ENGINEERING RESEARCH INSTITUTE
THE UNIVERSITY OF MICHIGAN
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Progress Report

DETERMINATION OF BLANK SIZES

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Project 2569

KELSEY-HAYES WHEEL COMPANY
DETROIT, MICHIGAN

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ABSTRACT

At the request of the Kelsey-Hayes Wheel Company of Detroit, members of the staff of the Engineering Research Institute of The University of Michigan investigated methods of obtaining the centroid of an area or the first moment of an area.

The sponsors desired such a method to reduce the work load and produce uniformity in determining the size of a metal blank from which piece parts in the form of volumes of revolution are made.

Four methods are presented in this report. Several others were considered but are not reported here since these four present the greatest advantages.

Each of the methods require one thing in common: a line drawing of the section of the volume of revolution. The methods are:

1. A direct improvement of the "manual" calculating method using ordinary measuring scales and a nomograph in the form of a drafting tool.

2. A fully or semi-automatic electronic analog computing method, using well-known circuitry.

3. A mechanical computing method using well-known linkages.

4. An optical method using principles of projection from or onto curved surfaces to convert the given area into another whose area in turn is the desired quantity.

OBJECTIVE

The purpose of the work described in this report was to investigate methods of obtaining the centroid of an area or the first moment of an area to reduce the work load and produce uniformity in determining the size of a metal blank from which piece parts in the form of volumes of revolution are made.

INTRODUCTION

STATEMENT OF THE PROBLEM

The volume of a surface of revolution made from a flat piece of metal is to be computed. The manufacturing process for the finished product will involve several operations during which the material will be drawn in a somewhat unpredictable manner. In general, however, a finished product drawing is available and it is to its specification that the product is held. It is assumed here and for the remainder of the report that the manufacturing process is of constant volume.

Well-known procedures exist to determine the volume of a surface of revolution by measuring the first moment of the area of the generating cross section about the axis of revolution.

The common way of computing the first moment of an area is to subdivide the given area into a number of areas, usually a large number if the area is at all complicated in its shape, for which the centroidal properties are known; then the pertinent dimensions are measured or read from a drawing and entered into a computing table. This is followed by an amount of straightforward work on a desk calculator.

This procedure required the operator to have a good knowledge of the subject and also permits a number of arbitrary ways in which different operators will subdivide a given area. These factors, amount of work, knowledge of the subject, and arbitrary procedure, combine to produce different answers to the same problem.

The purpose of the procedures presented in this report is to abbreviate the labor involved in the normal calculating method, even to replace it entirely with a method that, with the minimum of operator time or effort, can take a line drawing, operate on it, and present the final result.

GENERAL INTRODUCTION TO THE VARIOUS METHODS

To obtain the volume V generated by revolving the area A about the nonintersecting x-axis, the following theorem of Pappus or Guldinus is used:

$$V = 2\pi\bar{y}A,$$

where $\bar{y}$ is the location of the centroid of the area from the x-axis.

Basically, it is necessary to compute the first moment of the area, $\bar{y}A$, which need then only be multiplied by the appropriate constants to obtain the volume, weight, or mass.

The first moment of the area may be expressed either by integral calculus or by finite summations being written $\int ydA$ or ΣyA, respectively.

I. NOMOGRAPH METHOD OF CALCULATION MOMENTS OF AREAS*

The types of shapes which have provided the stimulus for this report are composed of rectangles, triangles, and portions of circles (in general, annuli). Furthermore, most complicated areas can, as an approximation, be decomposed into a finite number of combinations of the above-mentioned regular areas. The indicated product and sum can then be performed. However, the calculations involved require that basic data be provided which requires the knowledge and background of, at least, some basic notions from statics.

To eliminate the need for knowledge of statics and reduce the number of associated notions to a minimum, a tool that could as far as possible achieve this goal was sought. The element of area that requires more than just measuring a few designated lengths of straight lines is part of a circle. Consequently a nomograph was designed from which necessary numbers could be read while all remaining quantities could be obtained with a scale. All these data will be entered into a complementary calculating sheet.

The use of this tool or nomograph will then dictate to a very large extent the subdivision of an area and thus eliminate one of the objections to the conventional methods as described in the previous pages.

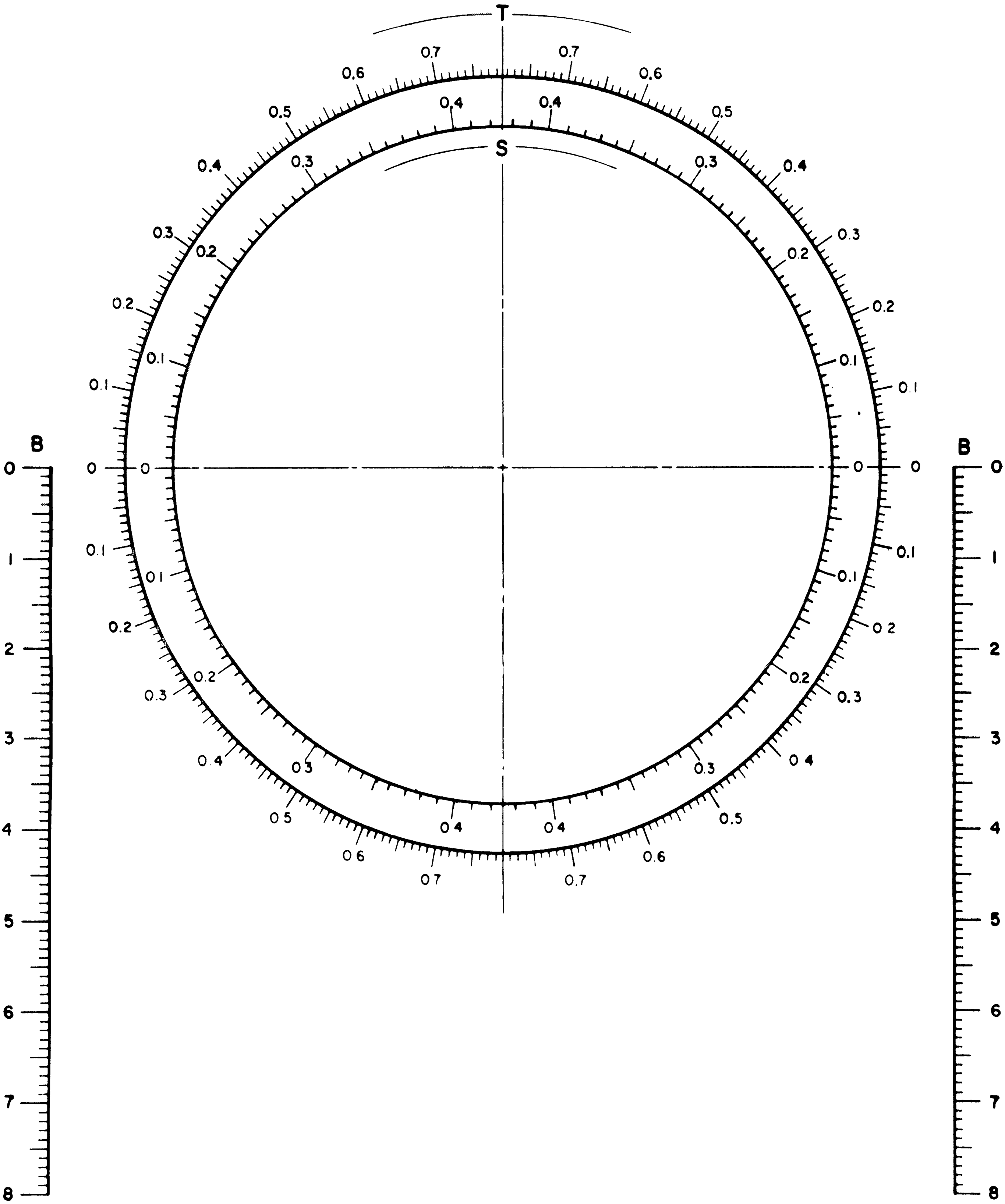
A diagram of such a tool is shown in Fig. 1.1. The form of the equation for the moment of the area to be used in this method is

$$\bar{y}A = \Sigma yA$$

THEORY

The fundamental notions which led to the form of the nomograph shown in Fig. 1.1 will be given in the following paragraphs. It is necessary to explain the relation between the conventional notations used for the calculation of moments of areas and those on the nomograph.

* By R. L. Hess



NOMOGRAPH FOR CONSTANTS USED IN DETERMINATION
OF MOMENT OF AREA OF CIRCULAR SECTORS AND ANNULII.

FIG. 1.1

The given area is to be subdivided in a systematic manner and certain references must be established for each of the elements. Fundamental to the whole operation is the identification of the axis of revolution which will hereafter be referred to as the Work Line. There is then associated with each type of element of area a Base Line, parallel to the Work Line, which is selected appropriately for rectangles, triangles, or sectors in such a manner that it, first, is easily established and, secondly, simplifies the consequent calculations.

The Rectangle.—For the rectangle of side lengths D and E shown in Fig. 1.2, the Base Line passes through the lowest corner. The distance C is the vertical intercept between the Base Line and a line drawn parallel to it through the highest corner. Then the area is DE and the centroid is located C/2 above the Base Line. B is the perpendicular distance between the Base Line and the Work Line (for all elements of areas). Finally, the first moment of this area about the Work Line is

$$\bar{y}A = \bar{y}(DE) = (B+\frac{C}{2})DE \qquad (1.1)$$

It should be noted that this result holds regardless of the angular orientation of the rectangle.

The Triangle.—The triangle shown in Fig. 1.3 is considered to be composed of two triangles having a common Base Line. The upper triangle having an area 1/2 JL while that of the lower is 1/2 KL. The centroidal distances from the Work Line are for the upper triangle.

$$B + \frac{J}{3} ,$$

and for the lower one

$$B - \frac{K}{3} .$$

Hence

$$\bar{y}A = 1/2\ JL(B+\frac{J}{3}) + 1/2\ KL(B-\frac{K}{3}), \qquad (1.2)$$

which can be more usefully grouped for computational purposes:

$$\bar{y}A = 1/2\ L[B(J+K) + 1/3(J^2-K^2)] . \qquad (1.3)$$

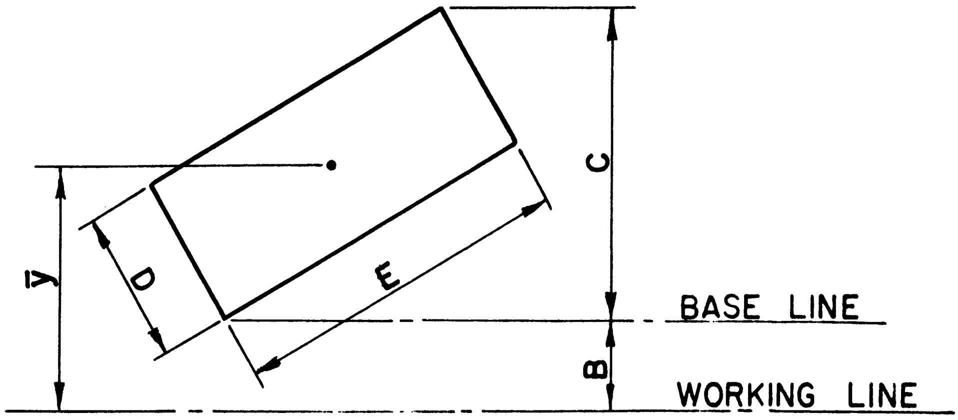


FIG. 1.2 THE RECTANGULAR ELEMENT

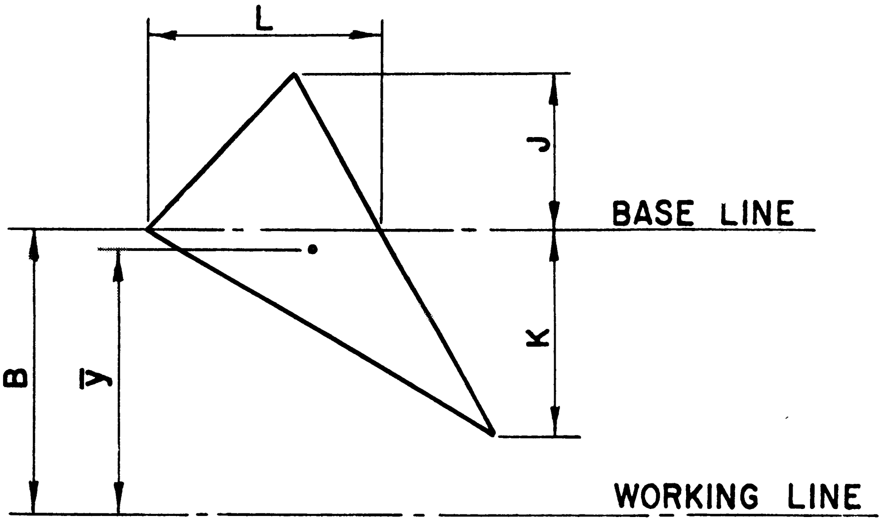


FIG. 1.3 THE TRIANGULAR ELEMENT

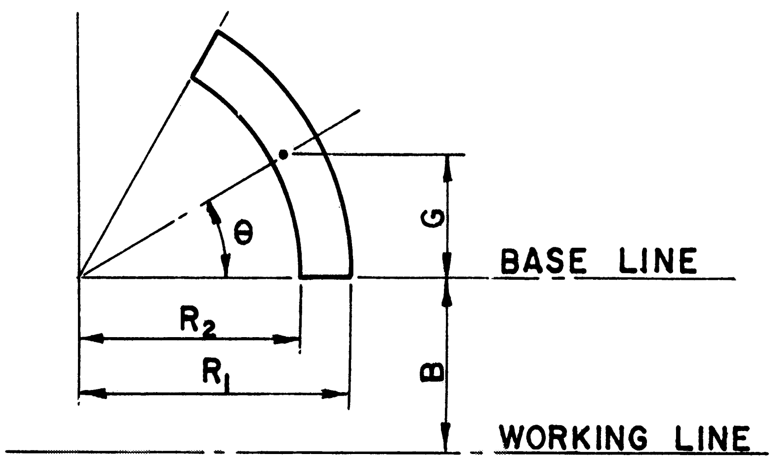


FIG. 1.4 THE ANNULUS SECTOR ELEMENT

The Annular Sector.—For a preliminary discussion of the annular sector consider the one shown in Fig. 1.4, where the lower enclosing radius coincides with the Base Line. The following quantities are defined with reference to Fig. 1.4:

R = radius of outer circle
r = radius of inner circle
$\bar{R}$ = radial distance to centroid
θ = half angle subtending the arc in radians
G = distance from Base Line to centroid
B = distance from Base Line to Work Line
A = area of annular sector

Then it follows that

$$G = \bar{R} \sin \theta \qquad (1.4)$$

And the first moment of the area is

$$\bar{y}A = \left[B(R^2 - r^2) \pm S(R^3 - r^3) \right] T, \qquad (1.5)$$

where

$$S = \frac{2}{3} \frac{\sin^2\theta}{\theta}, \qquad (1.6)$$

and

$$T = \theta \qquad (1.7)$$

The sign in Equation (1.5) is chosen plus if the area is as shown in Fig. 1.4, i.e., if the centroid lies above the Base Line, or minus if the centroid lies below the Base Line.

The radii of the sector would in general be specified or could be measured from the drawing; the quantities B, C, and D will be obtained from the nomograph shown in Fig. 1.1. C and D, representing functions of the angle θ only, are obtained by reading, from the scales, the intercept caused by the enclosing radial lines.

In an attempt to avoid, as much as possible, the use of algebraic negative numbers and thus avoiding the decision as to the specific location of the centroid of a sector, i.e., below or above the axis, the following rule is established.

Any sector is to be subdivided into component sector so that no one component sector will extend over the horizontal Base Line, and further that it will not subtend an angle greater than 90 degrees.

Then to expedite calculations, the above equations can be modified for sectors having an included angle of less than 90 degrees and with the lower enclosing radius not coincident with the Base Line. Such sectors are shown in Fig. 1.5(a) and (b). The B value will, of course, be constant, but let the radial intercepts of the S and T scale be designated S_U and T_U, and S_L and T_L as shown in Fig. 1.5. The first moment of the area then becomes

$$A\bar{y} = \left[B(R^2-r^2) \pm S_U(R^3-r^3)\right] T_U - \left[B(R^2-r^2) \pm S_L(R^3-r^3)\right] T_L$$

$$= B(R^2-r^2)(T_U-T_L) \pm (R^3-r^3)(S_UT_U-S_LT_L)$$

$$= M \pm N,$$

where again, the plus sign is used if the area lies above the Base Line, and the negative sign is used if the area lies below the Base Line.

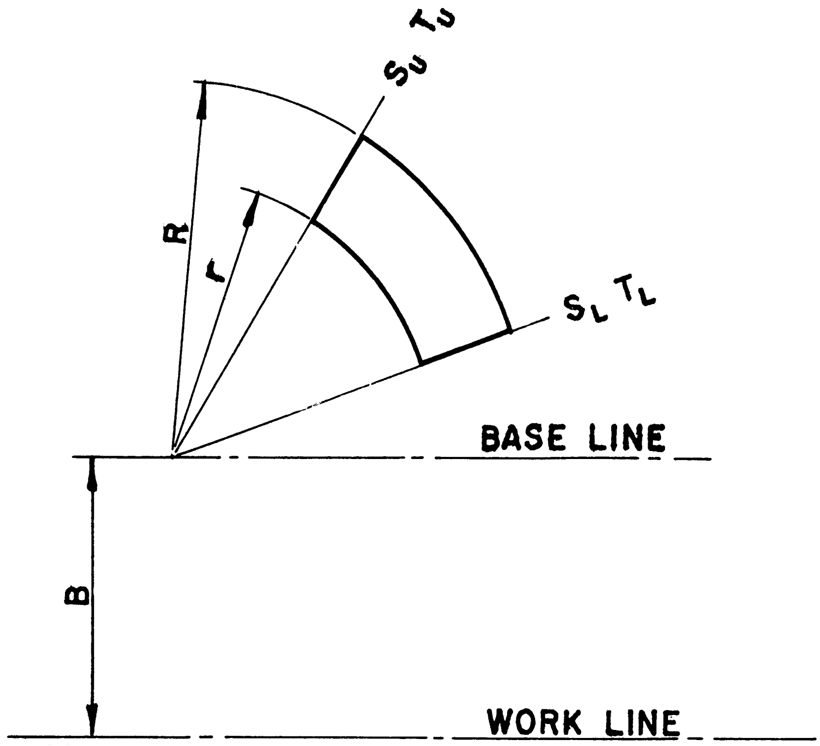
PREPARATION OF THE DRAWING FOR MEASURING PURPOSES

A typical section for which the sponsor desires to know the moment of the area is shown in Fig. 1.6. This area must now be subdivided into separate rectangles, triangles, and annuli. Lines are drawn in the drawing denoting the subdivision into these elements, thus establishing the requisite Base Lines as shown in the previous paragraphs and the lines which will make the intercept readings on the S and T scales of the template or nomograph.

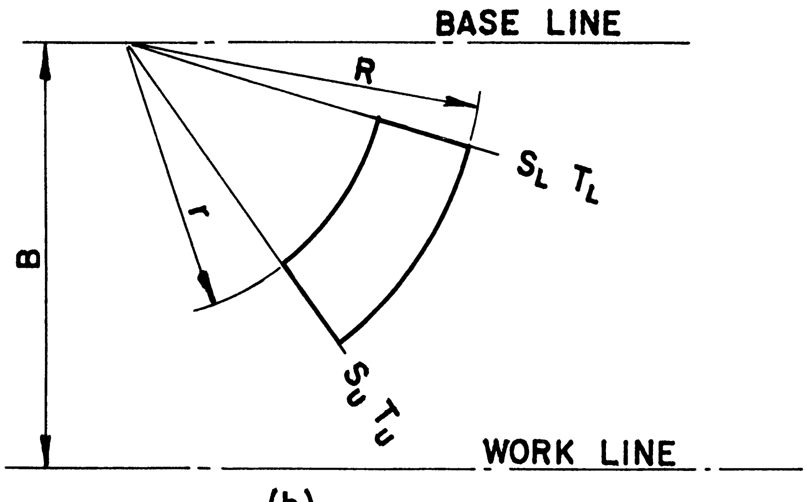
Dimensions of rectangles and triangles are measured or specified; radii are usually specified on the drawing.

All this information can be entered into a computation schedule, a copy of which is attached, which can be followed without any basic knowledge of statics to produce finally the first moment of the area.

Details and examples clarifying these procedures are presented in the Instruction Manual for Moments of Areas. This manual, together with the nomograph, is attached to this report.



(a)



(b)

FIG. 1.5 SECTOR NOT HAVING A RADIUS ON BASE LINE

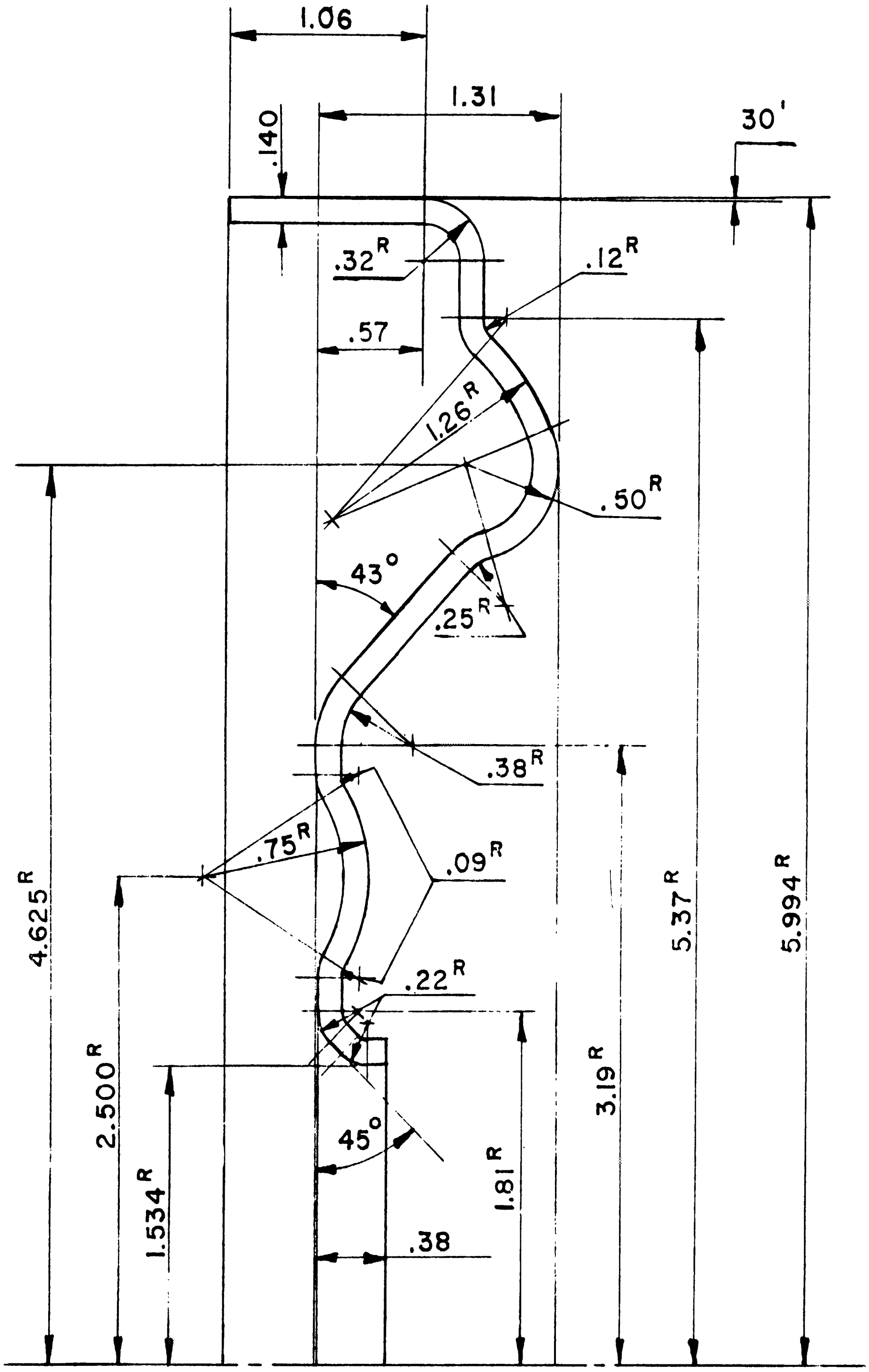


FIG. 1.6 TYPICAL AREA FOR WHICH $\bar{y}A$ IS TO BE COMPUTED

SUMMARY SHEET

ENTER TOTAL 1	
ENTER HALF OF TOTAL 2	
ENTER TOTAL 3	
ENTER TOTAL 4	
ENTER TOTAL 5	
TOTAL 6	

VOLUME = $2\pi \bar{y} A$ = 6.28 x ______ (ENTER TOTAL 6) = ______ IN^3

WORK SHEET FOR RECTANGULAR AREAS

DATA	B	C	D	E	$C/2$	DE	$B + C/2$	$\bar{y}A$
COLUMN	a	b	c	d	e	f	g	h
OPERATION	–	–	–	–	$\frac{1}{2}$(b)	(c)x(d)	(a)+(e)	(f)x(g)
ITEM								

TOTAL I =

WORK SHEET FOR TRIANGULAR AREAS

DATA	B	J	K	L	J + K	J^2	K^2	$J^2 - K^2$	B(J + K)	$\frac{1}{3}(J^2 - K^2)$	P	$2\bar{y}A$
COLUMN	a	b	c	d	e	f	g	h	i	j	k	l
OPERATION	—	—	—	—	(c)+(d)	(b)x(b)	(c)x(c)	(f)-(g)	(a)x(e)	$\frac{1}{3}$(h)	(i)+(j)	(d)x(k)
ITEM												

TOTAL 2 =

HALF OF TOTAL 2 =

WORK SHEET FOR AREAS OF CIRCULAR SECTORS AND ANNULII

DATA	B	R	r	S_U	T_U	S_L	T_L	R^2	R^3	r^2	r^3	$(R^2 - r^2)$	$T_U - T_L$	$B(R^2 - r^2)$	M	$(R^3 - r^3)$	$S_U T_U$	$S_L T_L$	$S_U T_U - S_L T_L$	N	+N	-N
COLUMN	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	s	t	u	v	w
OPERATION	—	—	—	—	—	—	—	(b)x(b)	(b)x(h)	(c)x(c)	(c)x(j)	(h)-(j)	(e)-(g)	(a)x(l)	(m)x(n)	(i)-(k)	(d)x(e)	(f)x(g)	(q)-(s)	(p)x(t)	+(u)	-(u)
ITEM																						

TOTAL 3 =

TOTAL 4 =

TOTAL 5 =

NOTE: USE COLUMN "v" FOR AREAS ABOVE WORKLINE
USE COLUMN "W" FOR AREAS BELOW WORKLINE

II. AN ELECTRONIC ANALOG METHOD OF OBTAINING THE FIRST MOMENT OF AN AREA*

The electronic analog computer permits the computation of the moment of the area by evaluating the integral Ay = ydA, where means integration around the whole region, area, concerned. This may be more suitably written (see Fig. 2.1) in the following two ways:

$$A\bar{y} = \oint \frac{1}{2} y^2 dx = \oint xydy.$$

Consider for the purpose of this report that the "Analog Computer" consists of a series of "black boxes" which will perform the required operations.** The simplest of these operations are addition, subtraction, multiplication by a constant, and integration. Other operations, which present more difficulty, are squaring, square root, multiplication of two dependent variables, etc. It should be understood at this point that the independent variable is time and all other variables will be voltages.

Generally the other components, resistors and capacitors, may be plugged into the operational amplifier, which is the core of the analog computer, to produce a circuit having the desired properties. Some examples of such circuits are shown in Fig. 2.2, where e_o refers to the output voltage and e_i to the input voltage. It should be noted that, in general, a sign change is involved in these operations; see, for example, the first three operations in Fig. 2.2.

Consider now the evaluation of the first moment of the area

$$A\bar{y} = \oint \frac{1}{2} y^2 dx.$$

Let the curve be drawn out on a sheet of paper which will be placed in an X-Y plotter. This device is, essentially, a two-axis voltmeter. The horizontal axis (x axis) is to be driven by a voltage proportional to the independent variable which is x in the problem and time for the computer, i.e., time on the computer corresponds to x. An operator will then turn a potentiometer in such a manner that the recording point of the X-Y plotter will follow the y coordinate of the designated curve. The output voltage on this potentiometer will then be proportional to y, which is then squared and integrated. Thus, in outline form the desired integral has been evaluated.

Figure 2.3 shows the block diagram for this operation. This diagram omits such items as scale factors, time constants inherent in certain operations, and excursion limits of the various devices. As long as these are constants they will not affect the general direction of the solution. The figure then represents, essentially, a flow diagram.

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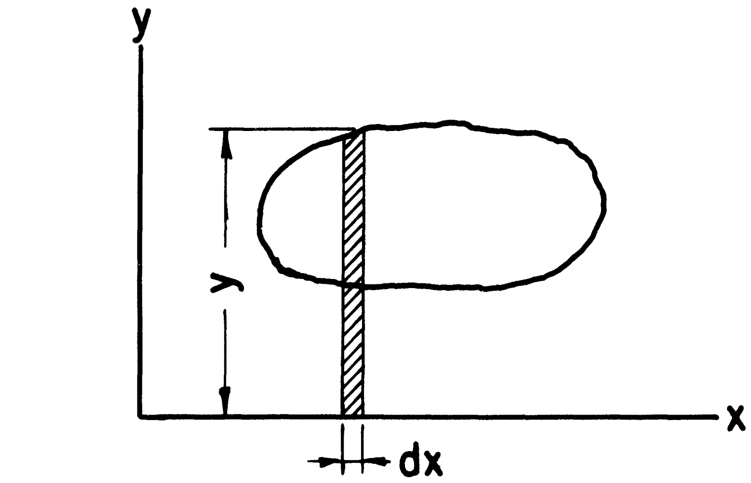
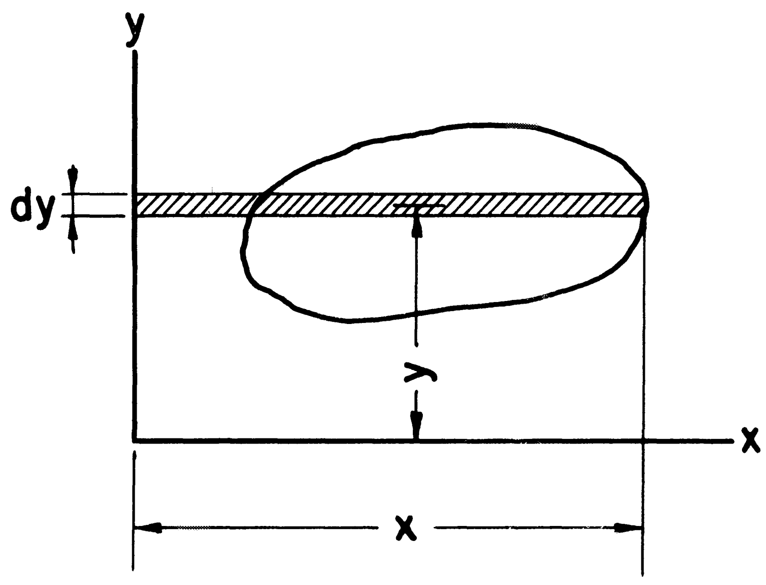
*By B. Herzog.

**For a detailed discussion of the analog computers, the reader is referred to any one of the several books published on the subject. An excellent introductory text is a book by C. L. Johnson, Analog Computer Techniques, McGraw-Hill Book Co., June, 1956.



SCHEME I: $A\bar{y} = \oint \frac{y}{2} \cdot y\,dx$



SCHEME II: $A\bar{y} = \oint y \cdot x\,dx$

FIG. 2.1

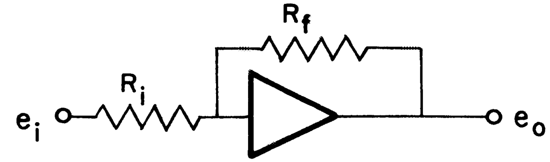
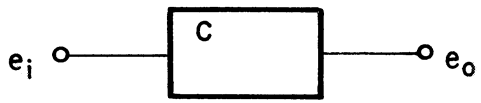
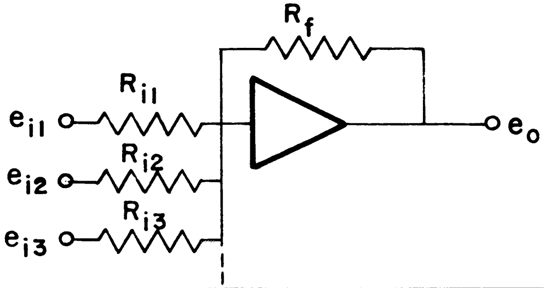
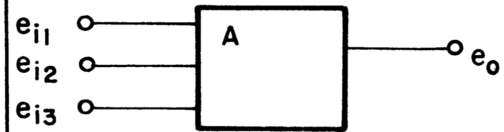
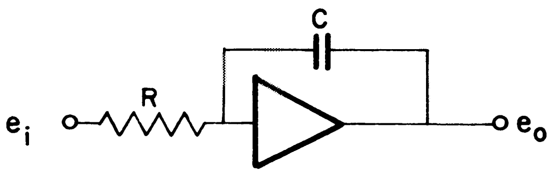
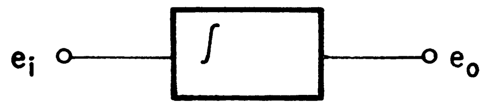
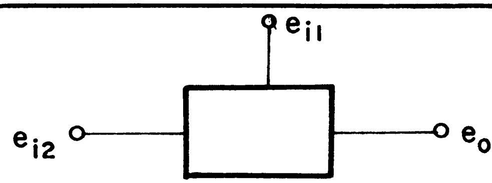
TRANSFER FUNCTION	CIRCUIT DIAGRAM	BLOCK DIAGRAM
MULTIPLICATION BY A CONSTANT $e_o = -\frac{R_f}{R_i} e_i$		
ADDITION $e_o = -\frac{R_f}{R_{i1}} e_{i1} - \frac{R_f}{R_{i2}} e_{i2} - \frac{R_f}{R_{i3}} e_{i3} \cdots$		
INTEGRATION $e_o = -\frac{1}{RC} \int e_i \, d\tau$		
MULTIPLICATION $e_o = \frac{e_{i1} \times e_{i2}}{k}$	SERVO MULTIPLIER	

FIG. 2.2

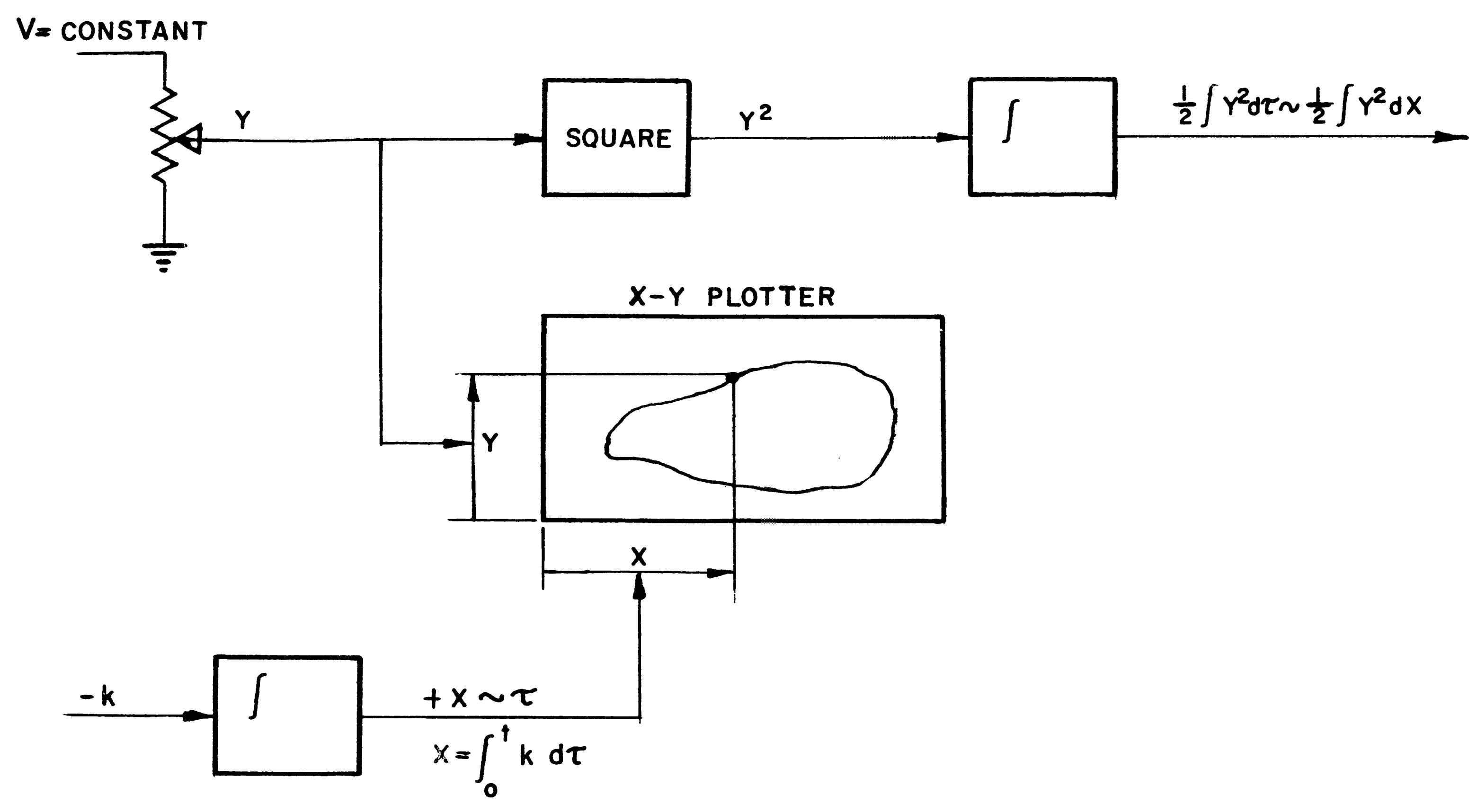


FIG. 2.3 BLOCK DIAGRAM FOR SCHEME I

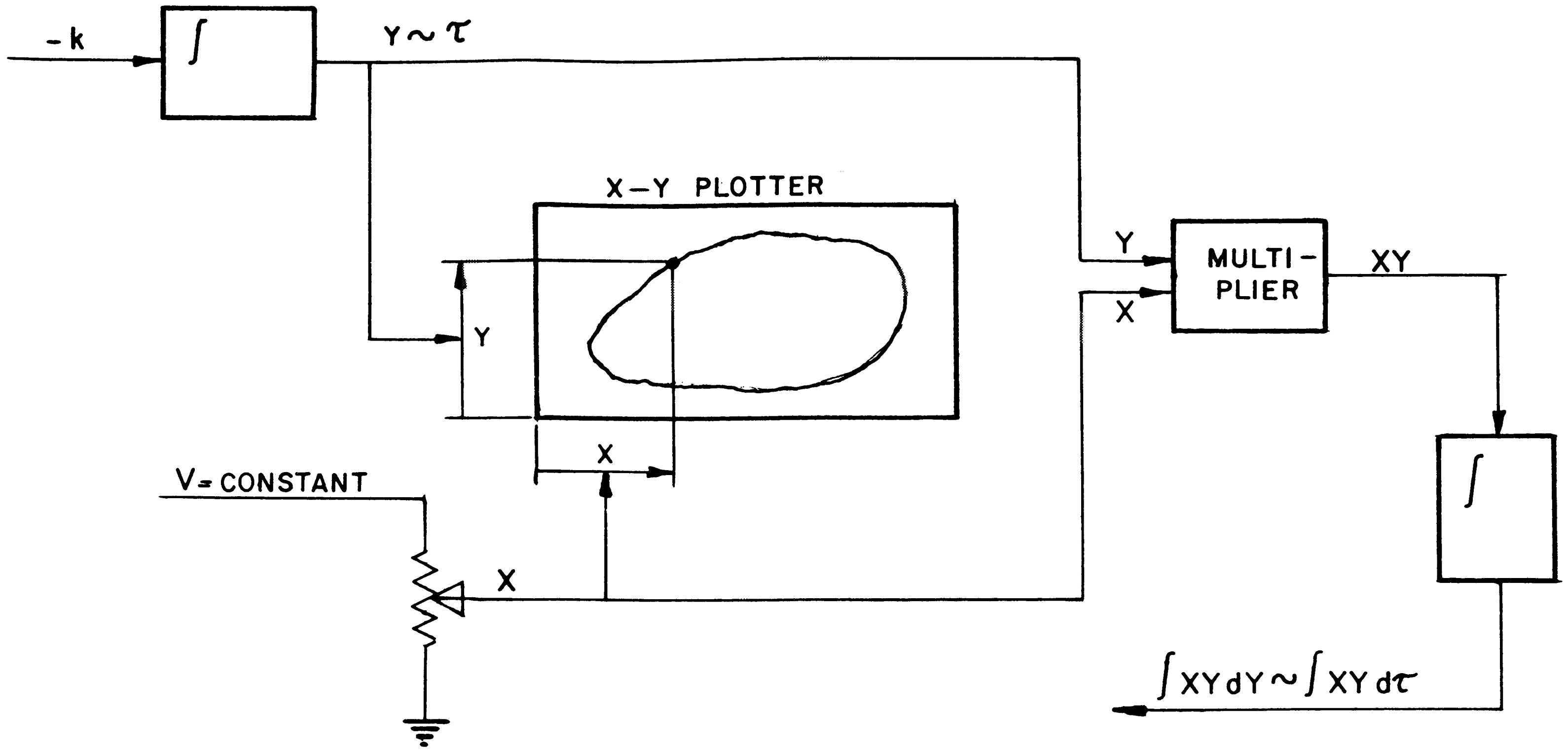


FIG. 2.4 BLOCK DIAGRAM FOR SCHEME II

In a similar manner the block diagram, Fig. 2.4, is obtained for the operation.

$$A\bar{y} = \oint xydy.$$

Here the variable y is made proportional to τ and the operator adjusts the potentiometer setting to be proportional to x.

The X-Y plotter then serves as a means of tracking the given curve. It is possible to obtain a curve-following device which will automatically track the curve and thus eliminate the need for an operator except to start the device.

The first method is better suited for an area extended essentially parallel to the x axis, while the second method will best serve an area extended along the y axis. There are several reasons for these preferences. First, it will take some amount of practice for an operator to become expert at adjusting the potentiometer setting to follow the curve. It is, of course, possible to have the voltage driving the independent variable axis increase slowly, thus permitting more time for the operator to make his adjustments. However, if the operation is extended over too long a time, errors introduced by drift of the electronic equipment may become significant.

These methods were tried for some elementary shapes. Equipment which was on hand in laboratories of the Engineering Mechanics Department was used. It should be remarked here that such equipment is mostly of the general utility type and not designed for this particular problem only. Consequently, some of the components were too sensitive for this purpose, while others could use some modification to be more accurate or stable for this particular use. Several of the components in themselves are rated as having tolerances of 1%. Thus, from a combination of these we would expect perhaps an accuracy within several percent.

A servo-driven potentiometer was used as a multiplying and squaring unit to complement the X-Y plotter and the general analog computer.

By Scheme I the first moment of the area of the triangle shown in Fig. 2.5 was obtained. The calculated result was 67.5 in.^{3}, while the computer obtained a value of 61.2, which is about 2% in error.

The first moment of the area of the rectangle of Fig. 2.6 was obtained by Scheme II. The calculated value was 96 in.^{3}, while that obtained by the computer was 95 in.^{3}, in error about 1%.

Under the conditions previously stated, these results are much better than were expected and it remains only to state that with equipment designed for the particular purpose better results should be possible.

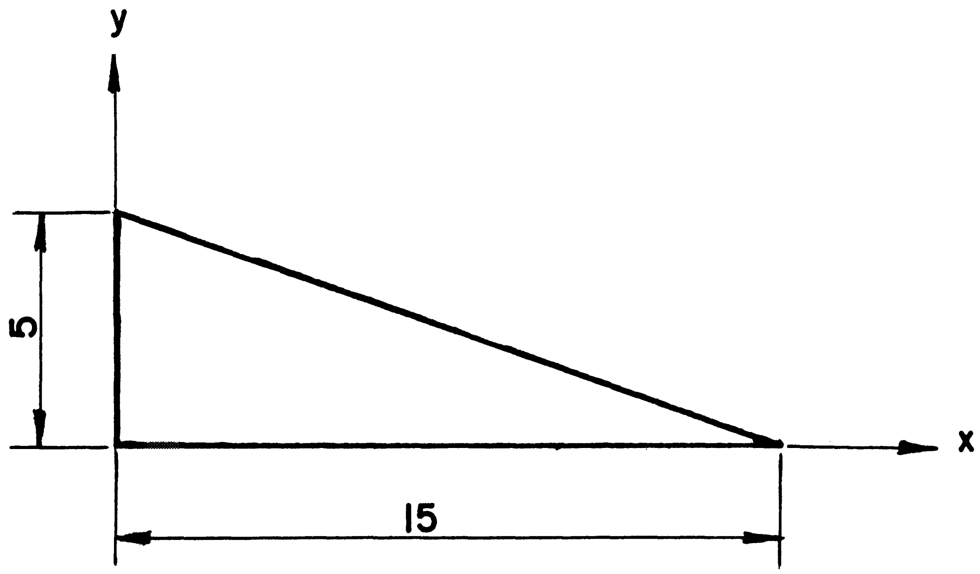


FIG. 2.5

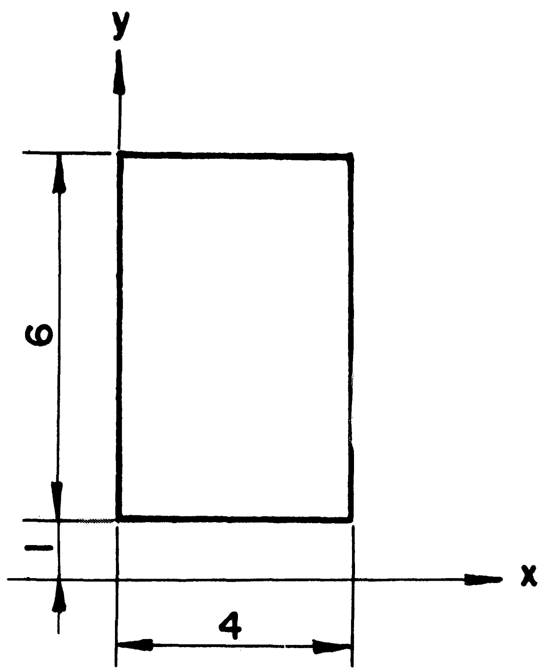


FIG. 2.6

III. THE MECHANICAL VOLUMETRIC PLANIMETER*

The design utilizes a conventional planimeter wheel mounted upon a linkage which produces the integration of the differential of the first moment of the area

$$\frac{1}{2}\, y dA.$$

Thus finally the moment of the area, when multiplied by 2π, which equals the volume of the body of revolution, is read out on the planimeter dial.

There are similar instruments commercially available,** but these are quite general in their utility. It is believed that the design presented here will be more specifically suitable.

The design is based upon the following expressions which are derived from the line diagram of Fig. 3.1.

A print of the curve in question is placed on the board on which the instrument is mounted. The pointer A on the arm AB of length L is used to trace around the outline of the curve. The arm BC is mounted on GH so that their intersection is always perpendicular. GH is permitted to travel in the x-direction through two bushings. The planimeter wheel D is mounted on arm CED. Member EF is pinned to CED at E so that distance CE equals distance BC. EF is constrained to slide through a bushing at B so that EF always remains perpendicular to AB.

Let the angle between AB and GH on the x axis be θ; the angle the plane of the wheel makes with the x-direction is 2θ. It follows that the center of the wheel will travel a distance $dx \cos 2\theta$ when the pointer A moves a distance dx. Then if the planimeter wheel has a radius r, it will turn through the angle $d\phi$ when the pointer moves a distance dx so that

$$d\phi = \frac{dx\cos 2\theta}{r} = \frac{dx}{r}(1-2\sin^2\theta) = \frac{dx}{r} - \frac{2\sin^2\theta}{r} .$$

Thus

$$\phi = \oint \frac{dx}{r} - \oint \frac{2\sin^2\theta dx}{r} .$$

But

$$\oint \frac{dx}{r} = 0,$$

and

$$\sin\theta = \frac{y}{L} .$$

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*By J. R. Pearson.

**Keuffel and Esser Co., Hoboken, N. J., Catalog Item 4270.
Eugene Dietzgen Co., Chicago, Ill., Catalog Item 1813A.

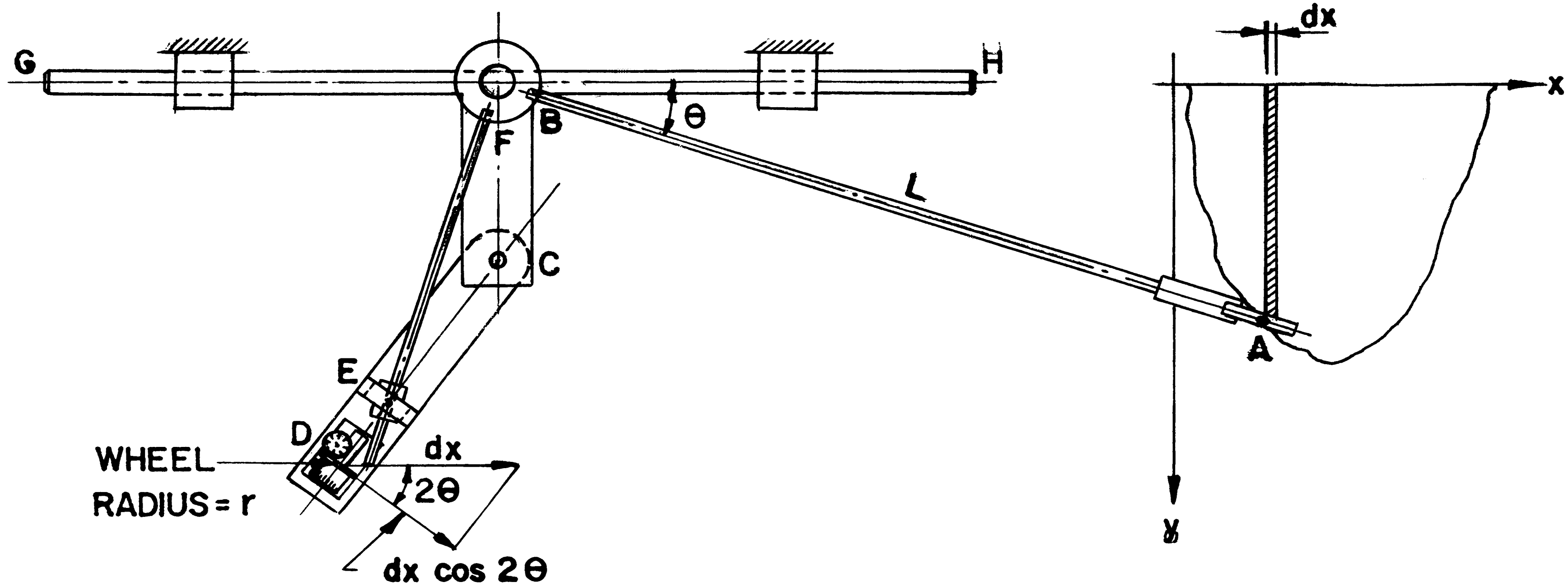


FIG. 3.1 SCHEMATIC OF MECHANICAL VOLUMETRIC PLANIMETER

Then $$\phi = \oint \frac{2y^2}{L^2 r} dx = \frac{4}{rL^2} \oint \frac{y^2}{2} dx \quad .$$

Finally $$\phi = \frac{4}{rL^2} \bar{y}A \quad .$$

The desired volume is

$$V = 2\pi \bar{y} A = \frac{2\pi r L^2}{4} \phi \quad .$$

It is apparent, then, that a scale factor or characteristic constant of the instrument exists which, by selecting appropriate values for r and L, will permit designing to a particular sensitivity.

Consider, for example, the following values:

$r = 0.500$ in.

$L = 20$ in.

The volume/revolution then becomes

$$2\pi \left(\frac{V}{\phi}\right) = \pi^2 r L^2 = 1975 \text{ in.}^3 \text{ / revolution} \quad .$$

This constant will, of course, be modified by manufacturing errors and the proper constant must be found by calibrating the instrument on an area for which the moment can be easily computed.

IV. OPTICAL METHODS OF PROJECTION AS AIDS TO EVALUATING THE FIRST MOMENT OF AN AREA*

An optical method using projection techniques offers some distinct advantages since only a few calculations are involved based upon ordinary planimeter measurements of areas.

A film negative of the original drawing is placed in a frame and its image projected on positive paper or on another piece of film. The negative of the original drawing may be placed in a frame of predetermined shape such that the image cast on the positive paper is the first moment of the area. Alternatively, the positive paper could be placed in a curved frame and have the image from a flat negative cast upon it with the same result. A

*By R. L. Hess and B. Herzog.

third method using a point source of light moving along a curved path can accomplish the desired effect by casting the image from a flat film on a flat positive paper. These methods are outlined in the following paragraphs.

Consider first the method using a curved surface to hold the positive paper. A photographic negative of the original drawing is placed on a frame in the xy plane (see Fig. 4.1). The positive paper is placed on the surface $z = f(x,y)$. Parallel light is used in the z direction to project the image of the area, on the drawing, on the film. The film is then developed, laid out on a board, and an ordinary planimeter is used to evaluate the resulting area which will be proportional to the moment of the area.

In choosing the surface $z = f(x,y)$ consider the integral for the first moment of the area in the form

$$A\bar{y} = \int \frac{y^2}{2}\, dx \ . \qquad (4.1)$$

Therefore, it is required that a projection be made which will be proportional (directly, preferably) to x along the x-direction and proportional to y^2 in the orthogonal direction. This may be accomplished if we define a coordinate s measured along the curve in Fig. 4.2, which is a section parallel to y-z plane. This coordinate will be perpendicular to x and s-x will be conventional Cartesian system when the film is laid out flat.

The quantity s must be of the form

$$s = ky^2 + k', \qquad (4.2)$$

where k and k' are arbitrary constants.

The equation for z which satisfies this condition is

$$z = \frac{y}{2}\sqrt{4k^2y^2-1} - \frac{1}{4k}\log_e(2ky + \sqrt{4k^2y^2-1}) + C, \qquad (4.3)$$

where C is an arbitrary constant. The above equation is restricted by

$$4ky^2 \geq 1. \qquad (4.4)$$

The selection of k, k', and C is governed by design requirements and scale factors. However, Eq. (4.4) places some definite restrictions which can be avoided by considering the following equation, where n is an arbitrary constant.

$$nA + \bar{y}A = \oint (n+y)dA \ , \qquad (4.5)$$

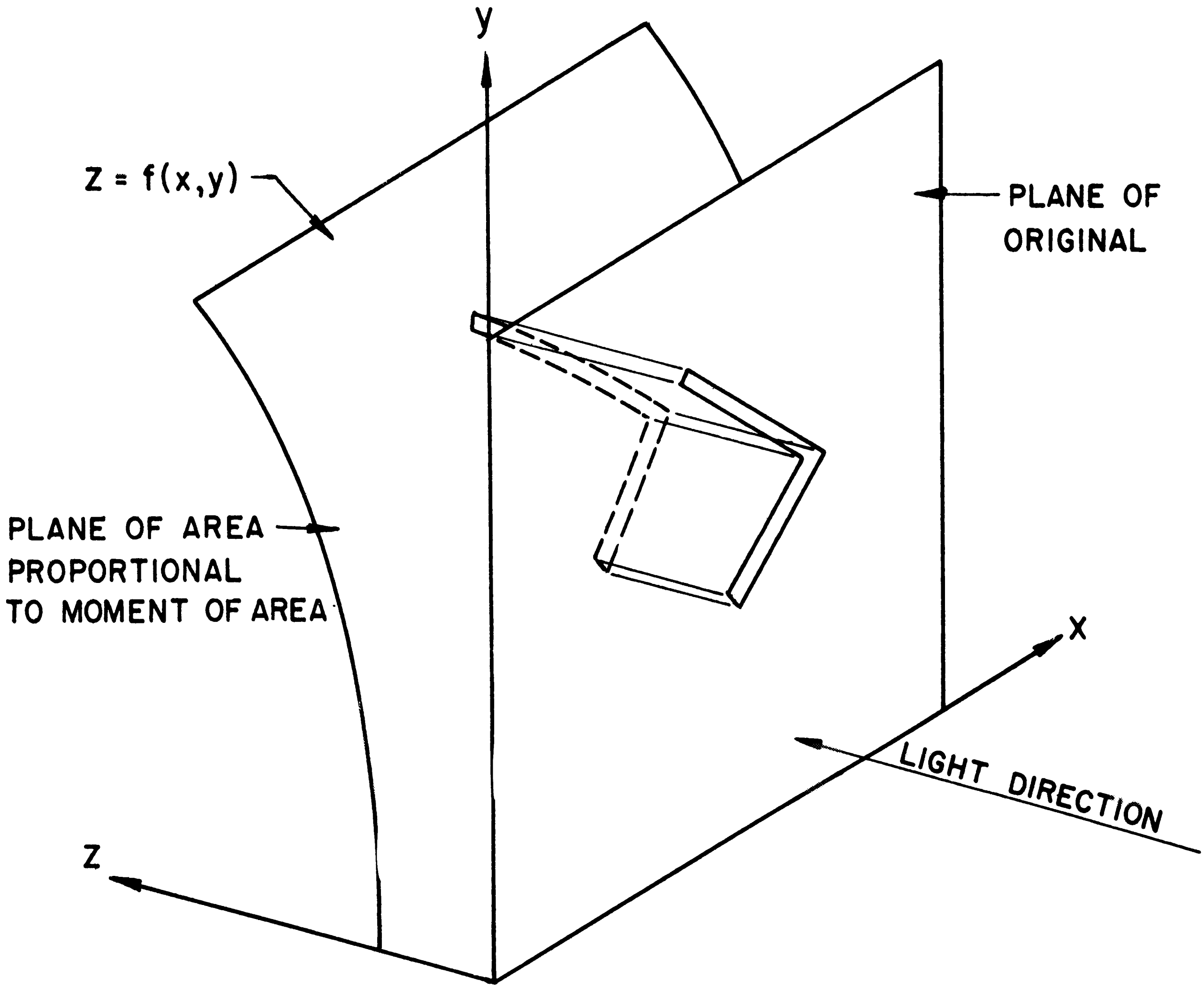

y
Z = f(x,y)
PLANE OF
ORIGINAL
PLANE OF AREA
PROPORTIONAL
TO MOMENT OF AREA
x
LIGHT DIRECTION
Z

FIG. 4.1

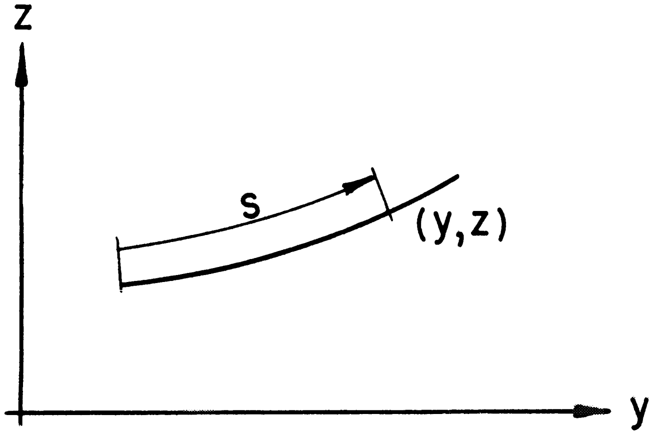

Z
S
(y,z)
y

FIG. 4.2

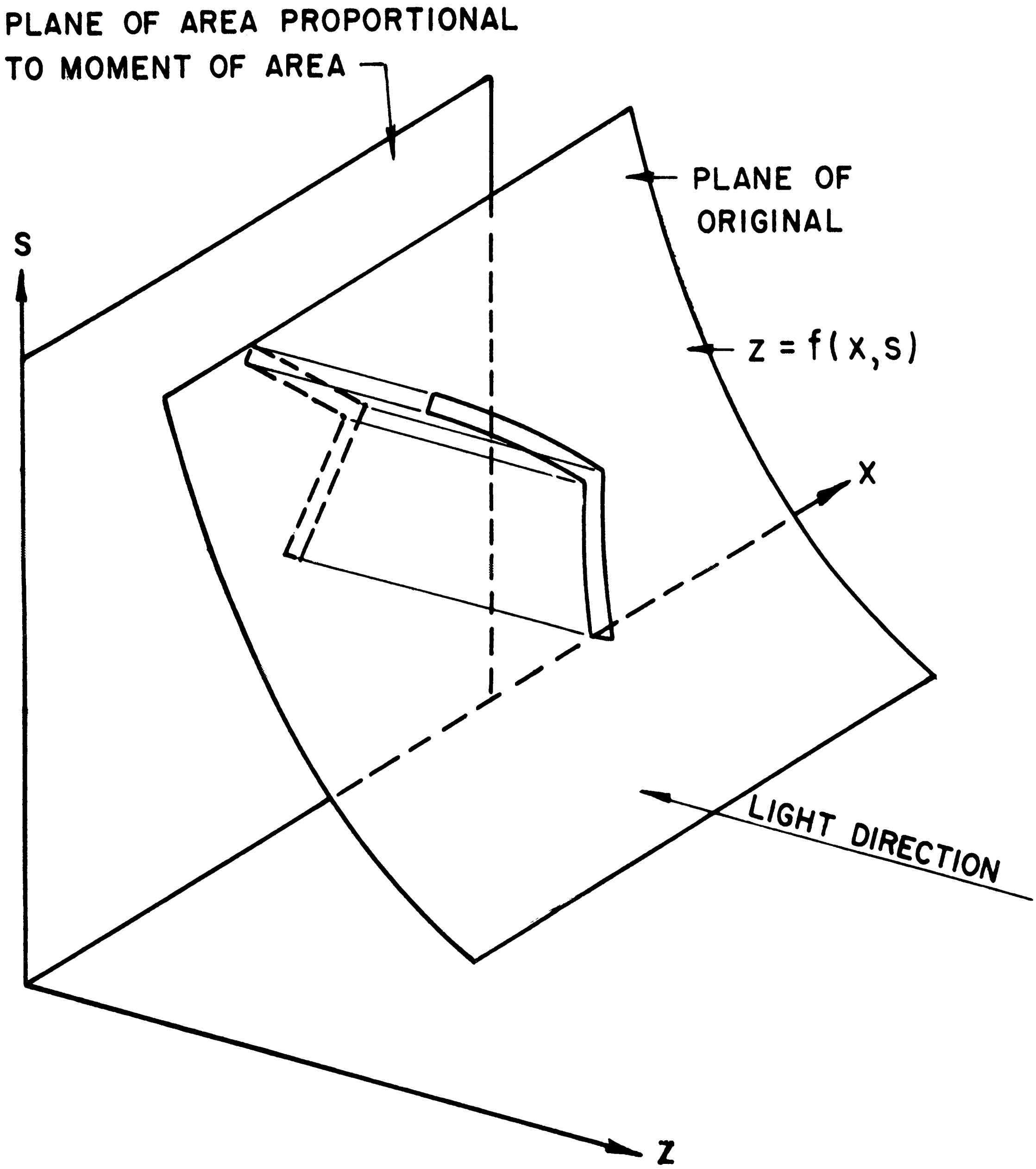

PLANE OF AREA PROPORTIONAL
TO MOMENT OF AREA
PLANE OF
ORIGINAL
S
z = f(x,s)
X
LIGHT DIRECTION
Z

FIG. 4.3

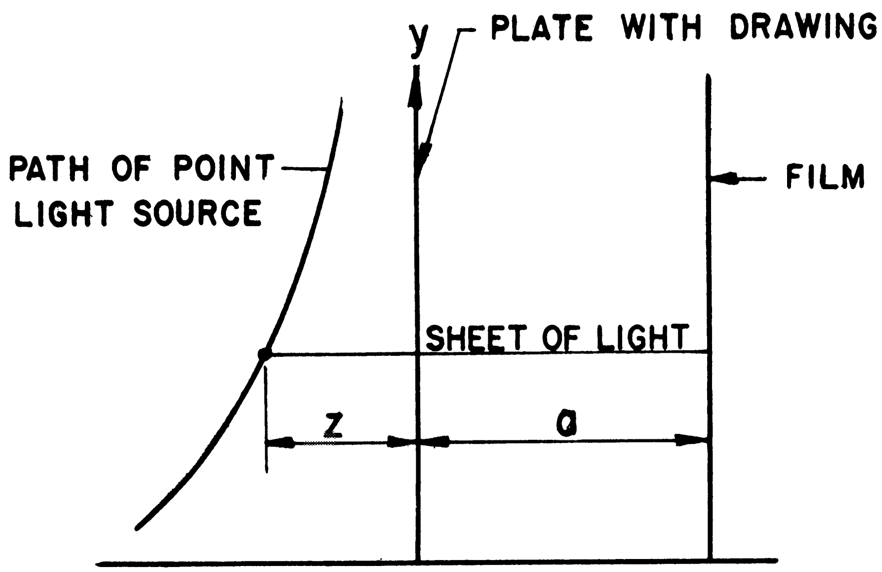

y
PLATE WITH DRAWING
PATH OF POINT LIGHT SOURCE
FILM
SHEET OF LIGHT
z
a

FIG. 4.4

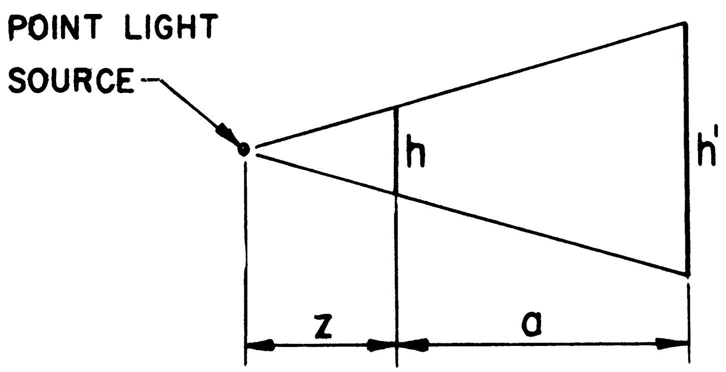

POINT LIGHT SOURCE
h
h'
z
a

FIG. 4.5

which is equivalent to transferring the axis. The above method can now be applied to the evaluation of the integral of Eq. (4.5). However, to obtain the first moment of the area now requires measuring both the original area and the projected area. This information combined with the knowledge of the constant n will allow the evaluation of the first moment of the area from the following equation:

$$\bar{y}A = A' - nA \;, \tag{4.6}$$

where A' is the magnitude of the projected area, i.e.,

$$A' = \oint (n + y)dA. \tag{4.7}$$

Similarly the negative of the original drawing may be placed on a surface and have its image cast on a flat plane (see Fig. 4.3). Here the y coordinates appear along the surface. In this case it is required that

$$y^2 = m\,s + m' \;, \tag{4.8}$$

where m and m' are arbitrary constants. The resulting equation for the surface is

$$z = \frac{m}{4}\log_e(ms + m') - s + D \;, \tag{4.9}$$

where D is an arbitrary constant.

For both of the above methods parallel light is required. This is most satisfactorally accomplished by using a double slit source and restricting it to a sheet in the y-z or s-z plane and having it traverse along the x axis.

A third method consists of a point source producing a sheet of light normal to the drawing and traveling along a curve in the y-z plane. The negative of the drawing is mounted in a flat plane parallel to a piece of positive paper also mounted flat. The light source travels along a curved path to satisfy the necessary transformation. A sketch of this arrangement is shown in Fig. 4.4.

The basis for this operation is the relation of Eq. (4.5), repeated for convenience:

$$nA + \bar{y}A = \oint (n + y)dA \;. \tag{4.5}$$

In Fig. 4.5 a section perpendicular to the y axis is shown. Let h represent the x intercept between the two boundaries of the area. Then h' is the projection of h onto the receiving plane located a distance a from the x-y plane. It follows then that

$$h' = \frac{z+a}{z} h , \tag{4.10}$$

where z is the distance of the point light source from the x-y plane. Let

$$h' = (n + y)h . \tag{4.11}$$

Rewriting the right-hand side of Eq. (4.5) and substituting Eq. (4.10) yields

$$(n + y)dA = \int_{y_1}^{y_2} (n + y)hdy = \int_{y_1}^{y_2} h'dy , \tag{4.12}$$

where y_1 and y_2 are the limits of integration. The result represented in the form of the last term of Eq. (4.12) may be obtained by the use of an ordinary planimeter.

Equation (4.11) provides, when combined with Eq. (4.10) and simplified, the equation of the path of the source of light, i.e.,

$$z = \frac{a}{n + y - 1} . \tag{4.13}$$

The constant n is provided here to prevent the denominator of (4.13) becoming zero.

Other modifications of the ideas outlined in the above paragraphs are possible. These methods were presented to explain some of the details of a projecting system. The advantages of such a system are that once the mechanical and optical components have been carefully built, the operator's only efforts will be developing the photographs and measuring the pertinent area with a planimeter.

APPENDIX

INSTRUCTION MANUAL

FOR

FIRST MOMENT OF AN AREA

INTRODUCTION

The method described in this manual for determination of the first moment of an area from a drawing of the cross section utilizes ordinary drafting procedures and a specially designed nomograph. The nomograph is shown in Fig. 1.

Procedure

The given area is subdivided into rectangles, triangles, sectors of a circle or annulus. The Work Line is selected - usually this is the center line of the object of which this area is the half cross section - and for each element of area a Base Line is selected according to the procedure outlined in the following paragraphs.

The first step is to take the line drawing and subdivide it into the rectangles, triangles, etc., by drawing dividing lines on the drawing, i.e., lines defining the edge of a rectangle where it joins a triangle or a sector of a circle, etc. Secondly, the Base Lines for each element of area are drawn as follows:

1. Rectangles (Fig. 2).—The Base Line is drawn through the lowest corner of the rectangle and parallel to the Work Line. For a rectangle having sides parallel to the Work Line the lower side becomes the Base Line.

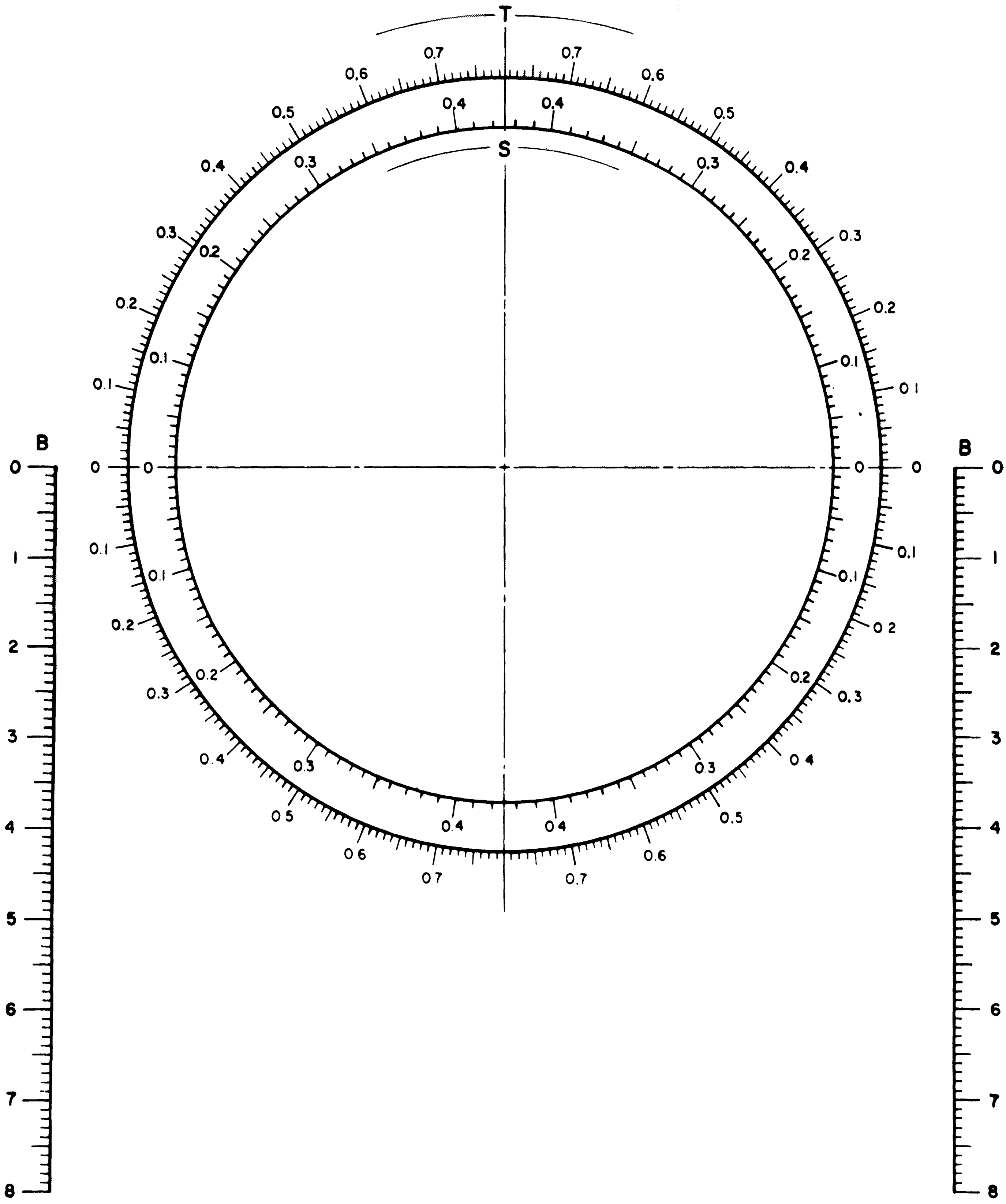
2. Triangles (Fig. 3).—The Base Line is drawn parallel to the Work Line through the vertex of the triangle that divides it into two triangles having a common side on the Base Line. Triangles having a side parallel to the Work Line will have that side as the Base Line.

3. Sectors of Circles or Annuli (Figs. 4 and 5).—All sectors to be used in the computation must have subtending angles of ninety degrees or less and lie in a single quadrant. All sectors shown in Fig. 5 satisfy these requirements. Many sectors will be encountered that do not qualify for these requirements and these must be further divided. Figure 4 presents such situations and shows the method of subdivision. Note that each of the resulting sectors is of the type to be found in Fig. 5.

After subdivision into the proper areas, each resulting element of area shall be given a number to identify it. In Figs. 2, 3, and 5 various dimensions have been indicated. These are the only dimensions required for the calculations. Three work sheets, one each for rectangles, triangles, and circular elements, have been prepared. Into these enter the dimensions indicated in Figs. 2, 3, and 5 and obtained by measurement from the given drawing. In many cases the drawing will be dimensioned and thus some of this information may be read directly from it. In any case, for rectangles and triangles an ordinary scale only is required, while for the circular elements the "Nomograph for Constants used in Determination of Moment of Area of Circular Sectors and Annulii" will provide all dimensions except the inside and outside radius, which may be obtained with a scale or from given dimensions.

This nonograph is placed on the drawing so that the center of the circular scale coincides with the center of the circular elements and the 0-0 (zero-zero) line coincides with the Base Line. The intercepts on the S and T scale will be read as outlined in Fig. 5 and the distance B from the Base Line to the Work Line is read from the B scale.

Having entered all the dimensions into the appropriate Work Sheets, the calculation may be started and carried out according to the order as given in the entry "Operation." The indicated totals are obtained and entered into the Summary Sheet where the indicated operations will yield the final result.



NOMOGRAPH FOR CONSTANTS USED IN DETERMINATION OF MOMENT OF AREA OF CIRCULAR SECTORS AND ANNULII.

FIG. 1

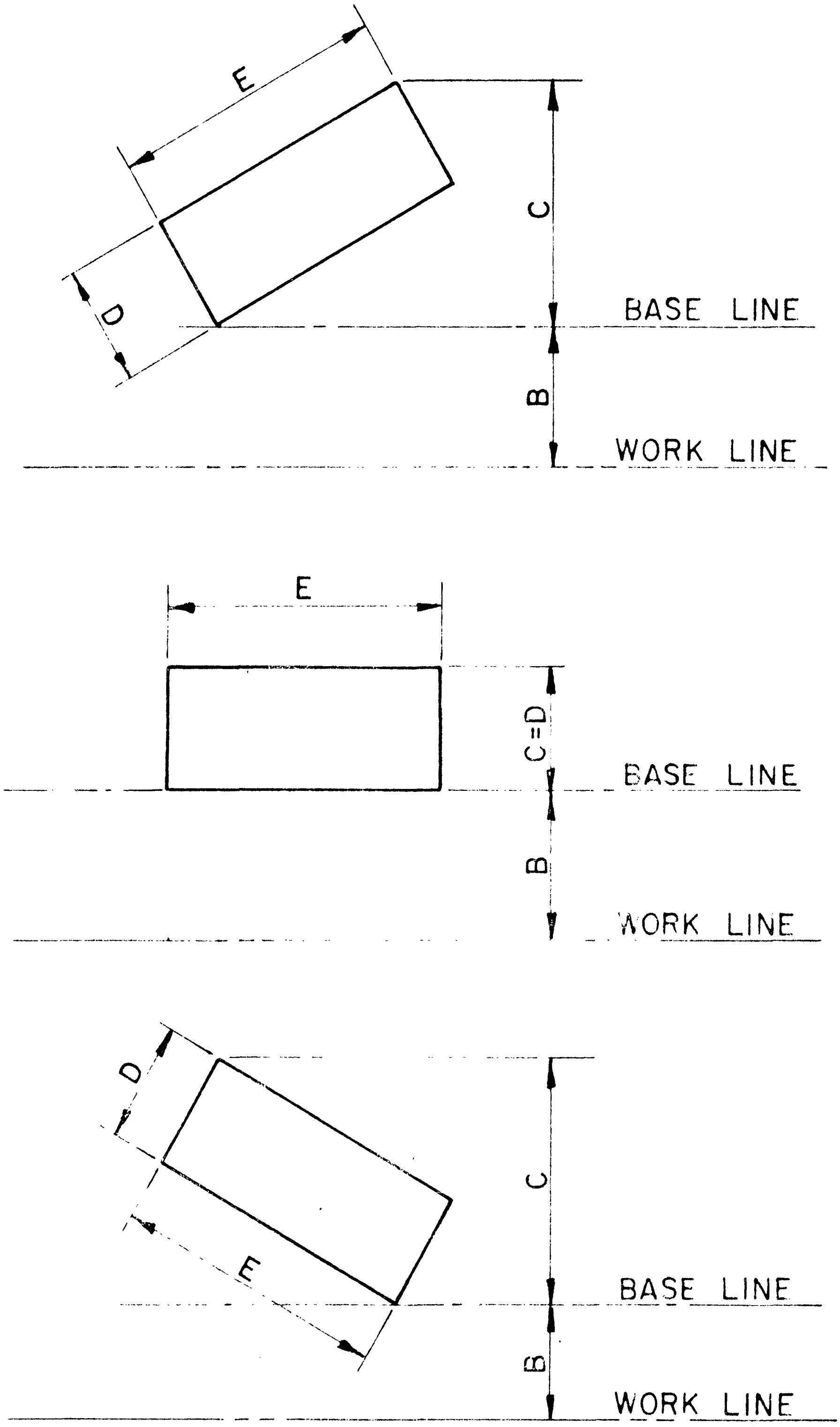


FIG. 2 DIMENSIONING FOR RECTANGLES

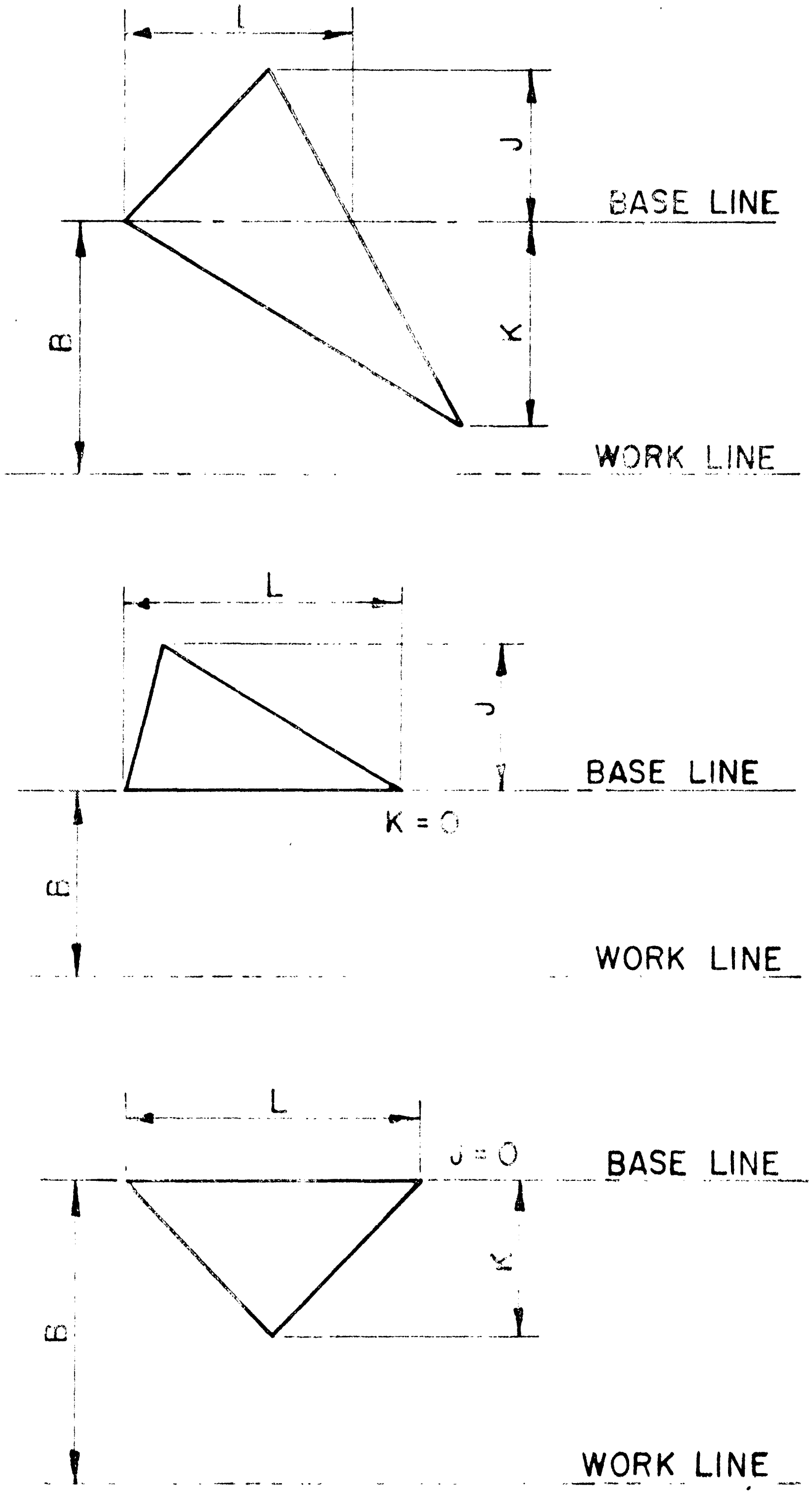


FIG. 3 DIMENSIONING FOR TRIANGLES

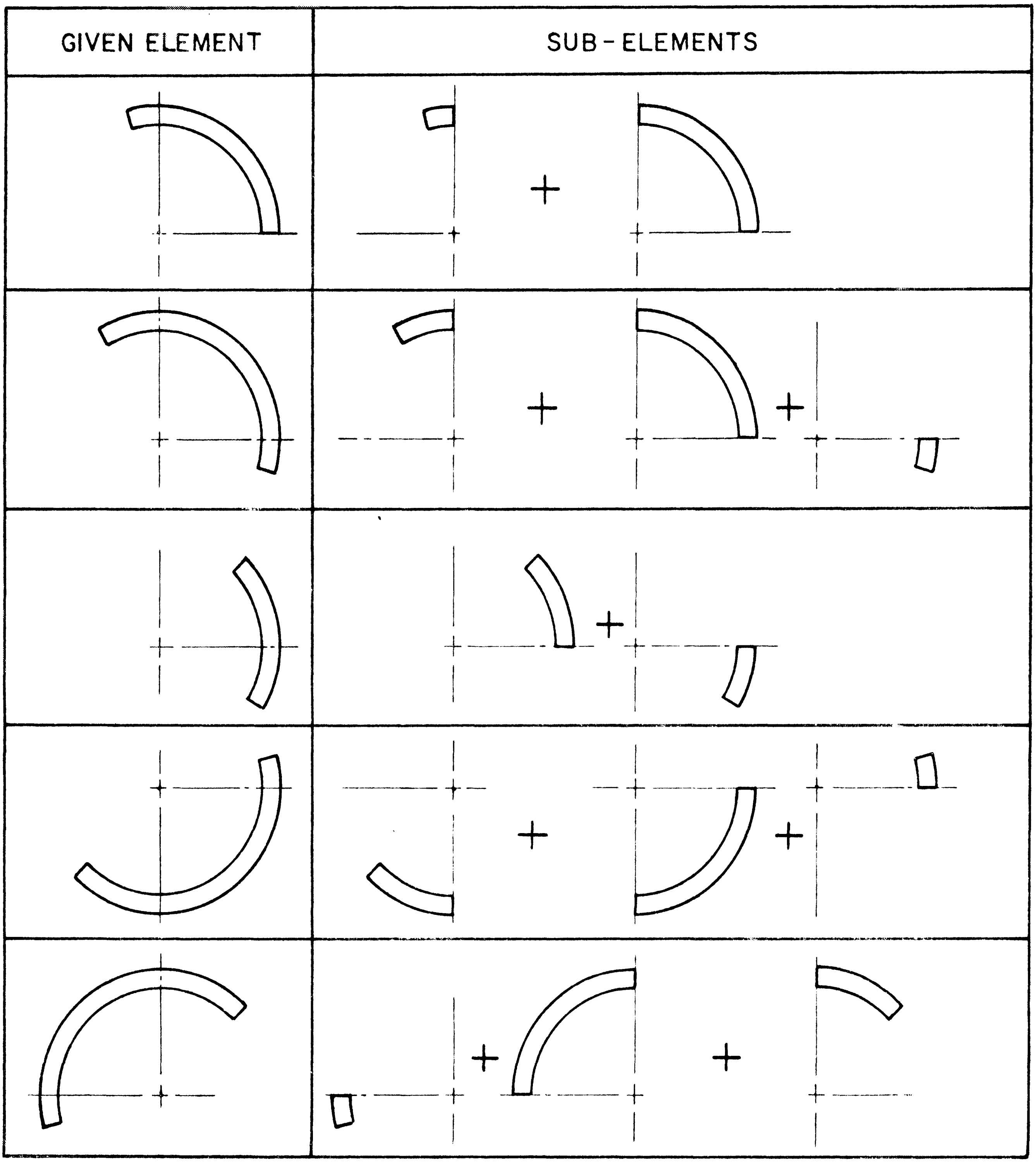


FIG 4. DETAILS OF DIVISION OF CIRCULAR ELEMENTS

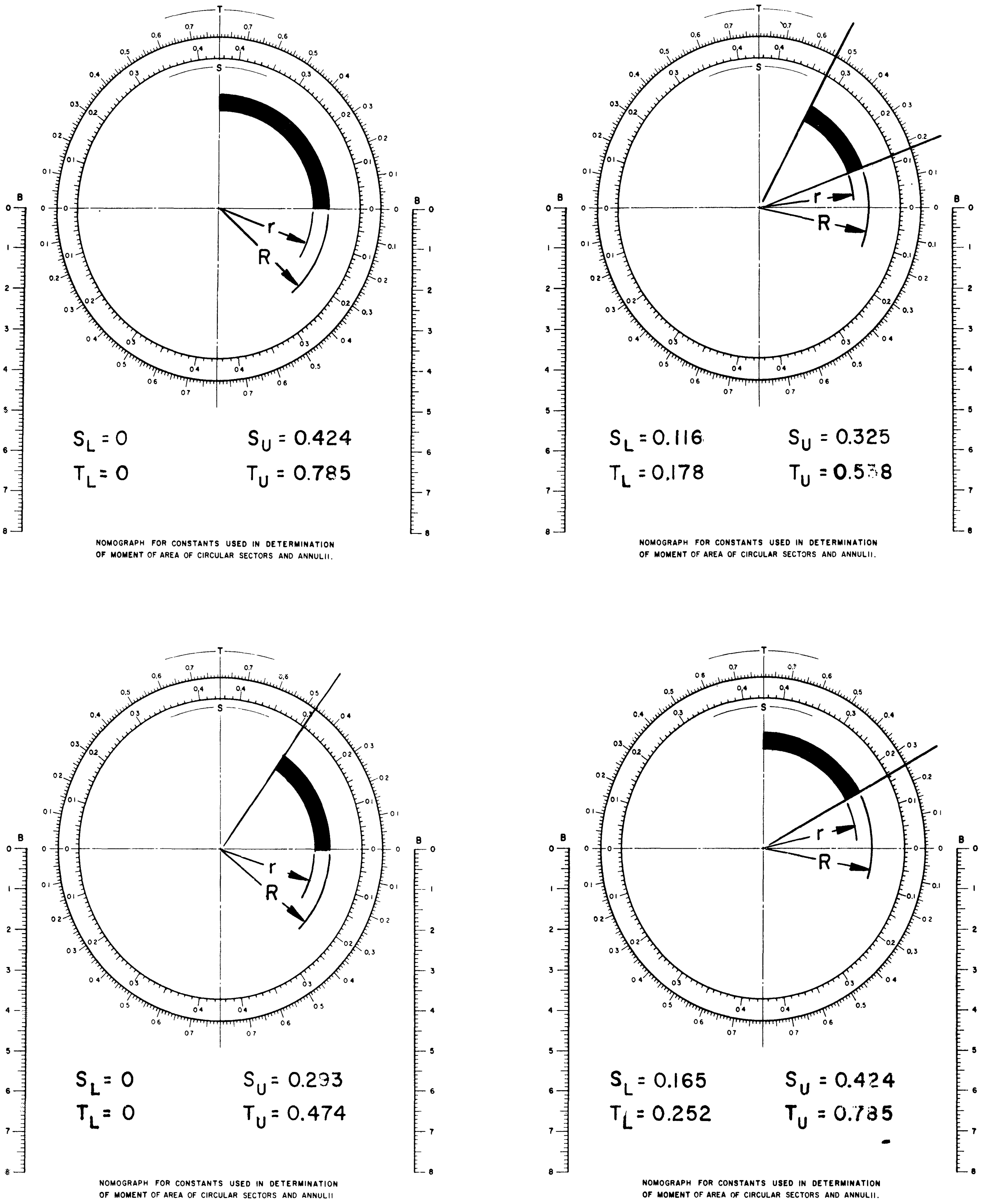


FIG. 5 DIMENSIONS FROM NOMOGRAPH FOR VARIOUS SECTOR ELEMENTS

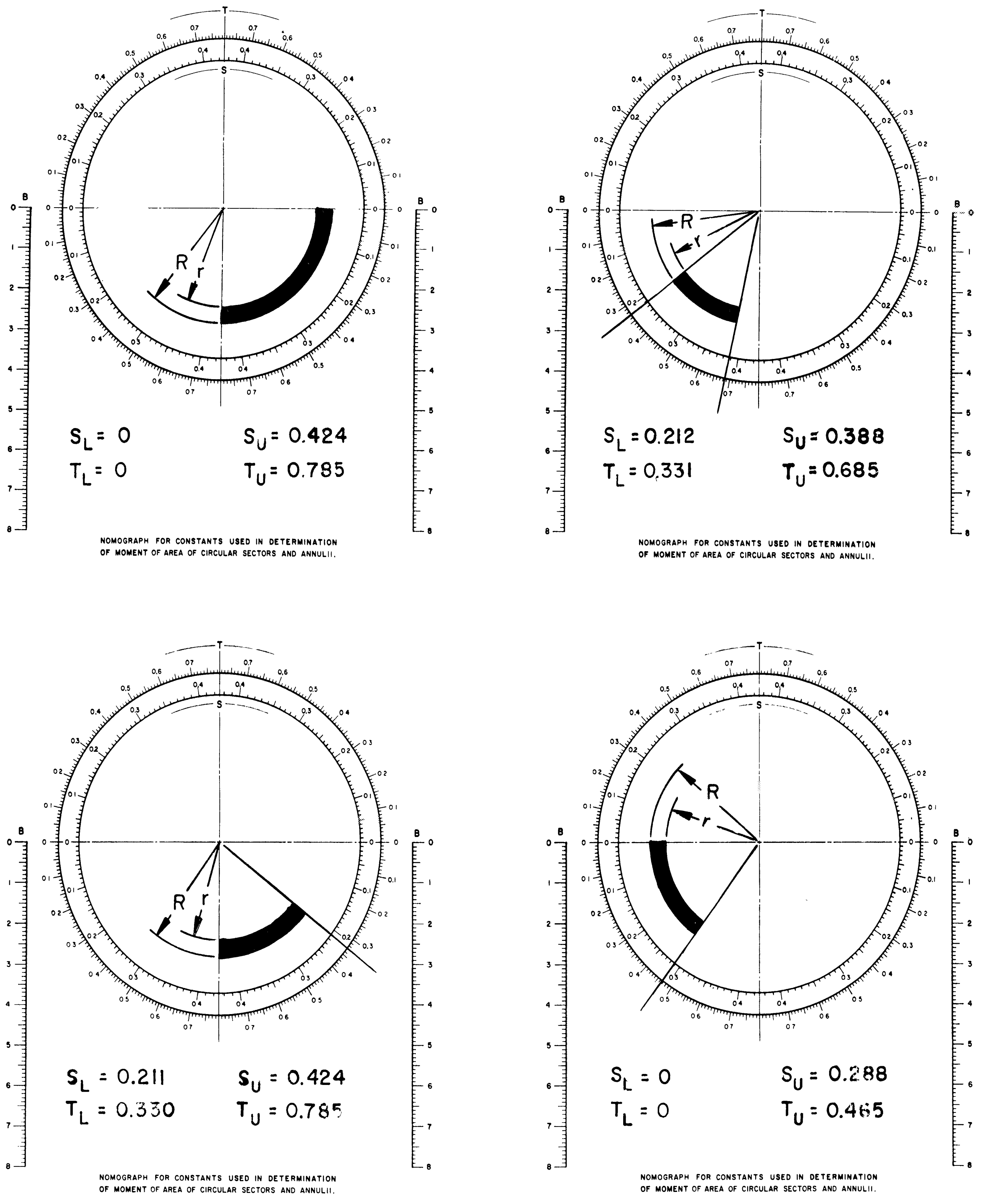


FIG. 5 DIMENSIONS FROM NOMOGRAPH FOR VARIOUS SECTOR ELEMENTS

Examples

1. Consider the area shown in Fig. 6. This can be divided as ten separate elements to qualify for the computation.

The Work Line and all the Base Lines are shown in Fig. 7. Data are entered into the work sheets.

2. Consider a typical area for the cross section of a wheel spider which is reproduced in Fig. 8. A complete subdivision of this area, with subdivision lines and Base Lines identified, is reproduced in Fig. 9. Data are entered into the work sheets and the computation carried to completion.

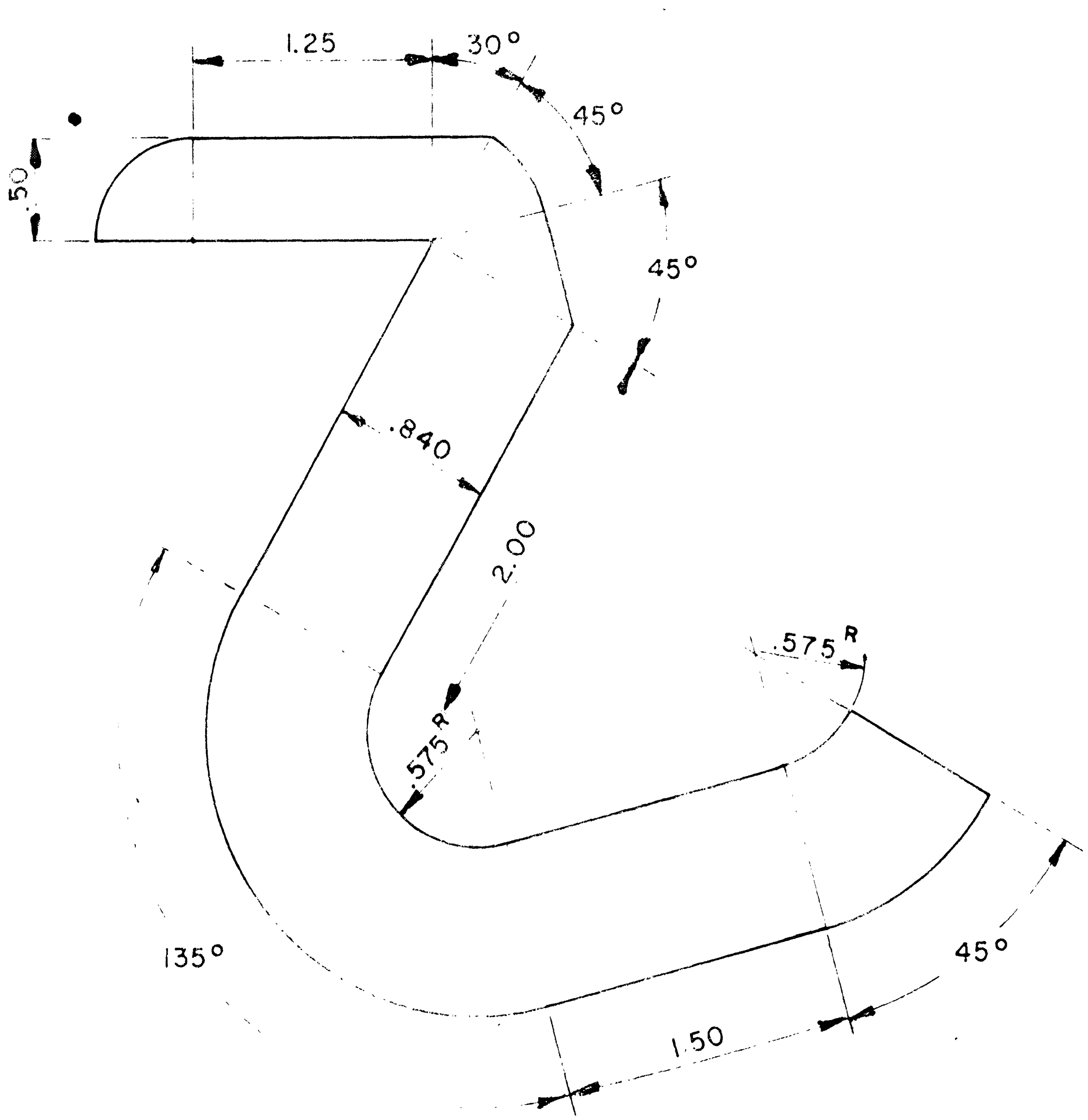


FIG. 6 SAMPLE AREA

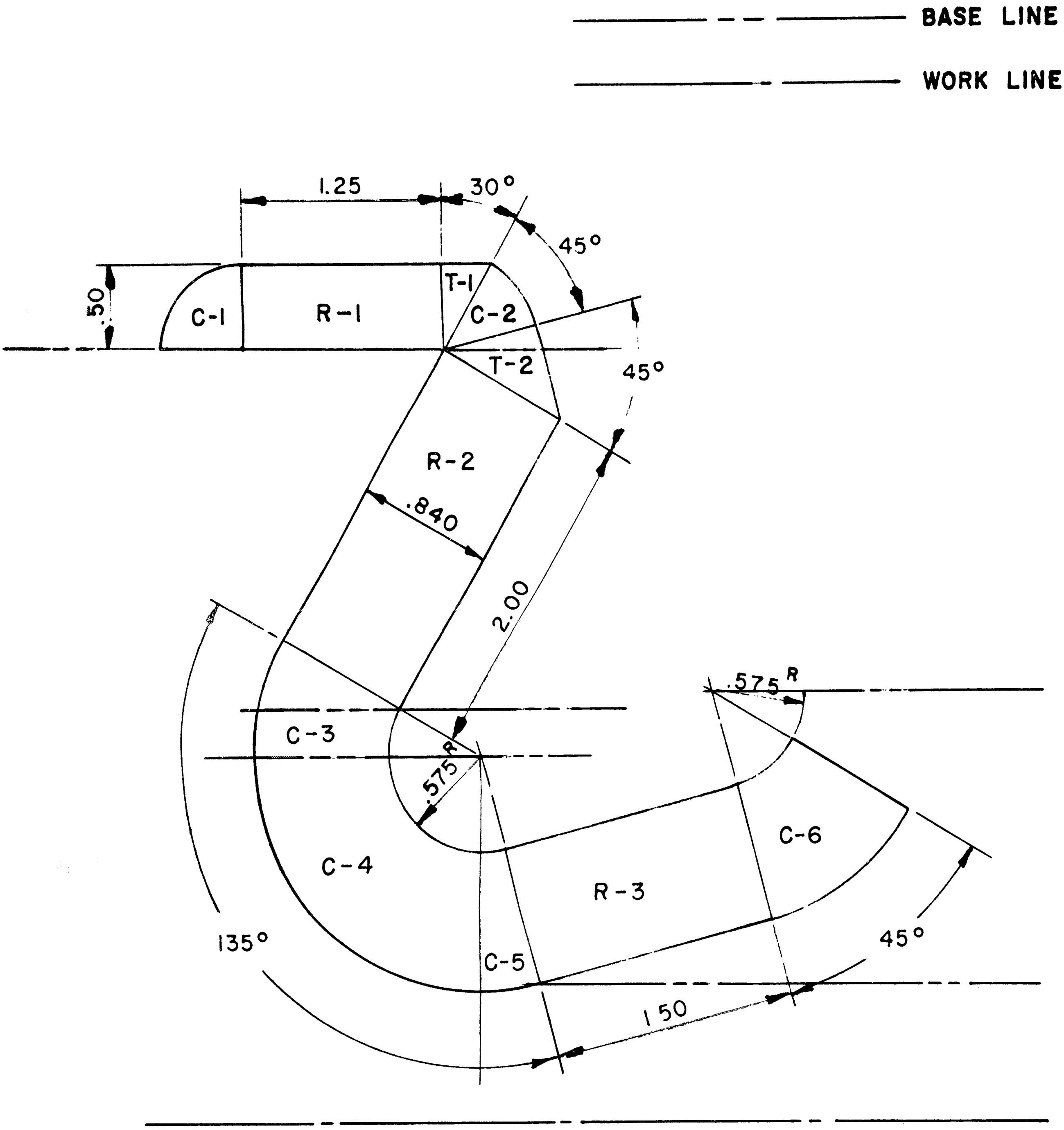


FIG. 7 SAMPLE AREA WITH WORK LINE AND BASE LINE SHOWN

SUMMARY SHEET*

ENTER TOTAL 1	10.83
ENTER HALF OF TOTAL 2	1.11
ENTER TOTAL 3	7.58
ENTER TOTAL 4	.20
ENTER TOTAL 5	-1.59
TOTAL 6	18.13

VOLUME = $2\pi\bar{y}A$ = 6.28 x 18.13 (ENTER TOTAL 6) = 114.0 $IN.^3$

*Refer to Fig. 7.

WORK SHEET FOR RECTANGULAR AREAS

DATA	B	C	D	E	C/2	DE	B+C/2	$\bar{y}$A
COLUMN	a	b	c	d	e	f	g	h
OPERATION	–	–	–	–	$\frac{1}{2}$(b)	(c)x(d)	(a)+(e)	(f)x(g)
ITEM								
R-1	4.65	.50	.50	1.25	.25	.625	4.90	3.06
R-2	2.47	2.17	.84	2.00	1.09	1.68	3.56	5.98
R-3	0.82	1.18	.84	1.50	.59	1.26	1.41	1.79

TOTAL I = 10.83

WORK SHEET FOR TRIANGULAR AREAS

DATA	B	J	K	L	J + K	J^2	K^2	$J^2 - K^2$	B(J + K)	$\frac{1}{3}(J^2 - K^2)$	P	$2\bar{y}A$
COLUMN	a	b	c	d	e	f	g	h	i	j	k	l
OPERATION	—	—	—	—	(c)+(b)	(b)x(b)	(c)x(c)	(f)-(g)	(a)x(e)	$\frac{1}{3}$(h)	(i)+(j)	(d)x(k)
ITEM												
T-1	5.15	0	.50	.30	.50	0	.25	-.25	2.57	-.083	2.49	.747
T-2	4.65	.14	.40	.60	.54	.019	.160	-.141	2.51	-.047	2.46	1.476

TOTAL 2 = 2.223

HALF OF TOTAL 2 = 1.111

WORK SHEET FOR AREAS OF CIRCULAR SECTORS AND ANNULII

DATA	B	R	r	S_U	T_U	S_L	T_L	R^2	R^3	r^2	r^3	(R^2-r^2)	T_U-T_L	$B(R^2-r^2)$	M	(R^3-r^3)	S_UT_U	S_LT_L	$S_UT_U-S_LT_L$	N	+N	-N
COLUMN	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	s	t	u	v	w
OPERATION	–	--	--	--	--	--	--	(b)x(b)	(b)x(h)	(c)x(c)	(c)x(j)	(h)-(j)	(e)-(g)	(a)x(l)	(m)x(n)	(i)-(k)	(d)x(e)	(f)x(g)	(q)-(s)	(p)x(t)	+(u)	-(u)
ITEM																						
C-1	4.65	.50	0	.424	.785	0	0	.25	.125	0	0	.25	.785	1.16	.910	.125	.333	0	.333	.042	.042	
C-2	4.65	.60	0	.326	.520	.080	.122	.36	.216	0	0	.36	.398	1.67	.665	.216	.187	.0098	.186	.040	.040	
C-3	2.25	1.40	.575	.173	.268	0	0	1.96	2.74	.330	.190	1.63	.768	3.67	.984	2.55	.046	0	.046	.117	.117	
C-4	2.25	1.40	.575	.424	.785	0	0	1.96	2.74	.330	.190	1.63	.785	3.67	2.88	2.55	.333	0	.333	.849		.849
C-5	2.25	1.40	.575	.424	.785	.378	.659	1.96	2.74	.330	.190	1.63	.126	3.67	.463	2.55	.333	.249	.084	.214		.214
C-6	2.60	1.40	.575	.380	.664	.174	.267	1.96	2.74	.330	.190	1.63	.397	4.23	1.68	2.55	.252	.047	.205	.523		.523

TOTAL 3 = 7.582

TOTAL 4 = .199

TOTAL 5 = -1.586

NOTE: USE COLUMN "V" FOR AREAS ABOVE WORKLINE
USE COLUMN "W" FOR AREAS BELOW WORKLINE

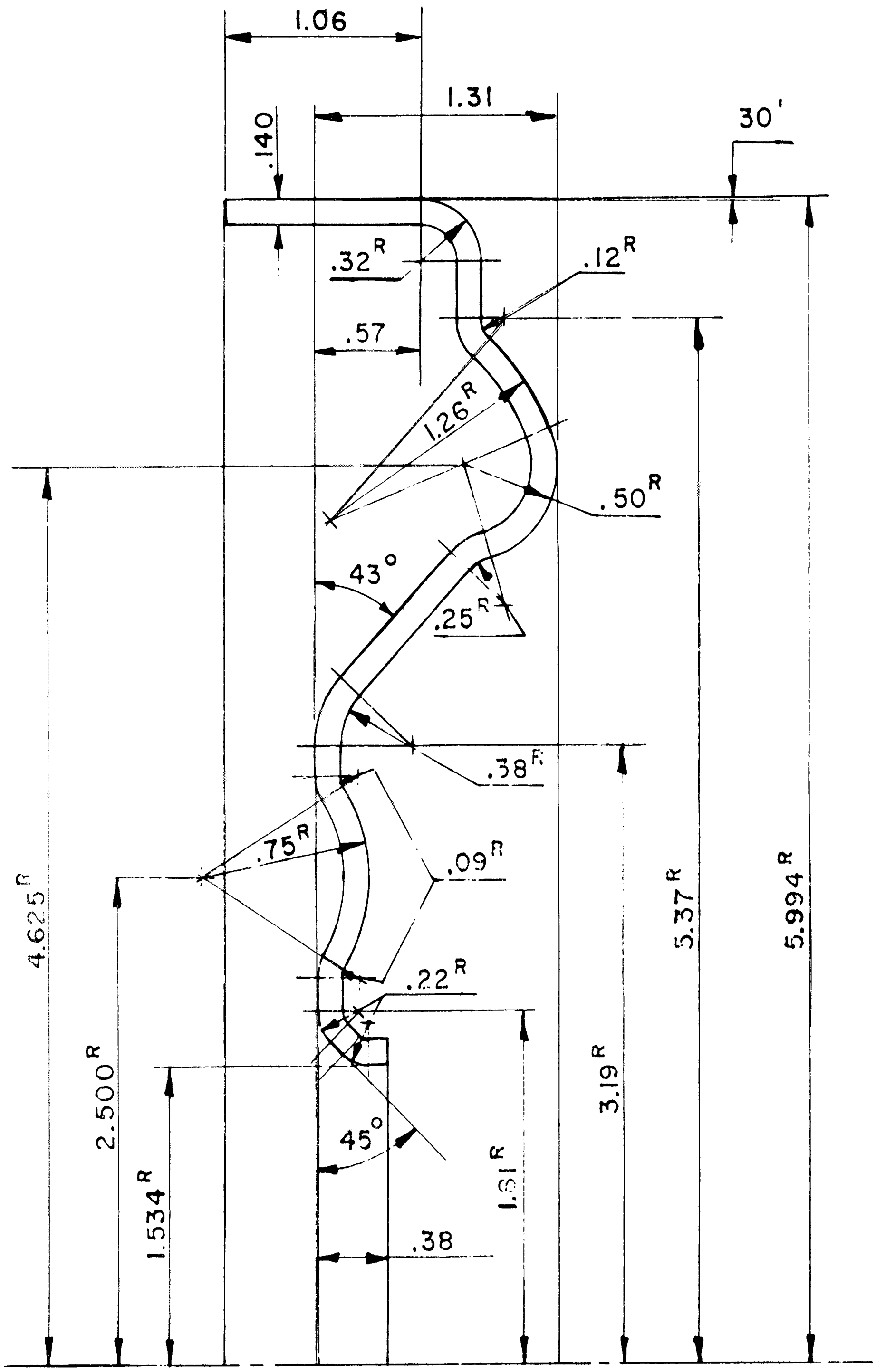


FIG. 8 TYPICAL WHEEL SPIDER CROSS SECTION

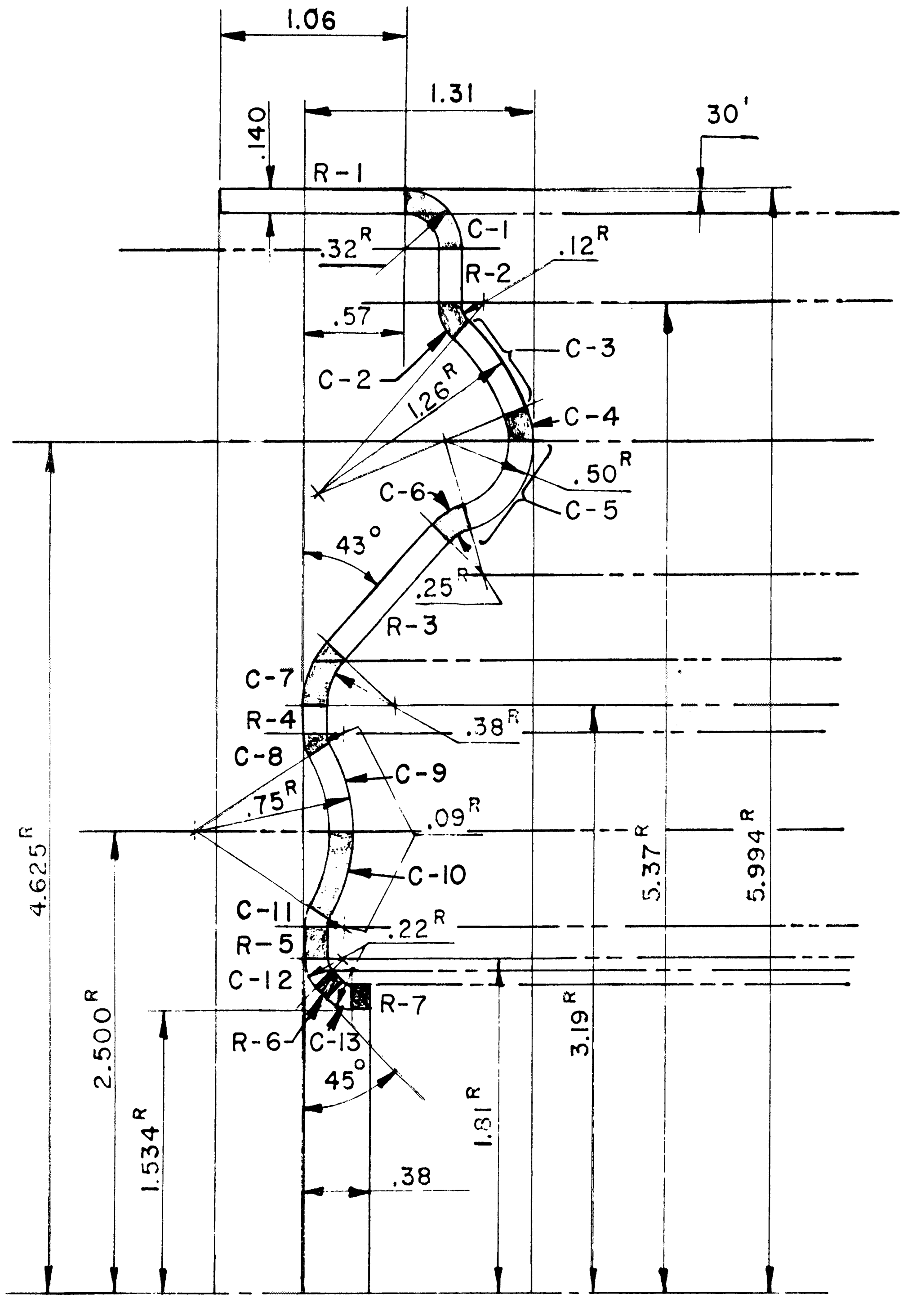


FIG. 9 WHEEL SPIDER CROSS SECTION SUBDIVIDED AND WITH WORKING LINE AND BASE LINES

SUMMARY SHEET*

ENTER TOTAL 1	1.740
ENTER HALF OF TOTAL 2	0
ENTER TOTAL 3	1.846
ENTER TOTAL 4	0.078
ENTER TOTAL 5	-0.026
TOTAL 6	3.638

VOLUME = $2\pi \bar{y} A$ = 6.28 x 3.638 (ENTER TOTAL 6) = 22.85 $IN.^3$

*Refer to Fig. 9.

WORK SHEET FOR RECTANGULAR AREAS

DATA	B	C	D	E	C/2	DE	B + C/2	$\bar{y}$A
COLUMN	a	b	c	d	e	f	g	h
OPERATION	—	—	—	—	$\frac{1}{2}$(b)	(c) x (d)	(a)+(e)	(f) x (g)
ITEM								
R-1	5.854	0.14	0.14	1.06	0.07	0.148	5.924	0.877
R-2	5.37	0.304	0.304	0.14	0.152	0.043	5.522	0.237
R-3	3.45	0.73	0.14	0.87	0.365	0.122	3.815	0.465
R-4	3.04	0.16	0.16	0.14	0.08	0.022	3.12	0.069
R-5	1.82	0.18	0.18	0.14	0.09	0.025	1.91	0.048
R-6	1.59	0.15	0.09	0.14	0.075	0.013	1.665	0.022
R-7	1.53	0.14	0.14	0.10	0.07	0.014	1.60	0.022

TOTAL I = 1.740

WORK SHEET FOR AREAS OF CIRCULAR SECTORS AND ANNULII

DATA	B	R	r	S_U	T_U	S_L	T_L	R^2	R^3	r^2	r^3	(R^2-r^2)	T_U-T_L	$B(R^2-r^2)$	M	(R^3-r^3)	$S_U T_U$	$S_L T_L$	$S_U T_U - S_L T_L$	N	+N	-N
COLUMN	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	s	t	u	v	w
OPERATION	—	—	—	—	—	—	—	(b)x(b)	(b)x(h)	(c)x(c)	(c)x(j)	(h)-(j)	(e)-(g)	(a)x(l)	(m)x(n)	(i)-(k)	(d)x(e)	(f)x(g)	(q)-(s)	(p)x(t)	+(u)	-(u)
ITEM																						
C-1	5.68	0.32	0.18	0.425	0.785	0	0	0.102	0.033	0.032	0.006	0.070	0.785	0.398	0.312	0.027	0.334	0	0.334	0.009	0.009	
C-2	5.37	0.26	0.12	0.248	0.396	0	0	0.068	0.018	0.014	0.002	0.054	0.396	0.290	0.115	0.016	0.112	0	0.112	0.002		0.002
C-3	4.34	1.26	1.12	0.262	0.418	0.126	0.190	1.588	2.001	1.254	1.404	0.334	0.228	1.450	0.331	0.597	0.110	0.024	0.086	0.051	0.051	
C-4	4.625	0.50	0.36	0.123	0.187	0	0	0.250	0.125	0.130	0.047	0.120	0.187	0.555	0.104	0.078	0.023	0	0.023	0.002	0.002	
C-5	4.625	0.50	0.36	0.374	0.650	0	0	0.250	0.125	0.130	0.047	0.120	0.650	0.555	0.361	0.078	0.243	0	0.243	0.019		0.019
C-6	3.90	0.39	0.25	0.372	0.641	0.240	0.376	0.152	0.059	0.063	0.016	0.089	0.265	0.347	0.092	0.043	0.238	0.090	0.148	0.006	0.006	
C-7	3.19	0.52	0.38	0.241	0.379	0	0	0.270	0.140	0.144	0.055	0.126	0.379	0.402	0.152	0.085	0.091	0	0.091	0.008	0.008	
C-8	3.03	0.23	0.09	0.184	0.285	0	0	0.053	0.012	0.008	0.001	0.045	0.285	0.136	0.039	0.011	0.052	0	0.052	0.001		0.001
C-9	2.50	0.75	0.61	0.185	0.285	0	0	0.563	0.422	0.372	0.227	0.191	0.285	0.478	0.136	0.195	0.053	0	0.053	0.001	0.001	
C-10	2.50	0.75	0.61	0.183	0.282	0	0	0.563	0.422	0.372	0.227	0.191	0.282	0.478	0.135	0.195	0.052	0	0.052	0.001		0.001
C-11	1.98	0.23	0.09	0.182	0.282	0	0	0.053	0.012	0.008	0.001	0.045	0.282	0.089	0.025	0.011	0.051	0	0.051	0.001	0.001	
C-12	1.81	0.22	0.13	0.250	0.395	0	0	0.048	0.011	0.017	0.002	0.031	0.395	0.056	0.022	0.009	0.099	0	0.099	0.001		0.001
C-13	1.76	0.22	0.13	0.425	0.785	0.250	0.394	0.048	0.011	0.017	0.002	0.031	0.391	0.055	0.022	0.009	0.334	0.099	0.235	0.002		0.002

TOTAL 3 = 1.846

TOTAL 4 = 0.078

TOTAL 5 = -0.026

NOTE: USE COLUMN "v" FOR AREAS ABOVE WORKLINE
USE COLUMN "w" FOR AREAS BELOW WORKLINE

COMMENTS REGARDING AN IDEAL METHOD FOR OBTAINING THE FIRST MOMENT OF AN AREA*

All the methods for obtaining the first moment of an area presented in this report have one feature in common: the aim to abbreviate the work involved. The accuracy of any of these methods depends, even at the outset, upon the accuracy of the line drawing of the cross section. Therefore, the most attention must be given to the drawing.

Consider Fig. 8, which is a reproduction of a typical drawing. A few minutes' study will reveal that dimensionally several of the elements of area are not completely specified. For example, see elements marked 4, 5, 6, 7, 11, 12, and 13 which lack one or more data.

What are the data required? For rectangles, the side dimensions and location dimensions will suffice. Location of the rectangle may, of course, be specified in many ways, e.g., locate any two corners. For the circular elements shown in Fig. 4, the following data are required: internal and external radii, horizontal and vertical location of center, and the angle the enclosing radii make, for example, with the horizontal axis (these can provide the subtended angle of the sector).

For a section to be consistent dimensionally, all this information must be available in one form or another. However, were it actually provided, then a calculating procedure could be designed that would yield any desired degree of accuracy. Such a procedure could be programmed for a desk calculator or a punch-card system for a high-speed digital computer.

The conclusion to be drawn is that a modification of drafting room procedure is necessary to provide sufficient data for a satisfactory calculating method and at the same time provide a dimensionally consistent drawing. It is further advisable to make the information on a drawing available in a specified form to facilitate the consequent procedures. A drawing has other purposes than to provide information for the calculation of the first moment of an area and might consequently be cluttered with dimensions which would needlessly confuse manufacturing efforts. However, an adjunct work drawing will be sufficient for the present purpose since the information is used infrequently once the results have been obtained.

Any other method must be expected to provide varied results unless the very same drawing and exactly similar procedures are utilized to obtain the first moment of an area.

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