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VIBRATIONS OF A CIRCULAR PLATE ABOUT A POST-BUCKLED STATIC STATE

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I. INTRODUCTION

Modern engineering structures have been experiencing a very rapid decrease in their "thickness" dimension as severe weight (and other) limitations have been imposed during recent years. As a consequence many structures are used in a post buckled state, that is to say the loads sustained are greater than those predicted in the usual "Euler column" sense. The motivation for this state of practice is obvious, however, a safe road to design remains to be paved. In addition, structures loaded in this manner are frequently expected to survive an environment of dynamic forces while subjected to these high static loads. The purpose of the present study is to determine the dynamic characteristics of such a structure. The results of this study are in the form of the natural frequencies and shapes of the modes of vibration of a circular plate as the function of a load parameter.

The free vibrations of elastic bodies or structures about an equilibrium configuration have been studied extensively. The natural frequency of vibration and the shape of the mode of vibration are the most important features which are obtained out of the solution of an eigenvalue problem.

If such a body or structure is first preloaded statically, then the resulting frequency and mode of vibration exhibit interesting features. In general, a tensile system of stresses or forces causes an increase in the frequency of vibration, while compressive forces serve to decrease the frequency of vibration. The initial loading affects the effective stiffness of the structure and in the case of a compressive loading the effect is such as to reduce the stiffness to zero, which is indicated by zero frequency of vibration, and thus buckling of the structure in the conventional sense occurs. Indeed, a dynamic approach to the evaluation of buckling loads predicts buckling when the frequency of small vibrations about the static configuration goes to zero.

The most straightforward example of such a problem is the lateral vibration of an elastic bar which is axially loaded.[1] The mode shape is sinusoidal for a simply supported bar, and the square of the frequency of vibration is linearly related to the axial force (or an associated loading parameter). Lurie[2] has discussed several examples related to vibration and structural stability and cites both theoretical and experimental results. He shows that, in general, within the framework of linear theories, whenever the mode shape of buckling and of vibration in the presence of axial loads is the same, then the interaction curve between the square of the frequency and some monotonic increasing load parameter will always be linear. Massonnet[3] discusses this same subject extensively, but frequently has to resort to approximate means, such as the Rayleigh-Ritz method, to solve the problem. These problems are solved within the framework of a linear theory.

The buckling problem is phrased as an eigenvalue problem of the linear theory where the eigenvalue is associated with the critical load and a buckling mode of undetermined amplitude is obtained. These results imply no lateral deflection (the trivial solution) or that buckling occurs suddenly and with uncontrolled amplitude. That this paradoxical situation never arises in reality is explained by the presence of some imperfections, either in the structure or in the loading system, which always insure that the structure deflects laterally as the load reaches the critical value. Even a crude experiment with a simple column shows that the structure does not collapse violently as the critical load is reached; however, the deflections do become large. The effects of these large deflections are not fully included in a linear theory. In order to discuss such phenomena more adequately, an improvement in the theory is made which results in nonlinear differential equations. In the case of a plat such equations were given by von Kármán.(4)

The solid circular plate is the structure to be investigated here. The linear equations of the classical theory of plates have been solved extensively.(5,6) The buckling of a circular plate was first studied by Bryan.(7) Federhofer(8) studied the problem of the vibrating clamped edge plate subjected to edge loads and presented extensive results of the interaction between compressive (and tensile) forces and the frequency of lateral vibration of the plate.*

There exist relatively few solutions for the nonlinear equations for plates, introduced in 1910 by von Kármán. The problem is particularly difficult for rectangular plates where several approximate methods have been introduced, in particular by Marguerre.(9) Bisplinghoff and Pian(10) treated the case of vibration of a rectangular plate of infinite length and some cases for plates of finite length. For the circular plate, however, several more solutions are available. Way(11) solved, by power series methods, the problem of a circular plate subjected to lateral load. Friedrichs and Stoker(12,13) used perturbation and

* During the course of the present investigation it was necessary to solve such equations for a simply supported plate. Results similar to those of Federhofer for the clamped edge plate are presented in the Appendix B.

power series methods to solve the problem of the simply supported circular plate subjected to compressive radial loading (in the plane of the plate). They treated only the axially symmetric case in a very exhaustive manner. A nonsymmetric version is beset by considerably more mathematical difficulties. The methods of these writers were applied by Bodner[14] to a clamped edge plate for the same type of loading. Bromberg[15] used the methods utilized by Friedrichs and Stoker to study the effect of very large lateral loads which give rise to certain instabilities. Keller and Reiss[16] applied numerical methods to the problem discussed by Friedrichs and Stoker. Similar problems are studied by Alexeev[17] and as a special case in a paper by Panov and Feodossiev.[18] Masur[19], in a paper published in 1958, utilized a stress function space together with a minimum energy principle to obtain a sequence of solutions with error estimates for the post-buckling behavior of a plate using the von Kármán equations.

In a recent paper, Massonnet[20] considered the effects of initial curvature on the natural frequencies of vibration of an edge-compressed, clamped edge, circular plate. He solves the static problem by the method of Friedrichs and Stoker and then assumes that the mode shape of vibration is the same as that of the static problem, and utilizing the Rayleigh-Ritz method obtains the approximate frequency of vibration.

The present study is concerned with the linearized vibrations of a circular plate relative to a static buckled configuration which is governed by the von Kármán equations. The plate is subjected to radial displacements which are the cause of the buckling and post buckling equilibrium behavior. Although these boundary conditions are different from those of Friedrichs and Stoker, they are nevertheless mathematically equivalent for the static problem. It is here possible, however, to treat the problem of nonsymmetric vibrations relative to a symmetric buckling or static configuration.

II. FORMULATION OF THE PROBLEM

For a preliminary consideration of the differential equations governing the present problem consider the xy plane of a cartesian coordinate system to be the middle plane of the plate. The z direction is the direction of the lateral deflection. Such a plate may be subjected to membrane forces in the plane of the plate and lateral loads in the z direction. The thickness of the plate is h. In the absence of body forces in the x and y directions there are two relevant differential equations due to von Kármán.[4] The Equation (2.1) represents the equation of lateral equilibrium while compatibility is expressed by the Equation (2.2).

$$D\Delta\Delta\bar{w} - c_{i\alpha}c_{j\beta}F,_{\alpha\beta}\bar{w},_{ij} = p \tag{2.1}$$

$$\Delta\Delta F \equiv F,_{iijj} = -\frac{Eh}{2} c_{i\alpha}c_{j\beta}\bar{w},_{\alpha\beta}\bar{w},_{ij}$$

$$\equiv - Eh(\bar{w},_{xx}\bar{w},_{yy} - \bar{w},_{xy}\bar{w},_{xy}) \tag{2.2}$$

where Δ represents the Laplacian operator, F is the Airy stress function, $\bar{w}$ is the lateral deflection of the plate and p is the load per unit area applied to the lateral surface of the plate. Further, the flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

where E is the Young's Modulus of Elasticity and ν is Poisson's Ratio, and

$$c_{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Inherent in the utilization of these equations is the inclusion in the strain displacement equations of nonlinear terms involving the derivatives of $\bar{w}$ with respect to x and y.

For a moving plate the inertia terms which are due to the motion of an element of the plate in the plane of the plate are neglected in comparison to those due to the lateral motion.* These inertia terms, which in actuality represent body forces, may, in this case then, be treated as the lateral load

$$p = -\rho h \frac{\partial^2 \bar{w}}{\partial t^2} \tag{2.3}$$

where ρ is the mass per unit volume.

Only small amplitude harmonic vibrations with respect to the static configuration of larger amplitude are considered. Consistent with this assumption the following partitioning of the stress function F, the displacements, strains and other quantities is proposed

$$\begin{aligned} F &= F^S + \epsilon^* F^D e^{i\omega t} \\ \bar{w} &= \bar{w}^S + \epsilon^* \bar{w}^D e^{i\omega t} \\ e_{ij} &= e_{ij}^S + \epsilon^* e_{ij}^D e^{i\omega t} \\ N_{ij} &= N_{ij}^S + \epsilon^* N_{ij}^D e^{i\omega t} \end{aligned} \tag{2.4}$$

where e_{ij} and N_{ij} are the cartesian components of the membrane strains and stresses, while ω is the circular frequency of vibration and ϵ^* is an arbitrary, small parameter.

* For a discussion to this point see references (21) and (22).

The membrane stresses N_{ij} are derivable from the stress function F by

$$N_{ij} = c_{i\alpha}c_{j\beta}F_{,\alpha\beta} \tag{2.5}$$

The membrane strains e_{ij} are related to the stresses by

$$e_{ij} = \frac{1}{Eh}[(1+\nu)N_{ij} - \nu N_{kk}\delta_{ij}] \tag{2.6}$$

where δ_{ij}, the Kronecker delta, has values

$$\delta_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Substitution of these quantities in Equations (2.1) and (2.2) and retaining only those terms which contain ϵ^* to the power of one or less yields two sets of differential equations, one governing the static problem and the other governing the dynamic problem. These are

$$\begin{aligned} D\Delta\Delta\bar{w}^S - c_{i\alpha}c_{j\beta}F^S_{,\alpha\beta}\bar{w}^S_{,ij} &= 0 \\ \Delta\Delta F^S &= -\frac{Eh}{2}c_{i\alpha}c_{j\beta}\bar{w}^S_{,\alpha\beta}\bar{w}^S_{,ij} \end{aligned} \tag{2.7}$$

$$\begin{aligned} D\Delta\Delta\bar{w}^D - c_{i\alpha}c_{j\beta}(F^S_{,\alpha\beta}\bar{w}^D_{,ij} + F^D_{,\alpha\beta}\bar{w}^S_{,ij}) &= \rho h\omega^2\bar{w}^D \\ \Delta\Delta F^D &= -Eh\, c_{i\alpha}c_{j\beta}\bar{w}^S_{,\alpha\beta}\bar{w}^D_{,ij} \end{aligned} \tag{2.8}$$

Since all detailed discussions of this plate are for a solid circular one, of outside radius R, the problem is rephrased in terms of the polar coordinates. The static configuration is assumed to be axially symmetric. However, permit a nonsymmetric dynamic configuration. In particular, all quantities are chosen in the following form* without any

* Henceforth, unless otherwise noted, a summation symbol not having the summation limits specified is intended to be summed over n from 0 to ∞.

significant loss of generality:

$$\bar{u}^S = \bar{u}^S(r) \qquad \bar{u}^D = \sum \bar{u}_n^D(r)\cos n\theta$$

$$\bar{v}^S = \bar{v}^S(r) = 0 \qquad \bar{v}^D = \sum \bar{v}_n^D(r)\sin n\theta + \left(\frac{\gamma}{R}\right)^2 B^*r$$

$$\bar{w}^S = \bar{w}^S(r) \qquad \bar{w}^D = \sum \bar{w}_n^D(r)\cos n\theta \tag{2.9}$$

$$F^S = F^S(r) \qquad F^D = \sum F_n^D(r)\cos n\theta$$

where the u and v variables represent the displacement components in the radial and tangential directions respectively.

Simultaneously it is advantageous to render all pertinent quantities in these equations dimensionless, and for this purpose let

$$x = \frac{r}{R}$$

$$U = \frac{R}{\gamma^2}\bar{u}^S \qquad u_n = \frac{R}{\gamma^2}\bar{u}_n^D$$

$$V = \frac{R}{\gamma^2}\bar{v}^S \qquad v_n = \frac{R}{\gamma^2}\bar{v}_n^D$$

$$W = \frac{\bar{w}^S}{\gamma} \qquad w^n = \frac{\bar{w}_n^D}{\gamma} \tag{2.10}$$

$$\Phi = \frac{F^S}{D} \qquad \phi^n = \frac{F_n^D}{D}$$

where $\gamma^2 = h^2/12(1-\nu^2)$.

By the use of the above expressions the differential equations for the static case become,

$$\nabla^4 W - \frac{1}{x}(\Phi' W')' = 0 \tag{2.11a}$$

and

$$\nabla^4 \Phi = -\frac{1}{2x}(W'W')' \tag{2.11b}$$

where $\nabla^2() = \frac{1}{x}[x(\;)']'$ and primes designate differentiation with respect to x.

The dynamic case for the n^{th} mode* is governed by

$$\left(\nabla^2 - \frac{n^2}{x^2}\right)^2 w^n - \frac{1}{x}\left(\Phi' w^{n\prime}\right)' + \frac{n^2}{x^2}\Phi'' w^n \qquad (2.12a)$$

$$- \frac{1}{x}\left(\phi^{n\prime} W'\right)' + \frac{n^2}{x^2}\phi^n W'' = \mu^n w^n$$

and $$\left(\nabla^2 - \frac{n^2}{x^2}\right)^2 \phi^n = - \left[\frac{1}{x}\left(W' w^{n\prime}\right)' - \frac{n^2}{x^2} W'' w^n\right] \qquad (2.12b)$$

where $$\mu^{(n)} = \frac{\rho h R^2 \omega_n^2}{D}$$

In order to be able to state the boundary conditions clearly, we rewrite all of the quantities involved, including moments, stresses and strains, in terms of the non-dimensional quantities. This is a matter of formal substitution and the results are listed below. As an example, consider in detail the radial strain and the radial stress. The radial strain for the dynamic configuration is defined as

$$e_{rr}^D = \frac{\partial \bar{u}^D}{\partial r} + \frac{\partial \bar{w}^S}{\partial r} \cdot \frac{\partial \bar{w}^D}{\partial r}$$

where as before the superscript D refers to the dynamic configuration and the superscript S refers to the static one. In terms of the dimensionless quantities define a strain as

$$e_{xx} = \sum \left(\frac{du_n}{dx} + \frac{dW}{dx}\frac{dw^n}{dx}\right)\cos n\theta = \sum e_{xx}^n \cos n\theta$$

and hence $e_{rr}^D = \left(\frac{\gamma^2}{R}\right) e_{xx}$

Consider now the definition for the radial stress component N_{rr}^S. The static membrane stress (as obtained from the stress function F^S), is

$$N_{rr} = \frac{1}{r}\frac{dF^S}{dr}$$

* Separation into modes is possible because of the assumed axial symmetry of the static solution.

The corresponding non-dimensional stress can be defined as

$$T_{xx} = \frac{1}{x}\frac{d\Phi}{dx}$$

and as a consequence the relationship among these two stresses is

$$N_{rr} = \frac{D}{R^2} T_{xx}$$

Similarly the various dimensional stresses and strains are as listed below with their relationship to the dimensionless quantities.

$$\begin{aligned}
N^S_{rr} &= \frac{D}{R^2}\,\frac{1}{x}\,\Phi' = \frac{D}{R^2} T_{xx} \\
N^S_{\theta\theta} &= \frac{D}{R^2}\,\Phi'' = \frac{D}{R^2} T_{\theta\theta} \\
N^S_{r\theta} &= 0 = T_{r\theta} \\
N^D_{rr} &= \frac{D}{R^2} \sum \left(\frac{1}{x}\,\phi^{n\prime} - \frac{n^2}{x^2}\,\phi^n\right) \cos n\theta = \frac{D}{R^2} \sum t^n_{xx} \cos n\theta \\
N^D_{\theta\theta} &= \frac{D}{R^2} \sum \phi^{n\prime\prime} \cos n\theta = \frac{D}{R^2} \sum t^n_{\theta\theta} \cos n\theta \\
N^D_{r\theta} &= \frac{D}{R^2} \sum n \left(\frac{\phi^n}{x}\right)' \sin n\theta = \frac{D}{R^2} \sum t^n_{x\theta} \sin n\theta
\end{aligned} \tag{2.13}$$

Consequently the stress strain relationships among the dimensionless quantities are:

$$\begin{aligned}
\bar{e}^S_{xx} &\equiv E_{xx} = T_{xx} - \nu T_{\theta\theta} = \frac{1}{x}\,\Phi' - \nu\Phi'' \\
\bar{e}^S_{\theta\theta} &\equiv E_{\theta\theta} = T_{\theta\theta} - \nu T_{xx} = \Phi'' - \frac{\nu}{x}\,\Phi' \\
\bar{e}^S_{x\theta} &= E_{x\theta} = 0 \\
e^n_{xx} &= t^n_{xx} - \nu t^n_{\theta\theta} = \frac{1}{x}\,\phi^{n\prime} - \frac{n^2}{x^2}\,\phi^n - \nu\phi^{n\prime\prime} \\
e^n_{x\theta} &= t^n_{\theta\theta} - \nu t^n_{xx} = \phi^{n\prime\prime} - \nu\left(\frac{1}{x}\,\phi^{n\prime} - \frac{n^2}{x^2}\,\phi^n\right) \\
e^n_{x\theta} &= \frac{E}{G}\, t^n_{x\theta} = 2(1+\nu)n\left(\frac{1}{x}\,\phi^{n\prime}\right)'
\end{aligned} \tag{2.14}$$

since $G = E/2(1+\nu)$.

Further the relationships among the bending moments are:

$$M^S_{rr} = -D(\tfrac{\gamma}{R})^2(W'' + \tfrac{\nu}{x}W') = D(\tfrac{\gamma}{R})^2 M_{xx}$$

$$M^S_{\theta\theta} = -D(\tfrac{\gamma}{R})^2(\tfrac{1}{x}W' + \nu W'') = D(\tfrac{\gamma}{R})^2 M_{\theta\theta}$$

$$M^S_{r\theta} = 0 = M_{x\theta}$$

$$M^D_{rr} = -D(\tfrac{\gamma}{R})^2 \sum [w^{n''} + \nu(\tfrac{1}{x}w^{n'} - \tfrac{n^2}{x^2}w^n)] \cos n\theta \qquad (2.15)$$

$$\equiv D(\tfrac{\gamma}{R})^2 \sum m^n_{xx} \cos n\theta$$

$$M^D_{\theta\theta} = -D(\tfrac{\gamma}{R})^2 \sum [\tfrac{1}{x}w^{n'} - \tfrac{n^2}{x^2}w^n + \nu w^{n''}] \cos n\theta$$

$$\equiv D(\tfrac{\gamma}{R})^2 \sum m^n_{\theta\theta} \cos n\theta$$

$$M^D_{r\theta} = D(1-\nu)(\tfrac{\gamma}{R})^2 \sum n(-\tfrac{w^{n'}}{x} + \tfrac{w^n}{x^2}) \sin n\theta$$

$$\equiv D(\tfrac{\gamma}{R})^2 \sum m^n_{x\theta} \sin n\theta$$

The appropriate boundary conditions for a circular plate must now be considered. It is convenient to consider the relevant conditions for the static equations apart from those associated with the differential equations governing the vibration motion. Let the plate be simply supported at its circumference. The usual interpretation of "simply supported" is to consider that both the deflection at the support and the radial resisting moment offered by the support are zero. However, a theory that includes membrane effects must in addition specify a restriction, at the boundaries, upon the membrane displacements or stresses. Thus for the static problem a radial displacement at the support is specified. Alternately, the radial membrane stress can be specified, which is indeed the manner in which Friedrichs and Stoker[13] chose to state their problem. However, since this investigation is primarily concerned with the dynamic problem a displacement condition appears more appropriate. The effects upon the static problem depending upon the nature of the membrane boundary condition specification are equivalent for either of the above cases. Physically the most realizable situation is to specify a zero radial displacement of the plate at the edge and then consider the effect of uniform heating of the plate (the supporting structure being assumed rigid in the usual sense as well as with respect to temperature changes). Thermal effects would require an additional term in Equation (2.2) and make the boundary condition homogeneous, which again is mathematically equivalent to the situation chosen here.

Thus the boundary conditions governing the solution of the static problem are

$$B_1(W) \equiv W(1) = 0 \tag{2.16}$$

$$B_2(W) \equiv \left(W'' - \frac{\nu}{x} W'\right)_{x=1} = 0 \tag{2.17}$$

and

$$\bar{u}^S(1) = -\lambda \bar{u}_E \tag{2.18}$$

Here $\bar{u}_E$ is the magnitude of the radial displacement which is required to cause the plate to buckle in the usual or linear, Euler, sense. The term λ is a parameter determining the extent to which the post buckling domain is penetrated. This third condition is conveniently rephrased in terms of the stress function Φ. Utilizing the relationship between

$\bar{u}^S$ and $e^S_{\theta\theta}$ for a symmetric configuration, one finds

$$\bar{u}^S = x\, e^S_{\theta\theta}$$

Consequently, by Equation (2.14)

$$B_3(\Phi) \equiv x[x(\tfrac{1}{x}\Phi')' + (1-\nu)(\tfrac{1}{x}\Phi')]_{x=1} = -\lambda U_E \qquad (2.19)$$

where $$U_E = \frac{R}{\gamma^2}\bar{u}_E$$

The boundary conditions for the dynamic equations require more careful consideration as a consequence of the permitted occurrence of non-symmetric modes. Again the notion of simple support implies that

$$B_1(w^n) = 0 \qquad (2.20)$$

and $$B_2(w^n) = 0 \qquad (2.21)$$

Consistent with Equation (2.9) the prescribed membrane displacements at the boundary take the form

$$u(1,\theta) = \sum A_n \cos n\theta$$
$$v(1,\theta) = \sum B_n \sin n\theta + B^* x \qquad (2.22)$$

where the A's and B's are specified.

It appears then that a specification of A_n and B_n provides the necessary number of conditions required of the problem. For the axially symmetric case, $n = 0$, v is identically zero (plus the possibility of rigid body motion) and just one condition needs to be imposed to specify the problem completely. However, a notable situation exists when $n = 1$. It can be shown** that in this case, also, just one additional condition is sufficient to specify the problem completely.

** This and other details for the case, $n = 1$, are carried out in detail in Appendix A.

The situation for n = 0 permits choosing the boundary condition which is similar with that for the static case, i.e. zero radial displacement, which in turn is synonymous with

$$\epsilon^{0}_{\theta\theta}(1) = 0 \qquad (2.23)$$

Furthermore, this same condition is admissible and sufficient for n = 1. These are the only cases for which computations are carried out. Consequently the third boundary condition used for the dynamic case is

$$B_4(\phi^n) \equiv e^{n}_{\theta\theta}(1) = [\phi^{n''} - \nu(\frac{1}{x}\phi^{n'} - \frac{n^2}{x^2}\phi^n)]_{x=1} = 0 \qquad (2.24)$$

with n = 0, 1.

III. THE PERTURBATION SOLUTION

In recalling the differential equations and boundary conditions governing the problem, we consider first the static case. It is required to solve the following differential equations

$$\nabla^4 W - \frac{1}{x} (\Phi^{*\prime} W')' = 0 \qquad (2.11a)$$

$$\nabla^4 \Phi^* = -\frac{1}{2x} (W'W')' \qquad (2.11b)$$

and the associated boundary conditions**

$$B_1(W) = 0 \qquad (2.16)$$

$$B_2(W) = 0 \qquad (2.17)$$

$$B_3(\Phi^*) = -\lambda U_E \qquad (2.18)$$

It is convenient here to partition the stress function such that

$$\Phi^* = \lambda \Phi_o + \Phi \qquad (3.1)$$

where the function Φ_o satisfies the differential equation

$$\nabla^4 \Phi_o = 0 \qquad (3.2)$$

and the boundary condition

$$B_3(\Phi_o) = -U_E \qquad (3.3)$$

Consequently, the function Φ satisfies the differential equation

$$\nabla^4 \Phi = -\frac{1}{2x} (W'W')' \qquad (3.4)$$

and the boundary condition

$$B_3(\Phi) = 0 \qquad (3.5)$$

** It should be understood that the regularity requirements at the origin are being considered implicitly.

The first of these equations will be recognized as the usual problem of plane elasticity whose well-known solution for the solid disk is

$$\Phi_o = -\frac{Tx^2}{2} \qquad (3.6)$$

where $$T = \frac{U_E}{1-\nu}$$

and where U_E is found in Equation (3.17).

In order to proceed to the topic of this thesis, it is first necessary to reproduce the solution previously obtained by Friedrichs and Stoker. There are two reasons that necessitate the repetition of this work. First, although these writers have completely solved this problem, their results are presented in such a form as not to permit direct application to the present work. Secondly, they were able to simplify the problem by a substitution of variables for W and for the stress function thereby reducing the order of the differential equations and making them directly integrable, at least in part. This is not possible here, since in the dynamic equations, due to the inertia term, w will appear explicitly and hence neither the integration nor the substitution of variables appears to be possible. Consequently, it is necessary to proceed with a solution in terms of W and Φ following, however, the example set forth by Friedrichs and Stoker.

Assume the functions W, Φ, and λ to be expandable in perturbation series:*

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* It can be shown that the other terms vanish.

$$W = \epsilon W_1 + \epsilon^3 W_3 + \epsilon^5 W_5 + \cdots\cdot$$
$$\Phi = \epsilon^2 \Phi_2 + \epsilon^4 \Phi_4 + \epsilon^6 \Phi_6 + \cdots\cdot \qquad (3.7)$$
$$\lambda = \lambda_o + \epsilon^2 \lambda_2 + \epsilon^4 \lambda_4 + \cdots\cdot$$

Here ϵ is the perturbation parameter which will be chosen as a monotone increasing function whose direct significance will be fixed at a later point in the development. It will eventually be related directly to the amount of deflection in this problem. On the other hand, it could represent the amount of radial displacement or, indeed, it could be any other monotone increasing function.

Substitution of these perturbation expansions in the differential equations and boundary conditions yields a sequence of differential equations, with associated boundary conditions, when coefficients of like powers of ϵ are equated. Consider now the various differential equations. It will be recognized that associated with ϵ^0 we obtain the equation whose solution is the function Φ_o.

For ϵ^1 the differential equation is

$$\nabla^4 W_1 - \frac{1}{x} \lambda_o (\Phi_o' W_1')' = 0 \qquad (3.8)$$

or alternately by Equation (3.6)

$$L_1(W_1) \equiv \nabla^4 W_1 + \lambda_o T \frac{1}{x} (x W_1')' = 0 \qquad (3.9)$$

and the boundary conditions are

$$B_1(W_1) = 0 \qquad (3.10)$$
$$B_2(W_1) = 0 \qquad (3.11)$$

This is seen to be the linear eigenvalue problem for the buckling of the plate subjected to compressive edge traction or displacement ($\lambda_o T$ represents the eigenvalue of the problem). There exists an infinity of eigenvalues and the solution to the problem may be represented as a development in the form of the associated modes or eigenfunctions* as

$$W_1 = \sum_{m=1}^{\infty} A_1^{(m)} H_m = \sum_{m=1}^{\infty} A_1^{(m)} [J_o(\alpha_m x) - J_o(\alpha_m)] \qquad (3.12)$$

This solution automatically satisfies the boundary condition B_1. The eigenfunctions must satisfy the associated characteristic equation which is obtainable from the boundary condition B_2. The characteristic equation for the m^{th} mode is

$$(1 + \nu) J_1 (\alpha_m) + \alpha_m J_2(\alpha_m) = 0 \qquad (3.13)$$

where $J_p(x)$ is the Bessel function** of the first kind and of order p and α_m is a root of the characteristic equation. There is an infinite number of such roots.

However, α_m is related to $\lambda_o^{(m)}$ and T through the differential equation by

$$\lambda_o^{(m)} T = \alpha_m^2 \qquad (3.14)$$

Consequently the characteristic equation determines $\lambda_o^{(m)} T$. Since the interest here centers around the first buckling mode, i.e. the symmetric one, only the lowest eigenvalue and its associated eigenfunction is

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* Superscripts in parentheses are intended to identify the variable and not to act as an exponent. Wherever possible, however, parentheses will be omitted whenever there is no possible confusion and will be included only to avoid confusion in isolated cases.

**In view of the varying defintions for the various Bessel functions, the ref. (23) is used as a standard throughout this thesis to avoid possible confusion.

considered. Consequently

$$W_1 = A_1^{(1)}[J_o(\alpha_1 x) - J_o(\alpha_1)] \tag{3.15}$$

and

$$\lambda_o^{(1)}T = \alpha_1^2 \tag{3.16}$$

It is convenient to choose

$$\lambda_o = \lambda_o^{(1)} = 1$$

whereby

$$T = \alpha_1^2$$

Furthermore, let

$$A_1^{(1)} = 1$$

This choice governs the selection of ϵ. Having obtained T, the value of U_E is now determined,

$$U_E = (1-\nu)T = (1-\nu)\alpha_1^2 \tag{3.17}$$

The above-mentioned eigenfunctions form a complete set of functions satisfying the boundary conditions at x = 1 and the regularity conditions at the center of the plate and may be utilized in developing expressions for other functions by the familiar expansion property of eigenfunctions. This property will be utilized extensively in the subsequent paragraphs.*

For ϵ^2 the differential equation for Φ_2 is obtained,

$$\nabla^4\Phi_2 = -\frac{1}{2x}(W_1'W_1')' \tag{3.18}$$

in which W_1 represents the function just obtained.

* For a detailed discussion of systems of eigenfunctions and their properties see ref. (24) or (25).

The associated boundary condition is

$$B_3(\Phi_2) = 0 \tag{3.19}$$

The equation is partially integrable and upon using the regularity conditions at the origin the following expression in closed form is obtained through the use of familiar recursion relations, reference (23),

$$\left(\frac{1}{x}\Phi_2'\right)' = -\frac{\alpha_1^2}{4x}\left[J_1^2(\alpha_1 x) - J_o(\alpha_1 x)\, J_2\,(\alpha_1 x)\right] \tag{3.20}$$

Further integration of this expression seems impossible except by substitution of an infinite series. When this is done and another integration performed, the result is

$$\frac{1}{x}\Phi_2' = -\frac{\alpha_1^2}{4}\sum_{r=0}^{\infty}\frac{(-1)^r(1+2r)!\,\left(\frac{x}{2}\right)^{2+2r}}{r!\,((2+r)!)^2(1+r)!} + C \tag{3.21}$$

Further integration is not necessary since the function Φ_2 will not be needed explicitly. The constant C can be determined from B_3.

For ϵ^3 the differential equation governing W_3 is obtained

$$L_1(W_3) = F_3(x) \tag{3.22}$$

where

$$\begin{aligned} F_3(x) &= \frac{1}{x}\lambda_2(\Phi_o'W_1')' + F_3^*(x) \\ &= -\frac{1}{x}\lambda_2 T(xW_1')' + F_3^*(x) \end{aligned} \tag{3.23}$$

and where

$$F_3^*(x) = \frac{1}{x}(\Phi_2'W_1')' \tag{3.24}$$

The associated boundary conditions are

$$B_1(W_3) = 0 \tag{3.25}$$

$$B_2(W_3) = 0 \tag{3.26}$$

The differential equation here is nonhomogeneous, but the associated homogeneous equation is identical with Equation (3.9). This homogeneous system has the nontrivial solution W_1. Hence the nonhomogeneous differential equation may have no solution or, if it has a solution, then this solution is not unique inasmuch as there can be added to it any arbitrary* multiple of W_1. In order that the above nonhomogeneous equation possess a solution, the right hand side must satisfy an orthogonality condition which is, in this case,

$$\int_0^1 F_3(x)\, H_1 x\, dx = 0 \tag{3.27}$$

This orthogonality condition serves to determine the coefficient λ_2,

$$\lambda_2 = - \frac{\int_0^1 F_3^* H_1 x\, dx}{T \int_0^1 x(xH_1')' H_1 dx} = \frac{\int_0^1 \Phi_2' H_1' H_1'\, dx}{T \int_0^1 x H_1' H_1' dx} \tag{3.28}$$

The particular solution can now be constructed and the procedure continued to determine further perturbation coefficients and functions. However, as will become apparent in the sequel, there is no need to pursue the solution of the equilibrium problem beyond this point.

Turning now to the subject of this thesis, which is the vibration of the plate in the presence of the static or initial configurations and the associated equilibrium system of stresses discussed in the previous paragraphs, it will be noted that the method of solution in the

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* While the choice is theoretically arbitrary, the specific value is selected on the basis of convenience of computation.

dynamic case has much in common with that used above and hence only the essential points are presented.

The equations governing the motion of the plate are (2.12a), (2.12b) and, as a matter of convenience, they are presented again. By substitution of (3.6), Equation (2.12a) is further simplified to

$$(\nabla^2 - \frac{n^2}{x^2})^2 w^n + \lambda_o T(\nabla^2 - \frac{n^2}{x^2}) w^n - \frac{1}{x}(\Phi' w^n)' + \frac{n^2}{x^2}\Phi'' w^n - \frac{1}{x}(\phi^{n\prime} W')' + \frac{n^2}{x^2}\phi^n W'' = \mu^n w^n \qquad (3.29)$$

$$(\nabla^2 - \frac{n^2}{x^2})^2 \phi^n = -[\frac{1}{x}(W' w^{n\prime})' - \frac{n^2}{x^2} W'' w^n] \qquad (2.12b)$$

As pointed out in Chapter II, the boundary conditions are

$$B_1(w^n) = 0 \qquad (2.21)$$

$$B_2(w^n) = 0 \qquad (2.22)$$

and

$$B_4(\phi^n) = 0 \qquad (2.24)$$

if attention is restricted to the cases

$$n = 0$$

and

$$n = 1 .$$

In general, the functions w, ϕ and μ are expanded in perturbation series utilizing the same parameter ϵ as in the static case.

$$\begin{aligned} w^n &= w_o^n + \epsilon^2 w_1^n + \cdots \\ \phi^n &= \epsilon\phi_1^n + \epsilon^3\phi_3^n + \cdots \\ \mu^n &= \mu_o + \epsilon^2\mu_2 + \cdots \end{aligned} \qquad (3.30)$$

Upon substitution of these perturbation expansions* in the differential

* The fact that w^n and μ^n are even expansions in ϵ and that ϕ^n is an odd expansion may be easily verified upon substitution in the relevant equations. For the sake of brevity these steps are omitted here.

equations and associated boundary conditions, a new sequence of differential equations is obtained whose solution will follow very similar methods to those presented for the static case. It will also become apparent that the partitioning of the stress function Φ^* makes this future work similar to the previous paragraphs.

For ϵ^0 the differential equation is

$$L_2(w_o^n) \equiv (\nabla^2 - \frac{n^2}{x^2})^2 w_o^n + \lambda_o T(\nabla^2 - \frac{n^2}{x^2}) w_o^n - \mu_o^n w_o^n = 0 \quad (3.31)$$

and the associated boundary conditions are

$$B_1(w_o^n) = 0 \quad (3.32)$$

$$B_2(w_o^n) = 0 \quad (3.33)$$

The solution of this equation is

$$w_o^n = \sum_{m=0}^{\infty} a_o^{(m)} w_o^{nm} \quad (3.34)$$

where

$$w_o^{nm} = J_n(\beta_2^{nm} x) - \frac{J_n(\beta_2^{nm})}{I_n(\beta_1^{nm})} \cdot I_n(\beta_1^{nm} x) \quad (3.35)$$

which immediately satisfies B_1 and where I_n is the modified Bessel function. This eigenvalue problem is governed by the characteristic equations obtained in the usual fashion from the boundary conditions, B_2.

$$\begin{aligned} &I_n(\beta_1^{nm})[(2n + 1 + \nu)\beta_2^{nm} J_{n+1}(\beta_2^{nm}) - (\beta_2^{nm})^2 J_{n+2}(\beta_2^{nm})] \\ &+ J_n(\beta_2^{nm})[(2n + 1 + \nu)\beta_1^{nm} I_{n+1}(\beta_1^{nm}) + (\beta_1^{nm})^2 I_{n+2}(\beta_1^{nm})] = 0 \end{aligned} \quad (3.36)$$

where β_1^{nm} and β_2^{nm} are related to $\lambda_o T$ (i.e., α_1) and μ^{nm} by

$$(\beta_2^{nm})^2 - (\beta_1^{nm})^2 = \alpha_1^2$$
$$\mu^{nm} = (\beta_1^{nm})^2(\beta_2^{nm})^2 \qquad (3.37)$$

The functions w^{nm} obey the usual orthogonality conditions which, in this case, are

$$\int_0^1 w^{nm} w^{rs} \, x \, dx = 0 \qquad (3.38)$$

$$\mu_{nm} \neq \mu_{rs}$$

and

$$\int_0^1 [(\nabla^2 - \frac{n^2}{x^2})^2 w_o^{nm} + \lambda_o T(\nabla^2 - \frac{n^2}{x^2}) w_o^{nm}] w_o^{rs} \, xdx = 0 \qquad (3.39)$$

It should be noted that, for a specific value of n, representing the number of nodal diameters, there exists an infinity of roots of the above system which are designated by the index m. The index m represents the number of nodal circles appearing in the vibration pattern of the plate. A similar system for the case of a clamped edge plate was solved by Federhofer in 1935.(8)

For ϵ^1 a differential equation governing the function ϕ_1^n is obtained which is

$$(\nabla^2 - \frac{n^2}{x^2})^2 \phi_1^n = - g_1(x) \qquad (3.40)$$

where

$$g_1(x) = \frac{1}{x} (W_1' w_o^{n\prime})' - \frac{n^2}{x^2} W_1'' w_o^n \qquad (3.41)$$

and the associated boundary conditions are

$$B_4(\phi_1^n) = 0 \qquad (3.42)$$

This is a problem of integration; however, the form of the function $g_1(x)$ is such that a numerical integration is evident except for n = 0. For ϵ^2 a differential equation governing the deflection function w_2^{nm} is obtained which is

$$L_2(w_2^{nm}) = f_2(x) \tag{3.43}$$

where

$$f_2(x) = \mu_2^{nm} w_o^{nm} + f_2^*(x) \tag{3.44}$$

and where

$$\begin{aligned} f_2^*(x) = & - \lambda_2 T(\nabla^2 - \frac{n^2}{x^2}) w_o^{nm} + \frac{1}{x} (\Phi_2' w_o^{nm'})' \\ & - \frac{n^2}{x^2} \Phi_2'' w_o^{nm} + \frac{1}{x} (\phi_1^{nm'} W_1')' - \frac{n^2}{x^2} \phi_1^{nm''} W_1 \end{aligned} \tag{3.45}$$

and where associated boundary conditions are

$$B_1(w_2^{nm}) = 0 \tag{3.46}$$

$$B_2(w_2^{nm}) = 0 \tag{3.47}$$

By the now familiar process, $f_2(x)$ must satisfy an orthogonality condition in order to obtain a solution for w_2 and out of this relation emerges μ_2^{nm}

$$\mu_2^{nm} = - \frac{\int_0^1 f_2^*(x) w_o^{nm} \, xdx}{\int_0^1 w_o^{nm} w_o^{nm} \, xdx} \tag{3.48}$$

For the special case, n = 0, numerical results are readily obtained. In particular the value of the rate of change of frequency with respect to the load parameter is desired in the neighborhood of the condition of linear buckling which is

$$\lim_{\epsilon \to 0} \frac{d\mu}{d\lambda} = \lim_{\epsilon \to 0} (\frac{d\mu}{d\epsilon})/(\frac{d\lambda}{d\epsilon}) = \mu_2/\lambda_2 \tag{3.49}$$

where μ_2 and λ_2 are defined by Equation (3.30).

In pursuing the analytical steps outlined previously for the symmetric mode of vibration it becomes apparent that ϕ^o becomes a multiple of Φ_2 and hence from Equation (3.48) and by the use of Equation (3.21)

$$\mu_2 = 2\lambda_2 T \frac{\int_o^1 xH_1'H_1' dx}{\int_o^1 xH_1 H_1 dx} \tag{3.50}$$

which is readily evaluated. Thus finally

$$\mu_2/\lambda_2 = 49.29 \tag{3.51}$$

In closing, then, it is evident that when the critical or linear buckling condition is reached the frequency of vibration is zero and as the plate proceeds into the post buckled condition the square of the frequency increases initially in a linear fashion with the buckling parameter.

Further calculations with the perturbation method become exceedingly cumbersome and are abandoned in favor of the power series method of the next chapter. However, the present result is exact as ϵ approaches zero since higher order expansions vanish for this condition.

IV. THE POWER SERIES METHOD

Another possible means of solving the system of differential equations presented here is to develop the solution in terms of a power series. Again we borrow the results of Friedrichs and Stoker for the solution of the static problem. The phrasing is slightly different and some of the numerical computations used are only minor variations of theirs.

The system of differential equations for the static case is

$$\nabla^4 W - \frac{1}{x}(\Phi' W')' = 0 \qquad (2.11a)$$

$$\nabla^4 \Phi = -\frac{1}{2x}(W'W')' \qquad (2.11b)$$

and the boundary conditions associated with them are

$$B_1(W_1) = 0 \qquad (2.16)$$

$$B_2(W_1) = 0 \qquad (2.17)$$

$$B_3(\Phi) = -\lambda U_E \qquad (2.19)$$

Let the functions W and Φ be expressed in power series of the coordinate x. These series are even functions in x.

$$W = \sum_{m=0}^{\infty} a_m x^{2m} \qquad (4.1)$$

$$\Phi = \sum_{m=0}^{\infty} b_m x^{2m} \qquad (4.2)$$

It is easy to show by substitution in Equation (2.11a) and (2.11b) that the coefficients a_0, a_1, b_0, and b_1 are, at this point, arbitrary and that all of the other coefficients of the two series will be given by the following recursion relationships.

$$a_m = \frac{1}{2m^2(m-1)} \sum_{i=1}^{m-1} i(m-i)\, a_i b_{m-i} \qquad (4.3)$$

$$m \geq 2$$

$$b_m = -\frac{1}{4m^2(m-1)} \sum_{i=1}^{m-1} i(m-i)\, a_i a_{m-i} \qquad (4.4)$$

In terms of the coefficients in the power series, the previously stated boundary conditions become

$$\sum_{0}^{\infty} a_m = 0 \qquad (4.5)$$

$$\sum_{1}^{\infty} m(2m - 1 + \nu)\, a_m = 0 \qquad (4.6)$$

$$\sum_{1}^{\infty} m(2m - 1 - \nu)\, b_m = -\frac{\lambda U_E}{2} \qquad (4.7)$$

It is evident that Equation (4.5) serves only to determine the coefficient a_0 after the others have been computed. The two equations (4.6) and (4.7) must be solved simultaneously for the values of a_1 and b_1. The value of the coefficient b_0 remains arbitrary and this is as might be expected since the stress function Φ can in general be varied with respect to as much as an arbitrary linear function of the cartesian coordinates without affecting the stresses.

Hence, in principle, once the value of λ is specified, it is possible to obtain all of the necessary coefficients in order to be able to describe the complete solution for the static case.

For the dynamic case the differential equations to be solved are

$$\left(\nabla^2 - \frac{n^2}{x^2}\right)^2 w^n - \frac{1}{x}\left(\Phi' w^{n\prime}\right)' + \frac{n^2}{x^2}\Phi^n w^n$$

$$- \frac{1}{x}\left(\phi^{n\prime} W'\right)' + \frac{n^2}{x^2}\phi^n W'' - \mu^n w^n = 0 \qquad (2.12a)$$

$$\left(\nabla^2 - \frac{n^2}{x^2}\right)^2 \phi^n = -\left[\frac{1}{x}\left(W' w^{n\prime}\right)' - \frac{n^2}{x^2} W'' w^n\right] \qquad (2.12b)$$

The associated boundary conditions* are

$$B_1(w^n) = 0 \qquad (2.20)$$

$$B_2(w^n) = 0 \qquad (2.21)$$

$$B_4(\phi^n) = 0 \qquad (2.24)$$

The solution of the above differential equations may be expressed as power series in x. Upon calculating with the power series by the usual methods, it becomes clear that the solutions which are not singular at the origin are

$$w^{(n)} = x^n \sum_{m=0}^{\infty} c_m^{(n)} x^{2m} \qquad (4.8)$$

and

$$\phi^{(n)} = x^n \sum_{m=0}^{\infty} d_m^{(n)} x^{2m} \qquad (4.9)$$

The recursion relationships for all values of n that evolve from this system are, after dropping the superscripts for c and d,

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* For $n \geq 2$ the number of boundary conditions increases to four.

$$c_m = \frac{1}{16m(m-1)(m+n)(m+n-1)} \{\mu^{(n)} c_{m-2} + (2m+n-2) \sum_{i=1}^{m} 2i(n+2m-2i)(a_i d_{m-i} + b_i c_{m-i}) - n^2 \sum_{i=1}^{m} 2i(2i-1)(a_i d_{m-i} + b_i c_{m-i})\} \quad m \geq 2 \tag{4.10}$$

$$d_m = -\frac{1}{16m(m-1)(m+n)(m+n-1)} \{(2m+n-2) \sum_{i=1}^{m} 2i(n+2m-2i)\, a_i c_{m-i} - n^2 \sum 2i(2i-1)\, a_i c_{m-i}\} \tag{4.11}$$

while the coefficients c_0, c_1, d_0, and d_1 remain, at this point, undetermined.

In terms of these power series, the boundary conditions for the dynamic problem assume the following form:

$$\sum_{m=0}^{\infty} c_m = 0 \tag{4.12}$$

$$\sum_{m=0}^{\infty} [(2m+n)(2m+n-1+\nu) - \nu n^2] c_m = 0 \tag{4.13}$$

$$\sum_{m=1}^{\infty} [(2m+n)(2m+n-1-\nu) + \nu n^2] d_m = 0 \tag{4.14}$$

It is readily verified by examination of the recursion relation in the cases when $n = 0$ and $n = 1$ that d_0 will not appear in the problem and this is again plausible in view of the remark made in connection with b_0.

The boundary conditions thus present a system of homogeneous linear algebraic equations in the three unknown coefficients c_0, c_1, and d_1, as well as of the eigenvalue μ. The explicit form of the characteristic determinental equation associated with this system is, by virtue of the form of the boundary conditions and the recursion relations, probably unattainable. In any case, no particular benefit would be derived from it since a numerical solution will have to be attempted and the appropriate algorithm can be stated in terms of these equations.

The solution of this system of equations is to be performed on a digital computer in the following manner. An arbitrary value, such as unity for example, is assigned to the coefficients c_0, c_1 and d_1. The remaining coefficients in the power series expansion are determined by the recursion relationship. A generic term in the series depends upon c_0, c_1 and d_1 and polynomials in terms of μ. Briefly then, each coefficient in the power series is the sum of three polynomials in μ, the coefficients of these polynomials being the constants c_0, c_1 and d_1 respectively. The various coefficients can be generated and their polynomial representation is substituted into the boundary equations (4.12), (4.13) and (4.14). Theoretically at least, each equation is composed of infinite series a_{ij} in μ having as their coefficients c_0, c_1 and d_1. In matrix notation

$$AB = 0 \tag{4.15}$$

where $$A = \{a_{ij}\} \qquad i,j = 1,2,3$$

and $$B = \begin{Bmatrix} c_0 \\ c_1 \\ d_1 \end{Bmatrix}$$

This system of equations will admit a nontrivial solution when the determinant of the coefficient matrix A is zero. This determinant is equivalent to a power series in μ whose roots represent the required eigenvalues.

Clearly the procedure here must be modified to truncate this process. The coefficients of the power series of w and ϕ are computed until a particular term becomes less than a designated value. However, convergence at this point alone is not necessarily the final test. Satisfying the boundary conditions yields an approximation to the a_{ij} in that they are now polynomials. It should be noted that the first neglected term, for instance, provides, in general, a contribution to all the previously computed coefficients in a_{ij}. However, the system was sufficiently convergent to insure convergence in this sense. The a_{ij} are polynomials in μ of order m/2 to the nearest integer when m terms are taken in the power series. It is quite clear then that the determinant when expanded is capable of yielding a polynomial whose order is three times that of the order of the a_{ij}. This determinant may be expanded algebraically, by the computer, producing a polynomial characteristic equation which must be solved for the roots, by standard methods.

Generally, about sixteen terms in the power series produced sufficiently convergent terms in the a_{ij} and it sufficed to take five or six terms in the characteristic equation to obtain the roots. For very large penetration of the post buckling domain the size of the numbers involved and the number of terms required for the necessary accuracy required a modification of this technique due to the limitations imposed by the available computer.

The modified procedure consisted of an iteration procedure one step earlier in the above process. The a_{ij} were evaluated using trial values for μ and then checking to see if the determinant is zero. Computing time of this iteration procedure was approximately eight times longer than by the first method.*

* Weinitschke(26) who faced a similar computation in connection with a problem in shells chose to use several power series expansion each being valid in part of the region and then matching them together. By this means fewer terms in each series were required for convergence.

V. RESULTS AND DISCUSSION

The penetration of the post buckling domain is measured by λ, a ratio of the edge displacement to that required for the initial instability. Friedrichs and Stoker[12,13] use a parameter λ_S based upon a stress ratio. For convenience the results here are expressed in terms of λ; Figures 4 and 5 show the relation between the two parameters. The main result offered here is the relation between μ (the squared frequency parameter) and λ. Figure 1 shows the relation for a symmetric mode ($n = 0$) and the first mode having a nodal diameter ($n = 1$). The details of this relation in the vicinity of $\lambda = 1$, the early stages of penetration of the post buckling domain, are shown in Figure 2. In Figure 3 the shapes of the modes of vibration are depicted. The data used in plotting Figures 1, 2 and 3 and further information are given in Table I.*

The fact that μ increases linearly with λ in the vicinity of initial instability for the symmetric mode as shown by the power series analysis is borne out by the perturbation analysis. From Figure 2 the numeric value of the slope is fifty which compares rather well with the results of Equation (3.49), i.e. 49.29. It is interesting to observe that after some increase in λ the frequency of the nonsymmetric mode is lower than that of the symmetric mode. Examination of the modes for the symmetric case shows that at a value of λ between 13 and 19 a nodal circle appears for the lowest frequency. Near this value of λ

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* All calculations are based upon the value of Poisson's ratio $\nu = .318$.

the frequency of the axially symmetric mode increases less rapidly and eventually becomes again less than that for n = 1. This behavior of the frequencies is reasonable inasmuch as the nonsymmetric mode is essentially inextensional while the symmetric mode is essentially extensional and consequently the frequencies of the symmetric mode can become greater than those for n = 1. Upon the appearance of the nodal circle in the n = 0 mode, this mode also becomes essentially inextensional and the frequency increases at a lesser rate while λ is increased. Under these circumstances the frequency can, and does, become less than that for n = 1. Apparently the frequency reaches an asymptotic value as λ is increased. It should be noted, however, that as λ becomes large the accuracy* of the results becomes less certain but in any case they also become less meaningful because by the time λ reaches values approaching ∞ the equilibrium configuration is in, what Friedrichs and Stoker call, the asymptotic range. In this range the plate has been stretched as a membrane except for a narrow boundary layer at the edge where large bending stresses occur. That a plate could reach such a state is subject to question on practical grounds, in particular, if one considers the effect of imperfections upon the behavior of the ideal plate considered here. Also the onset of plastic yielding or secondary buckling is likely to invalidate these somewhat academic results.

A very significant result is that, in the range of computations, the frequency of vibration does not return to zero for $\lambda > 1$. This implies

* Accuracy was determined by the amount the deflection at the edge differed from zero with respect to the magnitude of the largest deflection. For good results this amounted to one part in 10^8 while for poor results one part in 10^4.

that, at least within the limitations imposed here on the nature of the assumed vibration modes (i.e. small amplitude vibration), the positive definiteness of the potential energy and hence the stability of the buckled configuration is unchanged if only expansions up to the second power in the terms representing the additional neighboring deflections are included. It appears likely that this is true also in relation to higher modes. Consequently the experimentally observed phenomenon of secondary buckling[12] cannot be explained in terms of a simple branch point. Research delving into this matter is continuing and attention is directed at the possibility of a discontinuous snap-through to a position of lower potential energy. This becomes possible when the quadratic form introduced in reference (19) loses its positive definite character. This is indeed the case as pointed out in the paper, reference (19). The problem of secondary buckling therefore shows great similarity with that of buckling of certain types of shells.

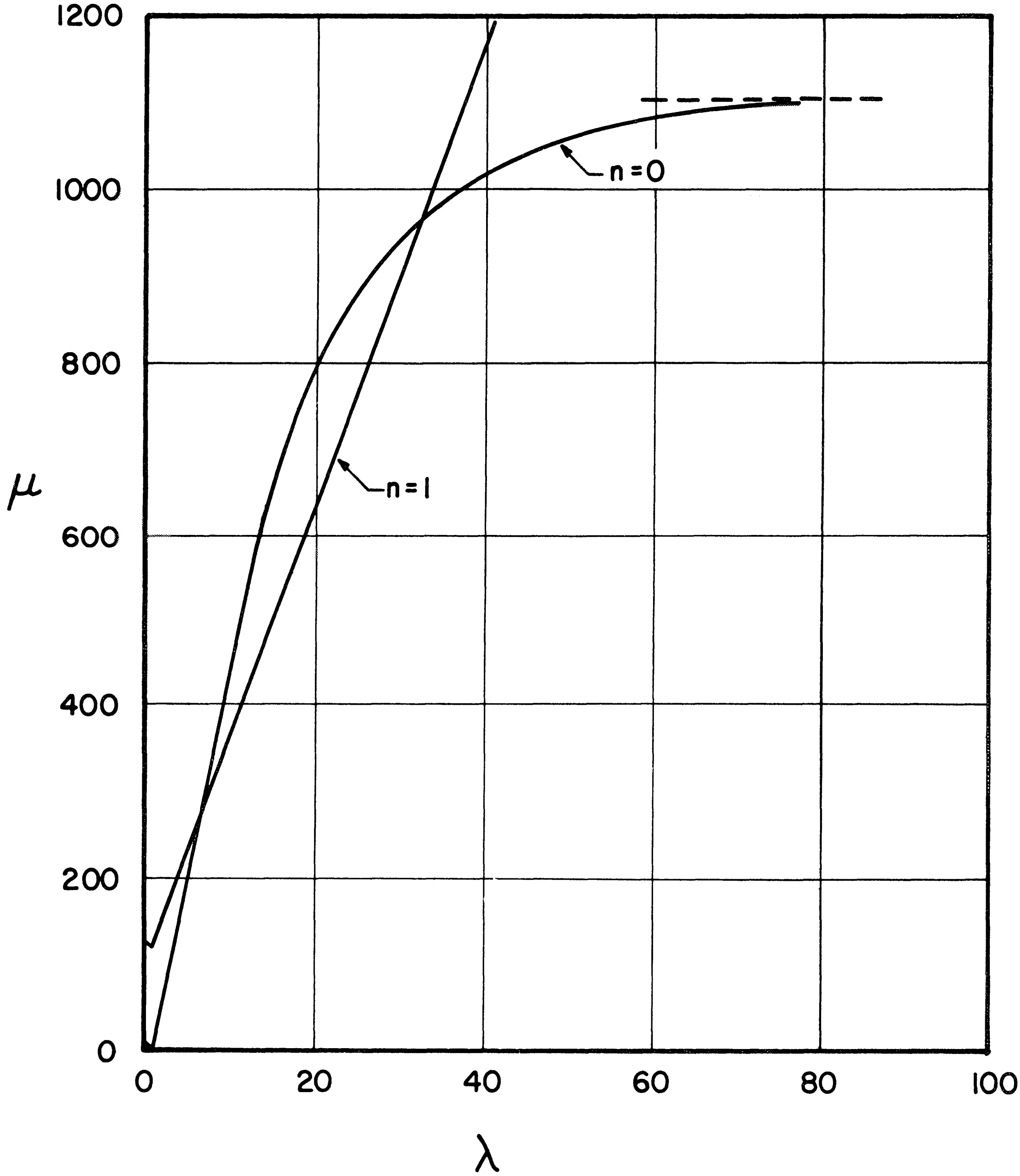

Figure 1. The Relation Between the Frequency Parameter μ and the Load Parameter λ.

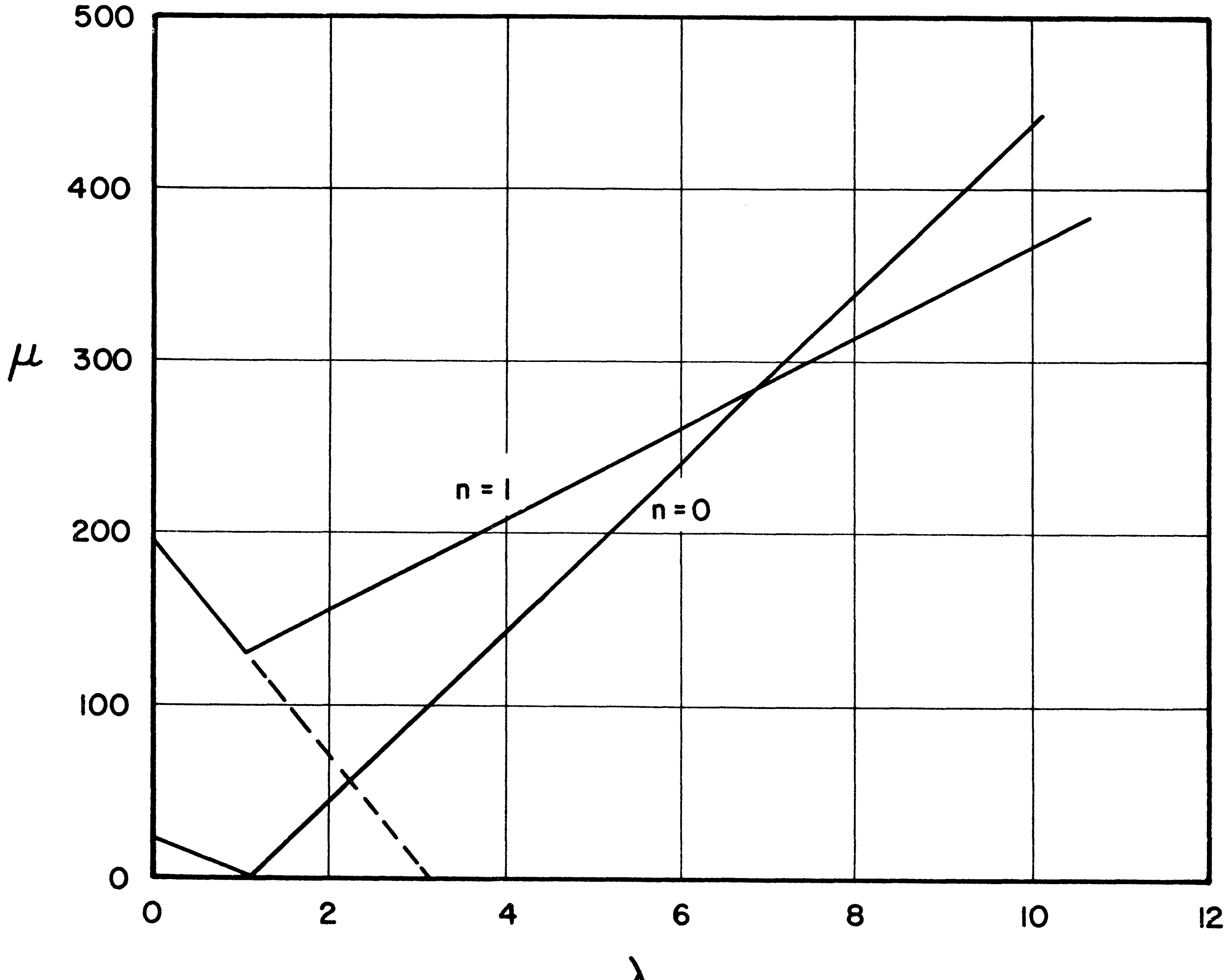

Figure 2. The Relation Between the Frequency Parameter μ and the Load Parameter λ for Small λ.

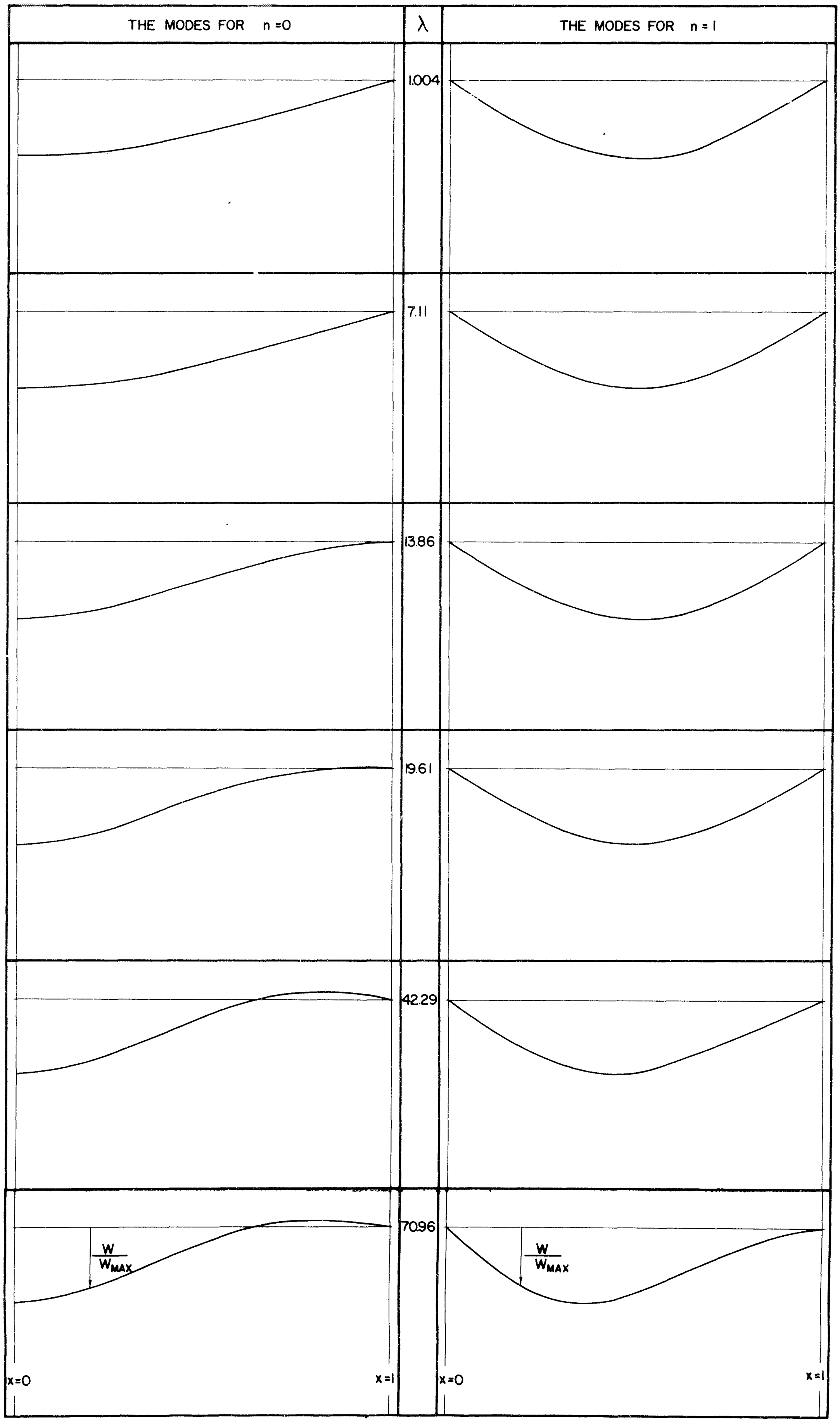

Figure 3. The Modes of Vibration Plotted for Half the Plate.

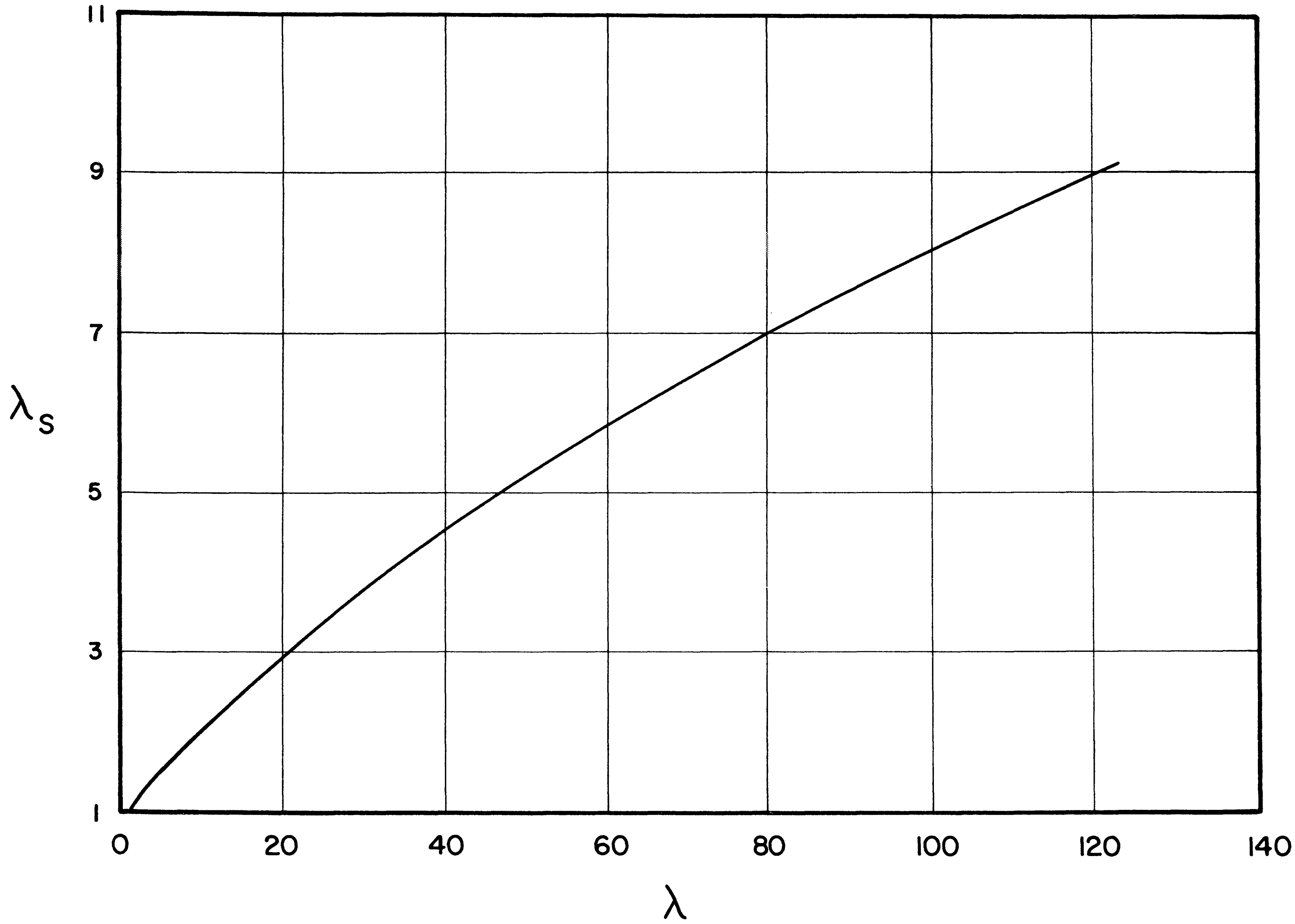

Figure 4. The Relation Between λ_S and λ.

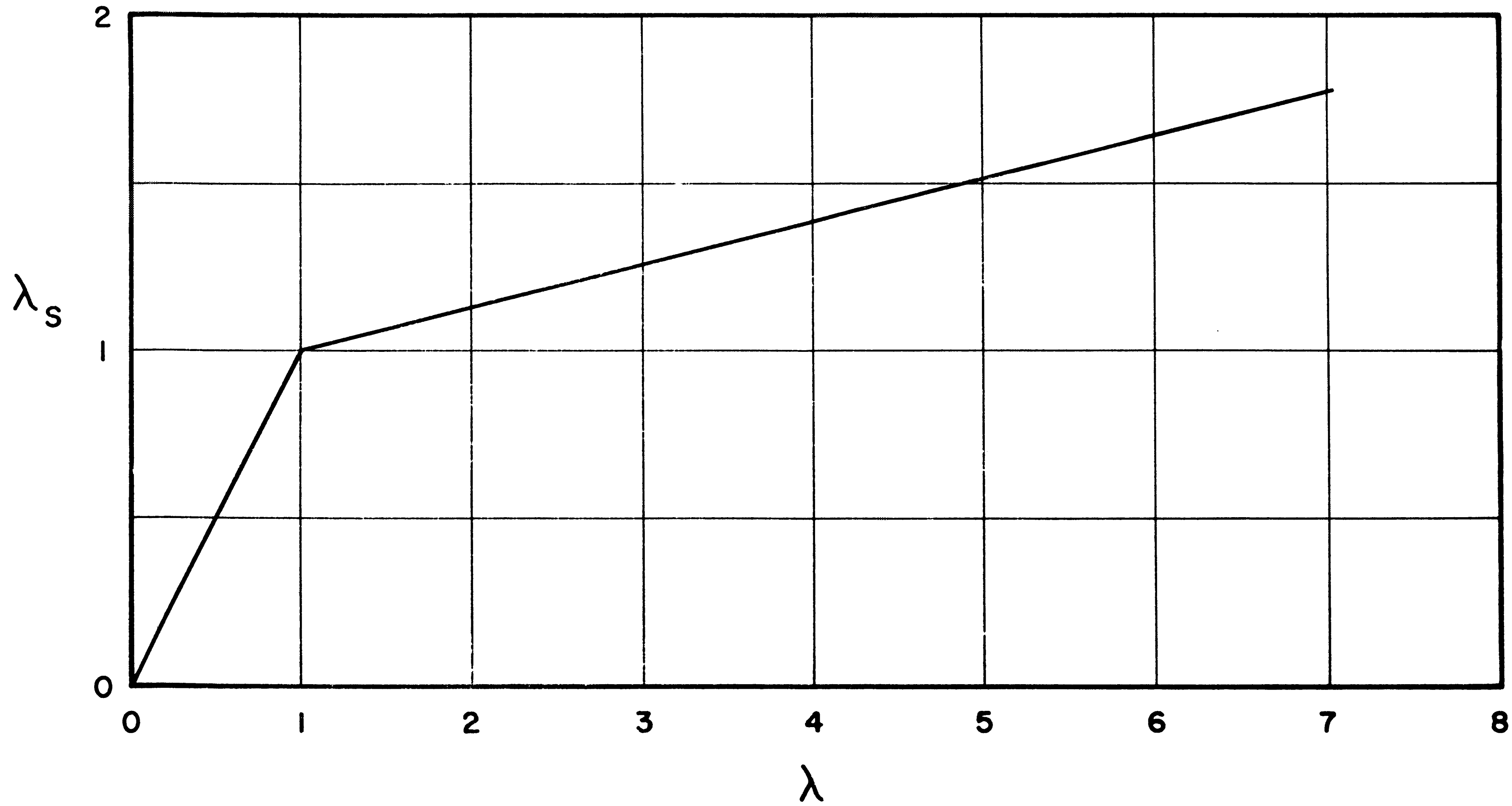

Figure 5. The Relation Between λ_S and λ for Small λ.

APPENDIX A

THE DYNAMIC BOUNDARY CONDITIONS FOR THE CASE n = 1

The boundary conditions for the case n = 1 require some special consideration. The condition of "simple" support requires that the lateral deflection as well as the radial bending moment at the outside edge be zero. To complete the boundary conditions, some requirements must be imposed on the membrane displacements or the membrane stresses (indirectly, through the stress function) or a combination of both. In general, two more boundary conditions are required, but due to the obvious symmetry in the case n = 0, only one is required beyond the implications of symmetry.

Consider now the details of the case n = 1. For simplicity, the superscripts are omitted inasmuch as this discussion is for a specific case.

The stress function for n = 1 is of the form

$$\phi = \psi(x) \cos \theta$$

while the lateral displacement is

$$\overline{w} = w(x) \cos \theta$$

From the strain displacement equation it follows that

$$\frac{\partial v}{\partial \theta} = xe_{\theta\theta} - u$$

and

$$\frac{\partial u}{\partial v} = e_{xx} - \frac{dW}{dx} \frac{\partial \overline{w}}{\partial x}$$

The stress strain equations, utilizing the expression of the stresses in terms of the stress function imply that

$$e_{xx} = [\frac{1}{x}\psi' - \frac{1}{x^2}\psi - \nu\psi''] \cos\Theta$$

and

$$e_{\Theta\Theta} = [\psi'' - \nu(\frac{1}{x}\psi' - \frac{1}{x^2}\psi] \cos\Theta$$

$$= [\psi'' - \nu(\frac{\psi}{x})'] \cos\Theta$$

where primes mean differentiation with respect to x.

Hence

$$\frac{\partial u}{\partial x} = [(\frac{1}{x}\psi)' - \nu\psi'' - W'w'] \cos\Theta \qquad (A.1)$$

and upon integration

$$u = [\frac{1}{x}\psi - \nu\psi' - \int W'w'dx] \cos\Theta + F_1(\Theta) \qquad (A.2)$$

Further

$$\frac{\partial v}{\partial \Theta} = x[\psi'' - \nu(\frac{\psi}{x})'] \cos\Theta - [\frac{1}{x}\psi - \nu\psi' - \int W'w'dx] \cos\Theta - F_1(\Theta) \qquad (A.3)$$

and thus

$$v = x[\psi'' - \nu(\frac{\psi}{x})'] \sin\Theta - [\frac{1}{x}\psi - \nu\psi - \int W'w'dx] \sin\Theta - F_2(\Theta) + G(x) \qquad (A.4)$$

where

$$\frac{dF_2}{d\Theta} = F_1 \qquad (A.5)$$

However,

$$2e_{x\theta} = \frac{1}{x}\frac{\partial u}{\partial \theta} + x\frac{\partial}{\partial x}\left(\frac{v}{x}\right) + \frac{1}{x}\frac{dW}{dx}\frac{\partial w}{\partial \theta} = -(1+\nu)\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \phi}{\partial x}\right)$$

Upon substitution of the appropriate quantities in the above expression and simplifying the following is obtained:

$$\left(\frac{3\psi}{x^2} + x\psi'' - \frac{3\psi'}{x} + W'w' - \frac{1}{x}W'w\right)\sin\theta + \frac{1}{x}\left(\frac{dF_1}{d\theta} + F_2\right) + x\left(\frac{G}{x}\right)' = 0 \qquad (A.6)$$

Equation (A.6) must hold for all x and θ and it can be shown that the required necessary and sufficient conditions are

$$x\left(\frac{3\psi}{x^2} + x\psi'' - \frac{3\psi'}{x} + W'w' - \frac{1}{x}W'w\right) = -2C \qquad (A.7)$$

$$\frac{dF_1}{d\theta} + F_2 = 3D_1 + 2C\sin\theta \qquad (A.8)$$

$$x^2\left(\frac{G}{x}\right)^1 = -3D_1 \qquad (A.9)$$

Equation (A.7) is satisfied if ψ satisfies the compatibility equation associated with the von Kármán system of equations. Equation (A.8) will be satisfied if F_2 satisfies

$$\frac{d^2F_2}{d\theta^2} + F_2 = 3D_1 + 2C\sin\theta \qquad (A.10)$$

The solution of Equation (A.10) is

$$F_2 = -C_1\cos\theta + C_2\sin\theta + 3D_1 - C\theta\cos\theta \qquad (A.11)$$

from which it follows, by Equation (A.5), that

$$F_1 = C_1 \sin \theta + C_2 \cos \theta + C(\theta \sin \theta - \cos \theta) \tag{A.12}$$

The solution of Equation (A.9) is

$$G = \frac{D_1}{x^2} + Dx \tag{A.13}$$

Consequently the displacement components become

$$u(1) = (f - \int W'w'dx + C_2) \cos \theta - C_1 \sin \theta + C(\theta \sin \theta - \cos \theta)$$

(A.14)

$$v(1) = (g + \int W'w'dx - C_2) \sin \theta - C_1 \cos \theta - 3D_1 + C\theta \cos \theta + \frac{D_1}{x^2} + Dx$$

where

$$f \equiv (\frac{1}{x}\psi - \nu\psi')_{x=1} = \psi(1) - \nu\psi'(1)$$

$$g \equiv [x\psi'' - (1-\nu)\frac{\psi}{x}]_{x=1} = \psi''(1) - (1-\nu)\psi(1)$$

and $f + g = \frac{1}{x}[\psi'' - \nu(\frac{\psi}{x})']_{x=1}$

Consistent with Equation (2.19) (and hence without any significant loss of generality) let

$$C_1 = 0$$

Single valued displacements and regularity conditions at the origin, require that $C = 0$ and $D_1 = 0$

For n = 1, from Equation (2.22)

$$u(1) = A_1 \cos \theta$$

$$v(1) = B_1 \sin \theta + B^* \tag{A.15}$$

Consequently $\qquad D = B^*$

and

$$f - \int W'w'dx + C_2 = A_1 \qquad (A.16)$$

$$g + \int W'w'dx - C_2 = B_1 \qquad (A.17)$$

In addition to the conditions imposed by the simple support at the edge it is sufficient to require that

$$f + g = 0 \qquad (A.18)$$

which by the use of (A.16) and (A.17) implies that

$$A_1 = -B_2 \qquad (A.19)$$

and it must also be noted that

$$e_{\theta\theta}(1) = f + g$$

and hence Equation (A.18) implies

$$e_{\theta\theta}(1) = 0 \qquad (A.20)$$

which is the boundary condition used in the main text.

For the power series approach

$$\psi \equiv \phi^{(1)} = \alpha_1 x + \alpha_2 \sum_{m=1}^{\infty} d_m x^{2m+3} \qquad (A.21)$$

where, for convenience of this discussion, α_1 and α_2 are the arbitrary constants and d_o is chosen as unity. It is easily shown that (f + g) does not depend upon α_1. Further, none of the stresses depend upon α_1. ~~Indeed~~, none of the stresses depend upon α_1, and further it can be shown that α_1 and C_2 can be combined as a single constant C_2^*. To demonstrate this, substitute the expression for ψ into f and g and the associated equations. Thus from Equation (A.16)

$$C_2 + (1-\nu)\alpha_2 + \alpha_2 \sum [1 - \nu(2m+3)]\, d_m - \int W'w'dx = A_1 \qquad (A.22)$$

is obtained and by the use of Equation (A.17) and

$$- C_2 - (1-\nu)\alpha_1 + \alpha_2 \sum [(2m+3)(2m+2) - (1-\nu)]\, d_m + \int W'w'dx = B_1 \qquad \text{(A.23)}$$

Let

$$C_2^* = C_2 + (1-\nu)\alpha_1$$

In terms of the power series solution

$$f + g = \alpha_2 \sum (2m+3-\nu)(2m+2)d_m$$

Consequently (f + g) is independent of α_1. Further, it can be seen that Equation (A.23) then serves to determine C_2^*.

APPENDIX B

THE VIBRATIONS OF THE SIMPLY SUPPORTED CIRCULAR PLATE WITH EDGE COMPRESSION (Linear Theory)

During the course of the present investigation, the linear problem of plate vibration in the presence of radial edge compression was solved. The same problem for a clamped edge plate was treated by Federhofer.[8] The boundary condition here is that of simple support. The relevant equations arose in connection with the perturbation solution and are repeated here for convenience.

The differential equation, after separation of variables, is similar to Equation (3.31),

$$(\nabla^2 - \frac{n^2}{x^2})^2 w_o^n + \lambda_o T(\nabla^2 - \frac{n^2}{x^2}) w_o^n - \mu_o^n w_o^n = 0 \qquad (B.1)$$

and the associated boundary conditions are

$$B_1(w_o^n) = 0 \qquad (3.32)$$

$$B_2(w_o^n) = 0 \qquad (3.33)$$

The solution to this equation is

$$w_o^n = \sum_{m=0}^{\infty} a_o^{(m)} w_o^{nm} \qquad (3.34)$$

where

$$w_o^{nm} = J_n(\beta_2^{nm} x) - \frac{J_n(\beta_2^{nm})}{I_n(\beta_1^{nm})} \cdot I_n(\beta_1^{nm} x) \qquad (3.35)$$

which immediately satisfies B_1. This eigenvalue problem is governed by the characteristic equations obtained in the usual fashion from the boundary conditions,

$$I_n(\beta_1^{nm})[(2n+1+\nu)\beta_2^{nm}J_{n+1}(\beta_2^{nm}) - (\beta_2^{nm})^2 J_{n+2}(\beta_2^{nm})] + J_n(\beta_2^{nm})[(2n+1+\nu)\beta_1^{nm}I_{n+1}(\beta_1^{nm}) + (\beta_1^{nm})^2 I_{n+2}(\beta_1^{nm})] = 0 \qquad (3.36)$$

where β_1^{nm} and β_2^{nm} are related to λ T(i.e., α) and μ^{nm} by

$$(\beta_2^{nm})^2 - (\beta_1^{nm})^2 = \alpha^2$$
$$\mu^{nm} = (\beta^{nm})^2 \; (\beta^{nm})^2 \qquad (B.2)$$

Since only the linear problem is considered here subscripts on λ and α are omitted (which in Chapter III had meaning in connection with the perturbation solution). For the linear buckling problem

$$\mu^{nm} = 0 \qquad (B.3)$$

and it may be shown this is equivalent to

$$\beta_1^{nm} = 0$$
$$\beta_2^{nm} = \alpha \qquad (B.4)$$

and Equation (3.36) becomes

$$(2n+1+\nu)J_{n+1}(\beta^{nm}) - \beta^{nm}J_{n+2}(\beta^{nm}) = 0 \qquad (B.5)$$

where n designates the number of nodal diameters and m the number of nodal circles.

The roots of the characteristic equations were obtained for several values of n and m by a high speed digital computer using a step-by-step searching method followed by an interval halving method until the roots were reproduced within specified limits In Table I and III roots of Equation (3.36) are tabulated for several values of n, m and α. For the axially symmetric mode the case of free vibration and of buckling was solved for a number of roots. These results are presented in Table IV.

TABLE I

FREQUENCY PARAMETERS μ FOR n = 0, n = 1 FOR
VARIOUS VALUES OF λ, λ_S

λ	λ_s	μ	
		n=0	n=1
1.00001	1.00000	0.000	131.410
1.00009	1.00001	0.004	131.412
1.00017	1.00003	0.009	131.414
1.00034	1.00005	0.017	131.419
1.00067	1.00010	0.033	131.427
1.00101	1.00015	0.050	131.435
1.00402	1.00059	0.198	131.511
1.00887	1.00129	0.437	131.632
1.01574	1.00229	0.775	131.804
1.02462	1.00359	1.213	132.026
1.03552	1.00517	1.750	132.299
1.04846	1.00705	2.387	132.623
1.06343	1.00923	3.124	132.998
1.08029	1.01168	3.954	133.420
1.09886	1.01437	4.869	133.886
1.41138	1.05927	20.258	141.735
1.98215	1.13931	48.371	156.161
2.91311	1.26498	94.298	179.920
4.15896	1.42499	155.952	212.109
4.42706	1.45835	169.243	219.088
6.31125	1.68353	262.640	268.598
7.11236	1.77494	302.177	289.862
9.57456	2.04258	421.501	355.833
11.25514	2.21547	499.317	401.278
13.86542	2.47114	610.406	472.235
19.60922	2.99070	794.910	629.174
29.17450	3.76396	948.097	886.156
33.88582	4.11459	988.571	1008.043
42.29802	4.70397	1035.067	1213.347
50.15607	5.22066	1061.999	1388.049
61.01744	5.89279	1091.762	1603.025
70.96683	6.47418	1122.798	1774.656

TABLE II

ROOTS OF THE CHARACTERISTIC EQUATION FOR VIBRATION OF THE CIRCULAR PLATE WITH EDGE COMPRESSION-LINEAR PROBLEM, n = 0.

α	β_1	β_2	$\mu^{1/2} = \beta_1 \beta_2$
m = 0			
0.0	2.2274580	2.2274580	4.961569
0.5	2.1656121	2.222583	4.813253
1.0	1.9663810	2.2060495	4.337934
1.3	1.7594769	2.1876378	3.849098
1.5	1.5675703	2.1696259	3.401041
1.6	1.4482550	2.1581109	3.125495
1.8	1.1342579	2.1275669	2.413210
2.0	0.5716016	2.0800789	1.188976
2.0600017	0.0	2.0600017	0.0
m = 1			
0.0	5.4535042	5.4535042	29.740708
0.5	5.4302508	5.4532215	29.612360
1.0	5.3598667	5.4523547	29.223895
1.3	5.2942696	5.4515402	28.861924
1.5	5.2404103	5.4508624	28.564755
1.6	5.2103511	5.4504824	28.398927
1.8	5.1437857	5.4496359	28.031759
2.0	5.0683233	5.4486605	27.615572
3.0	4.5398584	5.4415361	24.703803
4.0	3.6705019	5.4288658	19.926662
5.0	2.0548660	5.4057816	11.108157
5.3928115	0.0	5.3928115	0.0
m = 2			
0.0	8.6125308	8.6125308	74.175687
0.5	8.5979338	8.6124599	74.049359
1.0	8.5539953	8.6122493	73.669139
1.3	8.5133691	8.6120528	73.317583
1.5	8.4802543	8.6118938	73.031049
1.6	8.4618651	8.6118037	72.871920
1.8	8.4213943	8.6116133	72.521791
2.0	8.3759155	8.6113856	72.128238
3.0	8.0702978	8.6098611	69.484143
4.0	7.6216586	8.6075363	65.603703
5.0	7.0022717	8.6041740	60.248764
6.0	6.1602463	8.5993392	52.974048
7.0	4.9827378	8.5923034	42.813195
8.0	3.1059983	8.5817962	26.655044
8.5739849	0.0	8.5739849	0.0

TABLE III

ROOTS OF THE CHARACTERISTIC EQUATION FOR VIBRATION OF THE CIRCULAR PLATE WITH EDGE COMPRESSION-LINEAR PROBLEM, n = 1

α	β_1	β_2	$\mu^{1/2} = \beta_1 \beta_2$
m = 0			
0.0	3.7310427	3.7310427	13.9206793
0.5	3.6964372	3.7301003	13.788082
1.0	3.5904497	3.7271555	13.382351
1.3	3.4900374	3.7242934	12.997923
1.5	3.4061925	3.7218474	12.677329
1.6	3.3588249	3.7204441	12.496320
1.8	3.2523577	3.7172343	12.089775
2.0	3.1288098	3.7134150	11.618569
3.0	2.1309236	3.6797875	7.841346
3.6306551	0.0	3.6306551	0.0
m = 1			
0.0	6.9641241	6.9641241	48.499024
0.5	6.9460158	6.9639885	48.371974
1.0	6.8914081	6.9635843	47.988901
1.3	6.8407772	6.9632056	47.633738
1.5	6.7994036	6.9628938	47.343525
1.6	6.7763914	6.9627209	47.182122
1.8	6.7256276	6.9623320	46.826052
2.0	6.6684310	6.9618944	46.424912
3.0	6.2789499	6.9588226	43.694098
4.0	5.6882904	6.9538944	39.555770
5.0	4.8217636	6.9461792	33.492834
6.0	3.4752368	6.9337776	24.096519
6.9151434	0.0	3.6306551	0.0
m = 2			
0.0	10.138746	10.138746	102.794170
0.5	10.1262940	10.1386306	102.666754
1.0	10.0890692	10.1385068	102.288096
1.3	10.0546935	10.1383855	101.938358
1.5	10.0267117	10.1382912	101.653723
1.6	10.0111877	10.1382384	101.495807
1.8	9.9770459	10.1381185	101.148473
2.0	9.9387485	10.1379842	100.758874
3.0	9.6829959	10.1370810	98.157313
4.0	9.3130618	10.1357349	94.394726
5.0	8.8144723	10.1338503	89.324523
6.0	8.1635122	10.1312850	82.706868
7.0	7.3193063	10.1277957	74.128439
8.0	6.2028258	10.1229961	67.791182
10.1054066	0.0	10.1054066	0.0

TABLE IV

VALUES OF BUCKLING AND FREQUENCY ASSOCIATED PARAMETERS FOR n = 0

m	α	$\beta_1 = \beta_2$	$\mu^{1/2} = \beta_1 \beta_2$
0	2.0600017	2.2274580	4.9615690
1	5.3928115	5.4535041	29.740707
2	8.5739849	8.6125307	74.175685
3	11.733336	11.761687	138.33727
4	14.885062	14.907513	222.23394
5	18.033223	18.051816	325.86805
6	21.179422	21.195296	449.24056
7	24.324426	24.338273	592.35151
8	27.468644	27.480928	755.20139
9	30.612324	30.623355	937.78984
10	33.755613	33.765622	1140.1172
11	36.898613	36.907774	1362.1837
12	40.041390	40.049857	1603.9910
13	43.183997	43.191848	1865.5357
14	46.326466	46.333817	2146.8225

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