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Final Report

RESEARCH STUDY PERTAINING TO LOW-LEVEL WIND STRUCTURE

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ABSTRACT

TASK A: STATISTICAL DESIGNS

A system of equations is developed to calculate the atmospheric effect on a rocket along its flight trajectory. It is found that the major effect is contributed by low-frequency turbulence components. The rocket-influence function is defined which acts as a filter on the turbulent eddies operating along the rocket trajectory.

The magnitudes of many terms in the rocket equations are estimated. It is found that the total of turbulent effects is almost constant for any trajectory, whereas the mean wind effect is a definite function of a given rocket trajectory. Through a Fourier transform of the influence function, a correlation function is determined that would be observed by an object moving along a mean wind path. The binormal dispersion of the rocket is calculated by transforming frequency spectra into space spectra according to Taylor's hypothesis.

A prediction scheme is developed to reduce dispersion whereby the rocket launcher will be adjusted to the effects of low-level winds. The scheme is adjudged inferior to a least-squares technique which, however, is much more difficult to evaluate. The mean wind is less applicable to the actual mechanics of a rocket firing system than is the whole wind. The quality of prediction schemes may be improved by the use of Taylor's hypothesis at levels below 150 ft.

The combination of a least-squares prediction technique with Taylor's hypothesis for heights of interest holds promise of development in future work. The hypothesis could be tested below 1050 ft by analysis of data obtained from instrumentation on and between two television towers within a mile of each other.

The structure of low-tropospheric winds is considered from momentum-transfer theories and by analysis of the surface layers and turbulence regimes. The atmospheric parameters that affect the dispersion of the rocket trajectory after launching are reviewed.

TASKS B, C, AND D

Detailed investigations of anemometry, data-reduction systems, and wind-tunnel studies appropriate to this research were deferred in the interest of developing the statistical design.

OBJECTIVE

The object of the research is to analyze low-level wind structure as it pertains to dynamic wind loading.

PURPOSE

The purpose of this investigation is to analyze the problem of the determination of the structure of the wind in the lower layers of the atmosphere as it pertains to dynamic wind loading of objects. The resources of mathematical physics, meteorology, aerodynamics, and statistics are employed in the analysis.

The research program may be considered as consisting of four tasks, as follows:

TASK A: To produce one or more statistical designs for field experiments which will reveal the wind-flow features that are significant for dynamic loading problems.

TASK B: To evaluate existing or possible wind-measuring instruments, such as anemometers, gustometers, and bivanes, to determine their suitability for field use in measuring the three-dimensional large-scale structure of the atmosphere.

TASK C: To recommend one or more systems to reduce the data obtained by the sensing elements of the instruments to usable form.

TASK D: To assess the suitability of the wind tunnel as a device for simulating eddy structure over specified terrain features.

No continuous analysis of wind instruments of suitable response characteristics was made during the project. It was felt that detailed studies of appropriate anemometry should be deferred until adequate statistical designs had been developed to serve as a guide for such studies. Anemometers are mentioned briefly, however, in the Recommendations.

The analysis of statistical designs has not advanced far enough to warrant proceeding with studies of data-reduction systems and with wind-tunnel studies.

PUBLICATIONS, LECTURES, REPORTS, AND CONFERENCES

There have been no publications or lectures during the reporting period.

A number of informal meetings have been held since the Third Progress Report to elaborate the ideas developed in the summer conference of 1956 and to prepare this material in a form suitable for the Final Report.

FACTUAL DATA

TASK A: STATISTICAL DESIGNS—ROCKET RESPONSE TO ATMOSPHERIC TURBULENCE

by Ben Davidson and Leo J. Tick

1. DEVELOPMENT OF ROCKET EQUATIONS

It is instructive to consider the dynamic wind loading problem in coordinate systems which are standard for atmospheric turbulence work and then to rotate these standard coordinates to one which is convenient for calculating wind effects on rockets. The standard meteorological system defines the x axis in the direction of the mean wind, $\overline{u}$,* the y axis in the cross-wind direction, and the z axis vertically upward. The turbulent velocities in these directions are u', v', and w', respectively. The results of almost all atmospheric turbulence work are expressed in terms of u', v', and w' as described above. For this reason, it is desirable to maintain these familiar velocity components.

The natural coordinate system for a straight line ideal trajectory prior to burn-out is one where the x'' axis is along the flight trajectory, the z'' axis is normal to the flight trajectory, and the y'' axis is in the binormal direction. The wind relationships between the two systems are:

$$\begin{aligned} u'' &= [(\overline{u} + u') \cos A - v' \sin A] \cos E - w' \sin E \; ; \\ v'' &= (\overline{u} + u') \sin A + v' \cos A \; ; \text{ and} \\ w'' &= [(\overline{u} + u') \cos A - v' \sin A] \sin E + w' \cos E \; , \end{aligned} \tag{1}$$

where u'', v'', w'' are components of the wind in the x'', y'', and z'' directions. The angle A represents the angle between the x and x'' axis and E is the elevation angle of the rocket trajectory. We are assuming that the rocket trajectory is a straight line in space for distances of interest to us. The quantity $\overline{u}$ may vary with height. We also assume that below 150 meters there is no direction shear, so that $\overline{V} = 0$, and we are dealing with two sets of axes fixed in space.

*The notion of a mean wind is, at best, an elusive one and most attempts to formalize it have been, in the main, unsuccessful. We leave it here as mostly an intuitive notion, to be dealt with in each individual case.

The u'' component of wind affects the rocket by slight variations in drag. The v'' and w'' components of wind, the binormal and normal wind components, affect the rocket motion by introducing variations in angle of attack. We shall assume that the angle-of-attack effects are more important than the drag effects.

Let us now calculate atmospheric effects on the object along its flight trajectory. The integrated wind along the flight trajectory to burn-out s_b is

$$\int_{s_o}^{s_b} v''\,ds = \sin A \int_{s_o}^{s_b} \bar{u}\,(s \sin E)ds + \sin A \int_{s_o}^{s_b} u'ds + \cos A \int_{s_o}^{s_b} v'ds$$

$$\int_{s_o}^{s_b} w''ds = \cos A \sin E \left[\int_{s_o}^{s_b} \bar{u}\,(s \sin E)ds + \int_{s_o}^{s_b} u'ds\right] \tag{2}$$

$$- \sin A \sin E \int_{s_o}^{s_b} v'ds + \cos E \int_{s_o}^{s_b} w'ds$$

Over a long series of trials the average value of the integrals of the turbulent velocities is zero, by the nature of the mean, so that the only nonzero quantities are the integrals of the mean velocity along the trajectory. For individual trials, integrals along the trajectory are not zero. If we express the turbulent velocities in a sine series,*

$$v''(s) = \Sigma\, a_k \sin\left(\frac{2\pi k s}{s_b - s_o} + \delta_k\right) \tag{3}$$

The integrated wind along the path is then

$$\int v''ds = \Sigma\, a_k \int_{s_o}^{s_b} \sin\left(\frac{2\pi k s}{s_b - s_o} + \delta_k\right) ds$$

$$= \Sigma\, \frac{a_k}{2}\,\frac{(s_b - s_o)}{\pi k} \sin \pi k \left\{\sin\left[\frac{\pi k(s_b + s_o)}{s_b - s_o} + \delta_k\right]\right\} \approx \Sigma\, \frac{a_k}{2}\,(s_b - s_o)\,\frac{\sin^2 \pi k}{\pi k} \tag{4}$$

for $\delta_k = 0$. Aside from the phase relationship implicit in the oscillating term in brackets we see that if the amplitudes are constant (constant energy spectrum), contributions to nonzero values of the turbulent integrals are inversely proportional to wave number. The major contribution comes from relatively low wave number (long wave length) components.

*This is not a very meaningful representation for the larger wave lengths, but we only use it to discuss the high-frequency behavior.

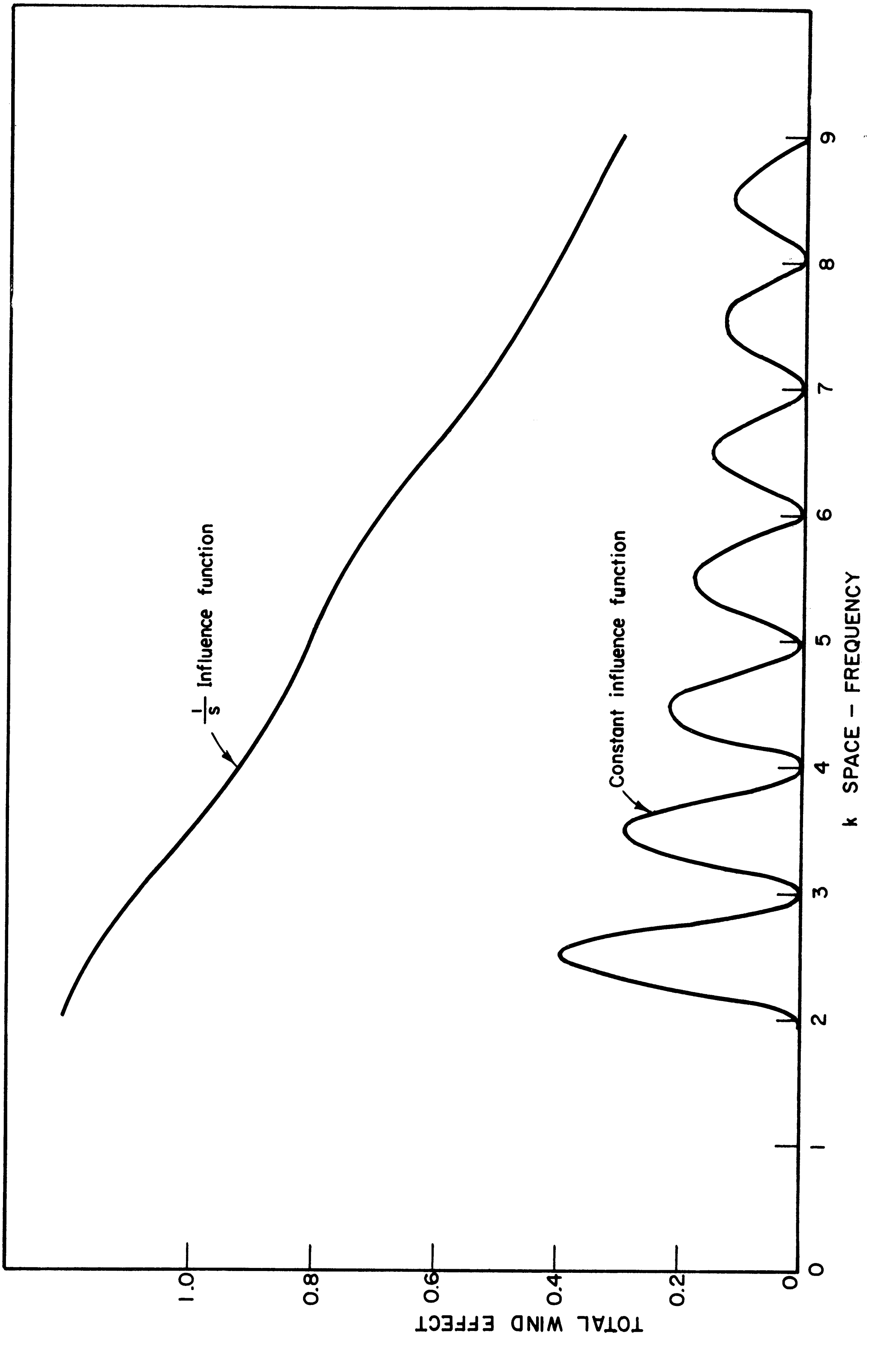

Fig. 1. Wind effects on a rocket for two influence functions—one constant and the other inversely proportional to distance from point of launching.

The previous result assumes that a turbulent wind impulse, no matter where applied, has an identical effect on the object. This will not usually be the case. The aerodynamics of the rocket and the rocket-velocity history along the trajectory will determine the relative effects of a gust at different points along the trajectory. Suppose we introduce a function $G(s)$ in Eq. (2) which expresses this dependence. $G(s)$ is usually called the "rocket influence function." We now have to integrate terms like

$$\int_{s_o}^{s_b} G(s)\, u'(s)ds = \sum \int_{s_o}^{s_b} G(s)\, a_k \sin\left(\frac{2\pi k s}{s_b - s_o} + \delta_k\right) ds \;. \tag{5}$$

Now the wave numbers which contribute to nonzero values of the integral are not so obvious. The function $G(s)$ acts like a filter. As an example we take $G(s)$ inversely proportional to s, which is saying that the turbulent wind effects are greatest in the beginning of the trajectory. The integral of Eq. (5) with $G(s) = 1/s$ is

$$\begin{aligned} \sum_k a_k \frac{(s_b \cdot s_o)}{2\pi} \Bigg\{ & \left[\mathrm{Si}\left(\frac{2\pi k}{s_b - s_o} s_b\right) - \mathrm{Si}\left(\frac{2\pi k}{s_b - s_o} s_o\right)\right] \cos \delta_k \\ & + \left[\mathrm{Ci}\left(\frac{2\pi k}{s_b - s_o} s_b\right) - \mathrm{Ci}\left(\frac{2\pi k}{s_b - s_o} s_o\right)\right] \sin \delta_k \Bigg\}, \end{aligned} \tag{6}$$

when Si and Ci are the so-called sine-integrals and cosine-integrals. For illustrative purposes we have plotted the integrals given by Eqs. (4) and (6) (assuming $\delta_k = 0$) in Fig. 1. It is evident that substantial contributions to nonzero values of the integral occur at higher wave numbers (shorter wave lengths) for the curve corresponding to Eq. (6) (influence function like 1/2) than for the curve corresponding to Eq. (4) (influence function like 1).

From this discussion it appears that one cannot fix the range of frequencies which should be measured or estimated without first knowing the character of the $G(s)$ function which will differ for various types of rockets.

We now calculate the mean square dispersion of the rocket at burn-out, assuming that the influence function $G(s)$ is symmetrical:

$$\begin{aligned} \overline{\theta_b^2} = \overline{\left[\int_{s_o}^{s_b} G(s) v'' ds\right]^2} &= \sin^2 A \, \overline{\left[\int_{s_o}^{s_b} G(s)\bar{u}(s \sin E) ds\right]^2} \\ &+ \sin^2 A \, \overline{\left[\int_{s_o}^{s_b} G(s) u' ds\right]^2} + \cos^2 A \, \overline{\left[\int_{s_o}^{s_b} G(s) v' ds\right]^2} \\ &+ 2 \sin A \cos A \, \overline{\int_{s_o}^{s_b} G(s) u' ds \int_{s_o}^{s_b} G(s) v' ds} \end{aligned} \tag{7a}$$

$$\overline{\theta_n^2} = \overline{\left[\int_{s_o}^{s_b} G(s)w''ds\right]^2} = \cos^2 A \sin^2 E \overline{\left[\int_{s_o}^{s_b} G(s)\overline{u}(s \sin E)ds\right]^2}$$

$$+ \cos^2 A \sin^2 E \overline{\left[\int_{s_o}^{s_b} G(s)u'ds\right]^2} + \sin^2 A \sin^2 E \overline{\left[\int_{s_o}^{s_b} G(s)w'ds\right]^2}$$

$$+ 2 \cos A \sin E \cos E \overline{\int_{s_o}^{s_b} G(s)w'ds \int_{s_o}^{s_b} G(s)v'ds}$$

$$- 2 \cos A \sin A \sin^2 E \overline{\int_{s_o}^{s_b} G(s)u'ds \int_{s_o}^{s_b} G(s)v'ds}$$

$$- 2 \sin A \sin E \cos E \overline{\int_{s_o}^{s_b} G(s)w'ds \int_{s_o}^{s_b} G(s)u'ds} \quad . \tag{7a}$$

Here θ represents the angular deviation from the windless trajectory of the rocket at burn-out in the binormal (θ_b) and normal (θ_n) directions, respectively.

	$\overline{\theta_b^2}$	$\overline{\theta_n^2}$	
$\left[\int_{s_o}^{s_b} G(s)\overline{u}(s \sin E)ds\right]^2$	$\sin^2 A$	$\cos^2 A \sin^2 E$	
$\iint G(s)\ G(s')\ R_{11}[\vec{x}(s)\ \vec{x}(s')]$	$\sin^2 A$	$\cos^2 A \sin^2 E$	
$\iint G(s)\ G(s')\ R_{22}[\vec{x}(s)\ \vec{x}(s')]$	$\cos^2 A$	$\sin^2 A \sin^2 E$	
$\iint G(s)\ G(s')\ R_{33}[\vec{x}(s)\ \vec{x}(s')]$		$\cos^2 E$	(7b)
$\iint G(s)\ G(s')\ R_{12}[\vec{x}(s)\ \vec{x}(s')]$	$2 \sin A \cos A$	$-2 \cos A \sin A \sin^2 E$	
$\iint G(s)\ G(s')\ R_{13}[\vec{x}(s)\ \vec{x}(s')]$		$-2 \sin A \sin E \cos E$	
$\iint G(s)\ G(s')\ R_{23}[\vec{x}(s)\ \vec{x}(s')]$		$2 \cos A \sin E \cos E$	

where the R terms represent the correlation of eddy wind components at two points s and s' along the line defined by the angles A and E. The subscripts 1,2,3 refer to the r, v, and w components of the turbulent wind, while $\overline{u}$ is the mean wind assumed to be a function of z (or s sin E) only.

With respect to a coordinate system fixed to the earth, the rocket trajectory is arbitrary, all values of $A(0 \leq A < 2\pi)$ and $E(0 \leq A < \pi)$ are permissible; the influence function $G(s)$ is a function of the aerodynamics of the particular rocket under consideration. A complete solution of the problem thus involves knowledge of appropriate correlation and influence functions along an infinite number of lines within the atmospheric boundary layer. Such an observational task is clearly impossible. We therefore turn to methods of characterizing and modeling the atmosphere.

2. ESTIMATION OF TERMS IN RESPONSE EQUATIONS

We shall attempt to do this by first summarizing all that is presently known concerning the structure of atmospheric mean and turbulent wind fields. Using this information coupled with an increasing set of minimum assumptions, we extract information as to the relative importance of the various terms in Eq. (7b) and suggest possible methods of improving the present rocket-launch system. It is convenient at this point to introduce a realistic influence function, and we take as a typical wind influence function that given by Hunter, Shef, and Black.[1] The dotted curve of Fig. 2 is a graph of this function, and for the moment all that concerns us is that $G(s)$ is a decreasing monotonic function of distance along the trajectory. We now cite some observa-

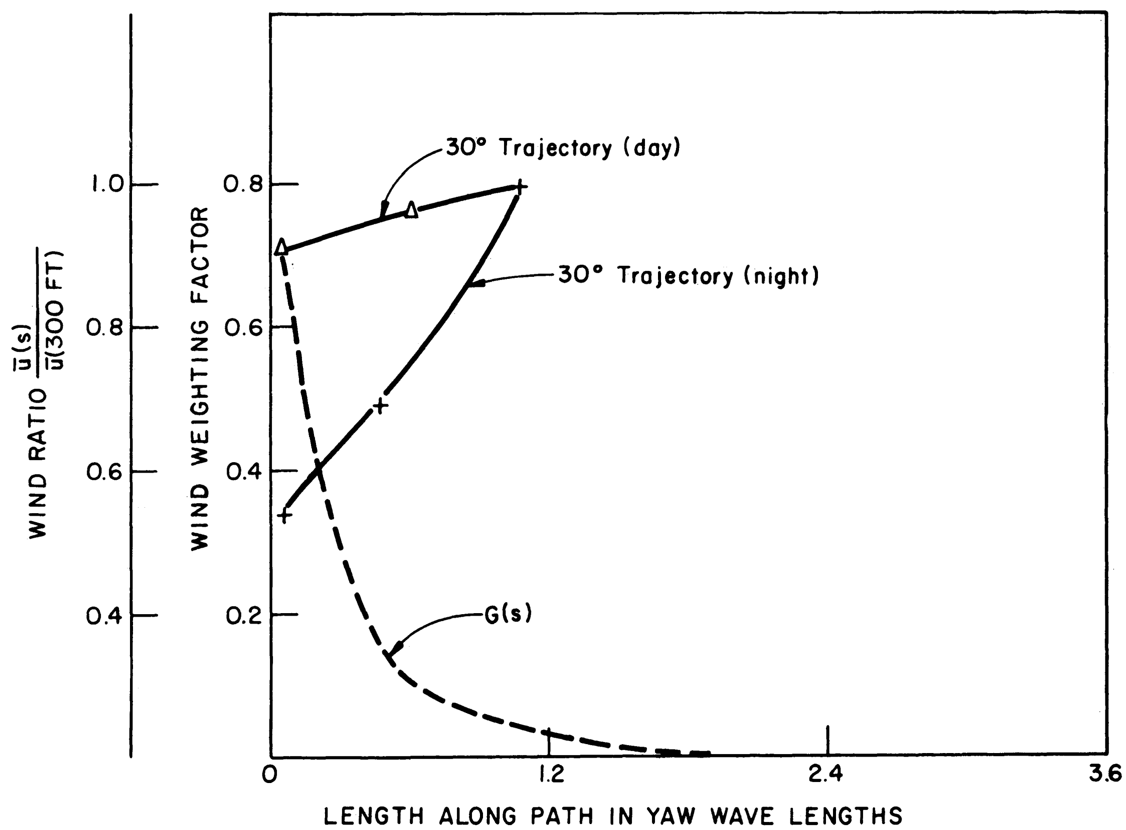

Fig. 2. Broken line: typical wind influence function as given by Hunter, Shef, and Black.[1] Solid lines: typical mean wind-speed ratios for day and night conditions at O'Neill, Neb. (For further details on the latter, see the discussion near the end of Section 4.)

tional evidence concerning the intensity of turbulence and the correlation structure of the turbulence at a fixed point. These data represent the results of observations at Brookhaven, Long Island, N. Y., Buchanan, N. Y., Round Hill, Mass., and O'Neill, Neb.[2,3,4] It is important to note that the values of gustiness vary with the averaging time used to define the mean wind; the appropriate averaging time for most of the data cited below is one hour.

TABLE I

TYPICAL VALUES OF GUSTINESS RATIOS AND ZERO SEPARATION CORRELATIONS FOR TYPICAL LAPSE CONDITIONS

Condition	$(u'^2)^{1/2}/\bar{u}$	$(v'^2)^{1/2}/\bar{u}$	$(w'^2)^{1/2}/\bar{u}$	$R_{12}(0)$	$R_{23}(0)$	$R_{13}(0)$
Lapse	0.2 - 0.5	0.2 - 0.5	0.07 - 0.3	0	0	$(0.3 \text{ to } 0.6)\sigma_w\sigma_u$

The lower values of the gustiness ratios are associated with relatively smooth terrain (O'Neill, Neb.), while the higher value has been observed at Buchanan, N. Y., under strong wind flow, over a ridge about 1.5 miles upwind. It is worth noting that under these conditions an anemometer, located 1/2 mile downwind of the ridge, registers almost pure turbulence; it is impossible to define a mean wind from the trace.

We now assume that the rocket is aimed to hit a target under zero wind conditions, i.e., that no wind correction is made. Not correcting for the mean wind introduces a systematic bias in the results so that the center of burst will not be around the target. Since $G(s)$ is a monotonic decreasing function of s, the integrals involving the hourly mean wind are

$$\leq \left[G(s_o) \int_{s_o}^{s_b} \bar{u}(s \sin E)ds\right]^2 \leq G^2(s_o)\bar{u}^2 (s_b - s_o)^2 .$$

In a similar fashion, the integrals involving identical subscript correlation functions must be

$$\leq \sigma_i^2 [G(s_o)]^2 .$$

Using the observed $R_{ij}(0)$ correlations, the integrals involving cross correlation functions are

$$\ll \sigma_i\sigma_j [G(s_o)]^2 (s_b - s_o)^2 .$$

The reason for the double inequality in the last relationship is because of the zero contribution from $R_{ij}(0)$ where $G(s)$ is largest. It is extremely unreasonable to expect $R_{ij}(s)$ terms to reach high values for $0 < s < s_b$, and even if they did, the character of the $G(s)$ function would minimize them.

We now compare

$$[\overline{u}(s_b)]^2 \quad \text{with} \quad \sigma_i^2 > \sigma_i\sigma_j \quad \text{or}$$

$$1 \quad \text{with} \quad \frac{\sigma_i^2}{\overline{u}^2} > \frac{\sigma_{ij}}{\sigma^2} \quad .$$

Using the values in Table I, we are comparing 1 with numbers ranging from 0.04 to 0.25. If for the moment we disregard the trigonometric terms in Eq. (7b), the above indicates that if no mean wind correction is made, the squared error due to mean wind speed is about 4 to 25 times that due to turbulence. The bias error due to mean wind (a more realistic number) is about 2 to 5 times that due to the standard deviation of the turbulent wind.

If we consider the trigonometric terms in Eq. (7b), we emerge with quite different results, depending on the angles A and E. For example, the binormal comparisons would then involve

$$\overline{u} \quad \text{with} \quad R_{11} + R_{22} > R_{12}$$

$$\sin^2 A \quad \text{with} \quad \frac{\sigma^2}{\overline{u}^2} > \frac{2 \sin A \cos A}{\overline{u}^2} \quad .$$

If the rocket is aimed in the direction of the mean wind, then the only contribution to the scatter in the binormal direction is that due to turbulent wind components. For the normal direction (assuming $A = 0$), we have

$$\begin{array}{cccc} \overline{u} & R_{11} & R_{33} & R_{23} \\ \sin^2 E & \sin^2 E \dfrac{\sigma_u^2}{\overline{u}^2} & \dfrac{\cos^2 E \, \sigma_w^2}{\overline{u}^2} & 2 \sin E \cos E \dfrac{\sigma_v\sigma_w}{\overline{u}^2} \end{array} \quad .$$

If we take a typical value of E, say 30°, then we find for the normal component

$$\begin{array}{cccc} \overline{u} & R_{11} & R_{33} & R_{23} \\ 0.25 & 0.25\,[0.04 + 0.25] \; + & 0.75 \; + & 0.86 \end{array} \quad .$$

The leading term in the squared error in the normal direction is the mean wind term. Comparing the normal and binormal scatter for this example, we find the squared error due to the mean wind in the normal direction is about 1 to 6 times that due to turbulence in the binormal direction. The realistic standard error would range from 1 to about 2.5. For elevation angles of 20° or less, and an azimuth angle of 0, the contribution of turbulence to the scatter in the binormal direction will exceed the error due to mean wind in the normal direction.

From the foregoing, it is apparent that generalizations about the error due to the relative importance of mean and turbulent wind are difficult to make without considering the path of the rocket with respect to the vertical and mean wind axis. Perhaps a more realistic criterion would be to accept a certain wind error and then to re-examine the problem in view of the acceptable wind error. For example, despite the fact that turbulent wind effects are of the same order as mean wind effects for the special trajectory $A = 0$, $E \neq 20^\circ$, it may be that the mean wind effect is itself negligible for this trajectory. In other words, Eq. (7b) implies that the total turbulent effects are almost constant for any trajectory, while the mean wind effect is very much a function of specified rocket trajectory.

3. IMPROVED ESTIMATES FOR HOMOGENEOUS FIELD

We may get a better estimate of the dispersion due to turbulence alone by introducing the notion of homogeneous and isotropic turbulence. In an homogeneous turbulent field, the mean quantities characterizing the field are independent of translations. It follows that the correlation tensors are functions only of the vector separation between points, and the spatial gradients of all mean quantities specifying the turbulence are zero. For example:

$$\begin{aligned} R_{ij}(\vec{r}) &= u_i\,(\vec{x} + \vec{r})\, u_j(\vec{x}) \\ \nabla\, \overline{u_i^2} &= 0 \quad . \end{aligned} \tag{8}$$

In an isotropic field the mean quantities characterizing the turbulence are independent of all rigid body rotations. It follows that (using the equation of continuity),

$$R_{ij}(\vec{r}) = f(r)\,[r_i r_j + \delta_{ij}] \qquad \delta_{ij} = 1 \quad i = j; \quad = 0 \quad i \neq j \tag{9}$$

and that

$$\overline{u'^2} = \overline{v'^2} = \overline{w'^2} \quad .$$

Now atmospheric turbulence is neither isotropic nor even homogeneous. In some restricted aspects, however, the atmosphere does behave like an homogeneous, isotropic medium, and any practical solution of the total problem must take advantage of the homogeneous or isotropic features of the atmosphere. We summarize now some of the knowledge which has been accumulated concerning the eddy structure in the lowest 100 m of the atmosphere. Most of these data are analyses in the frequency domain of three-component wind data obtained from meteorological towers located at Brookhaven, Long Island, N. Y., O'Neill, Neb., and at Round Hill, Mass.

Summarizing the data available at the present time and speaking in an average sense only, the following seems to represent the facts roughly:

A. Under unstable conditions:

1. $\overline{u'^2} \approx \overline{v'^2}$
2. $S_z(u') \approx S_z(v')$
3. $\frac{\partial}{\partial z} S_z(u') \approx \frac{\partial}{\partial z} S_z(v') \approx 0$
4. $\overline{[u'v'(0)]} = \overline{[v'w'(0)]} = 0$
5. $\overline{u'^2} \neq \overline{w'^2}$
6. $\overline{(u'w')}_z \neq 0$
7. $\frac{\partial}{\partial z} S_z(w') \neq 0$

B. Under stable conditions:

1. $\overline{u'^2} \approx \overline{4v'^2}$
2. Not enough spectra are available to warrant further statements.

Here the S_z is the Eulerian time spectrum, calculated from the readings of a fixed anemometer located at height z. During unstable conditions (clear sunlight hours), the atmosphere exhibits some features of isotropy (A1-4) or homogeneity (A1-5) in the horizontal components, but is clearly neither isotropic or homogeneous with respect to the vertical velocity (A5-7). If we refer back to Eq. (7), we see that the vertical velocity does not enter into the expression for the binormal at all, but does enter into the expression for the wind that acts normal to the trajectory. Moreover, we have already shown that terms involving cross correlations are likely to be quite a bit smaller than the terms involving identical subscripts in the correlation tensor. The conclusion clearly is that we may regard the turbulence entering into the binormal equation as at least homogeneous in the x and y directions while A3 implies some sort of homogeneity in the z direction for horizontal velocity components. Assuming that homogeneity for the horizontal velocity components exists in the x, y, and z directions, we may then write the correlation functions in the binormal component of Eq. (7b) as

$$R_{ij}(s) = R_{ij}\left\{[x(s) - x(s')]\,[y(s) - y(s')]\,[z(s) - z(s')]\right\} .$$

In this form the Fourier Transform of R_{ij} is the spectrum, and the binormal component of Eq. (7b) can be written as

$$\overline{\theta_b^2} = \iiint \sin^2 A\, \phi_{11} + \cos^2 A\, \phi_{22} + 2 \sin A \cos A\, \phi_{12}\, |\Gamma(k_1,k_2,k_3)|^2\, dk_1\, dk_2\, dk_3$$

$$+ \sin^2 A \left[\int_{s_o}^{s_b} \overline{u}\, (s \sin E) ds\right]^2 , \tag{10}$$

where ϕ_{ij} is the space-spectrum tensor of the turbulent velocity components, and Γ is the Fourier Transform of the influence function $G(s)$. For a rocket fired in the direction of the mean wind

$$\overline{\theta_b^2} = \iiint \phi_{22}\, |\Gamma(k_1,0,0)|^2\, dk_1 dk_2 dk_3$$

$$= \int \phi^*(k_1)\, |\Gamma(k_1,0,0)|^2\, dk_1 , \tag{11}$$

where

$$\phi^* = \iint \phi_{22}\, (k_1)\, dk_2\, dk_3$$

is the Fourier Transform of the one-dimensional lateral correlation function along a line parallel to the mean wind. This is the correlation function that would be observed by an aircraft flying along a mean wind path.

As was pointed out previously, the meteorological spectra available at the present time are in the frequency domain and are computed from the time history of the wind going by a fixed point. It is customary to employ Taylor's hypothesis to transform the time spectra into space spectra. The hypothesis states that

$$x = \overline{u}t \tag{12}$$

$$k_1\overline{u} = \omega ,$$

where $\overline{u}$ is the mean wind, k_1 the wave number parallel to the mean wind, and ω the time frequency. A frequency spectrum can be transformed into a space spectrum by dividing the abscissa and multiplying the ordinate by $\overline{u}$.

Considerable activity at Pennsylvania State University, Cornell Aeronautical Laboratory, New York University, Brookhaven National Laboratory, and Massachusetts Institute of Technology is now going on to determine the validity of Taylor's hypothesis for atmospheric levels of turbulence. The one report[5] which is available does not contradict the hypothesis.

We now transform a frequency spectrum into a space spectrum by use of Taylor's hypothesis. Using the Fourier Transform of the influence function given in Ref. 1, we calculate the binormal dispersion of the rocket. The relatively high wave-number portion of the spectrum was take from Ref. 2 while the low wave-number portion of the spectrum was obtained from Ref. 3.

We have plotted the space-equivalent spectrum obtained through use of Taylor's hypothesis in Fig. 3 for a typical wind speed of 5 m/sec together with the square of the admittance function given as a function of rocket-yaw wave length, which we assume to be 600 ft. The dispersion due to turbulence is the integral of the product of the two functions.

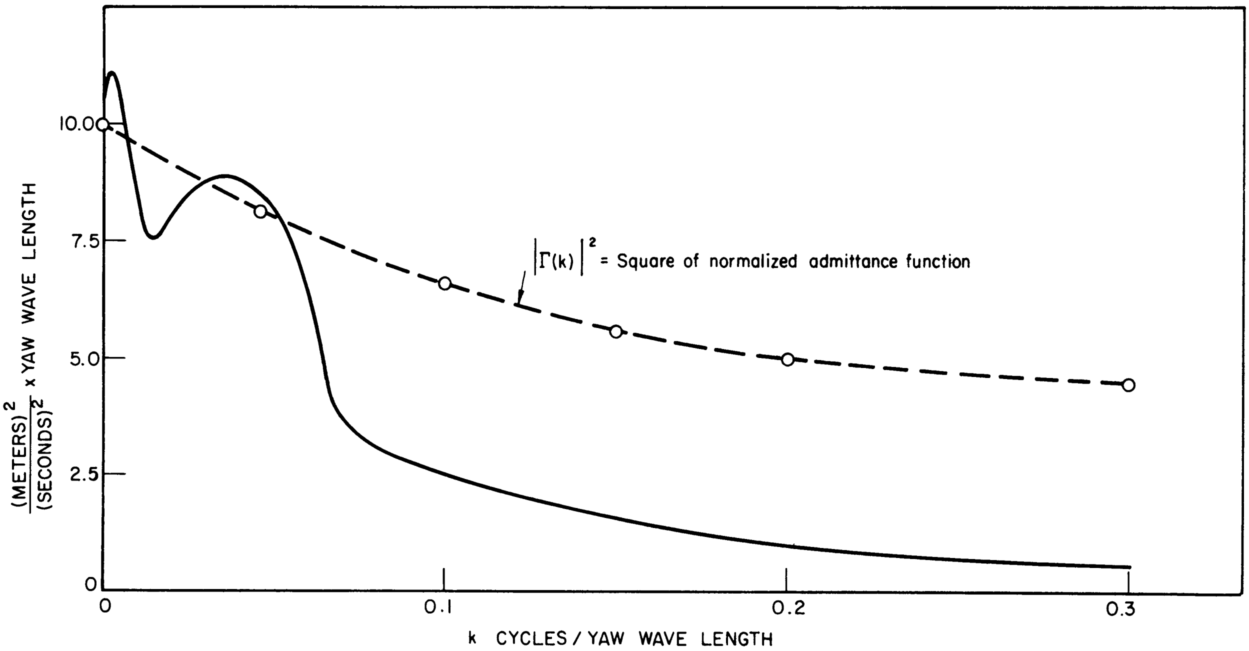

Fig. 3. Space spectrum for $\overline{u}$ = 5 m/sec.

Figure 3 is extremely instructive. It will be noted that, for the influence function used, the cutoff point for high wave numbers is determined not by the admittance function, but by the character of the turbulence spectrum. In other words, for relatively high wave numbers the spectrum goes to negligible values much sooner than does the admittance function.

The other point of major interest is that the value of the dispersion will be very much a function of the lower limit of integration of the product of the spectrum and the square of the admittance function. For example, if the lower limit of integration is k = 0.01 (corresponding to a mean wind averaging time of 60 min), the dispersion is about twice that for a lower limit k = 0.04 (corresponding to a mean wind averaging time of 15 min).

4. APPLICATION TO ROCKET FIRE SYSTEMS

The above suggests that the notion of a mean wind, while useful for a general approach to the problem and for estimates of orders of magnitude of possible dispersion, is of marginal utility in the actual mechanics of a rocket fire system. For example, the dispersion due to turbulence is a function of the averaging time used to define the mean wind. In Sections 2 and 3 the dispersion is estimated in terms of an averaging time of one hour. Application to a rocket fire system would assume that an hourly mean wind is predicted, that this wind is inserted into the rocket aiming system, and that after an hour of firing, the dispersion around the mean point of impact will be as given.

From an applied point of view, the mean wind inserted into the system may not be the true mean wind, and the dispersion will be centered not around the target but at some point distant from the target. Experience indicates roughly that the error in forecasting hourly mean winds is of the same magnitude as the turbulent variations about the hourly mean. For this reason, although the predicted dispersion may be correct, the actual point target may never be hit.

What is desirable in a rocket fire system is first of all a system which, after a series of firings, will insure that the center of scatter is around the target, and secondly that the dispersion around the target be as small as possible.

Restricting our discussion now to a horizontal trajectory along the mean wind, the first requirement is obviously satisfied by any system which continuously feeds a new aiming correction on the basis of current winds into the system as the rounds are fired. This has the virtue of closely approximating the true mean wind existing while the rounds are being fired. The second requirement is more difficult to satisfy.

To illustrate the possibilities of reducing the dispersion by a simple prediction scheme, we will assume that it takes a minimum of two minutes to make an adjustment in the launcher. We further assume that the turbulence is stationary and homogeneous in a horizontal plane and that we are justified in invoking Taylor's hypotheses.

If no wind correction is made, the rocket deviation is given by

$$\Theta_n = \int_{s_o}^{s_b} G(x)\, V(x)\, dx \quad , \tag{13}$$

where V(x) is the whole wind and where the trajectory is assumed to be horizontal into the wind. A simple prediction of Θ_n, say Θ_p, could be given by averaging the wind at launcher height over some time interval T and assuming this to hold along the entire path 2 minutes later.

Then

$$\theta_p = \int_{s_o}^{s_b} G(x)\, V_p(x)\, dx$$

and (14)

$$V_p(x) = \frac{1}{T}\int_{-T-2}^{-2} V(t)\, dt = \frac{1}{T\bar{u}}\int_{2\bar{u}}^{(T+2)\bar{u}} V(x)\, dx \quad .$$

Therefore the mean square deviation of the difference between the predicted impact and the true impact is given by

$$\overline{(\theta_n - \theta_p)^2} = \overline{\left\{\int_{s_o}^{s_b} G(x)\left[V(x) - \frac{1}{T}\int_{2\bar{u}}^{(T+2)\bar{u}} V(x')dx'\right] dx\right\}^2}$$

$$= \overline{\left[\int_{s_o}^{s_b} G(x)\, V^*(x)\, dx\right]^2}$$

$$= \int_{s_o}^{s_b}\int G(x)\, G(x')\, \overline{V^*(x)\, V^*(x')}\, dx\, dx' \quad . \qquad (15)$$

Now

$$\overline{V^*(x)\, V^*(x')} = R\,(x - x') + \frac{1}{T^2\bar{u}^2}\int_{2\bar{u}}^{(T+2)\bar{u}}\int R(y - y')\, dy\, dy'$$

$$- \frac{2}{T\bar{u}}\int_{2\bar{u}}^{(T+2)\bar{u}} R(x - y)\, dy$$

$$= \int\left\{e^{i(x-x')k}\, \frac{\sin^2 \frac{T}{2} k\bar{u}}{T^2 \frac{k}{4} \bar{u}^2} - \frac{2}{T}\, e^{ik\left[x - 2\bar{u} - \frac{T}{2}\bar{u}\right]}\, \frac{\sin \frac{T}{2}\bar{u}k}{\bar{u}k/2}\right\} s(k)\, dk \quad . \qquad (16)$$

Thus

$$\overline{(\theta_n - \theta_p)^2} = \int\left\{|\Gamma(k)|^2 + \frac{\Gamma^2(0)\, \sin^2 \frac{T}{2} k\bar{u}}{\bar{u}^2 T^2 \frac{k}{4}}\right.$$

$$\left. - \frac{2}{T}\,\Gamma(k)\,\Gamma(0)\, e^{-i\bar{u}k[2+T/2]}\, \frac{\sin \frac{T}{2}\bar{u}k}{\frac{\bar{u}k}{2}}\right\} s(k)\, dk \quad . \qquad (17)$$

We make the further assumption that the imaginary part of $\Gamma(k)$ is zero. This is a fairly good approximation for rough estimation purposes as the sine is small when G(x) is large. The imaginary part of $\Gamma(k)$ will grow with increasing k, but the spectrum decreases rapidly with increasing k. Hence Eq. (15) becomes

$$\overline{(\theta_n - \theta_p)^2} = \int \left\{ |\Gamma(k)|^2 + \frac{\Gamma^2(0)\ \sin^2 \frac{T}{2}\ 2\pi k\bar{u}}{(2\pi)^2\ \bar{u}^2 T^2\ \frac{k^2}{4}} \right.$$

$$\left. - 2\ |\Gamma(k)|\ \Gamma(0)\ \cos 2\pi\ \bar{u}k\ [2+T/2]\ \frac{\sin 2\pi \frac{T}{2}\ \bar{u}k}{2\pi T\ \frac{\bar{u}k}{2}} \right\} s(k)\ dk \quad . \tag{18}$$

Here k is cycles per meter if $\bar{u}$ is in m/sec, and T is in seconds. For T = 1 and 2 sec, and $\bar{u}$ = 5 m/sec, the results of this correcting scheme are given in Fig. 4, together with the corresponding wind velocity. As the filters are almost zero for long wave lengths, we see that these methods will reduce the bias, i.e., the center of impact can be made to coincide with the target. However, the filters rise extensively compared to the filter for an uncorrected rocket (the squared admittance function) which is given in Fig. 3. If we perform the integration, the square dispersion we get is almost 50% larger for the corrected case. A more meaningful measure is the square root, which makes the dispersion about 20% higher. Whether this is an improvement over no correction at all is dependent on the relative importance of scatter about the mean value as against the mean value not being at the target.

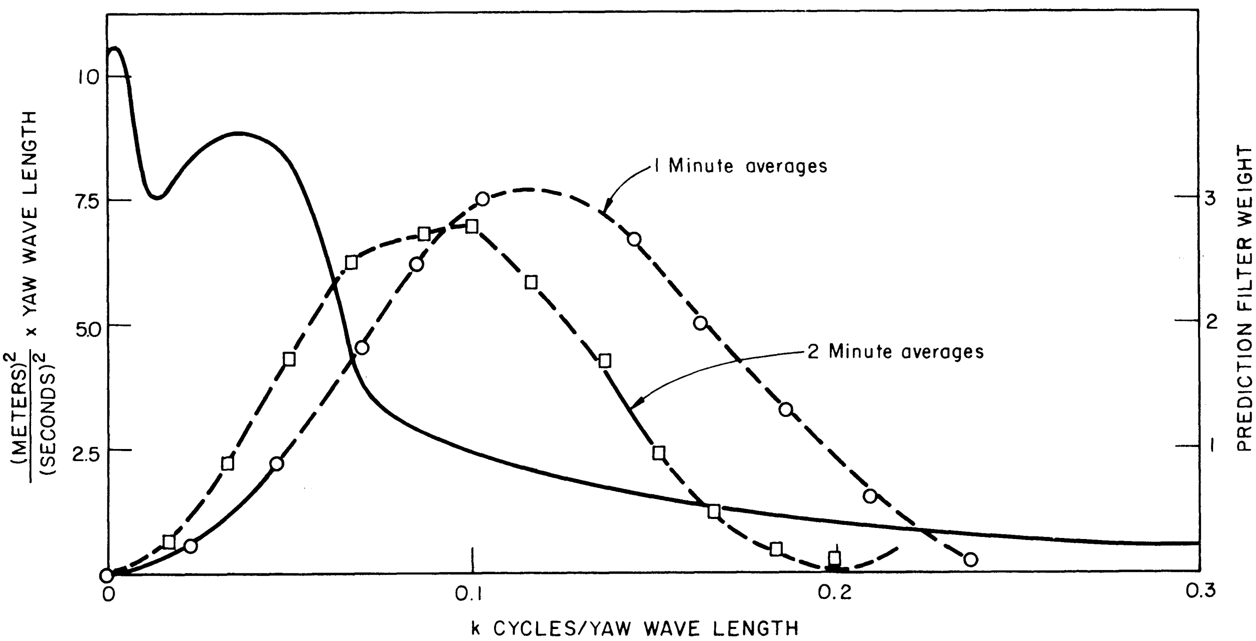

Fig. 4. Filters corresponding to a simple correction scheme.

It should be recognized that this is a simple and crude scheme. The best predictor would be the "least squares" one, whose characteristics are considerably harder to evaluate.

The major trouble with the simple scheme given above is that it weighs all the past observations equally in determining the predictor. If one used a weighted estimate of the mean so that the weights were decreasing for decreasing time, the cosine term in Eq. (18) would be damped for increasing k, and these would reduce the height of the peak of the dispersion filter, hence reducing the dispersion. It is possible to construct a wide variety of weighted means which would be good for different spectrum forms.

A more involved correction scheme for the horizontal trajectory-homogeneous turbulence case would attempt to use the observed wind 2 minutes upwind of the launcher as the aiming correction. In a wind speed of 5 m/sec this would mean a separation of 600 m. It is not known how much of an improvement this would give for anemometers located at s_o, about 50 ft above the ground. It is, however, fairly easy to devise such experiments and to estimate the reduction in scatter using such a simple system. The results would depend on the adequacy of Taylor's hypothesis for horizontal turbulence components 50 ft above the ground.

For a nonhorizontal trajectory, the variation of the mean or whole wind with height must be taken into account. The solid lines in Fig. 2 are plots of typical mean wind ratios for day and night conditions as observed at O'Neill, Neb. The horizontal scale for the dashed lines is in terms of s (s_o assumed to be 50 ft, s_b = 2000 ft). The original wind ratios are in terms of z, and we have used $z = s \sin E (E = 30°)$ in the plot. The relative mean wind shear is quite sensitive to E and the graphs should be replotted as E changes.

A rough correction scheme for this case is to find a value of u^* so that

$$u^* \int G(s)\, ds = \int \bar{u}\, (s \sin E)\, G(s)\, ds \quad .$$

Then the ratio $u^*/u(s_o)$ can be used to correct the prediction anemometer at launcher height s_o. Observed values of wind shear may be incorporated into the system as indicated.

5. CONCLUSIONS AND RECOMMENDATIONS

We should like to emphasize that all statements made previously are based on the assumed influence function given in Fig. 2. Our conclusions are that high-frequency turbulence components contribute very little to the dispersion, mostly because there is so little energy in horizontal turbulence components at high frequencies. The major contribution to rocket dispersion comes from low-frequency turbulent components. Aside from attempting to correct for vertical variation of mean wind, the notion of mean wind is difficult to apply to the actual mechanics of a rocket fire system. More meaningful results are obtained if one deals with whole winds. (This confirms the views in Ref. 6.)

In terms of standard meteorological notation, the mean deviation of the rocket from the target can be ascribed to mean wind. The dispersion around the point of mean deviation can be ascribed to turbulent winds. Because of the nature of the velocity fluctuations in atmospheric turbulence, the dispersion is due mainly to the R_{ii} components of the correlation tensor, while the contributions of the cross-correlation terms are considerably less. This is especially true for the influence function with which we have been working.

A rocket launching system should ensure that the center of the statistical scatter is on the target, and that the dispersion around the target is minimized. The simple prediction schemes used here satisfy the first requirement, but the dispersion is increased by 20% over what it would have been if the hourly mean wind had been known in advance.

An optimum technique would be a least-squares technique such as described in Ref. 7. This would require a lengthy series of computations to evaluate and it is doubtful that sufficient meteorological information is available to evaluate this technique generally. Application of Taylor's hypothesis to a prediction scheme would improve estimates to the extent that Taylor's hypothesis is correct for levels from 50 to 150 ft above the ground.

Our recommendations for future work in order of expected contribution to reduction in scatter are:

1) Investigation of the adequacy of Taylor's hypothesis for heights of interest.
2) Development of a model according to Taylor's hypothesis that also incorporates the vertical variation of the mean wind.
3) Development of more general models incorporating the vertical correlation structure of atmospheric turbulence.
4) Experimentation to evaluate these models.

The experimental setup for 1) is simple and would require stringing anemometers upwind of a potential firing site. Experimentation connected with more complicated models would involve more complicated experimental schemes such as those used in the Santa Barbara, Calif., project. Since towers are fixed on the earth's surface and therefore limit separation distances, we suggest a study of the potentialities of properly instrumented aircraft in conjunction with meteorological tower observations in verifying models under 2) or 3). Some data are available to check 1) and 2) from Santa Barbara, Brookhaven, and Cornell Aeronautical Laboratory; moreover, the Santa Barbara data can be used partially to construct models under 3).

We should like to emphasize that, in our opinion, the major return in reducing dispersion is likely to come from 1) coupled with a least-squares prediction technique, and that the improvements resulting from 2) and/or 3) are likely to be of secondary importance as long as the influence function in Fig. 2 is typical.

Acknowledgments

This analysis is an outgrowth of the conference held at The University of Michigan in Ann Arbor during July, 1956. Undoubtedly each participant has contributed in some form or other to this report. The following were present at that conference: E. Wendell Hewson, R. J. Deland, H. E. Reinhardt, G. Keller, M. A. Woodbury, and V. C. Liu. The responsibility for the contents of this section of the report, however, rests solely with the two authors.

TASK A: STATISTICAL DESIGNS—ROCKET TRAJECTORY DISPERSION BY ATMOSPHERIC DISTURBANCES

by Tse-Sun Chow

List of Symbols

a	Proportional constant
G	Geostrophic wind
g	Acceleration of gravity
K	Coefficient of eddy diffusivity; K_m coefficient of diffusivity for momentum; K_h coefficient of diffusivity for heat
k	Von Kármán's constant
ℓ	Mixing length
p	Pressure
Q_a	Heat flux at earth's surface
R	Gas constant; also $= K/u^* z_o$
S_n	Defined to be $(g\, z_o/u^{*2} \overline{T})\, (Q_a/u^*\, c_p\, \overline{\rho})$
T	Temperature
t	Time
V	Mean wind velocity; also, the complex wind velocity $u + iv$
u,v,w	Components of velocity
u^*	Defined to be $\sqrt{\tau_o/\overline{\rho}}$
x,y,z	Position coordinates
z	Height above the earth surface; z^*, a length proportional to the mean height of surface irregularities
ω	Angular speed of the rotation of earth
ϕ	Geocentric latitude

ρ	Density
μ	Proportional constant
σ	Angle between G and V_o (see figure)
λ	Coefficient of heat conductivity
ν	Kinematic viscosity
Θ	Potential temperature
η	Viscosity
τ	Shearing stress
ζ	$(z + z_o)/z_o$

1. STRUCTURE OF WIND IN THE LOWER LAYERS OF THE ATMOSPHERE

(a) Variation of wind with height based on theories of momentum transfer.—When the wind varies with the height, there is a transfer of momentum when one eddy travels from one level at one velocity to another level at a different velocity. Early investigators such as Schmidt[8] and Taylor[9] introduced the coefficient of eddy diffusivity K and established the net rate of gain of momentum per unit volume due to the eddying motion to be

$$\frac{\partial}{\partial z}\left(K\rho \frac{\partial \overline{V}}{\partial z}\right) ,$$

where ρ is the density and $\overline{V}$ is the average wind velocity at level z. According to Prandtl's development of these concepts, the coefficient of eddy diffusivity K may be expressed in terms of the vertical gradient of the wind velocity and a length ℓ, which he defines to be the mean vertical distance traversed by an eddy before it mixes with its environment. Because an eddy is a somewhat undefined entity, and the dynamical processes involved in the motion of an individual eddy traveling from one level to another are not clear, it is not possible to derive an expression for the mixing length ℓ based on theoretical considerations. Observations from experiments, however, tend to indicate that ℓ does not depend on the velocity but only on the distance from, and the nature of, the boundary surface.

Various assumptions have been made regarding the dependence of ℓ on the distance from, and the nature of, the boundary surface. Rossby and Montgomery[10] assume that over a rough surface and in an adiabatic atmosphere it can be written $\ell = a\,(z + z^*)$, where z^* is a length proportional to the mean height of the surface irregularities and a is a nondimensional constant. It has been observed by experiment, however, that a is not actually constant.

In 1946, Frost[11] put forward the hypothesis that $\ell = z^{1-m} z^{*m}$, where m is a nondimensional constant and lies between 0 and 1, if ℓ is to increase with both height above and the roughness of the surface. When this assumption is used in conjunction with the Prandtl's development, it is found that the coefficient of eddy diffusivity K is proportional to z^{1-m}. This expression is similar to that derived by Sutton[12] and Calder[13] based on different assumptions. The proportional constant can be evaluated numerically from meteorological measurements.

Based on the formulation that K is proportional to z^{1-m}, $0 \leq M \leq 1$, we have found it possible to solve the problem of the variation of wind with height in the atmospheric boundary layer. The geostrophic wind can be taken to be constant in the atmospheric boundary layer and the motion is assumed to be steady and two-dimensional. The boundary conditions are such that the wind should approach the geostrophic wind at a great height and also that the wind direction should coincide with its vertical derivative at the earth's surface. This is similar to the approach adopted by Taylor in solving the same problem with K assumed to be constant, a reproduction of which is given by Brunt.[14]

The details are given in Appendix A, and the final solution by Eq. (A-14). Numerical computation has to be carried out using different values of the parameters, and it may be possible to extend the consideration to three-dimensional turbulence, using an approach recently suggested by Davies.[15]

(b) Analysis of the atmospheric surface layer.—For the atmospheric surface layer which constitutes the lower part of the atmospheric boundary layer, the analysis is much more complicated due to the presence of convective currents resulting from the heat exchange at the earth's surface. As a result of these convective currents there are two kinds of turbulence in the atmospheric surface layer: convective turbulence and frictional turbulence produced by the friction at the earth's surface. Besides employing the dynamical equations of motion, the analysis calls for the use of the heat convection equation.

By starting with the four basic equations, the continuity equation, the equation of state, the Navier-Stokes equations, and the Fourier equation of heat convection, Businger,[16] in a recent study of the influence of the earth's surface on the atmosphere, is able to derive, after making various approximations, two transfer equations relating the vertical gradients of the mean velocity and the mean potential temperature with the shearing stress at the earth's surface, the flux of heat at the earth's surface, and the coefficient of eddy transfer for heat and momentum. For an adiabatic atmosphere in which the potential temperature is everywhere the same, the velocity profile is logarithmic. This is in good agreement with the experimental work of many investigators.[17] For the diabatic atmosphere further assumptions are required. Businger makes the formal distinction between convective turbulence and mechanical frictional turbulence and assumes that the total turbulence is the sum of the two parts. With these assumptions certain relations can be derived in terms of several non-dimensional parameters showing the variation of the coefficient of eddy diffusivity with height. With these relations the velocity and the temperature profiles in the atmospheric surface layer can be determined. A summary of such an analysis is presented in Appendix B. The experimental results regarding the variation of wind profile for the adiabatic and diabatic atmosphere have been summarized in Ref. 17.

Note: The atmospheric boundary layer extends to a height of about 3000 ft and the atmospheric surface layer to about 80 ft above the earth's surface.

2. GENERAL REMARKS ON TURBULENCE

The statistical part of the problem has already been presented by Hewson and Woodbury,[7] who derive the mean and second moment of the rocket dispersion in terms of velocity correlation coefficients. To visualize the simplest possible model, it is assumed that the turbulence is homogeneous and isotropic. Extension to the higher moments is immediate.

In the theory of turbulence one usually confines the discussion to homogeneous turbulence, which is a random motion whose statistical properties are independent of position in the fluid. To simplify the problem further, one ignores the directional preference of the statistical properties of the turbulent motion. This is the simplest case possible, and the turbulence is said to be isotropic. It should be remembered, however, that such cases are highly idealized, and can only be realized or approximated in an unbounded fluid extending theoretically to infinity in all directions. Thus, Kolmogoroff's theory of local isotropy asserts the existence of such a statistically steady, homogeneous, and isotropic turbulence of an unbounded fluid for a certain range of wave numbers, provided the Reynolds number of the flow is sufficiently high.

In practical problems such as this one where in particular the low-level wind structure is to be studied, the presence of a rigid boundary will probably make the turbulence nonhomogeneous. It is known in meteorology that the presence of the earth's surface will not affect the motion of the air at sufficient heights (above about 3000 ft) from the ground, i.e., outside the atmospheric boundary layer. The motion of the air in the low-level wind layer under consideration will lie well within the atmospheric boundary layer. Thus, because of the presence of the ground, there will be a variation of the mean velocity and the turbulent fluctuating motion with height, and there is a lack of homogeneity. There will also be effects of radiation, heat transfer to and from the soil, etc., which will further complicate the problem. These various aspects have already been discussed in Section 1.

3. RESPONSE OF THE ROCKET

In Ref. 18, the basic equations for the motion of the rocket have been derived in terms of the various aerodynamic coefficients. Within the framework of the linearized theory, it is possible to express the rocket dispersion due to a particular wind profile in the form of an integral over the time or space domain in terms of the rocket response functions due to gusts of unit impulse. This is the approach adopted by Hewson and Woodbury[7] in presenting the statistical aspect of the problem.

4. PROPOSED METHODS OF ATTACK ON THE PROBLEM

The pertinent problem is to determine the dispersion of the rocket trajectory after launching due to atmospheric disturbances. If we assume the statistical uniformity of such disturbances in the horizontal plane, the description of atmospheric disturbances can be given by

I. the variation of the average wind with height, and

II. the variation of the correlation functions with height (second order and higher moments) or their spectra by carrying out a statistical analysis.

When the average wind-velocity profile and the vertical variation of the correlation functions of the atmospheric disturbances are known, the dispersion of the rocket trajectory can be calculated (in I, the average value, in II, the higher moments, so that I and II give the complete probabilistic value) by making use of the rocket response function to gusts of unit impulse (Section 3).

By proceeding from the studies and correlating the results of Section 1 and 2 (also, Appendix A and B), the average wind-velocity profile in the lower layers of the atmosphere can be determined in terms of certain parameters which can be measured close to the ground. The relative significance of each parameter on the rocket dispersion can then be ascertained after numerical calculations showing the effect of each. On the other hand, the variation of the correlation functions with height has yet to be investigated. This can be carried out in a manner similar to that described in Appendix B. If this is not carried out one has to limit the analysis to homogeneous turbulence, which is probably too idealized for the case under consideration (Section 2).

Our proposed methods of attack on this problem are as follows:

I. Continuation and completion of the investigation of the variation of the average wind with height by proceeding from the studies in Section 1 and 2 (Appendix A and B).

II. Investigation of the vertical variation of the correlation functions with height by conducting an analysis similar to that in Appendix B.

III. Comparison of the results obtained in Tasks I and II with the experimental data already in existence. On the basis of this comparison, models embodying essential wind structure in the lower atmosphere pertinent to dynamic wind loading will be devised.

IV. Selection of a design for a field experiment or experiments to yield maximum information on rocket dispersion, to be followed by extensive numerical calculations. The design will include data-reduction recommendations.

V. Determination of anemometer types which have response characteristics suitable for field use.

APPENDIX A

VARIATION OF WIND WITH HEIGHT BASED ON CONSIDERATIONS OF MOMENTUM TRANSFER

It has been shown by Taylor that if K denotes the eddy diffusivity, then the rate of eddy transfer of x- and y-momentum per unit volume is

$$\frac{\partial}{\partial z}\left(K\rho \frac{\partial u}{\partial z}\right) \quad \text{and} \quad \frac{\partial}{\partial z}\left(K\rho \frac{\partial v}{\partial z}\right)$$

and the equations of motion will be, assuming the vertical component is zero,

$$\begin{cases} -2\omega \sin \phi \, v = -\dfrac{1}{\rho} \dfrac{\partial p}{\partial x} + \dfrac{1}{\rho} \dfrac{\partial}{\partial z}\left(K\rho \dfrac{\partial u}{\partial z}\right) , & \text{(A-1)} \\ 2\omega \sin \phi \, u = -\dfrac{1}{\rho} \dfrac{\partial p}{\partial y} + \dfrac{1}{\rho} \dfrac{\partial}{\partial z}\left(K\rho \dfrac{\partial v}{\partial z}\right) . & \text{(A-2)} \end{cases}$$

Writing $V = u + iv$, we can combine the above two equations into

$$\frac{d^2V}{dz^2} + \frac{1}{K}\frac{dK}{dz}\frac{dV}{dz} - (1+i)^2 \frac{\omega \sin \phi}{K} (V - G) = 0 , \qquad \text{(A-3)}$$

where G is the geostrophic wind.[14] An appropriate form of K is $aV_1^{1-n} z^{1-m}$ where V_1 is the value of the mean wind speed at some standard height z_1, and n,m are constants for a given turbulent state and may be evaluated numerically from meteorological measurements. We therefore write $K = \mu z^{1-m}$. By substituting this expression of K into (A-3), we get

$$\frac{d^2V}{dz^2} + \frac{1-m}{z}\frac{dV}{dz} + i^2(1+i)^2 \frac{\omega \sin \phi}{\mu} z^{m-1} (V - G) = 0 . \qquad \text{(A-4)}$$

We now put $V - G = ux^{\frac{m}{1+m}}$ and $z^{\frac{m+1}{2}} = x$ and the differential equation (A-4) in the new variables becomes

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left[\frac{4i^2(1+i)^2}{(m+1)^2} \frac{\omega \sin \phi}{\mu} - \left(\frac{m}{1+m}\right)^2 \frac{1}{x^2}\right] u = 0 , \qquad \text{(A-5)}$$

which is the standard Bessel differential equation. The solution of Eq. (A-4) is therefore

$$V - G = z^{m/2}\left\{A\, J_{\frac{m}{m+1}}\left[\frac{2i(1+i)}{m+1}\sqrt{\frac{\omega \sin \phi}{\mu}}\; z^{\frac{m+1}{2}}\right] + B\, J_{-\frac{m}{m+1}}\left[\frac{2i(1+i)}{m+1}\sqrt{\frac{\omega \sin \phi}{\mu}}\; z^{\frac{m+1}{2}}\right]\right\}, \tag{A-6}$$

assuming that $\frac{m}{m+1}$ is not an integer. Observations of many investigators[11] have shown that the value of m lies between 0 and 1. As stated by Frost,[11] $m = 1/7$ approximately under a condition of thermal equilibrium in wind tunnels and the same value may be taken for the atmosphere. Thus, using $m = 1/7$ as an illustration, we write (A-6) as

$$V - G = z^{\frac{1}{14}}\left\{A\, J_{\frac{1}{8}}\left[\frac{7(-1+i)}{4}\sqrt{\frac{\omega \sin \phi}{\mu}}\; z^{\frac{4}{7}}\right] + B\, J_{-\frac{1}{8}}\left[\frac{7(-1+i)}{4}\sqrt{\frac{\omega \sin \phi}{\mu}}\; z^{\frac{4}{7}}\right]\right\}. \tag{A-7}$$

To determine A and B, we first impose the condition that $V-G$ remain finite as $z \to \infty$. Using the asymptotic formulae of Bessel functions and making use of this condition, we get

$$A\, e^{\frac{\pi i}{8}} + B = 0 \quad . \tag{A-8}$$

We use this relation to eliminate B in (A-7) and obtain the solution of (A-6) as

$$V - G = A\, z^{\frac{1}{14}}\left\{J_{\frac{1}{8}}\left[\frac{7}{2}\sqrt{\frac{\omega \sin \phi}{2\mu}}\; e^{\frac{3\pi i}{4}}\; z^{\frac{4}{7}}\right] - e^{\frac{\pi i}{8}}\, J_{-\frac{1}{8}}\left[\frac{7}{2}\sqrt{\frac{\omega \sin \phi}{2\mu}}\; e^{\frac{3\pi i}{4}}\; z^{\frac{4}{7}}\right]\right\}, \tag{A-9}$$

so that at great heights we have

$$V - G \sim A\, z^{-\frac{3}{14}}\sqrt{\frac{1}{7\pi}\sqrt{\frac{2\mu}{\omega \sin \phi}}}\left(e^{\frac{\pi i}{4}} - 1\right) e^{-i\left(\frac{7z^{\frac{4}{7}}}{4}\sqrt{\frac{\omega \sin \phi}{\mu}} - \frac{5\pi}{16}\right) - \left(\frac{7z^{\frac{4}{7}}}{4}\sqrt{\frac{\omega \sin \phi}{\mu}}\right)} \quad . \tag{A-10}$$

We use the subscript o to denote the values of V, etc., as $z \to 0$. As $z \to 0$, we have

$$V_o - G \approx \frac{A\, e^{\frac{33\pi i}{32}}}{\Gamma\left(\frac{7}{8}\right)}\left(\frac{4}{7}\sqrt{\frac{2\mu}{\omega \sin \phi}}\right)^{\frac{1}{8}} \quad . \tag{A-11}$$

Also, as $z \to 0$, we have

$$\left(\frac{\partial V}{\partial z}\right)_o = \frac{4A}{7} \frac{2^{\frac{7}{8}}}{\Gamma\left(\frac{1}{8}\right)} \left(\frac{7}{2}\sqrt{\frac{\omega \sin \phi}{2\mu}}\, e^{\frac{3\pi i}{4}}\right)^{\frac{1}{8}} z^{-\frac{6}{7}} . \tag{A-12}$$

Following Taylor, we assume as $z \to 0$, V and $\partial V/\partial z$ are in the same direction, i.e., the slip is in the direction of the strain. Let the direction G be in the direction of the x-axis, and let σ be the angle between G and V_o. Then Fig. A-1 shows that

$$|A| = |V_o| \frac{\sin \sigma}{\sin\left(\frac{\pi}{16} - \sigma\right)} \Gamma\left(\frac{7}{8}\right)\left(\frac{7}{4}\sqrt{\frac{\omega \sin \phi}{2\mu}}\right)^{\frac{1}{8}} . \tag{A-13}$$

This determines the arbitrary constant A in terms of V_o. The solution can now be expressed as

$$V - G = e^{-\frac{3\pi}{32} i} V_o \frac{\sin \sigma}{\sin\left(\frac{\pi}{16} - \sigma\right)} \Gamma\left(\frac{7}{8}\right)\left(\frac{7}{4}\sqrt{\frac{\omega \sin \phi}{2\mu}}\right)^{\frac{1}{8}} z^{\frac{1}{14}} \left\{ J_{\frac{1}{8}}\left[\frac{7}{2}\sqrt{\frac{\omega \sin \phi}{2\mu}}\, e^{\frac{3\pi i}{4}} z^{\frac{4}{7}}\right] \right.$$

$$\left. - e^{\frac{\pi i}{8}} J_{-\frac{1}{8}}\left[\frac{7}{2}\sqrt{\frac{\omega \sin \phi}{2\mu}}\, e^{\frac{3\pi i}{4}} z^{\frac{4}{7}}\right]\right\} . \tag{A-14}$$

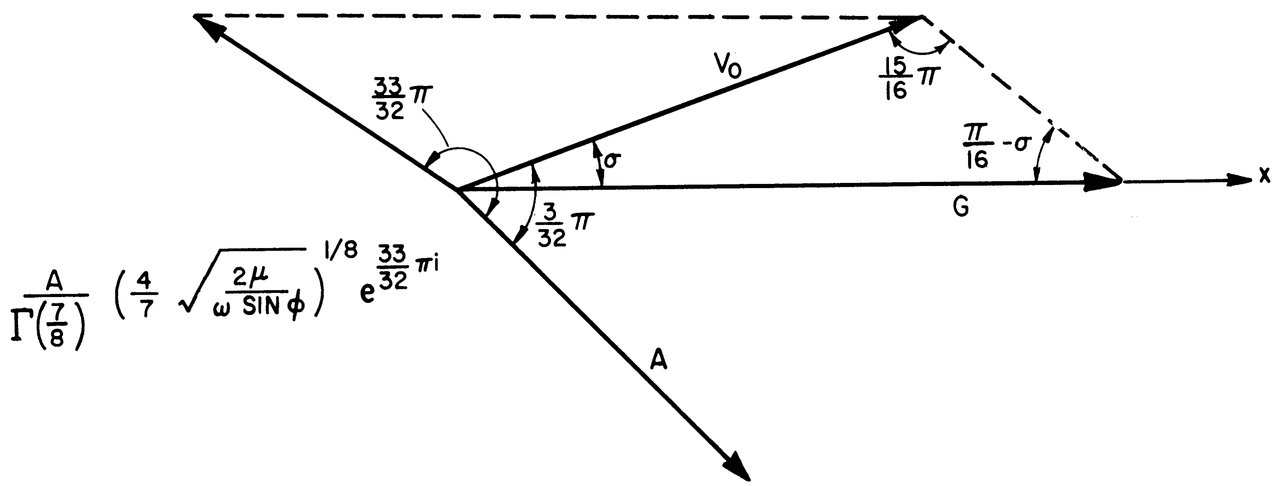

Fig. A-1. Relations among the geostrophic wind G, the surface wind V_o, and the arbitrary constant A, whereby A is expressed in terms of V_o.

APPENDIX B

ANALYSIS OF THE ATMOSPHERIC SURFACE LAYER

The equations that govern the structure of the atmospheric surface layer consist of the following:

(a) the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 , \qquad \text{(B-1)}$$

(b) the equation of state

$$p = \rho RT , \qquad \text{(B-2)}$$

(c) the Navier-Stokes equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= K_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad \text{(B-3)}$$

plus two similar equations in the v,w components, and

(d) the Fourier equation

$$\frac{\partial(\rho T)}{\partial t} + u\frac{\partial(\rho T)}{\partial x} + v\frac{\partial(\rho T)}{\partial y} + w\frac{\partial(\rho T)}{\partial z} = \frac{\lambda}{C_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) . \qquad \text{(B-4)}$$

Assuming that the velocity components, pressure, density, etc., can be split into a fluctuating part (indicated by a prime symbol) superimposed on the mean value (indicated by an upper bar), we can split each of the above equations, one for the mean state and the other for the fluctuating part. In many cases the equation for the fluctuating part can be left out of consideration. It will be assumed that the mean state motion is steady, and that the mass flow takes place in the x direction, with the z axis perpendicular to the horizontal plane, so that $\overline{\rho v} = \overline{\rho w} = 0$ and $\overline{\rho u} \neq 0$. The equation of continuity is seen to be reduced to, after integration,

$$\overline{\rho u} + \overline{\rho' u'} = f(z) . \qquad \text{(B-5)}$$

Calder[13] has shown that $p'/\bar{p} \ll \rho'/\bar{\rho}$ and $T'/\bar{T}$, so that the equation of state becomes $\rho'/\bar{\rho} = -T'/\bar{T}$. In the atmospheric surface layer the difference between the temperature T and the potential temperature Θ, which is defined to be the temperature the air would attain if brought adiabatically to a standard pressure, is small, so that the equation of state becomes

$$\frac{\rho'}{\bar{\rho}} = -\frac{\Theta'}{\bar{\Theta}} . \tag{B-6}$$

To simplify the Navier-Stokes equations, Businger assumed that in the atmospheric surface layer, $\overline{\rho' w'}$ remains practically constant and $\bar{\rho}\ \overline{u'w'} \gg \bar{w}\ \overline{\rho' u'}$, $\overline{\rho' u' w'}$, $\bar{\rho}\ \overline{w'^2} \gg \bar{\rho}\ \bar{w}^2$, $\overline{\rho' w'^2}$. The Navier-Stokes equations are then reduced to

$$\overline{u'w'} = -\frac{1}{\rho}\frac{\partial p}{\partial x} z + \nu \frac{\partial \bar{u}}{\partial z} - \nu \left(\frac{\partial \bar{u}}{\partial z}\right)_o , \tag{B-7}$$

$$\overline{\rho w'^2} = \bar{p}(0) - \bar{p}(z) - g \int_o^z \bar{\rho}\, d z , \tag{B-8}$$

after integration with respect to z and assuming that $\partial p/\partial x$ is constant, where the subscript o indicates the value at $z = 0$. Now $\nu(\partial \bar{u}/\partial z)_o$ is the shearing stress at the earth's surface and is equal to $\tau_o/\bar{\rho}$. In the atmospheric surface layer,

$$\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x} z$$

is small, and $\nu(\partial \bar{u}/\partial z)$ is also small compared with $\overline{u'w'}$ as soon as there is any turbulence. Thus for a first approximation we have

$$-\overline{u'w'} = \frac{\tau_o}{\bar{\rho}} . \tag{B-9}$$

Equation (B-9) implies that the vertical turbulent motion causes a pressure rise with regard to the static pressure in the free flow. Finally, the Fourier equation is reduced, after integration, to

$$\overline{\rho w T} = \frac{\lambda}{C_p}\frac{\partial \bar{T}}{\partial z} - \frac{\lambda}{C_p}\left(\frac{\partial \bar{T}}{\partial z}\right)_o . \tag{B-10}$$

Here $\lambda(\partial \bar{T}/\partial z)_o = Q_a$ is the heat flux at the surface of the earth. Also, $(\lambda/C_p)(\partial \bar{T}/\partial z)$ is the heat flux by pure conduction and is therefore negligible at the earth's surface. Furthermore, $\overline{\rho w T} \approx \bar{\rho}\ \overline{w'T'}$, so that we have

$$\overline{w'T'} = \frac{Q_a}{C_p\ \bar{\rho}} . \tag{B-11}$$

Thus from (B-6) we have

$$\overline{w'\theta'} = \frac{Q_a}{C_p\,\overline{\rho}} \quad . \tag{B-12}$$

The velocity profile in the atmospheric surface layer can not be determined.

A. THE ADIABATIC ATMOSPHERE

By means of the assumption of von Kármán,[19] which has also been proved by Hamel,[20]

$$\overline{u'w'} = -\,k^2 \frac{\left(\frac{\partial \overline{u}}{\partial z}\right)^4}{\left(\frac{\partial^2 \overline{u}}{\partial z^2}\right)^2} \quad ; \tag{B-13}$$

k being von Kármán's constant, we write (B-9) as

$$k\,\frac{\left(\frac{\partial \overline{u}}{\partial z}\right)^2}{\left(\frac{\partial^2 \overline{u}}{\partial z^2}\right)} = \sqrt{\frac{\tau_o}{\overline{\rho}}} \quad , \tag{B-14}$$

and upon integration we get

$$\frac{\partial \overline{u}}{\partial z} = \frac{1}{kz\sqrt{\overline{\rho}/\tau_o} + C} \quad . \tag{B-15}$$

Let $C = kz_o/u*$ where $u* = \sqrt{\tau_o/\overline{\rho}}$; then further integration gives

$$\frac{\overline{u}}{u*} = \frac{1}{k}\log\frac{z + z_o}{z_o} \quad , \tag{B-16}$$

which is a logarithmic profile.

B. THE DIABATIC ATMOSPHERE

When the atmosphere is diabatic, von Kármán's assumption (B-1) is no longer valid and we cannot solve (B-9). Instead, the two equations (B-9) and (B-12) have to be solved simultaneously. Following Taylor,[21] we write, in place of (B-9) and (B-12),

$$K_m \frac{\partial \bar{u}}{\partial z} = u^{*2} \quad , \tag{B-17}$$

$$K_h \frac{\partial \bar{\theta}}{\partial z} = - \frac{Q_a}{C_p \bar{\rho}} \quad . \tag{B-18}$$

Such a formulation is similar to that for the flux of momentum and heat in laminar flow. The unknowns are now K_m, K_h, $\bar{u}$, and $\bar{\theta}$. From the kinetic theory of gases, Held[22] pointed out that the two coefficients K_m, K_h are equal if the Prandtl number is 1 for an infinite number of degrees of freedom of the molecules. By writing $K_m = K_h = K$ we have three unknowns K, $\bar{u}$, θ but only two equations so that further assumptions have to be made.

In an adiabatic atmosphere the turbulence is entirely due to friction at the earth's surface. In the diabatic atmosphere the turbulence is due to mechanical friction and convective currents. We make a formal distinction between these two kinds of turbulence and assume that the total turbulence is the sum of frictional turbulence and convective turbulence. Based on this statement, Businger[16] formulates the following relation

$$\frac{K^2}{\ell^3} = \frac{K_f^2}{\ell_f^3} - g \frac{\ell \frac{\partial \bar{\theta}}{\partial z}}{\bar{T}} \tag{B-19}$$

from a consideration of the acceleration exercised on an eddy and the relation

$$\frac{K^2}{\ell^2} = \frac{K_f^2}{\ell_f^2} - g \frac{\ell^2 \frac{\partial \bar{\theta}}{\partial z}}{\bar{T}} \tag{B-20}$$

from a consideration of energy per unit mass. In these equations ℓ, ℓ_f are the mixing lengths of total turbulence and frictional turbulence, respectively. Furthermore, Businger formulates the relation

$$\frac{K}{K_f} = \left(\frac{\ell}{\ell_f}\right)^2 \quad . \tag{B-21}$$

Now by combining (B-15) and (B-17) we get

$$K_f = k\, u^* \,(z + z_o) \quad . \tag{B-22}$$

The solution of (B-17), (B-18), (B-19), and (B-21) gives

$$R = k^2 \zeta^2 S_n + \frac{1}{2} k\zeta \,[1 + (1 + 4k\, \zeta\, S_n)^{1/2}] \quad , \tag{B-23}$$

and the solution of (B-17), (B-18), (B-20), and (B-21) gives

$$R = k\zeta + S_n k^2\zeta^2 , \tag{B-24}$$

where $R = K/u^*z_o$, a dimensionless coefficient of eddy transfer, k is von Kármán's constant, $\zeta = (z + z_o)/z_o$, the dimensionless height, and

$$S_n = \frac{z_o g}{u^{*2}\,\overline{T}} \frac{Q_a}{u^*\,C_p\,\overline{\rho}} .$$

Thus, from the relations of R and ζ, S_n it is possible to find the velocity and temperature profiles. The practical significance of these results is that, if the profiles have once been determined for one given set of conditions involving a given value of S_n, then they are also determined for any other set of conditions which yield the same value of S_n.

TASK A: STATISTICAL DESIGNS—EVALUATION OF WIND DATA

The preliminary evaluation of the wind data obtained by Professor R. H. Sherlock and his associates was completed. It was subsequently decided not to use these data in the present research, so that the evaluation will not be presented here.

OVERALL CONCLUSIONS

Briefly stated, the overall conclusions are as follows:

1. Low-frequency turbulence makes the major contribution to rocket dispersion.

2. The concept of the whole wind is more meaningful and easier to apply in this problem than that of the mean wind.

3. A least-squares technique of prediction would be valuable but atmospheric information adequate to evaluate it may not be available yet.

4. Application of Taylor's hypothesis should improve estimates obtained by a prediction method.

RECOMMENDATIONS

Progress in this area is seriously hampered by the lack of information on the degree of applicability of Taylor's hypothesis. This hypothesis suggests that the structure of turbulence passing an area normal to the mean wind specifies the structure of the turbulence downwind from that area. There is obviously some degree of validity to this assumption, but precise information of the type needed for the present problem is lacking.

RECOMMENDATION 1.

It is therefore recommended that field studies be conducted to determine the range of validity of Taylor's hypothesis for the large-scale turbulence which is of primary importance in the present problem.

It is proposed that several TV and FM antenna towers closely grouped in an area of flat horizontal terrain on the northwest outskirts of Detroit be instrumented with anemometers and bivanes at several levels. The tower of WJBK-TV is 1050 ft high and has platforms at 300, 600, and 870 ft, which are reached by a small elevator. Temperature lapse-rate measurements are already being made at these levels on a routine basis. An almost identical tower, with platforms at the same heights, and owned by WWJ-TV, lies about one mile to the north-northeast of WJBK-TV. A shorter tower, 468 ft high, and operated by WLDM-FM, lies about one mile to the northeast of WJBK-TV; only the top 80 ft of this tower is energized, so that the portion up to 388 ft could be instrumented. Tentative approval for the instrumenting of each tower has been obtained.

The distance between the two high towers is sufficiently great to permit aliasing in spectra in the significant ranges unless intermediate observations are obtained. It is proposed to fly one or more kytoons between the two high towers when winds are NNE and SSW. A number of zero-lift balloons would be attached at intervals to the kytoon cable by elastic cords, to form one or more arrays of gustometers of the type described by Hewson.[23] After the validity of Taylor's hypothesis had been assessed by use of the tower instrumentation in conjunction with arrays of balloon gustometers, and the characteristics of the latter determined more fully, then the gustometers supported by kytoons would be used in rough terrain of various types and in various climatic regimes to determine how Taylor's hypothesis stands up under topographic and climatic conditions very different from those near Detroit.

A final step would be to devise, calibrate, and test simplified field apparatus that would provide more readily the basic information supplied by the balloon gustometers.

RECOMMENDATION 2

It is further recommended that concurrently with the above program the theoretical studies described in this report be continued along the lines which follow directly from the progress and findings to date.

REFERENCES

1. Hunter, M. W., Shef, A., and Black, D. V., "Some Aerodynamic Techniques in Design of Fin-Stabilized Free-Flight Missiles for Minimum Dispersion," J. of Aero. Sci., 23 (1956, 571-577.

2. Panofsky, H., Statistical Properties of the Vertical Flux and Kinetic Energy at 100 Meters, Scientific Report No. 2, AF 19(604)-166, The Penn. State Univ., Univ. Park, Pa., July, 1953.

3. Van der Haven, I., "Power Spectrum of Horizontal Wind Speed in the Frequency Range from 0.0007 to 900 Cycles per Hour," J. Met., 14 (April, 1957, 160-164.

4. "Great Plains Turbulence Field Program," to be published in 1957.

5. Panofsky, H., and Van der Haven, I., Structure of Small Scale and Middle Scale Turbulence at Brookhaven, AFCRC-TN-56-254, The Penn. State Univ., Univ. Park, Pa., 1955.

6. Shaffer, P. A., Jr., Wind Structure Study for Rockets, Final Report, Contract DA-36-039-SC-52599, North American Instruments, Santa Barbara, Calif., 1956.

7. Hewson, E. W., and Woodbury, M. A., Research Study Pertaining to Low-Level Wind Structure, The Univ. of Mich., Eng. Res. Inst. Report No. 2377-15-P, Ann Arbor, October, 1956.

8. Schmidt, W., "Der Massenaustausch bei der ungeordneten Stromung in freier Luft und seine Folgen," Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Wien, Ab. IIa, 126 (1917), 757-804.

9. Taylor, G. I., "Eddy Motion in the Atmosphere," Phil. Trans., A, 215 (1915), 1-26.

10. Montgomery, R. B., "Vertical Eddy Flux of Heat in the Atmosphere," J. Met., 5 (1948), 265-274.

11. Frost, R., "Turbulence and Diffusion in the Lower Atmosphere," Proc. Roy. Soc., A, 186 (1946), 20-35.

12. Sutton, O. G., "Wind Structure and Evaporation in a Turbulent Atmosphere," Proc. Roy. Soc., A, 146 (1934), 701-722.

REFERENCES
(Concluded)

13. Calder, K. L., "Eddy Diffusion and Evaporation in Flow over Aerodynamically Smooth and Rough Surfaces: A Treatment Based on Laboratory Laws of Turbulent Flow with Special Reference to Conditions in the Lower Atmosphere," Quart. Journ. Mech. and Appl. Math., 2 (1949), 153-176.

14. Brunt, D., Physical and Dynamical Meteorology, Cambridge Univ. Press, N. Y., 1944.

15. Davies, D. R., "Three Dimensional Turbulence and Evaporation in the Lower Atmosphere, I, II," Quart. Journ. of Mech. and Appl. Math., 3 (1950), 51-73.

16. Businger, J. A., Some Aspects of the Influence of the Earth's Surface on the Atmosphere, Koninklijk Nederland Meteorologisch Institut, 's Gravenhage, 1954.

17. Ellison, T. H., "Atmospheric Turbulence," Surveys in Mechanics, eds. Batchelor, G. K., and Davies, R. M., Cambridge Univ. Press, N.Y., 1956.

18. Rosser, J. B., Newton, R. R., and Gross, G. L., Mathematical Theory of Rocket Flight, McGraw-Hill Book Company, N. Y., 1947.

19. Von Kármán, Th., "Mechanische Ähnlichkeit und Turbulenz," Gesell. Wiss. Göttingen, Math.-Phys. Kl. Nachr., No. 8, 1930.

20. Hamel, G., "Streifenmethode und Ähnlichkeitsbetrachtungen zur turbulenten Bewegung," Akad. Wiss. Berlin, Kl. Math. naturwiss., Abhand. No. 8 (1943), 1-25.

21. Taylor, G. I., "Effect of Variation in Density on the Stability of Superposed Streams of Fluid," Proc. Roy. Soc., A, 132 (1931), 499-523.

22. Held, E. E. M., Standaarddictaat Gelijkvormigheidsleer, Utrecht, 1947.

23. Hewson, E. W., "A Method for Determining Atmospheric Turbulent Flow Parameters Throughout the Vertical Extent of the Boundary Layer," Geophysical Research Papers, No. 19 (1952), 165-170.