TRANSVERSE WHISTLER PROPAGATION

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ABSTRACT

Transpolar propagation of ion-electron whistler modes is investigated and shown, with the help of numerically computed rays, not to be useful for transmitting information.

The first two sections of the report comprise a summary of whistler phenomena and theory to provide a background for the present investigation. In Section III a specific ion-electron density distribution is introduced and applied to the appropriate Haselgrove ray tracing equations. In Section IV the numerical results for transpolar propagation are presented.

TRANSVERSE WHISTLER PROPAGATION

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I. Introduction to Whistlers (longitudinal mode)

The "whistler mode" is a term used to refer to the guiding of VLF (Very Low Frequency: $<$30 kc/s) radio waves by the lines of force of the earth's magnetic field, and is called such because of the characteristic change in pitch of the audio output of the receiver. If the audio signal is narrow-band, the whistler is musical; otherwise, it is described as a "swish". The word "p<u>e</u>ou" pronounced in a whisper is perhaps the best description of the whistler sound (Ref. 1). Ordinarily the pitch of a whistler will decrease as time increases. This feature suggests that this type of atmospheric radio noise may be due to the dispersion of a lightning pulse as it propagates over a long path in an ionized, and therefore conducting and dispersive, medium, such as the ionosphere. The lower frequencies in the impulse travel more slowly and therefore arrive later, producing the characteristic descending pitch.[+] Both Barkhausen (Ref. 1) and Eckersley (Ref. 2) recognized that whistlers probably originate in this fashion, but they were unable to suggest where such paths, extending possibly over several tens of thousands of kilometers, might exist in nature. Another contribution by Eckersley was his demonstration by means of magneto-ionic theory that at radio frequencies below

[+]The initial phase of a "nose whistler" to be described later, constitutes an exception to this statement.

the plasma and gyro frequencies the ionosphere is transparent to the extraordinary component of the radio wave so long as the ray direction and the direction of the geomagnetic field nearly coincide.

It remained for Storey in his monumental 1953 paper (Ref. 3) to show that such long dispersive paths are provided by the tendency of the longitudinal extraordinary component of VLF radio waves to follow the lines of force of the earth's magnetic field. Thus the VLF signals from lightning strokes and radio transmitters operating in this frequency band will travel to and fro along the geomagnetic field line connecting the source and the area in the other hemisphere of the earth (i.e., the so-called "conjugate point") where the line of force re-enters. The restriction of the signal path to the vicinity of the geomagnetic field line has resulted in the whistler mode also being termed the magneto-ionic duct.

The theoretical basis for the magneto-ionic duct may be outlined as follows (Refs 3,4). Suppose the radio wave circular frequency ω and the collisional frequenc ν are both much less than the product of the circular gyro-frequency G_e for electrons and the cosine of the angle ψ between the direction of the earth's magnetic field and the wave normal direction, and that $G_e \cos \psi$ is in turn much smaller than the ratio ω_e^2 / ω, (ω_e being the circular plasma frequency for electrons). Then the Quasi-Longitudinal approximation to the Appleton-Hartree formula for the refractive index of a magneto-ionic medium

$$n^2_{o,x} = 1 - \frac{\omega_e^2}{\omega(\omega + i\nu \pm G_e \cos\psi)} \qquad (1)$$

reduces at VLF to the approximate expression

$$n^2_{o,x} = \mp \frac{\omega_e^2}{\omega G_e \cos\psi} \qquad (2)$$

It is clear from this form for the square of the refractive index that the ordinary ray (corresponding to the minus sign) will be rapidly attenuated. The angle γ between the wave normal direction and the ray direction is given by the geometrical optics theory of a magneto-ionic medium as $\tan\gamma = -1/2\, \partial \ell n\, n^2 / \partial \psi$, or in the present instance

$$\tan\gamma = -\frac{1}{2}\tan\psi \qquad (3)$$

This result implies the inequality

$$|\beta| < \arctan\frac{1}{\sqrt{8}} = 19^{\circ}\, 28' \quad , \qquad (4)$$

for the angle $\beta = \gamma + \psi$ between the ray-direction and the field line. Thus in Storey's theory propagation of the VLF extraordinary component is restricted to a cone of half-angle 19° 28' about the geomagnetic field line. It should be emphasized that this is a geometrico-optical result, and that it contains no information of itself on how the ray path may behave within this cone. After several thousand kilometers the cross section of a cone with half-angle 19° 28' is rather considerable in extent, the diameter being approximately two thirds of the axial distance. Thus Storey in his paper gives a figure showing the geographical distribution of sferics (radio noise from nearby thunderstorms, in the form of "clicks") and whistlers. Lightning strokes which gave rise to whistlers at Cambridge, England, might have occurred as far away as the mid-Atlantic, although loud whistlers were detected only from lightning strokes within a 2000 km radius. Thus the end point of the whistler path can be a considerable distance away from the exact conjugate point of the source proper.

The group velocity of this type of transmission, can be obtained by using equation (2) (with the plus sign), in the expression $c/\left[\partial(n_x\omega)/\partial\omega\right]$. The time of propagation is then given by

$$T = \int_{Q}^{Q'} \frac{ds}{v_{gr}} \doteq \frac{1}{2c\sqrt{\omega}} \int_{Q}^{Q'} \frac{\omega_e}{\sqrt{G_e}} \, ds \quad ; \qquad (5)$$

where Q and Q' denote the source and conjugate points, respectively and ds is path length. It is clear from equation (5) that the higher the circular radio frequency ω, the shorter the travel time. Thus the higher frequencies in the lightning pulse should arrive first, and the pitch would descend during the reception of the whistler signal. The quantity

$$D = T f^{1/2} \quad , \qquad (6)$$

(f being the radio frequency) has been called by Storey the "dispersion" of the whistler. For whistlers only with descending tones, the "dispersion" depends upon the integrated electron density over the entire line of force. We thus have a means of estimating the charged particle density in the exosphere.

Another interesting feature of the "dispersion" D is its characteristically different behavior for "long" and "short" whistlers. Whistlers are classified as "long" or "short" according as they are or are not preceded by "clicks". The "long whistlers" thus originate near the receiving site and are dispersed over a round-trip path, whereas the "short whistler" is due to a sferic near the conjugate point in the opposite hemisphere and suffers only the dispersion appropriate to a one-way trip. Thus the dispersions of "long whistlers" are in the ratio 2:4:6:..., and for "short whistlers" in the ratio 1:3:5:... .

The peak field strength of a loud whistler is of the order of 1 millivolt per meter, and the ordinary loud whistler averages about 100 microvolts per meter (Ref. 5). The intensity usually peaks at about 3 to 5 kc/s, at which frequency the product of sferic energy and magneto-ionic guidance apparently reaches a maximum, and the low frequency cutoff is of the order of 1 to 2 kc/s (Ref. 5). The attenuation rate is estimated to be of the order of 10 db (and occasionally as low as 2 db) per round trip, although there are in addition insertion losses of the order of 25 db or so at the ends of the path, where the transmission passes through the highly-attenuating lower ionosphere. Apparent attenuation rates must not be regarded too rigidly, however, since the magneto-ionic duct may not be entirely passive. The sferic signal may be amplified by some type of growing wave interaction with the Van Allen "trapped electrons" or, as in the Gallet-Helliwell theory, with streams of incoming high-energy particles from the sun (Refs 5, 6, 7).

Storey also characterized three types of multiple whistlers: "whistler trains", "multiple-flash" type groups, and "whistler pairs", which may be distinguished one from another on the basis of the relative behavior of successive echo traces. For a full description of each of these types the interested reader is referred to Storey's original paper (Ref. 3). The same source may be referred to for a description of variation of whistler activity with time of day, season and "magnetic activity (Refs 3, 5).

Storey's theory predicted three effects: (1) that a short whistler would be preceded by a sferic in the vicinity of the conjugate point, (2) an absence of whistlers on the geomagnetic equator, and (3) the concentration of energy of a whistler in a terrestrial area whose radius is of the order of 2000 km. The existence of all three of these effects were soon verified, the second by Storey himself, and the others by Helliwell, Morgan and Allcock, and others (Refs. 5, 7, 8). In addition new whistler phenomena, such as the "nose whistler", in which there occur simultaneous rising and falling tones starting at the same frequency, were detected and given a theoretical basis (Refs 5, 8).

The simple time delay formula (5) will no longer hold if the wave frequency is not much smaller than the gyro-frequency. In that case the full formula for the group velocity in the quasi-longitudinal mode must be used, and we have instead of equation (5)

$$T = \frac{1}{2c\sqrt{\omega}} \int_{Q}^{Q'} \frac{\omega_e G_e}{(G_e - \omega)^{3/2}} \, ds \tag{7}$$

It follows from this expression that the dispersion should reverse above a radio frequency lying between one fourth of the minimum, over the path, of the gyro-frequency (this minimum frequency may be termed the "whistler MUF") and the gyro-frequency in the lower ionosphere. It is this effect which gives rise to the "nose". The "nose" frequency thus defines fairly well the gyro-frequency at the top of the path. The theory further indicates that the "nose frequency" should increase as the geomagnetic latitude becomes less, and this feature has been confirmed by observations at College, Alaska and at Seattle (Refs. 5, 7, 8).

Although whistlers had originally been thought to be confined to frequencies below 15 kc/s, more sensitive measurements demonstrated the existence of frequency components as high sometimes as 35 kc/s (Ref. 7). These observations led to the first confirmation, using man-made transmissions, of Storey's theory of the magneto-ionic duct. In this experiment, performed by Helliwell and Gehrels (Ref. 9) during January 1957, whistler-mode signals from station NSS (15.5 kc/s) at Annapolis, Md., were received at Cape Horn, a location some 1000 miles away from the conjugate point proper. The experimental results indicated that the magneto-ionic duct is open much more often than had been thought on the basis of whistler experience. In fact there was strong evidence that the magneto-ionic duct may be open nearly all the time (Ref. 9). On the basis of the fact that the "fringe area" whistler reception at Cape Horn was some 10 to 30 db below the sky wave from NSS, it was estimated that, in the immediate vicinity of the conjugate point, whistler signal strength may be of the same order as the sky wave signal strength. It was further noted in the experiment that the whistler signal suffered severe and relatively rapid fading as well as frequent echo splitting. This feature was attributed to multiple propagation paths with continually-varying relative phase. Such multiple paths could account for multiple whistlers such as the whistler pair, and could arise either through a multiplicity of properly-oriented initial wave normals or through field-aligned columns of denser ionization in the upper atmosphere.

A thorough discussion of other recent work may be found in the report by R.A. Helliwell (Ref. 5) and the Report on the XIIth General Assembly of URSI (Ref. 7).

II. The Ion-Electron Whistler Mode

Up until now the discussion has dealt entirely with what might be termed the longitudinal (or quasi-longitudinal) whistler mode, which can propagate only in a narrow cone about the geomagnetic field line. As long as magneto-ionic theory is assumed, i.e., as long as the only species supposed present are free electrons and neutral particles, it is clear from the Appleton-Hartree formula that only quasi-longitudinal propagation of the extraordinary component is possible. However, there are of course in the real ionosphere many species, ionized and not, in addition to the free electrons of magneto-ionic theory. Ordinarily these may safely be ignored in wave phenomena since their mobilities are negligible with respect to the free electrons. This neglect may not necessarily be justified in the case of protons (ionized hydrogen), due to the relatively small mass of the proton with respect to other, heavier ions (Refs 10, 11). This was recognized by Storey in 1956 although he assumed quasi-longitudinal propagation and so encountered proton contributions only for radio frequencies below 2 kc/s. This led Hines to consider ion-effects upon propagation in directions other than those near to the geomagnetic field line (Ref. 10). It turned out that new features did appear and that they extended higher in frequency, up to 10 kc/s or so.

Hines' analysis applied to propagation through a homogeneous ionized medium in the presence of a uniform magnetic field, neglecting collisions of the various species of particles with one another. It describes the "local" behavior of an electromagnetic wave in the proton-electron ionosphere. The parameters in Hines' refractive index formula are obtained from a previous generalization by him of magneto-ionic theory (Ref. 12):

$$\kappa_1 = 1 - \sum_s \frac{\omega_s^2}{\omega^2}$$

$$\kappa_2 = 1 + \sum_s \frac{\omega_s^2}{G_s^2 - \omega^2} \tag{8}$$

$$\kappa_3 = i \sum_s \frac{\omega_s^2 \, G_s}{\omega(G_s^2 - \omega^2)}$$

where $\omega_s = (N_s e_s^2 / \varepsilon_0 M_s)^{1/2}$ is the circular plasma frequency for the s th species, $G_s = e_s B_0 / M_s$ is the circular gyro-frequency (including sign) for the s th species, ω is the circular radio frequency, and the summation is extended over all s species involved. The refractive index n of the composite medium then turns out to be governed by the equation

$$\alpha_4 n^4 + \alpha_2 n^2 + \alpha_0 = 0 \quad , \tag{9}$$

where

$$\alpha_4 = \kappa_1 \cos^2 \psi + \kappa_2 \sin^2 \tag{10}$$

$$\alpha_2 = -\left\{ \kappa_1 \kappa_2 (1+\cos^2\psi) + (\kappa_2^2 + \kappa_3^2) \sin^2 \psi \right\} \tag{11}$$

$$\alpha_0 = \kappa_1 (\kappa_2^2 + \kappa_3^2), \tag{12}$$

and ψ is the angle between the geomagnetic field and the wave normal. The solution of the quadratic equation (9) for n^2 then yields the following expression for the square of the refractive index n in the composite medium

$$n^2 = -\left(\alpha_2 \mp \sqrt{\alpha_2^2 - 4\alpha_0 \alpha_4} \right) \Big/ (2\alpha_4) \tag{13}$$

In Hines' theory the coefficient α_4 accounts for the change in propagation characteristics as the direction of propagation is altered. In the classical theory of whistlers, which takes into account electrons only, κ_1 and κ_2 differ in sign, so that, as indicated by equation (10), one root of n^2 does become infinite for some values of ψ. Thus at angles ψ past this "cutoff angle", the classical whistler mode cannot propagate. However, when the effect of protons are included, χ_2 changes sign at the low-frequency end, and the preceding conditions no longer prevail. Thus there may no longer exist a "cutoff angle", and this generalized type of whistler mode, which we shall refer to as the "ion-electron whistler mode", can propagate in arbitrary directions.

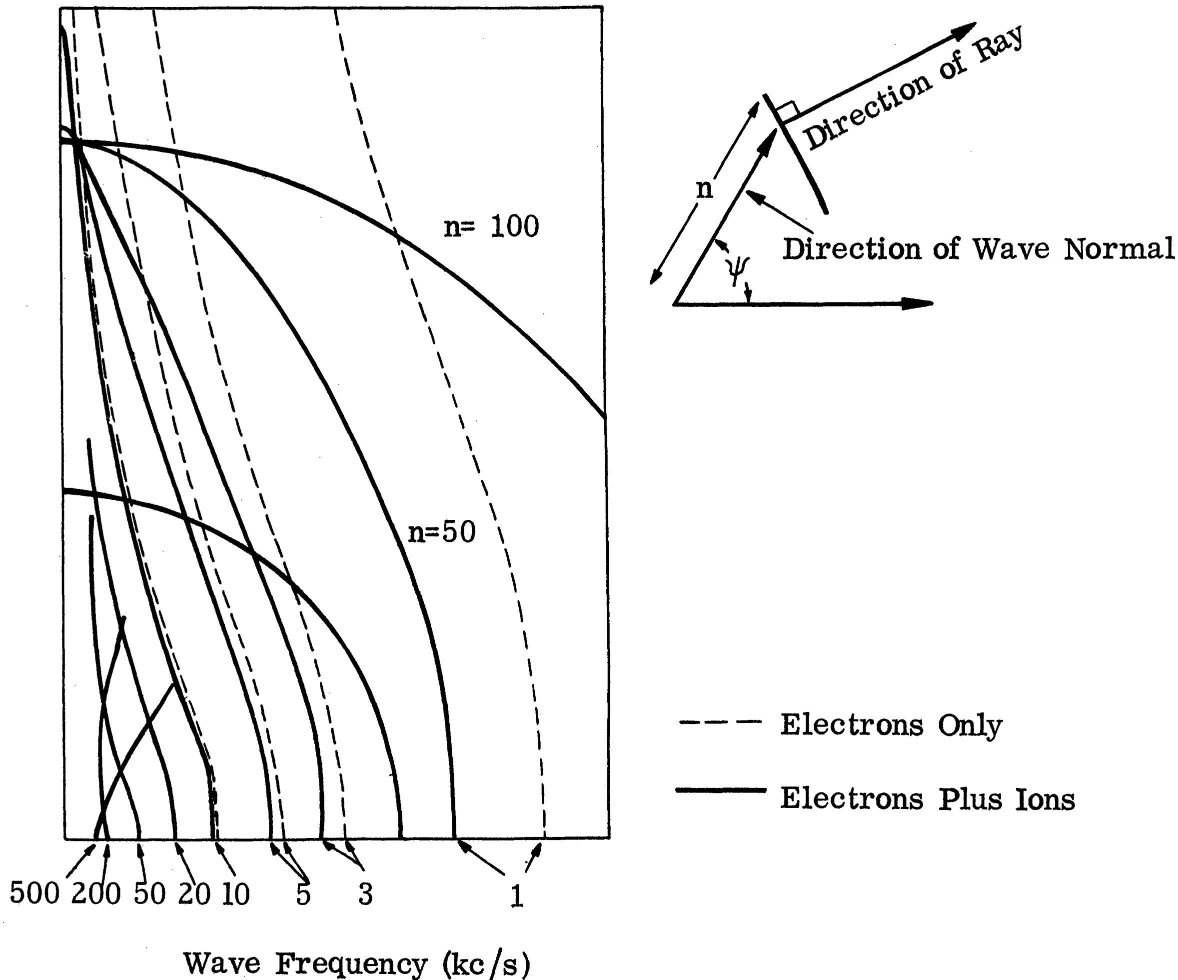
Hines shows a set of curves which depict the behavior of ray direction for a given value of ψ and wave frequency. These curves are reproduced here for convenience. Note that this set of curves has been constructed for a single pair of electron and ion gyro and plasma frequencies. The way these curves are to be used is indicated in the upper right hand corner of the figure. Lay off from the baseline the angle ψ between the local direction of the geomagnetic field and the wave normal, and extend the terminal side of this angle until it intersects the curve (solid for the ion-electron whistler, dashed for the classical type of whistler) appropriate to the radio frequency of interest. (The length of the terminal side of the angle measures the local value of the refractive index.) Next draw a perpendicular to this curve. This gives the resultant ray direction.

An examination of the figure shows that the actual ray direction may, for very low frequencies and wave normals nearly orthogonal to the direction of the ambient magnetic field, be nearly perpendicular to the direction predicted by classical whistler theory.

REFRACTIVE INDEX, n, AS A FUNCTION OF WAVE DIRECTION AND WAVE FREQUENCY WHEN

$G_e/2\pi = 918$ kc/s $\quad$ $G_1/2\pi = 0.5$ kc/s

$\omega_e/2\pi = 2143$ kc/s $\quad$ $\omega_1/2\pi = 50$ kc/s



(Reproduced from Hines, Ref. 10)

FIGURE 1

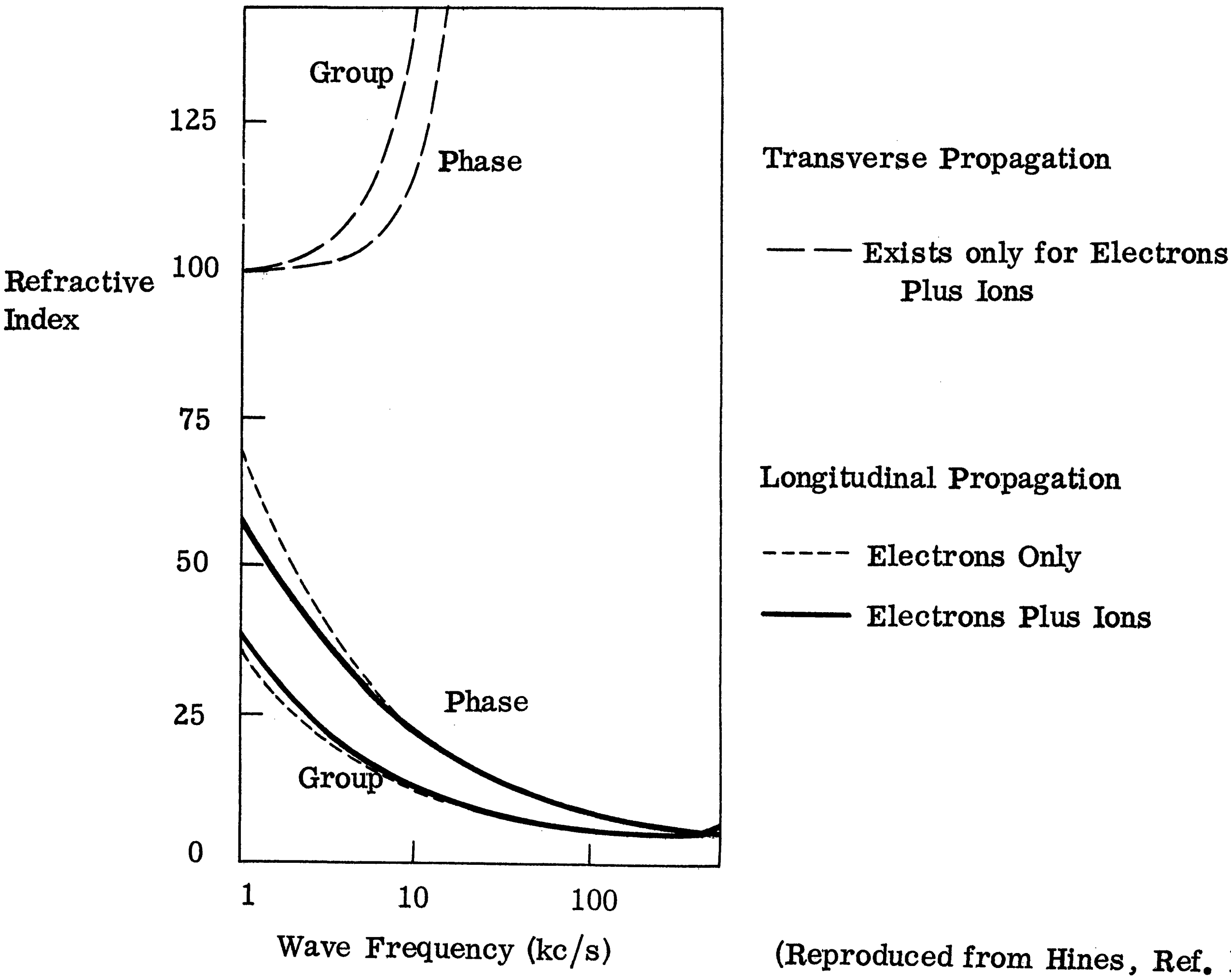
It should be noted, however, that the transverse mode exhibits no tendency towards guidance analogous to that of the classical whistler mode: there is no a priori theoretical argument to suggest that energy launched in a transverse direction would continue to propagate in an approximately transverse direction if it should encounter a change of magnetic field direction or variations of ionization density. This is revealed most clearly by examination of the 'n, θ' diagram: the ray directions for phase directions slightly off perpendicularity to the magnetic field, deviate markedly from the perpendicular direction except at the very lowest frequencies of interest. In fact, at the lowest frequencies, the tendency is for the refractive index to become isotropic and therefore incapable of producing guidance in any direction (including the classical longitudinal direction).

Hines also gives a figure showing the phase and group refractive indices for the two cases. This figure (Figure 2) is also reproduced here for convenience. The significance of this figure (which is also apparent from Figure 1) is that in the transverse mode (ion-electron case) the lower the frequency, the higher is the velocity of propagation, exactly opposite to the situation for the longitudinal whistler mode. The transverse mode thus appears to be an explanation for the rising whistler type of atmospheric (see Refs. 3 and 8 for a description of this type of atmospheric).

PHASE AND GROUP REFRACTIVE INDICES AS FUNCTIONS OF WAVE FREQUENCY WHEN

$G_e/2\pi=918$ kc/s $\qquad$ $G_1/2\pi=0.5$ kc/s

$\omega_e/2\pi=2143$ kc/s $\qquad$ $\omega_1/2\pi=50$ kc/s



(Reproduced from Hines, Ref. 10)

FIGURE 2

III. The Ion-Electron Whistler Mode in the Centered Dipole Field of the Earth

1. The Electron Density Profile

Previous ray-tracings for the whistler mode have assumed a constant or exponentially-decreasing electron density profile. The former describes high exospheric altitudes quite aptly, and the latter is appropriate to altitudes above the peak of the F-layer as well. However, the whistler path must also pass through the lower ionosphere, and in view of the relatively high values of refractive index and strong gradients there, the directions of the initial wave normals may undergo relatively large changes during this part of the transmission.

The characteristics of electron density variation with altitude that are important to whistler propagation may be summarized as follows:

1^{o} Below an altitude of about 80 km, the electron density N_{e/m^3} is essentially zero.

2^{o} The electron density then increases smoothly with altitude up to a value of about $1.6 \times 10^{12} \sqrt{\cos\theta}\ \sqrt{\cos\zeta}$ electrons per m^3 at the peak h_{f_oF2} of the F region. Here θ measures latitude and ζ the solar zenith angle. The square root of cosine θ dependence is intended to take account of the order of magnitude decrease in N_{e/m^3} between the geomagnetic equator and the poles. The square root of cos ζ expresses the classical dependence of electron density on solar zenith angle. To within the uncertainty in our knowledge of the process of solar production of ionospheric electrons, it may be assumed that the subsolar point is approximately on the geomagnetic equator. In that event the formula for cosine ζ reduces approximately to

$$\cos \zeta = \cos \theta \cos h,$$

h being the hour angle of the subsolar point as measured from the observer's meridian. Since the ionosphere is essentially uniform on the dark side of the earth we shall limit the variation of θ and h, arbitrarily setting these angles equal to 85^o whenever they exceed (in absolute value) the value 85^o. (We shall further assume in what follows that the sun is in the observer's meridian plane, i.e., h = 0.)

3^o Above h_{f_oF2}, N_{e/m^3} falls off in smooth (but otherwise unknown) fashion until a stable value somewhere between 500 and 1000 electrons per c.c., is attained in the outer reaches of the exosphere and the solar corona.

The above characteristics are embodied in the following analytic form for the electron density profile:

$$N_{e/m^3} = \begin{cases} 0 & \text{for } 6380 \leqslant r < 6460 \\ N_1 & \text{for } r \geqslant 6460, N_1 > 8\text{x}10^8, \text{ and } |\theta| \leqslant 85^o. \\ & \text{for } 85^o < \theta < 275^o, \text{take } \theta = 85^o \\ 8\text{x}10^8 & \text{for } N_1 < 8\text{x}10^8 \end{cases} \quad (1$$

where

$$N_1 = 4.35\text{x}10^{12} \cos\theta \sqrt{\cos\ h} \left(\frac{r-6460}{220}\right) \exp\left\{-\frac{r-6460}{220}\right\}$$

In these expressions θ is the geomagnetic latitude, h the hour angle (taken to be 85^o for $85^o < \zeta < 275^o$), and r is the radial distance in kilometers from the center of the earth. A plot of N_{e/m^3} for h=0 (i.e., the sun is in the merdian) is shown in Figure 3.

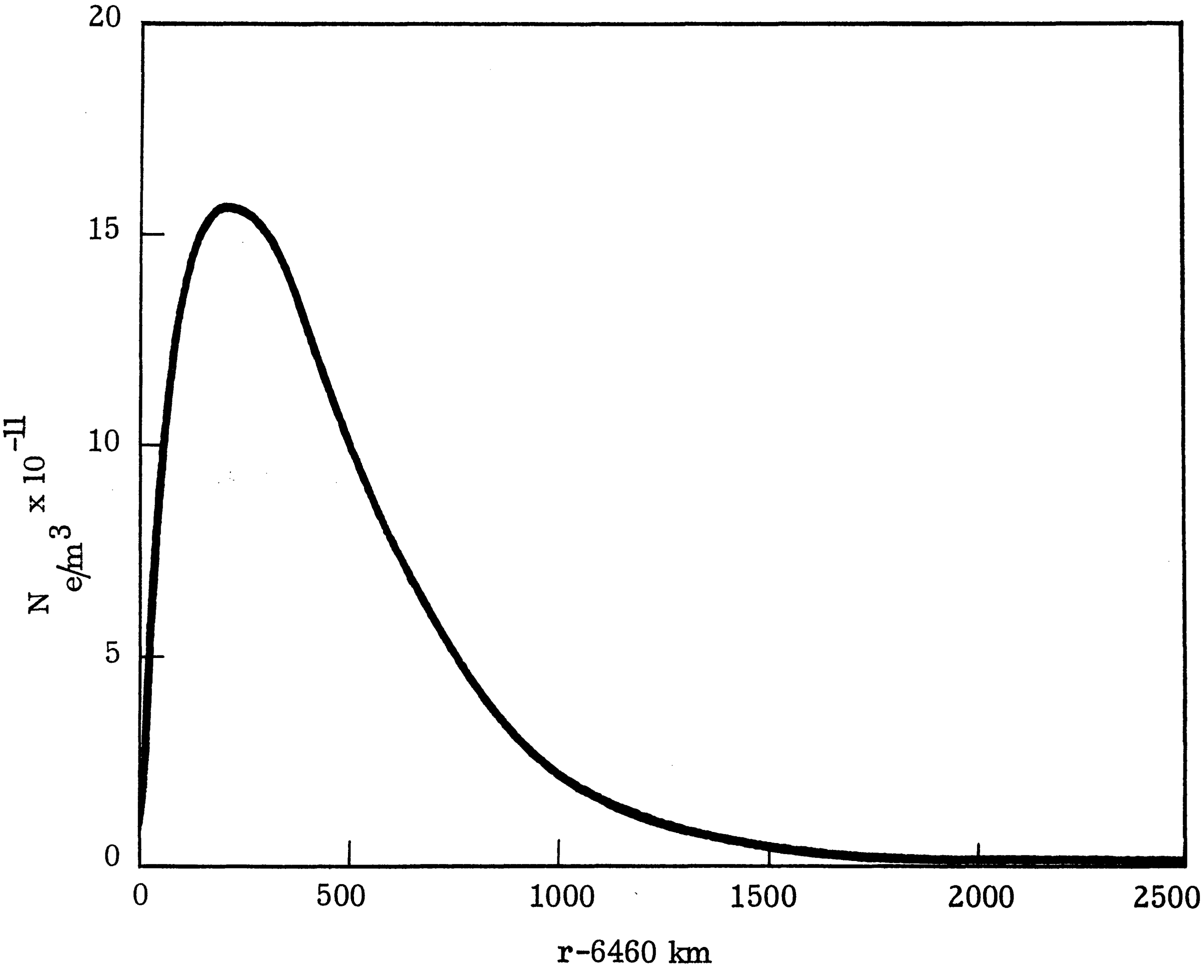


FIGURE 3 ASSUMED VARIATION OF ELECTRON DENSITY WITH ALTITUDE

The partial derivatives of N_{e/m^3} with respect to r and θ are required in the ray-tracing formulas, and are assembled here for future reference:

$$\frac{\partial N_{e/m^3}}{\partial r} = \begin{cases} 0, & \text{for } 6380 \leq r < 6460; \\ 1.98\text{x}10^{10}\cos\theta\sqrt{\cos h}\left(1-\frac{r-6460}{220}\right)\exp\left\{-\frac{r-6460}{220}\right\}, & \text{for } r \geq 6460, N_I \geq 8\text{x}10^8 \text{ and } |\theta| \leq 85^o. \text{ For } 85^o < \theta < 275^o, \text{ take } \theta = 85^o \\ 0, & N_I < 8\text{x}10^8 \end{cases} \tag{15}$$

and

$$\frac{\partial N_{e/m^3}}{\partial \theta} = \begin{cases} 0, & \text{for } 6380 \leq r < 6460; \text{ or } r > 6460 \text{ and } 85^o < \theta < 275^o \text{ or } N_I < 8\text{x}10^8 \\ -4.35\text{x}10^{12}\sin\theta\sqrt{\cos h}\left(\frac{r-6460}{220}\right)\exp\left\{-\frac{r-6460}{220}\right\}, & \text{for } r \geq 6460, N_I \geq 8\text{x}10^8, \text{ and } |\theta| \leq 85^o \end{cases} \tag{16}$$

Since only rays propagating in the plane of the magnetic meridian will be considered, it will be assumed in the sequel for the sake of symmetry that h=0, i.e., the local time is noon on the sunlit side of the earth.

2. The Haselgrove Equations in the Centered Dipole Field

Ray-tracing in the ionosphere is subject to the fundamental difficulty evident, for example, from the Appleton-Hartree formula, that the refractive index is in general a function of the angle the direction of propagation makes with the direction of the magnetic field as well as of the electron density, the magnetic field intensity, and the frequency. The refractive index is therefore an intrinsic property of the particular ray under consideration as well as of the medium.

Haselgrove (Ref. 14) has put Hamilton's differential equations for rays in such a medium in a form suitable for propagation around a spherical earth and has also written out the equations in the special case that the rays are confined to a plane. In the present calculations, we shall assume that the ray path is restricted to the plane of the magnetic meridian, and shall use the appropriate Haselgrove equations

$$\frac{dr}{dt} = \frac{1}{n}\left(\cos \chi_1 + \frac{1}{2} \frac{\partial \ln n^2}{\partial \psi} \sin \chi_1\right)$$

$$\frac{d\theta}{dt} = \frac{1}{nr}\left(\sin \chi_1 - \frac{1}{2} \frac{\partial \ln n^2}{\partial \psi} \cos \chi_1\right)$$

$$\frac{d\chi_1}{dt} = \frac{1}{2} \frac{1}{nr}\left\{\frac{\partial \ln n^2}{\partial \theta} \cos \chi_1 - \left(2 + r \frac{\partial \ln n^2}{\partial r}\right) \sin \chi_1\right.$$

In these equations,

r = radial distance from the center of the earth

θ = geomagnetic latitude (measured positive counterclockwise)

χ_1 = angle between the position vector and the wave normal

n = (real) refractive index

ψ = angle between the magnetic field and the wave normal

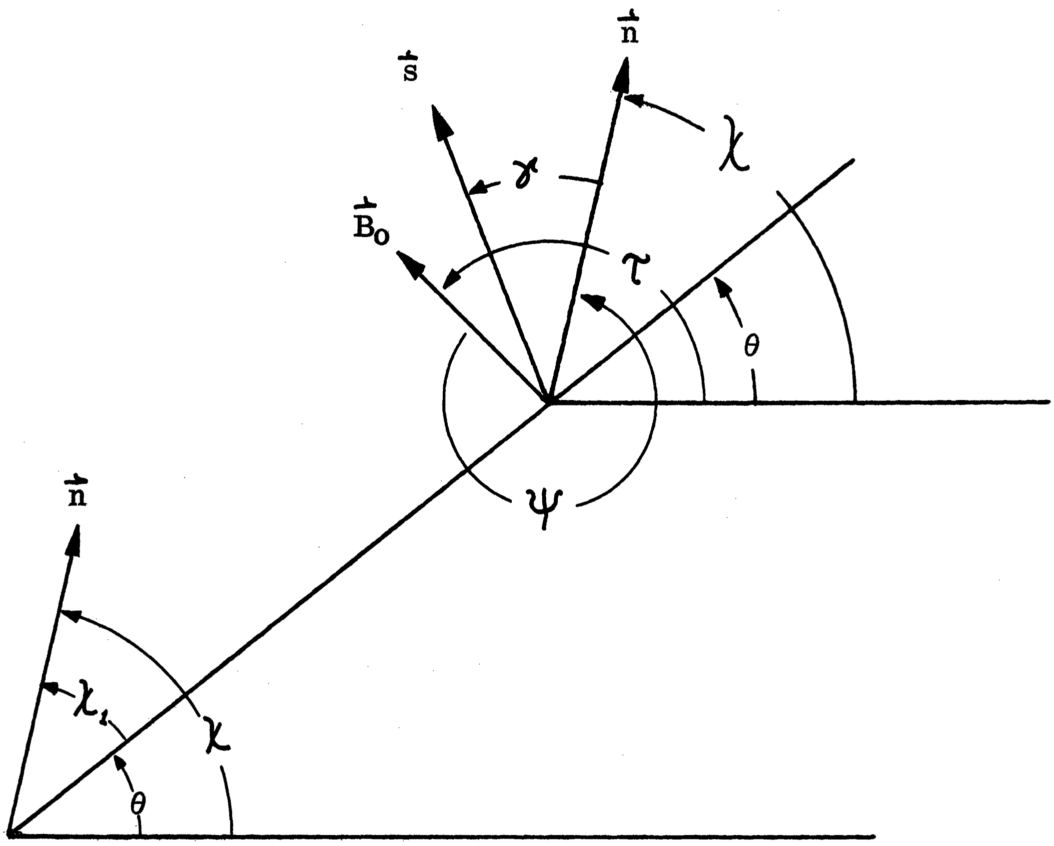


FIGURE 4 ANGULAR RELATIONS AMONG GEOMAGNETIC FIELD $\vec{B}_o$, WAVE NORMAL $\vec{n}$, AND RAY DIRECTION $\vec{s}$.

The next step is the incorporation of the centered dipole approximation for the geomagnetic field into the Haselgrove equations. The magnetic lines of force will thus be given by the equation

$$\rho = a \frac{\cos^2 \theta}{\cos^2 \theta_o} , \tag{18}$$

where ρ is the radial distance from the center of the earth to a point on the line of force, a is the radius of the earth, and θ_o is the geomagnetic latitude at that point where the field line re-enters the earth's surface. The relations among the various angles involved in this and the Haselgrove equations are depicted in Figure 4, wherein $\vec{B}_o$ represents the local magnetic field vector, $\vec{n}$ the wave normal, τ the angle between the positive polar axis (geomagnetic equator) and the vector $\vec{B}_o$, χ the angle between the polar axis and the wave normal $\vec{n}$, and the other quantities are defined just after equations (17).

Since $\chi = \chi_1 + \theta$, and $\psi = \chi - \tau (\text{mod } 2\pi)$, it follows that

$$\psi = \chi_1 + \arctan\left(\frac{1}{2} \cot\theta\right) . \tag{19}$$

To implement equations (17) it is necessary to specify a functional form for n^2, and its various derivatives, in terms of the variables r, θ and ψ. This is provided by equations (8) - (13). The parameters κ_i of equation (8) will be written out in the following section for a proton-electron medium.

3. The Refractive Index for a Proton-Electron Medium in the Presence of the Geomagnetic Field

The refractive index formula appropriate to a generalized magneto-ionic medium consisting of several species is defined by equations (8) - (13). Electrostatic equilibrium will be assumed, so that the number of positive ions (protons) will be equal to the number of negative ions (electrons):

$$N_+ = N_e \ .$$

Though this assumption is of course not strictly true, the use of the proton mass rather than the masses of the heavier species of ions in the expressions for plasma and gyro frequencies is not believed to cause any marked error in view of the relatively small change in effects with change in ion mass.

For two species, protons and electrons, and electrostatic equilibrium, the formulas for κ_1, κ_2, and κ_3 take the forms

$$\kappa_1 = 1 - \left(1 + \frac{1}{1837}\right) \frac{f_e^2}{f^2} = 1 - 8.104 \times 10^{-5} \, N_{e/m^3} \, f_{kc}^2 \qquad (20)$$

$$\kappa_2 = 1 + 3.18 \times 10^3 \, \frac{N_{e/m^3}}{G_e^2} \left(\frac{1}{1-(\omega/G_e)^2} + \frac{1837}{1-(1837\,\omega/G_e)^2} \right) \qquad (21)$$

$$\kappa_3 = i \, \frac{.506}{f_{kc}} \, \frac{N_{e/m^3}}{G_e} \left(\frac{1}{1(\omega/G_e)^2} - \frac{1}{1-(1837\,\omega/G_e)^2} \right) \qquad (22)$$

where N_{e/m^3} is the electron density per cubic meter, f_{kc} is the radio frequency in kilocycles/second, and G_e, etc. are given by

$$G_e = -5.027\text{x}10^6 \frac{\sqrt{1+3\sin^2\theta}}{(r/6380)^3} \tag{23}$$

$$(\omega/G_e)^2 = 1.563\text{x}10^{-6} f_{kc}^2 \frac{(r/6380)^6}{1+3\sin^2\theta} \tag{24}$$

$$(1837\,\omega/G_e)^2 = 5.273 f_{kc}^2 \frac{(r/6380)^6}{1+3\sin^2\theta} \quad . \tag{25}$$

We also note for future reference that

$$\frac{\partial G_e}{\partial r} \Big/ G_e = -3/r \tag{26}$$

$$\frac{\partial G_e}{\partial \theta} \Big/ G_e = 3 \frac{\sin\theta\cos\theta}{1+3\sin^2\theta} \quad . \tag{27}$$

The partial derivatives of $\ln n^2$ with respect to r, θ, ψ, and ω (see section 5) are required in the Haselgrove equations. These in turn involve the various partial derivatives of α_0, α_1, α_4 and κ_1, κ_2, κ_3 .

The hierarchy of partial derivatives required in the HASELGROVE equations are gathered in this appendix. The partial derivatives of $\ln n^2$ are expressed in terms of partial derivatives of the various α's, and these in turn are expressed in terms of the partial derivatives of the κ's. Many of the partial derivatives with respect to the various independent variables have the same form, and this feature has been capitalized upon below in writing down the various formulas. Thus, since

$$\frac{\partial \ln n^2}{\partial p} = \frac{\partial n^2}{\partial p} \Big/ n^2, \qquad (28)$$

we consider first of all $\frac{\partial n^2}{\partial p}$:

$$\frac{\partial n^2}{\partial p} = -\frac{1}{2}\left[1 + \frac{\alpha_2}{\sqrt{\alpha_2^2 - 4\alpha_o\alpha_4}}\right]\frac{\partial\left(\frac{\alpha_2}{\alpha_4}\right)}{\partial p} + \frac{\alpha_4 \frac{\partial\left(\frac{\alpha_o}{\alpha_4}\right)}{\partial p}}{\sqrt{\alpha_2^2 - 4\alpha_o\alpha_4}} \qquad (29)$$

<u>where</u> p <u>denotes</u> <u>any</u> <u>one</u> <u>of</u> <u>the</u> <u>four</u> <u>independent</u> <u>variables</u> r, θ, ψ, <u>or</u> ω. In turn $\frac{\partial n^2}{\partial p}$ requires

$$\frac{\partial\left(\frac{\alpha_2}{\alpha_4}\right)}{\partial p} = \frac{\;}{\alpha_4}\left(\frac{\partial \alpha_2}{\partial p} - \frac{\alpha_2}{\alpha_4}\cdot\frac{\partial \alpha_4}{\partial p}\right) \qquad (30)$$

$$\frac{\partial\left(\frac{\alpha_o}{\alpha_4}\right)}{\partial p} = \frac{1}{\alpha_4}\left(\frac{\partial \alpha_o}{\partial p} - \frac{\alpha_o}{\alpha_4}\cdot\frac{\partial \alpha_4}{\partial p}\right) \qquad (31)$$

We note that $\frac{\partial \alpha_o}{\partial p}$, $\frac{\partial \alpha_2}{\partial p}$, and $\frac{\partial \alpha_4}{\partial p}$ are now required. We can no longer represent all four independent variables with the same parameter, since, according to equations (10) - (12), these partial derivatives will have fundamentally different forms according as the independent variable is ψ or r, θ, or ω. We thus introduce a new parameter q to represent the latter set of independent variables:

$$q = \underline{\text{some one}} \text{ of } r, \theta, \underline{\text{or}}\ \omega, \qquad (32)$$

and treat the partial derivatives with respect to ψ separately:

$$\frac{\partial \alpha_o}{\partial \psi} = 0 \tag{33}$$

$$\frac{\partial \alpha_2}{\partial \psi} = \sin 2\psi (\kappa_1 \kappa_2 - \kappa_2^2 - \kappa_3^2) \tag{34}$$

$$\frac{\partial \alpha_4}{\partial \psi} = \sin\psi (\kappa_2 - \kappa_1). \tag{35}$$

However, the partial derivatives of α_o, α_2, α_4 with respect to r, θ, or ω can still be subsumed under a general form:

$$\frac{\partial \alpha_o}{\partial q} = 2 \kappa_1 \left(\kappa_2 \frac{\partial \kappa_2}{\partial q} + \kappa_3 \frac{\partial \kappa_3}{\partial q} \right) + (\kappa_2^2 + \kappa_3^2) \frac{\partial \kappa_1}{\partial q} \tag{36}$$

$$\frac{\partial \alpha_2}{\partial q} = - \left\{ (1+\cos^2\psi) \left(\kappa_1 \frac{\partial \kappa_2}{\partial q} + \kappa_2 \frac{\partial \kappa_1}{\partial q} \right) + 2 \left(\kappa_2 \frac{\partial \kappa_2}{\partial q} + \kappa_3 \frac{\partial \kappa_3}{\partial q} \right) \sin^2\psi \right\} \tag{37}$$

$$\frac{\partial \alpha_4}{\partial q} = \frac{\partial \kappa_1}{\partial q} \cos^2\psi + \frac{\partial \kappa_2}{\partial q} \sin^2\psi . \tag{38}$$

It is clear from equations (36)-(38) that we must now find expressions for the partial derivatives of κ_1, κ_2, κ_3 with respect to r, θ, and ω. The partial derivatives with respect to r and θ will have the same general form, and so may again be given in a typical form with a parameter u to denote independent variable:

$$u = \underline{\text{either}} \quad r \ \underline{\text{or}} \ \theta . \tag{39}$$

Then

$$\frac{\partial \kappa_1}{\partial u} = - \frac{8.104 \times 10^{-5}}{f_{kc}^2} \frac{\partial N_{e/m^3}}{\partial u} \tag{40}$$

where $\frac{\partial N_{e/m^3}}{\partial u}$ is given by (15) or (16) according as u = r or θ. Similarly

$$\frac{\partial \kappa_2}{\partial u} = \frac{3.18 \times 10^3}{G_e^2} \left\{ \frac{\partial N_{e/m^3}}{\partial u} \left(\frac{1}{1-(\omega/G_e)^2} + \frac{1837}{1-(1837\omega/G_e)^2} \right) - 2N_{e/m^3} \frac{\frac{\partial G_e}{\partial u}}{G_e^3} \left(\frac{1}{\left[1-(\omega/G_e)^2\right]^2} + \frac{1837}{\left[1-(1837\omega/G_e)^2\right]^2} \right) \right\} \tag{41}$$

and

$$\frac{\partial \kappa_3}{\partial u} = i \frac{.506}{f_{kc} G_e} \left\{ \left(\frac{\partial N_{e/m^3}}{\partial u} + N_{e/m^3} \frac{\frac{\partial G_e}{\partial u}}{G_e} \right) \left(\frac{1}{1-(\omega/G_e)^2} - \frac{1}{1-(1837\omega/G_e)^2} \right) - 2N_{e/m^3} \frac{\frac{\partial G_e}{\partial u}}{G_e} \left(\frac{1}{\left[1-(\omega/G_e)^2\right]^2} - \frac{1}{\left[1-(1837\omega/G_e)^2\right]^2} \right) \right\}, \tag{42}$$

where G_e, $\frac{\frac{\partial G_e}{\partial u}}{G_e}$, $(\omega/G_e)^2$ and $(1837\omega/G_e)^2$ are given by equations (23)-(27).

4. Extended Transverse Propagation

Despite the fact that no guidance exists, it is possible that for some electron-ion distributions refraction of the phase normal in an inhomogeneous ionosphere would be sufficient to bend the phase (and ray) direction so as to keep it perpendicular to the magnetic field, even when the magnetic field varies as drastically in space as it does in the earth's dipole field. In such circumstances, energy launched in the transverse direction would continue to propagate in the transverse direction over extended distances. This possibility has been considered by Hoffman who in fact finds a general form of refractive-index variation which would produce just such a behavior in the presence of the geomagnetic dipole field. Examination of the required form indicates that it would not ordinarily be expected to occur in nature over considerable distances.

5. Trans-polar Propagation.

Despite the conclusion that extended transverse propagation is not to be expected in nature over very great distances, the question remained open as to the possible existence of nearly-transverse propagation over distances sufficient to permit earth-to-earth trans-polar propagation, as tentatively postulated by one of us (W. C. Hoffman). This question can be answered with conviction only after detailed ray-tracing has been completed, and a program of ray-tracing was undertaken as a crucial phase of this project. This program took as a starting point modes launched horizontally from various heights above the north pole, and it followed the ray-path of those modes as they progressed in time. The crux of the computing program was to find if any horizontally-launched mode (i. e. transverse mode, at the pole) follows a ray path which brings it to the earth's surface. For practical application one would, in fact, like the ray to follow a path of monotonically decreasing altitude with a terminus in the northern hemisphere. If this were to happen earth-to-earth trans-polar propagation might be possible, for the ray path could equally well be followed backwards in time and its origin could be traced to a point on the earth if the refractive index were symmetrical about the pole (or possibly with various other electron-ion distributions). The reason for choosing horizontally launched modes for the investigation is that a priori one would expect that, if no horizontally launched mode comes down to earth, it is doubtful that modes launched above horizontal would return to earth. The computational results in the following section show that this reasoning is very likely incorrect because of the highly oscillatory nature of the ray paths. However, this very property of the paths makes them unsuitable for trans-polar propagation. It will be noted that the return to earth of modes launched in directions below horizontal could not establish earth-to-earth propagation, since such modes come from directions above horizontal and could not have originated on the earth unless the "above-horizontal" ray path also returns to earth.

6. Computations for Trans-polar Propagation.

Three rays were traced using the Haselgrove equations (17) and the index of refraction as given by equations (8) through (13). The computations were carried out on an IBM-704 digital computer using a Runge-Kutta integration scheme. The rays were launched horizontally over the north magnetic pole at 500 km altitude. Two were for 2 kc/s propagation (the most favorable case) and one ray for 8 kc/s. The results of these computations are shown in Figures 5 and 6. The rays are highly oscillatory near the poles and do not, in fact, return to earth. The 2 kc/s curves appear to continue indefinitely (the ray tracing was terminated arbitrarily). The 8 kc/s curve was terminated when n^2 became negative. The physical interpretation of negative n^2 is not completely settled in the literature but is commonly associated with high absorption.

It was felt that such ray paths would be quite unstable; that is, quite sensitive to variations in electron-ion distributions and launching directions. By launching right and left ($\psi = \pm 90^o$) using the unsymmetric number density distribution N, one example of the effect of different N distributions was obtained. It is seen that the oscillatory type of mode is maintained but the details are different.

Further indication of the lack of stability of these ray paths comes from computations made using a modification of the above index of refraction formulas corresponding to doubling K_3, the relatively unimportant "transverse Hall" component of the permittivity matrix. The same type of highly oscillatory rays occur, this time, for 2 kc/s, actually returning to earth in the southern hemisphere. It is clear that no practical application can be made of results which are so sensitive to input parameters which are not accurately known and are quite variable with time. One could not predict where, if at all, trans-polar rays would reach the earth. Hence no further computations of this type appeared warranted, particularly, when the rays corresponding to a theoretically self consistent permittivity tensor (not doubling K_3) did not even return to earth.

RAY PATHS OF MODES WHICH ARE TRANSVERSE ABOVE THE NORTH MAGNETIC POLE

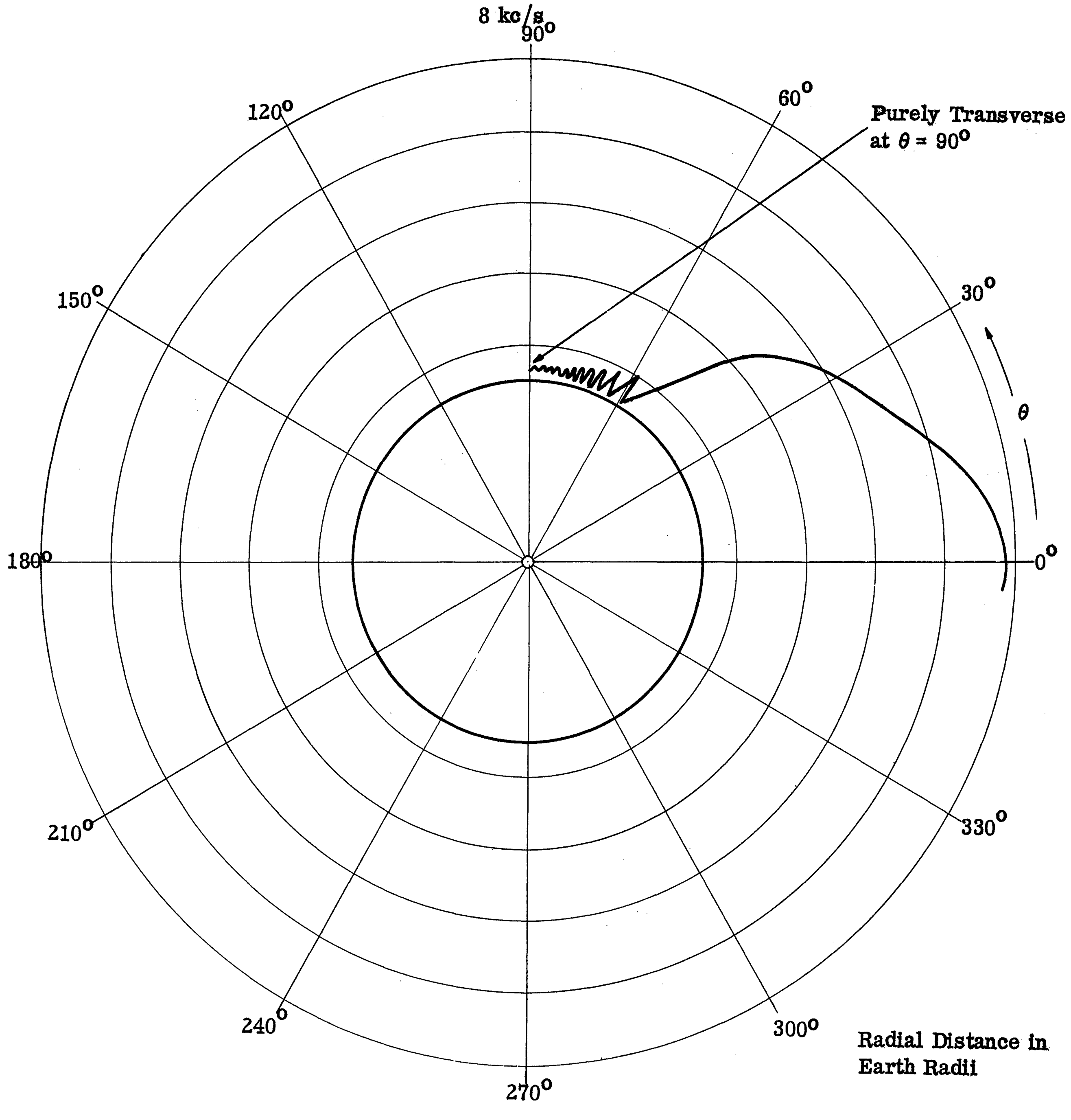


FIGURE 5

RAY PATHS OF MODES WHICH ARE TRANSVERSE ABOVE THE NORTH MAGNETIC POLE
2 kc/s

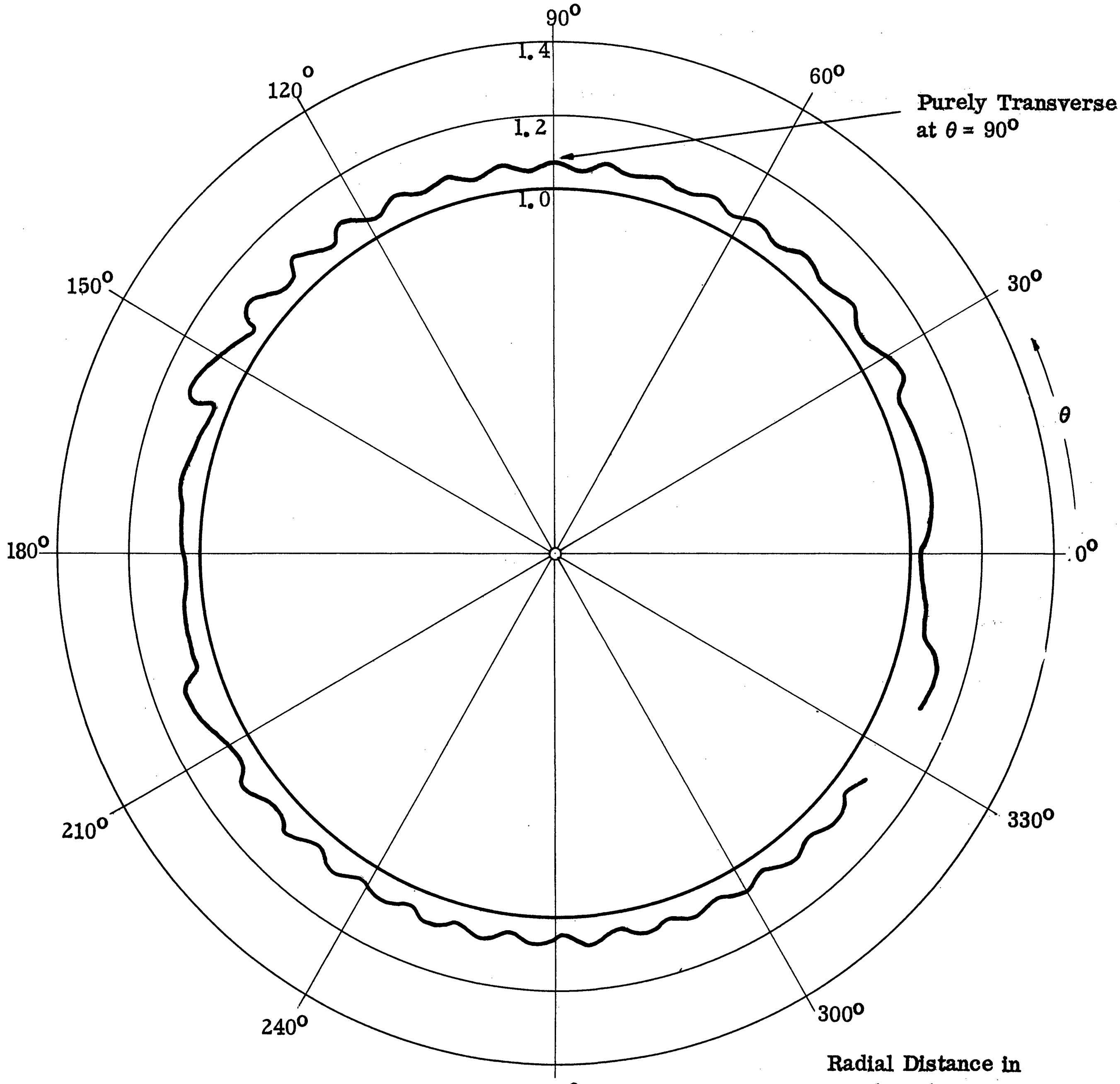


FIGURE 6

7. Further Difficulties.

Even if earth-to-earth trans-polar ray paths could have been reliably expected, at least two further difficulties which one would have had to face should be recognized.

(1) The energy must get into the proper mode. This is not easy, for sources below the ionosphere, since the tendency is for phase normals to turn nearly vertically on entry into the ionosphere (because of the high refractive index), and it would be only by chance that nearly-vertical phase propagation was the prerequisite for earth-to-earth propagation.

(2) The energy must not be dispersed too much in the course of its trans polar flight, if it is to be detected on its return to ground. This again is not easy, for dispersive effects will in general dominate, while focussing might possibly occur, its presence requires proof.

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