

## Forcing under Anti-Foundation Axiom: An expression of the stalks

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We introduce a new simple way of defining the forcing method that works well in the usual setting under FA, the Foundation Axiom, and moreover works even under Aczel's AFA, the Anti-Foundation Axiom. This new way allows us to have an intuition about what happens in defining the forcing relation. The main tool is H. Friedman's method of defining the extensional membership relation  $\in$  by means of the intensional membership relation  $\varepsilon$ .

Analogously to the usual forcing and the usual generic extension for FA-models, we can justify the existence of generic filters and can obtain the Forcing Theorem and the Minimal Model Theorem with some modifications. These results are on the line of works to investigate whether model theory for AFA-set theory can be developed in a similar way to that for FA-set theory.

Aczel pointed out that the quotient of transition systems by the largest bisimulation and transition relations have the essentially same theory as the set theory with AFA. Therefore, we could hope that, by using our new method, some open problems about transition systems turn out to be consistent or independent.

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### 1 Introduction

Whereas the Foundation Axiom, FA, is one of the axioms of ZFC, the standard set theory, there have been many criticisms of this axiom and many authors have proposed alternative axioms, among which one of the most mathematically integrated ones is Aczel's Anti-Foundation Axiom AFA from [1]. Although in this paper we do not enter into a philosophical argument whether FA is plausible or which axiom is appropriate, it is natural to ask whether model theory for these alternative axioms can be developed in a similar way to that for FA.

Recently, developing model theory for AFA-models has become of interest. Indeed, in [9], Viale considered Gödel operations and the constructible universe in this setting. In [5], Lazić and Roscoe conjectured that fundamental results for the forcing method, which are well known in the presence of FA, hold also in the presence of AFA and, in [7], Tzouvaras "proved" these results.<sup>1)</sup> Unfortunately, an error in Tzouvaras' work was found by the author of the present paper (see [8]). Thus it was still open until the publication of the present paper whether the forcing method can be used for constructing new models under AFA in the same manner as under FA. This paper gives a positive answer to this question and extends Tzouvaras' "answer" by eliminating AC from the assumption, with a shorter and more conceptual proof.

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<sup>1)</sup> However, the author of the present paper noticed Tzouvaras' paper only after submitting the first version of the present paper.

As is well known, the forcing method is one of the most important and fundamental tools to construct new models and to obtain consistency results in modern set theory. Moreover, as theory for the forcing method has been developed and various applications have been invented, the forcing method itself has become of interest in set theory, not only as a tool for consistency results.

Besides such various applications in set theory, the structure of forcing itself has become of interest, especially from sheaf or topos theoretical viewpoints. Indeed, forcing is one of the origins of topos theory and there are many topos theoretical investigations on forcing, e. g. [6, 2]. From sheaf theoretical viewpoints,  $V[G]$  looks like a stalk of a sheaf  $V^{\mathbb{P}}$ , while, from the set theoretical viewpoint,  $V[G] = \{\text{val}_G(\dot{a}) \mid \dot{a} \in V^{\mathbb{P}}\}$ . This expression of the stalks (essentially, Minimal Model Theorem) seems to depend heavily on FA, because we use the  $\in$ -induction to prove this fact in the usual setting with FA. Now it is natural to ask whether or not FA is inevitable or unreplaceable for this expression of the stalks of  $V^{\mathbb{P}}$  and this point is the most difficult place, where Tzouvaras committed an error. We will investigate how this expression depends on FA and show that this expression can be justified<sup>2)</sup> (but will not be proved) even under AFA.

Not only this expression, but also the usual definition of the forcing uses FA (especially in defining the forcing relation of atomic formulae) and for this reason the forcing method cannot apply directly to non-well-founded set theory.

Because the forcing method includes the two key theorems, Forcing Theorem, which states that the truth of the generic extensions can be approximated by means of the ground model, and Minimal Model Theorem, which states that the generic extension is the smallest model having the generic filter and containing the ground model, our purpose is to look for an analogous tool for new constructing models so that the corresponding results hold in non-well-founded set theory. For this purpose, we introduce a new simple way of defining the forcing method that works well in the usual setting under FA, and moreover works even under Aczel's AFA. Besides the contribution to non-well-founded set theory, it could also give us some new insight into the forcing method itself.

The key tool to avoid the assumption FA to define the forcing relation is a use of an intensional set theory and an interpretation from the usual extensional set theory to the intensional set theory. This method is from Friedman [3] and was originally introduced for the consistency proof of the axiom of extensionality (in other words, equiconsistency proof of the intensional and extensional set theories). In Section 2, this method will be explained.

In Section 3, we will define the forcing relation in this new way, and will prove that axioms of set theory with AFA are forced. The discussion in this section is parallel with the usual forcing theory, but the use of the interpretation from the intensional set theory makes the discussion simpler.

In Section 4, we will consider the method called generic extension. In contrast to generic extension under FA, the Forcing Theorem and Minimal Model Theorem do not seem to hold in general universes which contain the ground model as a transitive submodel. (Note that defining generic models does depend on the background universe.) However we will see that these theorems do hold under certain assumptions. In Sections 5 and 6, we will justify these assumptions. One of the justifications is known as "generic filter elimination" and will be done in the same way as that of the forcing under FA. The other is specific to the forcing under AFA. Here justification means establishing the consistency of the assumptions, which allows us to use the forcing method to construct new models and to obtain consistency and independence results, and the conservativeness of the assumptions, which allows us to use the forcing method to investigate the ground model  $V$  from the viewpoint of generic extensions  $V[G]$ .

Finally, in Section 7 we will investigate the relation between the usual forcing under FA and that under AFA. As a result, we will see that, if we assume AC, generic extensions do not change the class of ordinals as under FA and our generic models can be constructed by other well known methods, the usual generic extension and Aczel's construction, which have already been introduced. As a consequence, we obtain a positive solution to the conjecture in [5]. Although it seems that our new method is less useful in the presence of AC, this result means that generic extension works in a quite parallel way to with that in FA-set theory. In particular, the Forcing Theorem and Minimal Model Theorem we will establish in this paper show a direct relationship between the ground model and the generic models, and so these theorems have some practical use in finding and investigating new AFA-models even under AC. Whereas the utility of our method will undoubtedly become clear as model theory for AFA-models is developed, we will give one example which shows the utility.

<sup>2)</sup> The precise meaning of "justified" is explained below.

Moreover, as Aczel pointed out that the quotient of transition systems by the largest bisimulation and transition relations have the essentially same theory as the set theory with AFA (see [1, 5]), we could hope that, by using our new method, some open problems about transition systems will turn out to be consistent or independent.

In this paper, we use  $ZF_{\in}$  as a base theory.

**Definition 1.1** (Extensional set theory  $ZF_{\in}$ ) The language consists of two predicate symbols  $\in, =$  of arity 2. The axioms of  $ZF_{\in}$  are equality axioms and the following:

Extensionality Axiom:  $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$ .

Empty Set Axiom:  $\exists x \forall y (\neg y \in x)$ .

Pairing Axiom:  $\forall x \forall y \exists z (x \in z \wedge y \in z)$ .

Union Axiom:  $\forall x \exists y (\forall u \in x) (\forall v \in u) (v \in y)$ .

Powerset Scheme:  $\forall x \exists y \forall u (\exists v \in y) \forall s (s \in v \leftrightarrow s \in x \wedge F(s, u, x))$ , where  $v$  does not occur in  $F$ .

Infinity Scheme:  $\forall x \exists y (x \subset y \wedge (\forall u \in y) (\exists v F(u, v, x) \rightarrow (\exists v \in y) F(u, v, x)))$ , where  $y$  does not occur in  $F$ .

Notice that the Powerset Scheme implies the Separation Scheme and the usual formulation of Powerset Axiom. Indeed, putting  $F(s, u, x) \equiv G(s, x)$ , i. e. letting  $u$  be a dummy variable, the scheme for  $F$  becomes

$$\forall x \exists y (\exists v \in y) \forall s (s \in v \leftrightarrow s \in x \wedge G(s, x)),$$

which implies the Separation Scheme

$$\forall x \exists v \forall s (s \in v \leftrightarrow s \in x \wedge G(s, x)), \quad \text{where } v \text{ does not occur in } G.$$

Putting  $F(s, u, x) \equiv s \in u$ , the scheme becomes

$$\forall x \exists y \forall u (\exists v \in y) \forall s (s \in v \leftrightarrow s \in x \wedge s \in u),$$

which means  $\forall x \exists y \forall u (u \cap x \in y)$ . Moreover, the Infinity Scheme obviously implies the usual formulation of Infinity Axiom and the Collection Scheme (denoted by Coll):

Collection Scheme:  $\forall x \exists y (\forall u \in x) (\exists v F(v, u, x) \rightarrow (\exists v \in y) F(v, u, x))$ , where  $y$  does not occur in  $F$ .

It is easy to see that all axioms of  $ZF_{\in}$  are provable in the usual axiom system  $ZF$ . In what follows,  $ZF^{-}$  stands for  $ZF - FA$ . Notice also that  $ZF^{-} + \text{Coll} + \text{SDC} \vdash ZF_{\in}$ , where SDC stands for the Strong Dependent Choice Scheme, defined as follows:

Strong Dependent Choice Scheme:

$$\forall u \exists v F(x, u, v) \rightarrow \forall u \exists f : \text{function}(\text{dom}(f) = \omega \wedge f(0) = u \wedge (\forall n \in \omega) F(x, f(n), f(n+1))),$$

where  $f$  does not occur in  $F$ .

In fact, for

$$ZF^{-} + \text{Coll} + \text{SDC} \vdash ZF_{\in},$$

it suffices to show that the Infinity Scheme is deduced from  $ZF^{-} + \text{Coll} + \text{SDC}$ . Given  $x$  and a formula  $F(x, u, v)$ , consider the formula  $G(a, b, x) \equiv (\forall u \in a) (\exists v F(x, u, v) \rightarrow (\exists v \in b) F(x, u, v))$ . By Coll,  $\forall a \exists b G(a, b, x)$ . Then by SDC, we have a function  $f$  on  $\omega$  such that  $f(0) = u$  and that  $G(f(n), f(n+1), x)$  for all  $n \in \omega$ . Thus  $\bigcup \{f(n) \mid n \in \omega\}$  is required.

## 2 Intensional set theory $ZF_{\varepsilon}$

In this section, we will see that the extensional set theory  $ZF_{\in}$  and the corresponding intensional set theory  $ZF_{\varepsilon}$  are interpretable to each other and hence that they are equiconsistent. Using this interpretation, we can construct an extensional model (in other words, model of set theory with extensionality) from an intensional model (a model of set theory without extensionality) automatically. We will use this method for defining forcing in the next section.

**Definition 2.1** (Intensional set theory  $ZF_\varepsilon$ ) The language consists of a unique predicate symbol  $\varepsilon$  of arity 2. The axioms of  $ZF_\varepsilon$  are the following:

Empty Set Axiom:  $\exists x \forall y (\neg y \varepsilon x)$ .

Pairing Axiom:  $\forall x \forall y \exists z (x \varepsilon z \wedge y \varepsilon z)$ .

Union Axiom:  $\forall x \exists y (\forall u \varepsilon x) (\forall v \varepsilon u) (v \varepsilon y)$ .

Powerset Scheme:  $\forall x \exists y \forall u (\exists v \varepsilon y) \forall s (s \varepsilon v \leftrightarrow s \varepsilon x \wedge F(s, u, x))$ , where  $v$  does not occur in  $F$ .

Infinity Scheme:  $\forall x \exists y ((\forall u \varepsilon x) (u \varepsilon y) \wedge (\forall u \varepsilon y) (\exists v F(u, v, x) \rightarrow (\exists v \varepsilon y) F(u, v, x)))$ , where  $y$  does not occur in  $F$ .

Similarly, the Powerset Scheme implies the Separation Scheme:

$$\forall x \exists v \forall s (s \varepsilon v \leftrightarrow s \varepsilon x \wedge G(s, x)), \quad \text{where } v \text{ does not occur in } G,$$

and the Infinity Scheme implies the Collection Scheme:

$$\forall x \exists y (\forall u \varepsilon x) (\exists v F(v, u, x) \rightarrow (\exists v \varepsilon y) F(v, u, x)), \quad \text{where } y \text{ does not occur in } F.$$

The main lemma of this section is the following:

**Proposition 2.2** We can define  $\dot{\varepsilon}$  and  $\sim$  in  $ZF_\varepsilon$  such that

1.  $ZF_\varepsilon \vdash \forall x \forall y (x \dot{\varepsilon} y \leftrightarrow (\exists z \varepsilon y) (x \sim z))$ ,
2.  $ZF_\varepsilon \vdash \forall x \forall y (x \varepsilon y \rightarrow x \dot{\varepsilon} y)$ ,
3.  $ZF_\varepsilon \vdash \forall x \forall y (x \sim y \rightarrow (\forall u \varepsilon x) (u \dot{\varepsilon} y))$ , and
4.  $ZF_\varepsilon \vdash \forall x \forall y \forall z (x \sim x \wedge (x \sim y \leftrightarrow y \sim x) \wedge (x \sim y \wedge y \sim z \rightarrow x \sim z))$ .

The definitions of  $\dot{\varepsilon}$  and  $\sim$  and the proof of this proposition will be given later.

**Definition 2.3** First, define the Leibniz equality  $\stackrel{L}{\equiv}$ , by

$$x \stackrel{L}{\equiv} y \equiv \forall z (x \varepsilon z \rightarrow y \varepsilon z).$$

**Lemma 2.4**  $\stackrel{L}{\equiv}$  is a congruence for all formulae in  $ZF_\varepsilon$ .

*Proof.* Suppose  $x \stackrel{L}{\equiv} y$  and  $F(x)$ . Then, by the Pairing Axiom and the Separation Axiom Scheme, we have  $a$  such that  $x \varepsilon a$  and  $b$  such that  $\forall z (z \varepsilon b \leftrightarrow z \varepsilon a \wedge F(z))$ , which means  $x \varepsilon b$ . Thus  $x \stackrel{L}{\equiv} y$  implies  $y \varepsilon b$  and so  $F(y)$ . In particular, putting  $F(z) \equiv z \stackrel{L}{\equiv} x$ , we have  $x \stackrel{L}{\equiv} y \rightarrow y \stackrel{L}{\equiv} x$  and, putting  $F(z) \equiv z \stackrel{L}{\equiv} w$ , we have

$$x \stackrel{L}{\equiv} y \wedge x \stackrel{L}{\equiv} w \rightarrow y \stackrel{L}{\equiv} w. \quad \square$$

**Lemma 2.5** Every set is an (intensional) subset of some (intensionally) transitive set, i. e.

$$ZF_\varepsilon \vdash \forall a \exists b ((\forall x \varepsilon a) (x \varepsilon b) \wedge (\forall x \varepsilon b) (\forall y \varepsilon x) (y \varepsilon b)).$$

*Proof.* Given any  $u$ , by the Union Axiom and the Separation Axiom Scheme, we have  $v$  such that

$$\forall x (x \varepsilon v \leftrightarrow (\exists y \varepsilon u) (x \varepsilon y)).$$

Then by the Infinity Scheme with  $F(u, v, z) \equiv \forall x (x \varepsilon v \leftrightarrow (\exists y \varepsilon u) (x \varepsilon y))$ , given any  $a$ , where  $z$  is a dummy variable, we have  $c$  such that  $a \varepsilon c$  (more precisely  $(\forall z \varepsilon \{a\}) (u \varepsilon c)$ , where  $\{a\}$  denotes a set such that  $a \varepsilon \{a\}$ ) and  $(\forall u \varepsilon c) (\exists v \varepsilon c) (\forall y (y \varepsilon v \leftrightarrow (\exists x \varepsilon u) (y \varepsilon x)))$ . Again by the Union Axiom and the Separation Axiom Scheme, we have  $b$  such that  $\forall x (x \varepsilon b \leftrightarrow (\exists y \varepsilon c) (x \varepsilon y))$  and this  $b$  is required. In fact,  $x \varepsilon a \varepsilon c$  implies  $x \varepsilon b$  and if  $y \varepsilon x \varepsilon b$ , say  $(y \varepsilon) x \varepsilon u \varepsilon c$ , then there is  $v \varepsilon c$  such that  $\forall y' (y' \varepsilon v \leftrightarrow (\exists x' \varepsilon u) (y' \varepsilon x'))$ , in particular  $y \varepsilon v \varepsilon c$  and so  $y \varepsilon b$ .  $\square$

**Definition 2.6** For every set  $u$ , define  $\simeq_u$ ,  $D(u)$  and  $\sim$  by

$$\begin{aligned} x \simeq_u y &\equiv x \stackrel{L}{=} y \vee (\exists a \varepsilon u)(x \varepsilon a \wedge y \varepsilon a), \\ D(u) &\equiv \forall x \forall y (x \simeq_u y \rightarrow (\forall s \varepsilon x)(\exists t \varepsilon y)(s \simeq_u t)), \\ x \sim y &\equiv \exists u (D(u) \wedge x \simeq_u y). \end{aligned}$$

Obviously  $\simeq_u$  is reflexive and symmetric, and  $D(\emptyset)$  holds. Then, since  $x \simeq_\emptyset x$  and  $x \stackrel{L}{=} x$ , we have  $x \sim x$ . In what follows,  $a \approx \{b, c\}$  is an abbreviation for

$$\forall x (x \varepsilon a \leftrightarrow x \stackrel{L}{=} b \vee x \stackrel{L}{=} c).$$

**Lemma 2.7** If  $R(x, y)$  is a reflexive and symmetric bisimulation, i. e.

$$\forall x \forall y \forall s (s \varepsilon x \wedge R(x, y) \rightarrow (\exists t \varepsilon y) R(s, t)),$$

then  $\forall x \forall y (R(x, y) \rightarrow x \sim y)$ .

*Proof.* Suppose  $R(a, b)$ . By the Pairing Axiom we have  $c$  with  $a \varepsilon c$ ,  $b \varepsilon c$  and hence an (intensionally) transitive set  $A$  such that  $(\forall x \varepsilon c)(x \varepsilon A)$ , in particular  $a \varepsilon A$  and  $b \varepsilon A$ . By the Powerset Scheme we have  $B$  such that  $\forall z (z \varepsilon B \leftrightarrow (\forall s \varepsilon z)(s \varepsilon A))$  and, by the Separation Scheme, we have  $C$  such that

$$\forall z (z \varepsilon C \leftrightarrow z \varepsilon B \wedge (\exists x \varepsilon A)(\exists y \varepsilon A)(z \approx \{x, y\})).$$

Thus, by Pairing Axiom,  $(\forall x \varepsilon A)(\forall y \varepsilon A)(\exists z \varepsilon C)(z \approx \{x, y\})$ .

By the Separation Scheme, we have  $U$  such that

$$\forall z (z \varepsilon U \leftrightarrow z \varepsilon C \wedge (\forall x \varepsilon z)(\forall y \varepsilon z)(x \varepsilon A \wedge y \varepsilon A \wedge R(x, y))).$$

Then we have

$$(*) \quad \forall x \forall y (x \simeq_U y \leftrightarrow x \stackrel{L}{=} y \vee (x, y \varepsilon A \wedge R(x, y))).$$

Indeed, if  $x \simeq_U y$  and there is  $z \varepsilon U$  such that  $x \varepsilon z \wedge y \varepsilon z$ , then, by the definition of  $U$ , we have  $x \varepsilon A$ ,  $y \varepsilon A$  and  $R(x, y)$ . Conversely if  $x, y \varepsilon A \wedge R(x, y)$ , then, by Pairing Axiom and Separation Scheme we have  $z$  such that  $z \approx \{x, y\}$  and so  $z \varepsilon U$  since  $R(x, y)$  implies  $R(y, x)$ ,  $R(x, x)$  and  $R(y, y)$ , and  $z \varepsilon C$ .

It follows that  $a \simeq_U b$  and so it suffices to show that  $D(U)$ . Suppose  $x \simeq_U y$ , if  $x, y \varepsilon A$  and  $R(x, y)$ , then, by the assumption, for any  $s \varepsilon x$ , and so we have  $t \varepsilon y$  such that  $R(s, t)$ . Since  $A$  is (intensionally) transitive,  $s, t \varepsilon A$ , which means  $s \simeq_U t$ , by (\*). If  $x \stackrel{L}{=} y$ , then, for any  $s \varepsilon x$ ,  $s \varepsilon y$  with  $s \simeq_U s$ . Thus we obtain  $D(U)$ .  $\square$

**Lemma 2.8**  $ZF_\varepsilon \vdash \forall x \forall y (x \sim y \leftrightarrow (\forall s \varepsilon x)(\exists t \varepsilon y)(s \sim t) \wedge (\forall t \varepsilon y)(\exists s \varepsilon x)(s \sim t))$ .

*Proof.* Suppose  $x \sim y$ , say  $D(u)$  and  $x \simeq_u y$ . Since  $\simeq_u$  is symmetric, we have also  $y \simeq_u x$ . Then  $D(u)$  implies  $(\forall s \varepsilon x)(\exists t \varepsilon y)(s \simeq_u t)$  and  $(\forall t \varepsilon y)(\exists s \varepsilon x)(s \simeq_u t)$ , which again imply  $(\forall s \varepsilon x)(\exists t \varepsilon y)(s \sim t)$  and  $(\forall t \varepsilon y)(\exists s \varepsilon x)(s \sim t)$ .

For the converse, by the last lemma, it suffices to show that  $R$  is a reflexive and symmetric bisimulation, where

$$R(x, y) \equiv (\forall s \varepsilon x)(\exists t \varepsilon y)(s \sim t) \wedge (\forall t \varepsilon y)(\exists s \varepsilon x)(s \sim t).$$

The reflexivity and the symmetry are obvious. Suppose  $R(x, y)$  and  $s \varepsilon x$ . Then, by the definition of  $R$ , we have  $t \varepsilon y$  such that  $s \sim t$ . As we have just shown,  $R(s, t)$ . Thus  $R$  satisfies the hypothesis of the last lemma and so  $R(x, y) \rightarrow x \sim y$ .  $\square$

**Lemma 2.9**  $ZF_\varepsilon \vdash \forall x \forall y \forall z (x \sim x \wedge (x \sim y \leftrightarrow y \sim x) \wedge (x \sim y \wedge y \sim z \rightarrow x \sim z))$ .

*Proof.* As mentioned above,  $x \sim x$ . If  $x \sim y$ , say  $D(u)$  and  $x \simeq_u y$ , then  $y \simeq_u x$ , i. e.  $y \sim x$ , since  $\simeq_u$  is symmetric.

To show the transitivity, let  $R(x, z) \equiv \exists y (x \sim y \wedge y \sim z)$ . Then, by Lemma 2.7, it suffices to show that  $R$  is a bisimulation, since  $R$  is obviously reflexive and symmetric because so is  $\sim$ . Suppose  $R(x, z)$  and  $s \varepsilon x$ , say  $x \sim y$  and  $y \sim z$ . Since  $x \sim y$ , we have  $t \varepsilon y$  such that  $s \sim t$  by the last lemma, and  $y \sim z$  implies that there is  $u \varepsilon z$  such that  $t \sim u$ . Thus we have  $u \varepsilon z$  such that  $R(s, u)$ .  $\square$

**Definition 2.10** Now we define  $\dot{\varepsilon}$  as follows:

$$x \dot{\varepsilon} y \equiv (\exists u \varepsilon y)(x \sim u).$$

We now prove Proposition 2.2.

*Proof of Proposition 2.2.* Since 1. is immediate from the definition and 4. is the last lemma, it suffices to show 2. and 3. To show 2., suppose  $x \varepsilon y$ . Then  $x \sim x$  and  $x \varepsilon y$ , i. e.  $(\exists u \varepsilon y)(x \sim u)$ . To show 3., suppose  $x \sim y$  and  $u \varepsilon x$ . Then, by Lemma 2.8, we have  $v \varepsilon y$  such that  $u \sim v$ , i. e.  $u \dot{\varepsilon} y$ .  $\square$

**Lemma 2.11**  $ZF_\varepsilon$  can be interpreted in  $ZF_{\dot{\varepsilon}}$ .

*Proof.* Interpret  $\varepsilon$  in  $ZF_\varepsilon$  by  $\dot{\varepsilon}$  in  $ZF_{\dot{\varepsilon}}$ . Then all axioms clearly hold.  $\square$

**Proposition 2.12**  $ZF_{\dot{\varepsilon}}$  can be interpreted in  $ZF_\varepsilon$ .

*Proof.* Interpret  $x \varepsilon y$  and  $x = y$  in  $ZF_{\dot{\varepsilon}}$  by  $x \dot{\varepsilon} y$  and  $x \sim y$ .

First, we claim that equality axioms,

$$\forall x \forall u \forall v (v = u \wedge u \varepsilon x \rightarrow v \varepsilon x) \quad \text{and} \quad \forall x \forall y \forall u (x = y \wedge u \varepsilon x \rightarrow u \varepsilon y)$$

can be interpreted. Suppose  $v \sim u$  and  $u \dot{\varepsilon} x$ . Then there is  $w \varepsilon x$  with  $u \sim w$  and so  $v \sim w$ , which means  $v \dot{\varepsilon} x$ . Suppose  $x \sim y$  and  $u \dot{\varepsilon} x$ , say  $w \varepsilon x$  with  $u \sim w$ . Then, by  $x \sim y$ , by Proposition 2.2, 3. we have  $w' \varepsilon y$  with  $w \sim w'$  and hence  $u \sim w'$ , i. e.  $u \dot{\varepsilon} y$ . Thus  $\sim$  is congruent with all formulae from  $ZF_{\dot{\varepsilon}}$ .

Next we check the Extensionality Axiom. To show  $\forall z (z \dot{\varepsilon} x \leftrightarrow z \dot{\varepsilon} y) \rightarrow x \sim y$ , it suffices to show that  $R$  is a reflexive and symmetric bisimulation, where  $R(x, y) \equiv \forall z (z \dot{\varepsilon} x \leftrightarrow z \dot{\varepsilon} y)$ . Suppose  $R(x, y)$  and  $s \varepsilon x$ . Then  $s \dot{\varepsilon} x$  and so  $s \dot{\varepsilon} y$ , say  $t \varepsilon y$  and  $s \sim t$ . As we have just proved above,  $s \sim t$  implies  $\forall z (z \dot{\varepsilon} s \leftrightarrow z \dot{\varepsilon} t)$ , i. e.  $R(s, t)$ .

By the Empty Axiom of  $ZF_\varepsilon$ , we have  $x$  such that  $s \varepsilon x$  does not hold for any  $s$ . If  $y \dot{\varepsilon} x$ , then there is  $s \varepsilon x$  such that  $y \sim s$ , a contradiction. Thus  $\forall y \neg (y \dot{\varepsilon} x)$ .

To show the interpretability of the Pairing Axiom, take  $z$  such that  $x \varepsilon z$  and  $y \varepsilon z$  for the given  $x, y$  by the Pairing Axiom of  $ZF_\varepsilon$ . Then  $x \dot{\varepsilon} z$  and  $y \dot{\varepsilon} z$ .

To show the interpretability of the Union Axiom, take  $y$  such that  $(\forall u \varepsilon x)(\forall v \varepsilon u)(v \varepsilon y)$  for the given  $x$ . Suppose  $u' \dot{\varepsilon} x$  and  $v' \dot{\varepsilon} u'$ . Then by the definition of  $\dot{\varepsilon}$ , there is  $u \varepsilon x$  such that  $u' \sim u$  and hence  $v' \dot{\varepsilon} u$ . Again by the definition of  $\dot{\varepsilon}$ , there is  $v \varepsilon u$  such that  $v' \sim v$ . Now  $v \varepsilon u$  and  $u \varepsilon x$  imply  $v \varepsilon y$ , which implies  $v' \dot{\varepsilon} y$ .

To check that the Powerset Scheme can be interpreted, take  $y$  such that

$$\forall u (\exists v \varepsilon y) \forall s (s \varepsilon v \leftrightarrow s \varepsilon x \wedge \check{F}(s, u, x)),$$

where  $\check{F}$  is translated from the given  $F$ . Thus, for any  $u$ , we can take  $v \varepsilon y$  such that

$$\forall s (s \varepsilon v \leftrightarrow s \varepsilon x \wedge \check{F}(s, u, x)).$$

To show  $\forall s' (s' \dot{\varepsilon} v \leftrightarrow s' \dot{\varepsilon} x \wedge \check{F}(s', u, x))$ , suppose  $s' \dot{\varepsilon} v$ , say  $s' \sim s$  and  $s \varepsilon v$ . Then  $s \varepsilon x$  and  $\check{F}(s, u, x)$ .  $s \varepsilon x$  implies  $s' \dot{\varepsilon} x$  and  $\check{F}(s, u, x)$  implies  $\check{F}(s', u, x)$  since  $\sim$  is congruent with all formulae from  $ZF_{\dot{\varepsilon}}$ . For the converse, suppose  $s' \dot{\varepsilon} x \wedge \check{F}(s', u, x)$ . Then there is  $s \varepsilon x$  such that  $s' \sim s$  and hence  $\check{F}(s, u, x)$ , which implies  $s \varepsilon v$  and hence  $s' \dot{\varepsilon} v$ .

Finally to check the Infinity Scheme, for the given  $x$  take  $y$  such that

$$(\forall u \varepsilon x)(u \varepsilon y) \wedge (\forall u \varepsilon y)(\exists v \tilde{F}(u, v, x) \rightarrow (\exists v \varepsilon y)\tilde{F}(u, v, x)),$$

where  $\tilde{F}$  is translated from the given  $F$ . Then obviously  $\forall u(u \dot{\varepsilon} x \rightarrow u \dot{\varepsilon} y)$  and it remains to show that

$$(\forall u \dot{\varepsilon} y)(\exists v \tilde{F}(u, v, x) \rightarrow (\exists v \varepsilon y)\tilde{F}(u, v, x)).$$

For  $u \dot{\varepsilon} y$ , there is  $u' \varepsilon y$  with  $u \sim u'$  and hence  $\exists v \tilde{F}(u', v, x) \rightarrow (\exists v \varepsilon y)\tilde{F}(u', v, x)$ . By the congruency of  $\sim$ , we have  $\exists v \tilde{F}(u, v, x) \rightarrow (\exists v \varepsilon y)\tilde{F}(u, v, x)$ .  $\square$

**Corollary 2.13**  $\text{Cons}(\text{ZF}_\varepsilon) \leftrightarrow \text{Cons}(\text{ZF}_\varepsilon)$ .

Notice that the discussion in this section can be done in the intuitionistic logic. Thus we actually have

$$\text{Cons}(\text{IZF}_\varepsilon) \leftrightarrow \text{Cons}(\text{IZF}_\varepsilon),$$

where  $\text{IZF}_\varepsilon$  and  $\text{IZF}_\varepsilon$  are theories in intuitionistic logic, corresponding to  $\text{ZF}_\varepsilon$  and  $\text{ZF}_\varepsilon$  respectively.

### 3 Forcing under AFA

In this section, we assume Aczel's AFA, the *anti-foundation axiom*:

For any *pointed graph*  $(N, E, a)$ , i. e.  $E \subset N \times N$  and  $a \in N$ , there exists uniquely a *decoration*  $d : N \rightarrow V$ , i. e.  $d(b) = \{d(c) \mid \langle c, b \rangle \in E\}$  for any  $b \in N$ .

This means that any pointed graph represents exact one set. The key consequence of the unique existence of the decoration is that two pointed graphs which are bisimilar to each other represent the same set and, in particular, that two sets which are  $\varepsilon$ -bisimilar to each other are in fact same set. Here, two sets  $a$  and  $b$  are  $\varepsilon$ -bisimilar iff there is a relation  $R$  such that  $aRb$  and that

$$\forall x \forall y \forall z \exists w (z \in xRy \rightarrow zRw \in y) \wedge \forall x \forall y \forall w \exists z (xRy \ni w \rightarrow x \ni zRw).$$

It is well-known (see [1]) that there is the largest fixed point  $\Phi^{(\infty)}$  of any set-continuous operator, where an operator  $\Phi$  is *set-continuous* iff  $\Phi(X) = \bigcup\{\Phi(x) \mid x \in V \ \& \ x \subset X\}$  for any class  $X$ . This holds also in our system  $\text{ZF}_\varepsilon$ :

**Lemma 3.1** *For a set-continuous operator  $\Phi$ , there is the largest fixed point  $\Phi^{(\infty)}$  and if  $Y \subset \Phi(Y)$ , then  $Y \subset \Phi^{(\infty)}$  for any class  $Y$ .*

**Proof.** Let  $\Phi^{(\infty)} = \bigcup\{x \in V \mid x \subset \Phi(x)\}$ . To show  $\Phi^{(\infty)} \subset \Phi(\Phi^{(\infty)})$ , take  $s \in \Phi^{(\infty)}$ . Then there is  $x$  with  $x \subset \Phi(x)$  such that  $s \in x$ . Since  $x \subset \Phi^{(\infty)}$ , we have

$$s \in x \subset \Phi(x) \subset \bigcup\{\Phi(y) \mid y \in V \ \& \ y \subset \Phi^{(\infty)}\} = \Phi(\Phi^{(\infty)}).$$

Next suppose  $Y \subset \Phi(Y)$ . To show  $Y \subset \Phi^{(\infty)}$ , take  $y \in Y$ . Then, since  $y \in \Phi(Y)$  there is  $y' \subset Y$  such that  $y \in \Phi(y')$ . By Coll, for any  $x \subset Y$  there is  $x'$  such that

$$(\forall y \in x)(\exists y' \in x')(y' \subset Y \ \& \ y \in \Phi(y')).$$

Let  $z = (\bigcup x') \cap Y$ . Then  $x \subset \Phi(z)$ . By the Infinity Scheme, we have a set  $w$  such that  $\{\{y\}\} \subset w$  and that

$$(\forall x \in w \cap \mathfrak{P}(Y))(\exists z \in w)(x \subset \Phi(z) \ \& \ z \subset Y),$$

which implies  $\bigcup(w \cap \mathfrak{P}(Y)) \subset \Phi(\bigcup(w \cap \mathfrak{P}(Y)))$ . Thus  $x \subset \bigcup(w \cap \mathfrak{P}(Y)) \subset \Phi^{(\infty)}$  which means  $Y \subset \Phi^{(\infty)}$ .

Now  $\Phi^{(\infty)} \subset \Phi(\Phi^{(\infty)})$  implies  $\Phi(\Phi^{(\infty)}) \subset \Phi(\Phi(\Phi^{(\infty)}))$  and hence, by the fact we have just shown above,  $\Phi(\Phi^{(\infty)}) \subset \Phi^{(\infty)}$ .  $\square$

Now we define two basic notions for forcing theory:

1. for a poset  $\mathbb{P}$ , the largest fixed point  $V^{\mathbb{P}}$  of a set-continuous function  $\Phi^{\mathbb{P}}$ , where  $\Phi^{\mathbb{P}}(X) = \mathfrak{P}(X \times \mathbb{P})$ ;
2. the check function  $\check{\cdot}: V \longrightarrow V^{\mathbb{P}}$  such that  $\check{a} = \{\langle \check{b}, p \rangle \mid p \in \mathbb{P} \ \& \ b \in a\}$ . In fact, by the labeled anti foundation axiom which is a consequence of AFA [1], this condition defines the check function  $\check{\cdot}$ . Then, obviously,  $\text{ran}(\check{\cdot}) \subset \mathfrak{P}(\text{ran}(\check{\cdot}) \times \mathbb{P})$  and hence  $\text{ran}(\check{\cdot}) \subset V^{\mathbb{P}}$  by the maximality of  $V^{\mathbb{P}}$ . Notice that FA also allows us to define the check function.

According to the convention in set theory, we call an element of  $V^{\mathbb{P}}$  a  $\mathbb{P}$ -name and to denote names we use dotted symbols, e. g.  $\dot{x}$ .

**Definition 3.2** (Forcing relation) For  $p \in \mathbb{P}$  and  $\text{ZF}_{\varepsilon} + V(\cdot)$ -formula  $\varphi$  with parameters  $\dot{x}_1, \dots, \dot{x}_n \in V^{\mathbb{P}}$ , define  $p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$  recursively:

$$\begin{aligned} p \Vdash \dot{x}_1 \in \dot{x}_2 &\Leftrightarrow \{q \in \mathbb{P} \mid \langle \dot{x}_1, q \rangle \in \dot{x}_2\} \text{ is predense below } p; \\ p \Vdash (\varphi \wedge \psi)(\dot{x}_1, \dots, \dot{x}_n) &\Leftrightarrow p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n) \text{ and } p \Vdash \psi(\dot{x}_1, \dots, \dot{x}_n); \\ p \Vdash (\varphi \vee \psi)(\dot{x}_1, \dots, \dot{x}_n) &\Leftrightarrow \{q \in \mathbb{P} \mid q \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n) \text{ or } q \Vdash \psi(\dot{x}_1, \dots, \dot{x}_n)\} \text{ is dense below } p; \\ p \Vdash (\varphi \rightarrow \psi)(\dot{x}_1, \dots, \dot{x}_n) &\Leftrightarrow \text{for all } q \leq p, \text{ if } q \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n), \text{ then } q \Vdash \psi(\dot{x}_1, \dots, \dot{x}_n); \\ p \Vdash (\neg\varphi)(\dot{x}_1, \dots, \dot{x}_n) &\Leftrightarrow \text{there is no } q \leq p \text{ such that } q \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n); \\ p \Vdash \exists x \varphi(x, \dot{x}_1, \dots, \dot{x}_n) &\Leftrightarrow \{q \in \mathbb{P} \mid q \Vdash \varphi(\dot{x}, \dot{x}_1, \dots, \dot{x}_n) \text{ for some } \dot{x} \in V^{\mathbb{P}}\} \text{ is dense below } p; \\ p \Vdash \forall x \varphi(x, \dot{x}_1, \dots, \dot{x}_n) &\Leftrightarrow q \Vdash \varphi(\dot{x}, \dot{x}_1, \dots, \dot{x}_n) \text{ for all } q \leq p, \dot{x} \in V^{\mathbb{P}}; \\ p \Vdash V(\dot{x}) &\Leftrightarrow \{q \in \mathbb{P} \mid q \Vdash \dot{x} \sim \check{a} \text{ for some } a \in V\} \text{ is dense below } p. \end{aligned}$$

$A \subset \mathbb{P}$  is *predense below*  $p \in \mathbb{P}$  iff  $(\forall q \leq p)(\exists r \leq q)(\exists a \in A)(r \leq a)$ .

Extend the definition for  $\text{ZF}_{\varepsilon}$ -formulae via the interpretation as in Proposition 2.12, i. e. define

$$p \Vdash F(\dot{x}_1, \dots, \dot{x}_n)$$

by  $p \Vdash \tilde{F}(\dot{x}_1, \dots, \dot{x}_n)$ , where  $\tilde{F}$  is the interpreted formula from  $F$ .

The definition for atomic formulae is much easier than that of the usual forcing relation. The definitions for connectives and quantifiers are called ‘‘sheaf semantics’’ with the dense topology and hence the forcing relation respects the intuitionistic logic. In fact, it respects the classical logic:<sup>3)</sup>

**Lemma 3.3** *If  $\varphi_1(\vec{x}), \dots, \varphi_n(\vec{x}) \vdash_{\text{CL}} \psi(\vec{x})$  and if  $p \Vdash \varphi_i(\vec{x})$  for  $1 \leq i \leq n$ , then  $p \Vdash \psi(\vec{x})$ .*

*Proof.* We can replace  $\vdash_{\text{CL}}$  by  $\vdash_{\text{LJ+LEM}}$ . By induction on the proof of  $\varphi_1(\vec{x}), \dots, \varphi_n(\vec{x}) \vdash_{\text{LJ+LEM}} \psi(\vec{x})$ .

For all LJ-rules, this is obvious since the forcing relation is sheaf semantics over the dense topology. However the reader can prove easily these by induction on the derivation using the next lemma.

We now prove this in the case where the last rule is LEM-rule. Suppose  $p \not\Vdash (\varphi \vee \neg\varphi)(\vec{x})$ . Then

$$\{q \mid q \Vdash \varphi(\vec{x}) \text{ or } q \Vdash \neg\varphi(\vec{x})\} \text{ is not dense below } p,$$

i. e. there is  $r \leq p$  such that for any  $s \leq r$ ,  $s \not\Vdash \varphi(\vec{x})$  and  $s \not\Vdash \neg\varphi(\vec{x})$ . Then, by definition,  $r \Vdash \neg\varphi(\vec{x})$ , a contradiction.  $\square$

**Lemma 3.4** *If  $p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$  and  $q \leq p$ , then  $q \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$ . If  $\{q \in \mathbb{P} \mid q \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)\}$  is dense below  $p$ , then  $p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$ .*

*Proof.* These properties hold for all sheaf semantics generally but the reader can prove these by induction on  $\varphi$  easily.  $\square$

**Proposition 3.5** *For any  $\text{ZF}_{\varepsilon}$ -axiom  $\theta$  and  $p \in \mathbb{P}$ ,  $p \Vdash \theta$ . As a consequence,  $p \Vdash \theta$  for any  $\text{ZF}_{\varepsilon}$ -axiom  $\theta$ .*

<sup>3)</sup> In the following,  $\vdash_{\text{CL}}$  denotes the derivability of classical logic and  $\Gamma \vdash_{\text{LJ+LEM}} \varphi$  means that the sequent  $\Gamma \Rightarrow \varphi$  in the Gentzen-style sequent calculus LJ for intuitionistic logic with the law of excluded middle LEM as an axiom.



**Proof.** In what follows, for a set  $a$ , define  $\text{trcl}(a)$  as the least transitive set  $X$  with  $a \in X$ .

**Empty Axiom:** Since

$$\emptyset \in \mathfrak{P}(V^{\mathbb{P}} \times \mathbb{P}) = V^{\mathbb{P}}$$

and there are no  $\dot{x} \in V^{\mathbb{P}}$  and  $p \in \mathbb{P}$  such that  $p \Vdash \dot{x} \varepsilon \emptyset$ . Thus  $p \Vdash \forall y \neg(y \varepsilon \emptyset)$ .

**Pairing Axiom:** For  $\dot{x}, \dot{y} \in V^{\mathbb{P}}$ , define

$$\dot{z} = \text{up}(\dot{x}, \dot{y}) = \{\langle \dot{x}, p \rangle \mid p \in \mathbb{P}\} \cup \{\langle \dot{y}, p \rangle \mid p \in \mathbb{P}\} \in V^{\mathbb{P}}.$$

Then obviously  $p \Vdash \dot{x} \varepsilon \dot{z} \wedge \dot{y} \varepsilon \dot{z}$  for all  $p \in \mathbb{P}$ .

**Union Axiom:** For  $\dot{x} \in V^{\mathbb{P}}$ , define

$$\dot{y} = \{\langle \dot{z}, p \rangle \mid (\exists \dot{w} \in V^{\mathbb{P}})(\exists q \in \mathbb{P})(\langle \dot{z}, q \rangle \in \dot{w} \wedge \langle \dot{w}, p \rangle \in \dot{x})\}.$$

Then  $\dot{y} \in V^{\mathbb{P}}$ . Suppose  $\dot{a} \in V^{\mathbb{P}}$  and  $r \leq p$  with  $r \Vdash \dot{a} \varepsilon \dot{x}$  and suppose  $r' \leq r$  and  $r' \Vdash \dot{b} \varepsilon \dot{a}$ . Then

$$\{s \in \mathbb{P} \mid \langle \dot{a}, s \rangle \in \dot{x}\} \text{ is predense below } r$$

and  $\{q \in \mathbb{P} \mid \langle \dot{b}, q \rangle \in \dot{a}\}$  is predense below  $r'$ . Thus  $\{s \in \mathbb{P} \mid \langle \dot{b}, s \rangle \in \dot{y}\}$  is predense below  $r$ , i. e.  $r \Vdash \dot{b} \varepsilon \dot{y}$ , in particular  $r' \Vdash \dot{b} \varepsilon \dot{y}$ . Therefore  $r \Vdash \forall b(b \varepsilon \dot{a} \rightarrow b \varepsilon \dot{y})$  and so  $p \Vdash (\forall a \varepsilon \dot{x})(\forall b \varepsilon a)(b \varepsilon \dot{y})$  for all  $p \in \mathbb{P}$ .

**Powerset Scheme:** For  $\dot{x} \in V^{\mathbb{P}}$ , define

$$\dot{y} = \{\langle \dot{z}, p \rangle \in V^{\mathbb{P}} \times \mathbb{P} \mid (\forall q \in \mathbb{P})(q \Vdash (\forall s \varepsilon \dot{z})(s \varepsilon \dot{x}) \& (\exists \dot{u} \in V^{\mathbb{P}})(p \Vdash \forall s(s \varepsilon \dot{z} \leftrightarrow s \varepsilon \dot{x} \wedge F(s, \dot{u}, \dot{x})))\}.$$

Then  $\dot{y} \in V^{\mathbb{P}}$ . Notice that  $\dot{y}$  is a set by virtue of the intensional membership relation  $\varepsilon$ . Suppose  $p \in \mathbb{P}$  and  $\dot{u} \in V^{\mathbb{P}}$  and let  $\dot{z} = \{\langle \dot{s}, q \rangle \mid q \Vdash \dot{s} \varepsilon \dot{x} \wedge F(\dot{s}, \dot{u}, \dot{x})\}$ . Then  $\dot{z} \in V^{\mathbb{P}}$  and obviously  $p \Vdash \forall s(s \varepsilon \dot{z} \leftrightarrow s \varepsilon \dot{x} \wedge F(s, \dot{u}, \dot{x}))$  and so  $\langle \dot{z}, p \rangle \in \dot{y}$ , i. e.  $p \Vdash \dot{z} \varepsilon \dot{y}$ .

**Infinity Scheme:** By the Infinity Scheme, for any set  $X$  there is a set  $Y$  with  $X \subset Y$  such that, for all  $\dot{u} \in V^{\mathbb{P}}$  and  $p \in \mathbb{P}$  with  $\langle \dot{u}, p \rangle \in Y$ , if there is  $\dot{v} \in V^{\mathbb{P}}$  with  $p \Vdash F(\dot{v}, \dot{u}, \dot{x})$ , then there is  $\dot{v} \in V^{\mathbb{P}}$  with  $\langle \dot{v}, p \rangle \in Y$  and with  $p \Vdash F(\dot{v}, \dot{u}, \dot{x})$ . We denote this relation by  $R(X, Y)$  and  $\bar{Y} = \{\langle \dot{v}, q \rangle \mid (\exists p \geq q)(\langle \dot{v}, p \rangle \in Y)\}$ . Then we have  $\forall X \exists Z R'(X, Z)$ , where

$$R'(X, Z) \equiv \exists Y(R(X, Y) \wedge \bar{Y} = Z).$$

Again by the Infinity Scheme, for the given  $\dot{x} \in V^{\mathbb{P}}$ , we have  $W$  with  $\{\text{trcl}(\dot{x}) \times \mathbb{P}\} \subset W$  such that

$$(\forall X \in W)(\exists Z \in W)R'(X, Z).$$

Let  $W' = \{Y \in W \mid \bar{Y} = Y\}$ . Then  $\{\text{trcl}(\dot{x}) \times \mathbb{P}\} \subset W'$  and

$$(\forall X \in W')(\exists Z \in W')R'(X, Z).$$

Define  $X' = \bigcup W'$ . Then  $X'$  has the following closure properties: for any  $\dot{u} \in V^{\mathbb{P}}$  and  $p \in \mathbb{P}$  with  $\langle \dot{u}, p \rangle \in X'$ , if there is  $\dot{v} \in V^{\mathbb{P}}$  such that  $p \Vdash F(\dot{v}, \dot{u}, \dot{x})$ , then there is  $\dot{v} \in V^{\mathbb{P}}$  with  $\langle \dot{v}, p \rangle \in X'$  such that  $p \Vdash F(\dot{v}, \dot{u}, \dot{x})$ , and  $\langle \dot{u}, q \rangle \in X'$  for all  $q \leq p$ .

Define  $\dot{y} = X' \cap (V^{\mathbb{P}} \times \mathbb{P})$ . Then obviously  $\dot{y} \in V^{\mathbb{P}}$ .

Clearly, for all  $p \in \mathbb{P}$ ,  $p \Vdash \dot{s} \varepsilon \dot{x}$  implies the existence of  $q \in \mathbb{P}$  such that  $\langle \dot{s}, q \rangle \in \dot{x}$  and hence

$$\langle \dot{s}, p \rangle \in X' \cap (V^{\mathbb{P}} \times \mathbb{P}) = \dot{y}$$

which means  $p \Vdash \dot{s} \varepsilon \dot{y}$ .

Suppose  $p \Vdash \dot{u} \varepsilon \dot{y}$  and  $q \Vdash \exists v F(v, \dot{u}, \dot{x})$  for  $q \leq p$ . Then  $\{p' \in \mathbb{P} \mid \langle \dot{u}, p' \rangle \in \dot{y}\}$  is predense below  $p$  and so, by the closure of  $X'$ ,  $\{p' \leq p \mid \langle \dot{u}, p' \rangle \in \dot{y}\}$  is dense below  $p$ . Since

$$D = \{r \leq q \mid r \Vdash F(\dot{v}, \dot{u}, \dot{x}) \text{ for some } \dot{v} \in V^{\mathbb{P}}\}$$

is dense below  $q$ , we have

$$D' = \{r \leq q \mid \langle \dot{u}, r \rangle \in \dot{y} \ \& \ r \Vdash F(\dot{v}, \dot{u}, \dot{x}) \text{ for some } \dot{v} \in V^{\mathbb{P}}\}$$

is dense below  $q$ . For all  $r \in D'$ , since we have  $\dot{v} \in V^{\mathbb{P}}$  with  $\langle \dot{v}, r \rangle \in X'$  such that  $r \Vdash F(\dot{v}, \dot{u}, \dot{x})$ , we have

$$r \Vdash \dot{v} \varepsilon \dot{y} \wedge F(\dot{v}, \dot{u}, \dot{x}).$$

By the arbitrariness of  $r \in D'$ , we obtain  $q \Vdash (\exists v \varepsilon \dot{y})F(v, \dot{u}, \dot{x})$ . Thus

$$p \Vdash (\exists v)F(v, \dot{u}, \dot{x}) \rightarrow (\exists v \varepsilon \dot{y})F(v, \dot{u}, \dot{x}). \quad \square$$

Notice that, in the proof above, the formulae  $F$  in the axiom schemes can include the predicate  $V(\cdot)$  and hence that what we have proved just above is in fact stronger than the forced  $\text{ZF}_{\varepsilon}$ -axioms.

**Remark 3.6** As we notice in the part of the Powerset Axiom above, one of the most important advantages of defining the forcing via the interpretation is that  $\{\dot{x} \mid (\exists p \in \mathbb{P})(p \Vdash \dot{x} \varepsilon \dot{a})\}$  does not become a proper class. In usual, to avoid such a difficulty we use something like Scott's trick, bounding the rank to make it form a set. This trick looks irrelevant. By the interpretation in  $\text{ZF}_{\varepsilon}$ , we can avoid such an irrelevant trick.

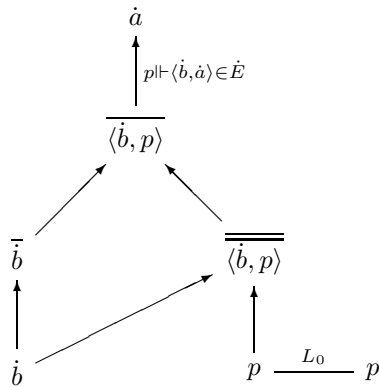
**Proposition 3.7** *Assume AFA. Then, for any  $p \in \mathbb{P}$ ,  $p \Vdash \text{AFA}$ .*

*Proof.* We prove  $p \Vdash \text{AFA}_1$  and  $p \Vdash \text{AFA}_2$ , where  $\text{AFA}_1$  states the existence of decorations for all pointed graphs and  $\text{AFA}_2$  states the uniqueness of them.

To show  $p \Vdash \text{AFA}_2$ , suppose that  $q \Vdash \text{“}\dot{a} \text{ and } \dot{b} \text{ are } \varepsilon\text{-bisimilar”}$ . Then, by Lemma 2.7 and Lemma 3.3, we have  $q \Vdash \dot{a} \sim \dot{b}$ , i. e.  $q \Vdash \dot{a} = \dot{b}$ .

Now we show  $p \Vdash \text{AFA}_1$ . Suppose  $q \Vdash \text{“}(\dot{N}, \dot{E}, \dot{c}) \text{ is a pointed graph”}$ . Define a labeled graph  $(N_0, E_0, L_0)$  as follows (see Figure 1):<sup>4)</sup>

$$\begin{aligned} N_0 &= \{\dot{b}, \overline{\dot{b}}, \overline{\langle \dot{b}, p \rangle}, \overline{\langle \dot{b}, p \rangle} \mid \dot{b} \in V^{\mathbb{P}} \ \& \ (\exists q \in \mathbb{P})(\langle \dot{b}, q \rangle \in \dot{N}) \ \& \ p \in \mathbb{P}\} \dot{\cup} \mathbb{P}, \\ E_0 &= \{(\dot{b}, \overline{\dot{b}}), (\overline{\dot{b}}, \overline{\langle \dot{b}, p \rangle}), (p, \overline{\langle \dot{b}, p \rangle}), (\overline{\dot{b}}, \overline{\langle \dot{b}, p \rangle}), (\overline{\langle \dot{b}, p \rangle}, \overline{\langle \dot{b}, p \rangle}) \mid \dot{b} \in V^{\mathbb{P}} \ \& \ p \in \mathbb{P}\} \\ &\quad \cup \{(\overline{\langle \dot{b}, p \rangle}, \dot{a}) \mid p \Vdash \langle \dot{b}, \dot{a} \rangle \in \dot{E}\}, \\ \text{dom}(L_0) &= \mathbb{P} \quad \text{and} \quad L_0(p) = p. \end{aligned}$$



**Fig. 1**

<sup>4)</sup>  $\dot{\cup}$  means disjoint union. Here, we may assume that  $\{\dot{b}, \overline{\dot{b}}, \overline{\langle \dot{b}, p \rangle}, \overline{\langle \dot{b}, p \rangle} \mid \dot{b} \in V^{\mathbb{P}} \ \& \ (\exists q \in \mathbb{P})(\langle \dot{b}, q \rangle \in \dot{N}) \ \& \ p \in \mathbb{P}\}$  and  $\mathbb{P}$  are disjoint.

By the labeled anti-foundation axiom, which is a consequence of AFA (see [1]), we have a labeled decoration

$$d : N_0 \longrightarrow V.$$

Then

$$\begin{aligned} d(\dot{a}) &= \{d(\overline{\langle \dot{b}, p \rangle}) \mid \dot{b} \in V^{\mathbb{P}} \cap N_0 \ \& \ p \Vdash \langle \dot{b}, \dot{a} \rangle \in \dot{E}\} \\ &= \{\{d(\overline{\dot{b}}), d(\overline{\langle \dot{b}, p \rangle})\} \mid \dot{b} \in V^{\mathbb{P}} \cap N_0 \ \& \ p \Vdash \langle \dot{b}, \dot{a} \rangle \in \dot{E}\} \\ &= \{\{\{d(\dot{b})\}, \{d(\dot{b}), d(p)\}\} \mid \dot{b} \in V^{\mathbb{P}} \cap N_0 \ \& \ p \Vdash \langle \dot{b}, \dot{a} \rangle \in \dot{E}\} \\ &= \{\{\{d(\dot{b})\}, \{d(\dot{b}), L_0(p)\}\} \mid \dot{b} \in V^{\mathbb{P}} \cap N_0 \ \& \ p \Vdash \langle \dot{b}, \dot{a} \rangle \in \dot{E}\} \\ &= \{\langle d(\dot{b}), p \rangle \mid \dot{b} \in V^{\mathbb{P}} \cap N_0 \ \& \ p \Vdash \langle \dot{b}, \dot{a} \rangle \in \dot{E}\}. \end{aligned}$$

Since  $\text{ran}(d \upharpoonright V^{\mathbb{P}}) \subset \mathfrak{P}(\text{ran}(d \upharpoonright V^{\mathbb{P}}) \times \mathbb{P})$ , we have  $\text{ran}(d \upharpoonright V^{\mathbb{P}}) \subset V^{\mathbb{P}}$ , i. e.  $d(\dot{a}) \in V^{\mathbb{P}}$  for all  $\dot{a} \in V^{\mathbb{P}} \cap N_0$ . In what follows, the name  $d(\dot{a})$  is denoted by  $\dot{a}^*$ .

Define a new name  $\dot{d} = \{\langle \text{op}(\dot{a}, \dot{a}^*), q \rangle \mid \dot{a} \in V^{\mathbb{P}} \cap N_0 \ \& \ q \in \mathbb{P}\}$ , where

$$\text{up}(\dot{b}, \dot{a}) = \{\langle \dot{b}, p \rangle, \langle \dot{a}, p \rangle \mid p \in \mathbb{P}\}, \quad \text{op}(\dot{b}, \dot{a}) = \text{up}(\text{up}(\dot{b}, \dot{b}), \text{up}(\dot{b}, \dot{a})).$$

We now claim that  $p \Vdash \dot{d}$  is a decoration of  $(\dot{N}, \dot{E}, \dot{c})$ . Because of  $p \Vdash \text{AFA}_2$ , it suffices to show that

$$p \Vdash \dot{d} \text{ is a bisimulation from } (\dot{N}, \dot{E}, \dot{c}) \text{ to } (\text{trcl}(\dot{d}(\dot{c})), \in, \dot{c}^*).$$

To show

$$p \Vdash \forall x \forall y \forall z (x \varepsilon y \wedge \langle z, y \rangle \varepsilon \dot{d} \rightarrow \exists w (\langle w, x \rangle \in \dot{d} \wedge \langle w, z \rangle \in \dot{E})),$$

suppose  $q \Vdash \dot{x} \varepsilon \dot{y} \wedge \langle \dot{z}, \dot{y} \rangle \varepsilon \dot{d}$ . Then  $q \Vdash \text{op}(\dot{z}, \dot{y}) \varepsilon \dot{d}$ , i. e.  $\dot{y} = \dot{z}^*$ . By  $q \Vdash \dot{x} \varepsilon \dot{y}$ ,  $\dot{x} = \dot{w}^*$  for some  $\dot{w}$  with

$$\{r \in \mathbb{P} \mid r \Vdash \langle \dot{w}, \dot{z} \rangle \in \dot{E}\} \text{ is predense below } q.$$

Thus  $q \Vdash \langle \dot{w}, \dot{z} \rangle \in \dot{E}$  and  $q \Vdash \langle \dot{w}, \dot{x} \rangle \in \dot{d}$ .

Conversely, to show

$$p \Vdash \forall x \forall y \forall w (\langle w, x \rangle \in \dot{E} \wedge \langle x, y \rangle \varepsilon \dot{d} \rightarrow \exists z (z \varepsilon y \wedge \langle w, z \rangle \in \dot{d})),$$

suppose

$$q \Vdash \langle \dot{w}, \dot{x} \rangle \in \dot{E} \wedge \langle \dot{x}, \dot{y} \rangle \varepsilon \dot{d}.$$

Then  $q \Vdash \langle \dot{x}, \dot{y} \rangle \varepsilon \dot{d}$ , i. e.  $\dot{y} = \dot{x}^*$ . By  $q \Vdash \langle \dot{w}, \dot{x} \rangle \in \dot{E}$ , we have  $\langle \dot{w}, q \rangle \in \dot{x} = \dot{y}$  and hence  $q \Vdash \dot{w} \in \dot{y}$  with obviously  $q \Vdash \langle \dot{w}, \dot{w} \rangle \in \dot{d}$ .

Thus we have proved that  $p \Vdash \text{AFA}_1$  for all  $p \in \mathbb{P}$ . □

**Remark 3.8** It is easy to deduce  $p \Vdash \text{FA}$  from  $\text{FA}$ : It suffices to show that  $p \Vdash \dot{a} \in \text{WF}$  for any  $p \in \mathbb{P}$  and  $\dot{a} \in V^{\mathbb{P}}$ , which is, by  $p \Vdash \text{ZF}_{\in}$ , equivalent to  $p \Vdash \dot{a} \subset \text{WF}$ , by induction on  $\dot{a}$ , where  $\text{WF}$  denotes the well-founded part of the universe. We prove that  $p \Vdash (\forall x \varepsilon \dot{a})(x \in \text{WF})$ . Suppose  $q \leq p$  and  $q \Vdash \dot{b} \varepsilon \dot{a}$ . Then  $\text{rank}(\dot{b}) < \text{rank}(\dot{a})$ , by the induction hypothesis, we have  $q \Vdash \dot{b} \in \text{WF}$ , which completes the proof.

If we assume  $\text{AC}$  in the ground model, then the Maximal Principle holds and  $\text{AC}$  is forced.

**Lemma 3.9** Assume  $\text{AC}$ . For  $p \in \mathbb{P}$ , any  $\text{ZF}_{\in}$ -formula  $\varphi$  and  $\dot{a}_1, \dots, \dot{a}_n \in V^{\mathbb{P}}$ , if  $p \Vdash \exists x \varphi(x, \dot{a}_1, \dots, \dot{a}_n)$ , then there is  $\dot{b} \in V^{\mathbb{P}}$  such that  $p \Vdash \varphi(\dot{b}, \dot{a}_1, \dots, \dot{a}_n)$ .

*Proof.* Assume  $p \Vdash \exists x \varphi(x, \dot{a}_1, \dots, \dot{a}_n)$ . By  $\text{Coll}$ , there is a set  $X$  such that

$$(\forall q \in \mathbb{P})((\exists \dot{x} \in V^{\mathbb{P}})(q \Vdash \varphi(\dot{x}, \dot{a}_1, \dots, \dot{a}_n)) \rightarrow (\exists \dot{x} \in V^{\mathbb{P}} \cap X)(q \Vdash \varphi(\dot{x}, \dot{a}_1, \dots, \dot{a}_n))).$$

By Zorn's Lemma we can choose maximal

$$W \subset \{ \langle \dot{x}, q \rangle \in (X \cap V^{\mathbb{P}}) \times \mathbb{P} \mid q \Vdash \varphi(\dot{x}, \dot{a}_1, \dots, \dot{a}_n) \}$$

such that for any distinct  $\langle \dot{x}, q \rangle, \langle \dot{y}, r \rangle \in W$  there is no  $s \leq q, r$ . Then  $D = \{ q \in \mathbb{P} \mid \exists \dot{x} (\langle \dot{x}, q \rangle \in W) \}$  is predense below  $p$ , since  $\{ q \in \mathbb{P} \mid (\exists \dot{x} \in V^{\mathbb{P}} \cap X)(q \Vdash \varphi(\dot{x}, \dot{a}_1, \dots, \dot{a}_n)) \}$  is dense below  $p$ . Define

$$\dot{b} = \{ \langle \dot{z}, s \rangle \mid (\exists \dot{x}, q) (\langle \dot{x}, q \rangle \in W)(s \leq q \ \& \ s \Vdash \dot{z} \varepsilon \dot{x}) \}.$$

Then obviously  $q \Vdash \dot{b} = \dot{x}$  for all  $\langle \dot{x}, q \rangle \in W$  and hence  $q \Vdash \varphi(\dot{b}, \dot{a}_1, \dots, \dot{a}_n)$ . Since  $D$  is predense below  $p$ , we have  $p \Vdash \varphi(\dot{b}, \dot{a}_1, \dots, \dot{a}_n)$ .  $\square$

**Proposition 3.10** *Assume AC. Then, for  $p \in \mathbb{P}$ ,  $p \Vdash \text{AC}$ .*

*Proof.* Suppose  $p \Vdash \emptyset \notin \dot{x}$ . Then, for any  $\dot{y}$  and  $q \leq p$  with  $q \Vdash \dot{y} \varepsilon \dot{x}$ ,  $q \Vdash \exists z (z \in \dot{y})$  and, by the Maximal Principle, there is  $\dot{z} \in V^{\mathbb{P}}$  such that  $q \Vdash \dot{z} \in \dot{y}$ . For  $\dot{y}$  and  $q \leq p$ , choosing  $\Phi(\dot{y}, q) \in V^{\mathbb{P}}$  such that if  $q \Vdash \dot{y} \varepsilon \dot{x}$ , then  $q \Vdash \Phi(\dot{y}, q) \in \dot{y}$ .

By Zorn's Lemma, we can take a maximal

$$W \subset \{ \langle \dot{y}, q \rangle \mid q \Vdash \dot{y} \varepsilon \dot{x} \}$$

such that for any distinct  $\langle \dot{y}_1, q_1 \rangle, \langle \dot{y}_2, q_2 \rangle \in W$ . Then there is no  $r \leq q_1, q_2$  with  $r \Vdash \dot{y}_1 = \dot{y}_2$ . Define  $\dot{f} \in V^{\mathbb{P}}$  as follows:

$$\dot{f} = \{ \langle \text{op}(\dot{y}, \Phi(\dot{y}, q)), q \rangle \mid \langle \dot{y}, q \rangle \in W \}.$$

Then obviously  $p \Vdash \dot{f}$  is a function on  $\dot{x}$  and  $p \Vdash (\forall y \varepsilon \dot{x})(\dot{f}(y) \in y)$ .  $\square$

## 4 Generic extensions under AFA

We now begin to describe the fundamental theorem of forcing, which is about the generic model. However, to treat a generic model, we need a certain assumption:

**Assumption 4.1** (Assumption 1) *We work in a model which contains  $V$  as a transitive submodel. Assume that AFA holds there and that there is a  $\mathbb{P}$ -generic filter  $G$  over  $V$ .*

The justification (the consistency and, moreover, the conservativeness) of this assumption will be discussed in the next section. Under this assumption, we define the generic model:

**Definition 4.2** For a  $\mathbb{P}$ -generic filter  $G$ , define the function  $\text{val}(\cdot, G)$  on  $V^{\mathbb{P}}$  and a class  $V[G]$  as follows:

$$\text{val}(\dot{a}, G) = \{ \text{val}(\dot{b}, G) \mid (\exists p \in G)(\langle \dot{b}, p \rangle \in \dot{a}) \}, \quad V[G] = \{ \text{val}(\dot{a}, G) \mid \dot{a} \in V^{\mathbb{P}} \}.$$

By AFA, the former condition determines the function  $\text{val}(\cdot, G)$  uniquely. Notice that  $\text{val}(\check{a}, G) = a$ .

Here is a problem: is  $\text{val}(\cdot, G)$  absolute for universes which have  $G$  and contain  $V$  as a transitive class? More precisely, if  $V \cup \{G\} \subset W, W'$ , then is  $\text{val}^W(\dot{a}, G) = \text{val}^W(\dot{b}, G)$  equivalent to  $\text{val}^{W'}(\dot{a}, G) = \text{val}^{W'}(\dot{b}, G)$ ? I. e., are the following two statements

$$\begin{aligned} W \models \text{“graph}(\dot{a}, G) \text{ and graph}(\dot{b}, G) \text{ are bisimilar”}, \\ W' \models \text{“graph}(\dot{a}, G) \text{ and graph}(\dot{b}, G) \text{ are bisimilar”} \end{aligned}$$

equivalent, where  $\text{Node}_{\text{graph}(\dot{a}, G)} = \text{trcl}(\dot{a}) \cap V^{\mathbb{P}}$  and  $(\dot{x}, \dot{y}) \in \text{Edge}_{\text{graph}(\dot{a}, G)}$  iff  $(\exists p \in G)(p \Vdash \dot{x} \varepsilon \dot{y})$ ? In general, the answer seems to be no (though the author has not proved it yet), and then it seems hopeless to obtain Forcing Theorem. This is what makes the forcing under AFA more difficult to deal with than under FA.

**Problem 4.3** *Does some analogous to the Minimal Model Theorem hold? If yes, then how should we define the minimal (generic) model and in which sense the model is minimal?*

One of the most promising candidates for generic model is, of course, the quotient model  $V^{\mathbb{P}}/G$ :

**Definition 4.4** For a  $\mathbb{P}$ -generic filter  $G$ , the quotient model  $V^{\mathbb{P}}/G$  is the (2-valued) model defined as follows:

$$\begin{aligned} |V^{\mathbb{P}}/G| &= V^{\mathbb{P}} / \sim_G, \\ \dot{a} \sim_G \dot{b} &\Leftrightarrow (\exists p \in G)(p \Vdash \dot{a} = \dot{b}), \\ V^{\mathbb{P}}/G \models \dot{a}/G \in \dot{b}/G &\Leftrightarrow (\exists p \in G)(p \Vdash \dot{a} \in \dot{b}), \end{aligned}$$

where  $\dot{a}/G$  denotes the equivalence class of  $\dot{a}$  with respect to  $\sim_G$ .

**Theorem 4.5** (Forcing Theorem) *Under the situation as above,  $V^{\mathbb{P}}/G \models \varphi(\dot{a}_1/G, \dots, \dot{a}_n/G)$  iff there is  $p \in G$  such that  $p \Vdash \varphi(\dot{a}_1, \dots, \dot{a}_n)$ . Thus  $V^{\mathbb{P}}/G \models \text{ZF}_\epsilon + \text{AFA}$ .*

*Proof.* The reader can prove these by induction on  $\varphi$  easily. □

Let us turn to the problem that in which model (or under which assumption) we should consider the Forcing Theorem and the Minimal Model Theorem. Now we assume that the universe we fixed as the outside of the ground model  $V$  contains the quotient model  $V^{\mathbb{P}}/G$  “essentially”, more precisely:

**Assumption 4.6** (Assumption 2)  $\text{val}(\dot{a}, G) = \text{val}(\dot{b}, G)$  iff there is  $p \in G$  such that  $p \Vdash \dot{a} = \dot{b}$ .

The justification of this assumption will also be discussed in a later section.

**Remark 4.7** Under FA, Assumption 4.6 actually holds because of the coincidence of the greatest and least fixed points for set-continuous maps: For a set-continuous map  $\Phi$ , let  $\Phi^{(\infty)}$  and  $\Phi_{(\infty)}$  denote the greatest fixed point  $\Phi^{(\infty)} = \bigcup \{x \mid x \in V, x \subset \Phi(x)\}$  and the least fixed point  $\Phi_{(\infty)} = \bigcap \{x \mid x \in V, x \supset \Phi(x)\}$  respectively. For a filter  $G$  of  $\mathbb{P}$ , define

$$\text{FEQ}_G = \{ \langle \dot{a}, \dot{b} \rangle \mid p \Vdash \dot{a} = \dot{b} \text{ for some } p \in G \}$$

and  $\Phi_G$  as follows:

$$\begin{aligned} \Phi_G(Y) = \{ \langle \dot{a}, \dot{b} \rangle \mid &\text{for all } \langle \dot{c}, q \rangle \in \dot{a} \text{ with } q \in G, \text{ there is } \langle \dot{d}, t \rangle \in \dot{b} \text{ with } t \in G \text{ such that } \langle \dot{c}, \dot{d} \rangle \in Y \text{ and} \\ &\text{for all } \langle \dot{d}, t \rangle \in \dot{b} \text{ with } t \in G, \text{ there is } \langle \dot{c}, q \rangle \in \dot{a} \text{ with } q \in G \text{ such that } \langle \dot{c}, \dot{d} \rangle \in Y \}. \end{aligned}$$

It is easy to see the set-continuity of  $\Phi_G$  and, under  $\text{AFA}_2$ ,  $\text{val}(\dot{a}, G) = \text{val}(\dot{b}, G)$  iff  $\langle \dot{a}, \dot{b} \rangle \in \Phi_G^{(\infty)}$ .

Thus Assumption 4.6 can be formulated as  $\text{FEQ}_G = \Phi_G^{(\infty)}$ . One can show without AFA that for a  $\mathbb{P}$ -generic filter  $G$ ,  $\Phi_G(\text{FEQ}_G) = \text{FEQ}_G$ , i. e.  $\text{FEQ}_G$  is a fixed point of  $\Phi_G$ .

Indeed,  $\Phi_G(\text{FEQ}_G) \supset \text{FEQ}_G$  is straightforward and, for the converse, if  $\langle \dot{a}, \dot{b} \rangle \notin \text{FEQ}_G$ , i.e. for any  $p \in G$ ,  $p \not\Vdash \dot{a} = \dot{b}$ , then, since  $p \Vdash_{\mathbb{P}} \text{LEM}$ , we have  $q \Vdash_{\mathbb{P}} \neg(\dot{a} = \dot{b})$  and so

$$q \Vdash_{\mathbb{P}} (\exists x \varepsilon \dot{a})(x \notin \dot{b}) \vee (\exists y \varepsilon \dot{b})(y \notin \dot{a}).$$

Now we may assume  $q \Vdash_{\mathbb{P}} (\exists x \varepsilon \dot{a})(x \notin \dot{b})$ . Thus

$$\{ r \leq q \mid r \Vdash \dot{c} \varepsilon \dot{a} \wedge \dot{c} \notin \dot{b} \text{ for some } \dot{c} \}$$
 is dense below  $q$

and so  $r \in G$  and  $\dot{c}$  such that  $r \Vdash \dot{c} \varepsilon \dot{a} \wedge \dot{c} \notin \dot{b}$ . We may assume  $\langle \dot{c}, s \rangle \in \dot{a}$  for some  $s \geq r$ . Since  $r \Vdash \neg(\dot{c} \in \dot{b})$ , for any  $\langle \dot{d}, t \rangle \in \dot{b}$  and  $u \leq t$  with  $u \leq r$ ,  $u \not\Vdash \dot{c} = \dot{d}$ . If additionally  $t \in G$  and  $\langle \dot{c}, \dot{d} \rangle \in \text{FEQ}_G$ , say  $v \Vdash \dot{c} = \dot{d}$  for  $v \in G$ , then we have  $u \leq t, r, v$  with  $u \in G$ , a contradiction. Thus  $\langle \dot{c}, \dot{d} \rangle \notin \Phi_G(\text{FEQ}_G)$ .

Therefore,  $\Phi_G^{(\infty)} \subset \text{FEQ}_G \subset \Phi_G^{(\infty)}$ . This implies  $\text{FEQ}_G = \Phi_G^{(\infty)}$  if  $\Phi_{G^{(\infty)}} = \Phi_G^{(\infty)}$  holds, which is a theorem of FA.

At last we now could reach the analogues of the key theorems for forcing under FA.

**Theorem 4.8** (Forcing Theorem and Minimal Model Theorem) *Under the assumptions above, the following hold:*

1.  $V[G]$  is the minimal model among the models having  $G$  and containing  $V$  as a transitive class.
2.  $V[G] \models \varphi(\text{val}(\dot{a}_1, G), \dots, \text{val}(\dot{a}_n, G))$  iff there is  $p \in G$  such that  $p \Vdash \varphi(\dot{a}_1, \dots, \dot{a}_n)$ , for any formula  $\varphi$ .

*Proof.* For the latter claim, it suffices to show that

$$\text{val}(\dot{a}, G) \mapsto \dot{a}/G$$

provides us an isomorphism  $V[G] \cong V^{\mathbb{P}}/G$  by the last theorem. The well-definedness is immediate from Assumption 4.6.

Suppose  $\text{val}(\dot{a}, G) \in \text{val}(\dot{b}, G)$ . Then we have  $\dot{c}$  such that  $\text{val}(\dot{a}, G) = \text{val}(\dot{c}, G)$  and  $\langle \dot{c}, p \rangle \in \dot{b}$  for some  $p \in G$  and therefore we have  $q \Vdash \dot{a} = \dot{c}$  for some  $q \in G$  and  $p \Vdash \dot{c} \in \dot{b}$ . Since  $G$  is a filter, we have  $s \in G$  such that  $s \leq p, q$  and hence  $s \Vdash \dot{a} \in \dot{b}$  and so  $\dot{a}/G \in \dot{b}/G$ .

Conversely suppose  $\dot{a}/G \in \dot{b}/G$ . Then  $p \Vdash \dot{a} \in \dot{b}$  for some  $p \in G$ . Thus  $p \Vdash \exists x(\dot{a} = x \wedge x \in \dot{b})$ , i. e.

$$\{q \leq p \mid q \Vdash \dot{a} = \dot{c} \wedge \dot{c} \in \dot{b} \text{ for some } \dot{c}\} \text{ is dense below } p.$$

Therefore we have  $q \in G$  such that  $q \Vdash \dot{a} = \dot{c} \wedge \dot{c} \in \dot{b}$  for some  $\dot{c}$ . Then

$$\text{val}(\dot{a}, G) = \text{val}(\dot{c}, G) \in \text{val}(\dot{b}, G).$$

We have completed the proof of the latter. For the former, it is obvious that for any model  $M$  of  $\text{ZF}_{\in} + \text{AFA}$  with  $G \in M$ ,  $V \subset M$  contains  $V[G]$  because of the absoluteness of decorations, and hence it remains to show that  $V[G] \cong V^{\mathbb{P}}/G$  itself is a model of  $\text{ZF}_{\in} + \text{AFA}$ . However, this is immediate from the latter (Forcing Theorem) and Propositions 3.5 and 3.7.  $\square$

## 5 The generic filter elimination

In this section, we try to justify Assumption 4.1 we needed in the previous section. The method we employ here is an analogue to one of the usual methods which we use under FA. As usual, we will have the conservativeness of Assumption 4.1 over the original theory.

Recall that, for a unary formula  $U(x)$ , the relativization to  $U$  is the syntactical transformation  $\varphi \mapsto \varphi^U$ , defined as follows:

$$\begin{aligned} (x \in y)^U &\equiv x \in y, & (x = y)^U &\equiv x = y, \\ (\varphi \square \psi)^U &\equiv \varphi^U \square \psi^U \quad (\text{where } \square \equiv \wedge, \vee, \rightarrow), & (\neg \varphi)^U &\equiv \neg(\varphi^U), \\ (\exists x \varphi)^U &\equiv \exists x(U(x) \wedge \varphi^U), & (\forall x \varphi)^U &\equiv \forall x(U(x) \rightarrow \varphi^U). \end{aligned}$$

According to the convention,  $U \vDash \varphi$  denotes  $\varphi^U$ .

**Lemma 5.1** For any  $p \in \mathbb{P}$  and  $\text{ZF}_{\in}$ -formula  $\varphi$ ,  $p \Vdash V \vDash \varphi(\check{a}_1, \dots, \check{a}_n)$  iff  $\varphi(a_1, \dots, a_n)$ .

*Proof.* We prove this by induction on  $\varphi$ . Notice that  $p \Vdash \check{a} \in \check{b}$  iff  $a \in b$ .

(1)  $\check{a}_1 = \check{a}_2$ : Define

$$R(a, b) \equiv (\exists p \in \mathbb{P})(p \Vdash V \vDash \check{a} \sim \check{b}).$$

Then  $R$  is an  $\in$ -bisimulation. In fact, if  $c \in a$  and  $R(a, b)$ , then  $p \Vdash \check{c} \in \check{a}$  and, by Lemma 2.8, Lemma 3.3 and Proposition 3.5,  $p \Vdash \exists x(x \in \check{b} \wedge \check{c} \sim x)$ , which means that there are  $q \leq p$  and  $\dot{x}$  such that  $q \Vdash \dot{x} \in \check{b} \wedge \check{c} \sim \dot{x}$ . Since  $q \Vdash \dot{x} \in \check{b}$ ,  $\dot{x} = \check{d}$  for some  $d \in b$  and so  $q \Vdash \check{d} \sim \check{c}$  which implies  $R(c, d)$ . The symmetry is obvious. Then, by  $\text{AFA}_2$  (which is also a consequence of FA),  $R(a_1, a_2)$  implies  $a_1 = a_2$ . Thus  $p \Vdash \check{a} = \check{b}$  implies  $a = b$  and the converse is obvious.

(2)  $\check{a}_1 \in \check{a}_2$ : Suppose  $p \Vdash V \vDash \check{a}_1 \in \check{a}_2$ , i. e.  $p \Vdash \exists x(x \in \check{a}_2 \wedge \check{a}_1 \sim x)$ . Then there are  $q \leq p$  and  $\dot{x}$  such that  $q \Vdash \dot{x} \in \check{a}_2 \wedge \check{a}_1 \sim \dot{x}$ . Since  $q \Vdash \dot{x} \in \check{a}_2$ ,  $\dot{x} = \check{b}$  for some  $b \in a_2$  and  $q \Vdash \check{b} \sim \check{a}_1$  which implies  $b = a_1$  by (1). Thus  $a_1 = b \in a_2$ . The converse is obvious.

(3) Connectives: We know that the forcing relation respects the classical logic and so  $\wedge$ - and  $\neg$ -cases would suffice. The  $\wedge$ -case is immediate from the definition. For the  $\neg$ -case, if  $\neg \varphi(a_1, \dots, a_n)$ , then, by the induction hypothesis,  $q \Vdash V \vDash \varphi(\check{a}_1, \dots, \check{a}_n)$  for all  $q \leq p$  and hence

$$p \Vdash \neg(V \vDash \varphi(\check{a}_1, \dots, \check{a}_n)),$$

i. e.  $p \Vdash V \vDash \neg \varphi(\check{a}_1, \dots, \check{a}_n)$ . If  $p \Vdash V \vDash \neg \varphi(\check{a}_1, \dots, \check{a}_n)$ , then  $p \Vdash V \vDash \varphi(\check{a}_1, \dots, \check{a}_n)$  and so, by the induction hypothesis, we have  $\neg \varphi(a_1, \dots, a_n)$ .

(4) Quantifiers: Similarly the  $\exists$ -case would suffice.

Suppose  $p \Vdash V \vDash \exists x \varphi(x, \check{a}_1, \dots, \check{a}_n)$ , i. e.

$$p \Vdash \exists x (V(x) \wedge V \vDash \varphi(x, \check{a}_1, \dots, \check{a}_n)).$$

Then there are  $q \leq p$  and  $\dot{x}$  such that  $q \Vdash V(\dot{x})$  and  $q \Vdash V \vDash \varphi(\dot{x}, \check{a}_1, \dots, \check{a}_n)$ . Again there are  $r \leq q$  and  $a \in V$  such that  $r \Vdash \check{a} = \dot{x}$  and hence  $r \Vdash V \vDash \varphi(\check{a}, \check{a}_1, \dots, \check{a}_n)$  which implies  $\varphi(a, a_1, \dots, a_n)$  by the induction hypothesis. Thus  $p \Vdash V \vDash \exists x \varphi(x, \check{a}_1, \dots, \check{a}_n)$  implies  $\exists x \varphi(x, a_1, \dots, a_n)$ .

For the converse, if  $\exists x \varphi(x, a_1, \dots, a_n)$ , say  $\varphi(a, a_1, \dots, a_n)$ , then by the induction hypothesis,

$$p \Vdash V \vDash \varphi(\check{a}, \check{a}_1, \dots, \check{a}_n).$$

Since  $p \Vdash V(\check{a})$ , by Lemma 3.3, we have  $p \Vdash \exists x (V(x) \wedge V \vDash \varphi(x, \check{a}_1, \dots, \check{a}_n))$ .  $\square$

**Lemma 5.2** For any  $p \in \mathbb{P}$ ,  $p \Vdash$  “ $V$  is a transitive class”.

*Proof.* What we must show is that  $p \Vdash \forall x \forall y (y \in x \wedge V(x) \rightarrow V(y))$  for all  $p \in \mathbb{P}$ . Suppose  $q \Vdash \dot{y} \in \dot{x}$  and  $q \Vdash V(\dot{x})$ . Then  $D = \{r \leq q \mid \exists a (r \Vdash \dot{x} = \check{a})\}$  is dense below  $q$  and for any  $r \in D$ , say  $r \Vdash \dot{x} = \check{a}$ ,  $r \Vdash \dot{y} \in \check{a}$ , i. e.  $r \Vdash \exists z (z \varepsilon \check{a} \wedge \dot{y} = z)$ . Thus  $D'(r) = \{s \leq r \mid (\exists \dot{z} \in V^{\mathbb{P}})(s \Vdash \dot{z} \varepsilon \check{a} \wedge \dot{y} = \dot{z})\}$  is dense below  $r$  for all  $r \in D$ . For  $s \in D(r)$ , say  $s \Vdash \dot{z} \varepsilon \check{a} \wedge \dot{y} = \dot{z}$ , then  $\dot{z} = \check{b}$  for some  $b \in a$  and hence  $s \Vdash \dot{y} = \check{b}$ . Therefore  $r \Vdash V(\dot{y})$  for all  $r \in D$  and hence  $q \Vdash V(\dot{y})$  by Lemma 3.4.  $\square$

We now introduce the canonical name  $\check{G}$  for the generic filter as follows:

$$\check{G} = \{\langle \check{p}, p \rangle \mid p \in \mathbb{P}\}.$$

**Lemma 5.3** For any  $p \in \mathbb{P}$ ,

$$p \Vdash \text{“}\check{G} \text{ is a } \check{\mathbb{P}}\text{-generic filter over } V\text{”}.$$

If additionally  $\mathbb{P}$  is not atomic, then  $p \Vdash \check{G} \subsetneq \check{\mathbb{P}}$ .

*Proof.* First we check that  $p \Vdash \check{G} \subset \check{\mathbb{P}}$  and  $p \Vdash \check{G} \neq \emptyset$ . To show these, it suffices to show that

$$p \Vdash \forall x (x \varepsilon \check{G} \rightarrow x \in \check{\mathbb{P}}) \quad \text{and} \quad p \Vdash \check{p} \in \check{G}.$$

The latter is obvious and, for the former, suppose  $p \Vdash \dot{x} \varepsilon \check{G}$ . Then  $\dot{x} = \check{q}$  for some  $q \in \mathbb{P}$  and so  $p \Vdash \dot{x} \in \check{\mathbb{P}}$ .

To show the upward closure  $p \Vdash \forall x \forall y (y \in \check{\mathbb{P}} \wedge x \varepsilon \check{G} \wedge x \check{\leq} y \rightarrow y \varepsilon \check{G})$ , it suffices to show

$$p \Vdash \forall x \forall y (y \varepsilon \check{\mathbb{P}} \wedge x \varepsilon \check{G} \wedge x \check{\leq} y \rightarrow y \varepsilon \check{G}).$$

Suppose

$$p \Vdash \dot{y} \varepsilon \check{\mathbb{P}} \wedge \dot{x} \varepsilon \check{G} \wedge \dot{x} \check{\leq} \dot{y}.$$

Then  $\dot{y} = \check{q}$  for some  $q \in \mathbb{P}$  and  $\dot{x} = \check{r}$  with  $\{r\}$  is predense below  $p$ . Since  $p \Vdash V \vDash \check{r} \check{\leq} \check{q}$ , we have  $r \leq q$  by Lemma 5.1, and so  $\{q\}$  is also predense below  $p$ . Thus  $p \Vdash \check{q} \varepsilon \check{G}$  and hence  $p \Vdash \dot{y} \varepsilon \check{G}$ .

To show the filterness  $p \Vdash \forall x \forall y (x \varepsilon \check{G} \wedge y \varepsilon \check{G} \rightarrow (\exists z \in \check{G})(z \check{\leq} x \wedge z \check{\leq} y))$ , it suffices to show

$$p \Vdash \forall x \forall y (x \varepsilon \check{G} \wedge y \varepsilon \check{G} \rightarrow (\exists z \in \check{G})(z \check{\leq} x \wedge z \check{\leq} y)).$$

Suppose  $p \Vdash \check{q} \varepsilon \check{G}$  and  $p \Vdash \check{r} \varepsilon \check{G}$ , i. e. both  $\{q\}$  and  $\{r\}$  are predense below  $p$ . Then

$$D = \{s \leq p \mid s \leq q \ \& \ s \leq r\}$$

is dense below  $p$ . For all  $s \in D$ ,  $s \Vdash \check{s} \check{\leq} \check{q} \wedge \check{s} \check{\leq} \check{r} \wedge \check{s} \varepsilon \check{G}$ , and therefore  $p \Vdash (\exists z \in \check{G})(z \check{\leq} \check{q} \wedge z \check{\leq} \check{r})$ .

Finally we claim the genericity

$$p \Vdash \forall D ("V(D) \wedge D \subset \check{\mathbb{P}} \wedge D \text{ is dense}" \rightarrow D \cap \check{G} \neq \emptyset).$$

Suppose  $p \Vdash "V(\dot{D}) \wedge \dot{D} \subset \check{\mathbb{P}} \wedge \dot{D} \text{ is dense}"$ . Then  $D' = \{q \leq p \mid q \Vdash \dot{D} = \check{E} \text{ for some } E\}$  is dense below  $p$ . For  $q \in D'$ , say  $q \Vdash \dot{D} = \check{E}$ , then  $q \Vdash "\check{E} \subset \check{\mathbb{P}} \wedge \check{E} \text{ is dense}"$  which is equivalent to

$$q \Vdash "V \vDash \check{E} \subset \check{\mathbb{P}} \wedge \check{E} \text{ is dense}".$$

Thus, by Lemma 5.1, we have that  $E \subset \mathbb{P}$  is dense. Since, for all  $r \in E$ ,  $r \Vdash \check{r} \varepsilon \check{G} \wedge \check{r} \varepsilon \check{E}$ , we have

$$q \Vdash \exists x (x \in \check{G} \wedge x \in \check{E}),$$

i. e.  $q \Vdash \exists x (x \in \check{G} \wedge x \in \dot{D})$ . By the arbitrariness of  $q \in D'$ , we have  $p \Vdash \exists x (x \in \check{G} \wedge x \in \dot{D})$ .  $\square$

Therefore, if we assume that we work in the forcing relations, then, by the previous two lemmata, Assumption 4.1 is satisfied and, by Lemma 5.1, this assumption does not change the theory of the ground model  $V$ .

The well known alternative way to justify the existence of a generic filter is by the reflection principle, which asserts that, for any finite sentences, there is a countable transitive model to which the sentences are absolute. However, to obtain such a *countable* transitive model, we need to assume AC and the countability is inevitable to construct generic filters in general.

## 6 The quotient model $V^{\mathbb{P}}/G$

Now let us turn to Assumption 4.6. Since we have established the conservativeness of Assumption 4.1 in the last section, what we must do is to establish that of Assumption 4.6 under Assumption 4.1. As mentioned above, Assumption 4.6 requires the fixed universe as the outside of  $V$  contains essentially  $V^{\mathbb{P}}/G$  and so it is natural to consider whether  $V^{\mathbb{P}}/G$  itself satisfies Assumption 4.6. If so, then the conservativeness of Assumption 4.6 is obtained.

**Lemma 6.1**  $V^{V^{\mathbb{P}}/G} = \{x \in |V^{\mathbb{P}}/G| \mid V^{\mathbb{P}}/G \vDash V(x)\}$  is isomorphic to  $V$  via  $a \mapsto \check{a}/G$ .

*Proof.* It is obvious that  $\check{a}/G \in V^{V^{\mathbb{P}}/G}$  for all  $a \in V$  since  $p \Vdash V(\check{a})$  for all  $p \in \mathbb{P}$ . Conversely suppose that  $\check{a}/G \in |V^{\mathbb{P}}/G|$  satisfies  $V^{\mathbb{P}}/G \vDash V(\check{a}/G)$ . Then there is  $p \in G$  such that  $p \Vdash V(\check{a})$  and hence

$$\{q \leq p \mid q \Vdash \check{a} = \check{b} \text{ for some } b \in V\} \text{ is dense below } p,$$

which implies that there are  $q \in G$  and  $b \in V$  such that  $q \Vdash \check{b} = \check{a}$ . Thus  $\check{a}/G = \check{b}/G$  for some  $b \in V$ .

Then

$$\check{a}/G = \check{b}/G \text{ iff there is } p \in G \text{ such that } p \Vdash \check{a} = \check{b} \text{ iff } a = b \text{ by Lemma 5.1.}$$

Similarly

$$V^{\mathbb{P}}/G \vDash \check{a}/G \in \check{b}/G \text{ iff there is } p \in G \text{ such that } p \Vdash \check{a} \in \check{b} \text{ iff } a \in b \text{ by Lemma 5.1.} \quad \square$$

In what follows, we identify  $V$  with  $V^{V^{\mathbb{P}}/G}$  via  $a \mapsto \check{a}/G$ .

**Lemma 6.2**  $V^{\mathbb{P}}/G \vDash \check{p}/G \in \check{q}/G$  iff  $p \in G$ , for all  $p \in V$ .

*Proof.* Suppose  $\check{p}/G \in \check{q}/G$ , say  $q \Vdash \check{p} \in \check{q}$  and  $q \in G$ . Thus we have  $\dot{x}$  such that  $r \Vdash \check{p} = \dot{x} \wedge \dot{x} \varepsilon \check{q}$  for some  $G \ni r \leq q$  and  $\dot{x} \in V^{\mathbb{P}}$ . Then  $\dot{x} = \check{s}$  such that  $\{s\}$  is predense below  $r$ , in particular  $s \in G$ . Since

$$r \Vdash \check{p} = \check{s},$$

we have  $p = s \in G$  by Lemma 5.1.

Conversely, since  $p \Vdash \check{p} \in \check{q}$ ,  $p \in G$  implies  $\check{p}/G \in \check{q}/G$ .  $\square$



**Proposition 6.3** In  $V^{\mathbb{P}}/G$ , for any  $\dot{a}_1, \dots, \dot{a}_n \in (V^{\mathbb{P}})^V$ , the Forcing Theorem holds:

$$\varphi(\text{val}(\dot{a}_1, \mathcal{G}/G), \dots, \text{val}(\dot{a}_n, \mathcal{G}/G)) \text{ iff, for some } p \in G, p \Vdash^V \varphi(\dot{a}_1, \dots, \dot{a}_n).$$

*Proof.* Now we have  $V^{\mathbb{P}}/G \models \varphi(\dot{a}_1/G, \dots, \dot{a}_n/G)$  iff there is  $p \in G$  such that  $p \Vdash^V \varphi(\dot{a}_1, \dots, \dot{a}_n)$  iff there is  $p$  such that

$$V^{\mathbb{P}}/G \models \check{p}/G \in \mathcal{G}/G \quad \text{and} \quad \check{p}/G \Vdash^V \varphi(\check{a}_1/G, \dots, \check{a}_n/G).$$

Since  $V^{\mathbb{P}}/G \models \mathcal{G}/G \subset \check{P}/G \subset V$ , by Lemma 6.1, these are equivalent to the statement which is completely in  $V^{\mathbb{P}}/G$ :

$$V^{\mathbb{P}}/G \models (\exists z \in \mathcal{G}/G)(z \Vdash^V \varphi(\check{a}_1/G, \dots, \check{a}_n/G)).$$

Therefore, we have the following equivalence defined in  $V^{\mathbb{P}}/G$ :

$$V^{\mathbb{P}}/G \models \varphi(\dot{a}_1/G, \dots, \dot{a}_n/G) \leftrightarrow (\exists z \in \mathcal{G}/G)(z \Vdash^V \varphi(\check{a}_1/G, \dots, \check{a}_n/G)).$$

Notice that, by Lemma 6.1, if  $\dot{a}/G \in |V^{\mathbb{P}}/G|$  with  $V^{\mathbb{P}}/G \models \dot{a}/G \in (V^{\mathbb{P}})^V$ , then  $\dot{a}/G = (\check{c})/G$  for some  $\check{c} \in V^{\mathbb{P}}$ . Thus it suffices to show the following lemma.  $\square$

**Lemma 6.4**  $V^{\mathbb{P}}/G \models \text{val}((\check{a})/G, \mathcal{G}/G) = \dot{a}/G$ .

*Proof.* Using the previous lemmata, we have the following equations which hold in  $V^{\mathbb{P}}/G$ :

$$\begin{aligned} \text{val}^{V^{\mathbb{P}}/G}((\check{a})/G, \mathcal{G}/G) &= \{\text{val}^{V^{\mathbb{P}}/G}(\check{b}/G, \mathcal{G}/G) \mid V^{\mathbb{P}}/G \models (\exists z \in \mathcal{G}/G)((\check{b}/G, z) \in \check{a}/G)\} \\ &= \{\text{val}^{V^{\mathbb{P}}/G}(\check{b}/G, \mathcal{G}/G) \mid V^{\mathbb{P}}/G \models \check{b}/G \in (V^{\mathbb{P}})^V \\ &\quad \wedge (\exists z \in \mathcal{G}/G)((\check{b}/G, z) \in \check{a}/G)\} \\ &= \{\text{val}^{V^{\mathbb{P}}/G}((\check{c})/G, \mathcal{G}/G) \mid V^{\mathbb{P}}/G \models (\exists z \in \mathcal{G}/G)(V \models ((\check{c}/G, z) \in \check{a}/G))\} \\ &= \{\text{val}^{V^{\mathbb{P}}/G}((\check{c})/G, \mathcal{G}/G) \mid (\exists r \in V)(V^{\mathbb{P}}/G \models \check{r}/G \in \mathcal{G}/G \\ &\quad \& V \models (\check{c}/G, \check{r}/G) \in \check{a}/G)\} \\ &= \{\text{val}^{V^{\mathbb{P}}/G}((\check{c})/G, \mathcal{G}/G) \mid (\exists r \in G)((\check{c}, r) \in \dot{a})\} \end{aligned}$$

and

$$\begin{aligned} \dot{a}/G &= \{\check{b}/G \mid (\exists p \in G)(p \Vdash \check{b} \in \dot{a})\} \\ &= \{\check{b}/G \mid (\exists p \in G)(\{q \in \mathbb{P} \mid (\exists \check{c} \in V^{\mathbb{P}})(q \Vdash \check{b} = \check{c} \varepsilon \dot{a})\} \text{ is dense below } p)\} \\ &= \{\check{b}/G \mid (\exists q \in G)(\exists \check{c} \in V^{\mathbb{P}})(q \Vdash \check{b} = \check{c} \varepsilon \dot{a})\} \\ &= \{\check{c}/G \mid (\exists q \in G)(\{r \in \mathbb{P} \mid (\check{c}, r) \in \dot{a}\} \text{ is predense below } q)\} \\ &= \{\check{c}/G \mid (\exists r \in G)((\check{c}, r) \in \dot{a})\}. \end{aligned}$$

Thus, by AFA<sub>2</sub> in  $V^{\mathbb{P}}/G$ , we have  $V^{\mathbb{P}}/G \models \text{val}((\check{a})/G, \mathcal{G}/G) = \dot{a}/G$ .  $\square$

**Corollary 6.5** In  $V^{\mathbb{P}}/G$ , for any  $\dot{a}, \dot{b} \in (V^{\mathbb{P}})^V$ ,

$$\text{val}(\dot{a}, \mathcal{G}/G) = \text{val}(\dot{b}, \mathcal{G}/G) \text{ iff there is } p \in \mathcal{G}/G \text{ such that } p \Vdash \dot{a} = \dot{b}.$$

*I. e.,  $V^{\mathbb{P}}/G$  satisfies Assumption 4.6.*

We have proved the consistency of Assumption 4.6 from that of Assumption 4.1. Moreover, Lemma 6.1 means that the conservativeness of Assumption 4.6 over the original theory follows from that of Assumption 4.1.

## 7 The relation to forcing under FA

As we remarked in several places in the previous sections, the way we have defined the forcing under AFA and obtained some results on it is applicable to the forcing under FA with small modifications, and that this way makes the definition (of forcing relation for atomic formulae) and the proofs quite simpler than those of the usual formulation.

Now it is natural to investigate the relation between the well-founded part of the generic extension (by forcing under AFA) and the generic extension (by forcing under FA) of the well-founded part of the ground model.

In this section, we assume  $\mathbb{P} \in \text{WF}$ . Note that it is not relevant to the exact poset  $\mathbb{P}$ , but the isomorphic type of  $\mathbb{P}$ . Thus, under AC, this assumption makes no change. Even in the absence of AC, this is a natural assumption for our purpose of this section.

In what follows,  $\text{Acz}$  denotes Aczel's model of AFA, i. e.

$$\{(G, E, a) \mid E \subset G \times G, a \in G\} / \equiv,$$

where  $(G, E, a) \equiv (G', E', a')$  iff there is an  $(E, E')$ -bisimulation  $R$  (in other words, a bisimulation  $R$  from  $E$  to  $E'$ ) such that  $aRa'$ . Here is a key lemma:

**Lemma 7.1** *Assume  $V \models \text{ZF}_\in + \text{AFA} + \text{AC}$ . Then  $V \cong \text{Acz}^{\text{WF}^V}$ .*

*Proof.* Define  $\Phi : V \rightarrow \text{Acz}^{\text{WF}^V}$  as follows: Letting  $a \in V$ , then, by AC, we have a cardinal  $\kappa$  and a bijection  $f : \text{trcl}(a) \cong \kappa$ . Define a pointed graph  $(\kappa, R, f(a))$ , where

$$R = \{\langle f(b), f(c) \rangle \in \kappa \times \kappa \mid b, c \in \text{trcl}(a) \ \& \ b \in c\}.$$

Put  $\Phi(a) = (\kappa, R, f(a)) / \equiv$ , the equivalent class of  $(\kappa, R, f(a))$ . This does not depend on the choice of  $\kappa, f$ .

Conversely define  $\Psi : \text{Acz}^{\text{WF}^V} \rightarrow V$  as follows: suppose  $a \in \text{Acz}^{\text{WF}^V}$ , say  $a$  is the equivalence class of a pointed graph  $(G, E, g)$  and put  $\Psi(a) = d(g)$ , where  $d$  is the unique decoration of the graph  $(G, E)$ . This does not depend on the choice of  $(G, E, g)$ .

It is obvious that  $\Phi$  and  $\Psi$  are mutually inverse. □

For a class  $\mathfrak{M}$ , define  $\mathfrak{M}[G]$  as follows:

$$\mathfrak{M}[G] = \{\text{val}(\dot{a}, G) \mid \dot{a} \in V^{\mathbb{P}} \cap \mathfrak{M}\}.$$

Corresponding to the fact that FA is forced in the usual forcing, we can easily obtain:

**Lemma 7.2**  *$\text{WF}^V[G]$  is a transitive submodel of  $\text{WF}^{V[G]}$ . Therefore  $\text{WF}^V \subset \text{WF}^{V[G]}$  and  $G \in \text{WF}^{V[G]}$ .*

*Proof.* By induction on  $\text{rank}(\dot{a})$  for  $\dot{a} \in \text{WF} \cap V^{\mathbb{P}}$ . To show  $p \Vdash \dot{a} \in \text{WF}$  for all  $p \in \mathbb{P}$ , it suffices to show

$$p \Vdash \forall x (x \varepsilon \dot{a} \rightarrow x \in \text{WF}).$$

Suppose  $q \leq p$  and  $q \Vdash \dot{b} \varepsilon \dot{a}$ . Then  $\dot{b} \in \text{WF} \cap V^{\mathbb{P}}$  with  $\text{rank}(\dot{b}) < \text{rank}(\dot{a})$  and hence  $q \Vdash \dot{b} \in \text{WF}$  by the induction hypothesis.

Suppose  $x \in \text{val}(\dot{y}, G)$  with  $\dot{y} \in \text{WF}^V \cap V^{\mathbb{P}}$ . Then there are  $\dot{x}$  and  $p \in G$  with  $x = \text{val}(\dot{x}, G)$  and  $\langle \dot{x}, p \rangle \in \dot{y}$ .  $\dot{y} \in \text{WF}^V$  implies  $\dot{x} \in \text{WF}^V$  and hence  $x \in \text{WF}^V[G]$ . □

**Lemma 7.3**  *$\text{WF}^V \cap V^{\mathbb{P}}$  is the least fixed point of  $\Phi^{\mathbb{P}}$ . I. e.,  $\dot{a} \in \text{WF}^V \cap V^{\mathbb{P}}$  iff  $\dot{a}$  is a  $\mathbb{P}$ -name in the usual sense (under FA). As a consequence,  $\text{WF}^V[G]$  is a generic extension of  $\text{WF}^V$  in the usual sense.*

*Proof.* Since  $\mathfrak{P}(V^{\mathbb{P}} \times \mathbb{P}) = V^{\mathbb{P}}$ ,  $\dot{a} \in V^{\mathbb{P}} \cap \text{WF}^V$  iff  $\dot{a}$  is a relation and, for any  $\langle x, p \rangle \in \dot{a}$ ,  $x \in V^{\mathbb{P}} \cap \text{WF}^V$  and  $p \in \mathbb{P}$ . This equivalence determines the class of  $\mathbb{P}$ -names in the usual sense in the universe satisfying FA. □

**Theorem 7.4** *Assume AC in the ground model  $V$ . Let  $G$  be a generic filter of  $\mathbb{P}$ . Then  $\text{WF}^{V[G]} = \text{WF}^V[G]$ . As a consequence,  $\text{ON}^{V[G]} = \text{ON}^V$ .*

**Proof.** By Lemma 7.2, it suffices to show that  $WF^{V[G]} \subset WF^V[G]$ . Suppose  $\text{val}(\dot{x}, G) \in WF^{V[G]}$ . In  $V$ , using AC take a set  $N \in WF^V$  and a bijection  $\varphi : \text{trcl}(\dot{x}) \cap V^{\mathbb{P}} \rightarrow N$ . Also define

$$\tilde{E} = \{ \langle \varphi(\dot{y}), \varphi(\dot{z}), p \rangle \mid \langle \dot{y}, p \rangle \in \dot{z} \ \& \ \dot{y}, \dot{z} \in \text{trcl}(\dot{x}) \cap V^{\mathbb{P}} \} \in WF^V.$$

Let

$$\tilde{E}_G = \{ \langle \varphi(\dot{y}), \varphi(\dot{z}) \rangle \mid (\exists p \in G) (\langle \dot{y}, p \rangle \in \dot{z}) \ \& \ \dot{y}, \dot{z} \in \text{trcl}(\dot{x}) \cap V^{\mathbb{P}} \}.$$

Then  $\tilde{E}_G = \{ \langle \varphi(\dot{y}), \varphi(\dot{z}) \rangle \mid (\exists p \in G) (\langle \varphi(\dot{y}), \varphi(\dot{z}), p \rangle \in \tilde{E}) \} \in WF^V[G]$  by the last lemma. Since

$$WF^{V[G]} \models \text{“val}(\dot{x}, G) \text{ is the transitive collapse of } (N, \tilde{E}_G)\text{”},$$

$WF^{V[G]} \models \text{“}(N, \tilde{E}_G) \text{ is well-founded graph”}$  and by the transitivity of  $WF^V[G]$  in  $WF^{V[G]}$ , we also have

$$WF^V[G] \models \text{“}(N, \tilde{E}_G) \text{ is well-founded graph”}.$$

Thus, there exists  $x \in WF^V[G]$  such that  $WF^V[G] \models \text{“}x \text{ is the transitive collapse of } (N, \tilde{E}_G)\text{”}$ , and so, by the absoluteness of collapsing,  $WF^{V[G]} \models \text{“}x \text{ is the transitive collapse of } (N, \tilde{E}_G)\text{”}$ . By the uniqueness of collapsing,

$$\text{val}(\dot{x}, G) = x \in WF^V[G].$$

Thus  $ON^{V[G]} = ON^{WF^{V[G]}} = ON^{WF^V[G]} = ON^{WF^V} = ON^V$ . □

**Corollary 7.5** *Assume AC. Then  $V[G] \cong \text{Acz}^{WF^V[G]}$ .*

**Proof.** By Proposition 3.10, Lemma 7.1 and the theorem,  $V[G] \cong \text{Acz}^{WF^{V[G]}} \cong \text{Acz}^{WF^V[G]}$ . □

This corollary means that our generic model and the generic model defined in [5, Appendix] coincide. By Theorem 4.8, we have Forcing Theorem and Minimal Model Theorem for their generic model under Assumptions 4.6, in particular, when the outside universe in which we are working is  $V^{\mathbb{P}}/G$ , while the conjecture there [5] claims that Forcing Theorem does hold without Assumption 4.6, i. e. in any outside universe  $W$  satisfying Assumption 4.1 with  $V \subset W$  (which implies  $WF^V \subset WF^W$ , by the absoluteness of wellfoundedness under AFA, established essentially in [9, Lemma 8]). However, by the usual Forcing Theorem, the generic models are absolute under FA, i. e.  $(WF^V[G])^{V^{\mathbb{P}}/G} \cong (WF^V[G])^W$ , which implies Forcing Theorem for  $\text{Acz}^{(WF^V[G])^W}$ . Thus we have obtained a positive answer to the conjecture.

Another meaning of this corollary is that, assuming AC, we can construct  $V[G]$  by methods which have already been introduced, usual generic extension for FA-models and Aczel’s construction, and so the new method we have just introduced looks useless for constructing new models at least under AC. Is it actually useless? No, even under AC. In fact, the Forcing Theorem and Minimal Model Theorem show the direct relation between the ground model and the generic models and so these theorems will have some practical value in investigating AFA-models even under AC.

As mentioned in Section 1, we give one example which shows the utility of such a direct relation: In [9], the Gödel operations under AFA and constructible universe  $J_{X_1}$  for AFA are considered and so it is easy to define the relatively constructible universes  $J_{X_1}[x]$  and  $J_{X_1}(x)$  for any set  $x$ . For a set  $x$ , consider the consistency of the statement “ $\exists y (y \notin J_{X_1}[x])$ ”, or “there is a set which is not constructible in  $x$ ”. Our forcing method immediately shows the consistency, in the exactly same way as in the presence of FA, i. e. taking a non-atomic poset  $\mathbb{P}$  and considering  $(J_{X_1}[x])[G]$  for a  $\mathbb{P}$ -generic filter  $G$  over  $J_{X_1}[x]$ . Note that this discussion does not require AC. On the other hand, without our forcing method, one needs to take a copy of the structure  $(\text{trcl}(x), \in)$  in the well-founded part, to extend it by the usual forcing method and to recover the non-well-founded part, a cumbersome procedure! Moreover, this procedure cannot be done without AC, because there is no guarantee that a copy of  $(\text{trcl}(x), \in)$  exists in the well-founded part.

This is only one example. As the forcing method is an inevitable part of model theory for FA-models, our forcing method will undoubtedly become an inevitable part of AFA-model theory.

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