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THE PLASMA TEST PARTICLE PROBLEM

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ABSTRACT

A charged test particle passing through an electron plasma is subject to a drag force. The calculation of the drag, when the test particle's velocity is maintained at a constant value, is referred to as the plasma test particle problem. Several calculations of the drag have been given, but all treatments have resulted in a logarithmically diverging drag expression. The purpose of this dissertation is to give a convergent and consistent treatment of the plasma test particle problem.

In Chapters II-V we review the previous treatments of the plasma test particle problem. These are the binary collision, linear hydrodynamic, dielectric constant, and linear Landau-Vlasov treatments. In each case the source of the logarithmic divergence in the drag expression is fully discussed.

In Chapter III we also describe a modified linear solution of the hydrodynamic equations. This solution leads to a convergent drag expression. The hydrodynamic description of the test particle problem is basically a macroscopic description. In Chapter VI we obtain a more detailed, microscopic, solution of the test particle problem in terms of the Landau-Vlasov equation. The method of solution is similar to the modified linear solution of the hydrodynamic equations used in Chapter III. A convergent expression for the drag is given that is exact for a slow moving test particle, and which retains the qualitative features of an exact solution when the test particle speed is greater than the mean thermal electron speed.

The remaining three chapters of the dissertation discuss the derivation of a spatially homogeneous electron plasma kinetic (transport) equation. The problem of deriving a kinetic equation that contains a collision term giving an irreversible approach to equilibrium is closely related to the test particle problem.

In Chapters VII and VIII we review, criticize, and clarify several electron plasma kinetic equations that have been frequently used. These kinetic equations contain the same logarithmic divergences encountered in the test particle problem. The source of these divergences is clearly shown.

In Chapter IX the convergent test particle theories of Chapters III and VI are further developed to give a convergent electron plasma kinetic equation.

CHAPTER I

INTRODUCTION

This dissertation is concerned with the theory of stopping power in a fully ionized plasma. Specifically, the problem is to calculate the drag force on a charged "test" particle which is constrained to move with constant velocity through the plasma.

The first stopping power theory was developed by N. Bohr[1] to treat the penetration of charged particles through unionized matter. In the Bohr theory the charged test particle is assumed to undergo successive binary collisions with the atoms comprising the material. The momentum transferred in a single collision is calculated in a first approximation by neglecting the recoil of the colliding particles. The Bohr expression for the drag force, due to collisions with electrons with density n_0 cm^{-3}, has the form

$$\vec{\mathcal{F}} = -\frac{e_T^2 \omega_p^2}{v_T^2} \ln\left(\frac{b_{max}}{b_{min}}\right) \frac{\vec{v}_T}{v_T} \tag{1.1}$$

where $\omega_p = \left(\frac{4\pi e^2 n_0}{m}\right)^{1/2}$ has the dimension of a frequency (later we will see that ω_p is the plasma frequency), $-e_T$ is the test particle charge, $-e$ the electrcon charge, m the electron mass, $\vec{v}_T$ the test particle velocity, b_{max}, b_{min} are appropriate maximum and minimum impact parameters.*

*The impact parameter is the distance of closest approach between two colliding particles when there is no interaction.

The limiting values of the impact parameter, b_{max} and b_{min}, are chosen to satisfy the following requirements. For a collision with an impact parameter b and velocity v_T the time of collision τ_c is of order of magnitude $\tau_c = b/v_T$. When the time of collision τ_c exceeds the period of electron orbital motion in an atom τ_e it can be shown that the momentum transfer is approximately zero. Therefore, collisions with impact parameter larger than $v_T \tau_e$ must not be included in (1.1).

A minimum impact parameter must be used to limit the momentum transfer in a single collision. The calculation giving (1.1) assumes the colliding particles move in straight trajectories, and this assumption leads to an infinite momentum transfer as $b \to 0$. For a heavy test particle, $m_T >> m$, the maximum possible momentum transfer occurs in a head on collision, $\Delta p_{max} = 2 m v_T$, and b_{min} must be chosen to satisfy this requirement. The form of expression (1.1) for the drag force is characteristic of all stopping power calculations.

The binary collision picture presented above is a microscopic description of the stopping power problem. A macroscopic description was developed by E. Fermi[2] in which the medium is thought of as a continuous dielectric material described by a dielectric constant $\mathcal{E}(k, \omega)$. In this way one can account for the polarization induced in the medium by the test particle. As a consequence of the polarization the test particle is shielded from interacting with distant regions of the medium. The drag force has the form (1.1) except that b_{max} is automatically specified by the polarization shielding. Unfortunately, the dielectric description does not introduce a short range cutoff b_{min}.

More recently stopping power theories have been extended with renewed vigor into the domain of plasma physics where one is interested, for example, in calculating the drag force on charged satellities in the ionosphere or the thermalization of fast charged particles injected into a trapping machine. In addition to these direct applications of the plasma test particle problem interest has centered on the related problem of deriving a kinetic equation with a collision term capable of producing an irreversible approach to thermodynamic equilibrium.

An extensive bibliography of stopping power calculations is contained in the paper of Y. Klimontovich and V. P. Silin, Chapter II of the book Plasma Physics by J. E. Drummond, McGraw-Hill (1961).

The simplest stopping power theories for a plasma are based on a classical and non-relativistic description. The non-relativistic limit is permissible for the majority of plasmas encountered in practice. On the other hand, a classical treatment of a system of interacting ions and electrons poses difficulties, since a classical plasma is unstable against collapse. To circumvent this difficulty in a classical framework it is necessary to introduce short range repulsive forces between the ions and electrons. A simpler method, used in this dissertation, is to neglect terms of the order of the ratio of the electron mass to the ion mass. In this case the plasma is represented by a model in which the electrons move in a uniform background of positive charge, the so called "electron plasma." In addition to using an electron plasma the simpler stopping power theories assume that the plasma is infinite in extent, and that no external electric or magnetic fields are present.

The force on a test particle in an electron plasma has been calculated under the assumption of binary collisions by Spitzer,[3] Rosenbluth, et al.,[4] Hubbard,[5] and Aono.[6] The result obtained by these authors for an infinitely heavy test particle, when v_T >> (average thermal velocity of electrons), is

$$\vec{f} = -\frac{e_T^2 \omega_p^2}{v_T^2} \ln\left[\frac{m v_T^2}{e e_T} b_{max}\right] \frac{\vec{v}_T}{v_T} \tag{1.2}$$

In deriving this result the two body dynamics have been included exactly, and a short range cutoff, b_{min} , is not needed.

The divergence of (1.2) when $b_{max} = +\infty$ is the reason one refers to the Coulomb potential as a "long range" potential. Actually, it is not correct to speak of pure binary interactions when the impact parameter b is greater than the average interparticle spacing $n_0^{-1/3}$. When a "many particle" or "collective" description of a plasma is used it can be shown that the field of a charge is shielded exponentially with a characteristic length R_D , the Debye length, defined as $R_D = \left(\frac{4\pi e^2 n_0}{\Theta}\right)^{-1/2}$. Here Θ is the kinetic temperature $k_B T$. Because of the shielding, collisions cannot occur beyond a Debye distance and so we should choose $b_{max} = R_D$. In effect this is a compromise; choosing $b_{max} = R_D$ overestimates the momentum transfer for those collisions with $n_0^{-1/3} < b < R_D$ since these are not complete binary collisions. The momentum transfer from collisions with $b > R_D$ are underestimated since they are not included.

The electron plasma dielectric constant has been used for the force calculation by Hubbard,[5] Aono,[6] Linhard,[7] and Thompson.[8] The dielectric constant takes into account the "polarization" of the plasma by the test particle, i.e., the shielding, and eliminates the large impact parameter divergence encountered in the binary collision method. The theory, however, is not appropriate for close collisions. The result when $v_T \gg \sqrt{\Theta/m}$ is

$$\vec{\mathcal{F}} = -\frac{e_T^2 \omega_p^2}{v_T^2} \ln\left[\frac{\epsilon}{\lambda} \frac{v_T}{\sqrt{\Theta/m}} \frac{R_D}{\xi_0}\right] \frac{\vec{v}_T}{v_T} \qquad (1.3)$$

where $\epsilon = 2.718\cdots$ is the base of the natural logarithms and $\ln\lambda = 0.577\cdots$ is Euler's constant. ξ_0 is a minimum cutoff distance at which the dielectric theory breaks down. It turns out that ξ_0 should be on the order of $\frac{e e_T}{\Theta}$. The upper cutoff distance $b_{max} \cong R_D$ arises automatically from the dielectric screening of the test particle.

Rosenbluth and Rostoker,[9] Gasiorowicz <u>et al.</u>,[10] and Vlasov[11] have treated the plasma test particle problem from the standpoint of statistical mechanics using the Landau-Vlasov equation (self consistent field approximation). This method correctly treats both short and long range collisions, but the non-linearity of the equation makes it necessary to look for an approximate solution. The approximation made by these authors consists of a linearization in which the test particle field is assumed to be a small quantity which disturbs the plasma slightly from a spatially uniform equilibrium distribution. This approximation is equivalent to using the electron plasma dielectric constant and hence still requires an arbitrary short range cutoff.

Several authors[5,6,11] have discussed the result of combining the binary collision method and the collective method into a single expression for the force. Their result for $V_T \gg \sqrt{\Theta/m}$ is

$$\vec{\mathcal{F}} = -\frac{e_T^2 \omega_p^2}{V_T^2} \ln\left[\frac{2\sqrt{\epsilon}}{\lambda}\left(\frac{V_T}{\sqrt{\Theta/m}}\right)^3 \frac{R_D}{R_L}\right]\frac{\vec{V}_T}{V_T} \qquad (1.4)$$

where $R_L = \frac{e e_T}{\Theta}$ is referred to as the Landau length, $\epsilon = 2.718\cdots$ is the base of the natural logarithms, and $\ln\lambda = 0.577\cdots$ is Euler's constant. This result, which depends on the joining of the binary collision picture for the close collisions and the dielectric picture for the distant collisions, does not give correctly the coefficient of R_D/R_L in the logarithm argument. Indeed, the use of a cutoff is still implicit in (1.4).

We note that the general form of the drag expression will involve a logarithm factor. The argument of the logarithm is a function of V_T, generally with a small numerical value, times the ratio of lengths R_D/R_L. Throughout this dissertation we assume the inverse ratio R_L/R_D is an extremely small number; this makes it a useful expansion parameter. The assumption that R_L/R_D is small is not a serious restriction. In Table I values of R_L/R_D are tabulated for several interesting plasmas.

TABLE I

TYPICAL VALUES OF R_L/R_D

Plasma Type	n_0 cm^{-3}	T °K	$n_0^{-1/3}$ cm	$R_L = e^2/\theta$ cm	R_D cm	R_L/R_D
Gaseous Nebula	10^3	10^4	10^{-1}	$1.67\ 10^{-8}$	70	$2.4\ 10^{-10}$
Ionosphere— F-Layer	10^5	10^3	$2.2\ 10^{-2}$	$1.67\ 10^{-7}$	2.2	$7.6\ 10^{-8}$
Gas Discharge	10^{14}	10^4	$2.2\ 10^{-5}$	$1.67\ 10^{-8}$	$2.2\ 10^{-4}$	$7.6\ 10^{-5}$
Hot Plasma	10^{14}	10^5	$2.2\ 10^{-5}$	$1.67\ 10^{-9}$	$7.0\ 10^{-4}$	$2.4\ 10^{-6}$

The purpose of this dissertation is to eliminate, in a consistent manner, the divergences of the earlier work. Part I is devoted to the test particle problem in which the drag force on a charged particle, constrained to move uniformly through an electron plasma, is calculated.

The first chapter of Part I, Chapter II, describes the binary collision calculation of the drag. Chapter III compares the binary collision treatment with a linearized hydrodynamic theory of the plasma. A combination of these results giving (1.4) is discussed. In addition, Chapter III contains a modified linearization of the hydrodynamic equations that leads to a convergent expression for the force without the use of arbitrary cutoffs. This convergent solution of the test particle problem (in the hydrodynamic limit) is used as a model for the solution of the Landau-Vlasov equation presented in Chapter VI. In Chapter IV we review the method of expressing the drag force in terms of a general dielectric constant and magnetic permeability. In Chapter V the dielectric constant for an electron plasma is used to compute the drag. In Chapter VI, the last chapter of Part I, we present a new solution of the Landau-Vlasov equation that gives the drag on a test particle without the use of auxiliary cutoffs.

Part II of the dissertation is concerned with the general transport equations for a spatially homogeneous electron plasma. Chapters VII and VIII review several approximate kinetic equations and discusses their relationship to one another. These approximate kinetic equations involve divergent integrals identical to those discussed in connection with the test particle problem. In Chapter IX the convergent test particle results of Chapter VI are further developed, and are shown to lead to a convergent kinetic equation.

PART I: TEST PARTICLE PROBLEM

CHAPTER II

BINARY COLLISION TREATMENT OF THE TEST PARTICLE PROBLEM

1. INTRODUCTION

The aim of the test particle problem is to calculate the drag force on a charged particle, charge $-e_T$, as it moves with constant velocity $\vec{v}_T$ through a plasma. We shall assume the plasma is composed of singly charged ions and electrons with equal densities. The extension to multiple ionizations and mixtures of different ion species is straightforward.

A similar problem occurs in the kinetic theory of unionized gases where one deals with low density gases and interparticle forces of short range (i.e., range of interparticle force $<$ (density)$^{-1/3}$). In this case the motion of an uncharged test particle will be influenced by collisions with the gas atoms. The number of collisions with single atoms per second is proportional to the gas density n_0. The number of collisions simultaneously with two gas atoms is proportional to n_0^2, and so on for the higher multiple collisions. When the density is low multiple collisions with two or more gas atoms exert a negligible influence on the test particle. If collisions with two or more gas atoms are neglected the force on the test particle arises from the accumulation of momentum transfer in successive two body, or binary collisions.

In this chapter we discuss the simple binary collision picture and its application to the plasma test particle problem.

On the one hand it is not clear that the assumption of binary collisions is sensible for a test particle in a plasma, since it is the "long range character" of the Coulomb potential which accounts for the unique properties of the plasma state. On the other hand, it would appear that the plasma particle closest to the test particle will be the dominant source for the force felt by the test particle, and in this sense a binary collision calculation for the close collisions should be, at least qualitatively, correct. Actually we shall see later (Chapter III), that the drag force resulting from the accumulation of many distant collisions is of the same order of magnitude as the drag resulting from the close collisions. For the time being, however, we consider only the close collisions for which we make the binary collision assumption.

In the next section the notion of a collision cross section is introduced in terms of the momentum transfer in a binary collision. Expressing the cross section in terms of the momentum transfer vector instead of the scattering angles, as is customary, proves convenient since the momentum transfer is invariant in a Galilean transformation.

In the last section the drag force on a test particle is calculated under the assumption of binary collisions. The expression for the force diverges logarithmically when it is extended to include arbitrarily distant collisions. Qualitative arguments are given for the introduction of a cutoff to exclude the distant collisions.

2. KINEMATICS OF BINARY COLLISIONS

The Hamiltonian for two particles interacting through a potential $\Phi(|\vec{r}_1 - \vec{r}_2|)$ can be written as

$$\mathcal{H} = \frac{P^2}{2(m_1+m_2)} + \frac{p^2}{2\mu} + \Phi(r) \tag{2.1}$$

where $\vec{r} = \vec{r}_2 - \vec{r}_1$, $\vec{P}$, is the momentum of the center of mass of particles 1 and 2, $\vec{p} = \mu(\vec{v}_2 - \vec{v}_1)$, and μ is the reduced mass, $\mu = \frac{m_1 m_2}{m_1 + m_2}$. The Hamiltonian implies

$$\begin{aligned} \frac{d\vec{P}}{dt} &= 0 \\ \frac{d}{dt}\vec{r} \times \vec{p} &= 0 \end{aligned} \tag{2.2}$$

The first of Eqs. (2.2) shows that the center of mass moves with constant velocity. The second equation gives the angular momentum integral

$$\mu\,(\vec{r}_2 - \vec{r}_1) \times (\vec{v}_2 - \vec{v}_1) = \vec{\ell} \tag{2.3}$$

where $\vec{\ell}$ is a constant vector. Equation (2.3) shows that the particles move in a plane whose normal is in the direction $\vec{\ell}$. This plane is referred to as the <u>orbital</u> or <u>scattering plane</u>. Since we assume $\Phi(r)$ tends to zero as $r \to \infty$ the Hamiltonian (2.1) together with a constant value of $\vec{P}$ implies that the relative velocity magnitude before collision equals the relative velocity magnitude after collision,

$$|\vec{v}_2 - \vec{v}_1| = |\vec{v}_2{}' - \vec{v}_1{}'| \tag{2.4}$$

where unprimed quantities refer to quantities before a collision and primes refer to quantities after a collision. The effect of a collision is to rotate the relative velocity vector in the orbital plane.

A collision of an m_2 particle with an m_1 particle having velocity $\vec{v}_1$ is specified by the relative velocity $\vec{v}_2 - \vec{v}_1$ and the angular momentum $\vec{\ell}$ of Eq. (2.3). These quantities are not independent, for (2.3) implies $\vec{\ell}$ is perpendicular to $\vec{v}_2 - \vec{v}_1$,

$$\vec{\ell} \cdot (\vec{v}_2 - \vec{v}_1) = 0$$

The two independent components of $\vec{\ell}$ are conveniently chosen as the angle by which $\vec{\ell}$ is rotated about $\vec{v}_2 - \vec{v}_1$, call it ϕ, and the magnitude of $\vec{\ell}$ which can be written in terms of the <u>impact parameter</u> b as

$$|\vec{\ell}| = \mu b \, |\vec{v}_2 - \vec{v}_1| \tag{2.5}$$

The physical interpretation of b is that it would be the distance of closest approach between the particles if there were no interaction.

Now consider a beam of m_2 particles with velocity $\vec{v}_2$ and uniform density to be incident on an m_1 particle with velocity $\vec{v}_1$. Each particle undergoes a binary collision with the m_1 particle. The parameters specifying a given binary collision are $\vec{v}_2$, b, ϕ. The <u>differential scattering cross section</u> is defined as the number of collisions per second with impact parameter between b and $b+db$ and ϕ between

ϕ and $\phi + d\phi$ divided by the incident flux of m_2 particles. This is clearly

$$b\,db\,d\phi \tag{2.6}$$

As we have seen, the affect of a collision is to rotate the relative velocity vector, $\vec{g} = \vec{v}_2 - \vec{v}_1$, in the orbital plane. Denoting the angle of rotation by θ the differential scattering cross section can also be written as

$$b\,db\,d\phi = I(\theta, g)\,d\Omega \tag{2.7}$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$ is an element of solid angle. The quantity $I(\theta, g)\,d\Omega$ is interpreted as the number of m_2 particles per second, whose relative velocity $\vec{g}$ is rotated through an angle θ into the solid angle $d\Omega$, divided by the incident flux of $(m_2, \vec{v}_2)$ particles.

Another form of the differential cross section, and for our purposes the most useful form, is obtained by expressing the differential cross section in terms of the momentum transferred in a collision. Let $\vec{q}$ denote the momentum given to particle 1 after a collision with particle 2. According to the conservation of momentum, the momentum given to particle 2 by particle 1 is $-\vec{q}$. The collision can be represented by

$$\begin{aligned} \vec{v}_1 &\longrightarrow \vec{v}_1{}' = \vec{v}_1 + \vec{q}/m_1 \\ \vec{v}_2 &\longrightarrow \vec{v}_2{}' = \vec{v}_2 - \vec{q}/m_2 \end{aligned} \tag{2.8}$$

Conservation of energy in an elastic collision requires

$$\vec{q}\cdot\vec{g} - \frac{q^2}{2\mu} = 0 \tag{2.9}$$

Subtracting the first of Eqs. (2.8) from the second gives

$$\frac{1}{\mu}\vec{q} = (\vec{v}_2 - \vec{v}_1) - (\vec{v}_2{}' - \vec{v}_1{}')$$

Taking the magnitude of each side relates the amount of momentum transferred to the angle through which the relative velocity vector is rotated in a collision

$$q = 2\mu g \sin \theta/2 \quad , \quad 0 \leq \theta \leq \pi \tag{2.10}$$

The angle ϕ specifies the rotation of the orbital plane $(\vec{l})$ about the direction $\vec{g}$ with respect to an arbitrary reference direction $\phi = 0$. Since $\vec{q}$ is in the orbital plane ϕ is also the azimuthal angle of $\vec{q}$ about $\vec{g}$ as polar axis. The polar angle ψ which $\vec{q}$ makes with $\vec{g}$ is given by the conservation of energy relation (2.9),

$$\cos\psi = \frac{q}{2\mu g} \quad , \quad 0 \leq q \leq 2\mu g$$

With the delta function identity

$$\int_{-1}^{1} \delta\left(\cos\psi - \frac{q}{2\mu g}\right) d(\cos\psi) = 1$$

and relation (2.10) the differential cross section (2.7) can be written as

$$\int_{\Omega}^{\Omega+\Delta\Omega} d\Omega\, I(\theta,q) = \int_{\vec{q}}^{\vec{q}+\Delta\vec{q}} d\vec{q}\, \frac{I(\theta,q)}{\mu^2 q}\, \delta\left(\vec{q}\cdot\vec{g} - \frac{q^2}{2\mu}\right)$$

$$= \int_{\vec{q}}^{\vec{q}+\Delta\vec{q}} d\vec{q}\, \frac{\omega(\vec{q},\vec{g})}{g} \tag{2.11}$$

where $d\vec{q} = q^2 \sin\psi\, d\psi\, d\phi\, dq$ is a three dimensional volume element in $\vec{q}$ space. The second equality defines the function $\omega(\vec{q},\vec{g})$. Since $\omega(\vec{q},\vec{g})$ contains a delta function its physical interpretation is sensible only upon integration. When integrated between $\vec{q}$ and $\vec{q}+\Delta\vec{q}$ the quantity $\frac{\omega(\vec{q},\vec{g})}{g}$ is the number of $(m_2,\vec{V}_2)$ particles per second, that lose momentum between $\vec{q}$ and $\vec{q}+\Delta\vec{q}$ upon collision with particle $(m_1,\vec{V}_1)$, divided by the incident flux of m_2 particles. Alternatively,

$$\int_{\vec{q}}^{\vec{q}+\Delta\vec{q}} d\vec{q}\, \omega(\vec{q},\vec{g})$$

is the number of $(m_2,\vec{V}_2)$ particles per second <u>per unit density</u>, that lose momentum between $\vec{q}$ and $\vec{q}+\Delta\vec{q}$ when colliding with particle $(m_1,\vec{V}_1)$.

The function $\omega(\vec{q},\vec{g})\,d\vec{q}$ is called the <u>collision volume</u>. Two important identities can be established for the collision volume. Before doing this, however, we will find it convenient, when we come to the Boltzmann equation in Chapter VIII, to indicate in $\omega(\vec{q},\vec{g})$ the specific binary collision under consideration. For the colision (2.8)

$$\begin{aligned} \vec{v}_1 &\longrightarrow \vec{v}_1 + \vec{q}/m_1 \\ \vec{v}_2 &\longrightarrow \vec{v}_2 - \vec{q}/m_2 \end{aligned} \tag{2.12}$$

the collision volume will be denoted by

$$\omega\left(\vec{v}_1, \vec{v}_2 \,\middle|\, \vec{q}/m_1, -\vec{q}/m_2\right) d\vec{q} \tag{2.13}$$

The manner in which the arguments of ω specify the collision (2.12) is evident.

The first identity satisfied by the collision volume results from the invariance of the momentum transfer under a Galilean transformation,

$$\omega\left(\vec{v}_1+\vec{V}_0, \vec{v}_2+\vec{V}_0 \,\middle|\, \vec{q}/m_1, -\vec{q}/m_2\right) = \omega\left(\vec{v}_1, \vec{v}_2 \,\middle|\, \vec{q}/m_1, -\vec{q}/m_2\right) \tag{2.14}$$

where $\vec{V}_0$ represents a uniform translation. In other words, ω depends on $\vec{v}_1$, $\vec{v}_2$ only through the difference $\vec{v}_2 - \vec{v}_1$.

The second identity involves the notion of an <u>inverse collision</u>. The inverse collision corresponding to the direct collision (2.12) is

$$\begin{aligned} \vec{v}_1 + \vec{q}/m_1 &\longrightarrow \vec{v}_1 \\ \vec{v}_2 - \vec{q}/m_2 &\longrightarrow \vec{v}_2 \end{aligned} \tag{2.15}$$

For a spherically symmetric potential, $\Phi(|\vec{r}_1 - \vec{r}_2|)$, the collision volume is the same for a direct and inverse collision; the second identity satisfied by the collision volume is

$$\omega(\vec{v}_1,\vec{v}_2 \mid \vec{q}/m_1, -\vec{q}/m_2) = \omega(\vec{v}_1+\vec{q}/m_1, \vec{v}_2-\vec{q}/m_2 \mid -\vec{q}/m_1, \vec{q}/m_2) \tag{2.16}$$

This completes our survey of the basic kinematics of binary collisions with spherically symmetric potentials. Of particular importance to us is the special case of a Coulomb potential

$$\Phi(|\vec{r}-\vec{r}_T|) = \frac{e\,e_T}{|\vec{r}-\vec{r}_T|} \tag{2.17}$$

This is the interaction potential of an electron with charge $-e$ at position $\vec{r}$ and a test particle of charge $-e_T$ at position $\vec{r}_T$. The differential cross section is the well known Rutherford cross section*

$$I(\theta,g) = \frac{e^2 e_T^2}{4\mu^2 g^4 \sin^4\theta/2} \tag{2.18}$$

where $\vec{g} = \vec{v}-\vec{v}_T$ and $\mu = \frac{m\,m_T}{m+m_T}$. Together with (2.10) and (2.11) we immediately obtain for the collision volume

$$\omega(\vec{v}_T,\vec{v} \mid \vec{q}/m_T, -\vec{q}/m)\,d\vec{q} = \frac{4e^2e_T^2}{q^4}\,\delta\left(\vec{q}\cdot\vec{g}-\frac{q^2}{2\mu}\right)d\vec{q} \tag{2.19}$$

3. BINARY COLLISION CALCULATION OF THE DRAG ON A TEST PARTICLE

Consider an infinite plasma with an average constant electron density n_0 and a velocity distribution $f(\vec{v})$ that is normalized to unity,

*Goldstein, H., Classical Mechanics, Addison-Wesley (1957).

$$\int d\vec{v}\, f(\vec{v}) = 1$$

The electron charge is $-e$ and the electron mass is m. We wish to calculate the force on a test particle with charge $-e_T$, mass m_T, and moving with constant velocity $\vec{v}_T$ through the electron plasma. It is further supposed that the test particle undergoes only binary collisions with the plasma electrons. The average force on the test particle is the time rate of change of momentum, i.e., the average momentum change in a single binary collision times the number of collisions per second,

$$\vec{\mathcal{F}}_{B.C.} = n_0 \int d\vec{v}\, f(\vec{v}) \int d\vec{q}\, \vec{q}\, \omega\left(\vec{v}_T, \vec{v} \,\middle|\, \vec{q}/m_T, -\vec{q}/m\right) \tag{2.20}$$

Introducing the collision volume for Coulomb interactions (2.19)

$$\vec{\mathcal{F}}_{B.C.} = 4e^2 e_T^2 n_0 \int d\vec{v}\, f(\vec{v}) \int d\vec{q}\, \frac{\vec{q}}{q^4}\, \delta\left(\vec{q}\cdot\vec{g} - q^2/2\mu\right) \tag{2.21}$$

where $\vec{g} = \vec{v} - \vec{v}_T$ and $\mu = \dfrac{m\, m_T}{m + m_T}$.

The integration over $\vec{q}$ is performed by using polar coordinates (q, $\eta = \cos\theta$, ϕ) with $\vec{g}$ as polar axis (i.e., $\vec{q}\cdot\vec{g} = qg\eta$). The components of $\vec{q}$ perpendicular to $\vec{g}$ involve $\sin\phi$ or $\cos\phi$ and integrate to zero. The remaining component parallel to $\vec{g}$ is

$$\int d\vec{q}\, \frac{\vec{q}}{q^4}\, \delta\left(\vec{q}\cdot\vec{g} - \frac{q^2}{2\mu}\right) = 2\pi \int_0^\infty dq \int_{-1}^{1} d\eta\, \frac{\eta}{q}\, \frac{\vec{g}}{g}\, \delta\left(qg\eta - q^2/2\mu\right)$$

The delta function gives a contribution only when $q \leq 2\mu g$,

$$= \frac{\pi}{\mu}\, \frac{\vec{g}}{g^3} \int_0^{2\mu g} dq\, \frac{1}{q}$$

The one remaining integral diverges logarithmically at the lower limit. To secure convergence the lower limit must be replaced by a lower cutoff q_{min} ,

$$= \frac{\pi}{\mu}\frac{\vec{g}}{g^3}\ln\left(\frac{2\mu g}{q_{min}}\right)$$

Inserting this result in (2.21),

$$\vec{\mathcal{J}}_{B.C.} = \frac{4\pi e^2 e_T^2 n_0}{\mu}\int d\vec{v}\, f(\vec{v})\frac{\vec{g}}{g^3}\ln\left(\frac{2\mu g}{q_{min}}\right) \qquad (2.22)$$

The divergence as $q_{min}\to 0$ corresponds to a large impact parameter divergence, $b_{max}\to\infty$. To find the connection between q_{min} and b_{max} consider a binary collision with an extremely small momentum transfer. By "small" one means that the momentum transfer is much less than the initial momentum, $q \ll \mu g$. In the limiting case $q \lll \mu g$ the colliding particles move along straight trajectories with constant speeds. The calculation of $\vec{q}$ in this so called <u>straight path</u> limit is quite simple. The component of $\vec{q}$ along the direction of motion $\vec{g}$ averages to zero. The perpendicular component of $\vec{q}$ is (see Fig. 1),

$$q = \int_{-\infty}^{\infty} dt\, F(r)\,\frac{b}{r}$$

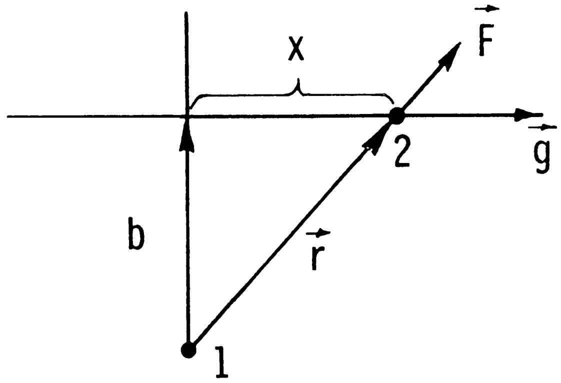

Fig. 1. Straight path collision in rest frame of particle T.

Making use of

$$F(r) = \frac{e e_T}{r^2}$$

$$dx = g\, dt$$

$$r^2 = x^2 - b^2$$

where b is the impact parameter, the momentum transfer is given by

$$q = \frac{2 e e_T}{g} \int_0^\infty dx \frac{b}{[x^2 + b^2]^{3/2}}$$

$$= \frac{2 e e_T}{g b}$$

Therefore, in the limit $q \ll \mu g$, we can replace q_{min} by a maximum impact parameter according to the relation

$$q_{min} = \frac{2 e e_T}{g\, b_{max}} \tag{2.23}$$

Inserting (2.23) for q_{min} in (2.22) the force integral becomes

$$\vec{\mathcal{F}}_{B.C.} = \frac{4\pi e^2 e_T^2 n_0}{\mu} \int d\vec{v} f(\vec{v}) \frac{\vec{g}}{g^3} \ln\left(\frac{\mu g^2 b_{max}}{e e_T}\right) \tag{2.24}$$

where $\vec{g} = \vec{v} - \vec{v}_T$. The integration over $\vec{v}$ does not diverge at $\vec{g} = 0$ provided we agree to perform the integration over the direction of $\vec{g}$ first.

The most interesting situation is when $f(\vec{v})$ is the Maxwell-Boltzmann distribution

$$f(\vec{v}) = f^{(0)}(\vec{v}) = \left(\frac{m}{2\pi\theta}\right)^{3/2} e^{-\frac{mv^2}{2\theta}} \tag{2.25}$$

where $\Theta = k_B T$ is the kinetic temperature. In this case there are two limiting forms of (2.24) that are easily discussed. For a fast test particle, $v_T >> \sqrt{\frac{\Theta}{m}}$, the quantity $|\vec{v}-\vec{v}_T|^2$ in the argument of the logarithm can be approximated by v_T^2, and the force (2.24) becomes

$$\vec{\mathcal{F}}_{B.C.} = \frac{4\pi e^2 e_T^2 n_0}{\mu} \ln\left(\frac{\mu v_T^2}{e e_T} b_{max}\right) \int d\vec{v}\, f^{(0)}(\vec{v}) \frac{\vec{v}-\vec{v}_T}{|\vec{v}-\vec{v}_T|^3} \qquad (2.26)$$

The remaining integral has the form of an "electric field" at $\vec{v}_T$ due to a charge distribution with density $-f^{(0)}(\vec{v})$. Since $f^{(0)}(\vec{v})$ is spherically symmetric this "electric field" is equivalent to the field produced when all the "charge" within a sphere of radius v_T is concentrated at the origin,

$$\int d\vec{v}\, f^{(0)}(\vec{v}) \frac{\vec{v}-\vec{v}_T}{|\vec{v}-\vec{v}_T|^3} = -\frac{\vec{v}_T}{v_T^3} \int_0^{v_T} d\vec{v}\, f^{(0)}(\vec{v}) \qquad (2.27)$$

Thus,

$$\vec{\mathcal{F}}_{B.C.} = -\frac{4\pi e^2 e_T^2 n_0}{\mu} \ln\left(\frac{\mu v_T^2}{\Theta} \frac{b_{max}}{R_L}\right) \frac{\vec{v}_T}{v_T^3} \int_0^{v_T} d\vec{v}\, f^{(0)}(\vec{v}) \qquad (2.28)$$

where we have introduced $R_L = \frac{e e_T}{\Theta}$, which has the dimensions of a length.

The second limiting case occurs for a slow test particle, $v_T << \sqrt{\Theta/m}$. In this case $|\vec{v}-\vec{v}_T| \approx v^2$. If we replace v^2 in the logarithm argument by its average value, $3\Theta/m$, expression (2.24) reduces to,

$$\vec{\mathcal{F}}_{B.C.} = -\frac{4\pi e^2 e_T^2 n_0}{\mu} \ln\left(\frac{3\mu}{m} \frac{b_{max}}{R_L}\right) \frac{\vec{v}_T}{v_T^3} \int_0^{v_T} d\vec{v}\, f^{(0)}(\vec{v}) \qquad (2.29)$$

with $R_L = \frac{e e_T}{\Theta}$. Approximating the last integral for $v_T << \sqrt{\Theta/m}$ gives

$$\vec{F}_{B.C.} = -\frac{4\pi e^2 e_T^2 n_0}{\mu}\sqrt{\frac{2}{\pi}}\left(\frac{m}{\theta}\right)^{3/2}\ln\left(\frac{3\mu}{m}\frac{b_{max}}{R_L}\right)\vec{V} \tag{2.30}$$

There still remains the problem of specifying the maximum impact parameter b_{max}. If we are to include the momentum transfer with each plasma electron then $b_{max} = +\infty$. However, the assumption of binary collisions is not appropriate when the impact parameter is on the order of the interparticle separation, $b \gtrsim n_0^{-1/3}$. When $b \gg n_0^{-1/3}$ a many particle description of the plasma must be used. The collective response of the plasma screens the test charge field beyond a Debye length, $R_D = \left(\frac{4\pi e^2 n_0}{\theta}\right)^{-1/2}$. For this reason the maximum impact parameter is often identified with the Debye length.[3,4,12,13]

Expression (2.24) gives the drag force due to binary collisions with plasma electrons. To include the effects of collisions with ions it is necessary to add terms similar to (2.24) with the appropriate reduced mass, ion charge, and ion distribution function. These terms are of order (electron mass) $\div$ (ion mass) smaller than (2.24) and are neglected in this dissertation.

The main objection to the binary collision picture is its inability to supply b_{max}. That it is at all necessary to introduce an upper cutoff is a peculiarity associated with the $1/r$ potential. The real difficulty, however, lies in the assumption of binary collisions. A correct treatment of the plasma-test particle interaction must allow for the test particle to interact with many plasma particles at one time. The result of many simultaneous interactions is a redistribution of positive and negative plasma charge around the test particle which screens the test particle field at distances greater than the Debye length, $R_D = \left(\frac{4\pi e^2 n_0}{\Theta}\right)^{-1/2}$, and this effectively limits the maximum impact parameter.

We conclude, then, that a consistent expression for the drag force that does not require an external cutoff can only be obtained through a many particle description of the plasma.

CHAPTER III

HYDRODYNAMIC TREATMENT OF THE TEST PARTICLE PROBLEM

1. INTRODUCTION

We have seen that the binary collision treatment of the test particle problem cannot give a consistent result for the drag without using a long range cutoff. It was necessary to insert a cutoff because the assumption of binary collisions does not allow for the simultaneous interaction of many particles. If we are to give a consistent derivation of the drag force we must use a description of the plasma that can account for its collective behavior. The simplest way to accomplish this is to treat the electrons and ions as simple fluids subject to body forces arising from their "self consistent" electric fields. We do this in Section 2 by adopting the Euler hydrodynamic equations modified to include the self consistent fields and the frictional forces arising from the relative streaming of the electron and ion fluids.

The modified Euler equations are non-linear and quite complicated. For the test particle problem we linearize these equations by treating the disturbance caused by the test particle as a small perturbation. When the test particle speed is greater than the average electron thermal speed the drag is due primarily to the electrons; the ions contribute small corrections of the order of the electron mass divided by the ion mass. This leads to use of the "electron plasma" model in which the ion motion is neglected and the electrons are assumed to move in a uniform background of positive charge.

The solution of the linearized equations describing the test particle motion in an electron plasma is straightforward. An expression for the drag is derived which does not diverge at large impact parameters. Screening effects introduce the Debye length as an effective maximum impact parameter. However, a new difficulty appears in the problem. The expression for the drag force still diverges logarithmically; this time at short distances due to an incorrect treatment of the plasma in the neighborhood of the test particle. The short range breakdown of the theory is traced to the linearization of the Euler equations. The linearization was based on the assumption that the test particle introduced only a small perturbation in the plasma, but this is not correct in regions sufficiently close to the test particle.

In Section 3 the results obtained from the linearized Euler equations are discussed and compared with the results of the binary collision treatment of Chapter II. A combination of the two treatments that has been discussed by Vlasov,[11] Hubbard,[5] and Aono[6] is reviewed.

In Section 4 we consider a modified linearization of the Euler equations which leads to an expression for the drag that does not require the use of cutoffs. The modified linearization uses as a zeroth approximation for the density the solution to the Euler equations when $\vec{v}_T = 0$, call it $n_0(|\vec{r}-\vec{r}_T|)$. If one assumes that the density about a moving test particle differs only slightly from $n_0(|\vec{r}-\vec{r}_T|)$ the correction may be treated as a perturbation,

$$n(\vec{r}-\vec{r}_T) = n_0(|\vec{r}-\vec{r}_T|)\left\{1 + h(\vec{r}-\vec{r}_T)\right\}$$

In the zeroth approximation the density, $n_0(|\vec{r}-\vec{r}_T|)$, vanishes at the test particle (proportional to a Boltzmann-Gibbs type factor $\exp\left\{-\frac{ee_T}{\theta}\frac{1}{|\vec{r}-\vec{r}_T|}\right\}$. The next approximation, $h(\vec{r}-\vec{r}_T)$, is determined by linearizing the Euler equations. Although the modified linearization breaks down near the test particle the effect is unimportant, since $n_0(|\vec{r}-\vec{r}_T|)$ vanishes in that region. The results are exact as $v_T \to 0$.

The solution of the modified linearization is used to calculate the drag force on the test particle. The drag is expressed in terms of an integral which is evaluated to lowest order in an expansion in powers of R_L/R_D. That the ratio of the Landau length, $R_L = \frac{ee_T}{\theta}$, to the Debye length, $R_D = \left(\frac{4\pi e^2 n_0}{\theta}\right)^{-1/2}$, is an appropriate expansion parameter is seen from Table I.

We present the modified linearization calculation in detail since it clarifies the more refined treatment of the test particle problem in Chapter VI.

2. LINEARIZATION OF THE HYDRODYNAMIC EQUATIONS

In a simple hydrodynamic treatment of the test particle problem we assume the test particle (charge $-e_T$, mass m_T) is moving with uniform velocity $\vec{v}_T$ through an infinite plasma. Both the ions (charge e, mass M) and electrons (charge $-e$, mass m) are assumed to satisfy a set of

Euler equations.* For electrons with density $n(\vec{r},t)$, velocity $\vec{u}_e(\vec{r},t)$, scalar pressure $p_e(\vec{r},t)$, and constant kinetic temperature $\theta = k_B T$,

$$mn\left[\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e \cdot \nabla \vec{u}_e\right] = -\nabla p_e - en\vec{E} + \vec{P}_{ei}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{u}_e) = 0 \tag{3.1}$$

$$p_e = \theta n$$

For ions with density $N(\vec{r},t)$, velocity $\vec{u}_i(\vec{r},t)$, scalar pressure $p_i(\vec{r},t)$, and constant kinetic temperature $\theta = k_B T$,

$$MN\left[\frac{\partial \vec{u}_i}{\partial t} + \vec{u}_i \cdot \nabla \vec{u}_i\right] = -\nabla p_i + eN\vec{E} + \vec{P}_{ie}$$

$$\frac{\partial N}{\partial t} + \nabla \cdot (N\vec{u}_i) = 0 \tag{3.2}$$

$$p_i = \theta N$$

In Eqs. (3.1)-(3.2) $\vec{E}$ is the electric field due to the charge densities eN, $-en$, and the test charge "density" $-e_T\delta(\vec{r}-\vec{r}_T)$, $\vec{r}_T = \vec{v}_T t$. These quantities are related by Poisson's equation

$$\nabla \cdot \vec{E} = 4\pi e\,(N-n) - 4\pi e_T\,\delta(\vec{r}-\vec{v}_T t) \tag{3.3}$$

The vector quantity $\vec{P}_{ei}$ is the force per unit volume exerted on the electrons by collisions with the ions. More specifically, $\vec{P}_{ei}$ arises from those collisions not already included in the macroscopic electric

*For a discussion of these equations as applied to a plasma see Ref. 3.

field $\vec{E}$. The corresponding quantity for the ions is $\vec{P}_{ie}$. The conservation of momentum assures us that $\vec{P}_{ei} + \vec{P}_{ie} = 0$. A simple collision picture gives

$$\vec{P}_{ei} = 2\mu n \nu_{ei} \left(\vec{u}_i - \vec{u}_e \right) = -\vec{P}_{ie} \tag{3.4}$$

where $\mu = \frac{mM}{m+M}$ and ν_{ei} is an effective collision frequency.

The conservation of energy equation, the last equation of (3.1) and (3.2), has been replaced by the simple ideal gas equation of state $p = \theta n$. The kinetic temperature $\theta = k_B T$ is assumed to be a constant, the same for both ions and electrons.*

Throughout this dissertation terms proportional to the ratio of electron mass to ion mass, m/M, are neglected. (The largest value of m/M occurs in a hydrogen plasma, $\frac{m}{M} = \frac{1}{1836}$.) To terms of order m/M the ion motion can be neglected, and the electrons are treated as moving in a uniform background of positive charge.

We can now proceed with the linearization of Eqs. (3.1)-(3.3). For the undisturbed values of the fluid variables we have

$$\begin{aligned} n &= N = n_0 \\ p_e &= n_0 \theta \\ \vec{u}_e &= 0 \end{aligned} \tag{3.5}$$

---

*If adiabatic changes are assumed, $p n^{-\gamma}$ is a constant along a streamline where γ is the ratio of specific heats. In the linearized theory the only modifications introduced by this refinement are (1) the "sound velocity," $\sqrt{\theta/M}$, is replaced by $\sqrt{\gamma\theta/m}$, and (2) the Debye length R_D is replaced by $\sqrt{\gamma}\, R_D$.

The test particle is assumed to introduce a small deviation from the undisturbed values,

$$
\begin{aligned}
n &= n_o + n' \\
p_e &= n_o\theta + p_e' \qquad\qquad (3.6) \\
\vec{u_e} &= \vec{u_e}
\end{aligned}
$$

Neglecting second and higher order quantities the linearized form of Eqs. (3.1)-(3.4) is

$$
\begin{aligned}
m n_o \frac{\partial \vec{u_e}}{\partial t} &= -\nabla p_e' - e n_o \vec{E} - 2\mu\nu n_o \vec{u_e} \\
\frac{\partial n'}{\partial t} + n_o \nabla\cdot\vec{u_e} &= 0 \qquad\qquad (3.7) \\
p_e' &= \theta n'
\end{aligned}
$$

$$\nabla\cdot\vec{E} = -4\pi e n' - 4\pi e_T\,\delta(\vec{r}-\vec{v_T}t) \qquad (3.8)$$

$$\nu_{ei} = \nu_{ie} = \nu \qquad (3.9)$$

It should be noted that in the neighborhood of the test particle $|\vec{E}| \approx \frac{e_T}{|\vec{r}-\vec{r_T}|^2}$, and the assumption that $\vec{E}$ is small is no longer valid. We return to this point in Section 4.

Taking the divergence of the first of Eqs. (3.7), and using the continunity equation together with Poisson's equation (3.8) one obtains

$$\left\{\frac{\partial^2}{\partial t^2} - a^2\nabla^2 + 2\nu\frac{\partial}{\partial t} + \omega_p^2\right\} n' = -\omega_p^2 \frac{e_T}{e}\,\delta(\vec{r}-\vec{v_T}t) \qquad (3.10)$$

where $\omega_p = \left(\frac{4\pi e^2 n_0}{m}\right)^{1/2}$ is the plasma frequency and $a = \sqrt{\theta/m}$ is the speed of sound in the electron fluid, $a^2 = \frac{dp_e'}{d(mn')}$.

We now make an approximation concerning the collision frequency ν . To obtain an estimate of the magnitude of ν we must first decide on what constitutes a collision. If we suppose t_q is the time necessary for the average force on an electron, $\vec{F}$, to produce a momentum change comparable to its initial momentum, i.e., $|\vec{F}|t_q = mu$, then a possible choice for the collision frequency is $\nu = t_q^{-1} = \frac{|\vec{F}|}{mu}$. Equation (2.30) gives an estimate for the force when u is small. With the substitutions $v_T = u$, $\mu = m$, $e_T = e$, and $b_{max} = R_D$ in (2.30) we find

$$\nu = \omega_p \frac{R_L}{R_D} \sqrt{\frac{2}{\pi}} \ln\left(\frac{3 R_D}{R_L}\right) \tag{3.11}$$

where $R_L = e^2/\theta$ is the Landau length and $R_D = \left(\frac{4\pi e^2 n_0}{\theta}\right)^{-1/2}$ is the Debye length. For most plasmas R_L/R_D is on the order of 10^{-5} to 10^{-12} (see Table I), and $\nu \ll \omega_p$. In Eq. (3.10) the collision frequency occurs through the term $2\nu\frac{\partial n'}{\partial t}$. The dominant frequency in $\partial n'/\partial t$ is near the natural frequency of the plasma, ω_p . Hence, $2\nu\frac{\partial n'}{\partial t} \sim 2\nu\omega_p \ll \omega_p^2$, and to a good first approximation we set $\nu = 0$ in Eq. (3.10),

$$\left\{ \frac{\partial^2}{\partial t^2} - a^2\nabla^2 + \omega_p^2 \right\} n' = -\omega_p^2 \frac{e_T}{e} \delta(\vec{r} - \vec{v}_T t) \tag{3.12}$$

The solution of (3.12) is straightforward and is sufficiently simple that an explicit expression for the test particle drag can be obtained. To proceed with the solution we first note that the time dependence of n' must occur in the form $n' = n'(\vec{r} - \vec{v}_T t)$. The operator

$\partial/\partial t$ is replaced by $-\vec{v}_T\cdot\nabla$. Introducing $\vec{\xi}=\vec{r}-\vec{v}_T t$ Eq. (3.12) becomes

$$\left\{ \left(\vec{v}_T\cdot\frac{\partial}{\partial\vec{\xi}}\right)^2 - a^2\frac{\partial^2}{\partial\vec{\xi}^2} + \omega_p^2 \right\} n'(\vec{\xi}) = -\omega_p^2\,\frac{e_T}{e}\,\delta(\vec{\xi}) \tag{3.13}$$

Taking the ξ_z axis along $\vec{v}_T$ and using the boundary condition that there be no disturbance in the fluid as $\xi_z \to +\infty$ there are two cases depending on whether v_T is greater than or less than the sound velocity.

When $v_T < a$

$$n'(\vec{\xi}) = -\frac{c\,e_T\,n_0}{m a\sqrt{a^2-v_T^2}}\;\frac{e^{-\frac{\omega_p}{a}\sqrt{\xi_\perp^2+\frac{\xi_z^2}{1-v_T^2/a^2}}}}{\sqrt{\xi_\perp^2+\frac{\xi_z^2}{1-v_T^2/a^2}}} \tag{3.14}$$

where $\xi_z = \dfrac{\vec{\xi}\cdot\vec{v}_T}{v_T}$ and $\xi_\perp^2=\xi^2-\xi_z^2$.

When $v_T=0$ the disturbed density n' is spherically symmetric about the test particle, and decays exponentially with a characteristic length $\dfrac{a}{\omega_p}=R_D$. If $0<v_T<a$ the density is squashed, symmetrically in $\pm\xi_z$, along the direction of motion of the test particle.

When $v_T>a$ the situation is quite different. The electron density n' is different from zero only in a conical region behind the test particle

$$n'(\vec{\xi}) = \begin{cases} -\dfrac{2c\,e_T\,n_0}{m a\sqrt{v_T^2-a^2}}\;\dfrac{\cos\left[\frac{\omega_p}{a}\sqrt{\frac{\xi_z^2}{v_T^2/a^2-1}-\xi_\perp^2}\right]}{\sqrt{\frac{\xi_z^2}{v_T^2/a^2-1}-\xi_\perp^2}} & ;\ \xi_z<0,\ \dfrac{|\xi_z|}{\sqrt{\frac{v_T^2}{a^2}-1}}>|\xi_\perp| \\[2ex] 0 & ;\ \text{elsewhere} \end{cases} \tag{3.15}$$

The electric field $\vec{E}(\vec{\xi})$ induced in the electron plasma is determined by the density deviation n'. The force on the test particle $\vec{\mathcal{F}}_{\mu}$, where the index "μ" means hydrodynamic or fluid approximation, is given in terms of this field evaluated at the test particle $\vec{\xi}=0$,

$$\begin{aligned} \vec{\mathcal{F}}_{\mu} &= -e_T \vec{E}(\vec{\xi}=0) \\ &= -e e_T \int d\vec{\xi}\, \frac{\vec{\xi}}{\xi^3}\, n'(\vec{\xi}) \end{aligned} \tag{3.16}$$

Inspection of (3.14) shows $\vec{\mathcal{F}}_{\mu}=0$ when $V_T < a$; there is no force on the test particle when it moves slower than the sound velocity.*

When the speed of the test charge exceeds the sound velocity the charge density induced in the fluid is confined to a cone behind the test particle. In this case there is a net force on the test particle given by

$$\vec{\mathcal{F}}_{\mu} = -e e_T \int_{\xi_0}^{\infty} \xi^2 d\xi \int_{-1}^{-\sqrt{1-a^2/V_T^2}} d\eta \int_0^{2\pi} d\phi\, n'(\vec{\xi})\, \frac{\vec{\xi}}{\xi^3} \tag{3.17}$$

where $n'(\vec{\xi})$ is given by (3.15) and polar coordinates (ξ, $\eta=\cos\theta$, ϕ) have been used for $\vec{\xi}$ with $\vec{V}_T$ as polar axis. The lower limit of the ξ integration is ξ_0 instead of zero. This is necessary since the force diverges logarithmically as $\xi_0 \to 0$. Physically the introduc-

---

*That the force is exactly zero is a consequence of the simple fluid model which does not allow for a distribution of plasma particle velocities. Equations (2.28) and (2.29) show the force to depend on those particles whose speed is less than V_T. In the fluid case the speed of a plasma particle is characterized by the sound velocity a.

tion of ξ_0 means the force calculated in (3.17) is due to the induced charge density outisde of a sphere of radius ξ_0 centered at the test particle.

Substituting (3.15) for $n'(\vec{\xi})$ the components of $\vec{\mathcal{F}}_{pl}$ perpendicular to $\vec{v}_T$ go out with the ϕ integration. Introducing $s = \sqrt{1-a^2/v_T^2}$ the component of $\vec{\mathcal{F}}_{pl}$ parallel to $\vec{v}_T$ reduces to

$$\vec{\mathcal{F}}_{pl} = -\mathcal{F}_{pl}\frac{\vec{v}_T}{v_T}$$

where

$$\mathcal{F}_{pl} = \frac{e_T^2\omega_p^2}{a v_T}\int_{\xi_0}^{\infty} d\xi\,\frac{1}{\xi}\int_s^1 d\eta\,\eta\,\frac{\cos\left[\xi\frac{\omega_p}{a s}\sqrt{\eta^2-s^2}\right]}{\sqrt{\eta^2-s^2}}$$

$$= \frac{e_T^2\omega_p^2 s}{v_T}\int_{\xi_0}^{\infty} d\xi\,\frac{\sin\left[\xi\frac{\omega_p}{v_T s}\right]}{\xi^2} \tag{3.18}$$

The integral can be performed* and the result is

$$\mathcal{F}_{pl} = \frac{e_T^2\omega_p^2}{v_T^2}\left\{\ln\left[\frac{\sqrt{v_T^2-a^2}}{\lambda\omega_p}\,\frac{1}{\xi_0}\right] + 1 - \sum_{l=1}^{\infty}\frac{(-)^l\left[\frac{\xi_0\omega_p}{\sqrt{v_T^2-a^2}}\right]^{2l}}{2l\,(2l+1)!}\right\} \tag{3.19}$$

where $\ln\lambda = 0.577\cdots$ is the Euler constant. If only the dominant terms are retained in the limit $\frac{\xi_0\omega_p}{\sqrt{v_T^2-a^2}} \ll 1$ the drag force is given by

$$\vec{\mathcal{F}}_{pl} = -e_T^2\omega_p^2\ln\left[\frac{\epsilon}{\lambda}\sqrt{\frac{v_T^2}{a^2}-1}\,\frac{R_D}{\xi_0}\right]\frac{\vec{v}_T}{v_T^3} + O(\xi_0^2) \tag{3.20}$$

---

*See e.g., Grobner, Integraltaflen, I 129 (5a,6e).

where $R_D = \left(\frac{4\pi e^2 n_0}{\theta}\right)^{-1/2}$ is the Debye length and $\epsilon = 2.718\cdots$ is the base of the natural logarithms.

It is interesting to compare (3.20) with a similar result obtained by Vlasov.[11] To evaluate the force Vlasov used the induced electric field evaluated at a distance ξ_z in front of the test particle ($\vec{\xi}_\perp = 0$). Then as $\xi_z \to 0$ be obtained*

$$\vec{\mathcal{F}}_{Vlasov} = -e_T^2 \omega_p^2 \ln\left[\frac{2\sqrt{\epsilon}}{\lambda} \frac{V_T}{a} \frac{R_D}{\xi_z}\right] \frac{\vec{V}_T}{V_T^3} + O(\xi_z^2 \ln \xi_z) \qquad (3.21)$$

3. DISCUSSION OF THE RESULTS

There is no reason to prefer either (3.20) or (3.21), since both involve an unspecified short range cutoff, ξ_0 or ξ_z. The reason these expressions diverge is directly related to a failure of the linearization approximation for the Euler equations. In the neighborhood of the test particle the electric field can no longer be considered as a small perturbation. In fact, when $\xi_0 \leq R_L = e e_T/\theta$ Eq. (3.18) shows that $n' \gtrsim n_0$, and the linearized theory is definitely not correct. A modified linearization that eliminates the divergence in $\vec{\mathcal{F}}_{fl}$ is discussed in Section 4.

If we must make a choice for the value of the lower cutoff, ξ_0 or ξ_z, it would be best to choose the Landau distance, $R_L = e e_T/\theta$, since (1) R_L is the minimum impact parameter that occurs in the binary collision theory, and (2) the linearized theory breaks down when $\xi_0 \leq R_L$.**

*In Vlasov's expression the factor $\sqrt{\epsilon}$ was incorrectly omitted from the logarithm argument.

**The use of R_L as a lower cutoff is adopted by the authors of Ref. 9,10.

The important feature of expression (3.20) or (3.21) is the demonstration that a collective description of the plasma, which is capable of describing Debye shielding, will automatically introduce an upper limit on the impact parameter of the order R_D, the Debye length.

To understand clearly the relationship between the hydrodynamic drag $\vec{\mathcal{F}}_{H}$ and the binary collision drag $\vec{\mathcal{F}}_{B.C.}$ it is helpful to think of the neighborhood of the test charge as divided into three domains (Fig. 2). Domain I consists of the spherical region about the test particle with radius R_L, the Landau length. Domain III consists of the region exterior to a sphere of radius R_D,* the Debye length, and domain II is the intermediate region extending from R_L to R_D.

The binary collision hypothesis assumes one plasma particle collides with the test particle at a given time and the momentum transferred in a given collision is that which results when the colliding particles approach one another from infinity and after colliding receed to infinity. This is clearly a gross approximation in a plasma, but is nearly satisfied in domain I. At distances on the order of $n_0^{-1/3}$ inaccuracies can be expected, and at distances on the order of R_D the situation is unfavorable for the binary collision assumption. Correspondingly, the fluid description applies to regions of plasma containing many particles, therefore, to domain III. It cannot be accepted at distances on the order of $n_0^{-1/3}$. Neither description of the plasma is appropriate for the intermediate region of domain II.

*For most plasmas of interest in inequality $R_L < n_0^{-1/3} < R_D$ is satisfied (see Table I).

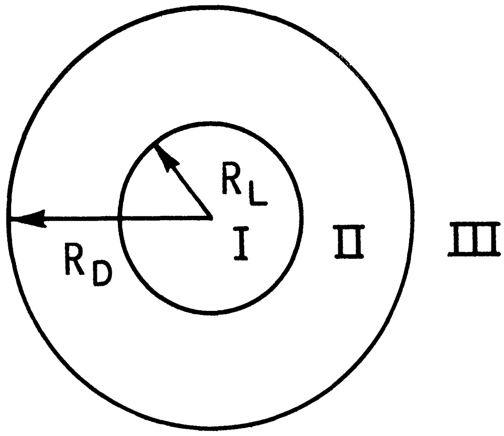

Fig. 2. Domains of interaction about a test particle.

Now it is evident that (2.28) with $b_{max} = R_D$ is the force expression when the binary collision treatment includes domain II as well as domain I,

$$\vec{\mathcal{F}}_{B.C.} = -e_T^2 \omega_p^2 \ln\left[\frac{v_T^2}{a^2} \frac{R_D}{R_L}\right] \frac{\vec{v}_T}{v_T^3} \qquad ; \; v_T >> a \qquad (3.22)$$

where $m_T >> m$ for a test particle with constant velocity. If, instead, the fluid description is used in domain II the force expression is (3.20) with $\xi_o = R_L$,

$$\vec{\mathcal{F}}_{fl} = -e_T^2 \omega_p^2 \ln\left[\frac{\epsilon}{\lambda} \frac{v_T}{a} \frac{R_D}{R_L}\right] \frac{\vec{v}_T}{v_T^3} \quad ; \; v_T >> a \qquad (3.23)$$

Since neither method can be extended to include all domains the obvious step is to try a combination of the two results. If the binary collision hypothesis is assumed correct in a spherical region about the test particle of radius L, $(R_L < L < R_D)$, and if the fluid approximation is assumed correct in the region exterior to L, then the force is given by a sum of (2.28) and (3.20) with $b_{max} = \xi_o = L$;

$$\vec{\mathcal{F}} = -e_T^2 \omega_p^2 \ln\left[\frac{\epsilon}{\lambda}\,\frac{v_T^3}{a^3}\,\frac{R_D}{R_L}\right]\frac{\vec{v}_T}{v_T^3} \quad ; \quad v_T \gg a \tag{3.24}$$

The interesting feature of this formula is that the arbitrary length does not appear explicitly. This result was first emphasized by Vlasov.[11] If Vlasov's drag expression (3.21) is used in place of (3.20) this gives the slightly different result

$$\vec{\mathcal{F}} = -e_T^2 \omega_p^2 \ln\left[\frac{2\sqrt{\epsilon}}{\lambda}\,\frac{v_T^3}{a^3}\,\frac{R_D}{R_L}\right]\frac{\vec{v}_T}{v_T^3} \quad ; \quad v_T \gg a \tag{3.25}$$

The last result has also been discussed by Hubbard[5] and Aono.[6] In domain III the last two authors describe the plasma by a one particle distribution function satisfying a linearized Landau-Vlasov equation, but in the limit $v_T \gg a$ their results agree with those of the simple fluid theory. It is clear from (3.24) and (3.25) that an ambiguity is still present in the drag force even though a cutoff does not appear explicitly.

To summarize, the binary collision calculation (2.28) diverges at large impact parameters, because of the peculiar character of the Coulomb potential and the neglect of collective interactions. The continuous fluid result (3.20) diverges at small impact parameters, because of an improper linearization of the Euler equations (3.1)-(3.3). A combination of these two results that includes contributions to the drag force from plasma in all regions about the test charge is given by (3.24) or by (3.25). The difference between these expressions results from the ambiguous definition of the drag in the fluid model; the cutoffs in (3.20) and (3.21) are necessary because we have not properly solved the test particle problem.

4. MODIFIED LINEARIZATION OF THE EULER EQUATIONS

We continue to treat the electron plasma by neglecting terms of order m/M. The Euler equations for a fluid of electrons moving in a uniform background of positive charge density n_0 are

$$mn\left\{\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right\} = -\nabla p - en\vec{E}$$

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = 0 \tag{3.26}$$

$$p = \theta n$$

where the energy equation has been simplified to the ideal gas equation of state, $p = \theta n$. The electric field is determined from Poisson's equation

$$\nabla\cdot\vec{E} = 4\pi e\,(n_0 - n) - 4\pi e_T\,\delta(\vec{r} - \vec{r}_T) \tag{3.27}$$

where $\vec{r}_T = \vec{v}_T t$.

The electron density in the neighborhood of the test particle must vanish because of the repulsive Coulomb forces. The assumption that the test particle introduces only a small perturbation from a uniform electron density n_0 is obviously not correct near the test particle. It is this failure of the linearization in Section 2 that led to a divergent drag expression (3.20).

When the test particle velocity is zero, $\vec{V}_T = 0$, one expects a time independent solution of (3.26) in the form of a Boltzmann-Gibbs factor

$$n(|\vec{r}-\vec{r}_T|) = n_0 \exp\left\{ -\frac{e e_T}{\Theta} \Phi(|\vec{r}-\vec{r}_T|) \right\} \tag{3.28}$$

If the test particle is moving the density (3.28) is distorted along the direction of motion. Let us suppose that this distortion can be described as a small perturbation

$$n(\vec{r},t) = n_0 e^{-R_L \Phi(|\vec{r}-\vec{r}_T|)} \left\{ 1 + h(\vec{r},t) \right\} \tag{3.29}$$

where $R_L = \frac{e e_T}{\Theta}$ and $\vec{r}_T = \vec{V}_T t$. The assumption that $h(\vec{r},t)$ and $\vec{E}(\vec{r},t)$ are small leads to a linear equation for $h(\vec{r},t)$ which is readily solved.

To obtain an equation for $\Phi(|\vec{r}-\vec{r}_T|)$ the density (3.28) is inserted into (3.26) with $\vec{V}_T = \vec{u} = 0$,

$$\nabla \Phi(|\vec{r}-\vec{r}_T|) = \frac{1}{e_T} \vec{E}$$

The divergence of this equation can be reduced, using (3.27), to an equation for Φ alone,

$$\nabla^2 \Phi(r) = 4\pi \frac{e}{e_T} n_0 \left[1 - e^{-R_L \Phi(r)} \right] - 4\pi \delta(\vec{r}) \tag{3.30}$$

This equation is highly non-linear. An approximate solution, obtained in Appendix A, is the familiar Debye-Hückel result

$$\Phi(r) = \frac{1}{r} e^{-\frac{r}{R_D}} \tag{3.31}$$

where $R_D = \left(\frac{4\pi e^2 n_0}{\theta}\right)^{-1/2}$ is the Debye length.

The linearized equation for $h(\vec{r},t)$ is obtained from (3.26) by using the definition (3.29) and neglecting quantities of second and higher order in Φ, $\vec{E}$, $\vec{u}$, h. The result is

$$m\frac{\partial \vec{u}}{\partial t} = -\theta\nabla\left[h - R_L\Phi(|\vec{r}-\vec{r}_T|)\right] - e\vec{E}$$

$$\frac{\partial}{\partial t}\left[h - R_L\Phi\right] = -\nabla\cdot\vec{u} \tag{3.32}$$

$$\nabla\cdot\vec{E} = -4\pi e n_0\left[h - R_L\Phi\right] - 4\pi e_T\delta(\vec{r}-\vec{r}_T)$$

Since we are looking for a solution of the form $h(\vec{r},t) = h(\vec{r}-\vec{r}_T)$ the operator $\partial/\partial t$ can be replaced by $-\vec{v}_T\cdot\nabla$. The divergence of the first equation can be reduced with the help of the remaining two equations to yield a single equation for $[h - R_L\Phi]$,

$$\left\{(\vec{v}_T\cdot\nabla)^2 - a^2\nabla^2 + \omega_p^2\right\}\left[h - R_L\Phi\right] = -\frac{4\pi e e_T}{m}\delta(\vec{r}-\vec{r}_T) \tag{3.33}$$

where $a^2 = \theta/m$, $R_D = \frac{a}{\omega_p} = \left(\frac{4\pi e^2 n_0}{\theta}\right)^{-1/2}$.

This equation is identical to (3.13) except for a constant factor of n_0 which is missing from the right hand side. We can immediately write down the solution for $h(\vec{r}-\vec{r}_T)$. When $v_T < a$, h is given by

$$h(\vec{\xi}) = R_L\Phi(\xi) - \frac{R_L}{\sqrt{1-v_T^2/a^2}}\,\frac{e^{-\frac{1}{R_D}\sqrt{\xi_\perp^2 + \frac{\xi_z^2}{1-v_T^2/a^2}}}}{\sqrt{\xi_\perp^2 + \frac{\xi_z^2}{1-v_T^2/a^2}}} \tag{3.34}$$

where $\vec{\xi} = \vec{r} - \vec{r}_T = \vec{\xi}_z + \vec{\xi}_\perp$ and $\vec{\xi}_z$ is the component of $\vec{\xi}$ in the direction of $\vec{v}_T$.

When $v_T > a$, h is given by

$$h(\vec{\xi}) = \begin{cases} R_L \Phi(\xi) - \dfrac{2R_L}{\sqrt{\frac{v_T^2}{a^2}-1}} \dfrac{\cos\left[\frac{1}{R_D}\sqrt{\frac{\xi_z^2}{v_T^2/a^2-1} - \xi_\perp^2}\right]}{\sqrt{\frac{\xi_z^2}{v_T^2/a^2-1} - \xi_\perp^2}} ; & \xi_z < 0,\ \dfrac{|\xi_z|}{\sqrt{\frac{v_T^2}{a^2}-1}} > \xi_\perp \\ R_L \Phi(\xi) ; & \text{elsewhere} \end{cases} \tag{3.35}$$

The validity of the linearized approximation for $h(\vec{\xi})$ is readily ascertained. When $v_T \ll a$ we have a good approximation for h, even in the neighborhood of the test particle ($\vec{\xi} \approx 0$). This is evident from (3.31) and (3.34), the two terms in $h(\vec{\xi})$ tend to cancel one another. As v_T increases the cancellation is less effective. When $v_T > a$ we run into two difficulties. First, the density correction $h(\vec{\xi})$ diverges on the cone $\xi_z < 0$; $|\xi_z| = \sqrt{\frac{v_T^2}{a^2}-1}\,|\vec{\xi}_\perp|$. This arises from an inadequacy of the continuous fluid model. In Chapter VI this difficulty is eliminated by describing the plasma in terms of a one particle distribution function.

Second, even if $h(\vec{\xi})$ is not evaluated on the cone, say $\vec{\xi}_\perp = 0$, it is large if evaluated near the origin, $|\xi_z| < R_L$. This behavior is a failure of the linearization approximation used to calculate h. The breakdown of the linearization is unimportant, however, because of the exponential factor, $\exp\{-R_L\Phi(\xi)\}$, in the density $n(\vec{\xi})$. If $|\vec{\xi}| \approx R_L$, then $R_L\,\Phi(\xi) \approx 1$, $h(\vec{\xi}) \approx O(1)$, and $\exp\{-R_L\Phi(\xi)\} \approx e^{-1}$. Thus, when

$|\vec{\xi}| \lesssim R_L$, the exponential factor overwhelms the misbehavior of $h(\vec{\xi})$.

The force on the test particle is $-e_T$ times the electric field evaluated at $\vec{\xi} = 0$,

$$\vec{\mathcal{F}} = e_T e \int d\vec{\xi}\, \frac{\partial}{\partial \vec{\xi}}\left(\frac{1}{\xi}\right) n(\vec{\xi})$$

$$= -e_T e n_0 \int d\vec{\xi}\, \frac{\vec{\xi}}{\xi^3}\, e^{-R_L \Phi(\xi)} \left\{1 + h(\vec{\xi})\right\} \tag{3.36}$$

When $v_T < a$, $h(\vec{\xi})$ is given by (3.34), and $\vec{\mathcal{F}} = 0$ since the integrand in (3.36) is an odd function of $\vec{\xi}$.

When $v_T > a$, $h(\vec{\xi})$ is given by (3.35). That part of $h(\vec{\xi})$ given by $R_L \Phi(\xi)$ cannot contribute to the force, because of its spherical symmetry about the test charge. To evaluate the remaining part use spherical polar coordinates (ξ, $\eta = \cos\theta$, ϕ) for $\vec{\xi}$ with $\vec{v}_T$ as the polar axis,

$$\vec{\mathcal{F}} = -\frac{2 e_T e n_0 R_L}{v_T/a} \int_0^\infty d\xi \int_S^1 d\eta \int_0^{2\pi} d\phi\, \frac{\vec{\xi}}{\xi^2}\, e^{-R_L \Phi(\xi)}\, \frac{\cos\left[\frac{\xi}{S R_D}\sqrt{\eta^2 - S^2}\right]}{\sqrt{\eta^2 - S^2}} \tag{3.37}$$

where $S = \sqrt{1 - a^2/v_T^2}$. The components of $\vec{\mathcal{F}}$ that are perpendicular to $\vec{v}_T$ involve $\sin\phi$, $\cos\phi$, and vanish when integrated over ϕ. The resultant force is a drag,

$$\vec{\mathcal{F}} = -\mathcal{F}\, \frac{\vec{v}_T}{v_T}$$

where

$$\mathcal{F} = \frac{4\pi e_T e n_0 R_L}{v_T/a} \int_0^\infty d\xi \, \frac{e^{-R_L \Phi(\xi)}}{\xi} \int_s^1 d\eta \, \frac{\eta \cos\left[\frac{\xi}{s R_D}\sqrt{\eta^2 - s^2}\right]}{\sqrt{\eta^2 - s^2}}$$

$$= \frac{e_T^2 \omega_p s}{v_T} \int_0^\infty d\xi \, \frac{e^{-R_L \Phi(\xi)}}{\xi^2} \sin\left(\frac{\omega_p}{s v_T}\xi\right) \tag{3.38}$$

In Appendix A the integral in (3.38) is evaluated to lowest order in R_L/R_D. The magnitude of the drag force to lowest order in R_L/R_D is

$$\mathcal{F} = \frac{e_T^2 \omega_p^2}{v_T^2} \ln\left[\frac{\epsilon}{\lambda^2}\sqrt{\frac{v_T^2}{a^2} - 1}\,\frac{R_D}{R_L}\right] + \mathcal{O}\left(\frac{R_L}{R_D}\right) \tag{3.39}$$

where $\epsilon = 2.718\cdots$ is the base of the natural logarithms and $\ln\lambda = 0.577\cdots$ is Euler's constant.*

This result would be identical to that obtained from the simple linearization method of Section 2, Eq. (3.20), if the cutoff is chosen as $\xi_0 = \lambda R_L$. The significant feature of the modified linearization is that it is no longer necessary to introduce a cutoff.

It was indicated in the discussion following equation (3.35) that the continuous fluid model leads to a divergent solution on the surface of a conical wake behind the test particle ($v_T > a$). For this reason the drag expression (3.39) is only a preliminary result. The point to keep in mind is that a linearization, similar to that used in this section, will also work for the test particle problem described in Chapter

*The order relation $\mathcal{O}(x)$ has the meaning $\lim_{x \to 0} \frac{1}{x}\mathcal{O}(x) =$ constant.

VI. In Chapter VI the electron plasma is described by a one particle distribution function satisfying the Landau-Vlasov equation. The drag expression obtained is nearly identical to (3.39); the logarithm argument is a more complicated function of v_T .

CHAPTER IV

TEST PARTICLE PROBLEM—CONTINUUM DESCRIPTION

1. INTRODUCTION

In the preceding chapter an attempt was made to calculate the response of a plasma to a test charge by considering the plasma as a continuous fluid. It was discovered that the plasma responded to a slow moving test charge by forming a charged "screening cloud" about the test particle, Eq. (3.14), while a fast moving test charge had this "screening cloud" distorted into a trailing wake, Eq. (3.15). This behavior is similar to that observed when a charged particle traverses a dielectric medium (i.e., polarization of the medium and Čerenkov radiation). On the basis of this similarity we are led to an investigation of the passage of a charged test particle through a continuous dielectric medium.

The force on a test particle was first calculated in terms of a dielectric constant $\varepsilon(\omega)$ by E. Fermi.[2] We extend the Fermi calculation to include spatial dispersion, that is, to include a dependence of the dielectric constant on the wave vector $\vec{k}$ as well as the frequency ω.

In Section 2 the necessary formalism, relating the electric and magnetic fields to arbitrary charge and current densities, is developed in the framework of the initial value problem. In Section 3 expressions for the drag force on a test particle are given in terms of an arbitrary dielectric constant. In Section 4 a simple example is investigated. In the following chapter the dielectric constant for an electron plasma is used in an explicit calculation of the plasma test particle drag.

2. THE MAXWELL EQUATIONS

The phenomenological Maxwell equation in Gaussian units are

$$\begin{aligned} \nabla\cdot\vec{D} &= 4\pi\rho_{ext} & \qquad \nabla\times\vec{E} &= -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} &= 0 & \qquad \nabla\times\vec{H} &= \frac{1}{c}\frac{\partial\vec{D}}{\partial t}+\frac{4\pi}{c}\vec{j}_{ext} \end{aligned} \tag{4.1}$$

where ρ_{ext} , $\vec{j}_{ext}$ are the "external" charge and current densities. The charge density and its corresponding current must satisfy a continuity equation,

$$\frac{\partial\rho_{ext}}{\partial t}+\nabla\cdot\vec{j}_{ext}=0 \tag{4.2}$$

To these equations must be added the so called <u>constitutive relations</u> connecting $\vec{D}$ with $\vec{E}$ and $\vec{H}$ with $\vec{B}$.

In this chapter linear constitutive equations connecting the Fourier amplitudes of the field vectors are used. In the initial value problem all fields are zero for negative times ($t<0$). The Fourier time transform of $\vec{E}(\vec{r},t)$ is

$$\vec{E}(\vec{r},\omega)=\frac{1}{2\pi}\int_0^\infty dt\,\vec{E}(\vec{r},t)\,e^{i\omega t}$$

The Fourier space-time transform of $\vec{E}(\vec{r},t)$ is

$$\tilde{\vec{E}}(\vec{k},\omega) = \frac{1}{2\pi}\int d\vec{r}\int_0^\infty dt\,\vec{E}(\vec{r},t)\,e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \tag{4.3}$$

which exists for $\vec{k}$ real and $\operatorname{Im}\omega$ sufficiently large and positive.* Using similar transforms for the remaining functions, the constitutive equations are

$$\begin{aligned}\tilde{\vec{D}}(\vec{k},\omega) &= \varepsilon(k,\omega)\,\tilde{\vec{E}}(\vec{k},\omega)\\ \tilde{\vec{B}}(\vec{k},\omega) &= \mu(k,\omega)\,\tilde{\vec{H}}(\vec{k},\omega)\end{aligned} \tag{4.4}$$

In this way the personality of a medium is cast into the functions $\varepsilon(k,\omega)$, the dielectric constant, and $\mu(k,\omega)$, the magnetic permeability. In general, ε and μ will be complex functions. It is the purpose of the next chapter to determine these functions for an electron plasma.

Note that the constitutive functions $\varepsilon(k,\omega)$ and $\mu(k,\omega)$ are assumed to depend only on the magnitude of $\vec{k}$ and not on its direction. This assumption is appropriate for an isotropic medium. In the next chapter we deal with a plasma that has an isotropic equilibrium velocity distribution, the Maxwell-Boltzmann distribution. In that case the plasma behaves as an isotropic dielectric material, hence the assumption that ε, μ depend on $|\vec{k}|$, ω is sufficient for our purposes.

*Complex Fourier transforms are discussed in Morse and Feshbach, Methods of Mathematical Physics, p. 468, McGraw-Hill (1958).

Actually, the calculations described are easily performed when ε , μ depend on $\vec{k}$. The results, however, are more complicated and will not be given.

The customary derivations of ε , μ for a plasma start with the Landau-Vlasov equation (see next chapter). In this case transverse and longitudinal electromagnetic fields have the same propagation characteristics, and only one scalar function is necessary to describe the plasma, i.e., only ε is necessary and $\mu(k,\omega) = 1$. If electron-electron correlations are included two scalar constitutive functions are necessary to describe the plasma. Both ε and μ are retained in this chapter.

In terms of Fourier amplitudes the phenomenological Maxwell equations (4.1) become

$$\begin{aligned} &\varepsilon\, \vec{k}\cdot\hat{\vec{E}} = -4\pi i\,\hat{\rho}_{ext} && \vec{k}\times\hat{\vec{E}} = \frac{\omega}{c}\hat{\vec{B}} \\ &\vec{k}\cdot\hat{\vec{B}} = 0 && \frac{1}{\mu}\vec{k}\times\hat{\vec{B}} = -\frac{\omega}{c}\varepsilon\hat{\vec{E}} - \frac{4\pi i}{c}\hat{\vec{j}}_{ext} \end{aligned} \tag{4.5}$$

The transformation of the continuity Eq. (4.2) is, remembering that in the initial value problem the external charge density is established at $t = 0$,

$$i\vec{k}\cdot\hat{\vec{j}}_{ext} - i\omega\hat{\rho}_{ext} - \frac{1}{2\pi}P(\vec{k}) = 0 \tag{4.6}$$

where

$$P(\vec{k}) = \int d\vec{r}\,\rho_{ext}(\vec{r},t=0)\,e^{-i\vec{k}\cdot\vec{r}} \tag{4.7}$$

is the Fourier space transform of the initial external charge density.

It is convenient to decompose the field vectors into components parallel and transverse to $\vec{k}$. The parallel, or longitudinal components, are denoted by a superscript "ℓ," and transverse components by a superscript "tr." The phenomenological Maxwell equations reduce to

$$\begin{aligned} &\varepsilon(k,\omega)\,\tilde{\vec{E}}^{\ell} = -4\pi i\,\tilde{\rho}_{ext}\,\frac{\vec{k}}{k^2} && \tilde{\vec{B}}^{\ell} = 0 \\ &\left[\varepsilon(k,\omega) - \frac{c^2k^2}{\omega^2\mu(k,\omega)}\right]\tilde{\vec{E}}^{tr} = -\frac{4\pi i}{\omega}\,\tilde{\vec{j}}^{tr}_{ext} && \tilde{\vec{B}}^{tr} = \frac{c}{\omega}\,\vec{k}\times\tilde{\vec{E}}^{tr} \end{aligned} \tag{4.8}$$

These equations express the fields in terms of the external sources and the constitutive functions of the medium.

A word should be said concerning the Fourier inversion of (4.8). Let us fix our attention on determining the temporal dependence of $\vec{E}^{\ell}$, for fixed real $\vec{k}$,

$$\vec{E}^{\ell}(\vec{k},t) = \frac{1}{2\pi}\int_{-\infty+i\gamma}^{+\infty+i\gamma} d\omega\,\frac{-4\pi i\,\tilde{\rho}_{ext}(\vec{k},\omega)\,\vec{k}}{k^2\,\varepsilon(k,\omega)}\,e^{-i\omega t} \tag{4.9}$$

where the path of integration in the complex ω-plane must lie above all singularities of the integrand. To evaluate the integral for $t<0$, close the path with an infinite semicircle in the upper half ω-plane. Since, by definition of the path, there are no poles within the contour $\vec{E}^{\ell}$ is zero as required for the initial value problem. To evaluate the integral for $t>0$ the path of integration can be extended* along

*This extension of the integration path requires justification for each choice of functions occurring in the integrand. We shall assume the procedure is generally valid.

an infinite semicircle in the lower half ω-plane. Contributions to $\vec{E}^{\ell}$ come from poles of the integrand. Poles in the upper half ω-plane give exponentially increasing fields while those poles on the real ω axis give harmonic contributions and poles in the lower half ω-plane give exponentially damped terms.

Poles of $\tilde{\rho}_{ext}$ arise for special choices of the external sources. The poles from zeros of the denominator govern the behavior of the medium as it is characterized by $\varepsilon(k,\omega)$. Since the condition $\varepsilon(k,\omega) = 0$ is just the condition necessary in order to have a non-zero field, $\vec{E}^{\ell}$, when the external charge density $\tilde{\rho}_{ext}$ is zero we see that

$$\varepsilon(k,\omega) = 0$$

is the "dispersion relation" for longitudinal fields. Correspondingly, the "dispersion relation" for transverse fields is

$$\varepsilon(k,\omega) - \frac{c^2 k^2}{\omega^2 \mu(k,\omega)} = 0$$

3. FORCE ON A TEXT PARTICLE

We now calculate the drag force on a test charge moving through the medium with constant velocity $\vec{v}_T$. Although we have hitherto considered the test particle as a point charge, in this section it is more perspicuous to ascribe to it an arbitrary spherically symmetric charge density $\rho_T(|\vec{r}|)$. If the test charge is located at the origin of the coordinate system at $t = 0$

$$\rho_{ext}(\vec{r},t) = \begin{cases} \rho_T(|\vec{r}-\vec{v}_T t|) & , \quad t>0 \\ 0 & , \quad t<0 \end{cases}$$

$$\vec{j}_{ext}(\vec{r},t) = \begin{cases} \rho_T(|\vec{r}-\vec{v}_T t|)\,\vec{v}_T & , \quad t>0 \\ 0 & , \quad t<0 \end{cases} \tag{4.10}$$

The Fourier space-time transforms are

$$\hat{\rho}_{ext}(\vec{k},\omega) = -\frac{\rho_T(k)}{2\pi i\left(\omega-\vec{k}\cdot\vec{v}_T\right)} \quad , \quad \mathrm{Im}\,\omega>0$$

$$\hat{\vec{j}}_{ext}(\vec{k},\omega) = -\frac{\rho_T(k)\,\vec{v}_T}{2\pi i\left(\omega-\vec{k}\cdot\vec{v}_T\right)} \quad , \quad \mathrm{Im}\,\omega>0 \tag{4.11}$$

where $\rho_T(k)$ is the Fourier space transform of the charge density,

$$\rho_T(k) = \int d\vec{r}\,\rho_T(|\vec{r}|)\,e^{-i\vec{k}\cdot\vec{r}} \tag{4.12}$$

The charge density is a real quantity which implies $\hat{\rho}_T^*(k) = \hat{\rho}_T(k)$ for real $\vec{k}$, where a "star" means complex conjugate.

The force on the test charge is*

$$\vec{\mathcal{F}} = \int d\vec{r}\,\rho_T(|\vec{r}-\vec{v}_T t|)\left\{\vec{E}(\vec{r},t) + \frac{\vec{v}_T}{c}\times\vec{B}(\vec{r},t)\right\} \tag{4.13}$$

Representing the functions in the integrand by their Fourier transforms

$$\vec{\mathcal{F}} = \frac{-1}{(2\pi)^6}\int d\vec{r}\int d\vec{k}_1\int_{-\infty+i\gamma_1}^{\infty+i\gamma_1} d\omega_1\int d\vec{k}_2\int_{-\infty+i\gamma_2}^{\infty+i\gamma_2} d\omega_2\,\frac{\rho_T(k_1)\,e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{r}-i(\omega_1+\omega_2)t}}{2\pi i\left(\omega_1-\vec{k}_1\cdot\vec{v}_T\right)}\left\{\hat{\vec{E}}(\vec{k}_2,\omega_2)+\frac{\vec{v}_T}{c}\times\hat{\vec{B}}(\vec{k}_2,\omega_2)\right\} \tag{4.14}$$

*It is not necessary to subtract the self field of the test particle provided we agree to define integrations over infinite ranges in the sense of Cauchy , $\int_{-\infty}^{\infty}dx = \lim_{L\to\infty}\int_{-L}^{L}dx$.

Changing orders of integration, using the identity

$$\frac{1}{(2\pi)^3}\int d\vec{r}\, e^{i\vec{k}\cdot\vec{r}} = \delta(\vec{k})$$

and preforming the $\vec{k}_1$, ω_1 integrations, which are straightforward

$$\vec{\mathcal{F}} = \frac{1}{(2\pi)^3}\int d\vec{k}\int_{-\omega+i\gamma}^{\omega+i\gamma} d\omega\, \rho_T(k)\, e^{-i(\omega-\vec{k}\cdot\vec{v}_T)t}\left\{\tilde{\vec{E}}(\vec{k},\omega) + \frac{\vec{v}_T}{c}\times\tilde{\vec{B}}(\vec{k},\omega)\right\} \tag{4.15}$$

The transforms of the field vectors are given by (4.8), (4.11), and (4.12). Both the longitudinal and transverse electric fields contribute to the force. Denoting these contributions by $\vec{\mathcal{F}}_{l}$ and $\vec{\mathcal{F}}_{tr}$ respectively, and the magnetic field term by $\vec{\mathcal{F}}_B$ we have

$$\vec{\mathcal{F}} = \vec{\mathcal{F}}_{l} + \vec{\mathcal{F}}_{tr} + \vec{\mathcal{F}}_B$$

where

$$\vec{\mathcal{F}}_{l} = \frac{4\pi i}{(2\pi)^3}\int d\vec{k}\int_{-\omega+i\gamma}^{\omega+i\gamma} d\omega\, \frac{\rho_T^2(k)\,\vec{k}\, e^{-i(\omega-\vec{k}\cdot\vec{v}_T)t}}{2\pi i\,(\omega-\vec{k}\cdot\vec{v}_T)\,k^2\,\mathcal{E}(k,\omega)}$$

$$\vec{\mathcal{F}}_{tr} = \frac{4\pi i}{(2\pi)^3}\int d\vec{k}\int_{-\omega+i\gamma}^{\omega+i\gamma} d\omega\, \frac{\rho_T^2(k)\,\omega\, e^{-i(\omega-\vec{k}\cdot\vec{v}_T)t}\left[k^2\vec{v}_T - (\vec{k}\cdot\vec{v}_T)\,\vec{k}\right]}{2\pi i\,(\omega-\vec{k}\cdot\vec{v}_T)\,k^2\left[\omega^2\mathcal{E}(k,\omega) - \dfrac{k^2c^2}{\mu(k,\omega)}\right]} \tag{4.16}$$

$$\vec{\mathcal{F}}_B = \frac{4\pi i}{(2\pi)^3}\int d\vec{k}\int_{-\omega+i\gamma}^{\omega+i\gamma} d\omega\, \frac{\rho_T^2(k)\, e^{-i(\omega-\vec{k}\cdot\vec{v}_T)t}\left[v_T^2\vec{k} - (\vec{k}\cdot\vec{v}_T)\vec{v}_T\right]}{2\pi i\,(\omega-\vec{k}\cdot\vec{v}_T)\left[\omega^2\mathcal{E}(k,\omega) - \dfrac{k^2c^2}{\mu(k,\omega)}\right]} = 0$$

The assumptions of an isotropic medium and a spherically symmetric test charge are responsible for the vanishing of the magnetic force, $\vec{\mathcal{F}}_B = 0$, since the integrand is then an odd function of the component of $\vec{k}$ perpendicular to $\vec{v}_T$. For the same reason the forces $\vec{\mathcal{F}}_\ell$ and $\vec{\mathcal{F}}_{tr}$ are parallel to $\vec{v}_T$. The force $\vec{\mathcal{F}}_\ell$ is the dominant drag force on a slow test particle, since the transverse force $\vec{\mathcal{F}}_{tr}$ is of order $(v_T/c)^2$ smaller than $\vec{\mathcal{F}}_\ell$. When $\varepsilon = \varepsilon(\omega)$, $\mu = 1$, $\vec{\mathcal{F}}_{tr}$ is the force resulting from the emission of Čerenkov radiation, when the speed of the test particle exceeds the phase velocity of light in the medium.

In the test particle problem it is customary to assume the medium is stable; i.e., there are no poles in the upper half ω -plane. There may, however, be poles on the real axis, and these require special care. One procedure is to extend the ω integration along an infinite semicircle in the lower half ω-plane, and to retain only contributions from poles on the real ω axis as representing the time asymptotic result. This procedure has the disadvantage that subsequent integration on $\vec{k}$ may alter the order of these poles, making it necessary to divide the $\vec{k}$ integration into suitably restricted ranges. An alternative procedure is to decompose $\vec{k}$ into $\vec{k}_\parallel$, $\vec{k}_\perp$ (the parallel and perpendicular components of $\vec{k}$ referring to $\vec{v}_T$), and to perform the $k_\parallel$ integration by completing the integration path along an infinite semicircle in the upper half $k_\parallel$-plane. We can assume $\mathrm{Im}\,\omega = \gamma$ is so small that all poles of the integrands in (4.16), in the upper half $k_\parallel$ -plane, occur at $v_T \,\mathrm{Im}\, k_\parallel > \gamma$. If this is the case the time asymptotic result comes from the pole at $\omega - k_{\parallel} v_T = 0$, all other poles containing exponentially decreasing factors $\exp\{-(v_T \mathrm{Im}\, k_{\parallel} - \gamma)t\}$.

For example, the time asymptotic force $\vec{\mathcal{F}}_{\ell}$ will be

$$\vec{\mathcal{F}}_{\ell} = -\frac{4\pi i}{(2\pi)^3} \frac{1}{v_T} \int d\vec{k}_{\perp} \int_{-\infty+i\gamma}^{\infty+i\gamma} d\omega \left[\frac{\rho_T^2(k)\, \vec{k}}{k^2 \mathcal{E}(k,\omega)} \right]_{k_{\parallel} = \omega/v_T} \tag{4.17}$$

Changing integration variables from ω to $k_{\parallel} v_T$

$$\vec{\mathcal{F}}_{\ell} = -\frac{4\pi i}{(2\pi)^3} \int d\vec{k}_{\perp} \int_{-\infty+i\gamma}^{\infty+i\gamma} dk_{\parallel} \frac{\rho_T^2(k)\, \vec{k}}{k^2 \mathcal{E}(k, \vec{k}\cdot\vec{v}_T)} \tag{4.18}$$

In this fashion (4.16) may be simplified to

$$\vec{\mathcal{F}}_{\ell} = -\frac{4\pi i}{(2\pi)^3} \int_{\Gamma} d\vec{k} \frac{\rho_T^2(k)\, \vec{k}}{k^2 \mathcal{E}(k, \vec{k}\cdot\vec{v}_T)}$$

$$\vec{\mathcal{F}}_{tr} = -\frac{4\pi i}{(2\pi)^3} \int_{\Gamma} d\vec{k} \frac{\rho_T^2(k)\, (\vec{k}\cdot\vec{v}_T)\left[k^2 \vec{v}_T - (\vec{k}\cdot\vec{v}_T)\vec{k}\right]}{k^2\left[(\vec{k}\cdot\vec{v}_T)^2 \mathcal{E}(k,\vec{k}\cdot\vec{v}_T) - \dfrac{k^2 c^2}{\mu(k,\vec{k}\cdot\vec{v}_T)}\right]} \tag{4.19}$$

where Γ means the path of integration for $k_{\parallel}$ passes above poles on the real axis.

4. EXAMPLE

To illustrate these results we work the problem considered by Tamm[15] in his explanation of Čerenkov radiation. Tamm calculated $\vec{\mathcal{F}}_{tr}$ for a medium whose constitutive functions are

$$\mathcal{E} = \mathcal{E}(\omega) \qquad \mu = 1$$

where $\mathcal{E}(\omega)$ is a real function of real ω . Using our formalism it is easy to calculate both $\vec{\mathcal{J}}_\ell$ and $\vec{\mathcal{J}}_{tr}$.

If $\mathcal{E}(\omega)=0$ for some ω the prescription given for treating the poles in $\mathcal{E}^{-1}(\omega)$ is to add to ω a small positive imaginary part, say $i\delta$. This corresponds to adding a small conductivity to the medium:

$$\mathcal{E}(\omega+i\delta) = \mathcal{E}(\omega) + i\delta \frac{d}{d\omega}\mathcal{E}(\omega) + \cdots$$

however, $\mathcal{E}(\omega) = \mathcal{E}(-\omega)$, and in the limit $\delta \to 0^+$ this is equivalent to

$$\mathcal{E}(\omega+i\delta) = \mathcal{E}(\omega) + i\,\frac{4\pi\sigma}{\omega}$$

where $\sigma \to 0^+$ is the conductivity.

Applying (4.19), and retaining ω as the integration variable in place of $k_\parallel v_T$,

$$\vec{\mathcal{J}}_\ell = -\frac{2e_T^2 i}{2\pi}\int_{-\infty}^{\infty} d\omega \int_0^{\infty} k_\perp dk_\perp \frac{\omega\, \vec{v}_T/v_T^3}{\left[k_\perp^2+\frac{\omega^2}{v_T^2}\right]\left[\mathcal{E}(\omega)+\frac{4\pi i\sigma}{\omega}\right]}$$

$$\vec{\mathcal{J}}_{tr} = -\frac{2e_T^2 i}{2\pi}\int_{-\infty}^{\infty} d\omega \int_0^{\infty} k_\perp dk_\perp \frac{\vec{v}_T}{c^2 v_T} \frac{\omega c^2 k_\perp^2}{\left[k_\perp^2+\frac{\omega^2}{v_T^2}\right]\left[\omega^2\left(\mathcal{E}+i\frac{4\pi\sigma}{\omega}\right)-\frac{c^2\omega^2}{v_T^2}-c^2k_\perp^2\right]} \tag{4.20}$$

Using the relation

$$\lim_{a\to 0^+} \frac{a}{[f(x)]^2+a^2} = \pi\,\delta[f(x)]$$

Eqs. (4.20) reduce to

$$\vec{\mathcal{F}}_\ell = -2e_T^2 \frac{\vec{v}_T}{v_T^3} \int_0^{k_M} k_\perp dk_\perp \int_0^\infty d\omega \, \frac{\omega \, \delta[\varepsilon(\omega)]}{k_\perp^2 + \omega^2/v_T^2}$$

$$\vec{\mathcal{F}}_{tr} = -\frac{e_T^2}{c^2} \frac{\vec{v}_T}{v_T} \int_{\substack{0 \\ \{\frac{v_T^2}{c^2}\varepsilon > 1\}}}^\infty d\omega \left[1 - \frac{c^2}{v_T^2 \varepsilon(\omega)}\right] \tag{4.21}$$

where it is necessary to insert an upper limit (k_M) in the $k_\perp$ integration, since the assumption of a dielectric constant that depends only on $\omega = k_\parallel v_T$ is an approximation to a real medium that cannot produce a natural cutoff in $k_\perp$.

The force $\vec{\mathcal{F}}_\ell$ receives contributions from those values of ω satisfying the "dispersion relation" $\varepsilon(\omega) = 0$; that is, the force results from the emission of longitudinal waves (polarization waves). The force $\vec{\mathcal{F}}_{tr}$ is Tamm's result for a particle emitting transverse waves (Čerenkov radiation). Note that $|\vec{\mathcal{F}}_{tr}|$ is of order $\frac{v_T^2}{c^2} |\vec{\mathcal{F}}_\ell|$, and can be neglected in a non-relativistic treatment of the test particle problem.

CHAPTER V

PLASMA DIELECTRIC CONSTANT AND FORCE ON A TEST PARTICLE

1. INTRODUCTION

The purpose of this chapter is to apply the drag formulae (4.19), which were obtained for an unspecified dielectric constant, to an electron plasma.*

The electron plasma dielectric constant is determined from the response of a plasma to an arbitrary external electric field.** If $\sigma(k,\omega)$ is the conductivity,

$$\vec{\tilde{j}}_{ind}(\vec{k},\omega) = \sigma(k,\omega)\,\vec{\tilde{E}} \tag{5.1}$$

then

$$\mathcal{E}(k,\omega) = 1 + \frac{4\pi i}{\omega}\,\sigma(k,\omega) \tag{5.2}$$

To determine $\sigma(k,\omega)$ the plasma is described by a distribution function $f(\vec{r},\vec{v},t)$ where

$$n_0 f(\vec{r},\vec{v},t)\,d\vec{r}\,d\vec{v}$$

is the number of electrons within $d\vec{r}$ of $\vec{r}$, and $d\vec{v}$ of $\vec{v}$, at time t. The distribution function satisfies the Landau-Vlasov equation

*As in the hydrodynamic calculation we confine our attention to an electron plasma, and assume the ions are distributed with a uniform density n_0.

**For a derivation of the electron plasma dielectric constant see Ref. 8.

$$\frac{\partial f(\vec{r},\vec{v},t)}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} - \frac{e}{m}\left(\vec{E}_{ext} + \vec{E}_{ind}\right)\cdot\frac{\partial f}{\partial \vec{v}} = 0 \tag{5.3}$$

$\vec{E}_{ext}$ is an external electric field, and $\vec{E}_{ind}$ is the induced (self consistent) electric field,

$$\vec{E}_{ind}(\vec{r},t) = e n_0 \int d\vec{v}' \int d\vec{r}' \frac{\partial \phi(|\vec{r}-\vec{r}'|)}{\partial \vec{r}} f(\vec{r}',\vec{v}',t) \tag{5.4}$$

where $\phi(r) = 1/r$. The induced current is

$$\vec{j}_{ind}(\vec{r},t) = -e n_0 \int d\vec{v}\, \vec{v} f(\vec{r},\vec{v},t) \tag{5.5}$$

Let

$$f(\vec{r},\vec{v},t) = f^{(0)}(\vec{v}) + h(\vec{r},\vec{v},t) \tag{5.6}$$

where

$$f^{(0)}(\vec{v}) = \left(\frac{m}{2\pi\theta}\right)^{3/2} e^{-\frac{mv^2}{2\theta}}$$

is the Maxwell-Boltzmann distribution function. Equation (5.3) is linearized by using (5.6) and neglecting products of the small quantities $\vec{E}_{ext}$ and $h(\vec{r},\vec{v},t)$. Using the linearized solution for h and Eqs. (5.1)-(5.5) the electron plasma dielectric constant is

$$\varepsilon(k,\omega) = 1 + \frac{\omega_p^2}{k^2}\int d\vec{v}\,\frac{\vec{k}\cdot\frac{\partial f^{(0)}(\vec{v})}{\partial \vec{v}}}{\omega - \vec{k}\cdot\vec{v}} \tag{5.7}$$

where $\omega_p = \left(\frac{4\pi e^2 n_0}{m}\right)^{1/2}$ is the plasma frequency.

In Section 2 the dielectric constant (5.7) is used to calculate the drag, $\vec{\mathcal{F}}_\ell$, associated with longitudinal fields. The expression for the force diverges logarithmically, and it is necessary to employ a short range cutoff to obtain a finite result. In Section 3 the source of this divergence is discussed.

Although the dielectric constant calculation emphasizes the relationship between the test particle problem in a plasma and the stopping power problem of unionized matter, it is an indirect approach to the plasma problem. We might as well solve (5.3) using the test particle as the external field

$$\frac{\partial f(\vec{r},\vec{v},t)}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} - \frac{e e_T}{m}\frac{\partial \phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}}\cdot\frac{\partial f}{\partial \vec{v}} - \frac{e}{m}\vec{E}_{ind}\cdot\frac{\partial f}{\partial \vec{v}} \tag{5.8}$$

where $\vec{r}_T = \vec{v}_T t$. The drag on a test particle is given by the electric field induced in the plasma evaluated at $\vec{r}_T$,

$$\vec{\mathcal{F}} = e_T e n_0 \int d\vec{r}\, \frac{\partial \phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}} \int d\vec{v}\, f(\vec{r},\vec{v},t) \tag{5.9}$$

If (5.6) is used in (5.8), and the result is linearized by neglecting products of the small quantities $\vec{E}_{ext}$, $h(\vec{r},\vec{v},t)$, the drag expression obtained from (5.9) is equivalent to the drag expression obtained from the dielectric constant. This treatment of the plasma test particle problem has been given by several authors.[9,10,14].

The connection between the dielectric plasma test particle problem and the hydrodynamic test particle problem of Chapter III should be clear. The hydrodynamic equations (3.1) represent the first three velocity moments of the Landau-Vlasov equation (5.8).* It follows that a solution of the Landau-Vlasov equation (5.8) contains more information than the corresponding solution of the hydrodynamic equations (3.1). The linearization of (5.8), described above, is analogous to the first linearization of the hydrodynamic equations in Chapter III, Section 2. In the next chapter a modified linearization of the Landau-Vlasov equation is given that is analogous to the modified linearization of the hydrodynamic equations in Chapter III, Section 4. The modified linearization of the Landau-Vlasov equation gives a convergent expression for the drag on a test particle.

2. DRAG ON A TEST PARTICLE

For a non-relativistic test particle we need only calculate the drag (4.19) associated with longitudinal fields,

$$\vec{\mathcal{F}}_{\ell} = -\frac{4\pi e_T^2 i}{(2\pi)^3}\int_{\Gamma} d\vec{k}\,\frac{\vec{k}}{k^2\,\varepsilon(k,\vec{k}\cdot\vec{v}_T)} \tag{5.10}$$

where $\varepsilon(k,\omega)$ is given by (5.7). Recall that Γ means the path of integration, for the component of $\vec{k}$ parallel to $\vec{v}_T$, passes <u>above</u> any singularities on the real axis. Furthermore, the derivation of (5.10) assumed that $\varepsilon(k,\omega)$ was analytically continued from $\operatorname{Im}\omega > 0$ to all parts of the finite ω-plane.

The analytic continuation of (5.7) has been put in a succinct form by Landau.[16] First the 3-fold integral in (5.7) is reduced to a single

*One must also assume a scalar pressure tensor, a zero heat flux vector, and a ratio of specific heats $\gamma = 1$; see Ref. 3.

integral,

$$\varepsilon(k,\omega) = 1 + \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} du \, \frac{dg(u)/du}{\frac{\omega}{k} - u} \quad , \ \mathrm{Im}\,\omega > 0 \tag{5.11}$$

where

$$g(u) = \int d\vec{v}\, f^{(0)}(\vec{v})\, \delta\left(u - \frac{\vec{k}\cdot\vec{v}}{k}\right)$$

$$= \left(\frac{m}{2\pi\theta}\right)^{1/2} e^{-\frac{mu^2}{2\theta}} \tag{5.12}$$

For the analytic continuation of (5.11) Landau writes

$$\varepsilon(k,\omega) = 1 + \frac{\omega_p^2}{k^2}\int_C du \, \frac{dg(u)/du}{\frac{\omega}{k} - u} \tag{5.13}$$

where the contour C is the real axis if $\mathrm{Im}\,\omega > 0$ and is shown in Fig. 3 when $\mathrm{Im}\,\omega \leq 0$.

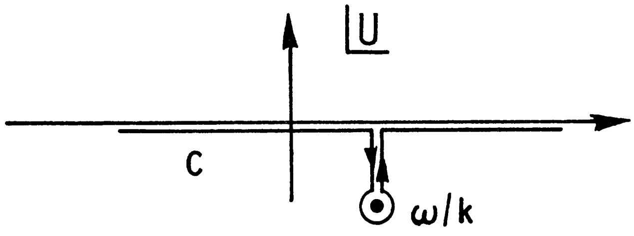

Fig. 3. Path of integration in complex u-plane.

Landau has also shown that the dispersion relation for longitudinal plasma waves, $\varepsilon(k,\omega) = 0$, has no solutions when $\mathrm{Im}\,\omega \geq 0$. In other words, we are dealing with a stable plasma.

Since the plasma is stable, expression (5.10) can be used with the $\vec{k}$ integration on the real axis,

$$\vec{\mathcal{F}}_\ell = -\frac{4\pi i e_T^2}{(2\pi)^3}\int d\vec{k}\,\frac{\vec{k}}{k^2\,\epsilon(k,\vec{k}\cdot\vec{v}_T)} \tag{5.14}$$

Introducing polar coordinates (k , $\eta = \cos\theta$, ϕ) for $\vec{k}$ with $\vec{v}_T$ as polar axis, the components of $\vec{k}$ perpendicular to $\vec{v}_T$ vanish when integrated over ϕ leaving only the parallel component,

$$\vec{\mathcal{F}}_\ell = -\mathcal{F}_\ell\,\frac{\vec{v}_T}{v_T}$$

where

$$\mathcal{F}_\ell = \frac{e_T^2 i}{\pi}\int_0^\infty dk\,k^2\int_{-1}^{1} d\eta\,\frac{\eta k}{k^2\,\epsilon(k,kv_T\eta)} \tag{5.15}$$

Replacing the dielectric constant with (5.13)

$$\mathcal{F}_\ell = \frac{e_T^2 i}{\pi}\int_0^\infty dk\int_{-1}^{1} d\eta\,\eta k\left\{1 - \frac{\omega_p^2\int_C du\,\frac{dg/du}{v_T\eta - u}}{k^2+\omega_p^2\int_C du\,\frac{dg/du}{v_T\eta - u}}\right\} \tag{5.16}$$

The first term is an odd function of η and vanishes; this term accounts for the self field of the test particle. To reduce the remaining term it is convenient to define a new function W ,

$$W\left(\eta v_T\sqrt{\tfrac{m}{2\theta}}\right) = R_D^2\omega_p^2\int_C du\,\frac{dg(u)/du}{\eta v_T - u} \tag{5.17}$$

Employing the definition of $g(u)$, (5.12), and using dimensionless variables we see that

$$W(t) = \frac{1}{2\sqrt{\pi}} \int_C d\chi \, \frac{\frac{de^{-\chi^2}}{d\chi}}{t-\chi}$$

or

$$W(t) = 1 - 2te^{-t^2}\int_0^t dx\, e^{x^2} + i\sqrt{\pi}\, te^{-t^2} \tag{5.18}$$

The function $W(t)$ is equal to $-\frac{1}{2}\frac{d}{dt}Z(t)$ where $Z(t)$ is the plasma dispersion function tabulated by Fried and Conte.* Graphs of the real and imaginary parts of $W(t)$ are shown in Figs. 4 and 5. The magnitude of $W(t)$ is never greater than unity, $|W(t)| \leq 1$.

Inserting (5.17) in the force expression (5.16), and making the change of variables $\eta = \dfrac{t}{v_T\sqrt{\frac{m}{2\theta}}}$, we obtain

$$\mathcal{F}_\ell = -\frac{2e_T^2\omega_p^2 i}{\pi v_T^2}\int_{-v_T\sqrt{\frac{m}{2\theta}}}^{v_T\sqrt{\frac{m}{2\theta}}} dt\, t\, W(t) \int_0^\infty k\,dk\, \frac{R_D^2}{k^2R_D^2 + W(t)} \tag{5.19}$$

The k integration diverges logarithmically at large k , and it is necessary to terminate the integration at some maximum value of k, denoted by k_M,

$$\mathcal{F}_\ell = -\frac{e_T^2\omega_p^2 i}{\pi v_T^2}\int_{-v_T\sqrt{\frac{m}{2\theta}}}^{v_T\sqrt{\frac{m}{2\theta}}} dt\, t\,W(t)\, \ln\left[\frac{(k_M R_D)^2 + W(t)}{W(t)}\right] \tag{5.20}$$

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*Fried, B. D., Conte, S. D., <u>The Plasma Dispersion Function</u>, Academic Press, Inc., New York (1961).

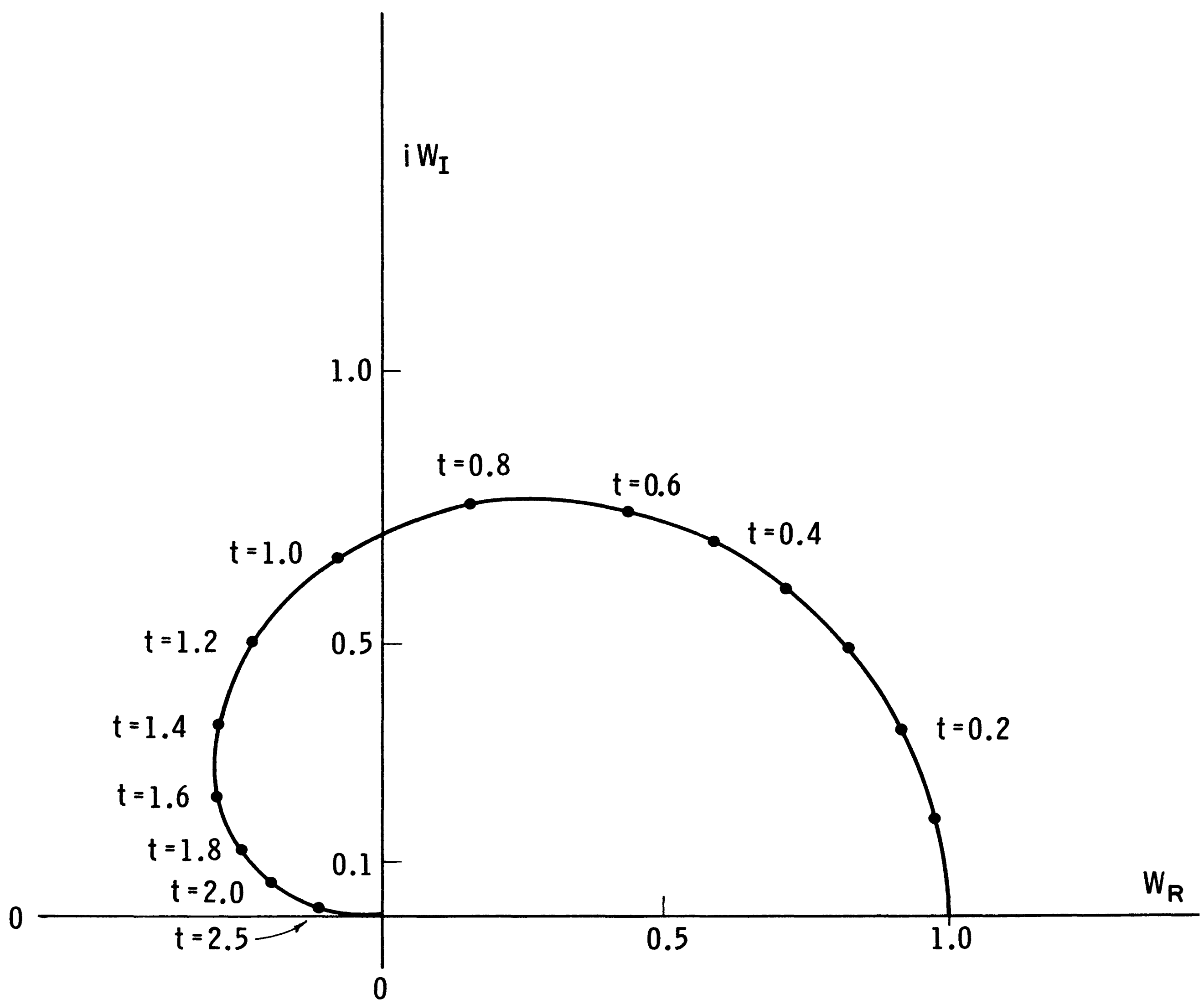

Fig. 4. Graph of $W = W_R + iW_I$ as a function of the parameter t.

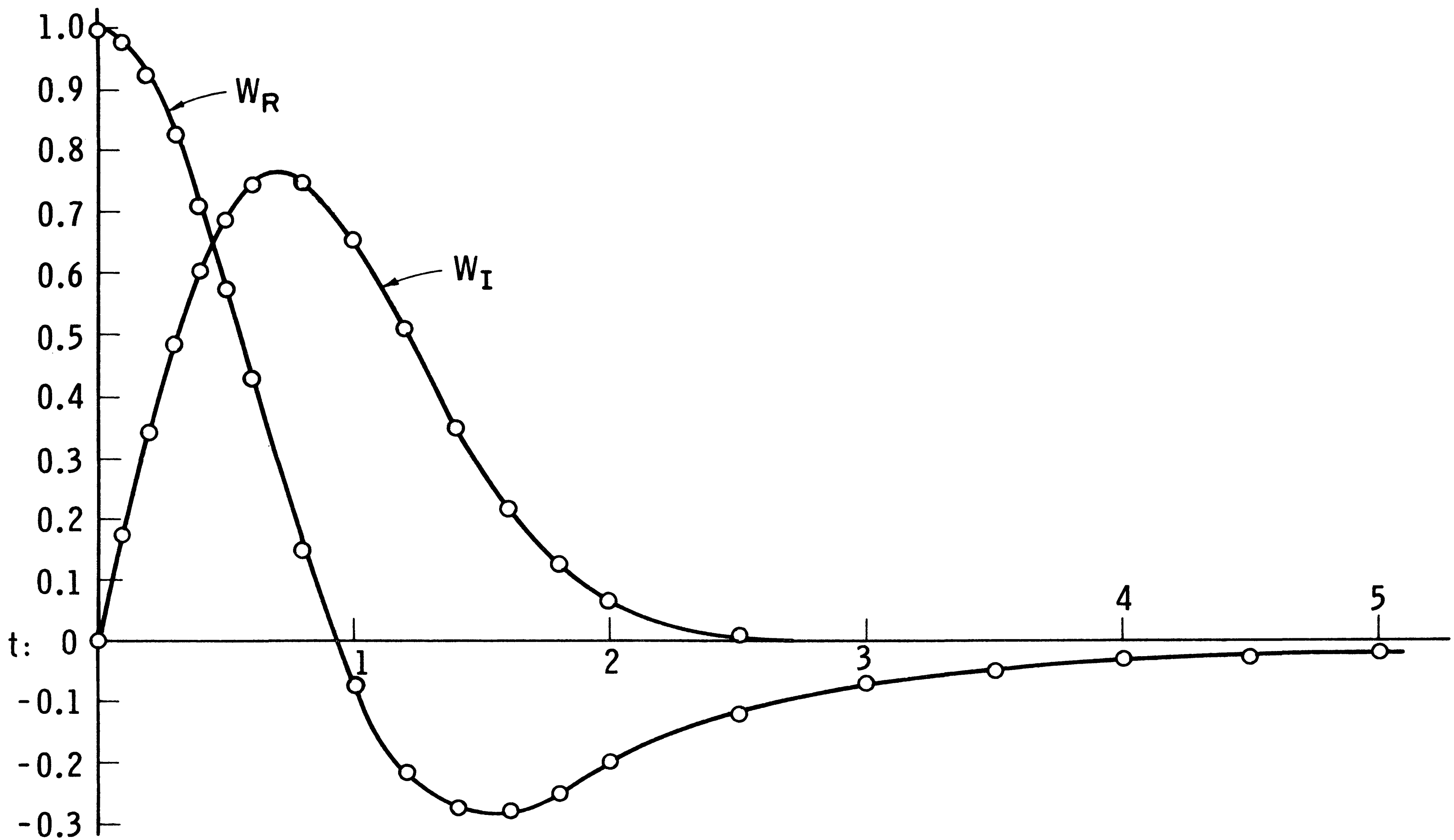

Fig. 5. Graph of $W_R(t)$ and $W_I(t)$.

A complete discussion of this divergence is given in Section 3. For the time being we note that large k corresponds to short distances from the test particle. The binary collision treatment of the test particle problem in Chapter II indicated that the proper cutoff at short distances is the Landau length, $R_L = \frac{e e_T}{\Theta}$. Thus k_M should be on the order of R_L^{-1} and from Table I we see that $k_M R_D \sim \frac{R_D}{R_L} >> 1$. Correspondingly, the logarithm in (5.20) can be approximated by $\ln (k_M R_D)^2$. The magnitude of the drag force is approximately

$$\mathcal{F}_\ell = -\frac{2}{\pi} \frac{e_T^2 \omega_p^2}{v_T^2} i \ln(k_M R_D) \int_{-v_T\sqrt{\frac{m}{2\Theta}}}^{v_T\sqrt{\frac{m}{2\Theta}}} dt \, t \, W(t) \tag{5.21}$$

The real part of $W(t)$ is an even function of t and does not contribute to the integral in (5.21). The imaginary part of $W(t)$ is obtained from (5.18) and yields

$$\mathcal{F}_\ell = \frac{e_T^2 \omega_p^2}{v_T^2} \ln(k_M R_D) \frac{2}{\sqrt{\pi}} \int_{-v_T\sqrt{\frac{m}{2\Theta}}}^{v_T\sqrt{\frac{m}{2\Theta}}} dt \, t^2 e^{-t^2} \tag{5.22}$$

Using the definitions of $g(u)$ and $f^{(0)}(\vec{v})$ we have

$$\frac{2}{\sqrt{\pi}} \int_{-v_T\sqrt{\frac{m}{2\Theta}}}^{v_T\sqrt{\frac{m}{2\Theta}}} dt \, t^2 e^{-t^2} = \int_0^{v_T} d\vec{v} \, f^{(0)}(\vec{v}) \tag{5.23}$$

where the three dimensional integration on the right hand side is over a sphere of radius v_T. Combining (5.23) with expression (5.22) the expression for the drag force is

$$\vec{\mathcal{F}}_\ell = -e_T^2\,\omega_p^2 \ln(k_M R_D)\,\frac{\vec{v}_T}{v_T^3}\int_0^{v_T} d\vec{v}\, f^{(0)}(\vec{v}) \tag{5.24}$$

This result has been obtained by Rostoker and Rosenbluth,[9] Gasiorowicz, Neuman, and Riddel,[10] and Rand.[14]

3. DISCUSSION OF THE LOGARITHMIC DIVERGENCE

It is clear that the linearization of (5.8) is not applicable for all $\vec{r}$, since E_{ext} is unbounded in the neighborhood of the test particle. It is this weakness of the linear solution which accounts for the divergence of the drag expression. To see this, consider the solution for $f(\vec{r},\vec{v},t)$ in the neighborhood of $\vec{r}_T$. For $\vec{r}$ sufficiently close to $\vec{r}_T$ the induced field is negligible compared to $\vec{E}_{ext}$, and Eq. (5.8) is approximately

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} - \frac{e e_T}{m}\,\frac{\partial \phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}}\cdot\frac{\partial f}{\partial \vec{v}} = 0 \tag{5.25}$$

This equation describes the motion of non-interacting electrons in the external field of the test particle.* This is nothing more than the binary collision description, and a force calculated from the solution of (5.25) must be identical to the binary collision expression (2.26). In (2.26) no divergence is associated with short distances; the long range divergence results from the neglect of collective interactions.

When $\vec{r}$ is near $\vec{r}_T$, the equation used to obtain the divergent

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*Ron and Kalman[13] give an explicit solution of (5.25).

drag expression is the linearized version of (5.25),

$$\frac{\partial h}{\partial t} + \vec{v} \cdot \frac{\partial h}{\partial \vec{r}} = \frac{e e_T}{m} \frac{\partial \phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}} \cdot \frac{\partial f^{(0)}}{\partial \vec{v}} \tag{5.26}$$

The solution of (5.26) in terms of $\vec{\xi} = \vec{r} - \vec{r}_T$ as $t \to \infty$ is

$$h(\vec{\xi}, \vec{v}) = \frac{e e_T}{m} \int_0^\infty d\tau \frac{\partial \phi(|\vec{\xi} - (\vec{v}-\vec{v}_T)\tau|)}{\partial \vec{\xi}} \cdot \frac{\partial f^{(0)}(\vec{v})}{\partial \vec{v}} \tag{5.27}$$

The time asymptotic distribution function in the linearized theory is then given by

$$f(\vec{\xi}, \vec{v}) = f^{(0)}(\vec{v}) + \frac{e e_T}{m} \int_0^\infty d\tau \frac{\partial \phi(|\vec{\xi} - (\vec{v}-\vec{v}_T)\tau|)}{\partial \vec{\xi}} \cdot \frac{\partial f^{(0)}(\vec{v})}{\partial \vec{v}} \tag{5.28}$$

The interpretation of this result is clear. The linearized theory develops the distribution function, through the first term of a Taylor series, about the Maxwell-Boltzmann distribution at $\vec{\xi} = +\infty$. The momentum given to a particle coming from $\vec{\xi} = +\infty$ to $\vec{\xi}$, according to the linear theory, is

$$\Delta \vec{p} = e e_T \int_0^\infty d\tau \frac{\partial \phi(|\vec{\xi} - (\vec{v}-\vec{v}_T)\tau|)}{\partial \vec{\xi}}$$

This is a "straight path" approximation, the particles move with constant velocities along a straight path parallel to $\vec{v} - \vec{v}_T$. As $\vec{\xi} \to 0$ the momentum change in the straight path calculation is infinite, and this accounts for the apparent divergence of the drag.

CHAPTER VI

A CONVERGENT TEST PARTICLE PROBLEM

1. INTRODUCTION

The aim of this chapter is to present a consistent solution of the test particle problem in an electron plasma from the viewpoint of kinetic theory. The kinetic theory description begins with the Landau-Vlasov equation (5.8),

$$\frac{\partial f(\vec{r},\vec{v},t)}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} - \frac{e e_T}{m}\frac{\partial \phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}}\cdot\frac{\partial f}{\partial \vec{v}}$$

$$- \frac{e^2 n_0}{m}\int d\vec{v}'\int d\vec{r}'\,\frac{\partial \phi(|\vec{r}-\vec{r}'|)}{\partial \vec{r}}\,f(\vec{r}',\vec{v}',t)\cdot\frac{\partial f}{\partial \vec{v}} = 0 \tag{6.1}$$

where $\phi(r) = 1/r$. We already know (Chapter V) that a straightforward linearization of (6.1), in which the test particle is assumed to introduce only a small perturbation from a <u>uniform</u> Maxwell-Boltzmann distribution, will not work. The situation is quite similar to the hydrodynamic treatment of the test particle problem in Chapter III. In that case a simple linearization of the Euler equations was not correct and led to a divergent drag expression. Further investigation of the hydrodynamic model suggested a modified linearization that eliminated the difficulties.

The modified linearization method can also be applied to the kinetic theory treatment. The method is based on the observation that it is better to start a perturbation solution not with a uniform density n_0, but with the density that would be prescribed by the Vlasov equation (6.1) if $\vec{v}_T = 0$. In particular, when $\vec{v}_T = 0$ we shall assume the electron velocity distribution is Maxwell-Boltzmann and that the spatial distribution has the form of a Boltzmann-Gibbs factor,

$$f(\vec{r},\vec{v}) = f^{(0)}(\vec{v}) \exp\left\{-R_L \Phi(|\vec{r}-\vec{r}_T|)\right\} \tag{6.2}$$

where $R_L = \frac{e e_T}{\Theta}$ and $f^{(0)}(\vec{v}) = \left(\frac{m}{2\pi\Theta}\right)^{3/2} e^{-\frac{mv^2}{2\Theta}}$. The "generalized potential" $\Phi(|\vec{r}-\vec{r}_T|)$ is determined from the Vlasov equation (6.1) with $\vec{v}_T = 0$. This Φ turns out to be identical to the generalized potential found in Chapter III.

When v_T is small we expect the distribution function to differ only slightly from (6.2). Therefore, we write

$$f(\vec{r},\vec{v},t) = \exp\left\{-R_L \Phi(|\vec{r}-\vec{r}_T|)\right\}\left\{f^{(0)}(\vec{v}) + h(\vec{r},\vec{v},t)\right\} \tag{6.3}$$

where $h(\vec{r},\vec{v},t)$ is a small perturbation. The Vlasov equation, when linearized in h, can be solved, and leads to a drag expression that does not require a cutoff. In fact, this result is exact as $v_T \to 0$, and is a good approximation even when v_T significantly exceeds the mean electron speed $\sqrt{\Theta/m}$.

In Sections 2 and 3 the equations for Φ and h are derived, and in Section 4 the solution for h is obtained and used in a calculation of the drag. Section 5 completes our work on the test particle problem with a discussion of the drag expression, and its relationship to the approximate drag expressions given in Chapters II and V.

2. THE GENERALIZED POTENTIAL

To obtain the function $\Phi(|\vec{r}-\vec{r}_T|)$ we look for the time independent solution of Eq. (6.1) in which the distribution function has the form (6.2). Dividing the equation obtained from (6.1) and (6.2) by the distribution function (6.2) yields

$$\vec{v}\cdot\left\{\frac{\partial}{\partial\vec{r}}\Phi(|\vec{r}-\vec{r}_T|)-\frac{\partial\phi(|\vec{r}-\vec{r}_T|)}{\partial\vec{r}}-\frac{e n_0}{e_T}\int d\vec{r}\,'\frac{\partial\phi(|\vec{r}-\vec{r}_T-\vec{r}\,'|)}{\partial\vec{r}}e^{-R_L\Phi(r')}\right\}=0 \qquad (6.4)$$

Since Eq. (6.4) must hold for all $\vec{v}$ the quantity in brackets is identically zero. Taking the divergence of the vector in brackets, and using the identity

$$\nabla^2\phi(\xi)=-4\pi\,\delta(\vec{\xi}) \qquad (6.5)$$

we find

$$\nabla^2\Phi(\xi)=-\frac{4\pi e n_0}{e_T}\left[e^{-R_L\Phi(\xi)}-1\right]-4\pi\,\delta(\vec{\xi}) \qquad (6.6)$$

The -1 in the **first term** on the right hand side of (6.6) accounts for the uniform background of positive charge. Equation (6.6) is identical to Eq. (A.1) of Appendix A. We found there that a good approximation to $\Phi(\xi)$ is the Debye-Hückel result

$$\Phi(\xi)=\frac{1}{\xi}e^{-\frac{\xi}{R_D}} \qquad (6.7)$$

where $R_D = \left(\frac{4\pi e^2 n_0}{\Theta}\right)^{-1/2}$ is the Debye length.

3. MODIFIED LINEARIZATION OF VLASOV EQUATION

When the test particle is moving with a constant velocity $\vec{v}_T$ we assume the distribution function can be written in the form (6.3),

$$f(\vec{r},\vec{v},t) = e^{-R_L \Phi(|\vec{r}-\vec{r}_T|)} \left\{ f^{(0)}(\vec{v}) + h(\vec{r},\vec{v},t) \right\} \tag{6.8}$$

where $\vec{r}_T = \vec{v}_T t$ and $h(\vec{r},\vec{v},t)$ is a small perturbation. Inserting (6.8) in the Vlasov equation (6.1), and neglecting products of the quantities Φ , h , and $e_T \partial\phi/\partial\vec{r}$ which are assumed small for purposes of linearization we obtain a linear equation for $h(\vec{r},\vec{v},t)$,

$$\frac{\partial h(\vec{r},\vec{v},t)}{\partial t} + \vec{v}\cdot\frac{\partial h}{\partial \vec{r}} - \frac{e e_T}{m}\frac{\partial \Phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}}\cdot\frac{\partial f^{(0)}}{\partial \vec{v}}$$

$$- \frac{e^2 n_0}{m}\int d\vec{v}'\int d\vec{r}'\,\frac{\partial \phi(|\vec{r}-\vec{r}'|)}{\partial \vec{r}}\, h(\vec{r}',\vec{v}',t)\cdot\frac{\partial f^{(0)}}{\partial \vec{v}} - R_L f^{(0)}(\vec{v})\,(\vec{v}-\vec{v}_T)\cdot\frac{\partial \Phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}} = 0 \tag{6.9}$$

Equation (6.4) has been used in the derivation of (6.9).

Although the linearization assumes that $e_T \frac{\partial \phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}}$ is a small quantity, this is violated when $\vec{r}$ is sufficiently close to $\vec{r}_T$. To see what effect this has on the distribution function we first inspect the terms in Eq. (6.9). The third and last terms of (6.9) combine to give

$$R_L f^{(0)}(\vec{v})\ \vec{v}_T\cdot\frac{\partial \Phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}} \tag{6.10}$$

The time dependence of the function $h(\vec{r},\vec{v},t)$ must be of the form $\vec{r}-\vec{v}_T t$ or $\vec{r}-\vec{r}_T$, and so $\frac{\partial h}{\partial t} = -\vec{v}_T\cdot\nabla h$. Furthermore, when $\vec{r}$ is close to $\vec{r}_T$ the integral term in (6.9) is unimportant compared to $\partial\Phi/\partial\vec{r}$ Thus, the dominate behavior of $h(\vec{r}-\vec{r}_T,\vec{v})$ is governed by the equation

$$(\vec{v}-\vec{v}_T)\cdot\frac{\partial h}{\partial\vec{r}} + R_L f^{(0)}(\vec{v})\ \vec{v}_T\cdot\frac{\partial}{\partial\vec{r}}\Phi(|\vec{r}-\vec{r}_T|) = 0$$

or

$$\frac{\partial}{\partial\vec{r}}\cdot\left\{(\vec{v}-\vec{v}_T)h(\vec{r}-\vec{r}_T,\vec{v}) + R_L\vec{v}_T f^{(0)}(\vec{v})\Phi(|\vec{r}-\vec{r}_T|)\right\} = 0 \qquad (6.11)$$

The order of magnitude of $h(\vec{r}-\vec{r}_T,\vec{v})$ can be obtained by comparison of the two terms in (6.11). If v_T is much less than the average value of v , i.e., $v_T \ll \sqrt{\frac{\theta}{m}}$, then h will be on the order of $f^{(0)}(\vec{v})$, and hence the linearization will breakdown, when $\vec{r}$ satisfies

$$\frac{v_T}{\sqrt{\frac{\theta}{m}}} R_L\Phi(|\vec{r}-\vec{r}_T|) \gtrsim 1$$

However, the distribution function goes exponentially to zero as $R_L\Phi(|\vec{r}-\vec{r}_T|)$ increases, (6.8), which means the failure of the linearized solution for h is negligible as v_T decreases.

If v_T is on the order of or greater than $\sqrt{\theta/m}$ the function $h(\vec{r}-\vec{r}_T,\vec{v})$ is of order $f^{(0)}(\vec{v})$ when

$$R_L\Phi(|\vec{r}-\vec{r}_T|) \gtrsim 1$$

This defines the limit of validity of the modified linearization; the exponential factor in the distribution function is reduced to e^{-1} when the perturbation h has attained a value comparable to the zeroth order term $f^{(0)}(\vec{v})$. In other words, the distribution function has started rapidly toward zero at the same value of $\vec{r}$ that the linearization breaks down. It can be concluded that the modified linearization is an excellent approximation to the electron distribution function when v_T is less than $\sqrt{\theta/m}$, it becomes exact as $v_T \to 0$. When v_T exceeds the mean electron speed $\sqrt{\theta/m}$ the modified linearization still yields the qualitative features of the distribution function.

4. DRAG ON A TEST PARTICLE

Once $h(\vec{r},\vec{v},t)$ has been determined the force on the test particle can be calculated as $-e_T$ times the electric field evaluated at $\vec{r}_T$,

$$\vec{F} = e e_T n_0 \int d\vec{r} \frac{\partial \phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}} e^{-R_L \bar{\Phi}(|\vec{r}-\vec{r}_T|)} \int d\vec{v} \left\{ f^{(0)}(\vec{v}) + h(\vec{r},\vec{v},t) \right\} \tag{6.12}$$

The first term does not contribute to the force, and we are left with

$$\vec{F} = e e_T n_0 \int d\vec{r} \frac{\partial \phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}} e^{-R_L \bar{\Phi}(|\vec{r}-\vec{r}_T|)} \int d\vec{v}\, h(\vec{r},\vec{v},t) \tag{6.13}$$

The integral in (6.13) can also be expressed as an integral over the Fourier space transforms of the functions in the integrand by using Parseval's relation. The result is

$$\vec{F} = \frac{e e_T n_0}{(2\pi)^3} \int d\vec{k}\, \vec{Q}(-\vec{k}) e^{i\vec{k}\cdot\vec{v}_T t} \int d\vec{v}\, \breve{h}(\vec{k},\vec{v},t) \tag{6.14}$$

where

$$\vec{Q}(\vec{k}) = \int d\vec{r} \frac{\partial \phi(r)}{\partial \vec{r}} e^{-R_L \Phi(r)} e^{-i\vec{k}\cdot\vec{r}} \tag{6.15}$$

and we have used $\vec{r}_T = \vec{v}_T t$.

To calculate the drag we must determine $\int d\vec{v}\, \hat{h}(\vec{k},\vec{v},t)$. The function $h(\vec{r},\vec{v},t)$ is a solution of (6.9). To be specific let us consider the initial value problem $h = 0$ for $t \leq 0$. Equation (6.9) is a linear equation for h and is easily handled if we introduce the Fourier space and time transform

$$\hat{h}(\vec{k},\vec{v},\omega) = \frac{1}{2\pi}\int_0^\infty dt \int d\vec{r}\, h(\vec{r},\vec{v},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \tag{6.16}$$

where t is integrated only over positive values for the initial value problem. The inverse transform is given by

$$h(\vec{r},\vec{v},t) = \frac{1}{(2\pi)^3}\int d\vec{k} \int_{-\infty+i\gamma}^{\infty+i\gamma} d\omega\, \hat{h}(\vec{k},\vec{v},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{6.17}$$

where γ is chosen such that the ω integration path lies above all singularities of the integrand in the complex ω-plane.

The Fourier transformation of (6.9) is

$$\begin{aligned}(\vec{k}\cdot\vec{v}-\omega)\hat{h}(\vec{k},\vec{v},\omega) &+ \frac{e e_T}{m}\hat{\Phi}(k)\frac{\vec{k}\cdot\frac{\partial f^{(0)}}{\partial \vec{v}}}{2\pi i\,(\omega-\vec{k}\cdot\vec{v}_T)} \\ &- \omega_p^2\frac{\vec{k}}{k^2}\cdot\frac{\partial f^{(0)}}{\partial \vec{v}}\int d\vec{v}'\,\hat{h}(\vec{k},\vec{v}',\omega) + R_L\hat{\Phi}(k) f^{(0)}(\vec{v})\frac{(\vec{v}-\vec{v}_T)\cdot\vec{k}}{2\pi i\,(\omega-\vec{k}\cdot\vec{v}_T)} = 0\end{aligned} \tag{6.18}$$

Rearranging and integrating over $\vec{v}$ gives

$$\int d\vec{v}\,\check{h}(\vec{k},\vec{v},\omega) = \frac{\check{\Phi}(k)}{2\pi i(\omega-\vec{k}\cdot\vec{v}_T)}\cdot\frac{\frac{ee_T}{m}\int d\vec{v}\frac{\vec{k}\cdot\frac{\partial f^{(0)}}{\partial\vec{v}}}{\omega-\vec{k}\cdot\vec{v}} + R_L\int d\vec{v}\frac{f^{(0)}(\vec{v})\,\vec{k}\cdot(\vec{v}-\vec{v}_T)}{\omega-\vec{k}\cdot\vec{v}}}{1+\frac{\omega_p^2}{k^2}\int d\vec{v}\frac{\vec{k}\cdot\frac{\partial f^{(0)}}{\partial\vec{v}}}{\omega-\vec{k}\cdot\vec{v}}} \tag{6.19}$$

$$= -\frac{(\vec{k}\cdot\vec{v}_T)\,\check{\Phi}(k)}{2\pi i\,\omega\,(\omega-\vec{k}\cdot\vec{v}_T)}\cdot\frac{R_L-\frac{ee_T}{m}\int d\vec{v}\frac{\vec{k}\cdot\frac{\partial f^{(0)}}{\partial\vec{v}}}{\omega-\vec{k}\cdot\vec{v}}}{1+\frac{\omega_p^2}{k^2}\int d\vec{v}\frac{\vec{k}\cdot\partial f^{(0)}/\partial\vec{v}}{\omega-\vec{k}\cdot\vec{v}}}$$

for $\mathrm{Im}\,\omega > 0$. This expression is more compactly written in terms of the function W defined in (5.17) and (5.18),

$$\int d\vec{v}\,\check{h}(\vec{k},\vec{v},\omega) = -\frac{R_L(\vec{k}\cdot\vec{v}_T)(kR_D)^2\,\check{\Phi}(k)}{2\pi i\,\omega\,(\omega-\vec{k}\cdot\vec{v}_T)}\;\frac{1-W\left(\frac{\omega}{k}\sqrt{\frac{m}{2\theta}}\right)}{(kR_D)^2+W\left(\frac{\omega}{k}\sqrt{\frac{m}{2\theta}}\right)} \tag{6.20}$$

This also gives the analytic continuation of (6.19).

The inverse Fourier time transform of this function is

$$\int d\vec{v}\,\check{h}(\vec{k},\vec{v},t) = \int_{-\infty+i\gamma}^{\infty+i\gamma} d\omega\, e^{-i\omega t}\int d\vec{v}\,\check{h}(\vec{k},\vec{v},\omega) \tag{6.21}$$

Since $(kR_D)^2+W\left(\frac{\omega}{k}\sqrt{\frac{m}{2\theta}}\right)$ is, aside from a factor $(kR_D)^2$, the function (5.13) it is never zero for $\mathrm{Im}\,\omega \geq 0$. There is no pole at $\omega = 0$. Therefore, the only pole of (6.20) for $\mathrm{Im}\,\omega \geq 0$ is at $\omega = \vec{k}\cdot\vec{v}_T$. As we are only interested in the time asymptotic results the poles for $\mathrm{Im}\,\omega < 0$ are of no concern. The time asymptotic value of (6.21) is therefore

$$\int d\vec{v}\,\hat{h}(\vec{k},\vec{v},t=\infty) = R_L (kR_D)^2\,\hat{\Phi}(k)\,e^{-i\vec{k}\cdot\vec{v}_T t}\left\{\frac{1-W\left(\frac{\vec{k}\cdot\vec{v}_T}{k}\sqrt{\frac{m}{2\theta}}\right)}{(kR_D)^2+W\left(\frac{\vec{k}\cdot\vec{v}_T}{k}\sqrt{\frac{m}{2\theta}}\right)}\right\} \tag{6.22}$$

Substitution of (6.22) in (6.14) yields the force on a test particle,

$$\vec{\mathcal{F}} = \frac{e e_T n_0 R_L}{(2\pi)^3}\int d\vec{k}\,\vec{Q}(-\vec{k})(kR_D)^2\,\hat{\Phi}(k)\left\{\frac{1-W\left(\frac{\vec{k}\cdot\vec{v}_T}{k}\sqrt{\frac{m}{2\theta}}\right)}{(kR_D)^2+W\left(\frac{\vec{k}\cdot\vec{v}_T}{k}\sqrt{\frac{m}{2\theta}}\right)}\right\} \tag{6.23}$$

where $\vec{Q}(\vec{k})$ is given by (6.15) and $\Phi(r)$ is given approximately by the Debye-Hückel potential (6.7). The Fourier transform of the Debye-Hückel potential is

$$\hat{\Phi}(k) = \frac{4\pi R_D^2}{1+(kR_D)^2} \tag{6.24}$$

In Appendix B the function $\vec{Q}$ is evaluated to lowest order in the small quantity R_L/R_D ,

$$\vec{Q}(\vec{k}) = -(2\pi)^2 i\,\frac{\vec{k}}{k^2}\,\mathrm{hei}_2\!\left(\sqrt{4R_L k}\right) + O\!\left(\frac{R_L}{R_D}\right) \tag{6.25}$$

where the function $\mathrm{hei}_\nu(x)$ is related to a Hankel function of imaginary argument[18]

$$\mathrm{her}_\nu(x) \pm i\,\mathrm{hei}_\nu(x) = H_\nu^{(1)}\!\left(x e^{\pm i\frac{3}{4}\pi}\right) \tag{6.26}$$

Using (6.24) and (6.25) in the force expression (6.23) gives

$$\vec{f} = \frac{e_T^2 i}{2\pi}\int d\vec{k}\,\frac{\vec{k}}{k^2}\,\mathrm{hei}_2\left(\sqrt{4R_L k}\right)\frac{(kR_D)^2}{1+(kR_D)^2}\left\{\frac{1-W\left(\frac{\vec{k}\cdot\vec{v}_T}{k}\sqrt{\frac{m}{2\theta}}\right)}{(kR_D)^2+W\left(\frac{\vec{k}\cdot\vec{v}_T}{k}\sqrt{\frac{m}{2\theta}}\right)}\right\}$$

$$(6.27)$$

$$= \frac{e_T^2 i}{2\pi}\int d\vec{k}\,\frac{\vec{k}}{k^2}\,\mathrm{hei}_2\left(\sqrt{4R_L k}\right)\left\{\frac{1}{1+(kR_D)^2} - \frac{W\left(\frac{\vec{k}\cdot\vec{v}_T}{k}\sqrt{\frac{m}{2\theta}}\right)}{(kR_D)^2+W\left(\frac{\vec{k}\cdot\vec{v}_T}{k}\sqrt{\frac{m}{2\theta}}\right)}\right\}$$

The first term of (6.27) is an odd function of $\vec{k}$ and gives no contribution. The remaining term is simplified with the use of polar coordinates (k , $\eta = \cos\theta$, ϕ) for $\vec{k}$ with $\vec{v}_T$ as polar axis. Only the component of $\vec{k}$ parallel to $\vec{v}_T$ remains after the ϕ integration, and one obtains the drag

$$\vec{f} = -f\,\frac{\vec{v}_T}{v_T}$$

where

$$f = e_T^2 i\int_{-1}^{1}\eta\,d\eta\int_0^\infty k\,dk\,\mathrm{hei}_2\left(\sqrt{4R_L k}\right)\frac{W\left(\eta v_T\sqrt{\frac{m}{2\theta}}\right)}{(kR_D)^2+W\left(\eta v_T\sqrt{\frac{m}{2\theta}}\right)} \qquad (6.28)$$

Changing variables to $t = \eta v_T\sqrt{\frac{m}{2\theta}}$ and $x = kR_D$ gives

$$f = \frac{2e_T^2\omega_p^2 i}{v_T^2}\int_{-v_T\sqrt{\frac{m}{2\theta}}}^{v_T\sqrt{\frac{m}{2\theta}}} t\,dt\int_0^\infty x\,dx\,\mathrm{hei}_2\left(\sqrt{4\frac{R_L}{R_D}x}\right)\frac{W(t)}{x^2+W(t)} \qquad (6.29)$$

This drag is identical to (5.19) except for the additional factor $-\pi\,\mathrm{hei}_2\left(\sqrt{4\frac{R_L}{R_D}x}\right)$. It is this factor which produces a convergent result since (Ref. 18, p. 333)

$$\mathrm{hei}_2(x) \underset{x\to\infty}{\backsimeq} -\sqrt{\frac{2}{\pi x}}\cos\left(\frac{x}{\sqrt{2}}+\frac{9}{8}\pi\right)e^{-\frac{x}{\sqrt{2}}}$$

Recalling that the real part of $W(t)$ is an even function of t, and the imaginary part is an odd function of t, see Eq. (5.18)], Eq. (6.29) reduces to

$$\mathcal{F} = -\frac{2e_T^2\omega_p^2}{v_T^2}\int_{-v_T\sqrt{\frac{m}{2\theta}}}^{v_T\sqrt{\frac{m}{2\theta}}} t\,dt \int_0^\infty dx\,\mathrm{hei}_2\left(\sqrt{4\frac{R_L}{R_D}x}\right)\frac{x^3 W_I}{[x^2+W_R]^2+W_I^2} \tag{6.30}$$

where $W = W_R + i W_I$.

The x integration in (6.30) has not been obtained in closed form. In Appendix B we obtain the lowest order term in an expansion in powers of R_L/R_D. The result is

$$\int_0^\infty dx\,\mathrm{hei}_2\left(\sqrt{4\frac{R_L}{R_D}x}\right)\frac{x^3W_I}{[x^2+W_R]^2+W_I^2} \backsimeq -\frac{1}{\pi}W_I\ln\left[\frac{\epsilon}{\lambda^2}\frac{R_D}{R_L}\right]+\frac{1}{2\pi}\mathrm{Im}\left(W\ln W\right)+O\left(\frac{R_L}{R_D}\right) \tag{6.31}$$

where $\ln\lambda = 0.577\cdots$ is Euler's constant, $\epsilon = 2.718\cdots$ is the base of the natural logarithms and "Im" means the imaginary part.

Thus, to lowest order in R_L/R_D the magnitude of the drag (6.30) is

$$\mathcal{F} = \frac{2}{\pi}\frac{e_T^2\omega_p^2}{v_T^2}\left\{ \ln\left[\frac{\epsilon}{\lambda^2}\frac{R_D}{R_L}\right]\int_{-v_T\sqrt{\frac{m}{2\theta}}}^{v_T\sqrt{\frac{m}{2\theta}}} t\,dt\,W_I(t) - \frac{1}{2}\int_{-v_T\sqrt{\frac{m}{2\theta}}}^{v_T\sqrt{\frac{m}{2\theta}}} t\,dt\,\mathrm{Im}(W\ln W)\right\} \tag{6.32}$$

Recalling the relation (5.23)

$$\frac{2}{\pi}\int_{-v_T\sqrt{\frac{m}{2\theta}}}^{v_T\sqrt{\frac{m}{2\theta}}} t\,dt\,W_I(t) = \int_0^{v_T} d\vec{v}\,f^{(0)}(\vec{v})$$

we can rewrite (6.32) in a more familiar form by taking the second term of (6.32) inside the logarithm. The drag on a test particle is

$$\vec{\mathcal{F}} = -e_T^2\omega_p^2 \ln\left[\frac{\epsilon}{\lambda^2}e^{\,\varsigma(v_T)}\frac{R_D}{R_L}\right]\frac{\vec{v}_T}{v_T^3}\int_0^{v_T} d\vec{v}\,f^{(0)}(\vec{v}) \tag{6.33}$$

where

$$\varsigma(\vec{v}_T) = \frac{-\frac{1}{\pi}\int_{-v_T\sqrt{\frac{m}{2\theta}}}^{v_T\sqrt{\frac{m}{2\theta}}} t\,dt\,\mathrm{Im}(W\ln W)}{\int_0^{v_T} d\vec{v}\,f^{(0)}(\vec{v})} \tag{6.34}$$

A numerical integration of the numerator of (6.34) results in $\exp\{\xi(v_T)\}$ as shown in Fig. 6. The limiting values of this function are

$$e^{\xi(v_T)} = e^{-1/2} = 0.6065 \quad , \quad v_T = 0$$

and

$$e^{\xi(v_T)} \simeq v_T\sqrt{\frac{m}{2\theta}} \quad , \quad v_T\sqrt{\frac{m}{2\theta}} \gg 3$$

As one would expect the logarithm argument increases as v_T increases; that is, the average distance of closest approach in a collision should be smaller than R_L while a decrease in the shielding efficiency, as v_T increases, leads to a long range cutoff greater than R_D .

5. DISCUSSION OF THE RESULTS

The discussion in Section 3 showed that the modified linearization gives exact results as $v_T \to 0$. When $v_T \gtrsim \sqrt{\frac{\theta}{m}}$ one still expects the present method to retain the qualitative features of an exact solution; an exact solution would alter the velocity dependence of the logarithm argument at large v_T .

It is interesting to compare the convergent force expression calculated in this chapter, Eq. (6.33), with the results of Chapter II, Eq. (2.28), and Chapter V, Eq. (5.22). In Chapter V we started with the Vlasov equation linearized about a spatially homogeneous equilibrium state, $f^{(0)}(\vec{v})$. Comparison of (5.19) with (6.29) shows that the only

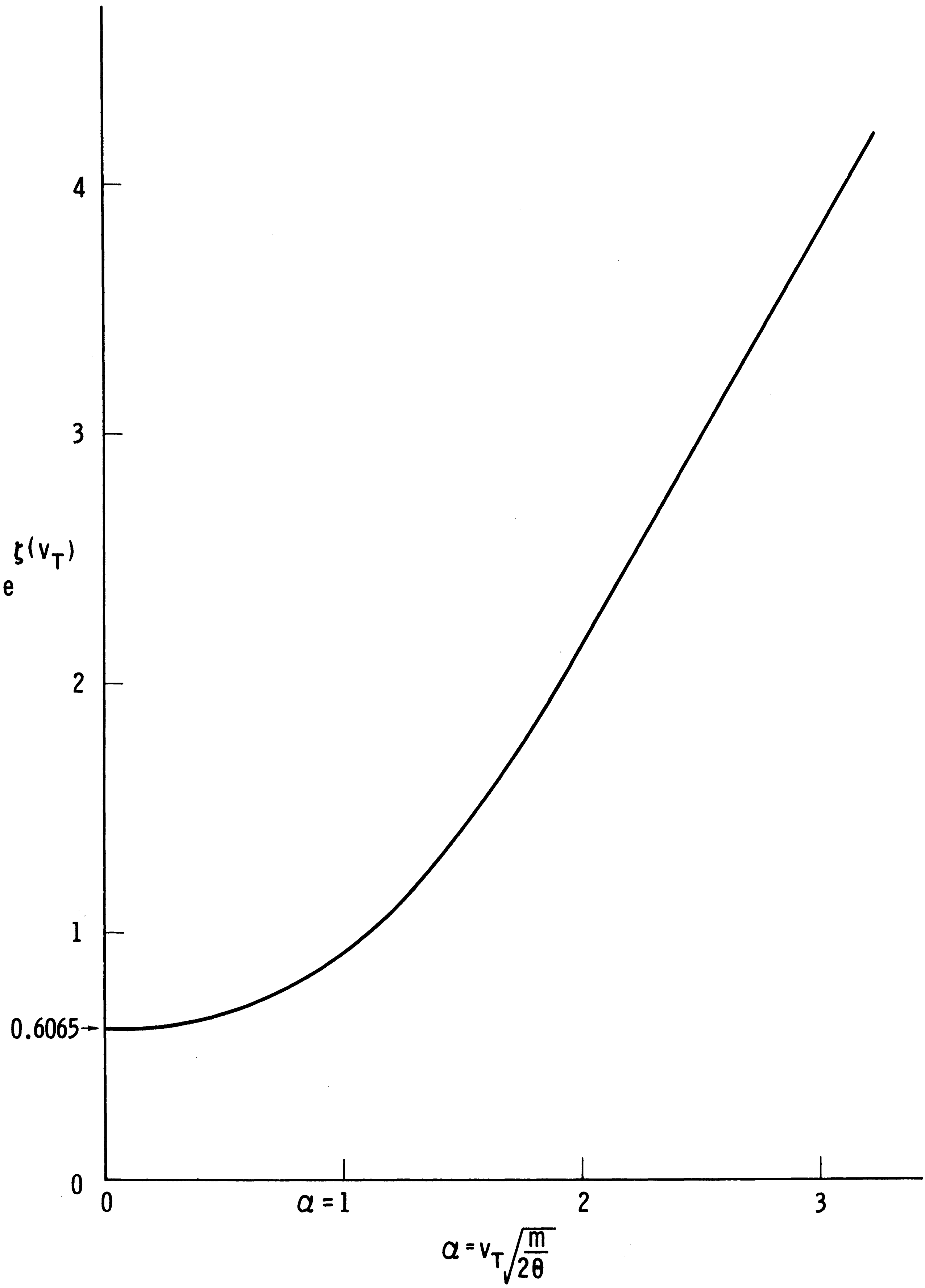

Fig. 6. Graph of $e^{\zeta(v_T)}$ as a function of $\alpha = v_T\sqrt{\frac{m}{2\theta}}$.

change introduced by the modified linearization is the introduction of the factor $-\pi\, \mathrm{hei}_2\left(\sqrt{4\frac{R_L}{R_D}\chi}\right)$ into the force integral. If this factor is developed in powers of $\frac{R_L}{R_D}\chi$,

$$-\pi\, \mathrm{hei}_2\left(\sqrt{4\frac{R_L}{R_D}\chi}\right) \simeq 1 - \frac{\pi}{4}\frac{R_L}{R_D}\chi + \cdots \tag{6.35}$$

the assumption that only the first term is needed when R_L/R_D is small would give the divergent drag expression (5.24). It is evident that $\frac{R_L}{R_D}\chi$ is not small for all values of χ , and the expansion of the hei_2 function is not correct.

To compare (6.33) with the binary collision treatment of Chapter II it is necessary to consider a solution of the Vlasov equation with the self consistent field term omitted (5.25),

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} - \frac{e e_T}{m}\frac{\partial \phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}}\cdot\frac{\partial f}{\partial \vec{v}} = 0 \tag{6.36}$$

As we saw in Chapter V a solution of this equation would give the binary collision force (2.26). On the other hand, suppose we look for a solution of the form

$$f(\vec{r},\vec{v},t) = e^{-R_L \Phi_{B.C.}(|\vec{r}-\vec{r}_T|)}\left\{ f^{(0)}(\vec{v}) + h_{B.C.}(\vec{r},\vec{v},t)\right\} \tag{6.37}$$

Following the same procedure used for the full Vlasov equation we find

$$\Phi_{B.C.}(|\vec{r}-\vec{r}_T|) = \frac{1}{|\vec{r}-\vec{r}_T|}$$

and

$$\int d\vec{v}\,\hat{h}_{B.C.}(\vec{k},\vec{v},t=\infty) = \left\{ R_L \hat{\phi}(k) + \frac{e e_T}{m} \hat{\phi}(k) \int_C d\vec{v} \frac{\vec{k}\cdot\frac{\partial f^{(0)}}{\partial \vec{v}}}{\vec{k}\cdot(\vec{v}-\vec{v}_T)} \right\} e^{-i\vec{k}\cdot\vec{v}_T t} \tag{6.38}$$

The force on the test particle is given by

$$\vec{\mathcal{F}}_{B.C.} = e e_T n_0 \int d\vec{r} \frac{\partial \phi(|\vec{r}-\vec{r}_T|)}{\partial \vec{r}} e^{-\frac{R_L}{|\vec{r}-\vec{r}_T|}} \int d\vec{v} \left\{ f^{(0)}(\vec{v}) + h_{B.C.} \right\} \tag{6.39}$$

The first term integrates to zero. Using the Fourier representations (6.25) and (6.38) the force becomes

$$\vec{\mathcal{F}}_{B.C.} = \frac{e e_T n_0}{(2\pi)^3} \int d\vec{k}\, \vec{Q}_{B.C.}(-\vec{k}) \left\{ R_L \hat{\phi}(k) + \frac{e e_T}{m} \hat{\phi}(k) \int_C d\vec{v} \frac{\vec{k}\cdot\frac{\partial f^{(0)}}{\partial \vec{v}}}{\vec{k}\cdot(\vec{v}-\vec{v}_T)} \right\} \tag{6.40}$$

where we have used $\vec{r}_T = \vec{v}_T t$. The second term can be expressed in terms of $W(t)$, (5.17), and we have

$$\vec{\mathcal{F}}_{B.C.} = \frac{e e_T n_0 R_L}{(2\pi)^3} \int d\vec{k}\, \vec{Q}_{B.C.}(-\vec{k})\, \hat{\phi}(k) \left\{ 1 - W\left(\frac{\vec{k}\cdot\vec{v}_T}{k} \sqrt{\frac{m}{2\theta}} \right) \right\} \tag{6.41}$$

This should be compared with (6.23) which is the corresponding result when the self consistent field is included. We see that the inclusion of interactions between plasma electrons has two consequences; (1) it replaces the pure Coulomb potential $\phi(r)$ by the shielded potential $\Phi(r)$, and (2) the plasma dielectric constant

$$\mathcal{E}(k, \vec{k}\cdot\vec{v}_T) = 1 + \frac{1}{(kR_D)^2} W\left(\frac{\vec{k}\cdot\vec{v}_T}{k}\sqrt{\frac{m}{2\theta}}\right)$$

appears in the denominator of the integrand in (6.23). Both changes reflect the effects of shielding and are responsible for the low k convergence of the force.

The remainder of this thesis is concerned with the development of general kinetic equations (i.e., equations describing the time evolution of the one particle distribution function). The customary approximations applied to the Liouville equation lead to the same apparent divergences which occur in the test particle problem. A generalization of the convergent test particle problem, as presented in this chapter, will lead to a convergent kinetic equation.

PART II: KINETIC EQUATION

CHAPTER VII

APPROXIMATE KINETIC EQUATIONS FOR AN ELECTRON PLASMA

1. INTRODUCTION

In Part II of this dissertation we concentrate on the problem of describing the time evolution of an electron plasma* which is not in equilibrium. Beginning with Section 3 we restrict our attention to the spatially homogeneous case owing to its greater tractability.

A complete description of a system of N electrons in a uniform background of positive charge and in a volume V is given by the N-particle distribution function $D_N(x_1,\cdots,x_N,t)$ where x stands for the six variables $(\vec{r},\vec{v})$ and

$$D_N(x_1,\cdots,x_N,t)\,dx_1\cdots dx_N \tag{7.1}$$

represents the probability that electron 1 is within dx_1 of x_1, etc. Since D_N is a probability we assume it to be normalized to unity. Furthermore, D_N must satisfy the Liouville equation

$$\frac{\partial D_N}{\partial t} + \sum_{i=1}^{N} \vec{v}_i\cdot\frac{\partial D_N}{\partial \vec{r}_i} - \frac{e^2}{m}\sum_{\substack{i,j=1\\(i\neq j)}}^{N} \frac{\partial \phi_{ij}}{\partial \vec{r}_i}\cdot\frac{\partial D_N}{\partial \vec{v}_i} = 0 \tag{7.2}$$

*We continue to use the electron plasma model in which the ions are replaced by a uniform distribution of positive charge. It was shown in Chapter III that this is equivalent to neglecting terms of order m/M the ratio of electron mass to ion mass.

Note that we have replaced the momenta $\vec{p}_i$ by $m\vec{v}_i$. The pair potential ϕ_{ij} is

$$\phi_{ij} = \frac{1}{|\vec{r}_i - \vec{r}_j|}$$

Instead of working with the N-particle distribution function, D_N, one usually introduces the reduced s-particle distribution functions f_s,

$$\frac{1}{V^s} f_s(x_1, \cdots, x_s, t) = \int \cdots \int dx_{s+1} \cdots dx_N \, D_N(x_1, \cdots, x_N, t) \tag{7.3}$$

The differential equations governing these functions are obtained by successive integrations of the Liouville equation (7.2).[19] The resulting set of equations ($s = 1, 2, \cdots, N-1$) is known as the B-B-G-K-Y hierarchy after its founders Bogoliubov, Born, Green, Kirkwood, and Yvon. We shall always assume the limit $N, V \to \infty$ but $n_0 = N/V$ finite. The first two members of the hierarchy ($s = 1, 2$) are

$$\frac{\partial f(x_1,t)}{\partial t} + \vec{v}_1 \cdot \frac{\partial f}{\partial \vec{r}_1} - \frac{e^2 n_0}{m} \int dx_2 \frac{\partial \phi_{12}}{\partial \vec{r}_1} \cdot \frac{\partial}{\partial \vec{v}_1} f_2(x_1, x_2, t) = 0 \tag{7.4}$$

and

$$\frac{\partial f_2(x_1,x_2,t)}{\partial t} + \vec{v}_1 \cdot \frac{\partial f_2}{\partial \vec{r}_1} + \vec{v}_2 \cdot \frac{\partial f_2}{\partial \vec{r}_2} - \frac{e^2}{m} \frac{\partial \phi_{12}}{\partial \vec{r}_1} \cdot \frac{\partial f_2}{\partial \vec{v}_1} - \frac{e^2}{m} \frac{\partial \phi_{21}}{\partial \vec{r}_2} \cdot \frac{\partial f_2}{\partial \vec{v}_2}$$
$$- \frac{e^2 n_0}{m} \int dx_3 \left[\frac{\partial \phi_{13}}{\partial \vec{r}_1} \cdot \frac{\partial}{\partial \vec{v}_1} + \frac{\partial \phi_{23}}{\partial \vec{r}_2} \cdot \frac{\partial}{\partial \vec{v}_2} \right] f_3(x_1, x_2, x_3, t) = 0 \tag{7.5}$$

The basic problem of non-equilibrium statistical mechanics is the development of a kinetic equation which describes the time evolution of the one particle distribution function

$$\frac{\partial f(x,t)}{\partial t} = A(x;f) \tag{7.6}$$

where A depends functionally on $f(x,t)$. The initial value of $f(x,t)$ would then be sufficient to determine its value at any other time. If f_2 were a known function of $f(x,t)$ Eq. (7.4) would be the desired kinetic equation. The equation determining f_2, however, involves f_3, and so on for the higher distribution functions which means we must ultimately solve the full Liouville equation for D_N. The point of introducing the reduced distribution functions is that it facilitates an approximation in which, for some s, the s-particle distribution function can be expressed in terms of the lower order distribution functions. As we will see this "truncation" of the hierarchy does <u>not</u> lead directly to a kinetic equation of the form (7.6). The remainder of this chapter is devoted to a review of the assumptions which <u>do</u> lead from the B-B-G-K-Y hierarchy to a kinetic equation.

The outline used in Part II is similar to the outline used in Part I. In Section 4 of this chapter we discuss a kinetic equation that emphasizes binary collisions. In Section 5 we discuss a kinetic equation that emphasizes distant collective interactions. In Section 6 a brief discussion is given of two recent "derivations" of electron plasma kinetic equations.

In the following chapter we show the relationship between equations of Sections 4 and 5. These electron plasma kinetic equations suffer from convergence difficulties identical to those encountered in the test particle problem. In Chapter IX, the last chapter, we derive a convergent kinetic equation by a method similar to the convergent test particle solution given in Chapter VI.

2. TRUNCATION OF THE B-B-G-K-Y HIERARCHY

One can always write the two particle distribution function as

$$f_2(x_1, x_2, t) = f(x_1, t) f(x_2, t) + P(x_1, x_2, t) \tag{7.7}$$

where $P(x_1, x_2, t)$ is the pair correlation function. If we truncate the hierarchy by neglecting pair correlations, i.e., $P = 0$, the B-B-G-K-Y hierarchy reduces to the Landau-Vlasov equation

$$\frac{\partial f(x_1,t)}{\partial t} + \vec{v}_1 \cdot \frac{\partial f}{\partial \vec{r}_1} - \frac{e^2 n_0}{m} \int dx_2 \frac{\partial \phi_{12}}{\partial \vec{r}_1} f(x_2,t) \cdot \frac{\partial f}{\partial \vec{v}_1} = 0 \tag{7.8}$$

This equation is satisfied by any time independent spatially uniform function $f(\vec{v})$, consequently it does not lead to an H-theorem. To have an H-theorem we must truncate the hierarchy at some other point.

The three particle distribution function can be written as

$$\begin{aligned} f_3(x_1, x_2, x_3, t) = {} & f(x_1,t) f_2(x_2, x_3, t) + f(x_2,t) f_2(x_1, x_3, t) \\ & + f(x_3,t) f_2(x_1, x_2, t) - 2 f(x_1,t) f(x_2,t) f(x_3,t) + T(x_1, x_2, x_3, t) \end{aligned} \tag{7.9}$$

where $T(x_1, x_2, x_3, t)$ is the three particle correlation function. If closure of the hierarchy is accomplished by neglecting three particle correlations, that is, by setting $T = 0$ then (7.4) and (7.5) become

$$\left\{\frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{r}_1}\right\} f(x_1, t) - \frac{e^2 n_0}{m} \int dx_2 \frac{\partial \phi_{12}}{\partial \vec{r}_1} \cdot \frac{\partial}{\partial \vec{v}_1} f_2(x_1, x_2, t) = 0 \tag{7.10}$$

and

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{r}_1} + \vec{v}_2 \cdot \frac{\partial}{\partial \vec{r}_2} - \frac{e^2}{m} \frac{\partial \phi_{12}}{\partial \vec{r}_1} \cdot \frac{\partial}{\partial \vec{v}_1} - \frac{e^2}{m} \frac{\partial \phi_{21}}{\partial \vec{r}_2} \cdot \frac{\partial}{\partial \vec{v}_2}\right. \\
&\quad \left. - \frac{e^2 n_0}{m} \int dx_3 \frac{\partial \phi_{13}}{\partial \vec{r}_1} f(x_3, t) \cdot \frac{\partial}{\partial \vec{v}_1} - \frac{e^2 n_0}{m} \int dx_3 \frac{\partial \phi_{23}}{\partial \vec{r}_2} f(x_3, t) \cdot \frac{\partial}{\partial \vec{v}_2}\right\} f_2(x_1, x_2, t) \\
&+ 2 \frac{e^2 n_0}{m} \int dx_3 f(x_3, t) \left\{\frac{\partial \phi_{13}}{\partial \vec{r}_1} \cdot \frac{\partial}{\partial \vec{v}_1} + \frac{\partial \phi_{23}}{\partial \vec{r}_2} \cdot \frac{\partial}{\partial \vec{v}_2}\right\} f(x_1, t) f(x_2, t) \\
&- \frac{e^2 n_0}{m} \frac{\partial f(x_1, t)}{\partial \vec{v}_1} \cdot \int dx_3 \frac{\partial \phi_{13}}{\partial \vec{r}_1} f_2(x_2, x_3, t) - \frac{e^2 n_0}{m} \frac{\partial f(x_2, t)}{\partial \vec{v}_2} \cdot \int dx_3 \frac{\partial \phi_{23}}{\partial \vec{r}_2} f_2(x_1, x_3, t) \\
&- \frac{e^2 n_0}{m} f(x_1, t) \int dx_3 \frac{\partial \phi_{23}}{\partial \vec{r}_2} \cdot \frac{\partial}{\partial \vec{v}_2} f_2(x_2, x_3, t) - \frac{e^2 n_0}{m} f(x_2, t) \int dx_3 \frac{\partial \phi_{13}}{\partial \vec{r}_1} \cdot \frac{\partial}{\partial \vec{v}_1} f_2(x_1, x_3, t) = 0
\end{aligned} \tag{7.11}$$

The neglect of three particle correlations is an <u>uncontrolled</u> approximation.

3. SPATIALLY HOMOGENEOUS PROBLEM

In the spatially homogeneous case $f(x, t) = f(\vec{v}, t)$, and the spatial dependence of $f_2(x_1, x_2, t)$ is through the difference $\vec{r}_1 - \vec{r}_2$. In the spatially homogeneous case (7.10) and (7.11) become

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} - \frac{e^2 n_0}{m}\int dx_2 \frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} f_2(x_1,x_2,t) = 0 \tag{7.12}$$

and

$$\left\{\frac{\partial}{\partial t} + \vec{v}_1\cdot\frac{\partial}{\partial \vec{r}_1} + \vec{v}_2\cdot\frac{\partial}{\partial \vec{r}_2} - \frac{e^2}{m}\frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} - \frac{e^2}{m}\frac{\partial \phi_{21}}{\partial \vec{r}_2}\cdot\frac{\partial}{\partial \vec{v}_2}\right\} f_2(x_1,x_2,t)$$

$$- \frac{e^2 n_0}{m}\frac{\partial f(\vec{v}_1,t)}{\partial \vec{v}_1}\cdot\int dx_3 \frac{\partial \phi_{13}}{\partial \vec{r}_1} f_2(x_2,x_3,t) - \frac{e^2 n_0}{m}\frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2}\cdot\int dx_3 \frac{\partial \phi_{23}}{\partial \vec{r}_2} f_2(x_1,x_3,t) \tag{7.13}$$

$$- \frac{e^2 n_0}{m} f(\vec{v}_1,t)\int dx_3 \frac{\partial \phi_{23}}{\partial \vec{r}_2}\cdot\frac{\partial}{\partial \vec{v}_2} f_2(x_2,x_3,t) - \frac{e^2 n_0}{m} f(\vec{v}_2,t)\int dx_3 \frac{\partial \phi_{13}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} f_2(x_1,x_3,t) = 0$$

Noting that the last two terms of (7.13) are proportional to the integral term in (7.12) we introduce the pair correlation function, (7.7),

$$f_2(x_1,x_2,t) = f(\vec{v}_1,t)\, f(\vec{v}_2,t) + P(x_1,x_2,t) \tag{7.14}$$

Inserting this into (7.13), P must be a solution of

$$\left\{\frac{\partial}{\partial t} + \vec{v}_1\cdot\frac{\partial}{\partial \vec{r}_1} + \vec{v}_2\cdot\frac{\partial}{\partial \vec{r}_2} - \frac{e^2}{m}\frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} - \frac{e^2}{m}\frac{\partial \phi_{21}}{\partial \vec{r}_2}\cdot\frac{\partial}{\partial \vec{v}_2}\right\} P(x_1,x_2,t)$$

$$- \left[\frac{e^2}{m}\frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} + \frac{e^2}{m}\frac{\partial \phi_{21}}{\partial \vec{r}_2}\cdot\frac{\partial}{\partial \vec{v}_2}\right] f(\vec{v}_1,t)\, f(\vec{v}_2,t) \tag{7.15}$$

$$- \frac{e^2 n_0}{m}\frac{\partial f(\vec{v}_1,t)}{\partial \vec{v}_1}\cdot\int dx_3 \frac{\partial \phi_{13}}{\partial \vec{r}_1} P(x_2,x_3,t) - \frac{e^2 n_0}{m}\frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2}\int dx_3 \frac{\partial \phi_{23}}{\partial \vec{r}_2} P(x_1,x_3,t) = 0$$

The Eq. (7.12) for the one particle distribution is then

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} - \frac{e^2 n_0}{m}\int dx_2 \frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} P(x_1,x_2,t) = 0 \tag{7.16}$$

Approximate solutions for $P(\vec{r}_1-\vec{r}_2,\vec{v}_1,\vec{v}_2,t)$ can be established in two cases. In Section 4 the small separation solution $P_{B.C.}$, for $|\vec{r}_1-\vec{r}_2| < R_L = e^2/\theta$, is shown to lead to the Boltzmann kinetic equation. In Section 5 we find that the large separation solution $P_{S.P.}$, for $|\vec{r}_1-\vec{r}_2| > R_D = \left(\frac{4\pi e^2 n_0}{\theta}\right)^{-1/2}$ leads to a kinetic equation that has recently attracted a great deal of attention.[20-22]

4. SMALL SEPARATION SOLUTION

When $|\vec{r}_1-\vec{r}_2| \lesssim R_L$ we expect the last two terms of (7.15) to play a negligible role compared to the terms containing the binary interaction ϕ_{12}. In this case we approximate the solution of (7.15) by $P_{B.C.}$ ($B.C.$ = binary collision) where

$$\left\{\frac{\partial}{\partial t}+\vec{v}_1\cdot\frac{\partial}{\partial \vec{r}_1}+\vec{v}_2\cdot\frac{\partial}{\partial \vec{r}_2}-\frac{e^2}{m}\frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1}-\frac{e^2}{m}\frac{\partial \phi_{21}}{\partial \vec{r}_2}\cdot\frac{\partial}{\partial \vec{v}_2}\right\}\left\{P_{B.C.}(x_1,x_2,t)-f(\vec{v}_1,t)f(\vec{v}_2,t)\right\} = 0 \tag{7.17}$$

The solution to this equation is obtained by integrating along the characteristics,

$$\left\{P_{B.C.}(x_1,x_2,t) - f(\vec{v}_1,t)f(\vec{v}_2,t)\right\}\Big|_{X_1,X_2,t-\tau}^{x_1,x_2,t} = 0$$

where X_1, X_2 are the coordinates at time $t-\tau$ that are equal to x_1, x_2 at time t. X_1, X_2 are determined from the equations of motion

$$\frac{d\vec{r}_1}{dt} = \vec{v}_1 \qquad \frac{d\vec{v}_1}{dt} = -\frac{e^2}{m}\frac{\partial\phi_{12}}{\partial\vec{r}_1}$$
$$\frac{d\vec{r}_2}{dt} = \vec{v}_2 \qquad \frac{d\vec{v}_2}{dt} = -\frac{e^2}{m}\frac{\partial\phi_{21}}{\partial\vec{r}_2} \tag{7.18}$$

with the initial-value conditions $x_1 = X_1$, $x_2 = X_2$ at time $t-\tau$. We have then

$$P_{B.C.}(x_1,x_2,t) = P_{B.C.}(X_1,X_2,t-\tau) + f(\vec{v}_1,t)f(\vec{v}_2,t) - f(\vec{V}_1,t-\tau)f(\vec{V}_2,t-\tau) \tag{7.19}$$

Up to this point we have carefully avoided the problem of boundary conditions. Now, however, we are confronted with precisely this problem; what value do we assign to $P_{B.C.}(X_1,X_2,t-\tau)$? Let us assume that there is no correlation between particles before they collide:

$$P_{B.C.}(X_1,X_2,t-\tau) \longrightarrow 0 \quad , \quad \tau \to +\infty \tag{7.20}$$

Note that through the boundary condition (7.20) a direction has been ascribed to time; it is this assumption which changes our previously reversible equation to an irreversible one. If τ is sufficiently large and positive the boundary condition (7.20) can be used to eliminate the first term on the right hand side of (7.19). The second term does not contribute to the integral in (7.16), and so the kinetic equation reduces to

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = -\frac{e^2 n_0}{m}\int dx_2 \frac{\partial\phi_{12}}{\partial\vec{r}_1}\cdot\frac{\partial}{\partial\vec{v}_1} f(\vec{V}_1,t-\tau)f(\vec{V}_2,t-\tau) \tag{7.21}$$

This does not have the required form of the kinetic equation (7.6) since the integral term is evaluated at time $t-\tau$. If $f(\vec{v},t)$ is sufficiently slowly varying, however, we can develop $f(\vec{v},t-\tau)$ in a Taylor series. Keeping only the first term we obtain

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = -\frac{e^2 n_0}{m}\int dx_2 \frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} f(\vec{v}_1,t)\, f(\vec{v}_2,t) \tag{7.22}$$

The next term in the Taylor series is proportional to the ratio τ/τ_f where τ is the duration of a collision, and τ_f is the characteristic time for changes in $f(\vec{v},t)$. Therefore (7.22) is valid if the distribution function $f(\vec{v},t)$ remains essentially constant during a time interval necessary for a collision to be completed.

Equation (7.22) can be cast into a more familiar form by employing a trick due to Bogoliubov.[19] The transformation is given in Appendix C; the result is

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = n_0\int d\vec{v}_2 \iint b\,db\,d\phi\, |\vec{v}_1-\vec{v}_2| \left\{ f(\vec{v}_1{}',t)\, f(\vec{v}_2{}',t) - f(\vec{v}_1,t)\, f(\vec{v}_2,t) \right\} \tag{7.23}$$

where $\vec{v}_1{}'$, $\vec{v}_2{}'$ are the velocities two particles would have before a Coulomb collision if they have the velocities $\vec{v}_1$, $\vec{v}_2$ afterwards. The right hand side of (7.23) is the Boltzmann collision integral.

We have shown that the small separation solution, $P_{B.C.}$, of (7.15) with the boundary condition (7.20) and the assumption that $f(\vec{v},t)$ is a slowly varying function of t reproduces the Boltzmann kinetic equation. Since the Boltzmann collision integral is based on the mechanism of binary collisions we know from Chapter II that divergences must occur at large impact parameters. This will be shown explicitly in the following chapter. For large separation, however, collective interactions are important and we must seek a new solution of (7.15).

5. LARGE SEPARATION SOLUTION

If $|\vec{r}_1 - \vec{r}_2| > R_D$ the direct interaction terms containing ϕ_{12} are negligible, but the integral terms in (7.15) must be retained. Denoting the solution in this "straight path" (S.P.) approximation by $P_{S.P.}$ the limiting form of (7.15) is

$$\left\{\frac{\partial}{\partial t}+\vec{v}_1\cdot\frac{\partial}{\partial \vec{r}_1}+\vec{v}_2\cdot\frac{\partial}{\partial \vec{r}_2}\right\}P_{S.P.}(x_1,x_2,t)-\frac{e^2}{m}\left[\frac{\partial\phi_{12}}{\partial\vec{r}_1}\cdot\frac{\partial}{\partial\vec{v}_1}+\frac{\partial\phi_{21}}{\partial\vec{r}_2}\cdot\frac{\partial}{\partial\vec{v}_2}\right]f(\vec{v}_1,t)f(\vec{v}_2,t)$$

(7.24)

$$-\frac{e^2 n_0}{m}\frac{\partial f(\vec{v}_1,t)}{\partial\vec{v}_1}\cdot\int dx_3\frac{\partial\phi_{13}}{\partial\vec{r}_1}P_{S.P.}(x_2,x_3,t)-\frac{e^2 n_0}{m}\frac{\partial f(\vec{v}_2,t)}{\partial\vec{v}_2}\cdot\int dx_3\frac{\partial\phi_{23}}{\partial\vec{r}_2}P_{S.P.}(x_1,x_3,t)=0$$

It should be noted that this equation can be obtained formally from the B-B-G-K-Y hierarchy of equations by developing the S-particle distribution functions in powers of a small parameter, $g=\frac{1}{n_0 R_D^3}=4\pi\frac{R_L}{R_D}$, proportional to the reciprocal of the number of particles in a "Debye sphere." An expansion in powers of g automatically truncates the hierarchy. To zeroth order in g the Landau-Vlasov equation (7.8) is obtained. To first order in g Eqs. (7.16) and (7.24) are reproduced.[19]

If in addition, one assumes that for times of interest the pair correlation function depends on time only through a functional dependence on $f(\vec{v},t)$, that is

$$P_{S.P.} = P_{S.P.}(x_1, x_2; f)$$

then

$$\frac{\partial P_{S.P.}}{\partial t} = \int d\vec{v}' \, \frac{\delta P_{S.P}(x_1,x_2;f)}{\delta f(\vec{v}',t)} \frac{\partial f(\vec{v}',t)}{\partial t} \tag{7.25}$$

is of order q^2 since both $P_{S.P.}$ and $\partial f/\partial t$ are of order q. Therefore, to order q , $\partial P_{S.P.}/\partial t$ can be dropped from (7.24), and we are left with

$$\left\{ \vec{v}_1 \cdot \frac{\partial}{\partial \vec{r}_1} + \vec{v}_2 \cdot \frac{\partial}{\partial \vec{r}_2} \right\} P_{S.P.}(x_1,x_2;f) - \frac{e^2}{m}\left[\frac{\partial \phi_{12}}{\partial \vec{r}_1} \cdot \frac{\partial}{\partial \vec{v}_1} + \frac{\partial \phi_{21}}{\partial \vec{r}_2} \cdot \frac{\partial}{\partial \vec{v}_2} \right] f(\vec{v}_1,t) f(\vec{v}_2,t)$$

$$- \frac{e^2 n_0}{m} \frac{\partial f(\vec{v}_1,t)}{\partial \vec{v}_1} \cdot \int dx_3 \frac{\partial \phi_{13}}{\partial \vec{r}_1} P_{S.P.}(x_2,x_3;f) - \frac{e^2 n_0}{m} \frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2} \cdot \int dx_3 \frac{\partial \phi_{23}}{\partial \vec{r}_2} P_{S.P.}(x_1,x_3;f) = 0 \tag{7.26}$$

As a boundary condition for $P_{S.P.}$ we assume there are no correlations before a collision, in agreement with (7.20),

$$P_{S.P.}(x_1, x_2; f) \longrightarrow 0$$

when

$$|\vec{r}_1 - \vec{r}_2| \longrightarrow \infty \quad , \quad (\vec{r}_1 - \vec{r}_2)\cdot(\vec{v}_1 - \vec{v}_2) < 0 \tag{7.27}$$

This is the equation considered by Guernsey,[20] Balescu,[21] and Lenard[22]. The solution has been treated in detail by these authors (the solution of a similar equation will be carried through in Chapter IX).

The solution of (7.26) with the boundary conditions (7.27) yields the kinetic equation

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = \frac{2n_0 e^4}{m^2}\frac{\partial}{\partial \vec{v}_1}\cdot\int d\vec{v}_2 \int d\vec{k}\, \frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_2-\vec{v}_1)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\cdot\left\{f(\vec{v}_2,t)\frac{\partial f(\vec{v}_1,t)}{\partial \vec{v}_1} - f(\vec{v}_1,t)\frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2}\right\} \tag{7.28}$$

where $\varepsilon(\vec{k},\omega)$ is the plasma dielectric constant (5.13) for the distribution $f(\vec{v},t)$,

$$\varepsilon(\vec{k},\omega) = 1 + \frac{\omega_p^2}{k^2}\int_C d\vec{v}\,\frac{\vec{k}\cdot\frac{\partial f(\vec{v},t)}{\partial \vec{v}}}{\omega - \vec{k}\cdot\vec{v}} \tag{7.29}$$

Equation (7.28) can also be case into the form of a Fokker-Planck equation

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = -\frac{\partial}{\partial \vec{v}_1}\cdot\left\{\left[\vec{A}(\vec{v}_1,t) + \frac{1}{2}\frac{\partial}{\partial \vec{v}_1}\cdot\overleftrightarrow{B}(\vec{v}_1,t)\right]f(\vec{v}_1,t)\right\} + \frac{1}{2}\frac{\partial}{\partial \vec{v}_1}\frac{\partial}{\partial \vec{v}_1}:\left\{\overleftrightarrow{B}(\vec{v}_1,t)\,f(\vec{v}_1,t)\right\} \tag{7.30}$$

where

$$\vec{A}(\vec{v}_1,t) = \frac{2n_0 e^4}{m^2}\int d\vec{v}_2 \int d\vec{k}\, \frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_2-\vec{v}_1)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\cdot\frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2} \tag{7.31}$$

$$\overleftrightarrow{B}(\vec{v}_1,t) = \frac{4n_0 e^4}{m^2}\int d\vec{v}_2 \int d\vec{k}\, \frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_2-\vec{v}_1)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\,f(\vec{v}_2,t) \tag{7.32}$$

The similarity of (7.30) with the Fokker-Planck equation extends only to its form of writing, since the coefficients (7.31) and (7.32) depend on $f(\vec{v},t)$. We shall refer to (7.30) as the Balescu, Guernsey, and Lenard equation (B-G-L equation).

The integrals (7.31) and (7.32) have a logarithmic divergence at large k which corresponds to the inappropriateness of $P_{S.P.}$ for small separations, $|\vec{r}_1-\vec{r}_2| \lesssim R_L$. In the next chapter we derive the kinetic equation (7.30)-(7.32) from another viewpoint which clearly demonstrates the nature of the divergence at short distances.

It will also be shown in the next chapter that when $\varepsilon(\vec{k},\omega)$ is replaced by 1 the kinetic equation (7.30) is identical to the limiting form of the Boltzmann equation (7.23) for large impact parameters (or small k). The appearance of $\varepsilon(\vec{k},\omega)$ is due to the inclusion of collective interactions, and modifies the simple binary collision result to eliminate the logarithmic divergence for large impact parameters (small k).

To summarize, the small separation solution of (7.15) leads to the Boltzmann kinetic equation which is based on the mechanism of binary collisions. This kinetic equation has a logarithmic divergence at large impact parameters. The large separation solution, or straight path approximation, of (7.15) includes collective effects, and leads to a kinetic equation which diverges when extended to small impact parameters. Clearly, a complete solution of (7.15) that includes terms dominant for both large and small separations would give a convergent kinetic equation. This observation has been made by Tchen[23] who formally integrated the pair correlation equation (7.15), and estimated the order of the various terms contributing to the kinetic equation (7.16).

6. ADDITIONAL KINETIC EQUATIONS

There recently have appeared two papers concerned with the elimination of the divergences occurring in the electron plasma kinetic equation. The first of these, due to Willis,[24] attempts to obtain a solution for $f_2(x_1, x_2, t)$ by employing an expansion about equilibrium. Willis considers the deviation from equilibrium to be an additive correction to the equilibrium state,

$$f_2 = f_{2_{equil.}} + f_2'$$

and proceeds to calculate f_2' (to first order in the plasma parameter $g = 4\pi R_L/R_D$) for the spatially homogeneous electron plasma. The entire analysis presented by Willis contains a number of errors, the most significant is the assertion that his term [3d'] is zero for the spatially homogeneous case. Indeed, his term [3d'] is not the Fourier transform of [3d] as stated. In fact, the term [3d] is proportional to the right hand side of his kinetic equation [2a], it cannot be zero if he is to have a non-trivial kinetic equation. When the appropriate corrections are made one discovers that this term gives the dominant contribution to f_2' . The kinetic equation determined from the correct f_2' does <u>not</u> converge.

The second paper due to Baldwin[25] exhibits a kinetic equation that contains both the large and small separation features of (7.17) and (7.24). Baldwin proposes to solve the f_2 equation (7.5) by a modified expansion in the plasma parameter. The essence of this method is to add and subtract a shielded potential ϕ_{12}^{λ}, λ denoting the shielding distance, to the pure Coulomb potential ϕ_{12},

$$\psi_{12} = \phi_{12}^{\lambda} + (\phi_{12} - \phi_{12}^{\lambda})$$

The first term in ψ_{12} is assumed to be of zeroth order in g while the second term is first order. The requirement that ϕ_{12}^{λ} reduces to ϕ_{12} for small separations between particles 1 and 2 means that $(\phi_{12} - \phi_{12}^{\lambda})$ vanishes at small separations, and can therefore be treated as first order in g for all values of $|\vec{r}_1 - \vec{r}_2|$. The remaining pair potentials ϕ_{1i}, ϕ_{2i} ($i \neq 1,2$) are cutoff below a small separation distance b_0, e.g.,

$$\phi_{1i} = \begin{cases} \dfrac{1}{|\vec{r}_1 - \vec{r}_2|} & , |\vec{r}_1 - \vec{r}_2| > b_0 \\[2ex] \dfrac{1}{b_0} & , |\vec{r}_1 - \vec{r}_2| \leq b_0 \end{cases} \tag{7.33}$$

Baldwin develops a solution for f_2 in powers of g, and obtains a convergent kinetic equation with two collision terms. The first term is a Boltzmann collision integral for the shielded potential ϕ_{12}^{λ}, and the second term is a B-G-L like term (7.28) in which $|\underline{k}|^{-2}$ is replaced by

$$\frac{1}{|\varepsilon|^2} - \frac{[\hat{\phi}^\lambda(k)]^2}{[\hat{\phi}(k)]^2}$$

the latter going to zero as $k \to \infty$. The specification of ϕ_{12}^λ is quite arbitrary, one could choose for example

$$\phi_{12}^\lambda(r) = \begin{cases} 1/r & , r < \lambda \\ 0 & , r > \lambda \end{cases}$$

The Baldwin procedure for obtaining the kinetic equation is analogous to the Vlasov method of combining the binary collision and continuous fluid pictures to obtain a solution of the test particle problem (Chapter III). In the test particle problem the short and long range solutions were joined at a distance intermediate between R_L and R_D from the test particle; Fig. 2, region II. In the Baldwin method the analogous procedure assumes that Ψ_{12} can be separated into two parts, ϕ_{12}^λ of zeroth order in g and ($\phi_{12}-\phi_{12}^\lambda$) of order g. The reason that the potentials (7.33) must be introduced with a cutoff b_0 is that certain integrals arise in the Baldwin method which logarithmically diverge as $b_0 \to 0$. In deriving his final kinetic equation Baldwin finds that these divergent integrals have cancelled one another.

The asymmetric treatment given the potentials ϕ_{12} and ϕ_{1i}, $(i \neq 1,2)$, and the cancelling of divergent integrals leaves this method with an artifical flavor. A consistent derivation of the spatially homogeneous electron plasma kinetic equation is not given by any of the methods reported in this chapter.

CHAPTER VIII

FURTHER DISCUSSION OF THE APPROXIMATE KINETIC EQUATIONS

1. INTRODUCTION

In this chapter we show more clearly the relationship between the Boltzmann equation, the Fokker-Planck equation, and the B-G-L equation (7.30).

In Section 2 the Boltzmann equation (7.23) is cast into the form of a generalized Fokker-Planck equation. The first two Fokker-Planck coefficients, the friction and diffusion coefficients, are calculated for the case of Coulomb interactions. In Section 3 an expansion of the coefficients is made in powers of the momentum transfer per binary collision. This expansion gives a generalization of a kinetic equation developed by Landau.[27]

Section 4 discusses in detail a straight path approximation of the Fokker-Planck coefficients. This gives an alternative, and perhaps more "physical," derivation of the B-G-L equation. The connection between the B-G-L equation and the Landau equation is immediately obvious. Furthermore, we will see that all of these kinetic equations suffer from the same logarithmic divergences encountered in the test particle calculations of Chapters II and V.

2. REDUCTION OF BOLTZMANN'S EQUATION TO A GENERALIZED FOKKER-PLANCK EQUATION

The Boltzmann equation (7.23) is

$$\frac{\partial f(\vec{v},t)}{\partial t} = n_0 \int d\vec{v}_1 \int d\sigma\, |\vec{v}-\vec{v}_1| \left\{ f(\vec{v}',t)\, f(\vec{v}_1',t) - f(\vec{v},t)\, f(\vec{v}_1,t) \right\} \tag{8.1}$$

where $d\sigma$ is the differential cross section for the binary collision

$$\vec{v}' \longrightarrow \vec{v}$$
$$\vec{v}_1' \longrightarrow \vec{v}_1$$

It is more convenient to express the collision in terms of the momentum transfer $\vec{q}$,

$$\begin{aligned} \vec{v} - \vec{q}/m &\longrightarrow \vec{v} \\ \vec{v}_1 + \vec{q}/m &\longrightarrow \vec{v}_1 \end{aligned} \tag{8.2}$$

where we are assuming the particles have equal masses. Furthermore, we can use the collision volume of Chapter II,

$$\omega\left(\vec{v}-\vec{q}/m, \vec{v}_1+\vec{q}/m \,\middle|\, \vec{q}/m, -\vec{q}/m\right) d\vec{q} \tag{8.3}$$

where it will be recalled that the arguments of ω indicate the binary collision (8.2), to rewrite the Boltzmann equation as

$$\frac{\partial f(\vec{v},t)}{\partial t} = n_0 \int d\vec{v}_1 \int d\vec{q}\, \omega\left(\vec{v}-\vec{q}/m, \vec{v}_1+\vec{q}/m \,\middle|\, \vec{q}/m, -\vec{q}/m\right) \left\{ f(\vec{v}-\vec{q}/m,t)\, f(\vec{v}_1+\vec{q}/m,t) - f(\vec{v},t)\, f(\vec{v}_1,t) \right\} \tag{8.4}$$

If in (8.4) we formally expand $\omega\left(\vec{v}-\vec{q}/m, \vec{v}_1+\vec{q}/m \,\middle|\, \vec{q}/m, -\vec{q}/m\right) f(\vec{v}-\vec{q}/m,t)$ as a function of $\vec{v}-\vec{q}/m$, about the point $\vec{v}$, we find

$$\frac{\partial f(\vec{v},t)}{\partial t} = \sum_{\ell=0}^{\infty} \frac{(-)^{\ell}}{\ell!} \frac{\partial^{\ell}}{\partial \vec{v} \cdots \partial \vec{v}} n_0 \int d\vec{v}_1 \int d\vec{q}\, \underbrace{\frac{\vec{q}}{m} \cdots \frac{\vec{q}}{m}}_{\ell}\, \omega\left(\vec{v}, \vec{v}_1 + \frac{\vec{q}}{m} \,\middle|\, \frac{\vec{q}}{m}, -\frac{\vec{q}}{m}\right) f(\vec{v},t) f\left(\vec{v}_1 + \frac{\vec{q}}{m}, t\right) \tag{8.5}$$

$$- n_0 \int d\vec{v}_1 \int d\vec{q}\, \omega\left(\vec{v}, \vec{v}_1 \,\middle|\, -\frac{\vec{q}}{m}, \frac{\vec{q}}{m}\right) f(\vec{v},t) f(\vec{v}_1,t)$$

where (2.16) has been used to rewrite the last term. The $\ell = 0$ term cancels with the last term when one makes the change of variables $\vec{v}_1 \rightarrow \vec{v}_1 - \vec{q}/m$ and $\vec{q} \rightarrow -\vec{q}$. Making the change of variables $\vec{v}_1 \rightarrow \vec{v}_1 - \vec{q}/m$ in the remaining terms yields

$$\frac{\partial f(\vec{v},t)}{\partial t} = \sum_{\ell=1}^{\infty} \frac{(-)^{\ell}}{\ell!} \frac{\partial^{\ell}}{\partial \vec{v} \cdots \partial \vec{v}} \left\{ f(\vec{v},t) \frac{1}{m^{\ell}} \langle \underbrace{\vec{q} \cdots \vec{q}}_{\ell} \rangle \right\} \tag{8.6}$$

where the coefficients are given by

$$\frac{1}{m^{\ell}} \langle \underbrace{\vec{q} \cdots \vec{q}}_{\ell} \rangle = n_0 \int d\vec{v}_1\, f(\vec{v}_1,t) \int d\vec{q}\, \omega\left(\vec{v}, \vec{v}_1 \,\middle|\, \frac{\vec{q}}{m}, -\frac{\vec{q}}{m}\right) \underbrace{\frac{\vec{q}}{m} \cdots \frac{\vec{q}}{m}}_{\ell} \tag{8.7}$$

Equation (8.6) has the form of a generalized Fokker-Planck equation.[26] This is a "generalization" of the ordinary Fokker-Plank equation because it includes velocity derivatives of all orders. In addition the Fokker-Planck coefficients are not constants but functionals of $f(\vec{v},t)$. In other words, the non-linear character of the Boltzmann equation (8.4) is retained in the Fokker-Planck equation through the coefficients (8.7) which are linear integral functions of $f(\vec{v},t)$

The generalized Fokker-Planck equation is most useful when the friction coefficient $\frac{1}{m}\langle\vec{q}\rangle$ and the diffusion coefficient $\frac{1}{m^2}\langle\vec{q}\vec{q}\rangle$ are so small that higher order coefficients can be neglected. That is, when the average force on a particle, $\langle\vec{q}\rangle$, is small the Boltzmann equation can be approximated by

$$\frac{\partial f(\vec{v},t)}{\partial t} = -\frac{\partial}{\partial\vec{v}}\cdot\left\{f(\vec{v},t)\frac{1}{m}\langle\vec{q}\rangle\right\} + \frac{1}{2}\frac{\partial^2}{\partial\vec{v}\partial\vec{v}}:\left\{f(\vec{v},t)\frac{1}{m^2}\langle\vec{q}\vec{q}\rangle\right\} \tag{8.8}$$

To apply (8.8) to an electron plasma we use the Coulomb collision volume (2.19). In that case

$$\omega(\vec{v},\vec{v}_1|\tfrac{\vec{q}}{m},-\tfrac{\vec{q}}{m})\,d\vec{q} = \frac{4e^4}{q^4}\,\delta\left[\vec{q}\cdot(\vec{v}-\vec{v}_1)+q^2/m\right]d\vec{q} \tag{8.9}$$

The Fokker-Planck coefficients are

$$\frac{1}{m^l}\langle\underbrace{\vec{q}\cdots\vec{q}}_{l}\rangle = \frac{4e^4n_0}{m^l}\int d\vec{v}_1 f(\vec{v}_1,t)\int d\vec{q}\,\frac{\overbrace{\vec{q}\cdots\vec{q}}^{l}}{q^4}\,\delta\left[\vec{q}\cdot(\vec{v}-\vec{v}_1)+q^2/m\right] \tag{8.10}$$

The friction coefficient has already been encountered in the binary collision treatment of the test particle problem. In that case we calculated $\langle\vec{q}\rangle$, the average force on a particle moving with a velocity $\vec{v}$, (2.24). The integration has a logarithmic divergence that can be approximated by cutting the integration off at a maximum impact parameter b_M (or minimum q) yielding

$$\frac{1}{m}\langle\vec{q}\rangle = -\frac{8\pi e^4 n_0}{m^2}\int d\vec{v}_1 f(\vec{v}_1,t)\frac{\vec{v}-\vec{v}_1}{|\vec{v}-\vec{v}_1|^3}\ln\left[\frac{mb_M}{2e^2}|\vec{v}-\vec{v}_1|^2\right]$$

Approximating $|\vec{v}-\vec{v}_1|^2$ in the logarithm argument by $6\theta/m$, which has little effect on the integral because of the insensitivity of the logarithm,

$$\frac{1}{m}\langle\vec{q}\rangle = -\frac{4\pi e^4 n_0}{m^2}\ln\left[\frac{3b_M\theta}{e^2}\right]^2\int d\vec{v}_1 f(\vec{v}_1,t)\frac{\vec{v}-\vec{v}_1}{|\vec{v}-\vec{v}_1|^3} \tag{8.11}$$

A similar approximation can be made for the diffusion coefficient which also diverges logarithmically. The dominant portion of this approximation when $b_M \gg e^2/\theta$ is

$$\frac{1}{m^2}\langle\vec{q}\vec{q}\rangle = \frac{4\pi e^4 n_0}{m^2}\ln\left[\frac{3b_M\theta}{e^2}\right]\int d\vec{v}_1 f(\vec{v}_1,t)\frac{1}{|\vec{v}-\vec{v}_1|}\left\{\mathbb{1}-\hat{e}\hat{e}\right\} \tag{8.12}$$

where $\mathbb{1}$ is the unit tensor, $\hat{e}=\dfrac{\vec{v}-\vec{v}_1}{|\vec{v}-\vec{v}_1|}$.

The Fokker-Planck equation (8.8) with the coefficients (8.11) and (8.12) has been investigated by Rosenbluth, MacDonald, and Judd.[4] These authors assume a maximum impact parameter equal to the Debye length,

$$b_M = R_D = \left(\frac{4\pi e^2 n_0}{\theta}\right)^{-1/2}$$

3. EXPANSION FOR SMALL MOMENTUM TRANSFER

If the average momentum transfer, $\langle\vec{q}\rangle$, is small then one expects the Boltzmann equation to be adequately approximated by the Fokker-Planck equation. Note, however, that only the average momentum transfer in a collision must be small and not that all collisions involve small momentum transfers.

Another approximation of the Boltzmann equation has been carried out by L. Landau.[27] Landau expands the Boltzmann collision integral in powers of $\vec{q}$, and retains only the lowest order non-vanishing terms which he writes in the Fokker-Planck form. This expansion is not equivalent to (8.6) as it involves an expansion of the collision cross section as well as the distribution functions. To obtain Landau's result it is necessary to introduce his expression for the "collision volume," which we denote by ω_L,

$$\omega_L\left(\vec{v}+\frac{1}{2}\frac{\vec{q}}{m},\vec{v}_1-\frac{1}{2}\frac{\vec{q}}{m}\middle|\frac{\vec{q}}{m},-\frac{\vec{q}}{m}\right)=\omega\left(\vec{v},\vec{v}_1\middle|\frac{\vec{q}}{m},-\frac{\vec{q}}{m}\right) \tag{8.13}$$

Since the "collision volume" is invariant under a uniform translation of the center of mass of the colliding particles, (2.16), we have

$$\omega_L\left(\vec{v}+\frac{1}{2}\frac{\vec{q}}{m},\vec{v}_1-\frac{1}{2}\frac{\vec{q}}{m}\middle|\frac{\vec{q}}{m},-\frac{\vec{q}}{m}\right)=\omega_L\left(\vec{v},\vec{v}_1-\vec{q}/m\middle|\vec{q}/m,-\vec{q}/m\right) \tag{8.14}$$

Using (8.13) and (8.14) the Fokker-Planck coefficients (8.7) are

$$\frac{1}{m^l}\langle\underbrace{\vec{q}\cdots\vec{q}}_{l}\rangle=n_0\int d\vec{v}_1\int d\vec{q}\,\omega_L\left(\vec{v},\vec{v}_1-\vec{q}/m\middle|\vec{q}/m,-\vec{q}/m\right)f(\vec{v}_1,t)\underbrace{\frac{\vec{q}}{m}\cdots\frac{\vec{q}}{m}}_{l}$$

Changing variables $\vec{v}_1\longrightarrow\vec{v}_1+\vec{q}/m$,

$$\frac{1}{m^l}\langle\underbrace{\vec{q}\cdots\vec{q}}_{l}\rangle=n_0\int d\vec{v}_1\int d\vec{q}\,\omega_L\left(\vec{v},\vec{v}_1\middle|\vec{q}/m,-\vec{q}/m\right)f(\vec{v}_1+\vec{q}/m,t)\underbrace{\frac{\vec{q}}{m}\cdots\frac{\vec{q}}{m}}_{l} \tag{8.15}$$

Expanding $f(\vec{v}_1+\vec{q}/m,t)$ in a Taylor series, and integrating by parts with the relation $\partial\omega_L/\partial\vec{v}_1=-\partial\omega_L/\partial\vec{v}$, which follows from (2.16), Eq. (8.15) reduces to

$$\frac{1}{m^\ell}\langle\underbrace{\vec{q}\cdots\vec{q}}_{\ell}\rangle = \sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial^k}{\partial\vec{v}\cdots\partial\vec{v}}n_0\int d\vec{v}_1 f(\vec{v}_1,t)\int d\vec{q}\,\omega_L(\vec{v},\vec{v}_1|\tfrac{\vec{q}}{m},-\tfrac{\vec{q}}{m})\underbrace{\frac{\vec{q}}{m}\cdots\frac{\vec{q}}{m}}_{\ell+k} \tag{8.16}$$

However, from (8.13) and (2.16) we observe that

$$\omega_L(\vec{v},\vec{v}_1|\vec{q}/m,-\vec{q}/m) = \omega_L(\vec{v},\vec{v}_1|-\vec{q}/m,\vec{q}/m)$$

Thus, $\omega_L(\vec{v},\vec{v}_1|\vec{q}/m,-\vec{q}/m)$ is an even function of $\vec{q}$; hence, only those terms with $\ell+k$ even will remain in (8.16),

$$\frac{1}{m^\ell}\langle\underbrace{\vec{q}\cdots\vec{q}}_{\ell}\rangle = \begin{cases} \displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k)!}\frac{\partial^{2k}}{\partial\vec{v}\cdots\partial\vec{v}}\cdot\vec{B}_{\ell+2k} & \text{if } \ell \text{ is even} \\ \displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\frac{\partial^{2k+1}}{\partial\vec{v}\cdots\partial\vec{v}}\cdot\vec{B}_{\ell+2k+1} & \text{if } \ell \text{ is odd} \end{cases} \tag{8.17}$$

where

$$\vec{B}_\ell = n_0\int d\vec{v}_1 f(\vec{v}_1,t)\int d\vec{q}\,\omega_L(\vec{v},\vec{v}_1|\tfrac{\vec{q}}{m},-\tfrac{\vec{q}}{m})\underbrace{\frac{\vec{q}}{m}\cdots\frac{\vec{q}}{m}}_{\ell} \tag{8.18}$$

This is the extension of Landau's Fokker-Planck coefficients to arbitrary ℓ . Landau retains only the lowest order coefficient $\overleftrightarrow{B}_2$ so that his kinetic equation is written

$$\frac{\partial f(\vec{v},t)}{\partial t} = -\frac{\partial}{\partial\vec{v}}\cdot\left\{f(\vec{v},t)\frac{\partial}{\partial\vec{v}}\cdot\overleftrightarrow{B}_2\right\} + \frac{1}{2}\frac{\partial^2}{\partial\vec{v}\partial\vec{v}}:\left\{f(\vec{v},t)\overleftrightarrow{B}_2\right\} \tag{8.19}$$

For Coulomb interactions the Landau coefficients can be obtained from expressions (8.9), (8.13), and (8.18)

$$\frac{1}{m}\langle\vec{q}\rangle = \frac{\partial}{\partial\vec{v}}\cdot\overleftrightarrow{B}_2 \qquad \frac{1}{m^2}\langle\vec{q}\vec{q}\rangle = \overleftrightarrow{B}_2 \tag{8.20}$$

$$\overleftrightarrow{B}_2 = \frac{4e^4 n_0}{m^2}\int d\vec{v}_1 \int d\vec{q}\, \frac{\delta[\vec{q}\cdot(\vec{v}-\vec{v}_1)]}{q^4}\,\vec{q}\vec{q}\, f(\vec{v}_1,t) \tag{8.21}$$

The $\vec{q}$ integration in (8.21) diverges at both large and small values of q . To obtain a finite result the range of integration must be limited to the finite interval (q_{min} , q_{max}),

$$\overleftrightarrow{B}_2 = \frac{4\pi e^4 n_0}{m^2}\ln\left(\frac{q_{max}}{q_{min}}\right)\int d\vec{v}_1 f(\vec{v}_1,t)\frac{1}{|\vec{v}-\vec{v}_1|}\left\{\mathbb{1}-\hat{e}_1\hat{e}_1\right\} \tag{8.22}$$

where $\mathbb{1}$ is the unit tensor and $\hat{e}_1 = \frac{\vec{v}-\vec{v}_1}{|\vec{v}-\vec{v}_1|}$. The divergence at small q corresponds to the large impact parameter divergence encountered in (8.12) and is a consequence of the long range $1/r$ potential. This is the same divergence that arose in connection with the binary collision treatment of the test particle problem in Chapter II. In the next section this low q divergence will be eliminated when we allow for collective plasma interactions. The large q divergence is not surprising since the expansion that led to (8.21) was an expansion in powers of q . In the next section it is shown that the Landau Fokker-Planck coefficients (8.20) are equivalent to straight path approximations of the Coulomb Fokker-Planck coefficients (8.11) and (8.12). In the straight path calculation colliding particles are assumed to move along straight trajectories with constant velocities. Under this assumption close collisions have unbounded momentum transfers, and the result is the large q divergence of (8.22).

4. STRAIGHT PATH APPROXIMATION

In a binary collision, in the limit of vanishing momentum transfer, the colliding particles move along straight trajectories with constant velocities. The use of the straight path, constant velocity, picture is known as the "straight path approximation." It will now be shown that (8.20) is the straight path approximation of the Fokker-Planck coefficients (8.11) and (8.12). However, instead of making the straight path approximation directly for pure Coulomb potentials it is instructive to consider the following generalization. We know from Chapters IV and V that the distant field of a test particle in a plasma is dielectrically screened. Since the straight path approximation is good for distant collisions, and since it assumes the colliding particles maintain constant velocities (the test particle assumption), it is natural to attempt the straight path calculation using dielectrically screened Coulomb potentials. It will be found that the Fokker-Planck kinetic equation derived in this way is precisely the B-G-L equation (7.30)-(7.32). Replacement of the dielectric constant $\varepsilon(\vec{k},\omega)$ by 1 corresponds to pure Coulomb interactions, and results in the Fokker-Planck kinetic equation (8.19)-(8.21).

To be explicit we consider the binary collision

$$\vec{v}_1 \rightarrow \vec{v}_1 + \vec{q}/m$$
$$\vec{v}_2 \rightarrow \vec{v}_2 - \vec{q}/m$$

between two particles of charge $-e$ and mass m. The momentum transfer, $\vec{q}$, is developed in powers of a small parameter* ϵ,

$$\vec{q} = \epsilon \vec{q}_1 + \epsilon^2 \vec{q}_2 + \cdots$$

where the first order term is the momentum transfer calculated under the assumption that the colliding particles move in straight paths with constant velocities,

$$\vec{q}_1 = -e \int_{-\infty}^{\infty} dt \, \vec{E}_{12}[\vec{r}(t)] \tag{8.23}$$

Here the electric field at particle 1 due to particle 2 is denoted by $\vec{E}_{12}$, and in the straight path limit the radius vector is (see Fig. 7)

$$\vec{r}(t) = \vec{b} + (\vec{v}_1 - \vec{v}_2)t \tag{8.24}$$

The impact parameter vector $\vec{b}$ is perpendicular to $\vec{v}_1 - \vec{v}_2$.

*The parameter ϵ is used to keep track of the order of the terms and will eventually be set equal to 1; ϵ is proportional to the scattering angle in the relative coordinate system.

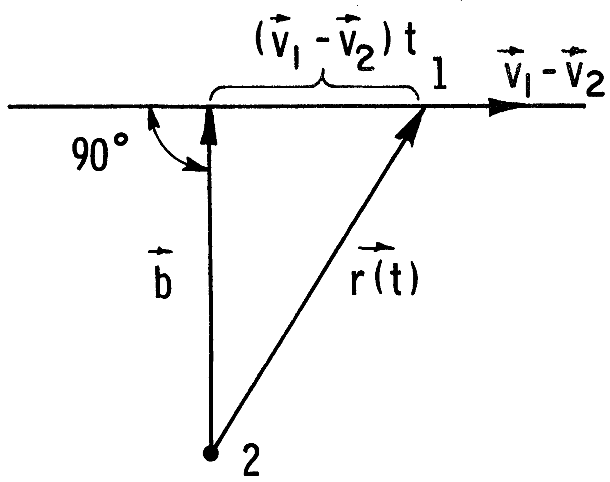

Fig. 7. Picture of straight path collision in rest system of particle 2.

It is convenient to introduce the Fourier representation of $\vec{E}(\vec{r})$,

$$\vec{E}(\vec{r}) = \frac{1}{(2\pi)^3}\int d\vec{k}\,\tilde{\vec{E}}(\vec{k})\,e^{i\vec{k}\cdot\vec{r}}$$

Putting this and (8.24) in (8.23) gives

$$\begin{aligned}\vec{q}_1 &= -\frac{e}{(2\pi)^3}\int d\vec{k}\,\tilde{\vec{E}}(\vec{k})\,e^{i\vec{k}\cdot\vec{b}}\int_{-\infty}^{\infty} dt\, e^{i\vec{k}\cdot(\vec{v}_1-\vec{v}_2)t} \\ &= -\frac{e}{(2\pi)^2}\int d\vec{k}\,\tilde{\vec{E}}(\vec{k})\,e^{i\vec{k}\cdot\vec{b}}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]\end{aligned} \tag{8.25}$$

To obtain the second order contribution to the momentum $\vec{q}$ we observe that for energy to be conserved in the collision requires

$$(\vec{v}_1 + \vec{q}/m)^2 + (\vec{v}_2 - \vec{q}/m)^2 = v_1^2 + v_2^2$$

or

$$\vec{q}\cdot(\vec{v}_1-\vec{v}_2) + \frac{1}{m}q^2 = 0 \tag{8.26}$$

To first order in ϵ (8.26) is

$$\vec{q}_1 \cdot (\vec{v}_1 - \vec{v}_2) = 0$$

According to (8.25) this condition is satisfied since $\vec{\tilde{E}}(\vec{k}) \propto \vec{k}$ for a point charge. The second order terms in (8.26) are

$$\vec{q}_2 \cdot (\vec{v}_1 - \vec{v}_2) + \frac{1}{m} q_1^2 = 0$$

which determines the component of $\vec{q}_2$ parallel to $\vec{v}_1 - \vec{v}_2$. The component of $\vec{q}_2$ perpendicular to $\vec{v}_1 - \vec{v}_2$ is of order ϵ^3. Therefore, to order ϵ^2, the momentum exchange in the straight path approximation is

$$\vec{q} = \vec{q}_1 - \frac{1}{m} q_1^2 \frac{\vec{v}_1 - \vec{v}_2}{|\vec{v}_1 - \vec{v}_2|^2} \tag{8.27}$$

The Fourier transform for the electric field of a point charge $-e$ in a dielectric medium, and moving with constant velocity $\vec{v}_2$, is obtained from (4.8)

$$\vec{\tilde{E}}(\vec{k}) = \frac{4\pi e i \vec{k}}{k^2 \varepsilon(\vec{k}, \vec{k}\cdot\vec{v}_2)} \tag{8.28}$$

Putting (8.28) in (8.25) gives for the lowest order momentum transfer

$$\vec{q}_1 = -\frac{4\pi e^2 i}{(2\pi)^2} \int d\vec{k} \, \frac{\vec{k} \, e^{i\vec{k}\cdot\vec{b}} \, \delta[\vec{k}\cdot(\vec{v}_1 - \vec{v}_2)]}{k^2 \varepsilon(\vec{k}, \vec{k}\cdot\vec{v}_2)} \tag{8.29}$$

Using (8.29) in (8.27) will give the momentum transfer to order ϵ^2.

The Fokker-Planck coefficients are

$$\frac{1}{m}\langle \vec{q} \rangle \qquad\qquad \frac{1}{m^2}\langle \vec{q}\,\vec{q} \rangle \tag{8.30}$$

The average $\langle \cdots \rangle$ of an arbitrary function of $\vec{q}$, say $T(\vec{q})$, is over the number of binary collisions per second,

$$\langle T(\vec{q}) \rangle = n_0 \int d\vec{v}_2 \int b\,db \int d\phi \; |\vec{v}_1 - \vec{v}_2| \, f(\vec{v}_2, t) \, T(\vec{q}) \tag{8.31}$$

The two dimensional integral $b\,db\,d\phi$ is over the plane perpendicular to $\vec{v}_1 - \vec{v}_2$, and can be written as an integral over all $\vec{b}$

$$b\,db\,d\phi = |\vec{v}_1 - \vec{v}_2| \, \delta\left[\vec{b}\cdot(\vec{v}_1 - \vec{v}_2)\right] d\vec{b} \tag{8.32}$$

Combining (8.27) and (8.29) with (8.30) and (8.31) gives for the Fokker-Planck coefficients to order ϵ^2,

$$\frac{1}{m}\langle \vec{q} \rangle_{S.P.} = \frac{n_0}{m} \int d\vec{v}_2 \, f(\vec{v}_2, t) \int d\vec{b} \; |\vec{v}_1 - \vec{v}_2|^2 \, \delta\left[\vec{b}\cdot(\vec{v}_1 - \vec{v}_2)\right] \left\{ \vec{q}_1 - \frac{q_1^2}{m} \frac{\vec{v}_1 - \vec{v}_2}{|\vec{v}_1 - \vec{v}_2|^2} \right\} \tag{8.23}$$

$$\frac{1}{m^2}\langle \vec{q}\,\vec{q} \rangle_{S.P.} = \frac{n_0}{m} \int d\vec{v}_2 \, f(\vec{v}_2, t) \int d\vec{b} \; |\vec{v}_1 - \vec{v}_2|^2 \, \delta\left[\vec{b}\cdot(\vec{v}_1 - \vec{v}_2)\right] \vec{q}_1 \vec{q}_1 \tag{8.34}$$

The subscript " S. P. " indicates the straight path approximation.

To calculate a representative term consider $\langle \vec{q}_1 \vec{q}_1 \rangle$ where $\vec{q}_1$ is given by (8.29)

$$\langle \vec{q}_1 \vec{q}_1 \rangle_{S.P.} = -\frac{e^4 n_0}{\pi^2} \int d\vec{v}_2 f(\vec{v}_2,t) |\vec{v}_1-\vec{v}_2|^2 \int d\vec{b}\, \delta[\vec{b}\cdot(\vec{v}_1-\vec{v}_2)] \times$$

$$\times \int d\vec{k}_1 \int d\vec{k}_2 \frac{\vec{k}_1 \vec{k}_2 e^{i\vec{b}\cdot(\vec{k}_1+\vec{k}_2)} \delta[\vec{k}_1\cdot(\vec{v}_1-\vec{v}_2)]\delta[\vec{k}_2\cdot(\vec{v}_1-\vec{v}_2)]}{k_1^2 k_2^2\, \varepsilon(\vec{k}_1,\vec{k}_1\cdot\vec{v}_2)\varepsilon(\vec{k}_2,\vec{k}_2\cdot\vec{v}_2)} \tag{8.35}$$

The $\vec{k}$ and $\vec{b}$ integrations are performed by separating the vectors into components parallel and perpendicular to $\vec{v}_1-\vec{v}_2$, e.g., $k_\parallel$, $\vec{k}_\perp$. Making use of the delta functions (8.35) reduces to

$$\langle \vec{q}_1 \vec{q}_1 \rangle_{S.P.} = -\frac{e^4 n_0}{\pi^2} \int d\vec{v}_2 f(\vec{v}_2,t) \frac{1}{|\vec{v}_1-\vec{v}_2|} \int d\vec{b}_\perp \int d\vec{k}_{\perp 1} \int d\vec{k}_{\perp 2} \frac{\vec{k}_{\perp 1} \vec{k}_{\perp 2} e^{i\vec{b}_\perp\cdot(\vec{k}_{\perp 1}+\vec{k}_{\perp 2})}}{k_{1\perp}^2 k_{2\perp}^2 \varepsilon(\vec{k}_{\perp 1},\vec{k}_{\perp 1}\cdot\vec{v}_2)\varepsilon(\vec{k}_{\perp 2},\vec{k}_{\perp 2}\cdot\vec{v}_2)} \tag{8.36}$$

The two dimensional integral over $\vec{b}_\perp$ yields $(2\pi)^2\delta[\vec{k}_{\perp 1}+\vec{k}_{\perp 2}]$, so that the $\vec{k}_{2\perp}$ integration is trivial and results in

$$\langle \vec{q}_1 \vec{q}_1 \rangle_{S.P.} = 4e^4 n_0 \int d\vec{v}_2 f(\vec{v}_2,t) \frac{1}{|\vec{v}_1-\vec{v}_2|} \int d\vec{k}_\perp \frac{\vec{k}_\perp \vec{k}_\perp}{k_\perp^4 |\varepsilon(\vec{k}_\perp,\vec{k}_\perp\cdot\vec{v}_2)|^2} \tag{8.37}$$

We have used the fact that $\varepsilon^*(\vec{k},\vec{k}\cdot\vec{v}_2) = \varepsilon(-\vec{k},-\vec{k}\cdot\vec{v}_2)$. The result (8.37) is more conveniently written with an integration over all $\vec{k}$,

$$\langle \vec{q}_1 \vec{q}_1 \rangle_{S.P.} = 4e^4 n_0 \int d\vec{v}_2 f(\vec{v}_2,t) \int d\vec{k} \frac{\vec{k}\vec{k}\, \delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^4 |\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \tag{8.38}$$

The remaining terms in (8.33) and (8.34) can be similarly reduced, and one finds for the Fokker-Planck coefficients in the straight path approximation,

$$\langle \vec{q}_1 \rangle_{S.P.} = 0$$

$$\frac{1}{m}\langle \vec{q} \rangle_{S.P.} = \frac{1}{m}\left\langle -\frac{q_1^2}{m} \frac{\vec{v}_1-\vec{v}_2}{|\vec{v}_1-\vec{v}_2|^2} \right\rangle = -\frac{4e^4 n_0}{m^2}\int d\vec{v}_2 f(\vec{v}_2,t)\frac{\vec{v}_1-\vec{v}_2}{|\vec{v}_1-\vec{v}_2|^2}\int d\vec{k}\frac{\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^2|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \tag{8.39}$$

$$\frac{1}{m^2}\langle \vec{q}\vec{q} \rangle_{S.P.} = \frac{1}{m^2}\langle \vec{q}_1\vec{q}_1 \rangle = \frac{4e^4 n_0}{m^2}\int d\vec{v}_2 f(\vec{v}_2,t)\int d\vec{k}\frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \tag{8.40}$$

To cast (8.39) into a more familiar form we need the following identity

$$2\frac{\vec{v}_1-\vec{v}_2}{|\vec{v}_1-\vec{v}_2|^2}\int d\vec{k}\frac{\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^2|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} = \left(\frac{\partial}{\partial\vec{v}_2}-\frac{\partial}{\partial\vec{v}_1}\right)\cdot\int d\vec{k}\frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \tag{8.41}$$

The proof of this relation will be found in Appendix D. With (8.41) and performing an integration by parts the friction coefficient (8.39) is

$$\begin{aligned}\frac{1}{m}\langle \vec{q} \rangle_{S.P.} = {} & \frac{2e^4 n_0}{m^2}\int d\vec{v}_2 \frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2}\cdot\int d\vec{k}\frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \\ & + \frac{\partial}{\partial\vec{v}_2}\cdot\frac{2e^4 n_0}{m^2}\int d\vec{v}_2 f(\vec{v}_2,t)\int d\vec{k}\frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\end{aligned} \tag{8.42}$$

Using the relations

$$\begin{aligned}\frac{1}{m}\langle \vec{q} \rangle &= \vec{A} + \frac{1}{2}\frac{\partial}{\partial\vec{v}_1}\cdot\overleftrightarrow{B} \\ \frac{1}{m^2}\langle \vec{q}\vec{q} \rangle &= \overleftrightarrow{B}\end{aligned} \tag{8.43}$$

Eqs. (8.40) and (8.42) give for $\vec{A}_{S.P.}$ and $\overleftrightarrow{B}_{S.P.}$

$$\vec{A}_{S.P.}(\vec{v}_1,t) = \frac{2e^4 n_0}{m^2}\int d\vec{v}_2 \frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2}\cdot\int d\vec{k}\,\frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \tag{8.44}$$

$$\overleftrightarrow{B}_{S.P.}(\vec{v}_1,t) = \frac{4e^4 n_0}{m^2}\int d\vec{v}_2\, f(\vec{v}_2,t)\int d\vec{k}\,\frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \tag{8.45}$$

When $\varepsilon(\vec{k},\omega)$ is the electron plasma dielectric constant (5.13), then (8.44), (8.45) are identical with the B-G-L results (7.31), (7.32). If $\varepsilon(\vec{k},\omega)$ is set equal to unity, corresponding to pure Coulomb interactions, then $\overleftrightarrow{B}_{S.P.}$ is the Landau coefficient $\overleftrightarrow{B}_2$, (8.21), and the Fokker-Planck coefficients are given by (8.20).

5. CONCLUSIONS

Both the Landau approximation of the Boltzmann equation and the B-G-L equation are straight path approximations. Collective interactions included in the B-G-L equation are responsible for the dielectric screening of the colliding particles. The screening effectively cuts off the Coulomb interaction beyond a Debye length, and eliminates the low k divergence associated with a pure Coulomb potential.

Both methods lead to divergent results when extended to large k (small impact parameters), since the straight path approximation necessarily lacks a cutoff for large momentum transfers.

The Boltzmann equation is inappropriate for a plasma, since it does not include collective interactions which limit the effective range of the Coulomb potential. On the other hand, expansion of the B-B-G-K-Y hierarchy in powers of the plasma parameter, $g = \frac{1}{n_0 R_D^3} = 4\pi \frac{R_L}{R_D}$, leads to the B-G-L equation which also possesses a logarithmic divergence. The breakdown in the expansion in powers of g occurs because terms proportional to $\frac{e^2}{m} \partial\phi_{12}/\partial\vec{r}$, are assumed to be of order g, but these terms can be arbitrarily large for sufficiently small $|\vec{r}_1 - \vec{r}_2|$. As we have seen the neglect of these "direct" interaction terms in the differential equation governing the 2-particle distribution (or correlation) function is equivalent to a straight path approximation in which colliding particles are assumed to be dielectrically screened.

In Chapters II and V of this dissertation we encountered the same difficulties with the test particle problem. A convergent solution to the test particle problem was obtained in Chapter VI by using a modified linearization of the Landau-Vlasov equation. The similarity of the general kinetic equation problem with the test particle problem suggests that a similar procedure be adopted for the solution of the 2-particle distribution function equation (7.13). This program is the subject of the next chapter where a convergent kinetic equation is derived and discussed.

CHAPTER IX

A CONVERGENT KINETIC EQUATION

1. INTRODUCTION

We have seen that an expansion of the B-B-G-K-Y hierarchy equations in powers of the plasma parameter, $g = \frac{1}{n_0 R_D^3} = 4\pi \frac{R_L}{R_D}$, is not consistent as it leads to a divergent kinetic equation. In this chapter we start with the truncated B-B-G-K-Y hierarchy (i.e., three particle correlation function equal to zero) for a spatially uniform electron plasma, and introduce a modified pair correlation function representing the deviation of the plasma from equilibrium. In particular, in place of (7.7) we write the 2-particle distribution function as

$$f_2(x_1,x_2,t) = C_2(|\vec{r}_1-\vec{r}_2|)\left\{f(\vec{v}_1,t)f(\vec{v}_2,t) + h_2(x_1,x_2,t)\right\} \tag{9.1}$$

where $C_2(|\vec{r}_1-\vec{r}_2|)$ is the equilibrium correlation function and $h_2(x_1,x_2,t)$ the modified correlation function describing the deviation of the plasma from equilibrium. This is not a linearization; although we will find that our results are best when the plasma is near equilibrium. In the equilibrium state $h_2 = 0$ and $f(\vec{v},t)$ is the Maxwell-Boltzmann distribution,

$$f^{(0)}(\vec{v}) = \left(\frac{m}{2\pi\theta}\right)^{3/2} e^{-\frac{mv^2}{2\theta}} \tag{9.2}$$

In Section 2 the equilibrium correlation function $C_2(|\vec{r}_1-\vec{r}_2|)$ is determined. We will find that a good approximation to $C_2(r)$ is $\exp\{-R_L\Psi(r)\}$ where $\Psi(r)$ is the Debye-Hückel potential, (6.7).

In Section 3 the modified correlation function $h_2(x_1,x_2,t)$ is calculated, to first order in the plasma parameter, $g = 4\pi\, R_L/R_D$, with the so called "adiabatic hypothesis" in which the 2-particle distribution function is assumed to depend on time only through a functional dependence on $f(\vec{v},t)$.

The approximate solution for $h_2(x_1,x_2;f)$ is used to obtain the kinetic equation for $f(\vec{v},t)$. This kinetic equation does not contain diverging integrals. A discussion of the result is given in Section 4. In Section 5 the kinetic equation is cast into the Fokker-Planck form. The Fokker-Planck coefficients are shown to converge, and are compared with the B-G-L coefficients, (7.31) and (7.32).

2. EQUILIBRIUM CORRELATION FUNCTION

The truncated B-B-G-K-Y hierarchy, in which three particle correlations are neglected

$$T(x_1,x_2,x_3,t) = 0 ,$$

for the spatially homogeneous electron plasma is

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} - \frac{e^2 n_0}{m}\int dx_2 \frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} f_2(x_1,x_2,t) = 0 \qquad (9.3)$$

and

$$\left\{\frac{\partial}{\partial t}+\vec{v}_1\cdot\frac{\partial}{\partial\vec{r}_1}+\vec{v}_2\cdot\frac{\partial}{\partial\vec{r}_2}-\frac{e^2}{m}\frac{\partial\phi_{12}}{\partial\vec{r}_1}\cdot\frac{\partial}{\partial\vec{v}_1}-\frac{e^2}{m}\frac{\partial\phi_{21}}{\partial\vec{r}_2}\cdot\frac{\partial}{\partial\vec{v}_2}\right\}f_2(x_1,x_2,t)$$

$$-\frac{e^2n_0}{m}\frac{\partial f(\vec{v}_1,t)}{\partial\vec{v}_1}\cdot\int dx_3\frac{\partial\phi_{13}}{\partial\vec{r}_1}f_2(x_2,x_3,t)-\frac{e^2n_0}{m}\frac{\partial f(\vec{v}_2,t)}{\partial\vec{v}_2}\cdot\int dx_3\frac{\partial\phi_{23}}{\partial\vec{r}_2}f_2(x_1,x_3,t) \qquad (9.4)$$

$$-\frac{e^2n_0}{m}f(\vec{v}_2,t)\int dx_3\frac{\partial\phi_{13}}{\partial\vec{r}_1}\cdot\frac{\partial}{\partial\vec{v}_1}f_2(x_1,x_3,t)-\frac{e^2n_0}{m}f(\vec{v}_1,t)\int dx_3\frac{\partial\phi_{23}}{\partial\vec{r}_2}\cdot\frac{\partial}{\partial\vec{v}_2}f_2(x_2,x_3,t)=0$$

The equilibrium solution is assumed to be

$$f(\vec{v},t)=f^{(0)}(\vec{v})=\left(\frac{m}{2\pi\theta}\right)^{3/2}e^{-\frac{mv^2}{2\theta}} \qquad (9.5)$$

and

$$f_2(x_1,x_2,t)=C_2(|\vec{r}_1-\vec{r}_2|)\,f^{(0)}(\vec{v}_1)\,f^{(0)}(\vec{v}_2) \qquad (9.6)$$

where the spatial part of the 2-particle equilibrium distribution function, $C_2(|\vec{r}_1-\vec{r}_2|)$, is determined from (9.3)-(9.6), with the boundary condition

$$C_2(|\vec{r}_1-\vec{r}_2|)\longrightarrow 1 \quad , \quad |\vec{r}_1-\vec{r}_2|\to\infty \qquad (9.7)$$

If we assume $C_2(r)$ has the form

$$C_2(r)=e^{-R_L\Psi(r)} \qquad (9.8)$$

where $R_L = e^2/\theta$ is the Landau length, then $\Psi(r)$ must be a solution of

$$\frac{d\Psi(r)}{dr} - \frac{d\phi}{dr} - \frac{1}{R_L R_D^2}\frac{d\phi}{dr} e^{-R_L\Psi(r)} \int_0^r dr' r'^2 \left[e^{-R_L \Psi(r')} - 1 \right] = 0 \tag{9.9}$$

An approximate solution for $\Psi(r)$ is obtained by expanding the last term in powers of $R_L\Psi(r)$. To lower order in the expansion, the solution that vanishes at infinity is the Debye-Hückel potential,

$$\Psi(r) = 1/r \, e^{-r/R_D} \tag{9.10}$$

where $R_D = \left(\frac{4\pi e^2 n_0}{\theta}\right)^{-1/2}$ is the Debye length. Therefore, $C_2(r)$ is approximately

$$C_2(r) = \exp\left\{-\frac{R_L}{r} e^{-\frac{r}{R_D}}\right\} \tag{9.11}$$

3. NON-EQUILIBRIUM SOLUTION

When the plasma is not in equilibrium, but still spatially homogeneous, we write the one and two particle distribution functions in the form

$$f(x,t) = f(\vec{v},t) \tag{9.12}$$

and

$$f_2(x_1,x_2,t) = C_2(|\vec{r}_1 - \vec{r}_2|)\left\{f(\vec{v}_1,t)f(\vec{v}_2,t) + h_2(x_1,x_2,t)\right\} \tag{9.13}$$

Using (9.13) in the kinetic equation (9.3) gives

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = \frac{e^2 n_0}{m}\int dx_2 \frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} C_2(|\vec{r}_1-\vec{r}_2|)\left\{f(\vec{v}_1,t)f(\vec{v}_2,t)+h_2(x_1,x_2,t)\right\} \tag{9.14}$$

Due to the symmetry of C_2 the first term does not contribute to the integral, and (9.14) reduces to

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = \frac{e^2 n_0}{m}\int dx_2 \frac{\partial \phi_{12}}{\partial \vec{r}_1} C_2(|\vec{r}_1-\vec{r}_2|)\cdot\frac{\partial}{\partial \vec{v}_1} h_2(x_1,x_2,t) \tag{9.15}$$

The equation for h_2 is obtained by inserting (9.13) in (9.4)

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t}+\vec{v}_1\cdot\frac{\partial}{\partial \vec{r}_1}+\vec{v}_2\cdot\frac{\partial}{\partial \vec{r}_2}-\frac{e^2}{m}\frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1}-\frac{e^2}{m}\frac{\partial \phi_{21}}{\partial \vec{r}_2}\cdot\frac{\partial}{\partial \vec{v}_2}\right\}C_2(|\vec{r}_1-\vec{r}_2|)h_2(x_1,x_2,t) \\
&-\frac{e^2}{m}\left\{\left[C_2(|\vec{r}_1-\vec{r}_2|)\frac{\partial \phi_{12}}{\partial \vec{r}_1}+n_0\int d\vec{r}_3\frac{\partial \phi_{13}}{\partial \vec{r}_1}C_2(|\vec{r}_2-\vec{r}_3|)\right]\cdot\frac{\partial}{\partial \vec{v}_1}\right. \\
&\qquad\left.+\left[C_2(|\vec{r}_1-\vec{r}_2|)\frac{\partial \phi_{21}}{\partial \vec{r}_2}+n_0\int d\vec{r}_3\frac{\partial \phi_{23}}{\partial \vec{r}_2}C_2(|\vec{r}_1-\vec{r}_3|)\right]\cdot\frac{\partial}{\partial \vec{v}_2}\right\}f(\vec{v}_1,t)f(\vec{v}_2,t) \\
&-\frac{e^2 n_0}{m}\frac{\partial f(\vec{v}_1,t)}{\partial \vec{v}_1}\cdot\int dx_3\frac{\partial \phi_{13}}{\partial \vec{r}_1}C_2(|\vec{r}_2-\vec{r}_3|)h(x_2,x_3,t)-\frac{e^2 n_0}{m}\frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2}\cdot\int dx_3\frac{\partial \phi_{23}}{\partial \vec{r}_2}C_2(|\vec{r}_1-\vec{r}_3|)h_2(x_1,x_3,t) \\
&+f(\vec{v}_1,t)f(\vec{v}_2,t)\left[\vec{v}_1\cdot\frac{\partial}{\partial \vec{r}_1}+\vec{v}_2\cdot\frac{\partial}{\partial \vec{r}_2}\right]C_2(|\vec{r}_1-\vec{r}_2|)+\left[C_2(|\vec{r}_1-\vec{r}_2|)-1\right]\frac{\partial}{\partial t}\left[f(\vec{v}_1,t)f(\vec{v}_2,t)\right]=0
\end{aligned} \tag{9.16}$$

where we have used (9.14) and the normalization $\int d\vec{v}\, f(\vec{v},t) = 1$.

The equation for $h_2(x_1,x_2,t)$ is quite complicated, nevertheless, it is possible to obtain h_2 to lowest order in the plasma parameter, $g = 4\pi \frac{R_L}{R_D}$. To carry out this expansion in (9.16) let ω_p^{-1} and R_D be units of time and length. The one particle distribution function $f(\vec{v},t)$ is of zero order in g while

$$h_2(x_1,x_2,t) \quad , \frac{e^2}{m}\frac{\partial\phi_{12}}{\partial\vec{r}_1}\cdot\frac{\partial}{\partial\vec{v}_1} \quad , R_L\frac{\partial\Psi}{\partial\vec{r}_1} \quad , \frac{\partial f(\vec{v}_1,t)}{\partial t}$$

are all assumed first order in g . Furthermore, the exponent of the equilibrium function C_2 is of order g so that

$$C_2(|\vec{r}_1-\vec{r}_2|) = 1 - R_L\Psi(|\vec{r}_1-\vec{r}_2|) + O(g^2)$$

If we make the additional assumption that $h_2(x_1,x_2,t)$ only depends on time functionally through $f(\vec{v},t)$, the "adiabatic hypothesis," then analogous to (7.25) we find that $\partial h_2/\partial t \propto O(g^2)$.

In view of the above remarks Eq. (9.16), to first order in the plasma parameter, becomes

$$\left\{\vec{v}_1\cdot\frac{\partial}{\partial\vec{r}_1}+\vec{v}_2\cdot\frac{\partial}{\partial\vec{r}_2}\right\}h_2(x_1,x_2) - \frac{e^2}{m}\left\{\frac{\partial\Psi_{12}}{\partial\vec{r}_1}\cdot\frac{\partial}{\partial\vec{v}_1}+\frac{\partial\Psi_{21}}{\partial\vec{r}_2}\cdot\frac{\partial}{\partial\vec{v}_2}\right\}f(\vec{v}_1,t)\, f(\vec{v}_2,t)$$

$$-\frac{e^2 n_0}{m}\frac{\partial f(\vec{v}_1,t)}{\partial\vec{v}_1}\cdot\int dx_3\frac{\partial\phi_{13}}{\partial\vec{r}_1}h_2(x_2,x_3) - \frac{e^2 n_0}{m}\frac{\partial f(\vec{v}_2,t)}{\partial\vec{v}_2}\cdot\int dx_3\frac{\partial\phi_{23}}{\partial\vec{r}_2}h_2(x_1,x_3) \tag{9.17}$$

$$-R_L f(\vec{v}_1,t)\, f(\vec{v}_2,t)\left\{\vec{v}_1\cdot\frac{\partial}{\partial\vec{r}_1}+\vec{v}_2\cdot\frac{\partial}{\partial\vec{r}_2}\right\}\Psi(|\vec{r}_1-\vec{r}_2|) = 0$$

where we have used the definition of $C_2(|\vec{r}_1-\vec{r}_2|)$. The time argument of h_2 has been omitted, since h_2 now depends on time functionally through $f(\vec{v},t)$. Except for the last term Eq. (9.17) is similar to the B-G-L equation (7.26), and is amenable to the same type solution. In the spatially homogeneous case under consideration $h_2(x_1,x_2)$ is a function of the relative coordinate $\vec{r}_1-\vec{r}_2$ so that (9.17) may be re-written in terms of $\vec{r}=\vec{r}_1-\vec{r}_2$,

$$(\vec{v}_1-\vec{v}_2)\cdot\frac{\partial}{\partial\vec{r}}h_2(\vec{r},\vec{v}_1,\vec{v}_2)-\frac{e^2}{m}\frac{\partial\Psi(r)}{\partial\vec{r}}\cdot\left(\frac{\partial}{\partial\vec{v}_1}-\frac{\partial}{\partial\vec{v}_2}\right)f(\vec{v}_1,t)f(\vec{v}_2,t)$$

$$-\frac{e^2n_0}{m}\frac{\partial f(\vec{v}_1,t)}{\partial\vec{v}_1}\cdot\int dx_3\frac{\partial\phi(|\vec{r}-\vec{r}_3|)}{\partial\vec{r}}h_2(-\vec{r}_3,\vec{v}_2,\vec{v}_3)$$

$$+\frac{e^2n_0}{m}\frac{\partial f(\vec{v}_2,t)}{\partial\vec{v}_2}\cdot\int dx_3\frac{\partial\phi(|\vec{r}-\vec{r}_3|)}{\partial\vec{r}}h_2(\vec{r}_3,\vec{v}_1,\vec{v}_3) \tag{9.18}$$

$$-R_L f(\vec{v}_1,t)f(\vec{v}_2,t)\ (\vec{v}_1-\vec{v}_2)\cdot\frac{\partial}{\partial\vec{r}}\Psi(r)=0$$

Observing that the integral terms in (9.18) are convolutions of $\partial\phi/\partial\vec{r}$ with h_2 we are naturally led to the introduction of Fourier space transforms

$$\tilde{h}_2(\vec{k},\vec{v}_1,\vec{v}_2)=\int d\vec{r}\,e^{-i\vec{k}\cdot\vec{r}}h_2(\vec{r},\vec{v}_1,\vec{v}_2) \tag{9.19}$$

and we note that

$$\tilde{h}_2^*(\vec{k},\vec{v}_1,\vec{v}_2) = \int d\vec{r}\, e^{-i\vec{k}\cdot\vec{r}} h_2(-\vec{r},\vec{v}_1,\vec{v}_2) \\ = \tilde{h}_2(-\vec{k},\vec{v}_1,\vec{v}_2) \tag{9.20}$$

The Fourier transformation of (9.18) is

$$\begin{aligned} &(\vec{v}_1-\vec{v}_2)\cdot\vec{k}\,\tilde{h}_2(\vec{k},\vec{v}_1,\vec{v}_2) - \frac{e^2}{m}\tilde{\Psi}(k)\,\vec{k}\cdot\left(\frac{\partial}{\partial\vec{v}_1}-\frac{\partial}{\partial\vec{v}_2}\right)f(\vec{v}_1,t)f(\vec{v}_2,t) \\ &- \frac{e^2 n_0}{m}\vec{k}\cdot\frac{\partial f(\vec{v}_1,t)}{\partial\vec{v}_1}\tilde{\phi}(k)\int d\vec{v}_3\,\tilde{h}_2^*(\vec{k},\vec{v}_2,\vec{v}_3) \\ &+ \frac{e^2 n_0}{m}\vec{k}\cdot\frac{\partial f(\vec{v}_2,t)}{\partial\vec{v}_2}\tilde{\phi}(k)\int d\vec{v}_3\,\tilde{h}_2(\vec{k},\vec{v}_1,\vec{v}_3) \\ &- R_L f(\vec{v}_1,t)f(\vec{v}_2,t)\tilde{\Psi}(k)\,\vec{k}\cdot(\vec{v}_1-\vec{v}_2) = 0 \end{aligned} \tag{9.21}$$

Before continuing with the solution of (9.21) it is helpful to look ahead and see what will be required for the evaluation of the kinetic equation. The kinetic equation (9.15) together with (9.11) is

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = \frac{e^2 n_0}{m}\frac{\partial}{\partial\vec{v}_1}\cdot\int d\vec{r}_2\frac{\partial\phi(|\vec{r}_1-\vec{r}_2|)}{\partial\vec{r}_1}e^{-R_L\Psi(|\vec{r}_1-\vec{r}_2|)}\int d\vec{v}_2\,h_2(\vec{r}_1-\vec{r}_2,\vec{v}_1,\vec{v}_2) \tag{9.22}$$

Introducing Fourier representations for the functions in the integrand,

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = -\frac{e^2 n_0}{(2\pi)^3 m}\frac{\partial}{\partial\vec{v}_1}\cdot\int d\vec{k}\,\vec{Q}(\vec{k})\int d\vec{v}_2\,\tilde{h}_2(\vec{k},\vec{v}_1,\vec{v}_2) \tag{9.23}$$

where

$$\vec{Q}(\vec{k}) = \int d\vec{r}\,\frac{\partial\phi(r)}{\partial\vec{r}}e^{-R_L\Psi(r)}e^{-i\vec{k}\cdot\vec{r}} \tag{9.24}$$

Since $\vec{Q}(\vec{k})$ is an odd function of $\vec{k}$, only the imaginary part of $\check{h}_2$, which according to (9.20) is also an odd function of $\vec{k}$, will contribute to (9.23)

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = -\frac{e^2 n_0 i}{(2\pi)^3 m}\frac{\partial}{\partial \vec{v}_1}\cdot \int d\vec{k}\,\vec{Q}(\vec{k})\,\mathrm{Im}\int d\vec{v}_2\,\check{h}_2(\vec{k},\vec{v}_1,\vec{v}_2) \tag{9.25}$$

In other words, the kinetic equation does not require complete knowledge of $\check{h}_2$ but only of $\mathrm{Im}\int d\vec{v}_2\,\check{h}_2(\vec{k},\vec{v}_1,\vec{v}_2)$.

Defining the function

$$H(\vec{v}_1) = n_0\int d\vec{v}_2\,\check{h}_2(\vec{k},\vec{v}_1,\vec{v}_2) \tag{9.26}$$

a solution of (9.21) for $\mathrm{Im}\,H(\vec{v}_1)$ is given in Appendix E. Using this solution in the kinetic equation (9.25) yields

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = -\frac{e^4 n_0 i}{8\pi^2 m^2}\frac{\partial}{\partial \vec{v}_1}\cdot\int d\vec{v}_2\int d\vec{k}\,\check{\Psi}(k)\left[1+\frac{1}{k^2R_D^2}\right]\frac{\vec{Q}(\vec{k})\,\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\times$$
$$\times\left(\frac{\partial}{\partial \vec{v}_1}-\frac{\partial}{\partial \vec{v}_2}\right)f(\vec{v}_1,t)f(\vec{v}_2,t) \tag{9.27}$$

where $\vec{Q}(\vec{k})$ is given by (9.24).

If we use the Debye-Hückel potential for $\Psi(r)$, (9.10), then

$$\check{\Psi}(k) = \frac{4\pi R_D^2}{1+k^2R_D^2}$$

and the kinetic equation (9.27) is

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = -\frac{e^4 n_0 i}{2\pi m^2}\frac{\partial}{\partial \vec{v}_1}\cdot\int d\vec{v}_2 \int d\vec{k}\, \frac{\vec{Q}(\vec{k})\vec{k}}{k^2}\frac{\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \times$$
$$\times \left\{ f(\vec{v}_2,t)\frac{\partial f(\vec{v}_1,t)}{\partial \vec{v}_1} - f(\vec{v}_1,t)\frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2} \right\} \tag{9.28}$$

It is interesting to note that if the equilibrium correlation function $C_2(r)$ is replaced by unity the function $\vec{Q}(\vec{k})$ is simply

$$\vec{Q} = \frac{4\pi i \vec{k}}{k^2}$$

and the kinetic equation reduces exactly to the B-G-L equation (7.28). It is important to retain the equilibrium correlation function, however, since it guarantees the convergence of the k integration.

It is shown in Appendix B, (B.6), that to order R_L/R_D

$$\vec{Q}(\vec{k}) = -(2\pi)^2 i \frac{\vec{k}}{k^2}\,\mathrm{hei}_2\left(\sqrt{4R_L k}\right) \tag{9.29}$$

The use of (9.29) in (9.28) gives the kinetic equation

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = -\frac{2\pi e^4 n_0}{m^2}\frac{\partial}{\partial \vec{v}_1}\cdot\int d\vec{v}_2 \int d\vec{k}\,\mathrm{hei}_2\left(\sqrt{4R_L k}\right)\frac{\vec{k}\vec{k}}{k^4} \times$$
$$\times \frac{\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\left\{ f(\vec{v}_2,t)\frac{\partial f(\vec{v}_1,t)}{\partial \vec{v}_1} - f(\vec{v}_1,t)\frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2} \right\} \tag{9.30}$$

4. DISCUSSION

Equation (9.30) is the spatially homogeneous electron plasma kinetic equation, in which the modified pair correlation function $h_2(x_1, x_2, t)$ and $\vec{Q}(\vec{k})$ have been calculated to first order in the plasma parameter $g = 4\pi R_L/R_D$. We have also used the "adiabatic hypothesis," the time dependence of h_2 appears through a functional dependence on $f(\vec{v}, t)$

The validity of (9.30) depends on the approximate solution for $h_2(x_1, x_2, t)$. In obtaining Eq. (9.18) for h_2 it was assumed that quantities like $R_L \Psi(r)$ are small, but this is certainly not correct for all values of r. An order of magnitude estimate for h_2, for small r, is easily obtained from (9.18). When r is small the integral terms in (9.18) are unimportant and h_2 is governed primarily by the equation

$$\frac{\partial}{\partial \vec{r}} \cdot \left\{ (\vec{v}_1 - \vec{v}_2) h_2 - \frac{e^2}{m} \Psi(r) \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) f(\vec{v}_1, t) f(\vec{v}_2, t) \right. \\ \left. - R_L \Psi(r) (\vec{v}_1 - \vec{v}_2) f(\vec{v}_1, t) f(\vec{v}_2, t) \right\} = 0 \tag{9.31}$$

From (9.31) we see that the approximate solution for h_2, which breaks down when $h_2 \approx f(\vec{v}_1, t) f(\vec{v}_2, t)$, fails for those values of r where

$$R_L \Psi(r) \gtrsim 1$$

However, the 2-particle distribution function (9.13) contains the factor $\exp\{-R_L \Psi(r)\}$. Therefore, the approximate solution for h_2 fails at the same time that the distribution function is rapidly going to zero. This case is marginal, and the kinetic equation (9.30) must be, at least, qualitatively correct. Certainly the lack of divergent integrals in (9.30) recommends it as an improvement over the B-G-L equation.

The previous discussion assumed the plasma to be arbitrarily far from the equilibrium state. However, if the plasma is near equilibrium, say

$$f(\vec{v},t) = f^{(0)}(\vec{v}) + \lambda h(\vec{v},t)$$

where $f^{(0)}(\vec{v})$ is the Maxwell-Boltzmann equilibrium distribution, (9.2), and λ is a measure of the deviation from equilibrium, then to zeroth order in λ the last two terms in (9.31) cancel, and one finds that $h_2 \approx f^{(0)}(\vec{v}_1) f^{(0)}(\vec{v}_2)$ for those values of r where

$$R_L \Psi(r) \gtrsim 1/\lambda \tag{9.32}$$

Thus, the approximate solution for h_2 is valid except for those values of r satisfying (9.32), and in this region the distribution function $f_2(x_1, x_2, t)$ is reduced by an exponential factor $e^{-1/\lambda}$. We conclude that the kinetic equation (9.30) becomes exact as the equilibrium state is approached.

5. THE FOKKER-PLANCK COEFFICIENTS

With some rearrangement Eq. (9.30) may be cast into the Fokker-Planck form

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} = -\frac{\partial}{\partial \vec{v}_1} \cdot \left\{ f(\vec{v}_1,t) \left[\vec{A}_c(\vec{v}_1,t) + \frac{1}{2} \frac{\partial}{\partial \vec{v}_1} \cdot \overleftrightarrow{B}_c(\vec{v}_1,t) \right] \right\} \tag{9.33}$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial \vec{v}_1 \partial \vec{v}_1} : \left\{ f(\vec{v}_1,t)\, \overleftrightarrow{B}_c(\vec{v}_1,t) \right\}$$

where

$$\vec{A}_c(\vec{v}_1,t) = -\frac{2\pi e^4 n_0}{m^2}\int d\vec{v}_2 \int d\vec{k}\, h_{ei_2}(\sqrt{4R_L k})\, \frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\cdot\frac{\partial f(\vec{v}_2,t)}{\partial\vec{v}_2} \tag{9.34}$$

$$\overleftrightarrow{B}_c(\vec{v}_1,t) = -\frac{4\pi e^4 n_0}{m^2}\int d\vec{v}_2 \int d\vec{k}\, h_{ei_2}(\sqrt{4R_L k})\, \frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\, f(\vec{v}_2,t) \tag{9.35}$$

The Fokker-Planck coefficients can be identified from (9.33) as

$$\begin{aligned} \frac{1}{m}\langle\vec{q}\rangle &= \vec{A}_c + \frac{1}{2}\frac{\partial}{\partial\vec{v}_1}\cdot\overleftrightarrow{B}_c \\ \frac{1}{m^2}\langle\vec{q}\vec{q}\rangle &= \overleftrightarrow{B}_c \end{aligned} \tag{9.36}$$

With the help of the identity (see Appendix D),

$$\begin{aligned} &\left(\frac{\partial}{\partial\vec{v}_2}-\frac{\partial}{\partial\vec{v}_1}\right)\cdot\int d\vec{k}\, h_{ei_2}(\sqrt{4R_L k})\,\frac{\vec{k}\vec{k}}{k^4}\,\frac{\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \\ &= 2\,\frac{\vec{v}_1-\vec{v}_2}{|\vec{v}_1-\vec{v}_2|^2}\int d\vec{k}\, h_{ei_2}(\sqrt{4R_L k})\,\frac{\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^2|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \end{aligned} \tag{9.37}$$

and the definitions of $\vec{A}_c$, $\vec{B}_c$ given by (9.34) and (9.35) the Fokker-Planck coefficients may be written in the form

$$\frac{1}{m}\langle\vec{q}\rangle = \frac{4\pi e^4 n_0}{m^2}\int d\vec{v}_2\, f(\vec{v}_2,t)\,\frac{\vec{q}}{q^2}\int d\vec{k}\, h_{ei_2}(\sqrt{4R_L k})\,\frac{\delta(\vec{k}\cdot\vec{q})}{k^2|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \tag{9.38}$$

and

$$\frac{1}{m^2}\langle\vec{q}\vec{q}\rangle = -\frac{2\pi e^4 n_0}{m^2}\int d\vec{v}_2\, f(\vec{v}_2,t)\left(\mathbb{1}-\frac{\vec{q}\vec{q}}{q^2}\right)\int d\vec{k}\, \mathrm{hei}_2(\sqrt{4R_L k})\, \frac{\delta(\vec{k}\cdot\vec{q})}{k^2|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \quad (9.39)$$

where $\vec{q}=\vec{v}_1-\vec{v}_2$ and $\mathbb{1}$ is the unit tensor. These Fokker-Planck coefficients differ from the B-G-L coefficients by the factor $-\pi\,\mathrm{hei}_2(\sqrt{4R_L k})$. Since $\mathrm{hei}_2(x)$ decays exponentially for large x the coefficients (9.38) and (9.39) do not diverge at large k.

APPENDIX A

APPROXIMATE SOLUTION FOR $\Phi(r)$

To obtain an approximate solution of Eq. (3.30),

$$\nabla^2 \Phi(r) = 4\pi \frac{e}{e_T} n_0 \left[1 - e^{-R_L \Phi(r)}\right] - 4\pi\delta(\vec{r}) \tag{A.1}$$

we introduce dimensionless variables

$$\vec{r} = R_D \vec{x} \quad , \quad \Phi(r) = \frac{1}{R_D} \Psi(x) \tag{A.2}$$

Equation (A.1) becomes

$$\nabla^2 \Psi(x) = \frac{R_D}{R_L}\left[1 - e^{-\frac{R_L}{R_D}\Psi(x)}\right] - 4\pi\delta(\vec{x}) \tag{A.3}$$

The ratio R_L/R_D is less than 10^{-5} for most plasmas (see Table I). Suppose we develop $\Psi(x)$ in a power series in R_L/R_D,

$$\Psi(x) = \Psi_0(x) + \frac{R_L}{R_D}\Psi_1(x) + \cdots$$

Using this expansion in (A.3), expanding the exponential term, and equating like powers of R_L/R_D,

$$\nabla^2 \Psi_0 - \Psi_0 = -4\pi\,\delta(\vec{x}) \tag{A.4}$$

$$\nabla^2 \Psi_1 - \Psi_1 = -\frac{1}{2}\Psi_0^{\,2} \tag{A.5}$$

The solution for $\psi_0(x)$ that vanishes at infinity is

$$\psi_0(x) = \frac{1}{\chi} e^{-\chi} \tag{A.6}$$

Thus, $\Phi(r)$ is approximately the Debye-Hückel result

$$\Phi(r) = \frac{1}{r} e^{-r/R_D} \tag{A.7}$$

This expansion cannot be carried further, since the expansion of $e^{-R_L\Phi}$, for all r, is not correct. Nevertheless (A.7) gives the asymptotic behavior of $\Phi(r)$, and since Φ must approach $1/r$ as $r \to 0$ (A.7) is an excellent approximation.

We now turn to evaluation of the integral

$$\int_0^\infty d\xi \, \frac{e^{-R_L\Phi(\xi)}}{\xi^2} \sin\left(\frac{\omega_p}{s v_T}\xi\right) \tag{A.8}$$

to lowest order in R_L/R_D. Inserting (A.7) for $\Phi(\xi)$, and changing to a dimensionless variable $\chi = \xi/R_L$, the integral (A.8) is

$$\frac{1}{R_L}\int_0^\infty d\chi \, \frac{e^{-\frac{1}{\chi}e^{-\frac{R_L}{R_D}\chi}}}{\chi^2} \sin\left(\frac{\omega_p R_L}{s v_T}\chi\right)$$

To lowest order in R_L/R_D it is permissible to replace the exponent $-\frac{1}{\chi}e^{-\frac{R_L}{R_D}\chi}$ by $-\frac{1}{\chi}$,

$$\frac{1}{R_L}\int_0^\infty d\chi \, \frac{e^{-\frac{1}{\chi}}}{\chi^2} \sin\left(\frac{\omega_p R_L}{s v_T}\chi\right) \tag{A.9}$$

The coefficient $\frac{\omega_p R_L}{s v_T}$ is equal to $\frac{a}{s v_T}\frac{R_L}{R_D}$. The integral (A.9) is

$$\frac{1}{R_L}\int_0^{\infty} dx\, \frac{e^{-\frac{1}{x}}}{x^2} \sin\left(\frac{a}{sv_T}\frac{R_L}{R_D}x\right) = \frac{1}{R_L}\chi\frac{d}{d\chi}\,\mathrm{kei}_0(\chi)\Big|_{\chi=2\sqrt{\frac{a}{sv_T}\frac{R_L}{R_D}}} \tag{A.10}$$

where $\mathrm{kei}_0(\chi)$ is Kelvin's function,* and the notation, $\Big|_{\chi=\chi_0}$, signifies that the function of χ is evaluated at $\chi=\chi_0$. For small values of the argument

$$\frac{d}{d\chi}\mathrm{kei}_0(\chi) \simeq -\frac{\chi}{4}\ln\left[\frac{\lambda^2}{\epsilon}\left(\frac{\chi}{2}\right)^2\right] + O(\chi^3)$$

where $\ln\lambda = 0.577\cdots$ is Euler's constant and $\epsilon = 2.718\cdots$ is the base of the natural logarithms. The integral (A.8) to lowest order in R_L/R_D is

$$-\frac{\omega_p}{sv_T}\ln\left[\frac{\lambda^2}{\epsilon}\frac{a}{sv_T}\frac{R_L}{R_D}\right] + O\left(\frac{R_L}{R_D}\right)$$

$$= \frac{\omega_p}{sv_T}\ln\left[\frac{\epsilon}{\lambda^2}\sqrt{\frac{v_T^2}{a^2}-1}\;\frac{R_D}{R_L}\right] + O\left(\frac{R_L}{R_D}\right) \tag{A.11}$$

*Watson, G. N., A Treatise on the Theory of Bessel Functions, p. 81, Cambridge (1948).

APPENDIX B

APPROXIMATION OF $\vec{Q}(\vec{k})$

Our first task is to obtain (6.25),

$$\vec{Q}(\vec{k}) = -(2\pi)^2 i \frac{\vec{k}}{k^2} \mathrm{hei}_2\left(\sqrt{4R_L k}\right) \tag{B.1}$$

to lowest order in R_L/R_D , when $\vec{Q}(\vec{k})$ is given by (6.15),

$$\vec{Q}(\vec{k}) = -\int d\vec{r}\, \frac{\vec{r}}{r^3} e^{-R_L \Phi(r)} e^{-i\vec{k}\cdot\vec{r}} \tag{B.2}$$

The form of $\Phi(r)$, to lowest order in R_L/R_D , is the Debye-Hückel result

$$\Phi(r) = \frac{1}{r} e^{-\frac{r}{R_D}}$$

but $\Phi(r)$ occurs in (B.2) only in the exponential, and to the first approximation in R_L/R_D we can replace $\Phi(r)$ by the Coulomb potential $1/r$,

$$\begin{aligned} \vec{Q}(\vec{k}) &= -\int d\vec{r}\, \frac{\vec{r}}{r^3} e^{-\frac{R_L}{r} - i\vec{k}\cdot\vec{r}} \\ &= -i \frac{\partial}{\partial \vec{k}} \int d\vec{r}\, \frac{1}{r^3} e^{-\frac{R_L}{r} - i\vec{k}\cdot\vec{r}} \end{aligned} \tag{B.3}$$

Using polar coordinates for $\vec{r}$ with $\vec{k}$ as polar axis

$$\vec{Q} = -i\frac{\partial}{\partial \vec{k}}\int_0^\infty dr\, \frac{e^{-\frac{R_L}{r}}}{r}\int_{-1}^{1} d\eta \int_0^{2\pi} d\phi\, e^{-ikr\eta}$$

$$= -4\pi i \frac{\partial}{\partial \vec{k}}\int_0^\infty dr\, \frac{e^{-R_L/r}}{kr^2}\sin kr \qquad \text{(B.4)}$$

Introducing the change of variable $x = kr$ (B.4) becomes

$$\vec{Q} = 4\pi i R_L \frac{\vec{k}}{k}\int_0^\infty dx\, \frac{e^{-\frac{R_L k}{x}}}{x^3}\sin x \qquad \text{(B.5)}$$

This integral is given in the Bateman tables* in terms of Hankel functions of imaginary argument, or by virtue of (6.26) we find

$$\vec{Q} = -(2\pi)^2 i \frac{\vec{k}}{k^2}\,\mathrm{hei}_2\left(\sqrt{4R_L k}\right) \qquad \text{(B.6)}$$

The second task is to evaluate, to lowest order in R_L/R_D, the integral (6.31),

$$\int_0^\infty dx\, \mathrm{hei}_2(2b\sqrt{x})\, \frac{x^3 W_I}{(x^2+W_R)^2+W_I^2} \qquad \text{(B.7)}$$

where $b = \sqrt{R_L/R_D}$. The simplest procedure for obtaining the dominant term as $R_L/R_D \to 0$ is to separate the range of integration into two parts $(0, 1/b)$ and $(1/b, \infty)$. In the first range $(0, 1/b)$ the argument of $\mathrm{hei}_2(2b\sqrt{x})$ satisfies the inequalities $0 < 2b\sqrt{x} \leqslant 2\sqrt{b} \ll 1$, where the last inequality follows on the assumption that $R_L/R_D \ll 1$.

*Bateman, Tables of Integral Transforms, I, p. 75, McGraw-Hill (1954).

Since the argument is always much less than unity we may expand

$$\mathrm{hei}_2(2b\sqrt{x}) \simeq -\frac{1}{\pi} + \frac{b^2}{4}x + \cdots \tag{B.8}$$

The contribution to (B.7) from the first range of integration is

$$-\frac{1}{\pi}\int_0^{1/b} dx \, \frac{x^3 W_I}{(x^2+W_R)^2+W_I^2} + \mathcal{O}(R_L/R_D) \tag{B.9}$$

$$= -\frac{1}{\pi}W_I \ln 1/b + \frac{1}{4\pi}W_I \ln(W_R^2+W_I^2) + \frac{1}{2\pi}W_R \tan^{-1}\frac{W_I}{W_R} + \mathcal{O}\left(\frac{R_L}{R_D}\right)$$

$$= -\frac{1}{\pi}W_I \ln 1/b + \frac{1}{2\pi}\operatorname{Im}(W \ln W) + \mathcal{O}(R_L/R_D) \tag{B.10}$$

where $W = W_R + iW_I$ is given by (5.16), and we have used the fact that $|W(t)| \leq 1 << 1/b$

In the second range of integration ($1/b$, ∞) we can again use $|W| \leq 1 << 1/b$ to expand the integrand of (B.7)

$$W_I \int_{1/b}^{\infty} dx \, \mathrm{hei}_2(2b\sqrt{x}) \frac{x^3}{(x^2+W_R)^2+W_I^2} \cong$$

$$W_I \int_{1/b}^{\infty} dx \, \mathrm{hei}_2(2b\sqrt{x}) \left\{ \frac{1}{x} - \frac{2W_R}{x^3} + \cdots \right\} \tag{B.11}$$

Making the variable change $2b\sqrt{x} = u$, the right hand side is

$$W_I \int_{2\sqrt{b}}^{\infty} du \, \mathrm{hei}_2(u) \left\{ \frac{1}{u} - \frac{32b^4 W_R}{u^5} + \cdots \right\}$$

The second term can be neglected. The first term can be carried out as an indefinite integral (Ref. 18, p. 319), and (B.11) becomes

$$\frac{W_I}{\sqrt{2}}\,\frac{1}{u}\left[\mathrm{her}_1(u)+\mathrm{hei}_1(u)\right]\Bigg|_{u=2\sqrt{b}}^{u=\infty} + O\left(\frac{R_L}{R_D}\right)$$

where $\mathrm{her}_1(u)$, $\mathrm{hei}_1(u)$, are defined by (6.26). There is no contribution from the upper limit $u=\infty$, and we are left with

$$-\frac{W_I}{\sqrt{2}}\,\frac{1}{2\sqrt{b}}\left[\mathrm{her}_1(2\sqrt{b})+\mathrm{hei}_1(2\sqrt{b})\right] + O\left(\frac{R_L}{R_D}\right) \tag{B.12}$$

Since $2\sqrt{b} \ll 1$ we expand the $\mathrm{her}_1(x)$ and $\mathrm{hei}_1(x)$ functions for small x to obtain (Ref. 18, p. 320),

$$\mathrm{her}_1(x)+\mathrm{hei}_1(x) \simeq -\frac{\sqrt{2}}{\pi}x\left\{\ln\frac{x\lambda}{2}-\frac{1}{2}+\mathcal{O}(x^2)\right\}$$

Therefore, (B.12) gives to lowest order in R_L/R_D,

$$\begin{aligned}&\frac{1}{\pi}W_I\left\{\ln\lambda^2 b-1+\mathcal{O}\left(\frac{R_L}{R_D}\right)\right\}\\ &=\frac{1}{\pi}W_I\ln\frac{\lambda^2}{\epsilon}b+\mathcal{O}\left(\frac{R_L}{R_D}\right)\end{aligned} \tag{B.13}$$

where $\ln\lambda = 0.577\cdots$ is Euler's constant and $\epsilon = 2.718\cdots$ is the base of the natural logarithms. Addition of (B.10) and (B.13) gives the dominant term for the integral (B.7) when $R_L/R_D \ll 1$,

$$\begin{aligned}\int_0^\infty dx\,\mathrm{hei}_2\left(\sqrt{4\frac{R_L}{R_D}x}\right)\frac{x^3 W_I}{(x^2+W_R)^2+W_I^2} &\simeq -\frac{1}{2\pi}W_I\ln\left[\frac{\epsilon}{\lambda^2}\frac{R_D}{R_L}\right]\\ &+\frac{1}{2\pi}\mathrm{Im}\left(W\ln W\right)+\mathcal{O}\left(R_L/R_D\right)\end{aligned} \tag{B.14}$$

APPENDIX C

TRANSFORMATION OF THE COLLISION INTEGRAL

To transform the integral (7.22),

$$-\frac{e^2 n_0}{m}\int dx_2 \frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} f(\vec{V}_1,t)\, f(\vec{V}_2,t) \tag{C.1}$$

we note that the Hamiltonian for the characteristic equations (7.18) is

$$H = p_1^2/2m + p_2^2/2m + e^2\phi(|\vec{r}_1-\vec{r}_2|) \tag{C.2}$$

where $\vec{p}_i = m\vec{v}_i$. The quantities $\vec{V}_1$, $\vec{V}_2$ are the "initial" velocities of two particles before a collision when they have the values $\vec{v}_1$, $\vec{v}_2$ at $\vec{r}_1$, $\vec{r}_2$. This means $f(\vec{V}_1,t)\, f(\vec{V}_2,t)$ is a constant of the motion for the collision orbit determined by H and passing through the state ($\vec{r}_1$, $\vec{r}_2$, $\vec{v}_1$, $\vec{v}_2$). Therefore the Poisson bracket of $f(\vec{V}_1,t)\, f(\vec{V}_2,t)$ with H must be zero,

$$\frac{e^2}{m}\left[\frac{\partial \phi_{12}}{\partial \vec{r}_1}\cdot\frac{\partial}{\partial \vec{v}_1} + \frac{\partial \phi_{21}}{\partial \vec{r}_2}\cdot\frac{\partial}{\partial \vec{v}_2}\right] f(\vec{V}_1,t)\, f(\vec{V}_2,t) = \left(\vec{v}_1\cdot\frac{\partial}{\partial \vec{r}_1}+\vec{v}_2\cdot\frac{\partial}{\partial \vec{r}_2}\right) f(\vec{V}_1,t)\, f(\vec{V}_2,t) \tag{C.3}$$

or

$$\text{L.H.S.} = -(\vec{v}_1-\vec{v}_2)\cdot\frac{\partial}{\partial \vec{r}_2} f(\vec{V}_1,t)\, f(\vec{V}_2,t)$$

where we have used the fact that $\vec{V}_1$, $\vec{V}_2$ must depend only on $\vec{r}_1-\vec{r}_2$ Using this identity in the integral (C.1), and noting that the second term on the left hand side of (C.3) vanishes when integrated by parts, we are left with

$$n_0 \int d\chi_2 \, (\vec{v}_1 - \vec{v}_2) \cdot \frac{\partial}{\partial \vec{r}_2} f(\vec{v}_1, t) f(\vec{v}_2, t) \tag{C.4}$$

Introducing cylindrical coordinates (z , b , ϕ) for $\vec{r}_2$ with $(\vec{v}_1 - \vec{v}_2)$ as the z axis we find

$$n_0 \int d\vec{v}_2 \iint b\,db\,d\phi \, |\vec{v}_1 - \vec{v}_2| \left\{ f(\vec{v}_1, t) f(\vec{v}_2, t) \right\} \Big|_{z=-\infty}^{z=+\infty}$$

From the definition of $\vec{v}_1$, $\vec{v}_2$ this is equal to

$$n_0 \int d\vec{v}_2 \iint b\,db\,d\phi \, |\vec{v}_1 - \vec{v}_2| \left\{ f(\vec{v}_1{}', t) f(\vec{v}_2{}', t) - f(\vec{v}_1, t) f(\vec{v}_2, t) \right\} \tag{C.5}$$

where $\vec{v}_1{}'$, $\vec{v}_2{}'$ are the velocities two particles would have before a Coulomb collision if they have the velocities $\vec{v}_1$, $\vec{v}_2$ afterward.

APPENDIX D

A USEFUL IDENTITY

To prove the relation (8.41),

$$\left(\frac{\partial}{\partial \vec{v}_2}-\frac{\partial}{\partial \vec{v}_1}\right)\cdot\int d\vec{k}\,\frac{\vec{k}\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}=2\frac{\vec{v}_1-\vec{v}_2}{|\vec{v}_1-\vec{v}_2|^2}\int d\vec{k}\,\frac{\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{k^2|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \tag{D.1}$$

we first observe that

$$\int d\vec{k}\,\frac{\vec{k}\vec{k}\,\delta(\vec{k}\cdot\vec{q})}{k^4|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}=\left(\mathbb{1}-\frac{\vec{q}\vec{q}}{q^2}\right)\frac{1}{2}\int d\vec{k}\,\frac{\delta(\vec{k}\cdot\vec{q})}{k^2|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} \tag{D.2}$$

where $\vec{q}=\vec{v}_1-\vec{v}_2$ and $\mathbb{1}$ is the unit tensor. Because of the delta function, the integrands in (D.2) can be symmetrized by replacing $\frac{1}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}$ by $\frac{1}{2}\left[\frac{1}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_1)|^2}+\frac{1}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\right]$. Operating on the left hand side of (D.2) with $\partial/\partial\vec{v}_2-\partial/\partial\vec{v}_1$ will give the left hand side of (D.1), call it $\vec{T}$,

$$\vec{T}=\tfrac{1}{2}\left(\frac{\partial}{\partial \vec{v}_2}-\frac{\partial}{\partial \vec{v}_1}\right)\cdot\int d\vec{k}\,\frac{\vec{k}\vec{k}\,\delta(\vec{k}\cdot\vec{q})}{k^4}\left[\frac{1}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_1)|^2}+\frac{1}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\right] \tag{D.3}$$

$$=\frac{1}{2}(-2)\int d\vec{k}\,\frac{\vec{k}\vec{k}}{k^4}\cdot\frac{\partial\delta(\vec{k}\cdot\vec{q})}{\partial\vec{v}_1}\left[\frac{1}{|\varepsilon|^2}+\frac{1}{|\varepsilon|^2}\right]+\frac{1}{2}\int d\vec{k}\,\frac{\vec{k}\vec{k}\,\delta(\vec{k}\cdot\vec{q})}{k^4}\cdot\left[\frac{\partial}{\partial\vec{v}_2}\frac{1}{|\varepsilon|^2}-\frac{\partial}{\partial\vec{v}_1}\frac{1}{|\varepsilon|^2}\right]$$

The last term is zero, because of the delta function. The first term can be rewritten as

$$\vec{T}=-\int d\vec{k}\,\frac{\vec{k}\,\delta'(\vec{k}\cdot\vec{q})}{k^2}\left[\frac{1}{|\varepsilon|^2}+\frac{1}{|\varepsilon|^2}\right] \tag{D.4}$$

where $\delta'(x) = \frac{d}{dx}\delta(x)$. Operating on the right hand side of (D.2) with $\partial/\partial\vec{v}_2 - \partial/\partial\vec{v}_1$, results again in $\vec{T}$,

$$\vec{T} = \left\{ \left(\frac{\partial}{\partial\vec{v}_2} - \frac{\partial}{\partial\vec{v}_1}\right)\left(\mathbb{1} - \frac{\vec{q}\vec{q}}{q^2}\right)\frac{1}{4}\int d\vec{k}\,\frac{\delta(\vec{k}\cdot\vec{q})}{k^2}\left[\frac{1}{|\varepsilon|^2} + \frac{1}{|\varepsilon|^2}\right] \right.$$
$$\left. + \frac{1}{4}\left(\mathbb{1} - \frac{\vec{q}\vec{q}}{q^2}\right)\cdot\left(\frac{\partial}{\partial\vec{v}_2} - \frac{\partial}{\partial\vec{v}_1}\right)\cdot\int d\vec{k}\,\frac{\delta(\vec{k}\cdot\vec{q})}{k^2}\left[\frac{1}{|\varepsilon|^2} + \frac{1}{|\varepsilon|^2}\right] \right. \tag{D.5}$$

In the last term the derivatives of the dielectric constants integrate to zero, because of the delta function, and the remainder is

$$-\frac{1}{2}\left(\mathbb{1} - \frac{\vec{q}\vec{q}}{q^2}\right)\circ\int d\vec{k}\,\frac{\vec{k}\,\delta'(\vec{k}\cdot\vec{q})}{k^2}\left[\frac{1}{|\varepsilon|^2} + \frac{1}{|\varepsilon|^2}\right]$$

$$= \frac{1}{2}\left(\mathbb{1} - \frac{\vec{q}\vec{q}}{q^2}\right)\cdot\vec{T}$$

where we have used (D.4). To reduce the first term of (D.5) we make use of

$$\frac{\partial}{\partial\vec{q}} = \hat{q}\frac{\partial}{\partial q} + \frac{1}{q}\frac{\partial}{\partial\hat{q}}$$

where $\hat{q} = \vec{q}/q$. Therefore,

$$\left(\frac{\partial}{\partial\vec{v}_2} - \frac{\partial}{\partial\vec{v}_1}\right)\left(\mathbb{1} - \frac{\vec{q}\vec{q}}{q^2}\right) = -2\frac{\partial}{\partial\vec{q}}\left(\mathbb{1} - \hat{q}\hat{q}\right)$$
$$= \frac{4}{q}\hat{q}$$
$$= 4\,\vec{q}/q^2$$

Equation (D.5) is then

$$\vec{T} = 2\frac{\vec{q}}{q^2}\int d\vec{k}\,\frac{\delta(\vec{k}\cdot\vec{q})}{k^2|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2} + \frac{1}{2}\left(\mathbb{1} - \frac{\vec{q}\vec{q}}{q^2}\right)\cdot\vec{T} \tag{D.6}$$

Contracting both sides with $\vec{q}$ yields

$$\vec{q} \cdot \vec{T} = 2 \int d\vec{k} \frac{\delta(\vec{k} \cdot \vec{q})}{k^2 \, |\varepsilon(\vec{k}, \vec{k} \cdot \vec{v}_2)|^2} \tag{D.7}$$

Therefore, from (D.6) and (D.7) we find

$$\vec{T} = 2 \frac{\vec{q}}{q^2} \int d\vec{k} \frac{\delta(\vec{k} \cdot \vec{q})}{k^2 \, |\varepsilon(\vec{k}, \vec{k} \cdot \vec{v}_2)|^2} \tag{D.8}$$

which is the desired relation, since $\vec{T}$ is the left hand side of (D.1).

APPENDIX E

SOLUTION FOR $\mathrm{Im}\,\mathcal{H}(\vec{v}_1)$

To obtain the solution of (9.21) for $\mathrm{Im}\,\mathcal{H}(\vec{v}_1)$ we use a method employed by Lenard.[22] In terms of the $\mathcal{H}(\vec{v})$ function Eq. (9.21) takes the form,

$$\mathcal{H}(\vec{v}_1)\left\{1+\frac{e^2 n_0}{m}\hat{\phi}(k)\int d\vec{v}_2\,\frac{\vec{k}\cdot\dfrac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2}}{\vec{k}\cdot(\vec{v}_1-\vec{v}_2)-ik\nu}\right\}=$$

$$+\frac{e^2 n_0}{m}\hat{\Psi}(k)\int d\vec{v}_2\,\frac{\vec{k}}{\vec{k}\cdot(\vec{v}_1-\vec{v}_2)-ik\nu}\cdot\left(\frac{\partial}{\partial\vec{v}_1}-\frac{\partial}{\partial\vec{v}_2}\right)f(\vec{v}_1,t)f(\vec{v}_2,t) \tag{E.1}$$

$$+\frac{e^2 n_0}{m}\hat{\phi}(k)\int d\vec{v}_2\,\frac{\vec{k}\,\mathcal{H}^*(\vec{v}_2)}{\vec{k}\cdot(\vec{v}_1-\vec{v}_2)-ik\nu}\cdot\frac{\partial f(\vec{v}_1,t)}{\partial\vec{v}_1}+R_L n_0\hat{\Psi}(k)f(\vec{v}_1,t)$$

The pole arising from division of (9.21) by $\vec{k}\cdot(\vec{v}_1-\vec{v}_2)$ has been prescribed by writing $\vec{k}\cdot(\vec{v}_1-\vec{v}_2)-ik\nu$ where the limit $\nu\to 0^+$ is implied. This prescription corresponds to the boundary condition (7.27),

$$h_2(x_1,x_2;f)\to 0 \tag{E.2}$$

when

$$|\vec{r}_1-\vec{r}_2|\to\infty\quad,\quad(\vec{r}_1-\vec{r}_2)\cdot(\vec{v}_1-\vec{v}_2)<0$$

The time reversible character of our equations is destroyed by this boundary condition.

If we insert the transform of the Coulomb potential, $\hat{\phi} = 4\pi/k^2$, in the left hand side of (E.1) we recognize the quantity in brackets as the electron plasma dielectric constant (5.13) for the distribution function $f(\vec{v},t)$,

$$\mathcal{E}(\vec{k},\vec{k}\cdot\vec{v}_1) \doteq 1 + \frac{\omega_p^2}{k^2}\int d\vec{v}_2 \frac{\vec{k}\cdot\frac{\partial f(\vec{v}_2,t)}{\partial \vec{v}_2}}{\vec{k}\cdot(\vec{v}_1-\vec{v}_2)-ik\nu}$$

or letting $\nu \to 0^+$

$$\mathcal{E}(\vec{k},\vec{k}\cdot\vec{v}_1) = 1 + \frac{\omega_p^2}{k^2}\mathcal{P}\int d\vec{v}_2 \frac{\vec{k}\cdot\frac{\partial f}{\partial \vec{v}_2}}{\vec{k}\cdot(\vec{v}_1-\vec{v}_2)} + i\frac{\pi\omega_p^2}{k^2}\int d\vec{v}_2\, \vec{k}\cdot\frac{\partial f}{\partial \vec{v}_2}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)] \tag{E.3}$$

where $\mathcal{P}$ means "principle value." Note that $\mathcal{E}(\vec{k},\vec{k}\cdot\vec{v}_1)$ in general depends on $\vec{k}$ as well as $\vec{k}\cdot\vec{v}_1$. It is only when $f(\vec{v},t)$ is symmetric in $\vec{v}$ that $\mathcal{E} = \mathcal{E}(|\vec{k}|,\vec{k}\cdot\vec{v}_1)$.

The solution of (E.1) for $\mathrm{Im}\, H(\vec{v}_1)$ is considerably simplified if we define the following quantities

$$\begin{aligned} &\vec{k}\cdot\vec{v}_1 = ku_1 \qquad\qquad \vec{k}\cdot\vec{v}_2 = ku_2 \\ &f(u,t) = \int d\vec{v}\, f(\vec{v},t)\,\delta\left[u-\frac{\vec{k}\cdot\vec{v}}{k}\right] \\ &H(u) = \int d\vec{v}\, H(\vec{v})\,\delta\left[u-\frac{\vec{k}\cdot\vec{v}}{k}\right] \end{aligned} \tag{E.4}$$

Using (E.3) and (E.4) expression (E.1) reduces to

$$\mathcal{H}(\vec{v}_1)\,\mathcal{E}(\vec{k},ku_1) = \frac{e^2 n_0}{m}\check{\Psi}(k)\int du_2 \frac{1}{u_1-u_2-i\nu}\left(\frac{\partial}{\partial u_1}-\frac{\partial}{\partial u_2}\right) f(\vec{v}_1,t)\, f(\vec{u}_2,t)$$

$$+\frac{e^2 n_0}{m}\check{\phi}(k)\,\frac{\partial f(\vec{v}_1,t)}{\partial u_1}\int du_2 \frac{\mathcal{H}^*(u_2)}{u_1-u_2-i\nu} + R_L n_0 \check{\Psi}(k)\, f(\vec{v}_1,t) \tag{E.5}$$

Multiplying (E.5) by $\delta\left[u_1-\frac{\vec{k}\cdot\vec{v}_1}{k}\right]$ and integrating with respect to $\vec{v}_1$ gives

$$\mathcal{H}(u_1)\,\mathcal{E}(\vec{k},ku_1) = \frac{e^2 n_0}{m}\check{\Psi}(k)\int du_2 \frac{1}{u_1-u_2-i\nu}\left(\frac{\partial}{\partial u_1}-\frac{\partial}{\partial u_2}\right) f(u_1,t)\, f(\vec{u}_2,t)$$

$$+\frac{e^2 n_0}{m}\check{\phi}(k)\,\frac{\partial f(u_1,t)}{\partial u_1}\int du_2 \frac{\mathcal{H}^*(u_2)}{u_1-u_2-i\nu} + R_L n_0 \check{\Psi}(k)\, f(u_1,t) \tag{E.6}$$

The terms involving $\mathcal{H}^*(u_2)$ can be eliminated by taking the following combination of (E.5) and (E.6)

$$\left\{\frac{\partial f(u_1,t)}{\partial u_1}\mathcal{H}(\vec{v}_1) - \frac{\partial f(\vec{v}_1,t)}{\partial u_1}\mathcal{H}(u_1)\right\}\mathcal{E}(\vec{k},ku_1) =$$

$$+\frac{e^2 n_0}{m}\check{\Psi}(k)\,\frac{\partial f(u_1,t)}{\partial u_1}\int du_2 \frac{1}{u_1-u_2-i\nu}\left(\frac{\partial}{\partial u_1}-\frac{\partial}{\partial u_2}\right) f(\vec{v}_1,t)\, f(u_2,t)$$

$$-\frac{e^2 n_0}{m}\check{\Psi}(k)\,\frac{\partial f(\vec{v}_1,t)}{\partial u_1}\int du_2 \frac{1}{u_1-u_2-i\nu}\left(\frac{\partial}{\partial u_1}-\frac{\partial}{\partial u_2}\right) f(u_1,t)\, f(u_2,t) \tag{E.7}$$

$$+ R_L n_0 \check{\Psi}(k)\left\{ f(\vec{v}_1,t)\frac{\partial f(u_1,t)}{\partial u_1} - f(u_1,t)\frac{\partial f(\vec{v}_1,t)}{\partial u_1}\right\}$$

After some rearrangement this becomes

$$\frac{\partial f(u_1,t)}{\partial u_1} H(\vec{v}_1) - \frac{\partial f(\vec{v}_1,t)}{\partial u_1} H(u_1) =$$

$$\frac{\hat{\Psi}(k)}{\mathcal{E}(\vec{k},ku_1)}\left[k_1 n_0 - \frac{e^2 n_0}{m}\int du_2 \frac{\partial f(u_2 t)/\partial u_2}{u_1-u_2-i\nu}\right]\left[f(\vec{v}_1,t)\frac{\partial f(u_1,t)}{\partial u_1} - f(u_1,t)\frac{\partial f(\vec{v}_1,t)}{\partial u_1}\right] \quad \text{(E.8)}$$

$$= \frac{k^2}{\mathcal{E}(\vec{k},ku_1)} \frac{\hat{\Psi}(k)}{4\pi}\left[1 + \frac{1}{k^2 R_D^2} - \mathcal{E}(\vec{k},ku_1)\right]\left[f(\vec{v}_1,t)\frac{\partial f(u_1,t)}{\partial u_1} - f(u_1,t)\frac{\partial f(\vec{v}_1,t)}{\partial u_1}\right]$$

From (E.8) we get the quantity $\operatorname{Im} H(\vec{v}_1)$ needed for the kinetic equation. If we use the fact, proven in Appendix F, that $H(u_1)$ is a real function we find

$$\operatorname{Im} H(\vec{v}_1) \frac{\partial f(u_1,t)}{\partial u_1} =$$

$$\operatorname{Im}\left[\frac{1}{\mathcal{E}(\vec{k},ku_1)}\right] \frac{k^2 \hat{\Psi}(k)}{4\pi}\left[1 + \frac{1}{k^2 R_D^2}\right]\left[f(\vec{v}_1,t)\frac{\partial f(u_1,t)}{\partial u_1} - f(u_1,t)\frac{\partial f(\vec{v}_1,t)}{\partial u_1}\right] \quad \text{(E.9)}$$

From the definition of $\mathcal{E}(\vec{k},ku_1)$, (E.3), we see that

$$\operatorname{Im}\frac{1}{\mathcal{E}(\vec{k},ku_1)} = -\frac{\pi\omega_p^2}{k^2}\frac{\partial f(u_1,t)/\partial u_1}{|\mathcal{E}(\vec{k},ku_1)|^2} \quad \text{(E.10)}$$

Equation (E.9) then reduces to

$$\operatorname{Im} H(\vec{v}_1) = -\frac{\omega_p^2}{4}\frac{\hat{\Psi}(k)\left[1+\frac{1}{k^2R_D^2}\right]}{|\mathcal{E}(\vec{k},ku_1)|^2}\left\{f(\vec{v}_1,t)\frac{\partial f(u_1,t)}{\partial u_1} - f(u_1,t)\frac{\partial f(\vec{v}_1,t)}{\partial u_1}\right\} \quad \text{(E.11)}$$

Inserting (E.11) in (9.25) and using (E.4) the kinetic equation is

$$\frac{\partial f(\vec{v}_1,t)}{\partial t} =$$

$$-\frac{e^4 n_{oi}}{8\pi^2 m^2}\frac{\partial}{\partial \vec{v}_1}\cdot\int d\vec{v}_2\int d\vec{k}\,\hat{\psi}(k)\left[1+\frac{1}{k^2 R_D^2}\right]\frac{\vec{Q}(\vec{k})\vec{k}\,\delta[\vec{k}\cdot(\vec{v}_1-\vec{v}_2)]}{|\varepsilon(\vec{k},\vec{k}\cdot\vec{v}_2)|^2}\cdot\left(\frac{\partial}{\partial\vec{v}_1}-\frac{\partial}{\partial\vec{v}_2}\right)f(\vec{v}_1,t)f(\vec{v}_2,t) \qquad \text{(E.12)}$$

where $\vec{Q}(\vec{k})$ is given by (9.24).

APPENDIX F

PROOF OF THE REALITY OF $\mathcal{H}(u)$

In order to show that $\mathcal{H}(u)$ is a real function we use the Wiener-Hopf method,[28] and pass into the complex plane through the following definitions

$$
\begin{aligned}
\mathcal{H}(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathcal{H}(u)\,du}{u-z} \\
P(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathcal{H}^{*}(u)\,du}{u-z} \\
f(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(u,t)\,du}{u-z} \qquad \text{(F.1)} \\
\mathcal{E}(z) &= 1-\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{\partial f(u,t)/\partial u}{u-z}\,du
\end{aligned}
$$

The function $\mathcal{H}(z)$, for example, will be an analytic function of z except on the real line, which is a branch cut, provided $\mathcal{H}(u)$ satisfies a Hölder condition on the real line.[28] A superscript "+" or "-" will denote the limiting values of the functions in (F.1) when z approaches the real axis from above or below respectively. For a typical function, say $\mathcal{H}(z)$, we have the Plemelj relations[28]

$$
\begin{aligned}
\mathcal{H}^{+}(u) - \mathcal{H}^{-}(u) &= \mathcal{H}(u) \\
\mathcal{H}^{+}(u) + \mathcal{H}^{-}(u) &= -i\,\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{\mathcal{H}(u')\,du'}{u'-u}
\end{aligned}
\qquad \text{(F.2)}
$$

Equation (E.6) for $\mathcal{H}(u)$ can therefore be written as

$$\left[\mathcal{H}^+ - \mathcal{H}^-\right]\left[\varepsilon^- - 1\right] =$$

$$\frac{k^2 \hat{\Psi}}{4\pi}\left\{f^-\left[\varepsilon^+ - \varepsilon^-\right] - \left[f^+ - f^-\right]\left[\varepsilon^- - 1\right]\right\} + \left[\varepsilon^+ - \varepsilon^-\right] P^- + R_L n_0 \hat{\Psi}\left[f^+ - f^-\right] \qquad \text{(F.3)}$$

The complex conjugate of Eq. (E.6) becomes

$$\left[P^+ - P^-\right]\left[\varepsilon^+ - 1\right] =$$

$$\frac{k^2 \hat{\Psi}}{4\pi}\left\{f^+\left[\varepsilon^+ - \varepsilon^-\right] - \left[f^+ - f^-\right]\left[\varepsilon^+ - 1\right]\right\} + \left[\varepsilon^+ - \varepsilon^-\right] \mathcal{H}^+ + R_L n_0 \hat{\Psi}\left[f^+ - f^-\right] \qquad \text{(F.4)}$$

Subtracting (F.4) from (F.3) yields, after collection of terms

$$\left[\varepsilon^+ - 1\right]\left[\mathcal{H}^+ - P^+\right] = \left[\varepsilon^- - 1\right]\left[\mathcal{H}^- - P^-\right] \qquad \text{(F.5)}$$

The left hand side is an analytic function in the upper half complex z plane while the right hand side is analytic in the lower half z plane. Equation (F.5) says they are equal at the cut along the real axis, and therefore $[\varepsilon - 1][\mathcal{H} - P]$ must be an entire analytic function. Since this function vanishes at $|z| = \infty$, Liouville's theorem* asserts that it is zero everywhere. In particular (since $\varepsilon \neq 1$)

$$\mathcal{H}(u) = P(u) = \mathcal{H}^*(u)$$

and this shows $\mathcal{H}(u)$ to be a real function of u.

---

*Liouville's theorem asserts: A function that is analytic and bounded in the whole plane must be a constant.

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