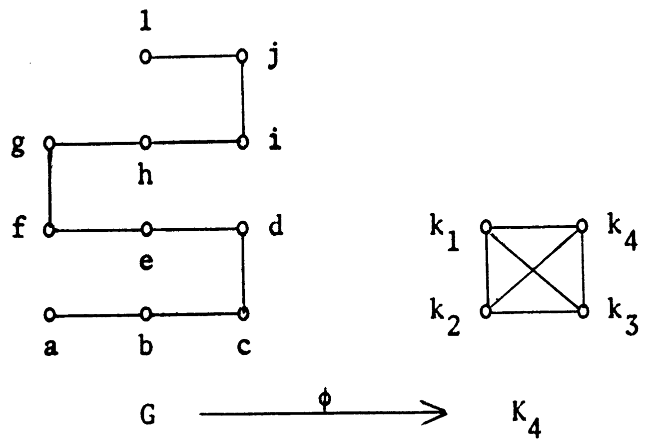


FIG. 4.

If a complete homomorphism is not of type-n for any n, it is said to be of type-e.

An example of a type-4 complete homomorphism is given in Figure 5; the subset {e,j} is a 4-basis of G; {c,i,l} is a 4-basis of G - {e,j}; {b,d,g} is a 4-basis of G - {e,j} - {c,i,l}; and what remains in G - {e,j} - {c,i,l} - {b,d,g} are the three isolated points a, f, and h.



$$\begin{aligned}\phi:\ & a,f,h \longrightarrow k_1\\ & b,d,g \longrightarrow k_2\\ & c,i,l \longrightarrow k_3\\ & e,j \longrightarrow k_4\end{aligned}$$

FIG. 5.

1.5. Two Invariants of a Graph.

Two interesting invariants of a graph arise out of considerations of the following question: Given a graph G, for what graphs H do there exist homomorphisms ϕ such that $H\phi = G$? A partial answer to this question is given by the following three propositions.

Proposition 1.5.1. For every connected graph G there exists a tree T and a homomorphism ϕ such that $T\phi = G$.

Proposition 1.5.2. For every connected graph G there exists a chain C and a homomorphism ϕ such that $C\phi = G$.

Proposition 1.5.3. For every connected graph G there exists a cycle P and a homomorphism ϕ such that $P\phi = G$.

It is natural in view of these last two propositions to make the following definitions.

The chain length of a connected graph G, ch(G), is the smallest number k for which there exists a chain C of length k and a homomorphism ϕ such that $C\phi = G$. Alternatively, ch(G) is the length of the shortest path which contains all the lines of G.'

The cycle length of a connected graph G, cy(G), is the smallest number k for which there exists a cycle P of length k and a homomorphism ϕ such that $P\phi = G$.

Proposition 1.5.4. For every connected graph G,

$ch(G) \leq cy(G) \leq 2ch(G)$.

Proof. Clearly $ch(G) \leq cy(G)$, for if $a_1, a_2, \ldots, a_k, a_1$ is any cycle P of length k which maps onto G under a homomorphism ϕ, then a chain $a'_1, a'_2, \ldots, a'_k$, b can be defined of the same length k which maps onto G under the homomorphism ϕ' for which $a'_i\phi' = a_i\phi$ and $b\phi' = a_1\phi$.

To show that $y(G) \leq 2ch(G)$, let C_k with points $c_1, c_2, \ldots, c_k, c_{k+1}$, be the shortest chain which maps under a homomorphism ϕ onto G. Then a cycle $c'_1, c'_2, \ldots, c'_{k+1}, d_k, d_{k-1}, \ldots, d_2, c'_1$ of length 2k can be defined which maps under the homomorphism ϕ' onto G, where $c'_i\phi' = c_i\phi$ and $d_i\phi' = c_i\phi$.||

The bounds given in *Proposition 1.5.4* are achieved in the following two cases; $ch(P_n) = cy(P_n) = n$, and $cy(C_n) = 2ch(C_n) = 2n$.

Proposition 1.5.5. For every connected graph G with q lines,

(i) $q \leq ch(G) \leq 2q-1$.

(ii) $q \leq cy(G) \leq 2q$.

Proof. (i) It is obvious that $q \leq ch(G)$, since the shortest chain which maps onto G must have at least as many lines as G.

That ch(G) $\leq$ 2q is an immediate consequence of the following theorem, quoted from Ore [16, p. 41]:

"*Theorem 3.1.4. In a finite connected graph it is always possible to construct a cyclic directed path passing through each edge once and only once in each direction.*"

(ii) The same arguments hold for cy(G).

Proposition 1.5.6. If G is a connected graph having q lines and in which every point has even degree, then ch(G) = cy(G) = q.

Proof. G is an Euler graph, i.e., a graph in which it is possible to find a cyclic path of lines such that each line in the graph appears once and only once.

Theorem 1.5.7. If G is a connected graph having q lines and exactly two points a,b of odd degree, then

$$(i) \quad ch(G) = q;$$

$$(ii) \quad cy(G) = q + d(a,b).$$

Proof. (i) See Ore [16, p. 40]. Such a graph has at least one path which passes through all the lines of G once and only once, which begins with point b and ends with point a. Hence, $ch(G) = q$.

(ii) Clearly, $cy(G) \leq q + d(a,b)$. A path corresponding to the one referred to in (i) above can obviously be extended to a cycle by an additional path, of length $d(a,b)$, from point a back to point b. In order to show that $cy(G) \nless q + d(a,b)$, let P_k with points $p_1, p_2, \ldots, p_k$ be the smallest cycle which maps onto G under the homomorphism ϕ_k, and let P_n be a cycle of length $n = q + d(a,b)$ which maps onto G under the homomorphism ϕ_n. Obviously, $k \leq n$. Consider the <u>multi-graph</u> G_k generated by P_k and the homomorphism ϕ_k, i.e., $V(G_k) = V(G)$, and there are as many lines between two points c,d in G_k as there are pairs of adjacent points p_i, p_j in P_k such that either $p_i\phi_k = c$ and $p_j\phi_k = d$, or $p_i\phi_k = d$ and $p_j\phi_k = c$. It follows that G_k has k lines. The degree of a point c in G_k equals the sum of the degrees of the points p_i for which $p_i\phi_k = c$. Since every point p_i has even degree in P_k, every point c has even degree in G_k.

Consider also the multi-graph G_n generated by P_n and the homomorphism ϕ_n; G_n has n lines. Clearly for every line (c,d) in G there is a line (c,d) in G_k and a line (c,d) in G_n. Hence, let us remove for each of the q lines in G a corresponding line in both G_k and G_n. Denote the resulting graphs by G_k-G and G_n-G, which have k-q and n-q = d(a,b) lines, respectively, where k-q ≤ d(a,b). Since the points a,b have even degree in G_k and odd degree in G, they have odd degree in G_k-G. And since they are the only two points of odd degree in G, they are the only two points of odd degree in G_k-G. It follows, therefore, by *Theorem 2.2.2*, Ore [16, p. 24], that since a,b are the only points of odd degree in G_k-G, a and b are connected in G_k-G. Hence G_k-G has at least d(a,b) lines, i.e., k-q ≥ d(a,b). Hence k-q = d(a,b), i.e., k = q + d(a,b).

Theorem 1.5.8. If G is a connected graph having q lines and 2n points of odd degree, n ≥ 2, and diameter δ, then

$$\text{(i)} \quad q + n-1 \le ch(G) \le q + (n-1)\delta;$$

$$\text{(ii)} \quad q + n \le cy(G) \le q + n\delta.$$

Proof. (i) That ch(G) ≤ q + (n-1)δ follows by virtue of *Theorem 3.1.2,* Ore [16, p. 40]. Such a graph G has a set of n line disjoint paths which contain all the lines of G once and only once. Let W_1, W_2, ..., W_n represent these n paths. A path which starts with the first point of W_1, which ends with the last point of W_n, and which connects the last point of W_i with the first point of W_{i+1} by a path of the length less than or equal to the diameter δ can always be constructed in G. Such a path clearly has length less than or equal to $q + (n-1)\delta$. It is a simple matter to construct a chain of the same length which maps onto this path. In order to demonstrate that $q + n-1 \leq ch(G)$, let $ch(G) = k$, let C_k with points $c_1, c_2, \ldots, c_{k+1}$ be a chain of length k, and let ϕ_k be a homomorphism which maps C_k onto G. Let G_k be the multigraph generated by C_k and the homomorphism ϕ_k, as in the proof of *Theorem 1.5.7.*

Since only points c_1 and c_{k+1} have odd degrees in C_k, at most two points have odd degrees in G_k. Obviously, G_k can be obtained from G by inserting additional lines between various pairs of adjacent points in V(G). The fewest number of additional lines which could possibly convert G, with 2n points of odd degree, into the multi-graph G_k, with at most two points of odd degree, is $(2n-2)/2 = n - 1$. This can be accomplished (if possible) by leaving two points of odd degree in G untouched by the addition of new lines, and by pairing-off the remaining 2n-2 points of odd degree into adjacent pairs and adding a new line between each pair, i.e., by adding n-1 new lines.

(ii) The proof for cy(G) is very similar to that for ch(G) and is omitted.||

It is interesting to compare the upper bounds given in *Proposition 1.5.5,* $ch(G) \leq 2q - 1$, and in *Theorem 1.5.8,* $ch(G) \leq q + (n-1)\delta$. There are cases in which each of the two bounds gives a better estimate than the other, as the examples in Figure 6 illustrate.

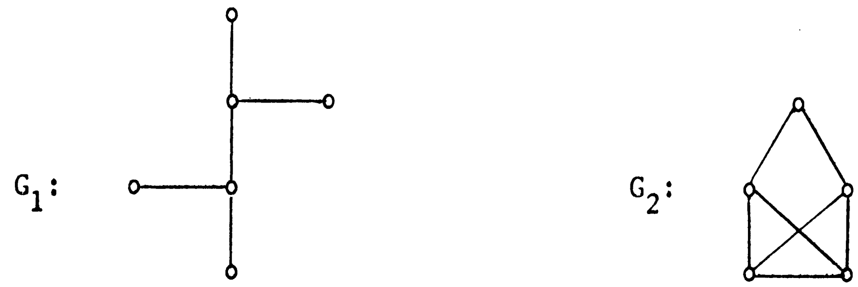


FIG. 6.

In G_1, $q = 5$, $\delta = 3$, and $n = 3$, i.e., G_1 has $2n = 6$ points of odd degree; hence $2q - 1 = 9$, while $q + (n-1)\delta = 11$. Note that $ch(G_1) = 7$. In G_2, $q = 7$, $\delta = 2$, and $n = 2$; hence $2q - 1 = 13$, while $q + (n-1)\delta = 9$. Note that $ch(G_2) = 8$. The examples in Figure 7 illustrate that the bounds given in *Theorem 1.5.8* can be achieved.

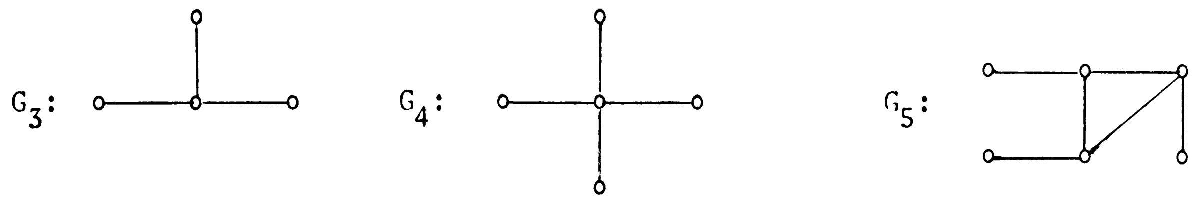


FIG. 7.

In G_3, $q = 3$, $\delta = 2$, and $n = 2$; hence $q + n-1 = 4 = ch(G_3)$. In G_4, $q = 4$, $\delta = 2$, and $n = 2$; hence $q + (n-1)\delta = 6 = ch(G_4)$, and $q + n\delta = 8 = cy(G_4)$. In G_5, $q = 6$, $\delta = 3$, and $n = 3$; hence $q + n = 9 = cy(G_5)$.

Corollary 1.5.9. (i) $ch(K_{2n}) = \binom{2n}{2} + n-1;$

(ii) $ch(K_{2n+1}) = \binom{2n+1}{2} = cy(K_{2n+1});$

(iii) $cy(K_{2n}) = \binom{2n}{2} + n.$

Proof. (i) and (iii) follow at once from *Theorem 1.5.8* and the observation that $\delta(K_n) = 1$, for all $n \geq 2$.

(ii) follows from *Proposition 1.5.6* since every K_{2n+1} is an Euler graph.

Lemma 1.5.10. Let ϕ be a homomorphism of a connected graph G onto a graph G'. Then (i) $ch(G) \geq ch(G')$;

(ii) $cy(G) \geq cy(G')$.

Proof. (i) Let ch(G) = k, and ϕ_k be a homomorphism which maps C_k onto G. Obviously, the product mapping $\phi_k\phi$ is a mapping of C_k onto G':

$$C_k \xrightarrow{\phi_k} G \xrightarrow{\phi} G'. \quad \text{Hence } ch(G') \le k = ch(G).$$

(ii) A similar argument also holds for cy(G).

1.6. Endomorphism-free Graphs.

A graph G is endomorphism-free iff every endomorphic image of G is isomorphic to G.

Theorem 1.6.1. If ϕ is an endomorphism of a graph G, then the chromatic number of $G\phi$ equals the chromatic number of G.

Proof. Obviously, $\chi(G\phi) \le \chi(G)$. But by *Corollary 1.3.4*, $\chi(G) \le \chi(G\phi)$.

Corollary 1.6.2. Every critical graph is endomorphism-free.

Inspection of all the graphs with six or fewer points shows that the converse to this corollary is true for these graphs.

Proposition 1.6.3. If a graph has at most six points and is endomorphism-free, then it is critical.

The graph in Figure 8, which is endomorphism-free and not critical, shows that the converse does not hold for all graphs. But Figure 8 has eight points, leaving the problem of whether there exists a seven point counter-example.

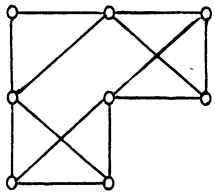


FIG. 8.

1.7. Homomorphisms for Functional Digraphs.

A functional digraph $G = \langle V, \rho \rangle$ is a graph whose adjacency relation ρ is a function from V into V; in other words, the adjacency relation is not necessarily symmetric and the out degree of every point equals one. In this section we will consider only full homomorphisms of functional digraphs onto functional digraphs; these mappings ϕ must satisfy the condition

(4) $a\phi = b\phi \Longrightarrow \rho(a)\phi = \rho(b)\phi.$

A homomorphism ϕ is prime iff in every expression of ϕ as a product $\phi_1\phi_2$, either ϕ_1 or ϕ_2 is an isomorphism.

Yoeli and Ginzburg [22] have given a complete characterization of this class of homomorphisms[1], which states that every such homomorphism (other than an isomorphism) can be expressed as a product over the following three types of prime[2] homomorphisms:

(i) ϕ_α is a homomorphism which maps a cycle P of length k onto a cycle P' of length k/p, where p is a prime divisor of k; for $a \varepsilon V(P)$, $(\rho^i(a))\phi_\alpha = (\rho^j(a))\phi_\alpha$ iff $i \equiv j \pmod{k/p}$;

(ii) ϕ_β is a homomorphism which identifies exactly two points a,b, $a\phi_\beta = b\phi_\beta$, for which $\rho(a) = \rho(b)$;

(iii) ϕ_γ is a homomorphism which identifies two cycles P and P' of the same length k, where for arbitrary $a \varepsilon P$, $a' \varepsilon P'$, $(\rho^n(a))\phi_\gamma = (\rho^n(a'))\phi_\gamma$, $n = 0, 1, 2, \ldots, k-1$.

Figures 9 - 11, with mappings marked ϕ_α, ϕ_β, and ϕ_γ, illustrate respectively the three types of prime homomorphisms.

---

[1] It is interesting to note that Yoeli and Ginzburg have incorporated the two conditions, (1) and (2), for a full homomorphism, into the one condition (5) $\rho' = \phi^{-1}\rho\phi$. Conditions (1) and (2) and condition (5) are clearly equivalent.

[2] Yoeli and Ginzburg call these elementary homomorphisms.

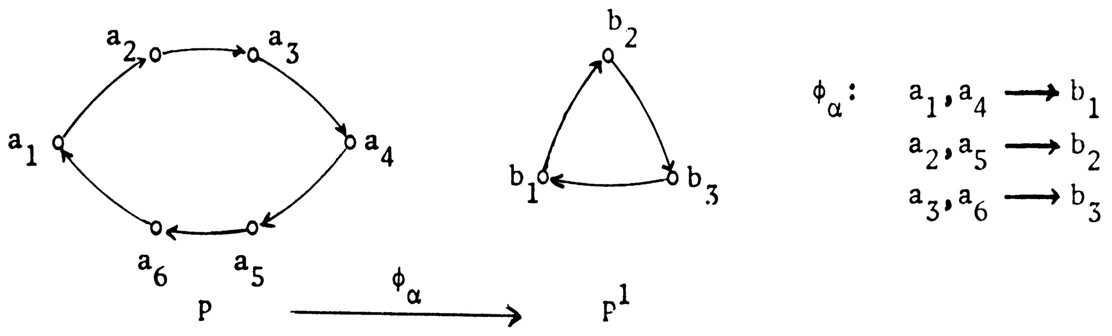


FIG. 9.

Figure 9 illustrates as well that *Lemma 1.3.9* does not hold for homomorphisms of functional digraphs; i.e., the homomorphism ϕ_α in Figure 9 cannot be represented as a product of elementary homomorphisms.

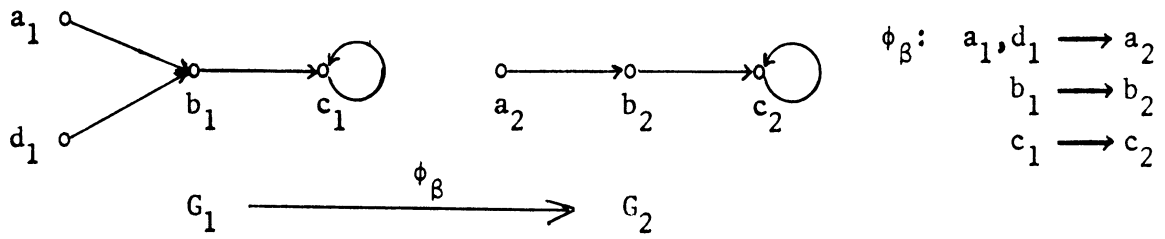


FIG. 10.

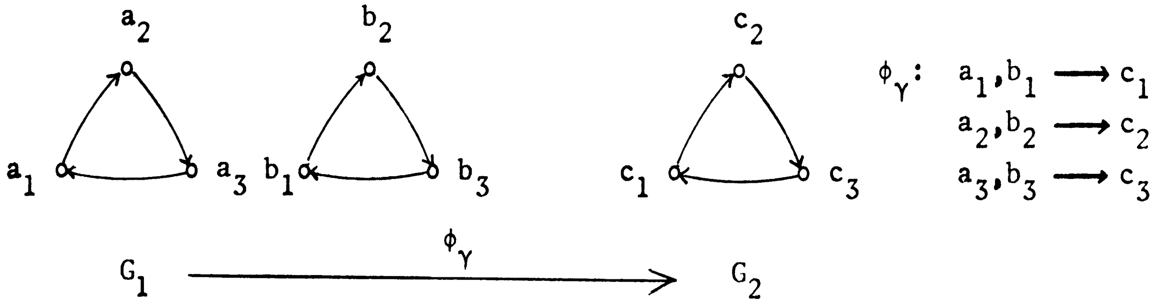


FIG. 11.

Yoeli and Ginzburg also show that provided G is a connected functional digraph, every series of prime homomorphisms of G onto G' has the same length.

1.8. Pathwise Homomorphisms.

An ordinary homomorphism ϕ of a graph $G = \langle V,\rho\rangle$ onto a graph $G' = \langle V',\rho'\rangle$ is a <u>pathwise homomorphism</u> iff it satisfies the following condition

(6) for every path $a'_{i_1}, a'_{i_2}, \ldots, a'_{i_n}$ in G' there exists a path $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ in G such that $a_{i_j}\phi = a'_{i_j}$, $j = 1, 2, \ldots, n$.

Note the similarity of condition (6) to condition (2), when the latter is rephrased as follows:

(2') for every path a',b' of <u>length one</u> in G' there exists a path a,b in G such that $a\phi = a'$ and $b\phi = b'$.

Pathwise homomorphisms were first defined by Robert McNaughton [14] for state transition graphs. The only result he obtained about these mappings was that the "order" of a pathwise homomorphic image G' of a state transition graph G cannot be greater than the "order" of G. The "order" of a state transition graph G will not be defined here; suffice it to say that it is a measure of the cyclic complexity of G. A notion similar to "order", which is frequently mentioned in graph theory, is what Ore [16, p. 67] calls circuit rank, $\gamma(G)$, the number of independent cycles. For any graph G with p points, q lines, and k connected components, $\gamma(G) = q - p + k$. It is natural therefore to state the following

Theorem 1.8.1. If ϕ is a pathwise homomorphism of a graph $G = \langle V,\rho\rangle$ onto a graph $G' = \langle V',\rho'\rangle$, then $\gamma(G) \geq \gamma(G')$.

Proof. We proceed by induction on the circuit rank of G.

Lemma 1.8.2. If ϕ is a pathwise homomorphism of a graph $G = \langle V,\rho\rangle$ onto a graph $G' = \langle V',\rho'\rangle$ and $\gamma(G) = 0$, then $\gamma(G') = 0$ also.

Proof. Suppose $\gamma(G') \geq 1$, i.e., G' contains at least one cycle P_k of length $k \geq 3$. Suppose G has p points. Certainly there exists a path $a'_{i_1}, a'_{i_2}, \ldots, a'_{i_{p+1}}$ (perhaps several times) around P_k of length p, with p+1 points. Since ϕ is a pathwise homomorphism there must exist a path $a_{i_1}, a_{i_2}, \ldots, a_{i_{p+1}}$ in G such that $a_{i_j}\phi = a'_{i_j}$. Any such path in G has p+1 points and therefore must contain at least one point which occurs twice.

Such a path therefore must have a sub-path of the form $\omega = a, a_{i_j}, \ldots, a_{i_k}, a.$ But since $\gamma(G) = 0$, G has no cycles and every path in G of the form ω must be such that $a_{i_j} = a_{i_k}$, and similarly $a_{i_{j+1}} = a_{i_{k-1}}$, etc., until finally the sub-path of the form ω must contain a sub-path either of the form $a_i a_i$ or $a_i a_j a_i$. But it cannot contain a sub-path of the form $a_i a_i$ since every two consecutive points of the corresponding path in G' are distinct. Also, it cannot contain a sub-path of the form $a_i a_j a_i$ since every three consecutive points of the corresponding path in G' are mutually distinct. Hence no path can exist in G which maps onto the path of length p in G' around the cycle P_k. Hence ϕ cannot be a pathwise homomorphism, contrary to hypothesis. The graph G' must therefore contain no cycles, hence $\gamma(G') = 0$.

Continuing with the proof of *Theorem 1.8.1*, assume it is true for all G, with $\gamma(G) = n$. We will show that it is also true for G with $\gamma(G) = n + 1$. Consider the sets of lines of G defined by $\phi^{-1}((a',b'))$, for all $(a',b') \epsilon \rho'$. Since $\gamma(G) = n + 1 \geq 1$, there must be a set $\phi^{-1}((a',b'))$ such that

$$\gamma(G - \phi^{-1}((a',b'))) < \gamma(G).$$

It can be shown then that ϕ is a pathwise homomorphism of $G - \phi^{-1}((a',b'))$

onto $G' - (a',b')$, and hence, by hypothesis

$$\gamma(G - \phi^{-1}((a',b'))) \geq \gamma(G' - (a',b')).$$

But since $\gamma(G) \geq \gamma(G - \phi^{-1}((a',b')) + 1$ and $\gamma(G' - (a',b')) + 1 \geq \gamma(G')$, it follows that $\gamma(G) \geq \gamma(G')$. $\|$

It is interesting to note that no analog of *Corollary 1.3.2* holds for pathwise homomorphisms. For example, the pentagon P_5, has <u>no</u> nontrivial pathwise homomorphisms; $\chi(P_5) = 3$, yet P_5 does not map under any pathwise homomorphism onto K_3. It remains an open question to determine those graphs which are simple with respect to pathwise homomorphisms.

One might also note in passing that *Lemma 1.3.9* does not hold for pathwise homomorphisms. The mapping ϕ_α in Figure 8, is a pathwise homomorphism which cannot be expressed as a product of elementary pathwise homomorphisms.

1.9. Strong Homomorphisms.

Let g be the characteristic function of the adjacency relation of a graph $G = \langle V,\rho\rangle$, i.e., $g(a,b) = 1$ iff $(a,b)\epsilon\rho$ and $g(a,b) = 0$ iff $(a,b)\notin\rho$.

A full homomorphism ϕ of a graph G onto a graph G' is a strong homomorphism iff ϕ satisfies the condition

(7) $g(a,b) = g'(a\phi,b\phi)$.

Condition (7) can also be expressed as

(7') $(a,b)\varepsilon\rho \Longrightarrow (a\phi,b\phi)\varepsilon\rho'$

$(a,b)\not\varepsilon\rho \Longrightarrow (a\phi,b\phi)\not\varepsilon\rho'$.

In words, condition (7) requires not only that two adjacent points in G map to two adjacent points in G', but also that two non-adjacent points in G map to two non-adjacent points in G'.

Theorem 1.9.1. If ϕ is a strong homomorphism of a graph $G = \langle V,\rho\rangle$ onto a graph $G' = \langle V',\rho'\rangle$, then there exists a subgraph H of G such that $H \simeq G'$.

Proof. Consider the inverse-image sets $\phi^{-1}(a')$, for $a'\varepsilon V'$. Pick, arbitrarily, a set of representatives, V_H, one point from each of these sets, and consider the full subgraph $H = \langle V_H,\rho_H\rangle$ of G generated by this set of representatives. The strong homomorphism ϕ when restricted to H is an isomorphism of H onto G'.

Corollary 1.9.2. If ϕ is a strong ordinary homomorphism of a graph G onto a graph G', then the chromatic number of G <u>*equals*</u> *the chromatic number of G'.*

Strong homomorphisms were first studied by Culik [3], who gave the following characterization of those graphs which have no non-trivial strong homomorphisms.

Theorem 1.9.3. (Culik) A graph G is simple with respect to strong homomorphisms iff for every two distinct points $a,b \in V(G)$ there exists a point $c \in V(G)$ such that

$$g(a,c) \neq g(b,c) \text{ or } g(c,a) \neq g(c,b).$$

Consider the equivalence relation $\equiv$ defined on V(G) by

$$a \equiv b \text{ iff } (\forall c)[g(a,c) = g(b,c) \wedge g(c,a) = g(c,b)],$$

or alternatively,

$$a \equiv b \text{ iff } [\rho(a) = \rho(b) \text{ and } \rho^{-1}(a) = \rho^{-1}(b)].$$

Lemma 1.9.4. If ϕ is a strong homomorphism of a graph G onto a graph G', then $a\phi = b\phi \Longrightarrow a \equiv b$.

Lemma 1.9.5. Given a graph $G = \langle V,\rho \rangle$ and two points $a,b \in V$ for which $a \equiv b$, a mapping ϕ which identifies a and b ($a\phi = b\phi$) and which is

1-1 on $V - \{a,b\}$ determines an elementary strong homomorphism of G.

Theorem 1.9.6. Every strong homomorphism of a graph G onto a graph G' can be expressed as a product of elementary strong homomorphisms.

Proof. This is a consequence of *Lemmas 1.9.4* and *1.9.5.*

Theorem 1.9.7. (Culik) Any given graph G has exactly one strong homomorphism ϕ_s onto a graph G_s, such that G_s is simple.[1]

Proof. From *Theorem 1.9.3,* and *Lemmas 1.9.4* and *1.9.5,* a graph G is simple iff the equivalence relation $\equiv$ on V(G) is the identity relation. The strong homomorphism ϕ_s is that mapping for which $a\phi = b\phi$ iff $a \equiv b$.

Corollary 1.9.9. (Sabidussi) Either $G \simeq G_s$ or G contains a subgraph H such that $H \simeq G_s$.[2]

Lemma 1.9.4 and the definition of the equivalence relation $\equiv$ suggest that one define the following two, slightly broader classes of homomorphisms. A full homomorphism ϕ of a graph G onto a graph G' is a <u>right homomorphism</u> iff ϕ satisfies the condition

(8) $a\phi = b\phi \Longrightarrow \rho(a) = \rho(b)$.

---

[1]Simple with respect to strong homomorphisms.

[2]*Theorem 1.9.1* was suggested by this result of Sabidussi [18, 19].

A full homomorphism ϕ of a graph G onto a graph G' is a left homomorphism iff ϕ satisfies the condition

(9) $a\phi = b\phi \Rightarrow \rho^{-1}(a) = \rho^{-1}(b)$.

Right homomorphisms appear to be of special interest in view of their relation to homomorphisms of finite automata, where for a given automaton two states a and b are said to be equivalent iff they satisfy a condition exactly parallel to condition (8).[1] Note that conditions (8) and (9) become identical when the graphs are assumed to be ordinary graphs; hence left, right and strong homomorphisms are all the same when defined for ordinary graphs. One might also note in passing the similarity between right homomorphisms and the prime homomorphisms of type ϕ_β in Section 1.7.

Lemma 1.9.4 and the definition of the equivalence relation $\equiv$ also suggest that one define an even broader class of homomorphisms.

A full homomorphism ϕ of a graph G onto a graph G' is semi-strong[2] iff ϕ satssfies the condition

(10) $a\phi = b\phi \Longrightarrow n(a) \subseteq n(b) \vee n(b) \subseteq n(a)$.

---

[1]cf. [14], [15].

[2]Strong homomorphisms satisfy the stronger condition that $a\phi = b\phi \Rightarrow n(a) = n(b)$.

Theorem 1.9.9. If ϕ is a ***semi-strong*** *homomorphism of a graph G onto a graph G', then there exists a subgraph H of G such that $H \simeq G'$.*

Proof. From each of the sets $\phi^{-1}(a')$, $a' \varepsilon V'$, pick one point which has maximum degree. The subgraph H generated by this set of representatives is isomorphic to G'. Obviously, $(a,b)\varepsilon\rho \Longrightarrow (a',b')\varepsilon\rho'$, where $a\phi = a'$, $b\phi = b'$, and a and b are the representatives chosen from $\phi^{-1}(a')$ and $\phi^{-1}(b')$, respectively.

Also, $(a',b')\varepsilon\rho' \Longrightarrow (a,b)\varepsilon\rho$, for if $(a',b')\varepsilon\rho'$, there must exist $(a'',b'')\varepsilon\rho$ such that $a''\phi = a'$ and $b''\phi = b'$. Since $n(a'') \subseteq n(a)$ and $b''\varepsilon n(a'')$, it follows that $b''\varepsilon n(a)$. By the symmetry of the neighborhood relation, $a\varepsilon n(b'')$. Since $n(b'') \subseteq n(b)$ it follows that $a\varepsilon n(b)$. Hence $(a,b)\varepsilon\rho$.

One can show without much difficulty that the results previously stated for strong homomorphisms can be stated in a very similar way for semi-strong homomorphisms. In particular, *Corollary 1.9.3*, *Theorem 1.9.6*, *Theorem 1.9.7*, and *Corollary 1.9.8* all hold when semi-strong is substituted for stong. In addition, *Theorem 1.9.3* can be restated as:

Theorem 1.9.10. A graph G is simple with respect to semi-strong homomorphisms iff for every two distinct points $a,b \in V(G)$, $n(a) \nsubseteq n(b)$ and $n(b) \nsubseteq n(a)$.

It is natural in view of *Theorem 1.9.9,* to ask if for a full homomorphism ϕ of a graph G onto a graph G' there exists a subgraph H of G such that $H \simeq G'$, is ϕ necessarily a semi-strong homomorphism? The answer is no; as shown by Figure 12.

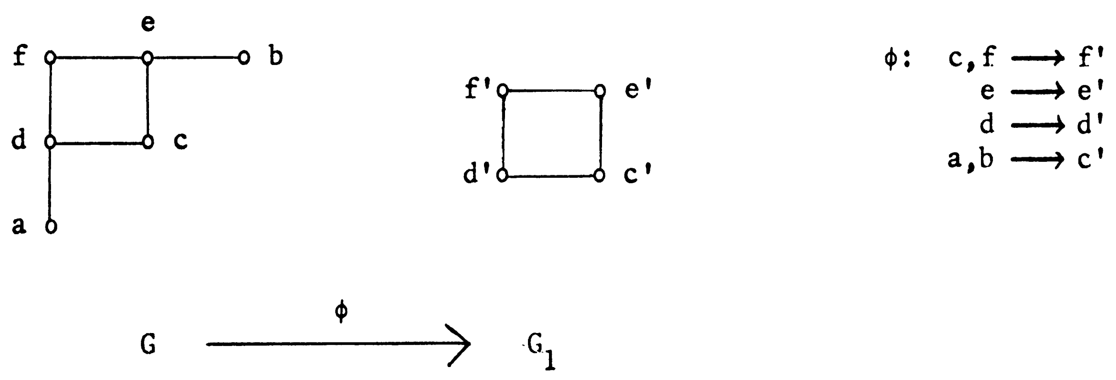


FIG. 12.

In Figure 12, $a\phi = b\phi$, yet neither $n(a) \subseteq n(b)$ nor $n(b) \subseteq n(a)$ holds.

2. CONTRACTIONS

2.1. General Definition.

A question arises in the literature on homomorphisms of graphs as to the interpretation of the homomorphic image of a pair of adjacent points $a,b \varepsilon V(G)$ which map to the same point $c' \varepsilon V(G')$. The definition of homomorphism given in Section 1.1, condition (1) in particular, specifies that the image is the point c' with a loop attached to it. However, Ore, Dirac, and several others choose to define homomorphisms for graphs for which the image of two adjacent points can be one point without a loop. These types of homomorphisms, which do not necessarily satisfy condition (1), can be defined as mappings ϕ which satisfy the following modification of condition (1):

(1m) $(a,b)\varepsilon\rho \Longrightarrow (a\phi,b\phi)\varepsilon\rho' \vee [a\phi = b\phi \wedge (a\phi,b\phi)\not\varepsilon\rho']$.

Following the terminology suggested by Dirac, Harary, and others, we will call a mapping ϕ of a graph $G = \langle V,\rho\rangle$ into a graph $G' = \langle V',\rho'\rangle$ which is a function from V into V' satisfying condition (1m) a <u>contraction.</u>

It is this distinction which is the basis for placing homomorphisms, mappings which satisfy condition (1), in one class, and contractions, mappings which do not necessarily satisfy condition (1) but which do satisfy condition (1m), in another class.

However, when considering homomorphisms of reflexive graphs, i.e., graphs whose adjacency relation is reflexive, the distinction we have made between homomorphisms and contractions disappears. For in considering reflexive graphs, a point is always considered to be adjacent with itself, and thus what distinguishes one graph from another is no longer its set of points and lines but rather the underlying graph, i.e., its set of points and lines minus its loops. In diagramming reflexive graphs it would seem therefore uncecessary to include all the loops. Hence, in Figure 13 for example, the homomorphism ϕ of G_1 onto G_2 could as well be represented by the homomorphism ϕ' of G_1' onto G_2'.

FIG. 13.

Forgetting, for a moment, that the points in G_1' and G_2' are assumed to be adjacent with themselves, it would seem that ϕ is a homomorphism, while ϕ' is a contraction, since it fails to satisfy condition (1). But ϕ' does satisfy condition (1) if we assume that the points are adjacent with themselves, and hence the distinction between the two mappings disappears.

2.2. Connected and Independent Contractions.

Ore [16] makes an interesting distinction between two types of contractions.[1] A contraction ϕ is <u>connected</u> iff the subgraphs of G generated by the sets of points $\phi^{-1}(a')$, for $a' \varepsilon V'$, are <u>connected</u>. A contraction ϕ is <u>independent</u> iff the subgraphs of G generated by the sets of points $\phi^{-1}(a')$, for $a' \varepsilon V'$, are <u>totally disconnected</u>. Ore's independent contractions coincide exactly with the ordinary homomorphisms when defined over the class of ordinary graphs.

Theorem 2.2.1. (Ore) Any contraction is the product of a connected and an independent contraction.

---

[1]Ore calls these mappings homomorphisms.

The <u>leaf composition graph</u> of a graph G is the graph whose points are the leaves of G; two points are adjacent iff the corresponding leaves are adjacent. Two points a,b belong to the same leaf iff there exists a sequence of cycles $P^1, P^2, \ldots, P^k$, $a \varepsilon V(P^1)$, $b \varepsilon V(P^k)$, and $V(P^i) \cap V(P^{i+1}) \neq \emptyset$.

Theorem 2.2.2. (Ore) Any graph G has a contraction onto its leaf composition graph G_1 such that the inverse image sets are the leaves of G. The graph G_1 is circuit free.

Theorem 2.2.2 implies, among other things, that every graph has at least one non-trivial tree as a contraction image. It follows also that since any two adjacent points can map to a single point, without a loop, every graph has K_1 as a contraction image. Dirac [5] gives the following definition of a contraction: a graph $G = \langle V, \rho \rangle$ can be <u>contracted</u> onto the graph $G' = \langle V', \rho' \rangle$ if there exists a function ϕ of V onto V' such that

(i) $(\forall a')[a' \varepsilon V' \Longrightarrow G - (G - \phi^{-1}(a'))$ is connected], and

(ii) $(\forall a')(\forall b')[a', b' \varepsilon V' \wedge (a', b') \varepsilon \rho' \Longrightarrow G$ contains at least one line joining a point of $\phi^{-1}(a')$ to a point of $\phi^{-1}(b')$].[1]

In words, G = G' or G' can be obtained from G by shrinking each of a set of connected subgraphs of G into a single point. It is clear from the definition that the contractions that Dirac considers are all connected. Several of the papers written about connected contractions, by Dirac [5], Wagner [21], and Halin [9], are attempts to clarify the following

Conjecture of H. Hadwiger: If the chromatic number of a graph G in n, then G has a connected contraction onto K_n.[2]

The conjecture is known to be true for values of $n \leq 4$; it has been shown that a proof for n = 5 would be equivalent to a proof of the famous Four Color Conjecture. A typical result directed towards this conjecture is the following: every ordinary graph with $n \geq 6$ points and at least 3n-5 lines has a connected contraction onto K_5 [6]. One property which is preserved under connected contractions but not under homomorphisms is planarity, i.e., if G is planar and ϕ is a connected contraction of G then $G\phi$ is also planar. The same cannot be said for ordinary homomorphisms.

---

[1] This is equivalent to condition (2) of Section 1.1.

[2] Note the similarity of this Conjecture to *Corollary 1.3.2.*

2.3. Simplicial Mappings.

In combinatorial topology, see Pontryagin [17] for example, a class of mappings known as <u>simplicial</u> mappings are defined for complexes and are shown to be approximations of continuous mappings of polyhedra. When interpreted for graphs simplicial mappings become, in one sense, approximations of contractions. Let a point be a 0-simplex, and let two points a,b and a line (a,b) between them be a 1-simplex. A mapping ϕ of a graph $G = \langle V,\rho\rangle$ <u>into</u> a graph $G' = \langle V',\rho'\rangle$ is <u>simplicial</u> iff

(i) ϕ is a function from V into V';

(ii) the image of a simplex in G is a simplex in G'.

It can be seen without much difficulty that condition (ii) is equivalent to condition (1m). If one were to extend this definition to get <u>onto</u> simplicial mappings, perhaps the most natural way would be to require that ϕ map the simplexes of G <u>onto</u> the simplexes of G'. The definition of a contraction is exactly this extension.

2.4. Homeomorphisms.

A class of mappings of graphs which closely resembles, but which

in fact overlaps, the class of contractions are the *homeomorphisms*.
Two ordinary graphs G and G' are said to be *homeomorphic* iff one can be obtained from the other by successive applications of the following two operations:

(i) replace a line $(a,b)\varepsilon\rho$ by two lines (a,c) and (c,b), c being a new point;

(ii) replace two lines (a,c), $(c,b)\varepsilon\rho$, where $(a,b)\not\varepsilon\rho$ and the point c has degree two, by a new line (a,b).

The pairs of graphs in Figure 14 are homeomorphic and serve to illustrate the difference between homeomorphisms and contractions.

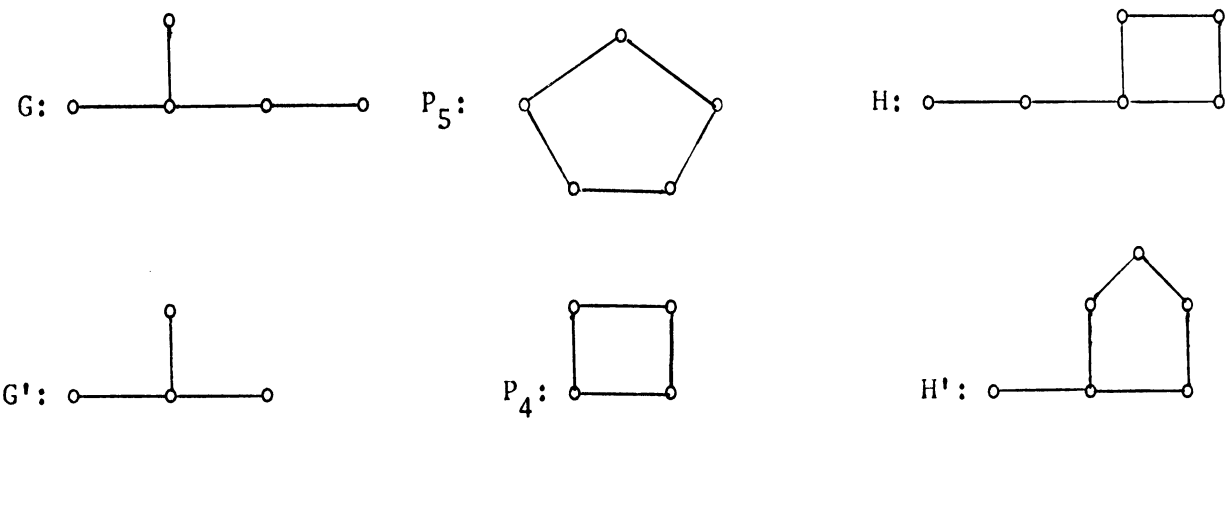


FIG. 14

In Figure 14, P_5 can be contracted onto P_4, yet no contraction maps H onto H', or H' onto H, and since K_3 can be contracted onto K_1 but K_3 and K_1 are not homeomorphic, it is clear that these two classes of mappings are distinct.

The most notable result concerning homeomorphisms is the now classical

Theorem of Kuratowski [13]: A graph G is planar iff G does not contain a subgraph homeomorphic to K_5 or K_{33}.

2.5. Strong Contractions.

Adam, Culik, and Pollak [1] have defined a class of mappings of graphs which very closely resembles the strong homomorphisms of Section 1.9; the definition of this class however contains an added condition which places the class in the class of contractions.

A mapping ϕ is a strong contraction[1] of an ordinary digraph[2] $G = \langle V,\rho\rangle$ onto an ordinary digraph $G' = \langle V',\rho'\rangle$ iff

(i) ϕ is a function from V onto V';

---

[1] Adam, Culik, and Pollak refer to such a mapping as "ein starker homomorphismus".
[2] An ordinary digraph, like an ordinary graph, has no loops or multilines.

(ii) $(a,b)\varepsilon\rho \wedge (b,a)\varepsilon\rho \Longleftrightarrow a\phi = b\phi \wedge (a\phi,b\phi)\not\varepsilon\rho'$;

(iii) $(a,b)\varepsilon\rho \wedge (b,a)\not\varepsilon\rho \Longrightarrow (a\phi,b\phi)\varepsilon\rho' \wedge (b\phi,a\phi)\not\varepsilon\rho'$;

(iv) $(a,b)\not\varepsilon\rho \wedge (b,a)\not\varepsilon\rho \Longrightarrow (a\phi,b\phi)\not\varepsilon\rho' \wedge (b\phi,a\phi)\not\varepsilon\rho'$.

It is assumed that points a and b in conditions (ii), (iii), and (iv) are distinct.

The authors show that an ordinary digraph has at most one strong contraction, and in the following theorem give a necessary and sufficient condition for a non-trivial strong contraction to exist.

Theorem 2.5.1. An ordinary digraph $G = \langle V,\rho\rangle$ *has a strong contraction iff for every triple a,b,c of distinct points*

(i) (a,b), (b,a), *and* $(a,c)\varepsilon\rho \Longrightarrow (b,c)\varepsilon\rho$;

(ii) (a,b), (b,a), *and* $(c,a)\varepsilon\rho \Longrightarrow (c,b)\varepsilon\rho$.

Corollary 2.5.2. If ϕ *is a strong contraction of an ordinary digraph G onto an ordinary digraph G', then there exists a subgraph H of G such that* $H \simeq G'$.

A proof of this corollary can be constructed that is essentially the same as the proof of *Theorem 1.9.1*, which is the corresponding statement for strong homomorphisms.

3. RELATIONAL HOMOMORPHISMS

3.1. General Definition.

The concept of a homomorphism (and a contraction), of a graph $G = \langle V,\rho\rangle$ into a graph $G' = \langle V',\rho'\rangle$, generally speaking, consists of a function ϕ from $V(G)$ into $V(G')$ which satisfies certain conditions. These conditions are usually imposed so that certain properties of the graph G will be preserved in G' by the function ϕ.

A natural generalization of this concept is that of a "relational homomorphism," between two graphs G and G', in which a binary relation is defined, $\psi \subseteq VxV'$, which satisfies certain conditions.

The idea of considering relations as homomorphisms between algebraic systems has occurred to at least two authors, some of whose definitions and results along this line are summarized in this section. We will refer, in general, to the mappings defined in this section as relational homomorphisms.

3.2. Weak Homomorphisms.

Yoeli and Ginzburg [8] and Yoeli [23] have defined a weak homomorphism for partial and complete algebras which can, in a natural way, be rephrased for directed graphs. Once this is done, a development exactly analogous to that in [23] can be carried out which illustrates the definition of weak homomorphism and some of its properties for directed graphs. The remainder of this section is just such a development; it contains nothing original and is simply a translation of the first part of [23] into graph theoretical terminology.

Every directed graph $G = \langle V,\rho \rangle$, including those with loops and multi-lines, can be expressed, generally in many ways, as a union of graphs of the form $G_i = \langle V,\rho_i \rangle$, where $\cup\rho_i = \rho$. Alternatively, every directed graph can be put in the form $G = \langle V,\{\rho_i\} \rangle$, where $\cup\rho_i = \rho$. In particular, each ρ_i can be required to be a many-one relation, i.e., a partial function from V to V. A digraph is out-regular of degree k iff it can be put in the form $G = \langle V,\{\rho_1,\rho_2,\ldots,\rho_k\} \rangle$, where each ρ_i is a function from V into V. Equivalently, a digraph G is out-regular of degree k iff the out-degree of every point of G equals k.

A binary relation $\psi \subseteq V x V'$ is a <u>weak homomorphism</u>[1] of a digraph $G = \langle V, \{\rho_i\} \rangle$ <u>into</u> a digraph $G' = \langle V', \{\rho_i'\} \rangle$, i = 1, 2, ..., m iff ψ satisfies the following two conditions

(11) $\psi^{-1}(V') = V$, and

(12) $\psi^{-1}\rho_i \subseteq \rho_i'\psi^{-1}$, for every i = 1, 2, ..., m; in words, if

$a \varepsilon \rho_i^{-1}(V)$ and $a\psi a'$, then $a' \varepsilon \rho_i'^{-1}$ and $\rho_i(a)\psi\rho_i'(a')$.

If in addition the binary relation ψ is a <u>function</u> from V onto V' satisfying the condition

(13) $\psi\rho_i' = \rho_i\psi$, for all i,

then ψ is a full homomorphism of G onto G'.[2]

The following three lemmas correspond exactly to *Lemmas 1, 2,* and *3* of Yoeli [23].

Lemma 3.2.1. If ψ is a weak homomorphism of a digraph G into a digraph G', and ψ' is a weak homomorphism of G' into a digraph G", then

---

[1] Note that the definition requires that a 1-1 correspondence exist between $\{\rho_i\}$ and $\{\rho_i'\}$.

[2] Yoeli calls ψ, in this case, a <u>strong homomorphism</u>. It can be seen however that condition (13) requires that the mapping be a full homomorphism, as defined in Section 1.1, of the digraph G onto the digraph G'.

$\psi\psi'$ is a weak homomorphism of G into G''.

Lemma 3.2.2. A mapping ψ of a digraph G onto a digraph G' is a full homomorphism iff both ψ and ψ^{-1} are weak homomorphisms.

Lemma 3.2.3. If ψ is a weak homomorphism of an out-regular digraph G onto a digraph G', then ψ^{-1} is a weak homomorphism of G' onto G. For additional discussion of weak homomorphisms the reader is referred to Yoeli.

In Figure 15, ψ_1 is an example of a weak homomorphism of G_1 onto G_2, where $\psi_1 = \{(a,c),(a,e),(b,d),(b,f)\}$.

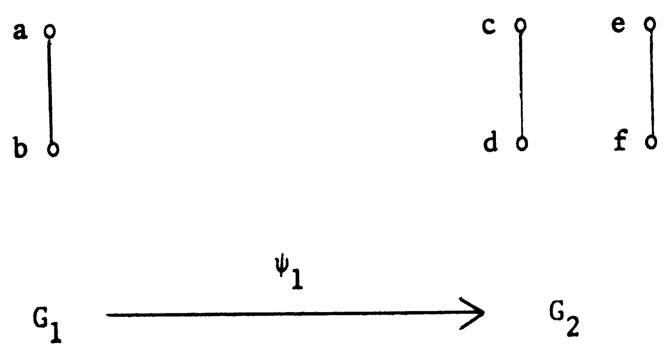


FIG. 15.

Note also that ψ_1^{-1} is a full homomorphism of G_2 onto G_1.

3.3. Weak Isomorphism.

A binary relation $\psi \subseteq V \times V'$ is a *weak full homomorphism* of a graph $G = \langle V, \rho \rangle$ *onto* a graph $G' = \langle V', \rho' \rangle$ iff in addition to satisfying condition (11) it satisfies conditions

(14) $\psi(V) = V'$, and

(15) $\psi(\rho) = \rho'$.

A weak full homomorphism of a graph G onto a graph G' is a *weak isomorphism* iff $\psi^{-1}(\rho') = \rho$. Consider Figure 16, where

$\psi = \{(a,a'),(a,a''),(b,b'),(b,b'')\}$ is a full weak homomorphism of G onto G'.

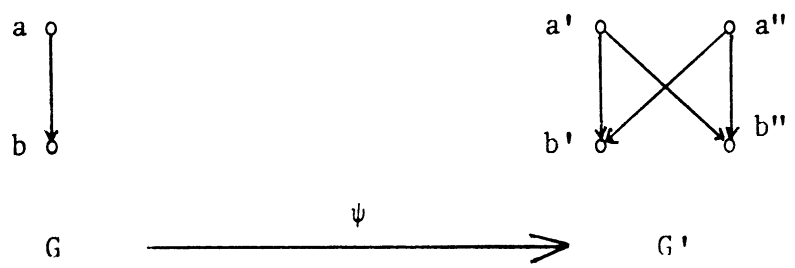


FIG. 16.

Since $\psi^{-1}(\rho') = \rho$, it follows that G and G' are weakly isomorphic.

Thatcher [20] has studied both weak full homomorphisms and weak isomorphisms, and offers the following explanation for considering intuitively G and G' to be 'isomorphic.'

Two graphs G and G' are elementarily equivalent iff they are indistinguishable using the first-order predicate calculus without equality, i.e., no sentence in this language is true of G without its also being true of G', and conversely.

Thatcher states that if two graphs G and G' are weakly isomorphic then they are elementarily equivalent. He also believes that the converse is true for finite graphs, i.e., if two finite graphs, G and G', are elementarily equivalent then they are weakly isomorphic. Since in the first order predicate calculus without equality one loses the ability to count, one cannot distinguish whether two graphs have the same number of points. It follows, therefore, that among other things all complete digraphs are weakly isomorphic.[1] Thatcher suggests as an interesting problem the study of these weak isomorphism types of graphs. The following results shed some light on this problem.

---

[1] It is not true however that all ordinary complete graphs are weakly isomorphic.

Lemma 3.3.1. If ϕ is a strong homomorphism[2] *of a graph G onto a graph G', then G and G' are weakly isomorphic, i.e., ϕ^{-1} is a weak homomorphism of G' onto G.*

Proof. Obviously ϕ is a weak homomorphism of G onto G', and as such $\phi(V) = V'$ and $\phi(\rho) = \rho'$. It follows therefore that $\phi^{-1}(V') = V$. It suffices to show that $\phi^{-1}(\rho') = \rho$. First,

$(a',b')\varepsilon\rho' \Longrightarrow (\phi^{-1}(a'),\phi^{-1}(b'))\varepsilon\rho$, for if $a\varepsilon\phi^{-1}(a')$ and $b\varepsilon\phi^{-1}(b')$, then $(a,b)\varepsilon\rho$, since $g(a,b) = g'(a\phi,b\phi) = g'(a',b') = 1$. Secondly, $(a',b')\not\varepsilon\rho' \Longrightarrow (\phi^{-1}(a'),\phi^{-1}(b'))\not\varepsilon\rho$, by the same reasoning.

Theorem 3.3.2. Two graphs G and G' are weakly isomorphic iff they have isomorphic simple (with respect to strong homomorphisms) images.

Proof. First, suppose that G and G' are weakly isomorphic with respect to the full weak homomorphism ψ, i.e.,

$$\begin{tikzcd} G \arrow[r, "\phi_s"] \arrow[d, bend right, "\psi"'] & G_s \\ G' \arrow[r, "\phi'_s"] \arrow[u, bend right, "\psi^{-1}"'] & G'_s \end{tikzcd},$$

where ϕ_s and ϕ'_s are strong homomorphisms of G and G' onto the simple graphs

---

[1] cf. Section 1.9.

G_s and G'_s, respectively. It suffices to show that $\psi\phi'_s$ is a strong homomorphism of G onto G'_s. The isomorphism of G_s and G'_s then follows as a result of *Theorem 1.9.7.*

We must show that $\psi\phi'_s$ is a well defined mapping of V(G) onto $V(G'_s)$, where ψ is a full weak homomorphism of G onto G' and ϕ'_s is a strong homomorphism of G' onto G'_s (and G'_s is simple). We do this by showing that

$$\text{(i)}\quad a\psi a'_1 \wedge a\psi a'_2 \Longrightarrow n(a'_1) = n(a'_2), \text{ and}$$

$$\text{(ii)}\quad n(a'_1) = n(a'_2) \Longrightarrow a'_1\phi'_s = a'_2\phi'_s.$$

Let us prove, (i) $a\psi a'_1 \wedge a\psi a'_2 \Longrightarrow n(a'_1) = n(a'_2)$. Suppose $n(a'_1) \neq n(a'_2)$. Then either $\rho'(a'_1) \neq \rho'(a'_2)$ or $\rho^{-1}(a'_1) \neq \rho'^{-1}(a'_2)$. If $\rho'(a'_1) \neq \rho'(a'_2)$ then

$(\exists d')[((a'_1,d')\epsilon\rho' \wedge (a'_2,d')\not\epsilon\rho') \vee ((a'_1,d')\not\epsilon\rho' \wedge (a'_2,d')\epsilon\rho')]$.

Suppose $(a'_1,d')\epsilon\rho' \wedge (a'_2,d')\not\epsilon\rho'$; then $(\exists c)(\exists d)[(c,d)\epsilon\rho \wedge c\psi a'_1 \wedge d\psi d']$.

But since ψ^{-1} is a full weak homomorphism of G' onto G,

$(a'_1,d')\epsilon\rho' \Longrightarrow (a,d)\epsilon\rho \Longrightarrow (a'_2,d')\epsilon\rho'$, hence we have a contradiction.

A similar argument shows that $(a'_1,d')\not\epsilon\rho' \wedge (a'_2,d')\epsilon\rho'$ cannot hold either, hence $\rho'(a'_1) = \rho'(a_2)$, must hold.

The same argument just used to show that $\rho'(a'_1) = \rho'(a'_2)$ can be used to show that $\rho'^{-1}(a_1') = \rho'^{-1}(a_2')$. Hence $n(a_1') = n(a_2')$ must hold. In order to see that (ii) $n(a_1') = n(a_2') \Longrightarrow a_1'\phi_s' = a_2'\phi_s'$, one need only refer back to the definition of the equivalence relation $\equiv$ which determines the strong homomorphism ϕ_s'.

Hence $\psi\phi_s'$ is a well defined mapping of $V(G)$ onto $V(G_s')$; it is a simple matter to verify that the mapping satisfies the condition

$$g(a,b) = g_s'(a\psi\phi_s', b\psi\phi_s').$$

The mapping $\psi\phi_s'$ is therefore a strong homomorphism of G onto G_s'.

Second, suppose that G_s and G_s' are isomorphic,

$$\begin{tikzcd}
G \arrow[dr, "\phi_s"] & \\
& G_s \arrow[ul, bend right, "\phi_s^{-1}"'] \arrow[dl, bend left, "\phi_s'^{-1}"] \\
G' \arrow[ur, "\phi_s'"] &
\end{tikzcd},$$

where for convenience we suppose that ϕ_s and ϕ_s' are strong homomorphisms of G and G' onto G_s. It suffices to show that $\phi_s\phi_s'^{-1}$ is a full weak homomorphism of G onto G', and similarly that $\phi_s'\phi_s^{-1}$ is a full weak homomorphism of G' onto G. But this follows immediately as a result of *Lemma 3.3.1* and the following

Lemma 3.3.3. If ψ *is a full weak homomorphism of* G *onto* G'*, and* ψ' *is a full weak homomorphism of* G' *onto* G''*, then* $\psi\psi'$ *is a full weak homomorphism of* G *onto* G''*.*

SUMMARY AND SUGGESTED PROBLEMS

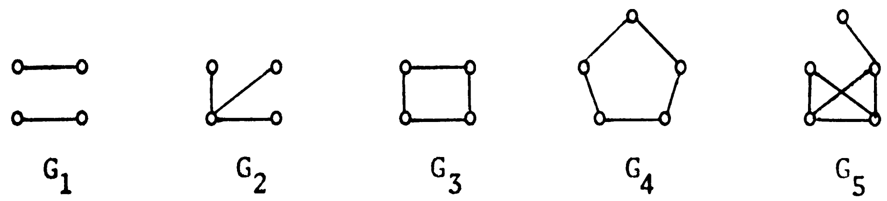
Several definitions have been given of homomorphisms for various classes of graphs and a few results have been stated or derived for each. The nature of the results gives an indication of the kinds of properties of graphs one can effectively study by means of homomorphisms, and gives an indication of the methods one might use in studying properties of the homomorphisms themselves.

Several open problems and problem areas have been suggested in the paper, and several new ones can be stated, among which are the following:

1) How does the set of (elementary) homomorphic images of a given graph G characterize G? It can be shown, for example, that the set of all non-trivial homomorphic images of an ordinary graph G, together with the multiplicities associated with each image, i.e., the number of distinct ways the particular image can be formed under a homomorphism, completely characterize the graphs with five points or less which are not complete graphs.

2) Of the 52 ordinary graphs with five points or less, 31 have

one elementary homomorphic image (up to isomorphism). Can one easily characterize those graphs G such that all of the elementary homomorphic images of G are isomorphic? The following five graphs have this property



3) Geert Prins, in a letter to the author, expressed an interest in studying homomorphisms of graphs via elementary homomorphisms. What in general can one say about elementary homomorphisms? For example, one could state the following

Proposition: If $G\varepsilon$ is that elementary homomorphic image of G obtained by identifying points $a,b\varepsilon V(G)$, and the chromatic number of G is n, then the chromatic number of $G\varepsilon$ is n iff points a and b can be colored identically in some n-coloring of G.

4) What sort of graph theoretical results are obtainable from Keisler's logical formulation of the properties of a graph which are preserved under full homomorphisms? For example, a brief explanation of Keisler's formulation by Professor Robert Ritchie led to the following observation:

Proposition. If ϕ is a full homomorphism of an ordinary graph G onto a graph G', and $G = G_1 + G_2$, i.e., G is the join of G_1 and G_2, then $G' = G_1\phi + G_2\phi$.

It is believed that almost all of the results derived using Keisler's formulation could be derived very easily without using it.

5) In regard to pre-images of a given graph, how close to a complete graph, in some sense, can a pre-image of a given graph be? For example, if an ordinary graph G has p points and q lines, define its density as $q/\binom{p}{2}$. It would seem, at first glance, that since every ordinary graph G maps onto a complete graph,

of density 1, that the density of a given graph G would be smaller than the density of every homomorphic image of G. But this is not the case as the following example illustrates:

$$\phi: \begin{array}{rcl} a & \longrightarrow & a' \\ b,e & \longrightarrow & b' \\ c,f & \longrightarrow & c' \\ d & \longrightarrow & d' \end{array}$$

$$G \xrightarrow{\quad\phi\quad} G'$$

Note that the density of G is 8/15, while the density of $G\phi = G'$ is 3/6.

6) For any class of homomorphisms, what graphs are endomorphism-free?

7) Which graphs are simple with respect to pathwise homomorphisms?

8) Are all prime homomorphisms of functional digraphs, pathwise homomorphisms as well? It appears so.

9) Characterize Prins' type-2 complete homomorphisms. This problem appears to be very difficult.

10) Relative to the study of type-n (complete) homomorphisms several

problems naturally arise among which are the following:

a) Does there exist for every integer n a type-n (complete) homomorphism?

b) Is the order of a type-n complete homomorphism greater than or equal to n?

c) Can one state an Interpolation Theorem for type-n complete homomorphisms, i.e., given a type-n complete homomorphism of a graph G, of order m, what can be asserted about type-k complete homomorphisms of G for $k \leq n$.

d) If the length of the longest chain in a graph G is n, it would seem that G could not have a type-n+1 complete homomorphism. Relate this length n to the maximum type-n complete homomorphism of G.

11) Any rule for producing a non-identity partition of the points of an arbitrary graph G determines a full homomorphism in the following sense:

Let $V_1, V_2, \ldots, V_m$ be a partition of V(G), and let x_i

denote the chromatic number of the full subgraph generated by the subset V_i. The mapping which maps each subgraph respectively onto K_i determines a full homomorphism of G.

If we denote the resulting graph in this process by $G\phi$ and apply the same partition-mapping process to $G\phi$, $G\phi^2$, etc., we might ultimately map G onto a complete graph. It would be interesting to study those processes which map a given graph G ultimately onto a complete graph and to examine the relationship between the chromatic numbers of G and the complete graph.

Two such processes have already been mentioned; one, of taking partitions determined by 1-bases, and two, of taking partitions determined by n-bases. A third process would be that of considering partitions induced by the <u>associated numbers</u> of the points of a graph.

12) How can the concept of complete homomorphism be applied to directed graphs?

13) The homomorphisms of functional digraphs onto functional digraphs

all satisfy the condition

$$\text{(i)} \quad od(a) = od(a\phi).$$

It is suggested therefore that one study full homomorphisms which satisfy this condition, or the following condition

$$\text{(ii)} \quad id(a) = id(a\phi),$$

or both; in particular, study full homomorphisms of ordinary graphs which satisfy

$$\text{(iii)} \quad deg(a) = deg(a\phi).$$

Furthermore, functional digraphs are a sub-class of the class of out-regular digraphs; hence, the suggestion of studying homomorphisms of out-regular and in-regular digraphs and regular ordinary graphs, satisfying conditions (i), (ii), and (iii), respectively, is raised.

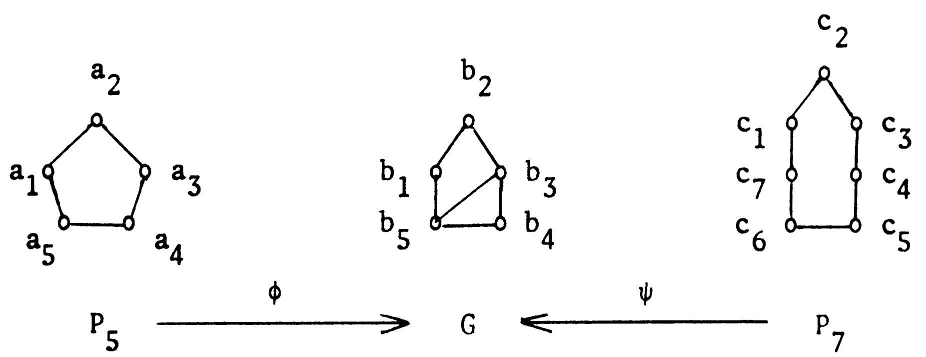
14) Y. Give'on has suggested that one study <u>epigenic</u> homomorphisms of directed graphs; homomorphisms for which the inverse image of bases (generating sets) are bases. It can be seen, for

example, that any homomorphism of a weakly connected digraph onto a strongly connected digraph is not epigenic.

15) Relative to *Proposition 1.5.5*, it seems that with the exception of K_2 one can strengthen the inequality to $ch(G) \leq 2q - 2$.

16) Relative to the question raised at the beginning of Section 1.5, one can ask the following, slightly different, and much more interesting question: for a given graph G what is the length of the shortest cycle P for which there exists a homomorphism ϕ which maps P onto G? The homomorphism ϕ need not be a full homomorphism. Suppose we call this the circuit length of G, and denote it by ci(G).

The following example illustrates that ci(G) and cy(G) are, in fact, distinct invariants; a little reflection shows further that $ci(G) \leq cy(G)$.



$$\begin{array}{llll}
\phi: & a_1 \longrightarrow b_1 & \qquad \psi: & c_1 \longrightarrow b_1 \\
 & a_2 \longrightarrow b_2 & & c_2 \longrightarrow b_2 \\
 & a_3 \longrightarrow b_3 & & c_3, c_6 \longrightarrow b_3 \\
 & a_4 \longrightarrow b_4 & & c_4 \longrightarrow b_4 \\
 & a_5 \longrightarrow b_5 & & c_5, c_7 \longrightarrow b_5
\end{array}$$

The mapping ϕ is a homomorphism of P_5 onto G; the mapping ψ is a full homomorphism of P_7 onto G. Hence, $ci(G) = 5$, while $cy(G) = 7$.

It can also be seen that if a graph G has p points then $ci(G) = p$ iff G is Hamiltonian.

The parameter ci(G) also can be seen to have a lot of relevance to solutions of the famous Traveling Salesman Problem. It would be interesting in view of these observations to investigate the extent to which the methods used to study cy(G) can be used to study ci(G).

APPENDIX OF DEFINITIONS

In a graph $G = \langle V,\rho \rangle$

... the <u>out degree</u> of a point $b \varepsilon V$, $od(b) = |\rho(b)|$;

... the <u>in degree</u> of a point $b \varepsilon V$, $id(b) = |\rho^{-1}(b)|$;

... the <u>degree</u> of a point $b \varepsilon V$, $deg(b) = |n(b)| = |\rho(b) \cup \rho^{-1}(b)|$;

... a <u>loop at a point b</u> is a line of the form (b,b);

... a <u>path of length n</u> from a point <u>a</u> to a point <u>b</u> is a sequence of points $a_{i_0}, a_{i_1}, \ldots, a_{i_n}$ such that $(a_{i_j}, a_{i_{j+1}}) \varepsilon \rho$, for $0 \leq j \leq n-1$, $a_{i_0} = a$, and $a_{i_n} = b$;

... the <u>distance</u> from point a to point b, $d(a,b)$, is the length of the shortest path from a to b;

... the <u>associated number</u> of a point $b \varepsilon V$ is max $d(b,c)$, for all $c \varepsilon V$.

... the <u>diameter</u> of G, $\delta(G)$, is max $d(a,b)$, for $a,b \varepsilon V$.

... an <u>independent set of points</u> is a set $V_1 \subseteq V$ such that for every $a,b \varepsilon V_1$, $(a,b) \notin \rho$;

... a <u>maximal</u> independent set of points is an independent set V_1 such that for no independent set V_2 is $V_1 \subset V_2$;

... the <u>point independence number</u>, $\alpha_0(G)$, is max $|V_1|$, where V_1 is an independent set of points;

... the <u>point covering number</u>, $\beta_0(G)$, is min $|V_2|$, where $V_2 \subseteq V$ is such that for every $(a,b) \epsilon \rho$, either $a \epsilon V_2$ or $b \epsilon V_2$.

... the <u>subgraph generated by a set of points</u> V', is the graph $G = \langle V', \rho' \rangle$ where $\rho' = \{(a,b) : a,b \epsilon V' \wedge (a,b) \epsilon \rho\}$

An ordinary graph $G = \langle V, \rho \rangle$

... is a <u>complete graph on n points</u> iff it has the form

$$V = \{k_1, k_2, \dots, k_n\}$$

$$\rho = \{(k_i, k_j) : i \neq j\};$$

... is a <u>cycle of length n</u> iff it has the form

$$V = \{p_1, p_2, \dots, p_n\}$$

$$\rho = \{(p_i, p_{i+1}), (p_{i+1}, p_i) : 1 \leq i \leq n-1\} \cup \{(p_1, p_n), (p_n p_1)\};$$

... is a <u>chain of length n</u> iff it has the form

$$V = \{c_1, c_2, \dots, c_{n+1}\}$$

$$\rho = \{(c_i, c_{i+1}), (c_{i+1}, c_i) : 1 \leq i \leq n\};$$

... is <u>connected</u> iff for every pair a,b of distinct points there is a path in G from point a to point b;

... is a <u>tree</u> iff it is connected and contains no cycles as subgraphs;

... is <u>critical of degree k</u> iff $\chi(G) = k$ and

$(\forall(a,b)\varepsilon\rho)\ [\chi(G-(a,b)) = k - 1]$; in words, the chromatic number of any graph obtained from G by removing any line, or point, of G is less than the chromatic number of G;

... is isomorphic to K_{33} iff it has the form

$$V = \{a_1,a_2,a_3,b_1,b_2,b_3\}$$

$$\rho = \{(a_i,b_j),(b_j,a_i) : 1 \leq i,\ j \leq 3\};$$

... has <u>multi-lines</u> iff ρ is considered to contain at least two occurences of at least one pair (a,b);

... is a <u>multi-graph</u> iff it contains multi-lines;

... is the <u>join</u> of graphs $G_1 = \langle V_1,\rho_1\rangle$ and $G_2 = \langle V_2,\rho_2\rangle$ iff

$V = V_1 \cup V_2$, $\rho = \rho_1 \cup \rho_2 \cup \{(a,b) |\ a\varepsilon V_1,\ b\varepsilon V_2\}$

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The works of Hartmanis and Stearns, Krohn and Rhodes, Yoeli and Ginzburg, and Zeiger amploy demonstrate the usefulness of homomorphisms in studying decompositions of finite automata. Yoeli and Ginzburg's approach is slightly different from the others in that it is more concerned with aspects of the state transition graphs of finite automata. In their paper, "On Homomorphic Images of Transition Graphs," they give a complete characterization of the class of homomorphisms of the graphs which correspond to input-free automata. (U)

This paper was motivated by an interest in extending these results of Yoeli and Ginzburg in the direction of a characterization of the class of homomorphisms of graphs which correspond to arbitrary finite automata. It is a review and a classification of most of the published definitions and results on mappings of graphs which have been called homomorphisms. The paper contains, in addition, several new results and several new definitions of homomorphisms of graphs. (U)

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