Technical Report

HOMOMORPHISMS OF GRAPHS

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Background: The Logic of Computers Group of the Communication Sciences Program of The University of Michigan is investigating the application of logic and mathematics to the design of computing automata. The use of graph-theoretical techniques in the study of automata forms a part of this investigation.

Condensed Report Contents:

The works of Hartmanis and Stearns, Krohn and Rhodes, Yoeli and Ginzburg, and Zeiger amply demonstrate the usefulness of homomorphisms in studying decompositions of finite automata. Yoeli and Ginzburg's approach is slightly different from the others in that it is more concerned with aspects of the state transition graphs of finite automata. In their paper, "On Homomorphic Images of Transition Graphs," they give a complete characterization of the class of homomorphisms of the graphs which correspond to input-free automata.

This paper was motivated by an interest in extending these results of Yoeli and Ginzburg in the direction of a characterization of the class of homomorphisms of graphs which correspond to arbitrary finite automata. It is a review and a classification of most of the published definitions and results on mappings of graphs which have been called homomorphisms. The paper contains, in addition, several new results and several new definitions of homomorphisms of graphs.

For Further Information: The complete report is available in the major Navy technical libraries and can be obtained from the Defense Documentation Center. A few copies are available for distribution by the author.
PREFACE

The works of Hartmanis and Stearns [10], Krohn and Rhodes [12], Yoeli and Ginzburg [22], and Zeiger [24] amply demonstrate the usefulness of homomorphisms in studying decompositions of finite automata. Yoeli and Ginzburg's approach is slightly different from the others in that it is more concerned with aspects of the state transition graphs of finite automata. In their paper, "On Homomorphic Images of Transition Graphs" [22], they give a complete characterization of the class of homomorphisms of the graphs which correspond to input-free automata.

This paper was motivated by an interest in extending these results of Yoeli and Ginzburg in the direction of a characterization of the class of homomorphisms of graphs which correspond to arbitrary finite automata. It is a review and a classification of most of the published definitions and results on mappings of graphs which have been called homomorphisms. The paper contains, in addition, several new results and several new definitions of homomorphisms of graphs, which are found primarily in Sections 1.3 — 1.6, 1.8, and 1.9.
The mappings of graphs which are defined in this paper are divided into three classes; Homomorphisms, Contractions, and Relational Homomorphisms. The distinction between the first two classes, while not very important, is a very natural one to make, and can best be understood after the class of Homomorphisms has been defined and studied in Section 1. It is for this reason that the discussion of this distinction is postponed until Section 2, where Contractions are defined and examined.

It is likely that there are definitions and results on mappings of graphs which have recently been published but which are not referenced here. The author would welcome any information concerning such omissions. For the reader not well versed in the terminology of graph theory, terms which are undefined in the text are defined in the Appendix.

Proofs of statements which are quite easily constructed have, in many instances, been omitted, it is hoped without any loss of content or continuity.

The author is especially indebted to Professor Frank Harary, of the Mathematics Department, and to Yehoshafat Give'on, of the Logic of
Computers Group, for their many critical comments, suggestions, and sharp reading of the manuscript, which greatly improved the organization and presentation of definitions, results, and proofs.
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NOTATION

Of a graph \( G = \langle V, \rho \rangle \):

\( V \) or \( V(G) \) denotes the set of points of \( G \);

\( \rho \) denotes the adjacency relation of \( G \), \( \rho \subseteq V \times V \);

\( \rho(a,b) \) or \( (a,b) \) means there is a line from point \( a \) to point \( b \) in \( G \);

\( \overline{\rho}(a,b) \) means there is no line from point \( a \) to point \( b \) in \( G \);

\[ \rho(a) = \{ b : (a,b) \in \rho \} \]

\[ \rho^{-1}(b) = \{ a : (a,b) \in \rho \} \]

\[ n(a) = \rho(a) \cup \rho^{-1}(a) \]

\( \chi(G) \) denotes the chromatic number of \( G \).

\( K_n \) denotes the complete graph on the \( n \) points, \( k_1, k_2, \ldots, k_n \).

\( P \) denotes a cycle; \( P_n \), a cycle of length \( n \) with points \( p_1, p_2, \ldots, p_n \).

\( C \) denotes a chain; \( C_n \), a chain of length \( n \) with points \( c_1, c_2, \ldots, c_{n+1} \).
1. HOMOMORPHISMS

1.1. General Definitions.

A graph \( G = \langle V, \rho \rangle \) is a relational system whose set is the finite set \( V \) of points of \( G \) and whose set of relations consists of one binary relation \( \rho \subseteq V \times V \), the adjacency relation. A graph \( G' = \langle V', \rho' \rangle \) is a subgraph of \( G \), written \( G' \subseteq G \), iff \( V' \subseteq V \) and \( \rho' \subseteq \rho \); \( G' \) is a full subgraph of \( G \) iff for every \( a, b \in V' \), \((a, b) \in \rho \implies (a, b) \in \rho' \).

Let \( \phi \) be a function from \( V \) into \( V' \), and define the extension \( \phi : V \times V \rightarrow V' \times V' \), by

\[(a, b) \phi = (a \phi, b \phi).\]

If \( \phi \) satisfies the condition

\[(1) \ \ \rho \phi \subseteq \rho', \text{ or equivalently, } (a, b) \in \rho \implies (a \phi, b \phi) \in \rho',\]

then \( \phi \) is a homomorphism of \( G \) into \( G' \). If \( \phi \) also satisfies the condition

\[(2) \ \ \phi (a \phi, b \phi) \phi \rightarrow (\exists c) (\exists d) [c \phi = a \phi \land d \phi = b \phi \land (c, d) \in \rho],\]

then \( \phi \) is a full homomorphism of \( G \) into \( G' \). From the definition of homomorphism it follows that the product of homomorphisms is itself a homomorphism, i.e., if \( G, G', \) and \( G'' \) are graphs and \( \phi : G \rightarrow G' \) and

---

1 Subgraphs and full subgraphs correspond in Ore's terminology to subgraphs and section graphs, and in Berge's terminology to partial subgraphs and subgraphs, respectively.
ψ : G' → G" are homomorphisms, then the product mapping φψ, for which

\[ aφψ = (aφ)ψ \quad \text{and} \quad (a,b)φψ = ((a,b)φ)ψ = (aφ,bφ)ψ = ((aφ)ψ,(bφ)ψ) = (aφψ,bφψ), \]

is a homomorphism of G into G". It follows also from (1) that if

Gφ = \langle Vφ, ρφ \rangle is the image of G under the homomorphism φ then Gφ is a subgraph of G', and if φ is a full homomorphism then Gφ is a full subgraph of G'. If, in addition, φ maps V onto V' then φ is said to be a (full) homomorphism of G onto G'. Note that if φ is a full homomorphism of G onto G' then ρφ = ρ'.

In the case G' ⊆ G and φ is a (full) homomorphism of G into G',

φ is said to be a (full) endomorphism of G; when Gφ = G' = G, φ is an automorphism of G. A full homomorphism φ of a graph G onto a graph G' is an isomorphism when φ is 1-1 from V onto V', in which case G and G' are said to be isomorphic, written \( G \cong G' \).

Figure 1 illustrates some of these definitions. The homomorphisms φ_1 and φ_3 are full endomorphisms of G onto the full subgraphs G_1 and G_3, respectively. The endomorphism φ_2 is not full, since G_2 is not a full subgraph of G. Note that G_2 and G_3 are isomorphic. Note also that
\[ G_1 \xrightarrow{\phi_1} G \xrightarrow{\phi_2} G_2 \]

\[ G_3 \xrightarrow{\phi_3} G \]

\[ \phi_1: a, d \to d \quad \phi_2: a \to d \quad \phi_3: a \to a \]
\[ b, c \to c \quad b \to c \quad b \to b \]
\[ e \to e \quad c, e \to e \quad c, e \to c \]
\[ f \to f \quad d, f \to f \quad d, f \to d \]

**FIG. 1.**

\[ G_{\phi_1}\phi_1 = G_{\phi_1} \text{ and } G_{\phi_3}\phi_3 = G_{\phi_3}, \text{ but } G_{\phi_2}\phi_2 \neq G_{\phi_2}. \]

1.2. Properties Preserved under Homomorphisms.

H. J. Keisler has written a paper [11] which is concerned essentially with a logical formulation of the properties of a relational system which are preserved under two classes of homomorphisms, which he
calls "strong homomorphisms"\(^1\) and "retracts". Using our terminology for
graphs, these "strong homomorphisms" are full homomorphisms for which the
function is onto, and "retracts" are endomorphisms which act as the identity
mapping on the image, i.e., \(G\phi = G'\) is a subgraph of \(G\) and \(b\phi = b\) for every
\(b \in V'\). It follows from this definition that "retracts" are those endo-
morphisms which are idempotent, i.e., for which \(G\phi^2 = G\phi\). Two theorems are
stated in the paper, one for each class of mappings, to the effect that
a property of a relational system is preserved if and only if it is logically
equivalent to a sentence of a particular form in the first order predicate
calculus with identity. Although these results obviously can be applied to
homomorphisms of graphs, they have not been as yet.

1.3. Complete Homomorphisms.

An ordinary graph is a graph whose adjacency relation is symmetric
and irreflexive; stated in other words, a graph is ordinary if it is undirected
and has no loops or multi-lines. In this section, and in Sections 1.4, 1.5,

\(^1\)Later in the paper, Section 1.9, we will define a different class of
mappings and call them strong homomorphisms.
and 1.6, we will consider only full homomorphisms of ordinary graphs onto ordinary graphs; these mappings $\phi$ must satisfy the condition

$$a\phi = b\phi \implies (a,b)\notin \rho,$$

and we will call them **ordinary homomorphisms.**$^1$ We will, however, omit the word "ordinary" when referring to graphs and homomorphisms in these four sections.

A homomorphism of a graph $G$ onto a graph $G'$ is called **complete** iff $G' = K_n$, for some $n$. A complete homomorphism of a graph $G$ onto $K_n$ is said to be of **order** $n$. A **coloring** of a graph $G$ is an assignment of colors to the points of $G$ such that no two adjacent points are assigned the same color. More formally, an **$n$-coloring** of a graph $G = \langle V, \rho \rangle$ is a function $n$ from $V$ onto $N$, where $N$ is the set of natural numbers (colors) $\{1, 2, \ldots, n\}$, satisfying the condition that $(a,b)\notin \rho \implies a_n \neq b_n$. An $n$-coloring $n$ is **complete** iff for every $i,j$, $i \neq j$, $\exists a \exists b [a_n = i \land b_n = j \land (a,b)\notin \rho]$.

The **chromatic number** of a graph $G$ is the smallest number $n$ such that $G$ has an $n$-coloring.

---

$^1$Full homomorphisms satisfying this condition are called **independent** by Ore, cf. Section 2.2.
Theorem 1.3.1. If \( \eta : V \rightarrow N \) is a (complete) \( n \)-coloring of graph \( G = \langle V, e \rangle \), then \( \eta \) determines a (complete) homomorphism \( \phi_\eta \) of \( G \) onto \( K_n \), and conversely.

Proof. If \( \eta_n = i \), let \( a_{\eta_i} = k_i \cdot \| \)

The next eight corollaries follow more or less directly from this theorem; the proofs of these corollaries are quite simple and are omitted.

Corollary 1.3.2. (Ore) If the chromatic number of a graph \( G \) is \( n \), then \( G \) has a complete homomorphism of order \( n \).

While this corollary defines the order of a particular complete homomorphism that every graph must have, a given graph may have several other complete homomorphisms of different orders. Figure 2 illustrates this possibility; the graph \( G \) has complete homomorphisms of orders 2, 3, and 4.

Corollary 1.3.3. (Prins) The smallest number \( s \) for which a given graph \( G \) has a complete homomorphism \(^1\) of order \( s \) is the chromatic number of \( G \).

Corollary 1.3.4. If \( \phi \) is a homomorphism of a graph \( G \), then \( \chi(G) \leq \chi(G\phi) \).

---

\(^1\)In a manuscript, "Complete Contractions of Graphs," dated February 1963, Prins refers to these mappings as complete contractions.
Corollary 1.3.5. If \( G \) is any graph and \( S \) is any independent set of points of \( G \), then \( \chi(G) \leq 1 + \chi(G-S) \).

Corollary 1.3.6. (Dirac)\(^1\) If \( G \) is a critical graph and \( S \) is

\(^1\) c.f. Dirac [4].
any independent set of points of $G$, then $\chi(G) = 1 + \chi(G-S)$.

Corollary 1.3.7. If $G$ is any graph having $n$ points and point
independence number $\alpha_0$, then $\chi(G) \leq 1 + n - \alpha_0$.

It is interesting to note in passing the relationship of this
result to the result stated in Berge [2, p. 37] and Ore [16, p. 225] which
asserts that

$$n/\alpha_0 \leq \chi(G).$$

A graph $G$ is **simple** iff every homomorphic image of $G$ is isomorphic
to $G$.

Corollary 1.3.8. A graph $G$ is simple$^1$ iff $G = K_n$, for some $n$.

A homomorphism $\epsilon$ of a graph $G$ is **elementary** iff for two points
$a, b \in V(G)$, $a \epsilon = b \epsilon$ and $\epsilon$ is 1-1 on $V(G)$ - \{a, b\}.

Lemma 1.3.9. Every ordinary homomorphism of a graph $G$ onto
a graph $G'$ can be expressed as a product of elementary homomorphisms.

As an immediate consequence of Corollary 1.3.5. we have

Corollary 1.3.10. The chromatic number of an elementary hom-

---

$^1$Simple with respect to ordinary homomorphisms.
morphic image of \( G \) is at most one greater than the chromatic number of \( G \).

And because of Corollary 1.3.6 we have

**Corollary 1.3.11.** Every elementary homomorphic image of a critical graph \( G \) has the same chromatic number as \( G \).

**Theorem 1.3.12.** (Prins) If the chromatic number of a graph \( G \) is \( s \), and \( G \) has a complete homomorphism of order \( t \), then for all \( n \),

\[ s \leq n < t, \] \( G \) has a complete homomorphism of order \( n \).

**Proof.** The theorem follows directly from Lemma 1.3.9 and

**Corollary 1.3.10.** Consider any homomorphism \( \phi \) which maps \( G \) onto \( K_t \).

By Lemma 1.3.9, \( \phi \) can be expressed as a product \( \epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_m} \) of elementary homomorphisms. Let \( G_1 = G \epsilon_{i_1} \), \( G_2 = G_1 \epsilon_{i_2} \), \ldots, \( G_m = G_{m-1} \epsilon_{i_m} = K_t \).

By Corollary 1.3.10 we know that the chromatic number of \( G_i \) is at most one greater than the chromatic number of \( G_{i-1} \). It follows therefore that

if \( \chi(G) = s \) and \( s \leq n < t \), then there exists at least one \( G_i \) whose chromatic number is \( n \). By Corollary 1.3.2, this \( G_i \) has a complete homomorphism \( \phi_i \) onto \( K_n \). Hence \( G \) has a complete homomorphism \( \epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_1} \phi_{i_1} \) onto \( K_n \).

**Prins**, in considering complete homomorphisms, made an interesting
distinction between two types. **Type-1** complete homomorphisms of a graph $G$ are obtained in the following manner. Take any maximal independent set of points $V_1$ in $V(G)$. Remove the points in $V_1$ from $G$ and take a second maximal independent set of points $V_2$ in $V(G-V_1)$. Iterate this process until $V(G)$ has been completely depleted, and has been partitioned into sets $V_1, V_2, \ldots, V_m$. This partition defines a complete homomorphism $\phi$ of $G$ onto $K_m$: $a \phi = k_i$ iff $a \in V_i$. If a complete homomorphism is not of type-1 it is said to be of **type-2**.

Prins has shown that the homomorphisms stated in **Corollary 1.3.2** and **Theorem 1.3.12** can be required to be of type-1, i.e., if the chromatic number of $G$ is $s$ and if $G$ has a type-1 complete homomorphism of order $t$, then for every $n$, $s \leq n < t$, $G$ has a type-1 complete homomorphism of order $n$. Prins has attempted to characterize the type-2 complete homomorphisms, but has not yet succeeded.

While **Corollary 1.3.3** establishes the minimum order $s$, it remains an open question to determine a decent description of the maximum order $t$ of all complete homomorphisms of a given graph $G$. The following result, which
extends Corollary 1.3.7, establishes a bound for this number $t$.

**Theorem 1.3.13.** Let $G$ be a graph having $n$ points and point

independence number $\alpha_0$, and let $t$ be the largest order of all complete

homomorphisms of $G$. Then $\chi(G) \leq t \leq n + 1 - \alpha_0$.

**Proof.** Let $\phi$ be a complete homomorphism of $G$ onto $K_t$, where $t$ is

maximum, and consider the partition of $V(G)$ determined by the sets

$\phi^{-1}(k_i)$, $i = 1, 2, \ldots, t$. Let $V_1$ be any maximal independent set of $G$

containing $\alpha_0$ points, and consider where among the sets $\phi^{-1}(k_i)$ the points

in $V_1$ lie. Three possibilities exists; $\phi^{-1}(k_i)$ contains no points of $V_1$, $\phi^{-1}(k_i)$ contains some points of $V_1$ and some points of $V-V_1$, or $\phi^{-1}(k_i)$ contains only points of $V_1$. Note, however, that at most one set $\phi^{-1}(k_i)$ can contain only members of $V_1$. It follows therefore that we can pick representatives from at least $t-1$ of these sets which are not members of $V_1$, and therefore the total number of points in $G$ minus these $t-1$ representatives leaves at least the $\alpha_0$ points of $V_1$ remaining. Thus

$$n - (t-1) \geq \alpha_0$$

$$n + 1 - \alpha_0 \geq t.$$
Corollary 1.3.14. \( t \leq \beta_0(G) + 1 \), where \( \beta_0(G) \) is the point cover number of \( G \).

Proof. This follows since \( n = \alpha_0(G) + \beta_0(G) \).\(^1\)

Perhaps the most natural bound for the largest order \( t \) of all complete homomorphisms of a given graph \( G \), having \( q \) lines, is the largest integer \( r \) such that

\[
\binom{r}{2} \leq q.
\]

As one might expect, however, there are cases in which each of the two bounds, \( r \) and \( n+1-\alpha_0 \), gives a better estimate than the other, as the examples in Figure 3 illustrate.

\[\begin{align*}
\text{G}_1: & \quad \begin{array}{c}
\includegraphics[width=0.3\textwidth]{G_1}
\end{array} \\
\text{G}_2: & \quad \begin{array}{c}
\includegraphics[width=0.3\textwidth]{G_2}
\end{array}
\end{align*}\]

FIG. 3.

In \( G_1 \), \( n = 9 \), \( q = 8 \), and \( \alpha_0 = 8 \), hence \( r = 4 \), \( n+1-\alpha_0 = 2 \), while \( t = 2 \).

\(^1\)cf. Gallai [7].
If $G_2$, $n = 10$, $q = 9$, and $\alpha = 5$, hence $r = 4$, $n+1-\alpha = 6$, while $t = 4$.

1.4. Type-$n$ Homomorphisms.

An **$n$-basis** of a graph $G = \langle V, \rho \rangle$ is a subset of points, $V_n \subseteq V$, satisfying

(i) $a, b \in V_n \implies d(a, b) > n$;

(ii) $c \notin V_n \implies d(c, V_n) \leq n$.

A Prins' type-1 complete homomorphism (cf. Section 1.3) can be described in terms of an iterated subtraction from a graph of maximal independent sets of points. Since from the definition of an $n$-basis it follows that maximal independent sets of points are 1-bases, the suggestion naturally arises of considering complete homomorphisms of type-$n$, mappings which can be described in terms of an iterated subtraction from a graph of $n$-bases.

Even though the process of obtaining a series of $n$-bases from a given graph does not always define a complete homomorphism, the fact that some complete homomorphisms which are not of Prins' type-1 can be expressed
as complete homomorphisms of type-n seems to justify the following definition.

A full homomorphism $\phi$ of a graph $G = \langle V, e \rangle$ onto a graph $G' = \langle V', e' \rangle$ is of type-n iff there exists an ordering of the sets $\phi^{-1}(a')$, for $a' \in V'$, say $V_1, V_2, \ldots, V_k$, such that

(i) $V_1$ is an n-basis of $G$, and

(ii) for $i \geq 2$, $V_i$ is an n-basis of $G - \bigcup_{j=1}^{i-1} V_j$,

and for no $m$, $0 < m < n$, is $\phi$ of type-$m$.

Note that not every complete homomorphism is of type-n, for some n. An example of this is the complete homomorphism given in Figure 4; neither of the sets $\{a, d\}$, $\{c, e\}$ or $\{b, f\}$ is an n-basis of $G$, for any n.

![Diagram](image)

FIG. 4.
If a complete homomorphism is not of type-\(n\) for any \(n\), it is said to be of type-e.

An example of a type-4 complete homomorphism is given in Figure 5; the subset \(\{e,j\}\) is a 4-basis of \(G\); \(\{c,i,1\}\) is a 4-basis of \(G - \{e,j\}\); \(\{b,d,g\}\) is a 4-basis of \(G - \{e,j\} - \{c,i,1\}\); and what remains in \(G - \{e,j\} - \{c,i,1\} - \{b,d,g\}\) are the three isolated points \(a\), \(f\), and \(h\).

\[
\begin{array}{c}
\text{a} \\
\downarrow \\
\text{b} \\
\downarrow \\
\text{c} \\
\downarrow \\
\text{d} \\
\downarrow \\
\text{e} \\
\downarrow \\
\text{f} \\
\downarrow \\
\text{g} \\
\downarrow \\
\text{i} \\
\downarrow \\
\text{j}
\end{array}
\]

\[
\begin{array}{c}
k_2 \\
\downarrow \\
k_3 \\
\downarrow \\
k_4 \\
\downarrow \\
k_1
\end{array}
\]

\[\phi: \begin{array}{c}
a,f,h \rightarrow k_1 \\
b,d,g \rightarrow k_2 \\
c,i,1 \rightarrow k_3 \\
e,j \rightarrow k_4
\end{array}\]

**FIG. 5.**

1.5. Two Invariants of a Graph.

Two interesting invariants of a graph arise out of considerations of the following question: Given a graph \(G\), for what graphs \(H\) do there exist
homomorphisms \( \phi \) such that \( H\phi = G \)? A partial answer to this question is given by the following three propositions.

**Proposition 1.5.1.** For every connected graph \( G \) there exists a tree \( T \) and a homomorphism \( \phi \) such that \( T\phi = G \).

**Proposition 1.5.2.** For every connected graph \( G \) there exists a chain \( C \) and a homomorphism \( \phi \) such that \( C\phi = G \).

**Proposition 1.5.3.** For every connected graph \( G \) there exists a cycle \( P \) and a homomorphism \( \phi \) such that \( P\phi = G \).

It is natural in view of these last two propositions to make the following definitions.

The **chain length** of a connected graph \( G \), \( ch(G) \), is the smallest number \( k \) for which there exists a chain \( C \) of length \( k \) and a homomorphism \( \phi \) such that \( C\phi = G \). Alternatively, \( ch(G) \) is the length of the shortest path which contains all the lines of \( G \).'

The **cycle length** of a connected graph \( G \), \( cy(G) \), is the smallest number \( k \) for which there exists a cycle \( P \) of length \( k \) and a homomorphism \( \phi \) such that \( P\phi = G \).
Proposition 1.5.4. For every connected graph $G$,

$$
ch(G) \leq cy(G) \leq 2ch(G).
$$

Proof. Clearly $ch(G) \leq cy(G)$, for if $a_1, a_2, \ldots, a_k, a_1$ is any cycle $P$ of length $k$ which maps onto $G$ under a homomorphism $\phi$, then a chain $a_1', a_2', \ldots, a_k'$, $b$ can be defined of the same length $k$ which maps onto $G$ under the homomorphism $\phi'$ for which $a_i' \phi' = a_i \phi$ and $b \phi' = a_1 \phi$.

To show that $y(G) \leq 2ch(G)$, let $C_k$ with points $c_1, c_2, \ldots, c_k, c_{k+1}$ be the shortest chain which maps under a homomorphism $\phi$ onto $G$. Then a cycle $c_1', c_2', \ldots, c_{k+1}', d_k, d_{k-1}, \ldots, d_2, c_1'$ of length $2k$ can be defined which maps under the homomorphism $\phi'$ onto $G$, where $c_i' \phi' = c_i \phi$ and $d_i \phi' = c_i \phi$.

The bounds given in Proposition 1.5.4 are achieved in the following two cases; $ch(P_n) = cy(P_n) = n$, and $cy(C_n) = 2ch(C_n) = 2n$.

Proposition 1.5.5. For every connected graph $G$ with $q$ lines,

(i) $q \leq ch(G) \leq 2q-1$.

(ii) $q \leq cy(G) \leq 2q$.

Proof. (i) It is obvious that $q \leq ch(G)$, since the shortest chain which maps onto $G$ must have at least as many lines as $G$. 

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That \( \text{ch}(G) \leq 2q \) is an immediate consequence of the following theorem, quoted from Ore [16, p. 41]:

"Theorem 3.1.4. In a finite connected graph it is always possible to construct a cyclic directed path passing through each edge once and only once in each direction."

(ii) The same arguments hold for \( \text{cy}(G) \).

Proposition 1.5.6. If \( G \) is a connected graph having \( q \) lines and in which every point has even degree, then \( \text{ch}(G) = \text{cy}(G) = q \).

Proof. \( G \) is an Euler graph, i.e., a graph in which it is possible to find a cyclic path of lines such that each line in the graph appears once and only once.

Theorem 1.5.7. If \( G \) is a connected graph having \( q \) lines and exactly two points \( a, b \) of odd degree, then

\[
(i) \quad \text{ch}(G) = q;
\]

\[
(ii) \quad \text{cy}(G) = q + d(a, b).
\]

Proof. (i) See Ore [16, p. 40]. Such a graph has at least one path which passes through all the lines of \( G \) once and only once, which begins
with point b and ends with point a. Hence, \( \text{ch}(G) = q \).

(ii) Clearly, \( \text{cy}(G) \leq q + d(a,b) \). A path corresponding to the
one referred to in (i) above can obviously be extended to a cycle by an
additional path, of length \( d(a,b) \), from point a back to point b. In order
to show that \( \text{cy}(G) \neq q + d(a,b) \), let \( P_k \) with points \( p_1, p_2, \ldots, p_k \) be
the smallest cycle which maps onto \( G \) under the homomorphism \( \phi_k \), and let \( P_n \)
be a cycle of length \( n = q + d(a,b) \) which maps onto \( G \) under the homomorphism
\( \phi_n \). Obviously, \( k \leq n \). Consider the multi-graph \( G_k \) generated by \( P_k \) and the
homomorphism \( \phi_k \), i.e., \( V(G_k) = V(G) \), and there are as many lines between
two points \( c,d \) in \( G_k \) as there are pairs of adjacent points \( p_i, p_j \) in \( P_k \) such
that either \( p_i \phi_k = c \) and \( p_j \phi_k = d \), or \( p_i \phi_k = d \) and \( p_j \phi_k = c \). It follows that
\( G_k \) has \( k \) lines. The degree of a point \( c \) in \( G_k \) equals the sum of the degrees
of the points \( p_i \) for which \( p_i \phi_k = c \). Since every point \( p_i \) has even degree
in \( P_k \), every point \( c \) has even degree in \( G_k \).

Consider also the multi-graph \( G_n \) generated by \( P_n \) and the homomorphism
\( \phi_n \); \( G_n \) has \( n \) lines. Clearly for every line \((c,d)\) in \( G \) there is a line \((c,d)\)
in \( G_k \) and a line \((c,d)\) in \( G_n \). Hence, let us remove for each of the \( q \) lines
in $G$ a corresponding line in both $G_k$ and $G_n$. Denote the resulting graphs by $G_k-G$ and $G_n-G$, which have $k-q$ and $n-q = d(a,b)$ lines, respectively, where $k-q \leq d(a,b)$. Since the points $a, b$ have even degree in $G_k$ and odd degree in $G$, they have odd degree in $G_k-G$. And since they are the only two points of odd degree in $G$, they are the only two points of odd degree in $G_k-G$. It follows, therefore, by Theorem 2.2.2, Ore [16, p. 24], that since $a, b$ are the only points of odd degree in $G_k-G$, $a$ and $b$ are connected in $G_k-G$. Hence $G_k-G$ has at least $d(a,b)$ lines, i.e., $k-q \geq d(a,b)$. Hence $k-q = d(a,b)$, i.e., $k = q + d(a,b)$.

Theorem 1.5.8. If $G$ is a connected graph having $q$ lines and $2n$ points of odd degree, $n \geq 2$, and diameter $\delta$, then

(i) $q + n-1 \leq ch(G) \leq q + (n-1)\delta$;

(ii) $q + n \leq cy(G) \leq q + n\delta$.

Proof. (i) That $ch(G) \leq q + (n-1)\delta$ follows by virtue of Theorem 3.1.2, Ore [16, p. 40]. Such a graph $G$ has a set of $n$ line disjoint paths which contain all the lines of $G$ once and only once. Let $W_1, W_2, \ldots, W_n$ represent these $n$ paths. A path which starts with the first
point of $W_1$, which ends with the last point of $W_n$, and which connects the
last point of $W_i$ with the first point of $W_{i+1}$ by a path of the length less
than or equal to the diameter $\delta$ can always be constructed in $G$. Such a path
clearly has length less than or equal to $q + (n-1)\delta$. It is a simple matter
to construct a chain of the same length which maps onto this path. In order
to demonstrate that $q + n - 1 \leq \text{ch}(G)$, let $\text{ch}(G) = k$, let $C_k$ with points
$c_1, c_2, \ldots, c_{k+1}$ be a chain of length $k$, and let $\phi_k$ be a homomorphism which
maps $C_k$ onto $G$. Let $G_k$ be the multigraph generated by $C_k$ and the homomorphism
$\phi_k$, as in the proof of Theorem 1.5.7.

Since only points $c_1$ and $c_{k+1}$ have odd degrees in $C_k$, at most two
points have odd degrees in $G_k$. Obviously, $G_k$ can be obtained from $G$ by
inserting additional lines between various pairs of adjacent points in $V(G)$.
The fewest number of additional lines which could possibly convert $G$, with
$2n$ points of odd degree, into the multi-graph $G_k$, with at most two points of
odd degree, is $(2n-2)/2 = n - 1$. This can be accomplished (if possible) by
leaving two points of odd degree in $G$ untouched by the addition of new lines,
and by pairing-off the remaining $2n-2$ points of odd degree into adjacent pairs
and adding a new line between each pair, i.e., by adding n-1 new lines.

(ii) The proof for \( \text{cy}(G) \) is very similar to that for \( \text{ch}(G) \) and is omitted. ||

It is interesting to compare the upper bounds given in Proposition 1.5.5, \( \text{ch}(G) \leq 2q - 1 \), and in Theorem 1.5.8, \( \text{ch}(G) \leq q + (n-1)\delta \).

There are cases in which each of the two bounds gives a better estimate than the other, as the examples in Figure 6 illustrate.

![Diagram of graphs G1 and G2](image)

FIG. 6.

In \( G_1 \), \( q = 5 \), \( \delta = 3 \), and \( n = 3 \), i.e., \( G_1 \) has 2n = 6 points of odd degree; hence \( 2q - 1 = 9 \), while \( q + (n-1)\delta = 11 \). Note that \( \text{ch}(G_1) = 7 \).

In \( G_2 \), \( q = 7 \), \( \delta = 2 \), and \( n = 2 \); hence \( 2q - 1 = 13 \), while \( q + (n-1)\delta = 9 \).

Note that \( \text{ch}(G_2) = 8 \). The examples in Figure 7 illustrate that the bounds given in Theorem 1.5.8 can be achieved.
In $G_3$, $q = 3$, $\delta = 2$, and $n = 2$; hence $q + n - 1 = 4 = \text{ch}(G_3)$. In $G_4$, $q = 4$, $\delta = 2$, and $n = 2$; hence $q + (n-1)\delta = 6 = \text{ch}(G_4)$, and $q + n\delta = 8 = \text{cy}(G_4)$.

In $G_5$, $q = 6$, $\delta = 3$, and $n = 3$; hence $q + n = 9 = \text{cy}(G_5)$.

**Corollary 1.5.9.** (i) $\text{ch}(K_{2n}) = \binom{2n}{2} + n - 1$;

(ii) $\text{ch}(K_{2n+1}) = \binom{2n+1}{2} = \text{cy}(K_{2n+1})$;

(iii) $\text{cy}(K_{2n}) = \binom{2n}{2} + n$.

**Proof.** (i) and (iii) follow at once from Theorem 1.5.8 and the observation that $\delta(K_n) = 1$, for all $n \geq 2$.

(ii) follows from Proposition 1.5.6 since every $K_{2n+1}$ is an Euler graph.

**Lemma 1.5.10.** Let $\phi$ be a homomorphism of a connected graph $G$ onto a graph $G'$. Then (i) $\text{ch}(G) \geq \text{ch}(G')$;

(ii) $\text{cy}(G) \geq \text{cy}(G')$. 

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Proof. (i) Let ch(G) = k, and \( \phi_k \) be a homomorphism which maps 
\( C_k \) onto G. Obviously, the product mapping \( \phi_k \phi \) is a mapping of \( C_k \) onto \( G' \):

\[
C_k \xrightarrow{\phi_k} G \xrightarrow{\phi} G'.
\]

Hence \( ch(G') \leq k = ch(G) \).

(ii) A similar argument also holds for \( cy(G) \).

1.6. Endomorphism-free Graphs.

A graph G is endomorphism-free iff every endomorphic image of G is isomorphic to G.

Theorem 1.6.1. If \( \phi \) is an endomorphism of a graph G, then the chromatic number of \( G^\phi \) equals the chromatic number of G.

Proof. Obviously, \( \chi(G^\phi) \leq \chi(G) \). But by Corollary 1.3.4, \( \chi(G) \leq \chi(G^\phi) \).

Corollary 1.6.2. Every critical graph is endomorphism-free.

Inspection of all the graphs with six or fewer points shows that the converse to this corollary is true for these graphs.

Proposition 1.6.3. If a graph has at most six points and is endomorphism-free, then it is critical.

The graph in Figure 8, which is endomorphism-free and not critical,
shows that the converse does not hold for all graphs. But Figure 8 has eight points, leaving the problem of whether there exists a seven point counter-example.

![Diagram](image)

**FIG. 8.**

1.7. Homomorphisms for Functional Digraphs.

A **functional digraph** $G = \langle V, \rho \rangle$ is a graph whose adjacency relation $\rho$ is a function from $V$ into $V$; in other words, the adjacency relation is not necessarily symmetric and the **out degree** of every point equals one. In this section we will consider only full homomorphisms of functional digraphs onto functional digraphs; these mappings $\phi$ must satisfy the condition

$$(4) \quad a\phi = b\phi \implies \rho(a)\phi = \rho(b)\phi.$$  

A homomorphism $\phi$ is prime iff in every expression of $\phi$ as a product $\phi_1\phi_2$, either $\phi_1$ or $\phi_2$ is an isomorphism.
Yoeli and Ginzburg [22] have given a complete characterization of this class of homomorphisms\(^1\), which states that every such homomorphism (other than an isomorphism) can be expressed as a product over the following three types of prime\(^2\) homomorphisms:

(i) \(\phi_\alpha\) is a homomorphism which maps a cycle \(P\) of length \(k\) onto a cycle \(P'\) of length \(k/p\), where \(p\) is a prime divisor of \(k\); for \(\alpha \in V(P)\), \((\rho^i(\alpha))\phi_\alpha = (\rho^j(\alpha))\phi_\alpha\) iff \(i \equiv j (\mod k/p)\);

(ii) \(\phi_\beta\) is a homomorphism which identifies exactly two points \(a, b, a\phi_\beta = b\phi_\beta\), for which \(\rho(a) = \rho(b)\);

(iii) \(\phi_\gamma\) is a homomorphism which identifies two cycles \(P\) and \(P'\) of the same length \(k\), where for arbitrary \(\alpha \in P, a' \in P'\),

\((\rho^n(\alpha))\phi_\gamma = (\rho^n(a'))\phi_\gamma, n = 0, 1, 2, \ldots, k-1.\)

Figures 9 - 11, with mappings marked \(\phi_\alpha, \phi_\beta,\) and \(\phi_\gamma\), illustrate respectively the three types of prime homomorphisms.

\(^1\)It is interesting to note that Yoeli and Ginzburg have incorporated the two conditions (1) and (2), for a full homomorphism, into the one condition (5) \(\rho^* \phi = \phi \rho\). Conditions (1) and (2) and condition (5) are clearly equivalent.

\(^2\)Yoeli and Ginzburg call these elementary homomorphisms.
Figure 9 illustrates as well that Lemma 1.3.9 does not hold for homomorphisms of functional digraphs; i.e., the homomorphism $\phi_\alpha$ in Figure 9 cannot be represented as a product of elementary homomorphisms.
Yoeli and Ginzburg also show that provided G is a connected functional
digraph, every series of prime homomorphisms of G onto G' has the same
length.

1.8. Pathwise Homomorphisms.

An ordinary homomorphism \( \phi \) of a graph \( G = \langle V, \rho \rangle \) onto a graph
\( G' = \langle V', \rho' \rangle \) is a pathwise homomorphism iff it satisfies the following
condition

\[(6) \text{ for every path } a'_1, a'_2, \ldots, a'_n \text{ in } G' \text{ there exists a path } a_1, a_2, \ldots, a_n \text{ in } G \text{ such that } a'_{ij} \phi = a_{ij}, \ j = 1, 2, \ldots, n.\]

Note the similarity of condition (6) to condition (2), when the latter is
rephrased as follows:

\[(2') \text{ for every path } a', b' \text{ of length one in } G' \text{ there exists a path } a, b \text{ in } G \text{ such that } a \phi = a' \text{ and } b \phi = b'.\]

Pathwise homomorphisms were first defined by Robert McNaughton [14] for state
transition graphs. The only result he obtained about these mappings was
that the "order" of a pathwise homomorphic image G' of a state transition
graph G cannot be greater than the "order" of G. The "order" of a state

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transition graph G will not be defined here; suffice it to say that it is
a measure of the cyclic complexity of G. A notion similar to "order",
which is frequently mentioned in graph theory, is what Ore [16, p. 67]
calls circuit rank, \( \gamma(G) \), the number of independent cycles. For any graph
G with \( p \) points, \( q \) lines, and \( k \) connected components, \( \gamma(G) = q - p + k \).

It is natural therefore to state the following

Theorem 1.8.1. If \( \phi \) is a pathwise homomorphism of a graph

\[ G = \langle V, \rho \rangle \text{ onto a graph } G' = \langle V', \rho' \rangle, \text{ then } \gamma(G) \geq \gamma(G'). \]

Proof. We proceed by induction on the circuit rank of \( G \).

Lemma 1.8.2. If \( \phi \) is a pathwise homomorphism of a graph \( G = \langle V, \rho \rangle \)
onto a graph \( G' = \langle V', \rho' \rangle \) and \( \gamma(G) = 0 \), then \( \gamma(G') = 0 \) also.

Proof. Suppose \( \gamma(G') \geq 1 \), i.e., \( G' \) contains at least one cycle

\( P_k \) of length \( k \geq 3 \). Suppose \( G \) has \( p \) points. Certainly there exists a path

\[ a_1', a_2', ..., a_{p+1}' \]  

(perhaps several times) around \( P_k \) of length \( p \), with

\( p+1 \) points. Since \( \phi \) is a pathwise homomorphism there must exist a path

\[ a_1, a_2, ..., a_{p+1} \]  
in \( G \) such that \( a_i \phi = a_i' \). Any such path in \( G \) has

\( p+1 \) points and therefore must contain at least one point which occurs twice.
Such a path therefore must have a sub-path of the form \( \omega = a, a_i, \ldots, a_i, a \).

But since \( \gamma(G) = 0 \), G has no cycles and every path in G of the form \( \omega \) must be such that \( a_i = a_i \), and similarly \( a_i = a_i \), etc., until finally the sub-path of the form \( \omega \) must contain a sub-path either of the form \( a_i a_i \) or \( a_i a_i \). But it cannot contain a sub-path of the form \( a_i a_i \) since every two consecutive points of the corresponding path in \( G' \) are distinct. Also, it cannot contain a sub-path of the form \( a_i a_i a_i \) since every three consecutive points of the corresponding path in \( G' \) are mutually distinct. Hence no path can exist in G which maps onto the path of length \( p \) in \( G' \) around the cycle \( P_k \). Hence \( \phi \) cannot be a pathwise homomorphism, contrary to hypothesis. The graph \( G' \) must therefore contain no cycles, hence \( \gamma(G') = 0 \).

Continuing with the proof of Theorem 1.8.1, assume it is true for all G, with \( \gamma(G) = n \). We will show that it is also true for G with \( \gamma(G) = n + 1 \). Consider the sets of lines of G defined by \( \phi^{-1}((a', b')) \), for all \( (a', b') \in \rho' \). Since \( \gamma(G) = n + 1 \geq 1 \), there must be a set \( \phi^{-1}((a', b')) \) such that

\[
\gamma(G - \phi^{-1}((a', b'))) < \gamma(G).
\]

It can be shown then that \( \phi \) is a pathwise homomorphism of \( G - \phi^{-1}((a', b')) \).
onto $G' - (a', b')$, and hence, by hypothesis

$$\gamma(G - \phi^{-1}((a', b'))) \geq \gamma(G' - (a', b')).$$

But since $\gamma(G) \geq \gamma(G - \phi^{-1}((a', b'))) + 1$ and $\gamma(G' - (a', b')) + 1 \geq \gamma(G')$, it follows that $\gamma(G) \geq \gamma(G')$. ||

It is interesting to note that no analog of Corollary 1.3.2 holds for pathwise homomorphisms. For example, the pentagon $P_5$, has no nontrivial pathwise homomorphisms; $\chi(P_5) = 3$, yet $P_5$ does not map under any pathwise homomorphism onto $K_3$. It remains an open question to determine those graphs which are simple with respect to pathwise homomorphisms.

One might also note in passing that Lemma 1.3.9 does not hold for pathwise homomorphisms. The mapping $\phi_\alpha$ in Figure 8, is a pathwise homomorphism which cannot be expressed as a product of elementary pathwise homomorphisms.

1.9. Strong Homomorphisms.

Let $g$ be the characteristic function of the adjacency relation of a graph $G = \langle V, E \rangle$, i.e., $g(a, b) = 1$ iff $(a, b) \in E$ and $g(a, b) = 0$ iff $(a, b) \notin E$. 

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A full homomorphism \( \phi \) of a graph \( G \) onto a graph \( G' \) is a **strong homomorphism** iff \( \phi \) satisfies the condition

\[(7) \quad g(a,b) = g'(a\phi, b\phi).\]

Condition (7) can also be expressed as

\[(7') \quad (a,b)\in\rho \implies (a\phi, b\phi)\in\rho'.\]

\[(a,b)\not\in\rho \implies (a\phi, b\phi)\not\in\rho'.\]

In words, condition (7) requires not only that two adjacent points in \( G \) map to two adjacent points in \( G' \), but also that two non-adjacent points in \( G \) map to two non-adjacent points in \( G' \).

**Theorem 1.9.1.** If \( \phi \) is a strong homomorphism of a graph \( G = \langle V, \rho \rangle \) onto a graph \( G' = \langle V', \rho' \rangle \), then there exists a subgraph \( H \) of \( G \) such that \( H \cong G' \).

**Proof.** Consider the inverse-image sets \( \phi^{-1}(a') \), for \( a' \in V' \). Pick, arbitrarily, a set of representatives, \( V_H \), one point from each of these sets, and consider the full subgraph \( H = \langle V_H, \rho_H \rangle \) of \( G \) generated by this set of representatives. The strong homomorphism \( \phi \) when restricted to \( H \) is an isomorphism of \( H \) onto \( G' \).
Corollary 1.9.2. If \( \phi \) is a strong ordinary homomorphism of a graph \( G \) onto a graph \( G' \), then the chromatic number of \( G \) equals the chromatic number of \( G' \).

Strong homomorphisms were first studied by Culik [3], who gave the following characterization of those graphs which have no non-trivial strong homomorphisms.

Theorem 1.9.3. (Culik) A graph \( G \) is simple with respect to strong homomorphisms iff for every two distinct points \( a, b \in V(G) \) there exists a point \( c \in V(G) \) such that

\[
g(a,c) \neq g(b,c) \text{ or } g(c,a) \neq g(c,b).
\]

Consider the equivalence relation \( \equiv \) defined on \( V(G) \) by

\[a \equiv b \text{ iff } (\forall c)[g(a,c) = g(b,c) \land g(c,a) = g(c,b)],\]

or alternatively,

\[a \equiv b \text{ iff } [\rho(a) = \rho(b) \text{ and } \rho^{-1}(a) = \rho^{-1}(b)].\]

Lemma 1.9.4. If \( \phi \) is a strong homomorphism of a graph \( G \) onto a graph \( G' \), then \( a \phi = b \phi \iff a \equiv b \).

Lemma 1.9.5. Given a graph \( G = \langle V, \rho \rangle \) and two points \( a, b \in V \) for which \( a \equiv b \), a mapping \( \phi \) which identifies \( a \) and \( b \) (\( a \phi = b \phi \)) and which is
Theorem 1.9.6. Every strong homomorphism of a graph \( G \) onto a graph \( G' \) can be expressed as a product of elementary strong homomorphisms.

Proof. This is a consequence of Lemmas 1.9.4 and 1.9.5.

Theorem 1.9.7. (Culik) Any given graph \( G \) has exactly one strong homomorphism \( \phi_s \) onto a graph \( G_s \), such that \( G_s \) is simple.\(^1\)

Proof. From Theorem 1.9.3, and Lemmas 1.9.4 and 1.9.5, a graph \( G \) is simple iff the equivalence relation \( \equiv \) on \( V(G) \) is the identity relation. The strong homomorphism \( \phi_s \) is that mapping for which \( a \phi = b \phi \) iff \( a \equiv b \).

Corollary 1.9.9. (Sabidussi) Either \( G \cong G_s \) or \( G \) contains a subgraph \( H \) such that \( H \cong G_s \).\(^2\)

Lemma 1.9.4 and the definition of the equivalence relation \( \equiv \) suggest that one define the following two, slightly broader classes of homomorphisms. A full homomorphism \( \phi \) of a graph \( G \) onto a graph \( G' \) is a right homomorphism iff \( \phi \) satisfies the condition

\[(8) \quad a \phi = b \phi \implies \rho(a) = \rho(b).\]

\(^1\)Simple with respect to strong homomorphisms.

\(^2\)Theorem 1.9.1 was suggested by this result of Sabidussi [18, 19].
A full homomorphism \( \phi \) of a graph \( G \) onto a graph \( G' \) is a **left homomorphism** iff \( \phi \) satisfies the condition

\[(9) \quad a \phi = b \phi \implies \rho^{-1}(a) = \rho^{-1}(b).\]

Right homomorphisms appear to be of special interest in view of their relation to homomorphisms of finite automata, where for a given automaton two states \( a \) and \( b \) are said to be equivalent iff they satisfy a condition exactly parallel to condition (8).\(^1\) Note that conditions (8) and (9) become identical when the graphs are assumed to be ordinary graphs; hence left, right and strong homomorphisms are all the same when defined for ordinary graphs. One might also note in passing the similarity between right homomorphisms and the prime homomorphisms of type \( \phi_B \) in Section 1.7.

*Lemma 1.9.4* and the definition of the equivalence relation \( \equiv \) also suggest that one define an even broader class of homomorphisms.

A full homomorphism \( \phi \) of a graph \( G \) onto a graph \( G' \) is **semi-strong**\(^2\) iff \( \phi \) satisfies the condition

\[(10) \quad a \phi = b \phi \implies n(a) \subseteq n(b) \lor n(b) \subseteq n(a).\]

---

\(^1\)cf. [14], [15].

\(^2\)Strong homomorphisms satisfy the stronger condition that \( a \phi = b \phi \implies n(a) = n(b)\).
Theorem 1.9.9. If \( \phi \) is a semi-strong homomorphism of a graph \( G \) onto a graph \( G' \), then there exists a subgraph \( H \) of \( G \) such that \( H \cong G' \).

Proof. From each of the sets \( \phi^{-1}(a') \), \( a' \in V' \), pick one point which has maximum degree. The subgraph \( H \) generated by this set of representatives is isomorphic to \( G' \). Obviously, \( (a,b) \in \phi \rightarrow (a',b') \in \phi' \), where \( a \phi = a' \), \( b \phi = b' \), and \( a \) and \( b \) are the representatives chosen from \( \phi^{-1}(a') \) and \( \phi^{-1}(b') \), respectively.

Also, \( (a',b') \in \phi' \rightarrow (a,b) \in \phi \), for if \( (a',b') \in \phi' \), there must exist \( (a'',b'') \in \phi \) such that \( a'' \phi = a' \) and \( b'' \phi = b' \). Since \( n(a'') \leq n(a) \) and \( b'' \in n(a'') \), it follows that \( b'' \in n(a) \). By the symmetry of the neighborhood relation, \( a \in n(b') \). Since \( n(b'') \subseteq n(b) \) it follows that \( a \in n(b) \). Hence \( (a,b) \in \phi \).

One can show without much difficulty that the results previously stated for strong homomorphisms can be stated in a very similar way for semi-strong homomorphisms. In particular, Corollary 1.9.3, Theorem 1.9.6, Theorem 1.9.7, and Corollary 1.9.8 all hold when semi-strong is substituted for strong. In addition, Theorem 1.9.3 can be restated as:
Theorem 1.9.10. A graph $G$ is simple with respect to semi-strong homomorphisms iff for every two distinct points $a, b \in V(G)$, $n(a) \notin n(b)$ and $n(b) \notin n(a)$.

It is natural in view of Theorem 1.9.9, to ask if for a full homomorphism $\phi$ of a graph $G$ onto a graph $G'$ there exists a subgraph $H$ of $G$ such that $H \cong G'$, is $\phi$ necessarily a semi-strong homomorphism? The answer is no; as shown by Figure 12.

\[
\begin{array}{c}
\text{a} \quad \text{b} \\
\text{d} \\
\text{f} \\
\end{array} \quad \begin{array}{c}
\text{c} \\
\text{e} \\
\end{array} \quad \begin{array}{c}
\text{d'} \\
\text{c'} \\
\end{array} \\
\]
\[
\begin{array}{c}
\text{e'} \\
\end{array} \\
\text{f'} \\
\]

$\phi: c, f \rightarrow f'$
$e \rightarrow e'$
$d \rightarrow d'$
$a, b \rightarrow c'$

$G \xrightarrow{\phi} G_1$

FIG. 12.

In Figure 12, $a \phi = b \phi$, yet neither $n(a) \subseteq n(b)$ nor $n(b) \subseteq n(a)$ holds.
2. CONTRACTIONS

2.1. General Definition.

A question arises in the literature on homomorphisms of graphs as to the interpretation of the homomorphic image of a pair of adjacent points \( a, b \in V(G) \) which map to the same point \( c' \in V(G') \). The definition of homomorphism given in Section 1.1, condition (1) in particular, specifies that the image is the point \( c' \) with a loop attached to it. However, Ore, Dirac, and several others choose to define homomorphisms for graphs for which the image of two adjacent points can be one point without a loop. These types of homomorphisms, which do not necessarily satisfy condition (1), can be defined as mappings \( \phi \) which satisfy the following modification of condition (1):

\[
(1m) \quad (a, b) \in \rho \Rightarrow (a_\phi, b_\phi) \in \rho' \lor [a_\phi = b_\phi \land (a_\phi, b_\phi) \notin \rho'].
\]

Following the terminology suggested by Dirac, Harary, and others, we will call a mapping \( \phi \) of a graph \( G = \langle V, \rho \rangle \) into a graph \( G' = \langle V', \rho' \rangle \) which is a function from \( V \) into \( V' \) satisfying condition (1m) a contraction.

It is this distinction which is the basis for placing homomorphisms, mappings which satisfy condition (1), in one class, and contractions, mappings
which do not necessarily satisfy condition (1) but which do satisfy condition (1m), in another class.

However, when considering homomorphisms of reflexive graphs, i.e., graphs whose adjacency relation is reflexive, the distinction we have made between homomorphisms and contractions disappears. For in considering reflexive graphs, a point is always considered to be adjacent with itself, and thus what distinguishes one graph from another is no longer its set of points and lines but rather the underlying graph, i.e., its set of points and lines minus its loops. In diagramming reflexive graphs it would seem therefore unnecessary to include all the loops. Hence, in Figure 13 for example, the homomorphism \( \phi \) of \( G_1 \) onto \( G_2 \) could as well be represented by the homomorphism \( \phi' \) of \( G'_1 \) onto \( G'_2 \).

\[ a_1 \quad b_1 \quad a_2 \]
\[ G_1 \xrightarrow{\phi} G_2 \]

\[ a_1 \quad b_1 \quad a_2 \]
\[ G'_1 \xrightarrow{\phi'} G'_2 \]

FIG. 13.
Forgetting, for a moment, that the points in $G_1^1$ and $G_2^1$ are assumed to be adjacent with themselves, it would seem that $\phi$ is a homomorphism, while $\phi'$ is a contraction, since it fails to satisfy condition (1). But $\phi'$ does satisfy condition (1) if we assume that the points are adjacent with themselves, and hence the distinction between the two mappings disappears.

2.2. Connected and Independent Contractions.

Ore [16] makes an interesting distinction between two types of contractions.\(^1\) A contraction $\phi$ is connected iff the subgraphs of $G$ generated by the sets of points $\phi^{-1}(a')$, for $a' \in V'$, are connected. A contraction $\phi$ is independent iff the subgraphs of $G$ generated by the sets of points $\phi^{-1}(a')$, for $a' \in V'$, are totally disconnected. Ore's independent contractions coincide exactly with the ordinary homomorphisms when defined over the class of ordinary graphs.

*Theorem 2.2.1.* (Ore) Any contraction is the product of a connected and an independent contraction.

\(^1\)Ore calls these mappings homomorphisms.
The **leaf composition graph** of a graph $G$ is the graph whose points are the leaves of $G$; two points are adjacent iff the corresponding leaves are adjacent. Two points $a,b$ belong to the same leaf iff there exists a sequence of cycles $p_1, p_2, \ldots, p_k, a \in V(p_i), b \in V(p_k)$, and $V(p_i) \cap V(p_{i+1}) \neq \emptyset$.

**Theorem 2.2.2. (Ore)** Any graph $G$ has a contraction onto its leaf composition graph $G_1$ such that the inverse image sets are the leaves of $G$. The graph $G_1$ is circuit free.

Theorem 2.2.2 implies, among other things, that every graph has at least one non-trivial tree as a contraction image. It follows also that since any two adjacent points can map to a single point, without a loop, every graph has $K_1$ as a contraction image. Dirac [5] gives the following definition of a contraction: a graph $G = \langle V, \rho \rangle$ can be **contracted** onto the graph $G' = \langle V', \rho' \rangle$ if there exists a function $\phi$ of $V$ onto $V'$ such that

(i) $(\forall a') (a' \in V' \implies G - (G - \phi^{-1}(a'))) \text{ is connected}$, and

(ii) $(\forall a') (\forall b') (a', b' \in V' \wedge (a', b') \in \rho') \implies G \text{ contains at least one line joining a point of } \phi^{-1}(a')$ to
a point of $\phi^{-1}(b')]$.\footnote{This is equivalent to condition (2) of Section 1.1.}

In words, $G = G'$ or $G'$ can be obtained from $G$ by shrinking each of a set of connected subgraphs of $G$ into a single point. It is clear from the definition that the contractions that Dirac considers are all connected. Several of the papers written about connected contractions, by Dirac [5], Wagner [21], and Halin [9], are attempts to clarify the following

**Conjecture of H. Hadwiger:** If the chromatic number of a graph $G$ in $n$, then $G$ has a connected contraction onto $K_n$.\footnote{Note the similarity of this Conjecture to Corollary 1.3.2.}

The conjecture is known to be true for values of $n \leq 4$; it has been shown that a proof for $n = 5$ would be equivalent to a proof of the famous Four Color Conjecture. A typical result directed towards this conjecture is the following: every ordinary graph with $n \geq 6$ points and at least $3n-5$ lines has a connected contraction onto $K_5$ [6]. One property which is preserved under connected contractions but not under homomorphisms is planarity, i.e., if $G$ is planar and $\phi$ is a connected contraction of $G$ then $G\phi$ is also planar.

The same cannot be said for ordinary homomorphisms.
2.3. Simplicial Mappings.

In combinatorial topology, see Pontryagin [17] for example, a class of mappings known as simplicial mappings are defined for complexes and are shown to be approximations of continuous mappings of polyhedra. When interpreted for graphs simplicial mappings become, in one sense, approximations of contractions. Let a point be a 0-simplex, and let two points a, b and a line (a, b) between them be a 1-simplex. A mapping \( \phi \) of a graph \( G = \langle V, \rho \rangle \) into a graph \( G' = \langle V', \rho' \rangle \) is simplicial iff

(i) \( \phi \) is a function from \( V \) into \( V' \);

(ii) the image of a simplex in \( G \) is a simplex in \( G' \).

It can be seen without much difficulty that condition (ii) is equivalent to condition (I'm). If one were to extend this definition to get onto simplicial mappings, perhaps the most natural way would be to require that \( \phi \) map the simplexes of \( G \) onto the simplexes of \( G' \). The definition of a contraction is exactly this extension.

2.4. Homeomorphisms.

A class of mappings of graphs which closely resembles, but which
in fact overlaps, the class of contractions are the homeomorphisms.

Two ordinary graphs \( G \) and \( G' \) are said to be homeomorphic iff one can be obtained from the other by successive applications of the following two operations:

(i) replace a line \((a,b)\in\rho\) by two lines \((a,c)\) and \((c,b)\),

\( c \) being a new point;

(ii) replace two lines \((a,c)\), \((c,b)\in\rho\), where \((a,b)\notin\rho\)

and the point \( c \) has degree two, by a new line \((a,b)\).

The pairs of graphs in Figure 14 are homeomorphic and serve to illustrate the difference between homeomorphisms and contractions.

![Diagram](image)

**FIG. 14**
In Figure 14, \( P_5 \) can be contracted onto \( P_4 \), yet no contraction maps \( H \) onto \( H' \), or \( H' \) onto \( H \), and since \( K_3 \) can be contracted onto \( K_1 \) but \( K_3 \) and \( K_1 \) are not homeomorphic, it is clear that these two classes of mappings are distinct.

The most notable result concerning homeomorphisms is the now classical

**Theorem of Kuratowski [13]:** A graph \( G \) is planar iff \( G \) does not contain a subgraph homeomorphic to \( K_5 \) or \( K_{33} \).

2.5. Strong Contractions.

Adam, Culik, and Pollak [1] have defined a class of mappings of graphs which very closely resembles the strong homomorphisms of Section 1.9; the definition of this class however contains an added condition which places the class in the class of contractions.

A mapping \( \phi \) is a strong contraction\(^1\) of an ordinary digraph\(^2\)

\[ G = (V, \rho) \] onto an ordinary digraph \( G' = (V', \rho') \) iff

(i) \( \phi \) is a function from \( V \) onto \( V' \);

\(^1\)Adam, Culik, and Pollak refer to such a mapping as "ein starker homomorphismus".
\(^2\)An ordinary digraph, like an ordinary graph, has no loops or multilines.
(ii) \((a,b) \epsilon \rho \land (b,a) \epsilon \rho \Longleftrightarrow a \phi = b \phi \land (a \phi, b \phi) \epsilon \rho';\)

(iii) \((a,b) \epsilon \rho \land (b,a) \epsilon \rho \Longrightarrow (a \phi, b \phi) \epsilon \rho' \land (b \phi, a \phi) \epsilon \rho';\)

(iv) \((a,b) \epsilon \rho \land (b,a) \epsilon \rho \Longrightarrow (a \phi, b \phi) \epsilon \rho' \land (b \phi, a \phi) \epsilon \rho'.\)

It is assumed that points \(a\) and \(b\) in conditions (ii), (iii), and (iv) are distinct.

The authors show that an ordinary digraph has at most one strong contraction, and in the following theorem give a necessary and sufficient condition for a non-trivial strong contraction to exist.

**Theorem 2.5.1.** An ordinary digraph \(G = \langle V, E \rangle\) has a strong contraction iff for every triple \(a, b, c\) of distinct points

(i) \((a,b), (b,a),\) and \((a,c) \epsilon \rho \Longrightarrow (b,c) \epsilon \rho;\)

(ii) \((a,b), (b,a),\) and \((a,a) \epsilon \rho \Longrightarrow (a,b) \epsilon \rho.\)

**Corollary 2.5.2.** If \(\phi\) is a strong contraction of an ordinary digraph \(G\) onto an ordinary digraph \(G',\) then there exists a subgraph \(H\) of \(G\) such that \(H \preceq G'.\)

A proof of this corollary can be constructed that is essentially the same as the proof of Theorem 1.9.1, which is the corresponding statement for strong homomorphisms.
3. RELATIONAL HOMOMORPHISMS

3.1. General Definition.

The concept of a homomorphism (and a contraction), of a graph \( G = \langle V, p \rangle \) into a graph \( G' = \langle V', p' \rangle \), generally speaking, consists of a function \( \phi \) from \( V(G) \) into \( V(G') \) which satisfies certain conditions. These conditions are usually imposed so that certain properties of the graph \( G \) will be preserved in \( G' \) by the function \( \phi \).

A natural generalization of this concept is that of a "relational homomorphism," between two graphs \( G \) and \( G' \), in which a binary relation is defined, \( \psi \subseteq V \times V' \), which satisfies certain conditions.

The idea of considering relations as homomorphisms between algebraic systems has occurred to at least two authors, some of whose definitions and results along this line are summarized in this section. We will refer, in general, to the mappings defined in this section as relational homomorphisms.
3.2. Weak Homomorphisms.

Yoeli and Ginzburg [8] and Yoeli [23] have defined a weak homomorphism for partial and complete algebras which can, in a natural way, be rephrased for directed graphs. Once this is done, a development exactly analogous to that in [23] can be carried out which illustrates the definition of weak homomorphism and some of its properties for directed graphs. The remainder of this section is just such a development; it contains nothing original and is simply a translation of the first part of [23] into graph theoretical terminology.

Every directed graph $G = \langle V, \rho \rangle$, including those with loops and multi-lines, can be expressed, generally in many ways, as a union of graphs of the form $G_i = \langle V, \rho_i \rangle$, where $U\rho_i = \rho$. Alternatively, every directed graph can be put in the form $G = \langle V, \{\rho_i\} \rangle$, where $U\rho_i = \rho$. In particular, each $\rho_i$ can be required to be a many-one relation, i.e., a partial function from $V$ to $V$. A digraph is out-regular of degree $k$ iff it can be put in the form $G = \langle V, \{\rho_1, \rho_2, \ldots, \rho_k\} \rangle$, where each $\rho_i$ is a function from $V$ into $V$.

Equivalently, a digraph $G$ is out-regular of degree $k$ iff the out-degree
of every point of $G$ equals $k$.

A binary relation $\psi \subseteq V \times V'$ is a weak homomorphism\(^1\) of a digraph

$G = \langle V, \{\rho_i\} \rangle$ into a digraph $G' = \langle V', \{\rho'_i\} \rangle$, $i = 1, 2, \ldots, m$ iff $\psi$

satisfies the following two conditions

(11) $\psi^{-1}(V') = V$, and

(12) $\psi^{-1}_i \subseteq \rho_i \psi^{-1}$, for every $i = 1, 2, \ldots, m$; in words, if

$\alpha \in \rho_i^{-1}(V)$ and $\alpha \psi a'$, then $\alpha' \in \rho_i^{-1}$ and $\rho_i(\alpha) \psi (a')$.

If in addition the binary relation $\psi$ is a function from $V$ onto $V'$ satisfying the condition

(13) $\psi \rho'_i = \rho_i \psi$, for all $i$,

then $\psi$ is a full homomorphism of $G$ onto $G'$\(^2\).

The following three lemmas correspond exactly to Lemmas 1, 2, and 3 of Yoeli [23].

Lemma 3.2.1. If $\psi$ is a weak homomorphism of a digraph $G$ into

a digraph $G'$, and $\psi'$ is a weak homomorphism of $G'$ into a digraph $G''$, then

\(^1\)Note that the definition requires that a 1-1 correspondence exist between \{\rho_i\} and \{\rho'_i\}.

\(^2\)Yoeli calls $\psi$, in this case, a strong homomorphism. It can be seen however that condition (13) requires that the mapping be a full homomorphism, as defined in Section 1.1, of the digraph $G$ onto the digraph $G'$.
\( \psi' \) is a weak homomorphism of \( G \) into \( G' \).

**Lemma 3.2.2.** A mapping \( \psi \) of a digraph \( G \) onto a digraph \( G' \) is a full homomorphism iff both \( \psi \) and \( \psi^{-1} \) are weak homomorphisms.

**Lemma 3.2.3.** If \( \psi \) is a weak homomorphism of an out-regular digraph \( G \) onto a digraph \( G' \), then \( \psi^{-1} \) is a weak homomorphism of \( G' \) onto \( G \). For additional discussion of weak homomorphisms the reader is referred to Yoeli.

In Figure 15, \( \psi_1 \) is an example of a weak homomorphism of \( G_1 \) onto \( G_2 \), where \( \psi_1 = \{(a,c),(a,e),(b,d),(b,f)\} \).

\[
\begin{array}{ccc}
| \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
\begin{array}{c}
\text{a} \\
\text{b}
\end{array} & \hspace{1cm} & \hspace{1cm}
\begin{array}{c}
\text{c} \\
\text{d}
\end{array} \quad \quad \quad \begin{array}{c}
\text{e} \\
\text{f}
\end{array}
\end{array}
\]

\[
G_1 \quad \quad \quad \psi_1 \quad \quad \quad G_2
\]

**FIG. 15.**

Note also that \( \psi_1^{-1} \) is a full homomorphism of \( G_2 \) onto \( G_1 \).
3.3. Weak Isomorphism.

A binary relation $\psi \subseteq V \times V'$ is a weak full homomorphism of a graph $G = \langle V, \rho \rangle$ onto a graph $G' = \langle V', \rho' \rangle$ iff in addition to satisfying condition (11) it satisfies conditions (14) $\psi(V) = V'$, and (15) $\psi(\rho) = \rho'$.

A weak full homomorphism of a graph $G$ onto a graph $G'$ is a weak isomorphism iff $\psi^{-1}(\rho') = \rho$. Consider Figure 16, where 

$\psi = \{(a, a'), (a, a''), (b, b'), (b, b'')\}$ is a full weak homomorphism of $G$ onto $G'$.

![Diagram](image)

FIG. 16.

Since $\psi^{-1}(\rho') = \rho$, it follows that $G$ and $G'$ are weakly isomorphic.

Thatcher [20] has studied both weak full homomorphisms and weak isomorphisms,
and offers the following explanation for considering intuitively $G$ and $G'$ to be 'isomorphic.'

Two graphs $G$ and $G'$ are **elementarily equivalent** iff they are indistinguishable using the first-order predicate calculus without equality, i.e., no sentence in this language is true of $G$ without its also being true of $G'$, and conversely.

Thatcher states that if two graphs $G$ and $G'$ are weakly isomorphic then they are elementarily equivalent. He also believes that the converse is true for finite graphs, i.e., if two finite graphs, $G$ and $G'$, are elementarily equivalent then they are weakly isomorphic. Since in the first order predicate calculus without equality one loses the ability to count, one cannot distinguish whether two graphs have the same number of points. It follows, therefore, that among other things all complete digraphs are weakly isomorphic.\(^1\) Thatcher suggests as an interesting problem the study of these weak isomorphism types of graphs. The following results shed some light on this problem.

\(^1\)It is not true however that all ordinary complete graphs are weakly isomorphic.
Lemma 3.3.1. If \( \phi \) is a strong homomorphism\(^2\) of a graph \( G \) onto

a graph \( G' \), then \( G \) and \( G' \) are weakly isomorphic, i.e., \( \phi^{-1} \) is a weak

homomorphism of \( G' \) onto \( G \).

Proof. Obviously \( \phi \) is a weak homomorphism of \( G \) onto \( G' \), and as

such \( \phi(V) = V' \) and \( \phi(\rho) = \rho' \). It follows therefore that \( \phi^{-1}(V') = V \).

It suffices to show that \( \phi^{-1}(\rho') = \rho \). First,

\[
(a', b') \in \rho' \implies (\phi^{-1}(a'), \phi^{-1}(b')) \in \rho,
\]

for if \( a \in \phi^{-1}(a') \) and \( b \in \phi^{-1}(b') \),

then \( (a, b) \in \rho \), since \( g(a, b) = g'(a', b') = g'(a', b') = 1 \). Secondly,

\[
(a', b') \notin \rho' \implies (\phi^{-1}(a'), \phi^{-1}(b')) \notin \rho,
\]

by the same reasoning.

Theorem 3.3.2. Two graphs \( G \) and \( G' \) are weakly isomorphic iff

they have isomorphic simple (with respect to strong homomorphisms) images.

Proof. First, suppose that \( G \) and \( G' \) are weakly isomorphic with

respect to the full weak homomorphism \( \psi \), i.e.,

\[
\begin{aligned}
G & \xrightarrow{\phi_S} G'_S \\
G' & \xleftarrow{\psi} \psi^{-1} \\
G' & \xrightarrow{\phi'_S} G'_S,
\end{aligned}
\]

where \( \phi_S \) and \( \phi'_S \) are strong homomorphisms of \( G \) and \( G' \) onto the simple graphs

\[\text{cf. Section 1.9.}\]
$G_S$ and $G'_S$, respectively. It suffices to show that $\psi^i_S$ is a strong homomorphism of $G$ onto $G'_S$. The isomorphism of $G_S$ and $G'_S$ then follows as a result of Theorem 1.3.7.

We must show that $\psi^i_S$ is a well defined mapping of $V(G)$ onto $V(G'_S)$, where $\psi$ is a full weak homomorphism of $G$ onto $G'$ and $\phi^i_S$ is a strong homomorphism of $G'$ onto $G'_S$ (and $G'_S$ is simple). We do this by showing that

(i) $a \psi a_1^i \land a \psi a_2^i \implies n(a_1^i) = n(a_2^i)$, and

(ii) $n(a_1^i) = n(a_2^i) \implies a_1^i \phi^i_S = a_2^i \phi^i_S$.

Let us prove, (i) $a \psi a_1^i \land a \psi a_2^i \implies n(a_1^i) = n(a_2^i)$. Suppose $n(a_1^i) \neq n(a_2^i)$.

Then either $\rho'(a_1^i) \neq \rho'(a_2^i)$ or $\rho^-1(a_1^i) \neq \rho'^-1(a_2^i)$. If $\rho'(a_1^i) \neq \rho'(a_2^i)$ then

$(\exists d')[(a_1^i, d') \epsilon \rho' \land (a_2^i, d') \not\epsilon \rho') \lor ((a_1^i, d') \not\epsilon \rho' \land (a_2^i, d') \epsilon \rho')]$.

Suppose $(a_1^i, d') \epsilon \rho' \land (a_2^i, d') \not\epsilon \rho'$; then $(\exists c)(\exists d)[(c, d) \epsilon \rho \land c \psi a_1^i \land d \not\epsilon d']$.

But since $\psi^{-1}$ is a full weak homomorphism of $G'$ onto $G$, $(a_1^i, d') \epsilon \rho' \implies (a, d) \epsilon \rho \implies (a_2^i, d') \epsilon \rho'$, hence we have a contradiction.

A similar argument shows that $(a_1^i, d') \not\epsilon \rho' \land (a_2^i, d') \epsilon \rho'$ cannot hold either, hence $\rho'(a_1^i) = \rho'(a_2^i)$, must hold.

The same argument just used to show that $\rho'(a_1^i) = \rho'(a_2^i)$ can be
used to show that $\rho^{-1}(a_1) = \rho^{-1}(a_2)$. Hence $n(a_1) = n(a_2)$ must hold.

In order to see that (ii) $n(a_1) = n(a_2) \implies a_1^{\phi_s} = a_2^{\phi_s}$, one need only refer back to the definition of the equivalence relation $\equiv$ which determines the strong homomorphism $\phi_s$.

Hence $\psi\phi_s$ is a well defined mapping of $V(G)$ onto $V(G')$; it is a simple matter to verify that the mapping satisfies the condition

$$g(a, b) = g'(a\psi\phi_s, b\psi\phi_s').$$

The mapping $\psi\phi_s$ is therefore a strong homomorphism of $G$ onto $G'$. Second, suppose that $G_s$ and $G_s'$ are isomorphic,

where for convenience we suppose that $\phi_s$ and $\phi_s'$ are strong homomorphisms of $G$ and $G'$ onto $G_s$. It suffices to show that $\phi_s\phi_s'^{-1}$ is a full weak homomorphism of $G$ onto $G'$, and similarly that $\phi_s'\phi_s^{-1}$ is a full weak homomorphism of $G'$ onto $G$. But this follows immediately as a result of Lemma 3.3.1 and
the following

Lemma 3.3.3. If \( \psi \) is a full weak homomorphism of \( G \) onto \( G' \),

and \( \psi' \) is a full weak homomorphism of \( G' \) onto \( G'' \), then \( \psi' \) is a full weak

homomorphism of \( G \) onto \( G'' \).
SUMMARY AND SUGGESTED PROBLEMS

Several definitions have been given of homomorphisms for various classes of graphs and a few results have been stated or derived for each. The nature of the results gives an indication of the kinds of properties of graphs one can effectively study by means of homomorphisms, and gives an indication of the methods one might use in studying properties of the homomorphisms themselves.

Several open problems and problem areas have been suggested in the paper, and several new ones can be stated, among which are the following:

1) How does the set of (elementary) homomorphic images of a given graph G characterize G? It can be shown, for example, that the set of all non-trivial homomorphic images of an ordinary graph G, together with the multiplicities associated with each image, i.e., the number of distinct ways the particular image can be formed under a homomorphism, completely characterize the graphs with five points or less which are not complete graphs.

2) Of the 52 ordinary graphs with five points or less, 31 have
one elementary homomorphic image (up to isomorphism). Can one easily characterize those graphs $G$ such that all of the elementary homomorphic images of $G$ are isomorphic?

The following five graphs have this property

3) Geert Prins, in a letter to the author, expressed an interest in studying homomorphisms of graphs via elementary homomorphisms.

What in general can one say about elementary homomorphisms?

For example, one could state the following

Proposition: If $G_\varepsilon$ is that elementary homomorphic image of $G$ obtained by identifying points $a, b \in V(G)$, and the chromatic number of $G$ is $n$, then the chromatic number of $G_\varepsilon$ is $n$ iff points $a$ and $b$ can be colored identically in some $n$-coloring of $G$. 

-58-
4) What sort of graph theoretical results are obtainable from
Keisler's logical formulation of the properties of a graph
which are preserved under full homomorphisms? For example,
a brief explanation of Keisler's formulation by Professor Robert
Ritchie led to the following observation:

Proposition. If $\phi$ is a full homomorphism of an ordinary graph $G$
onto a graph $G'$, and $G = G_1 \ast G_2$, i.e., $G$ is the join of $G_1$ and
$G_2$, then $G' = G_1\phi \ast G_2\phi$.

It is believed that almost all of the results derived using
Keisler's formulation could be derived very easily without using
it.

5) In regard to pre-images of a given graph, how close to a complete
graph, in some sense, can a pre-image of a given graph be?

For example, if an ordinary graph $G$ has $p$ points and $q$ lines,
define its density as $q/\binom{p}{2}$. It would seem, at first glance,
that since every ordinary graph $G$ maps onto a complete graph,
of density 1, that the density of a given graph $G$ would be smaller than the density of every homomorphic image of $G$. But this is not the case as the following example illustrates:

$$\phi: \begin{array}{c}
e \\
a, b, c, d \\
a, b, c, d' \\
G \\
\end{array} \quad \Rightarrow \quad \begin{array}{c}e' \\
f' \\
a', b', c', d' \\
G' \end{array}$$

Note that the density of $G$ is $8/15$, while the density of $G\phi = G'$ is $3/6$.

6) For any class of homomorphisms, what graphs are endomorphism-free?

7) Which graphs are simple with respect to pathwise homomorphisms?

8) Are all prime homomorphisms of functional digraphs, pathwise homomorphisms as well? It appears so.

9) Characterize Prins' type-2 complete homomorphisms. This problem appears to be very difficult.

10) Relative to the study of type-n (complete) homomorphisms several
problems naturally arise among which are the following:

a) Does there exist for every integer $n$ a type-$n$ (complete) homomorphism?

b) Is the order of a type-$n$ complete homomorphism greater than or equal to $n$?

c) Can one state an Interpolation Theorem for type-$n$ complete homomorphisms, i.e., given a type-$n$ complete homomorphism of a graph $G$, of order $m$, what can be asserted about type-$k$ complete homomorphisms of $G$ for $k \leq n$.

d) If the length of the longest chain in a graph $G$ is $n$, it would seem that $G$ could not have a type-$n+1$ complete homomorphism. Relate this length $n$ to the maximum type-$n$ complete homomorphism of $G$.

11) Any rule for producing a non-identity partition of the points of an arbitrary graph $G$ determines a full homomorphism in the following sense:

Let $V_1, V_2, \ldots, V_m$ be a partition of $V(G)$, and let $x_i$
denote the chromatic number of the full subgraph generated by the subset $V_i$. The mapping which maps each subgraph respectively onto $K_i$ determines a full homomorphism of $G$.

If we denote the resulting graph in this process by $G\phi$ and apply the same partition-mapping process to $G\phi$, $G\phi^2$, etc., we might ultimately map $G$ onto a complete graph. It would be interesting to study those processes which map a given graph $G$ ultimately onto a complete graph and to examine the relationship between the chromatic numbers of $G$ and the complete graph.

Two such processes have already been mentioned; one, of taking partitions determined by 1-bases, and two, of taking partitions determined by $n$-bases. A third process would be that of considering partitions induced by the associated numbers of the points of a graph.

12) How can the concept of complete homomorphism be applied to directed graphs?

13) The homomorphisms of functional digraphs onto functional digraphs
all satisfy the condition

(i) \( \text{od}(a) = \text{od}(a^\phi) \).

It is suggested therefore that one study full homomorphisms which satisfy this condition, or the following condition

(ii) \( \text{id}(a) = \text{id}(a^\phi) \),

or both; in particular, study full homomorphisms of ordinary graphs which satisfy

(iii) \( \text{deg}(a) = \text{deg}(a^\phi) \).

Furthermore, functional digraphs are a sub-class of the class of out-regular digraphs; hence, the suggestion of studying homomorphisms of out-regular and in-regular digraphs and regular ordinary graphs, satisfying conditions (i), (ii), and (iii), respectively, is raised.

14) Y. Give’on has suggested that one study epigenic homomorphisms of directed graphs; homomorphisms for which the inverse image of bases (generating sets) are bases. It can be seen, for
example, that any homomorphism of a weakly connected digraph onto a strongly connected digraph is not epigenic.

15) Relative to Proposition 1.5.5, it seems that with the exception of $K_2$ one can strengthen the inequality to $\text{ch}(G) \leq 2q - 2$.

16) Relative to the question raised at the beginning of Section 1.5, one can ask the following, slightly different, and much more interesting question: for a given graph $G$ what is the length of the shortest cycle $P$ for which there exists a homomorphism $\phi$ which maps $P$ onto $G$? The homomorphism $\phi$ need not be a full homomorphism. Suppose we call this the circuit length of $G$, and denote it by $\text{ci}(G)$.

The following example illustrates that $\text{ci}(G)$ and $\text{cy}(G)$ are, in fact, distinct invariants; a little reflection shows further that $\text{ci}(G) \leq \text{cy}(G)$.
The mapping $\phi$ is a homomorphism of $P_5$ onto $G$; the mapping $\psi$

is a full homomorphism of $P_7$ onto $G$. Hence, $ci(G) = 5$, while $cy(G) = 7$.

It can also be seen that if a graph $G$ has $p$ points then $ci(G) = p$ iff $G$ is Hamiltonian.

The parameter $ci(G)$ also can be seen to have a lot of relevance to solutions of the famous Traveling Salesman Problem.

It would be interesting in view of these observations to investigate the extent to which the methods used to study $cy(G)$ can be used to study $ci(G)$. 
APPENDIX OF DEFINITIONS

In a graph \( G = \langle V, \rho \rangle \)

... the \textbf{out degree} of a point \( b \in V \), \( \text{od}(b) = |\rho(b)| \);

... the \textbf{in degree} of a point \( b \in V \), \( \text{id}(b) = |\rho^{-1}(b)| \);

... the \textbf{degree} of a point \( b \in V \), \( \text{deg}(b) = |n(b)| = |\rho(b) \cup \rho^{-1}(b)| \);

... a \textbf{loop at a point} \( b \) is a line of the form \( (b, b) \);

... a \textbf{path of length} \( n \) from a point \( a \) to a point \( b \) is a sequence of points \( a_i, a_{i+1}, \ldots, a_n \) such that \((a_i, a_{i+1})\in \rho\), for \( 0 \leq i \leq n - 1 \), \( a_0 = a \), and \( a_n = b \);

... the \textbf{distance} from point \( a \) to point \( b \), \( d(a, b) \), is the length of the shortest path from \( a \) to \( b \);

... the \textbf{associated number} of a point \( b \in V \) is \( \max d(b, c) \), for all \( c \in V \).

... the \textbf{diameter} of \( G \), \( \delta(G) \), is \( \max d(a, b) \), for \( a, b \in V \).

... an \textbf{independent set of points} is a set \( V_1 \subseteq V \) such that for every \( a, b \in V_1 \), \( (a, b) \notin \rho \);

... a \textbf{maximal} independent set of points is an independent set \( V_1 \) such that
for no independent set \( V_2 \) is \( V_1 \subseteq V_2 \);

... the point independence number, \( \alpha_0(G) \), is max \( |V_1| \), where \( V_1 \) is an independent set of points;

... the point covering number, \( \beta_0(G) \), is min \( |V_2| \), where \( V_2 \subseteq V \) is such that for every \((a,b)\in\rho\), either \( a\in V_2 \) or \( b\in V_2 \).

... the subgraph generated by a set of points \( V' \), is the graph \( G = \langle V', \rho' \rangle \)

where \( \rho' = \{(a,b) : a, b\in V' \land (a,b)\in\rho\} \)

An ordinary graph \( G = \langle V, \rho \rangle \)

... is a complete graph on \( n \) points iff it has the form

\[
V = \{k_1, k_2, \ldots, k_n\}
\]

\[
\rho = \{(k_i, k_j) : i \neq j\};
\]

... is a cycle of length \( n \) iff it has the form

\[
V = \{p_1, p_2, \ldots, p_n\}
\]

\[
\rho = \{(p_i, p_{i+1}), (p_{i+1}, p_i) : 1 \leq i \leq n-1\} \cup \{(p_1, p_n), (p_n, p_1)\};
\]

... is a chain of length \( n \) iff it has the form

\[
V = \{c_1, c_2, \ldots, c_{n+1}\}
\]

\[
\rho = \{(c_i, c_{i+1}), (c_{i+1}, c_i) : 1 \leq i \leq n\};
\]
... is **connected** iff for every pair $a, b$ of distinct points there is a path in $G$ from point $a$ to point $b$;

... is a **tree** iff it is connected and contains no cycles as subgraphs;

... is **critical of degree** $k$ iff $\chi(G) = k$ and

$(\forall (a, b) \in \rho) [\chi(G-(a, b)) = k - 1]$; in words, the chromatic number of any graph obtained from $G$ by removing any line, or point, of $G$ is less than the chromatic number of $G$;

... is **isomorphic to** $K_{3,3}$ iff it has the form

$$V = \{a_1, a_2, a_3, b_1, b_2, b_3\}$$

$$\rho = \{(a_i, b_j), (b_j, a_i) : 1 \leq i, j \leq 3\};$$

... has **multi-lines** iff $\rho$ is considered to contain at least two occurrences of at least one pair $(a, b)$;

... is a **multi-graph** iff it contains multi-lines;

... is the **join** of graphs $G_1 =$ $\langle V_1, \rho_1 \rangle$ and $G_2 = \langle V_2, \rho_2 \rangle$ iff

$$V = V_1 \cup V_2, \rho = \rho_1 \cup \rho_2 \cup \{(a, b) | a \in V_1, b \in V_2\}$$
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HOMOMORPHISMS OF GRAPHS

Technical Report

Hedetniemi, Stephen

Qualified requesters may obtain copies of this report from DDC.

The works of Hartmanis and Stearns, Krohn and Rhodes, Yoeli and Ginzburg, and Zeiger employ demonstrate the usefulness of homomorphisms in studying decompositions of finite automata. Yoeli and Ginzburg's approach is slightly different from the others in that it is more concerned with aspects of the state transition graphs of finite automata. In their paper, "On Homomorphic Images of Transition Graphs," they give a complete characterization of the class of homomorphisms of the graphs which correspond to input-free automata. (U)

This paper was motivated by an interest in extending these results of Yoeli and Ginzburg in the direction of a characterization of the class of homomorphisms of graphs which correspond to arbitrary finite automata. It is a review and a classification of most of the published definitions and results on mappings of graphs which have been called homomorphisms. The paper contains, in addition, several new results and several new definitions of homomorphisms of graphs. (U)
1. graph theory
2. homomorphisms