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SELECTIVE-MODULATION AUTOMATIC DIRECTION FINDER

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ABSTRACT

In the selective-modulation automatic direction finder, the signals from the E-W and N-S antennas are modulated at different audio frequencies, passed through a common receiver, and separated in synchronous detectors before being applied to the plates of a cathode ray tube. The system gives instantaneous sense and bearing. The report is a treatment of this system considering optimization of system parameters, balance requirements, accuracy and sensitivity. Means are suggested for observing very short duration transmissions.

SELECTIVE-MODULATION AUTOMATIC DIRECTION FINDER

1. INTRODUCTION

The system under consideration is a single-channel selective-modulation ADF. The reasons for considering a system of this type are that it gives instantaneous sense as well as bearing. The fast response could lend itself to the reception of short duration or "burst" transmissions. The system appears to have been originally proposed by C. W. Earp.[1] Successful work on equipment of this nature has been done in Great Britain by R. F. Cleaver[2] and in this country by Stiber, et.al.[3]

2. GENERAL DESCRIPTION

The operation of the system can be seen from Figures 1, 2 and 3. Referring to Figure 2, for a vertically polarized plane wave of angular frequency ω incident at angle θ the antenna voltages are given by:

$$E_{sense} = E_c h_s \operatorname{Sin} \omega\tau$$

$$E_N = E_c h \operatorname{Sin} \left(\omega\tau + \frac{2\pi d}{\lambda} \operatorname{Sin} \theta\right)$$

$$E_E = E_c h \operatorname{Sin} \left(\omega\tau + \frac{2\pi d}{\lambda} \operatorname{Cos} \theta\right)$$

$$E_S = E_c h \operatorname{Sin} \left(\omega\tau - \frac{2\pi d}{\lambda} \operatorname{Sin} \theta\right)$$

$$E_W = E_c h \operatorname{Sin} \left(\omega\tau - \frac{2\pi d}{\lambda} \operatorname{Cos} \theta\right)$$

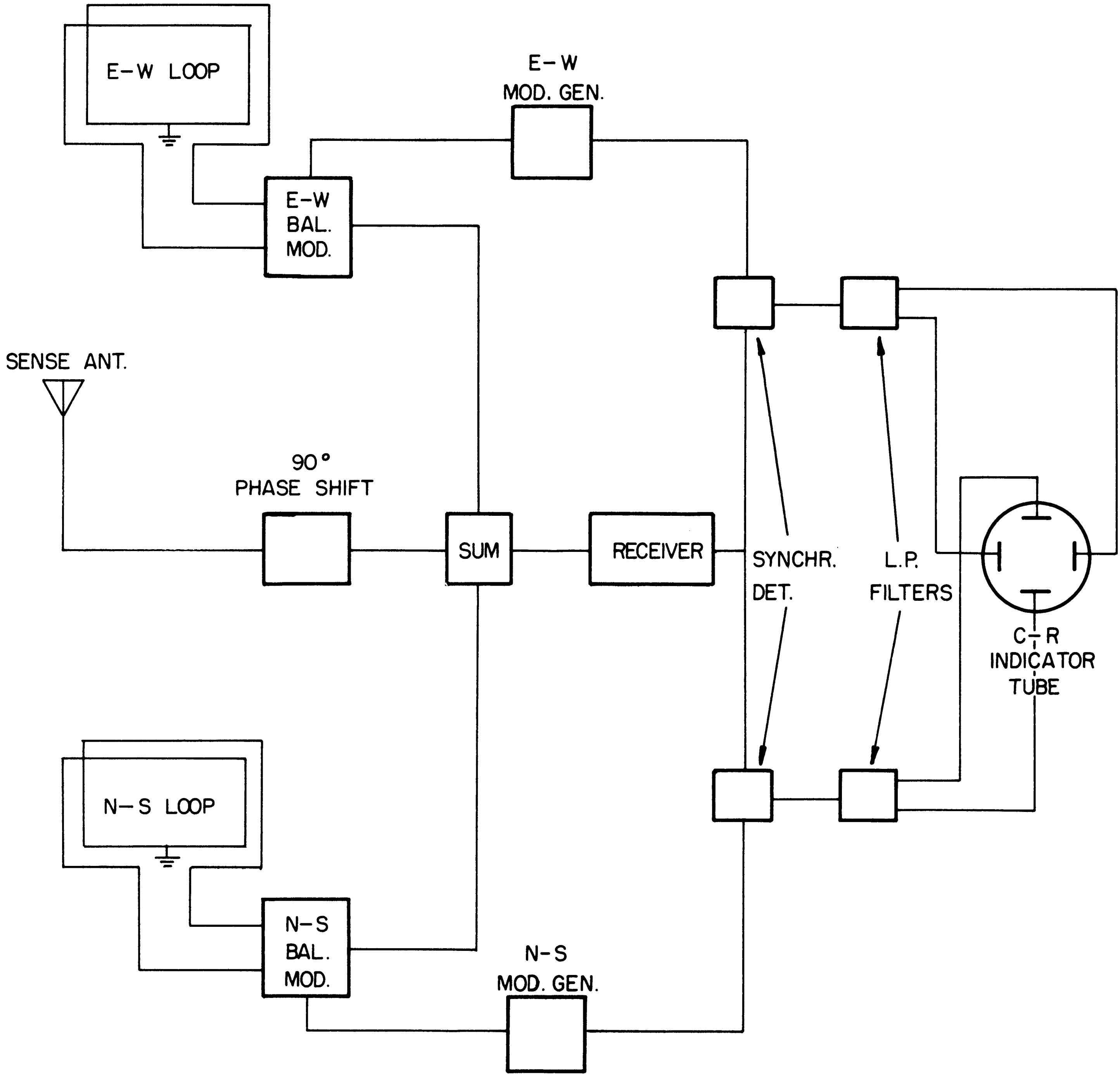

FIG.1 BLOCK DIAGRAM OF SELECTIVE-MODULATION ADF

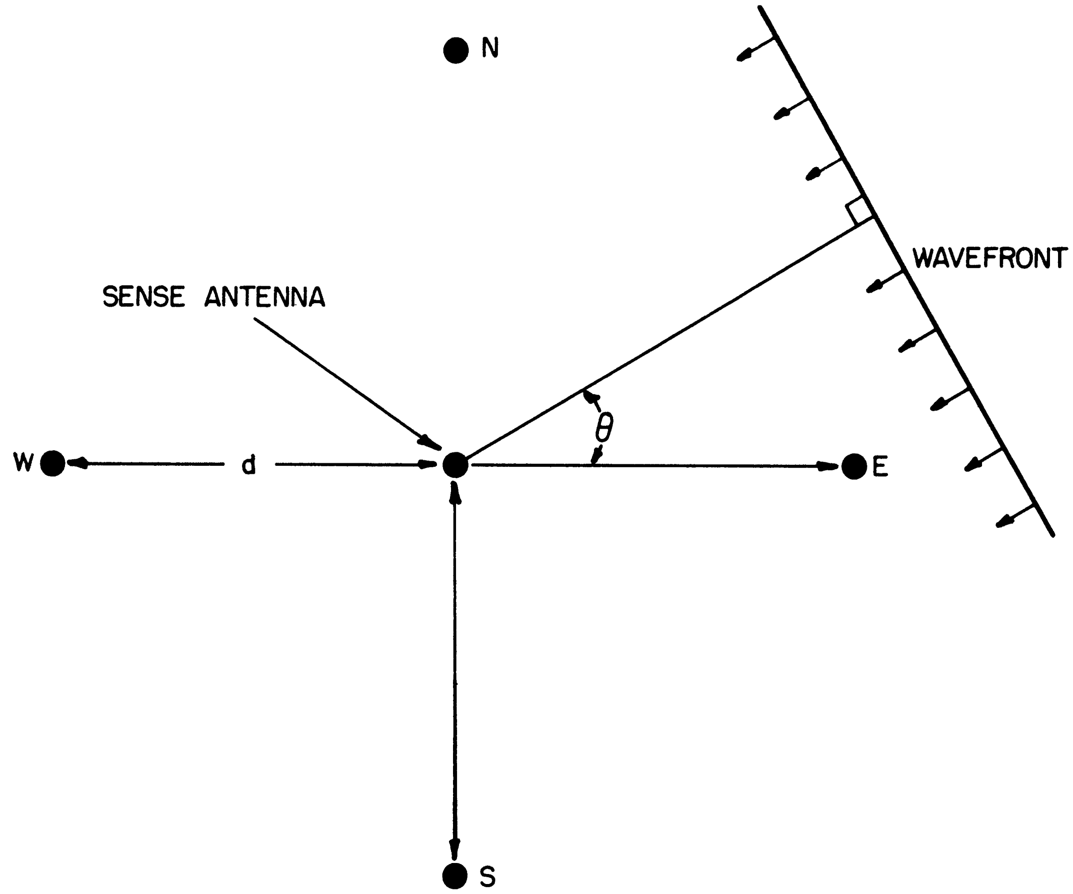

FIG. 2 TOP VIEW OF ANTENNA SYSTEM, HORIZONTAL MEMBERS NOT SHOWN

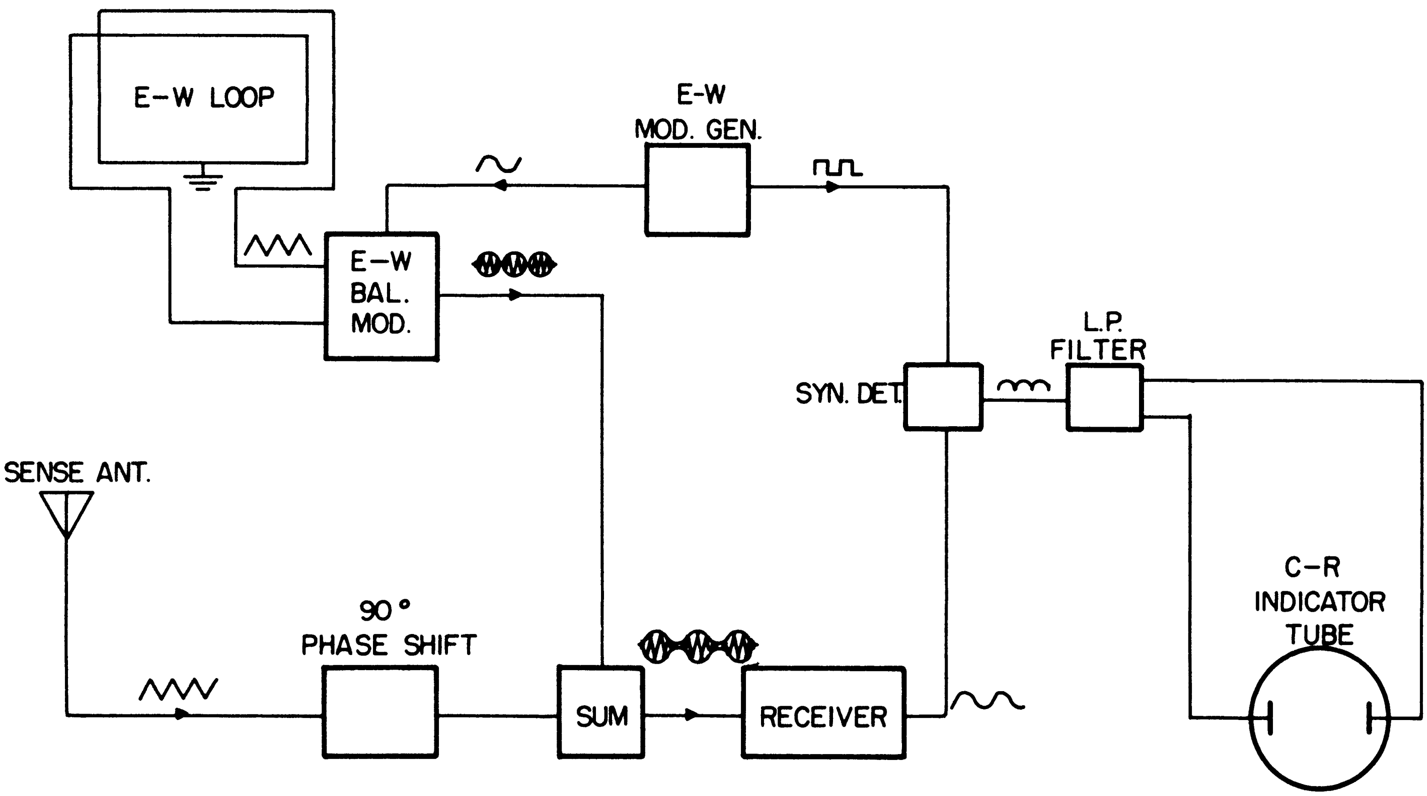

FIG. 3 E-W CHANNEL SHOWING SIGNAL AT VARIOUS POINTS

where E_c is the magnitude of the incident electric field vector and h is the effective antenna height. **It is the differences between the voltages induced in opposite antennas which are used.** These are given below,

$$E_N - E_S = E_{NS} = \frac{4\pi dE_c h}{\lambda} \text{ Cos } \Theta \text{ Cos } \omega\tau$$

$$E_E - E_W = E_{EW} = \frac{4\pi dE_c h}{\lambda} \text{ Sin } \Theta \text{ Cos } \omega\tau$$

where the usual assumption is made that $2\pi d/\lambda \ll 1$. The signals from the E-W and N-S antennas are fed into their respective balanced modulators. The periodic modulating signals are produced in the two blocks marked "MOD. GEN." It is the difference between the E-W and N-S modulating signals that allows post detection separation of the two channels and from this the name selective modulation is derived.

The outputs of the balanced modulators consist of amplitude-modulated, double-sideband, suppressed carrier signals. The RF from the sense antenna is shifted 90° in phase and added to the E-W and N-S modulator outputs. This is indicated by the block marked SUM (Fig. 1). This sense voltage supplies the carrier, so that the output of the summer consists of a standard AM signal, modulated by the sum of the E-W and N-S MOD. GEN. outputs. The sense voltage must, of course, be at least as large as the sum of the modulator outputs if overmodulation is to be prevented.

The output of the summer is (see Fig. 1):

$$E_o = AE_c \frac{4\pi dh}{\lambda}\left[(F_1 \text{ Cos } \Theta + F_2 \text{ Sin } \Theta) - h\ B\right] \text{ Cos } \omega\tau \quad (1)$$

where F_1 and F_2 are the modulating signals and A, B are constants of proportionality. It can be seen that reversing the direction of arrival of the wavefront changes ($F_1 \cos \Theta + F_2 \sin \Theta$) to ($-F_1 \cos \Theta - F_2 \sin \Theta$). It is this property which allows the automatic determination of sense by comparing the sign of this term with reference signals from the MOD. GEN.

The output of the summer goes to a receiver where it is amplified and detected. The AF output of the receiver is fed to two synchronous detectors synchronized to the respective MOD. GEN. signals. The outputs of these detectors are passed through two identical low-pass filters and applied to the respective plates of a cathode ray tube. The position of the CR spot gives the instantaneous relative bearing of the signal.

Though not shown in the figures, a means is provided to periodically short the deflection plates of the cathode ray tube together so that the actual display consists of a radial line pointing in the desired direction.

While loop antennas are shown for illustration purposes in Figs. 1 and 3, it is, of course, realized that a 4-element Adcock could be used just as well.

3.1 Modulating Signals

The optimum modulating signal would at first appear to be one which samples the antenna continuously, that is, a square wave. However, this places certain requirements on the minimum useable bandwidth in the receiver. The desire for a narrow bandwidth would suggest the use of a pure sine wave. This reduces the output of the synchronous detector by a factor of $\frac{\int_0^{\pi} \sin x \, dx}{\int_0^{\pi} dx} = \frac{2}{\pi}$. But a calculation (see App. I) shows that increasing the receiver bandwidth to accept successively higher order harmonics in the Fourier expansion of a square wave (only odd harmonics are present) increases the RMS noise faster than the signal. Another advantage of the sine wave over the square wave is that one does not have the large number of intermodulation products generated in the synchronous detectors, making cross-channel interference much less of a problem. One might also apply a square wave to the balanced modulators and reject the harmonics in a narrow band IF. This, however, places an upper bound on the maximum allowable bandwidth and the system would suffer accordingly on FM signals which generally require a wider bandwidth for their reception than the few hundred cycles which would be allowed by this scheme. From the above it appears that pure sine wave modulation has the advantage and will be assumed in all further discussion.[1]

There is, however, an advantage to be gained in applying square wave synchronizing signals to the synchronous detectors (which in this case are merely multipliers). This makes the requirement of balanced operation less of a problem since the tubes are always operating in a linear region during the "on" portion of the cycle.

Thus, the MOD. GEN. could be square wave sources. Well-shielded AC choppers of the kind often found in DC amplifiers would perhaps provide compact, reliable sources. Between the source and the antenna balanced modulator a low-pass filter to reject all but the fundamental would than be needed. The signal at various points for one channel is shown in Figure 3.

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1 The use of a sine wave places more stringent requirements on the balanced modulators than would square wave signals (see, footnote, page 15.)

The selection of the modulation frequencies is based on several requirements. For adequate sampling of extremely short pulses, a high frequency would be desirable, while bandwidth considerations suggest something quite low. The principal determining factor is the minimum isolated burst which is anticipated. For example, suppose it is desired to receive a single isolated pulse 20 milliseconds long. It seems reasonable, for accurate bearings, to require that the minimum modulation frequency be such that the signal exists during four periods. This gives a frequency of 200 cycles. The low-pass filter cutoff must be at least as high as 1/.020, or 50 cycles, and 150 cycles would probably be a better figure. In order to avoid cross-channel interference the other modulation frequency must be chosen so that when multiplied by the 200 cycle signal and its odd harmonics in the synchronous detectors, the resulting products do not fall in the 150 cycle lp pass band. It must, therefore, be greater than 200 + 150, or 350 cycles, but less than 600 - 150, or 450 cycles. The above considerations do not, of course, guarantee that the operator will be able to observe such a short duration signal. This problem will be taken up in a later section.

3.2 Balanced Modulators

The requirements on these units are (1) that the E-W and N-S units be essentially identical (see section V), (2) a good degree of isolation, say 30 db or better, should exist between input and output during cutoff, (3) that a reasonable noise figure be realized.

The first requirement might be met by balancing potentiometers or, if this means should prove insufficient or undesirable, by selection of matched tubes.

The second requirement, if not met, has the effect of reducing the gain and thus indirectly affects(3). By reasonable noise figure is meant something on the order of 10db or better. A means of meeting these objectives might be a grounded-grid configuration such as that shown in Figure 4.

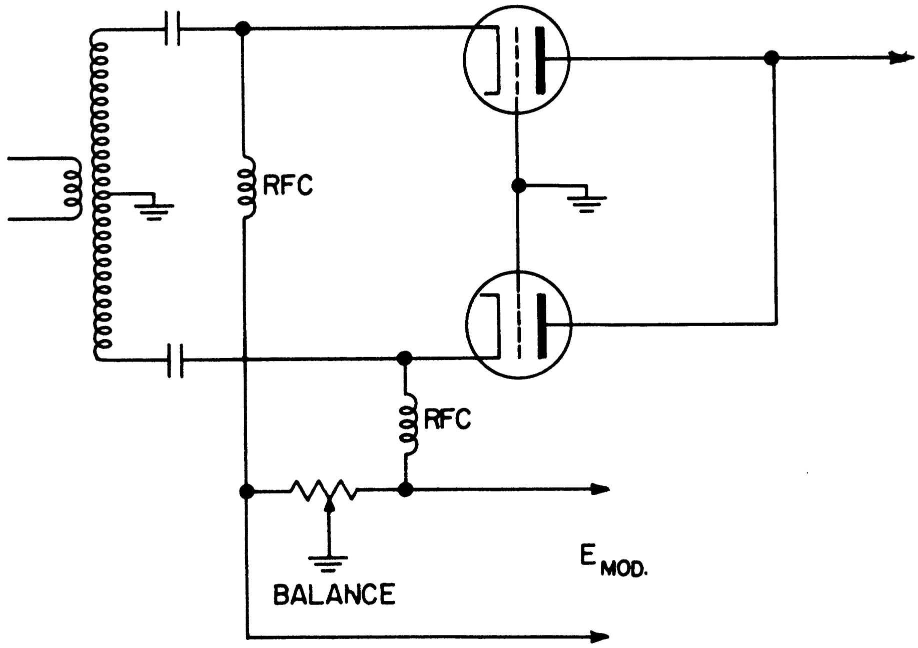

FIG. 4 BALANCED MODULATOR

3.3 Receiver

The receiver is a sensitive superheterodyne with the following requirements:

1. The IF bandwidth is quite small. For a maximum modulating frequency on the order of 400 cycles, 1 kc would be a good figure. Tuning accuracy becomes quite important with such a narrow band. It would be advantageous to be able to select a broader bandwidth on the stronger signals to provide more faithful listening-in, and for wideband signals such as FM.

2. Something on the order of a flat topped or "maximally flat" IF response is needed to ensure that both E-W and N-S signals are amplified an equal amount.

3. The phase shift must be linear with frequency over the pass band. One problem common to all DF's of the balanced modulator or goniometer type is that the sidebands do not all undergo the same phase shift.[4,8] The problem here is not so much the fact of unequal phase shift but that the amount is usually unknown and variable across the IF pass band. By selecting the bandwidth to meet other requirements, and inserting an equalizer so that phase is linear with frequency over the pass band, the resulting signal is a delayed replica of the original. Since the delay is known and constant, passing the signal from the modulation signal generators to the balanced modulators through phase shifters removes the effect. In a rotating goniometer system, one would merely rotate the CR tube the required number of degrees. To avoid stringent stability requirements on the modulation signal generators the time delay of the modulation signal generator low-pass filters should be constant over the band of expected frequency deviation.

4. The receiver must be linear over a wide dynamic range since near a bearing of 90 degrees the ratio of the signals from the two antennae is a very large number.

5. The receiver-detector must be a linear envelope detector. Power-law devices are out of the question because of the interchannel mixing and bearing inaccuracies they would introduce.

3.4 Low Pass Filter

For long duration signals (1 second or more) the filter should have a fairly narrow bandwidth, say 5-10 cycles or less. This removes the jitter due to modulation on the received carrier and markedly improves the S/N. The display S/N is given by

$$\frac{S}{N}_{Displ.} = \frac{S}{N}_{Rec.} \times \frac{B_{IF}}{B_{LP}} \qquad (2)$$

where B_{IF} is the IF bandwidth and B_{LF} is the low-pass bandwidth. For a low pass of 5 cycles and a receiver of IF of 1 kc this gives an improvement factor of 200, or 14:1. For signals of duration one second or longer, 5 cycles would be entirely adequate for B_{LF}. For general useage a somewhat wider bandwidth would be desirable. Forty cycles would probably be adequate for most conditions, such as keyed CCW, etc. However, for long duration noisy signals a five-cycle cutoff should be available to the operator.

For extremely short duration signals a fairly high cutoff is needed. Something on the order of 150 cycles was suggested for 20 millisecond bursts. This presents certain other problems.

For cutoffs less than one half the difference between the two modulation frequencies the Gaussian noise signals applied to the two pairs of CR plates are completely independent of each other. For cutoffs higher than this value this is not so (see Fig. 5). To illustrate, suppose the two modulation frequencies are 200 and 400 cycles and the filters cut off at 150 cycles. Then the input spectrum from 50 to 350 cycles will, after multiplication by the 200 cycle signal, fall in the filter pass band in one channel. Likewise, that portion lying between 250 and 550 cycles will be passed in the other (400 cycle) channel. Thus the noise voltage represented by spectral components in the 250-350 cycle band will be re-

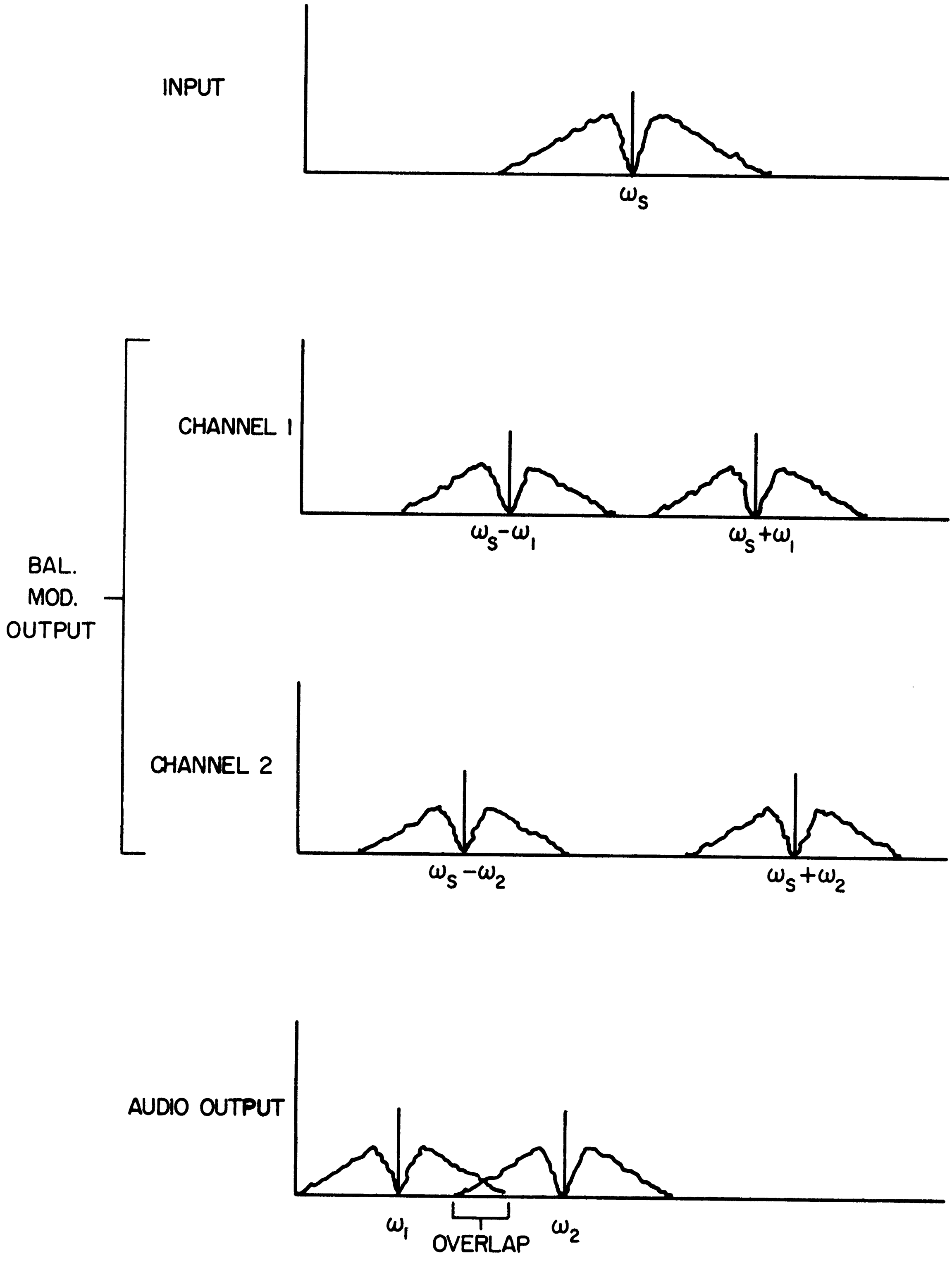

FIG. 5

SIGNAL & NOISE AT SAMPLING POINTS IN THE SYSTEM presented in both channels. The probability density function for the display spot is (see App. II):

$$p(R,\Theta) = \frac{\text{EXP}\left\{ - \dfrac{(\sigma_1^{\ 2} + \sigma_o^{\ 2}) \left[R^2 \quad 1 - \dfrac{\sigma_o^{\ 2} \cos\,(\omega_1 - \omega_2)t}{2(\sigma_1^{\ 2} + \sigma_o^{\ 2})} \quad \sin 2\,\Theta \right]}{2 \left[(\sigma_1^{\ 2} + \sigma\,_1^2 \ - \dfrac{\sigma_o^{\ 4} \cos^2\,(\omega_1 - \omega_2)\ t}{4} \right]^2} \right\}}{2\pi \cdot \sqrt{(\sigma_1^{\ 2} + \sigma_1^{\ 2}) - \dfrac{\sigma_o^{\ 4} \cos^2\,(\omega_1 - \omega_2)\ t}{4}}} \qquad (3)$$

where $\sigma_1^{\ 2}$ is the mean square noise in the 50-250 cycle band, assumed equal to that in the 350-550 cycle band; $\sigma_o^{\ 2}$ is the mean square noise in the band 250-350 cycles; and ω_1 and ω_2 are 2π times the respective modulation frequencies. R,Θ are the polar coordinates of the spot. It is seen that, in general, the density is not uniform about the origin because of the term involving $\sin\ 2\Theta$. However, by sampling at those times at which $\cos(\omega_1 - \omega_2)t = 0$ it can be made so. When this is done the density function becomes:

$$p(R,\Theta) = \frac{\text{EXP}\left\{ - \dfrac{R^2}{2(\sigma_1^{\ 2} + \sigma_1^{\ 2})} \right\}}{2\pi\ (\sigma_1^{\ 2} + \sigma_1^{\ 2})} \qquad (4)$$

which is a standard 2-dimensional Gaussian distribution centered at the origin. A means for accomplishing this might be to mix the modulation generator signals in a crystal diode and synchronizing the spot flyback generator to the difference output or a submultiple thereof.

3.5 Flyback Circuit

There are no stringent requirements on the rate at which the plates are shorted together except for short duration signals when the considerations of the previous paragraph and section 4 apply. It should be sufficiently rapid (say 50 cycles/s or higher) so that the operator observes no objectionable flicker. It is questionable whether retrace blanking would be of any value since the retrace will not differ appreciable from the trace. As to methods of accomplishing the flyback, Stiber[3] describes one and Cleaver[2] another, either of which would probably be adequate.

4. SHORT DURATION SIGNALS

4.1 The Problem

The problem here is one of decision. Since the signal will not, in general, be on the display long enough for a human observer to make a reliable yes or no decision, an automatic means must be provided.

4.2 A Proposed Solution

The following possibility presents itself. The first requirement is to obtain information of the spot position. This could be done by observing the shorting currents which flow at the instant the plates are shorted together, and deriving a voltage proportional to the radial distance of the spot at the shorting instant. The method of indicating the signal's existence will be described by an example. Let us suppose one is sampling the radial distance, R, of the spot from the center of the screen at a rate of 200/sec. It is desired to determine the existence of a signal of 20 milliseconds duration or longer. The choice will be based on the average of three samples, hereafter denoted by R_a. In the absence of a signal the probability distribution of this average is:

$$P(R_a > X) = \text{EXP}\left(\frac{-3x^2}{2\sigma^2}\right) \quad (5)$$

where σ is the RMS noise voltage. The 3 occurs because of the three-sample average, giving an effective RMS noise of $\sigma/\sqrt{3}$. When signal and noise are both present the probability distribution is:

$$P(R_a > x) = \frac{3}{\sigma 2} \int_x^\infty I_o\left(\frac{3RP}{2}\right) \text{EXP}\left[- \frac{(R^2 + P^2)}{2\sigma^2} \; 3\right] RdR \quad (6)$$

where P is the signal magnitude and I_o is the modified Bessel function of the first kind. The above expression cannot be evaluated in closed form but it has been tabulated.[6]

If one chooses to say that a signal exists when $R_a > 3.65\sigma$ then the false alarm probability from eq. 5 is $e^{-20} = 2 \times 10^{-9}$. This corresponds to one false alarm every $\frac{2 \times 10^9}{3600} \times \frac{1}{200} = 700$ hours.

The detection or intercept probability will be a function of P/σ, the RMS S/N. For $\frac{P}{\sigma} = 5.43$ we have an effective RMS S/N, for the average, of $\sqrt{3}\, P/\sigma = 9.4$. Using the tables in Marcum[6] we find a detection probability of .999. Assuming a low-pass cutoff of 200 cycles, and a receiver bandwidth of 1 kc, this would give a receiver RMS S/N of $5.43\sqrt{\frac{200}{1000}} = 2.43$ which is the required S/N ratio for a detection probability of .999 when the false alarm probability is 2×10^{-9}.

4.3 Implementation

The question remains as to how the decision scheme could be implemented in practice. One way might be to use an RC circuit charged by the deflection plate shorting current. Its time constant would be chosen such that the potential across it exceeded a predeterminined level if the average of 3 successive pulses exceeded some fixed value. When this potential reached the proper value it would actuate a switch in the deflection plate circuit, increasing the time constant to a value of several seconds, and perhaps sounding an audible warning device of some nature. The spot would now decay slowly enough to the center of the screen to enable the operator to observe a bearing. One could parallel the short-burst CR tube with a tube not having this provision so as to provide uninterrupted service on long duration signals.

5. INSTRUMENTAL ACCURACY

It is important that the total gain undergone by the E-W and N-S signals (herein referred to as channel A and channel B) be as nearly the same as is practicable.

The bearing error in the absence of noise is:

$$\epsilon = \text{Tan}^{-1} A(1 + \Delta) - \text{Tan}^{-1} \frac{A}{B} = \text{Tan}\left[\frac{A/B}{1 + (1 + \Delta)\frac{A^2}{B^2}}\right] \qquad (7)$$

$$= \text{Tan}^{-1}\left[\frac{\text{Tan}\,\theta}{1 + (1 + \Delta)\,\text{Tan}\,\theta}\right]$$

where θ is the true bearing and Δ represents the difference between the gain in channel A and the gain in channel B. By setting the derivative with respect to θ of both sides equal to zero it can be shown that maximum ϵ occurs when:

$$\text{Tan}\,\theta = \frac{1}{\sqrt{1 + \Delta}}$$

giving: (8)

$$\epsilon_{max} = \text{Tan}^{-1} \frac{1}{2\sqrt{1 + \Delta}}$$

For one-degree bearing accuracy the two channel gains must be equal to within $\pm$ .32 db. Assume the receiver to be flat within .1 db , a not unreasonable figure. The balanced modulators and synchronous detectors then cannot differ in gain by more than about .22 db. By the use of balancing pots and/or matched tubes this figure can be met, and with proper design the drift after warmup[1] will stay within these limits.[9]

Though the above work has considered solely CW or, at most, keyed CW, the system should operate satisfactorily on voice, provided significant spectral components do not beat with the two modulation frequencies or their odd harmonics to provide components in the LP passband. A simple low-pass filter preceeding the synchronous detectors might be found desirable.

As to other types of modulation such as SSB, DSB, or FM, the operation should be substantially unaffected provided the IF bandwidth is sufficiently broad to accept the full deviation. This gives the system a decided advantage over doppler-type units where the difficulty of obtaining bearings increases considerably with SSB.

6. SENSITIVITY

Following Cleaver,[3] the sensitivity will be defined as that CW signal strength which will give a bearing deflection due to noise of $\pm$ 5 degrees about the true bearing when the depth of modulation is adjusted to 60 per cent. It will be said (somewhat arbitrarily, it is true) that this condition occurs when the probability that the spot falls within a 10-degree

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1 The paper by Cleaver is a description of a unit of this type which was built for the British Navy and is considered worthwhile reading for anyone interested in further work along this line. The equipment described in this paper had an overall drift after warmup within $\pm$ 1/2 degree over a 30 hour period.

sector centered about the true bearing is .95. This condition occurs when the RMS S/N of the output of the low-pass filter is $16\sqrt{2}$ (see App. III).

Using an approximate analysis given by Busignies and Dishal[7] it is found that for an LP cut off of 5 cycles and an IF bandwidth of 1 kc the sensitivity S is:

$$S = 252\sqrt{KTR_g \cdot 10^{\left(\frac{NF}{10}\right)}} \qquad (9)$$

where R_g is the source resistance, K is Boltzmann's constant, T the temperature in degrees Kelvin and NF is the overall receiver noise figure in db. For T equal to 295^0K, R_g equal to 200 ohms, and a noise figure of 10 db, this gives a sensitivity of .8μv.

7. CONCLUSIONS

The system appears capable of providing a fairly versatile direction finding unit. It gives instantaneous visual sense as well as bearing indication. It is well suited to the addition of remote indicator units at distances up to about a mile without repeaters.[2] With some additional development, means could be provided for observing extremely short burst transmissions. It has reasonable sensitivity. It should perform equally well on AM, SSB, or FM.

The system is not capable of resolving two signals occuring simultaneously within the receiver pass band as is possible with aural null systems. However, if the interfering signal is discontinuous, and its presence is known, the inherent speed of the system should enable reasonable bearing to be taken during those times it is absent.

If the modulation on the received signal has significant spectral components very close to the modulating frequencies, a "jitter" will occur in the display. This would be most troublesome on voice. Whether it would be objectional or not could best be settled by experiment. One solution would be to allow the operator to select either fast or slow modulation depending on the particular operational requirements, slow for voice, and fast for keyed CW, very short transmissions, etc.

A fair amount of engineering time and skill must be expended in designing drift-free modulators and detectors, since proper operation depends on the stability of these units.

If portability rather than sensitivity is the dominant requirement, the use of square wave modulation with diode switch modulators[10] in conjunction with transistor circuitry could lead to a fairly compact unit.

APPENDIX I

BANDWIDTH AND NOISE CONSIDERATIONS IN SQUARE VERSUS SINE WAVE MODULATION

Assume a basic switch rate f and a receiver IF bandwidth B equal to twice the maximum harmonic present plus a guard band αf.

Denote the signal envelope by E_s and let N harmonics of a square wave be present.

Then:

$$E_s = C \sum_{n=1}^{N} \frac{\sin (2n - 1)\, 2\pi f \tau}{2n - 1}$$

and therefore

$$B = (2N - 1)f + \alpha f$$

where C is some constant.

Denote the synchronous detector RMS output by E_o.

$$E_o = \frac{KC}{\pi f} \sum_{1}^{N} \frac{1}{(2n - 1)^2}$$

Where K is some constant of proportionality. Denote the RMS noise by E_n.

$$E_n = D\sqrt{B} = D\sqrt{f}\sqrt{2N - 1 + \alpha}$$

Where D is some constant.

Then:

$$\frac{E_n\,(N + 1)}{E_n\,(N)} = \sqrt{\frac{2(N+1)-1+\alpha}{2N-1+\alpha}} = \sqrt{1 + \frac{2}{2N-1+\alpha}}$$

and:

$$\frac{E_o\,(N+1)}{E_o\,(N)} = \frac{\sum_{n=1}^{N+1} \frac{1}{(2n-1)^2}}{\sum_{n=1}^{N} \frac{1}{(2n-1)^2}} = 1 + \frac{1}{(2N+1)^2}\left(\frac{1}{\sum_{n=1}^{N} \frac{1}{(2n-1)^2}}\right) < \frac{E_n\,(N+1)}{E_n\,(N)} \qquad \begin{matrix} N \geq 1 \\ \alpha \leq 7.5 \end{matrix}$$

Thus it is seen that the noise increases faster than the signal provided the guard band does not exceed 7.5f.

APPENDIX II

PROBABILITY DENSITY FOR THE DISPLAY

As discussed in the section on the low-pass filter, for high cut-offs, the noise signals applied to the N-S and E-w CR plates are not independent. Assuming only Gaussian noise present, the E-W and N-S voltages (here denoted by E_1 and E_2 respectively) can be represented in the form:

1) $$E_1 = N_1 + N_o = N_1 + \sum_{n=1}^{N} c_m \cos\left[(m\Delta\omega t - \omega_1 t) - \varphi_m\right]$$

2) $$E_2 = N_2 + \sum_{n=1}^{N} c_m \cos\left[(m\Delta\omega t - \omega_2 t) - \varphi_m\right]$$

where N_1 and N_2 are independent Gaussian random signals of standard deviation σ_1^2, and the φ_m are distributed randomly from zero to 2π. The second terms are identical in 1 and 2 except for a shift in the spectrum. The notation used is that in Rice.[5]

$$c_n^2 = 2W(f_n)\,\Delta f$$

where W(f) is the power spectrum. The two-dimensional probability density function for E_1 and E_2 is:

3) $$\frac{1}{2\pi\sqrt{\bar{E}_1^2\,\bar{E}_1^2 - (\bar{E}_1\bar{E}_2)^2}}\ \mathrm{EXP}\left[\frac{-E_1^2\,\bar{E}_2^2 - E_2^2\,\bar{E}_1^2 + 2\,\bar{E}_1\bar{E}_2\ \ E_1E_2}{2(\bar{E}_1^2\,\bar{E}_2^2 - (E_1E_2)^2)}\right]$$

where the bars denote averages.

4) $$\bar{E}_1^2 = \sigma_1^2 + \sigma_o^2 = \bar{E}_2^2$$

5) $$\overline{E_1E_2} = \text{ave}\left(\sum_1^N c_n \cos\left[(\omega_n - \omega_1)t - \varphi_n\right]\right)\left(\sum_1^N c_k \cos\left[(\omega_k - \omega_2)t - \varphi_k\right]\right)$$

$$= \text{ave} \sum_{n=1}^N c_n \sum_{k=1}^N \frac{c_k}{2} \cos\left[(\omega_n - \omega_1)\, t - \varphi_n\right]\left\{\cos\left[(\omega_k - \omega_1)t - \varphi_k\right]\cos(\omega_1 - \omega_2)t\right.$$

$$\left. - \sin\left[(\omega_k - \omega_1)t - \varphi_k\right]\sin(\omega_1 - \omega_2)t\right\} = \frac{\cos(\omega_1 - \omega_2)t}{2}\ \sigma_o^2$$

Letting $E_1 = x$ and $E_2 = y$.

6) $$p(x,y) = \frac{1}{2\pi\sqrt{(\sigma_1^2 + \sigma_1^2)^2 - \sigma^4 \dfrac{\cos^2(\omega_1 - \omega_2)t}{4}}}$$

$$\text{EXP}\left[-\frac{(\sigma_1^2 + \sigma_o^2)(x^2 + y^2) - y\sigma_o^2 \cos(\omega_1 - \omega_2)t}{2\left[(\sigma_1^2 + \sigma_o^2)^2 - \dfrac{\sigma_o^4 \cos^2(\omega_1 - \omega_2)t)}{4}\right]}\right]$$

or in polar coordinates

7) $$p(R,\theta) = \frac{\text{EXP}\left[-\dfrac{(\sigma_1^2 + \sigma_o^2)R^2\left[1 - \dfrac{\sigma_o^2 \cos(\omega_1 - \omega_2)t}{2(\sigma_1^2 + \sigma_o^2)}\ \sin 2\theta\right]}{2\left[(\sigma_1^2 + \sigma_o^2)^2 - \dfrac{\sigma_o^4 \cos^2(\omega_1 - \omega_2)t}{4}\right]}\right]}{2\pi\sqrt{(\sigma_1^2 + \sigma_o^2)^2 - \dfrac{\sigma_o^4 \cos^2(\omega_1 - \omega_2)\, t}{4}}}$$

which is the probability density for the spot position on the cathode ray tube in the presence of noise only. When a signal is present of magnitude E_s at an angle φ one would merely substitute $(x - E_s \cos\varphi)$ and $(y - E_s \sin\varphi)$ for x and y respectively in equation (6).

APPENDIX III

PROBABILITY DISTRIBUTION FOR A SECTOR OF ARC

Given a symmetric two-dimensional Gaussian probability density $\left\{ \mathrm{EXP}\ \frac{1}{2\sigma^2} \left[(x-A)^2 + y^2 \right] \right\}$ centered at (A,o). The problem is to find the probability distribution for a circular sector of angle 2α symmetric about the positive x axis. Since the integrals involved cannot be evaluated in closed form an approximate analysis adequate for the purposes of this report will be given.

In cylindrical coordinates the probability density function is:

$$p(R,\theta) = \mathrm{EXP}\left[\frac{1}{2\sigma^2} \left(R^2 - 2RA\cos\theta + A^2 \right) \right]$$

σ^2 = standard deviation

$$p(\theta) = \int_0^\infty p\ (R,\theta) R dR = \frac{1}{2\pi} \left\{ e^{\frac{-k2}{2}} + \cos\theta e^{\frac{-k^2 \sin^2\theta}{2}} \left[k\frac{\sqrt{\pi}}{\sqrt{2}} + \sqrt{2}k \int_0^{\frac{k\cos\theta}{\sqrt{2}}} e^{-x^2} \right] \right\} dx$$

$k = \frac{A}{\sigma}$

$$P(-\alpha<\theta<\alpha) = \int_{-\alpha}^{\alpha} p(\theta)\ d\theta = \frac{\alpha}{\pi}\ e^{\frac{-k^2}{2}} + \frac{1}{2(2\pi)} \int_{-\sin\alpha}^{\sin\alpha} e^{\frac{-k^2y^2}{2}} dy$$

$$+ \frac{k}{\sqrt{2}\ \pi} \int_{-\sin\alpha}^{\sin\alpha} e^{\frac{-k^2y^2}{2}} dy \int_0^{\frac{k}{2}\sqrt{1-y^2}} e^{-x^2} dx$$

$$= \frac{\alpha}{\pi}\ e^{\frac{-k^2}{2}} + \frac{k}{2\pi} \int_0^{\sin\alpha} e^{\frac{-k^2y^2}{2}} dy + \frac{2}{\pi} \int_0^{\frac{k\sin\alpha}{\sqrt{2}}} e^{-y^2} dy \int_0^{\frac{k^2}{2} - x^2} e^{-x^2} dx$$

The last term is integrated over the shaded region shown in Figure 6 .

Suppose one were to approximate the integral by one over the rectangular region

$$0 \leq y \leq \frac{k \sin\alpha}{\sqrt{2}}, \quad 0 \leq x \leq \frac{k}{\sqrt{2}} .$$

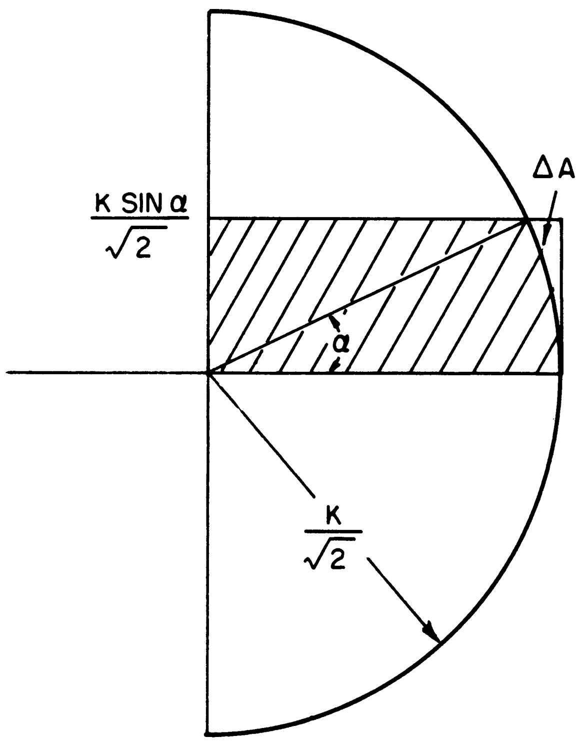

FIG. 6 PROBABILITY DENSITY FOR A SECTOR OF ARC

The error would be less than the area ΔA times the maximum value of the function in ΔA or:

$$\epsilon < \frac{2e^{\frac{-k^2}{2}}}{\pi} \Delta A = e^{\frac{-k^2}{2}} \frac{k^2}{\pi} \left[\left(\sin\alpha - \frac{\sin 2}{4} - \frac{\alpha}{2} \right) \right]$$

Thus the probability is approximately

$$P(-\alpha < \theta < \alpha) = \frac{\alpha}{\pi} e^{\frac{-k^2}{2}} + \frac{k}{\sqrt{2\pi}} \int_0^{\mathrm{Sin}\alpha} e^{\frac{-k^2 y^2}{2}} + \frac{2}{\pi} \int_0^{\frac{k}{\sqrt{2}}} e^{-x^2} dx \int_0^{\frac{k \sin\alpha}{\sqrt{2}}} e^{-y^2} dy$$

all of which are tabulated functions.

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