ON THE KINETIC THEORY OF THE LORENTZ GAS

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ABSTRACT

A study is made of the nonequilibrium behavior of the classical Lorentz gas. The particle distribution is found to satisfy a dynamical equation that generalizes the Lorentz-Boltzmann equation. The first term in a density expansion gives the Enskog generalization of the Lorentz-Boltzmann collision integral.

The analysis is carried out by expanding the system streaming operator in a cluster expansion and a Husimi-like expansion. Use of the binary collision expansion allows an interpretation of the terms in the collision operators.

The higher density collision operator terms are shown to diverge in the long time limit. The divergence is removed by a ring diagram technique, and a simple prescription is given for the general renormalized terms.

A formula is obtained for the Lorentz gas conductivity using the renormalized collision operator.

The one dimensional Lorentz gas particle distribution is completely evaluated. Renormalized and unrenormalized calculations are carried out for the wind-tree and the hard disks models, and the nonanalyticity of the latter model is exhibited.
I. HISTORICAL BACKGROUND

Recent nonequilibrium\textsuperscript{1-7} theories of dense gases attempt a fundamental derivation of the Boltzmann equation replacing the original physical, intuitive argument\textsuperscript{8,9} by an analytical treatment based on the Liouville equation using the Gibbs ensemble theory.\textsuperscript{10,11} These theories have two goals: (1) to verify the Stosszahlansatz* from the mechanics, and (2) to generalize the binary collision integral to an arbitrary number of particles.

A related nonequilibrium problem concerns the Lorentz gas consisting of a large number of fixed scattering centers and a small number of non-interacting scattered particles. Lorentz's application\textsuperscript{12} of the Boltzmann gas theory to the kinetic theory of electron motion in a metal led to the Lorentz-Boltzmann equation,

\[
\frac{\partial}{\partial t} f(\vec{r}, \vec{p}; t) + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{r}} f(\vec{r}, \vec{p}; t) + \frac{\vec{F}}{\vec{p}} \cdot \frac{\partial}{\partial \vec{p}} f(\vec{r}, \vec{p}; t) = \left[ \frac{\partial}{\partial t} f(\vec{r}, \vec{p}; t) \right]_{\text{coll}},
\]

(1.1)

where \(f(\vec{r}, \vec{p}; t)\) is the distribution of electrons. The right-hand side of Eq. (1.1) is the change in the distribution per unit time due to collisions with the heavy scatterers in the metal. Lorentz wrote this term guided by the Boltzmann gas theory, using a Stosszahlansatz to obtain the number of binary collisions (single encounters with the scatterers),

*For a clear exposition of the Stosszahlansatz see Reference 9, page 78.
\[
\left[ \frac{\partial}{\partial t} f(\vec{r}, \vec{p}; t) \right] \bigg|_{\text{coll}} = n \int I \nu d\Omega \left[ f(\vec{r}, \vec{p}'; t) - f(\vec{r}, \vec{p}; t) \right], \tag{1.2}
\]

where \( n \) is the density of scatterers, \( I \) is the differential cross section, \( \nu \) is the constant speed of the electrons (elastic scattering), \( d\Omega \) is the infinitesimal solid angle describing the point of collision, and \( \vec{p}' \) is the restituting momentum. Lorentz used Eq. (1.1) as a basis for a calculation of the conductivity. A fundamental derivation of the Lorentz-Boltzmann equation parallels that of the Boltzmann equation: to wit a mechanical verification of the Stosszahlansatz and a generalization of the binary collision integral to higher order in the density.

The first mechanical derivations of the Boltzmann equation based on the Liouville equation were due to Kirkwood\(^2\) and to Born and Green.\(^1\) Using the BBGKY hierarchy\(^9\) of equations, obtained from the Liouville equation by projection onto a subspace of the \( 6N \) dimensional ensemble phase space, they verified the Boltzmann collision integral. These derivations essentially replace the assumption of molecular chaos\(^8,9,11\) with the equivalent assumption of neglecting correlations among the gas particles at the initial time, and replace the intuitive, physical diagram\(^8,9\) of the collision between two gas particles by the mechanics of the Liouville equation. The Kirkwood and the Born and Green theories did not provide a scheme for generalizing the Boltzmann collision integral. Their techniques could be applied to the Lorentz gas as well.

Bogoliubov\(^3\) presented a different approach giving a scheme for a systematic generalization of the binary collision integral. As with the pre-
vious theories, Bogoliubov begins with the BBGKY hierarchy; however, he used a different replacement of the molecular chaos condition by restricting the time dependence of the m-particle distributions to that of the single particle distribution (this is an expression of the Gibbs idea of coarse-graining, a necessary requirement for molecular chaos), and by removing correlations at an infinite time in the past.

Choh and Uhlenbeck\textsuperscript{13} used the Bogoliubov theory to obtain a formal expression for the triple collision integral and to obtain the Enskog expression\textsuperscript{14} for the binary collision integral taking into account the spatial dependence of the collision. This technique could be applied to the Lorentz gas giving an expression for the collision integral with two scatterers and the spatially dependent single scatterer integral.

Many derivations have appeared since the early theories, e.g., those of M. S. Green,\textsuperscript{4} Cohen,\textsuperscript{5} and Ford.\textsuperscript{6} Cohen\textsuperscript{15} discussed the Kirkwood and Bogoliubov theories demonstrating their agreement up to and including the first correction to the Boltzmann collision integral. Green\textsuperscript{16} formally verified Bogoliubov's theory by demonstrating that under certain conditions the m-particle distributions depend on time only through the time dependence of the single particle distribution, and gave a simpler prescription for obtaining the higher collision integrals than the Bogoliubov theory.

Cohen,\textsuperscript{5} following the expansion techniques of the equilibrium theory of gases, used Ursell\textsuperscript{17} and Husimi\textsuperscript{18} expansions to obtain formal expressions for the m-particle collision integrals. Ford\textsuperscript{6} has developed an elegant technique for obtaining the general collision integral which avoids the
use of the BBGKY hierarchy. Other derivations include the works of Prigogine,7 Zwanzig,19 van Leeuwen and Zip,20 Weinstock,21 Swenson,22 etc.

Until recently23-26 it was thought that the generalized collision integrals and generalized transport coefficients were finite and formed a decreasing series in powers of $n \sigma^3$, where $n$ is the particle density and $\sigma$ is the radius of the gas particles. It is now accepted that the density expansions in the nonequilibrium theory of dense gases and of the Lorentz gas are invalid in the long time limit because the successive terms form a power series in time.

The divergence carries over into the calculation of transport coefficients as well. The higher density terms in Zwanzig's expansion18 of transport coefficients diverge in almost the same way as the collision integrals. Sengers22 computed the first density correction to the viscosity showing that in the first Enskog approximation the triple collision contribution diverges logarithmically for two-dimensional hard spheres.

The exact behavior of the divergence has only been computed in a few cases Sengers,22 and Haines.24 Some estimates based on the phase space volume show that in three dimensions the m-particle collision integral diverges23 as $\ln t$ for $m = 4$ and as $t^{m-4}$ for $m > 4$, and the m-particle term in the auto-correlation function diverges25 as $t^{m-4} \ln t$ for $m \geq 4$.

Kawasaki and Oppenheim27 were the first authors to apply the ring diagram technique of Montroll and Mayer28 to the generalized transport coefficients of a dense gas by summing the most divergent parts. They estimated that their first ring diagram term (first renormalized term) is a logarithmic
function of the density and therefore that the transport coefficients are nonanalytic functions of the density. Weinstock\textsuperscript{25} and Swenson\textsuperscript{29} also proposed that the divergence of the density power series gives rise to a nonanalytic, convergent expression for the transport coefficients.

Van Leeuwen and Weijland\textsuperscript{30} computed the diffusion coefficient of a Lorentz gas showing that the first ring diagram contributes a term proportional to the logarithm of the density in the inverse diffusion coefficient. They also found that the coefficient of the logarithmic term is identical with the coefficient of the first divergent term. Fujita\textsuperscript{31} has carried out a calculation of the transport coefficients for a quantum mechanical Lorentz gas, and found divergent terms using a weak short-range potential. Fujita's first ring diagram term does not contain a logarithmic density dependence.

This dissertation treats the Lorentz gas, obtaining the general term in the formal dynamical equation for the particle distribution. A simple representation for the general collision integral which is a direct generalization of the Lorentz-Boltzmann collision integral, Eq. (1.2), is found. Some model calculations are carried out demonstrating the divergence of the density series of the collision integrals. The ring diagram technique is used to sum terms of any order in the divergence, and is transformed into a simple form which demonstrates the term-by-term convergence of the renormalized series. Some calculations are carried out for the renormalized terms of the collision integrals demonstrating that the specific functional form depends on the physical model chosen for the scatterers. A formula
is obtained for the conductivity of the Lorentz gas, and the close relationship between the collision integral and the inverse transport coefficient is demonstrated. Some renormalized calculations are carried out for the conductivity.
A. THE LORENTZ GAS

We will treat the Lorentz gas and the problem of obtaining a convergent generalization of the Lorentz-Boltzmann equation\(^{12}\) because it is simpler than the nonequilibrium theory of dense gases, and yet it contains most of the features that would interest us in the dense gas problem. The Lorentz gas is a multiple scattering system consisting of one point particle which scatters with a large number, \(N\), of fixed elastic hard core scatterers.

We denote the coordinates of the scattering centers by \(\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_N\) or simply by the set of labels, 1, 2, ..., \(N\), and the position and momentum of the particle by \(\vec{r}\) and \(\vec{p}\). The particle motion is governed by the classical Hamilton equations with Hamiltonian,

\[
\mathcal{H}(1, 2, \ldots, N) = \frac{\vec{p}^2}{2m} + \sum_{i=1}^{N} \sqrt{(\vec{r} - \vec{\xi}_i)^2},
\]

(2.1)

where \(\mathcal{H}\) is symmetric in the \(N\) scatterer labels. The potential, \(V\), which depends only on the magnitude of the distance between the particle and the scattering center has the form,

\[
V(r) = \begin{cases} 
\infty & r < \sigma \\
0 & r > \sigma 
\end{cases},
\]

(2.2)

where \(\sigma\) is the radius of the hard core. The fact that the potential is not finite presents no problems since it does not appear in any final calculations; it does appear in formal manipulations where it may be thought of as the well-behaved potential, \((\sigma/r)^n\), with the limit \(n \to \infty\) taken after the formal manipulations have been carried out.
The particle distribution, \( D(l_2 \ldots N, \hat{r}, \hat{p}; t) \), determines the probability that the particle position and momentum are in the phase volume \( d\hat{r}d\hat{p} \) about \( \hat{r}, \hat{p} \) for a given configuration of scatterers, and is normalized to unity,

\[
\int d\hat{r} \int d\hat{p} \ D(12 \ldots N, \hat{r}, \hat{p}; t) = 1
\]  

(2.3)

We assume that the distribution, \( D(l_2 \ldots N, \hat{r}, \hat{p}; t) \), satisfies the Liouville equation,

\[
\frac{\partial}{\partial t} D = \{ H, D \}
\]  

(2.4)

where in general the Poisson bracket of two functions \( A(\hat{r}, \hat{p}) \) and \( B(\hat{r}, \hat{p}) \) is,

\[
\{ A, B \} = \frac{\partial A}{\partial \hat{r}} \cdot \frac{\partial B}{\partial \hat{p}} - \frac{\partial B}{\partial \hat{r}} \cdot \frac{\partial A}{\partial \hat{p}}
\]  

(2.5)

The formal solution of the Liouville equation may be written as a linear operator acting on the initial distribution,

\[
D(12 \ldots N, \hat{r}, \hat{p}; t) = S(12 \ldots N; t)D(12 \ldots N, \hat{r}, \hat{p}; 0).
\]  

(2.6)

The system streaming operator, \( S(l_2 \ldots N; t) \), is that operator which transforms the initial distribution into the distribution at time \( t \).

Substitution of the solution given by Eq. (2.6) into the Liouville equation shows that the streaming operator obeys the operator equation,

\[
\frac{\partial}{\partial t} S(l_2 \ldots N; t) = \{ H(l_2 \ldots N), S(l_2 \ldots N; t) \}
\]  

(2.7)

which has the formal solution,

*The streaming operator is frequently written in the form, \( S_{-t}(l_2 \ldots N) \), where \( S_{-t}(l_2 \ldots N) = S(l_2 \ldots N; t) \).
\[ S(12 \cdots N; t) = \exp \left( t \left\{ H(12 \cdots N), \{ \right\} \right) \right), \quad (2.8) \]

where this operator when acting on any function \( F(\vec{r}, \vec{p}) \) represents the series,

\[
F + t \left\{ H, F \right\} + \frac{t^2}{2!} \left\{ H, \left\{ H, F \right\} \right\} + \cdots \quad (2.9)
\]

A further aspect of the streaming operator results from considering how it acts on the particle canonical variables \( \vec{r}, \vec{p} \). We denote by \( \vec{R}_t, \vec{P}_t \) the canonical variables obtained from \( S(12 \cdots N; t) \) acting on \( \vec{r}, \vec{p} \),

\[
(\vec{R}_t, \vec{P}_t) = S(12 \cdots N; t)(\vec{r}, \vec{p}) \quad (2.10)
\]

One can verify (using a simple Hamiltonian and the exponential representation of Eq. (2.8), that \( \vec{R}_t, \vec{P}_t \) are those variables which evolve into \( \vec{r}, \vec{p} \) in a time \( t \) under the influence of the system Hamiltonian. This viewpoint of the streaming operator will be used extensively in the succeeding chapters. An equivalent view is that \( \vec{R}_t, \vec{P}_t \) are the particle position and momentum obtained from \( \vec{r}, \vec{p} \) by carrying the motion backward for a time \( t \) under the influence of the system Hamiltonian. Using this representation of the streaming operator, we may write the solution of the Liouville equation as,

\[
\mathcal{D}(12 \cdots N, \vec{r}, \vec{p}; t) = \mathcal{D}(12 \cdots N, \vec{R}_t, \vec{P}_t; 0). \quad (2.11)
\]

It may be verified that the right-hand side of Eq. (2.11) does indeed satisfy the Liouville equation by using the chain rule for differentiation.
and the operator equation for $S(12 \ldots N; t)$.

B. THE PARTICLE DISTRIBUTION $f(\vec{r}, \vec{p}; t)$

We previously introduced the particle distribution, $D(12 \ldots N, \vec{r}, \vec{p}; t)$, which in general depends on the positions of the $N$ scattering centers and may therefore be considered a microscopic quantity. To obtain the particle distribution which is independent of the scattering center positions, the macroscopic particle distribution $f(\vec{r}, \vec{p}; t)$, we introduce the distribution of scatterers, $P(12 \ldots N)$, which is symmetric in the $N$ labels and has unit normalization,

$$\int \cdots \int \, d\vec{s}_1 \cdots d\vec{s}_N \, P(12 \ldots N) = 1 \quad (2.12)$$

We note that the scatterer distribution is time independent and may be specified in any manner corresponding to the particular type of multiple scattering system we wish to study.

The $m$-scatterer distribution (reduced distributions),

$$P^{(m)}(12 \ldots m) = \frac{N!}{(N-m)!} \int \cdots \int \, d\vec{s}_{m+1} \cdots d\vec{s}_N \, P(12 \ldots N) \quad (2.13)$$

describe the distribution of $m$ scatterers independent of the positions of the remaining scatterers. $P^{(m)}$ is normalized to the number of $m$-tuples, $N!/(N-m)!$, which can be chosen from the totality of scatterers.

We specify the particle distribution $f(\vec{r}, \vec{p}; t)$ as the product of the distribution of scatterers, $P(12 \ldots N)$, multiplied by the conditional particle distribution, $D(12 \ldots N, \vec{r}, \vec{p}; t)$, integrated over all scatterer
positions,
\[
f'(\vec{r}, \vec{p}; t) = \int \int d\vec{s}_1 \cdots d\vec{s}_N \quad \mathcal{P}(\cdots N) \quad D(\cdots N, \vec{r}, \vec{p}; t) . \tag{2.14}
\]
f'(\vec{r}, \vec{p}; t) determines the probability that the particle has position and momentum in the infinitesimal phase volume d\vec{s}d\vec{p} about \vec{r}, \vec{p} independent of the scatterer positions, and is normalized to unity,
\[
\int \int d\vec{r} d\vec{p} \quad f'(\vec{r}, \vec{p}; t) = 1 . \tag{2.15}
\]
The particle distribution may be written using the initial value solution of the Liouville equation, involving the system streaming operator,
\[
f'(\vec{r}, \vec{p}; t) = \int \int d\vec{s}_1 \cdots d\vec{s}_N \quad \mathcal{P}(\cdots N) \quad S(\cdots N; t) \quad D(\cdots N, \vec{r}, \vec{p}; 0) . \tag{2.16}
\]
We observe that Eq. (2.16) is exact in the sense that no approximations or assumptions have been invoked other than the assumption that the system is described by the Liouville equation. However in making the contraction from a complete description of the system (where one specifies the positions of all the scattering centers in the initial distribution, \(D(l_2 \cdots N, \vec{r}, \vec{p}; 0)\)) to a closed description involving only the particle variables \(\vec{r}\) and \(\vec{p}\) we must make a statistical assumption on the initial distribution \(D(l_2 \cdots N, \vec{r}, \vec{p}; 0)\). We assume that the initial distribution, \(D(l_2 \cdots N, \vec{r}, \vec{p}; 0)\), is given by the initial particle distribution, \(f(\vec{r}, \vec{p}; 0)\), independent of the positions of the scattering centers, multiplied by the condition that the particle is excluded from the hard core interior of each scatterer,
\[ D(\cdot N, \vec{r}, \vec{p}; 0) = \prod_{i=1}^{N} \Theta( |\vec{r} - \vec{x}_i| - \sigma) f(\vec{r}, \vec{p}; 0), \]  

(2.17)

where the unit step function \( \Theta(|\vec{r} - \vec{x}_i| - \sigma) \) is one if \( |\vec{r} - \vec{x}_i| > \sigma \) and zero otherwise. By excluding the particle from the scattering center interiors we avoid infinite energy. With this assumption we obtain a closed description with the particle distribution at time \( t \), \( f(\vec{r}, \vec{p}; t) \), given by a time-dependent operator acting on the initial value distribution, \( f(\vec{r}, \vec{p}; 0) \),

\[ f(\vec{r}, \vec{p}; t) = \int \cdots d\vec{x}_N \cdots d\vec{x}_1 P(\cdot N) S(\cdot N; t) \cdot \prod_{i=1}^{N} \Theta( |\vec{r} - \vec{x}_i| - \sigma) f(\vec{r}, \vec{p}; 0). \]  

(2.18)

C. THE BINARY COLLISION EXPANSION

Since the hard core binary interaction between particle and scatterers makes a description in terms of sequences of single encounters natural and succinct, we expand the system streaming operator, \( S(12 \ldots N; t) \), in the binary collision expansion first developed by Lee and Yang,\(^{32}\) but in the form used by Zwanzig.\(^{19}\) We will expand the Laplace transform of the streaming operator thereby replacing cumbersome manyfold convolution integrals by products of Laplace-transformed operators.

The Laplace transform of a function \( F(t) \) is denoted by \( \mathcal{F}_s \),

\[ \mathcal{F}_s = \int_0^\infty dt \ e^{-st} F(t). \]  

(2.19)

Taking the Laplace transform of Eq. (2.18), and interchanging the time integral with the integral over the \( N \) scatterer coordinates, we find the formula for the Laplace-transformed particle distribution, \( f_s(\vec{r}, \vec{p}) \), as an
operator acting on the initial distribution,
\[
\begin{align*}
  f_s(\vec{r}, \vec{p}) &= \int \cdots \int d\vec{x}_1 \cdots d\vec{x}_N \ p(1 \cdots N) \ S_s(1 \cdots N) \cdot \\
  & \quad \cdot \prod_{i=1}^{N} \Theta(|\vec{r}_i - \vec{R}_{ii}| - \sigma) \ f(\vec{r}, \vec{p} ; 0) ,
\end{align*}
\]  
(2.20)

where \( S_s(1 \cdots N) \) denotes the Laplace transform of the system streaming operator. This result, Eq. (2.20), could also be obtained by taking the Laplace transform of the Liouville equation and then repeating the steps leading to Eq. (2.18). A formal representation of the Laplace-transformed streaming operator follows from the exponential representation of Eq. (2.8),
\[
S_s(1 \cdots N) = \left( s - \left\{ \mathcal{H}(1 \cdots N), \ \right\} \right)^{-1}.
\]
(2.21)

The binary collision operator, \( B(i) \), is defined in terms of the free streaming operator, \( S_s \),
\[
S_s = \left( s + \frac{\vec{p}}{m} \cdot \frac{d}{d\vec{r}} \right)^{-1},
\]
(2.22)

and the single-scatterer streaming operator, \( S_s(i) \),
\[
S_s(i) = \left( s + \frac{\vec{p}}{m} \cdot \frac{d}{d\vec{r}} - \left\{ \mathcal{V}(1 \cdots N), \ \right\} \right)^{-1},
\]
(2.23)

by the following equation,
\[
B(i) = S_s^{-1} \left[ S_s(i) - S_s \right].
\]
(2.24)

If we solve Eq. (2.24) for the potential operator \( \mathcal{V}(1 \cdots N) \), in terms of the binary collision operator,
\[
-\left\{ \sqrt{(1 - \frac{\vec{x}_i}{\vec{x}_j})}, \right\} = \left[ (1 + B(i))^{-1} \right] S_s^{-1}, \tag{2.25}
\]

then its insertion into formula (2.21), where
\[
\left\{ \mathcal{H}(1 \cdots N), \right\} = -\frac{2}{m} \frac{\partial}{\partial \vec{r}} + \sum_{i=1}^{N} \left\{ \sqrt{(1 - \frac{\vec{x}_i}{\vec{x}_j})}, \right\}, \tag{2.26}
\]
gives us a functional relationship between the system streaming operator and the binary collision operator,
\[
S_s(1 \cdots N) = S_s \left( 1 + \sum_{i=1}^{N} \left[ (1 + B(i))^{-1} \right] \right)^{-1}. \tag{2.27}
\]

This closed form expression for the binary collision expansion is shown in Appendix A to have the expanded form,
\[
S_s(1 \cdots N) = S_s \left[ 1 + \sum_{i=1}^{N} B(i) + \sum_{i,j=1}^{N} B(i) B(j) + \sum_{i,j,h=1}^{N} B(i) B(j) B(h) + \cdots \right]. \tag{2.28}
\]

Considering the sequence of labels corresponding to a product of binary collision operators, the expansion, Eq. (2.28), is a sum over all sequences of labels (from the set 12 ... N) occurring only once in succession. We may contrast this with the interaction expansion which is the sum over all sequences of labels occurring once or more than once in succession, and which reduces to the binary collision expansion by collecting together all products of operators having the same label.

This result, Eq. (2.28), may be generalized to include an arbitrary external force using the collision operator for interaction with the force.
(no scatterer interaction), $B(F)$, which is defined as,

$$B(F) = S_s^{-1} \left[ S_s(F) - S_s \right], \quad (2.29)$$

where $S_s(F)$ is the streaming operator,

$$S_s(F) = \left( s + \frac{\hat{\mathbf{F}} \cdot \hat{\mathbf{p}}}{m} + \frac{\hat{\mathbf{F}} \cdot \hat{\mathbf{p}}}{\gamma^2} \right)^{-1}. \quad (2.30)$$

The system streaming operator in the presence of an external force, $S_s(12 \ldots N,F)$, is given by Eq. (2.28) with the sum on labels $12 \ldots N$ now including the label $F$.

D. DENSITY EXPANSION OF THE PARTICLE DISTRIBUTION

The density expansion of the Laplace-transformed particle distribution, $f_s(\mathbf{r},\mathbf{p})$, may be obtained by grouping together all terms in the binary collision expansion of $S_s(12 \ldots N)$, Eq. (2.28), that contain a given number of labels, say $m$. We let $R(12 \ldots m)$ denote the sum of all products of binary collision operators whose sequences of labels contain each of the labels $1,2,\ldots,m$ at least once, but occurring only once in succession. $R(12 \ldots m)$ is therefore obtained from $S_s^{-1} S_s(12 \ldots N)$ by selecting all terms containing the $m$ labels $12 \ldots m$ and only these labels. This selection process represents a partitioning of the set $(12 \ldots N)$ into subsets $(12 \ldots m)$. Since $S_s(12 \ldots N)$ contains terms with any number of labels from zero to $N$, it may be written as the sum over all $R$ operators whose labels are the subsets of the set $(12 \ldots N)$,

$$S_s^{-1} S_s(1 \ldots N) = \sum_{\mathcal{A} \subset (1 \ldots N)} R(\mathcal{A}), \quad (2.31)$$
where \[ \sum_{A \subset \{12 \ldots N\}}^\mathbb{A} \] denotes the sum over all subsets, A, of the set \( \{12 \ldots N\} \), and \( R(A_0) = 1 \) where \( A_0 \) is the empty set.

Since Eq. (2.31), called the cluster expansion, is valid for any set \( \{12 \ldots N\} \) it is equivalent to the following series of expansions,

\[ S^{-1}_s S_s(1) = R(1) + 1, \]  
(2.32a)

\[ S^{-1}_s S_s(12) = R(12) + R(1) + R(2) + 1, \]  
(2.32b)

\[ S^{-1}_s S_s(123) = R(123) + R(12) + R(13) + R(23) + \]
\[ + R(1) + R(2) + R(3) + 1. \]  
(2.32c)

By successively inverting these equations we obtain the expansion of the \( R \) operators in terms of the streaming operators,

\[ R(1) = S^{-1}_s S_s(1) - 1, \]  
(2.33a)

\[ R(12) = S^{-1}_s S_s(12) - S^{-1}_s S_s(1) - S^{-1}_s S_s(2) + 1, \]  
(2.33b)

\[ R(123) = S^{-1}_s S_s(123) - S^{-1}_s S_s(12) - S^{-1}_s S_s(13) - S^{-1}_s S_s(23) + \]
\[ + S^{-1}_s S_s(1) + S^{-1}_s S_s(2) + S^{-1}_s S_s(3) - 1, \]  
(2.33c)

or we may use the powerful technique of the M"obius inversion formula\(^{33}\) to directly invert Eq. (2.31),

\[ R(12\ldots m) = \sum_{A \subset \{12\ldots m\}}^\mathbb{A} (-)^{m-V(A)} S^{-1}_s S_s(A), \]  
(2.34)

where \( V(A) \) is the number of elements in the set \( A \).
We note however that the R operators are succinctly specified as sums of products of binary collision operators as given previously. The first few R operators appear as follows,

\[ R(1) = B(1) \]  \hspace{1cm} (2.35a)  

\[ R(12) = \sum_{\text{perm}} \left[ B(1)B(2) + B(1)B(2)B(1) + \cdots \right], \]  \hspace{1cm} (2.35b)  

\[ R(123) = \sum_{\text{perm}} \left[ B(1)B(2)B(3) + B(1)B(2)B(1)B(3) + B(1)B(2)B(3)B(1) + \cdots \right], \]  \hspace{1cm} (2.35c)  

where the sum over symmetric permutations.

We now insert the cluster expansion of the streaming operator \( S_s(12...N) \) into the formula for \( f_s(\vec{r},\vec{p}) \) and relabel the terms,

\[ f_s(\vec{r},\vec{p}) = \sum_{m=0}^{N} \frac{1}{m!} \int \cdots \int d\vec{r}_m \cdots d\vec{r}_N \ S_s R(12\cdots N) \cdot \]  

\[ \cdot \frac{N!}{(N-m)!} \left( \int d\vec{r}_{m+1} \cdots d\vec{r}_N \right) \ P(12\cdots N) \cdot \]  

\[ \cdot \prod_{i=m+1}^{N} \Theta(1 - |\vec{r}_i - \vec{p}_i| - \sigma) \ f(\vec{r},\vec{p} ; \sigma), \]  \hspace{1cm} (2.36)  

with the factor \( N!/m!(N-m)! \) coming from the number of ways of relabeling the terms containing \( m \) labels in the \( R \) operator. We neglect the \( r \) dependence of the integral over the \( N-m \) scatterer coordinates of the distribution \( P(12...N) \sum_{i=m+1}^{N} \Theta(1 - |\vec{r}_i - \vec{p}_i| - \sigma) \) and denote it by the reduced distribution of Eq. (2.13),

\[ P^{(m)}(1\cdots m) = \frac{N!}{(N-m)!} \int \cdots \int d\vec{r}_{m+1} \cdots d\vec{r}_N \ P(1\cdots N) \prod_{i=m+1}^{N} \Theta(1 - |\vec{r}_i - \vec{p}_i| - \sigma). \]  \hspace{1cm} (2.37)
The analysis can be carried through without this approximation; however it is reasonable for a system of random scatterers.

The density expansion of $f^\rightarrow(\vec{r},\vec{p})$ is obtained by inserting Eq. (2.37) into Eq. (2.36),

$$f^\rightarrow_s(\vec{r},\vec{p}) = \sum_{m=0}^{N} \frac{1}{m!} \int \cdots \int d\vec{r}_1 \cdots d\vec{r}_m \ P^{(m)}(1\cdots m) \ S_s R(1\cdots m) \cdot \prod_{i=1}^{m} \Theta(l\vec{r}_i - \vec{s}_i - \sigma) f(\vec{r}_i, \vec{p}_i; 0),$$

where $P^{(m)}(12\cdots m)$ is proportional to $n^m$ where $n$ is the density of scatterers.

We denote the operator that gives $f^\rightarrow_s(\vec{r},\vec{p})$ when acting on the initial distribution $f(\vec{r},\vec{p}; 0)$ as the evolution operator, $\Gamma_s$,

$$f^\rightarrow_s(\vec{r},\vec{p}) = \Gamma_s f(\vec{r},\vec{p}; 0).$$

(2.39)

$\Gamma_s$ is the sum of the m-scatter evolution operators, $\Gamma_s^{(m)}$,

$$\Gamma_s = \sum_{m=0}^{N} \Gamma_s^{(m)},$$

(2.40)

where,

$$\Gamma_s^{(m)} = \frac{1}{m!} \int \cdots \int d\vec{r}_1 \cdots d\vec{r}_m \ S_s R(1\cdots m) \prod_{i=1}^{m} \Theta(l\vec{r}_i - \vec{s}_i - \sigma).$$

(2.41)

The density expansion of the time-dependent particle distribution, $f(\vec{r},\vec{p}; t)$, may also be obtained using the cluster expansion of the time-dependent streaming operator $S_s(12\cdots N; t)$,

$$S_s(12\cdots N; t) = \sum_{A \subseteq (12\cdots N)} U(A; t),$$

(2.42)
by inserting it in formula (2.15),
\[
\begin{align*}
\tilde{f}(\bar{r}, \bar{\vec{r}}; t) &= \sum_{m=0}^{N} \frac{1}{m!} \int d\bar{r}_1 \cdots d\bar{r}_m \ p^{(m)}_{(1 \cdots m)}(1 \cdots m; t) \cdot \prod_{i=1}^{m} \Theta(1 \bar{r}_i - \bar{\vec{r}}) \ f(\bar{r}, \bar{\vec{r}}; \bar{0}) .
\end{align*}
\] (2.43)

We note that the time-dependent clusters are given in terms of the streaming operators by the inverted cluster expansion,
\[
\begin{align*}
U(t) &= S(t) , \\
U(1; t) &= S(1; t) - S(t) , \\
U(12; t) &= S(12; t) - S(1; t) - S(2; t) + S(t) ,
\end{align*}
\] (2.44a, b, c)

with the general formula,
\[
U(1 \cdots m; t) = \sum_{A \subset \{1 \cdots m\}} (-1)^{m-|A|} S(A; t) .
\] (2.45)

The time-dependent cluster, \(U(12 \cdots m; t)\), is related to the R operator by the Laplace transform,
\[
S_{R} (1 \cdots m) = \int_{0}^{\infty} dt \ e^{-st} U(1 \cdots m; t) .
\] (2.46)

The time dependent evolution operator, \(\Gamma(t)\), gives \(f(\bar{r}, \bar{\vec{r}}; t)\) when acting on the initial distribution, and is the sum over the m-scatterer evolution operators, \(\Gamma^{(m)}(t)\), given by,
\[
\begin{align*}
\Gamma^{(m)}(t) &= \frac{1}{m!} \int d\bar{r}_1 \cdots d\bar{r}_m \ p^{(m)}_{(1 \cdots m)}(1 \cdots m; t) \prod_{i=1}^{m} \Theta(1 \bar{r}_i - \bar{\vec{r}} - \bar{0}) .
\end{align*}
\] (2.47)
III. THE FORMAL DYNAMICAL EQUATION

A. THE PRODUCT EQUATION

We first present a derivation of the Lorentz gas dynamical equation which is similar to the methods of Cohen\textsuperscript{5} and Ford\textsuperscript{6}. The derivation, starting from the virial expansion of $f(\vec{r}, \vec{p}; t)$, Eq. (2.43), gives a formal prescription for the m-scatterer collision integral thereby generalizing the Lorentz-Boltzmann equation. Our procedure will be to develop the inverse expansion to Eq. (2.43) expressing the initial distribution as a density expanded operator acting on $f(\vec{r}, \vec{p}; t)$, and then to insert this in the expression for the time derivative of $f(\vec{r}, \vec{p}; t)$.

The inversion is carried out by means of the Husimi-like expansion, named after the expansion first used by Husimi\textsuperscript{18} in the equilibrium theory of gases. The Husimi-like operators, $\mathcal{V}(12...m; t)$, are defined in terms of the script cluster operators, $\mathcal{V}(12...m; t) = U(-t)U(12...m; t)$, and the reduced scatterer distributions, $P^{(m)}(12...m)$, by the following expansion,

\begin{align}
\mathcal{P}^{(1)}(1) \mathcal{V}(1; t) &= \mathcal{V}(1; t) \hspace{1cm} \text{(3.1a)} \\
\mathcal{P}^{(2)}(12) \mathcal{V}(12; t) &= \mathcal{V}(12; t) + \sum_{\rho \in \text{perm}} \mathcal{V}(1; t) \mathcal{V}(2; t) \hspace{1cm} \text{(3.1b)} \\
\mathcal{P}^{(3)}(123) \mathcal{V}(123; t) &= \mathcal{V}(123; t) + \sum_{\rho \in \text{perm}} \mathcal{V}(1; t) \mathcal{V}(2; t) \mathcal{V}(3; t) + \\
&\hspace{1cm} \sum_{\rho \in \text{perm}} \mathcal{V}(1; t) \mathcal{V}(23; t) + \sum_{\rho \in \text{perm}} \mathcal{V}(1; t) \mathcal{V}(2; t) \mathcal{V}(3; t) \hspace{1cm} \text{(3.1c)}
\end{align}

*In this chapter we suppress the product of $\Theta$-functions, $\Theta(\left| r - \vec{r}_i \right| - \sigma)$, keeping in mind that they appear to the right of each cluster operator.
\[ P^{(m)}(1\ldots m, t) = \sum_{\pi^{\text{ord}} \in \Pi^{\text{ord}}(1\ldots m)} \sum_{\rho \in \Pi^{\text{ord}}} \prod_{B \in \pi^{\text{ord}}} \mathcal{V}(B; t), \] (3.1m)

where \( \sum_{\rho \in \Pi^{\text{ord}}} \) is the sum over all permutations among but not within the blocks (arguments) of the \( V \)-operators, \( \sum_{\pi^{\text{ord}} \in \Pi^{\text{ord}}(1\ldots m)} \) is the sum over all order preserving partitions of the set \( (1\ldots m) \) into disjoint sets called blocks, and \( \Pi^{\text{ord}} \) is the ordered product of \( V \)-operators whose arguments are the blocks of the partition \( \pi^{\text{ord}} \).

The set of Eqs. (3.1) successively define the Husimi-like operators, \( V(1\ldots m; t) \). Inverting these equations (term-by-term or using the Möbius inversion formula), we obtain the inverse Husimi-like expansion,

\[ \mathcal{V}(1; t) = P^{(1)}(1) U(1; t), \] (3.2a)

\[ \mathcal{V}(12; t) = P^{(2)}(12) U(12; t) - \sum_{\rho \in \Pi} P^{(1)} U(1; t) P^{(1)} U(2; t), \] (3.2b)

\[ \mathcal{V}(1\ldots m; t) = \sum_{\pi^{\text{ord}} \in \Pi^{\text{ord}}(1\ldots m)} \sum_{\rho \in \Pi^{\text{ord}}} (-)^{\mathcal{V}(\pi)-1} \prod_{B \in \pi^{\text{ord}}} P(B) U(B; t), \] (3.2m)

where \( \mathcal{V}(\pi) \) is the number of blocks in the partition \( \pi \). Performing the integral over scatterer positions in the \( U \) and \( V \)-operators in Eq. (3.1), we obtain the Husimi-like expansion of \( U(-t) \Gamma^{(m)}(t) \),

\[ U(-t) \Gamma^{(1)}(t) = \mathcal{V}_1, \] (3.3a)

\[ U(-t) \Gamma^{(2)}(t) = \mathcal{V}_2 + \mathcal{V}_1^2, \] (3.3b)

\[ U(-t) \Gamma^{(m)}(t) = \sum_{\pi \in \Pi} \prod_{b \in \pi} \mathcal{V}(b; t), \] (3.3m)

where \( \mathcal{V}_m(t) \) denotes the integral,

\[ \mathcal{V}_m(t) = \frac{1}{m!} \prod_{i=1}^{m} \int d\tilde{\xi}_i \ldots d\tilde{\xi}_m \mathcal{V}(1\ldots m; t), \] (3.4)
\[ \sum_{\pi \in \mathcal{P}} \text{is the sum over all partitions of the number } m \text{ into summands, and } \prod_{b \in \pi} \text{is the product over the summands } b. \text{ The numerical coefficient in Eqs.}(3.3) \]
is unity since the number* of permutations between the blocks of \( n_1, n_2, \ldots, n_k \)
labels, divided by \( m! = (n_1! \cdots n_k)! \) is equal to \( (n_1! n_2! \cdots n_k!)^{-1} \), there-
fore giving the \( 1/m! \) coefficient in Eq.(3.4). Upon integration over the
scatterer labels, the sum over order preserving partitions of Eq. (3.1)
becomes the sum over the number of labels in the blocks \( B \) or the sum over
the set of partitions of a number into summands taking into account the
order of distinct summands.

We take the thermodynamic limit of \( \Gamma(t) \) by letting the number of scatter-
erers, \( N \), approach infinity while keeping the density of scatterers, \( n \),
constant. Use of the thermodynamic limit on the Husimi-like expansion
yields the simple result that the inverse of \( \Gamma(t) \) is an expansion in Husimi-
like operators,
\[
\Gamma(t) = U(t) \left[ 1 + \sum_{m=1}^{\infty} \sum_{\pi \in \mathcal{P}} \prod_{b \in \pi} V_b(t) \right] = \\
= U(t) \left( 1 - \sum_{m=1}^{\infty} V_m(t) \right)^{-1}
\]
(3.5)

Thus the Husimi-like expansion, in the thermodynamic limit is the inverse
of the sum over Husimi-like operators. This simple result allows us to
write the initial particle distribution as a sum of Husimi-like operators
acting on the particle distribution at time \( t \),

*The number of permutations between blocks of labels is equal to the total
number of permutations divided by the number of permutations of the labels
within the blocks.
\[ f(\vec{r}, \vec{p}; t) = \left[ 1 - \sum_{m=1}^{\infty} V_m(t) \right] U(-t) f(\vec{r}, \vec{p}; t) \]  

Equation (3.6) demonstrates the property of the Husimi-like expansion of giving an operator whose inverse is a density series. This property was first used by Zwanzig in obtaining the density expansion of the inverse of the transport coefficients, and has also been used by Kawasaki- and Oppenheimer and van Leeuwen-Weijland.  

We obtain the product equation by substituting the inverted series, Eq. (3.6), into the expression for the partial time derivative of \[ U(-t) f(\vec{r}, \vec{p}; t), \]

\[ \frac{\partial}{\partial t} U(-t) f(\vec{r}, \vec{p}; t) = \frac{\partial}{\partial t} \left( U(-t) \frac{\Gamma(t)}{} \right)^{-1} U(-t) f(\vec{r}, \vec{p}; t), \]  

where,

\[ \frac{\partial}{\partial t} U(t) f(\vec{r}, \vec{p}; t) = U(t) \left[ \frac{\partial}{\partial t} + \vec{p} \cdot \frac{\partial}{\partial \vec{r}} \right] f(\vec{r}, \vec{p}; t). \]  

Multiplying Eq. (3.7) on the left by \( U(t) \) and using Eq. (3.8), we find the product equation with the collision integral, \( \Lambda(t) \), as a linear operator acting on \( f(\vec{r}, \vec{p}; t) \)

\[ \frac{\partial}{\partial t} f(\vec{r}, \vec{p}; t) + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{r}} f(\vec{r}, \vec{p}; t) = \Lambda(t) f(\vec{r}, \vec{p}; t), \]  

where the collision operator is formally given as

\[ \Lambda(t) = U(t) \frac{\partial}{\partial t} \left( U(-t) \frac{\Gamma(t)}{} \right)^{-1} U(-t), \]  

or using the m-scatterer operators \( \Gamma^{(m)}(t) \) and \( V_m \).
\[ \Lambda(t) = U(t) \frac{\partial}{\partial t} \left[ U(-t) \sum_{m=1}^{\infty} V_m(t) \right] U(-t). \]  

(3.10b)

We decompose the complete collision operator \( \Lambda(t) \) into the sum over m-scatterer collision operators \( \Lambda^{(m)}(t) \),

\[ \Lambda(t) = \sum_{m=1}^{\infty} \Lambda^{(m)}(t), \]  

(3.11)

by collecting together all products from Eq. (3.10b) involving \( m \) and only \( m \) scatterers.

\[ \Lambda^{(1)}(t) = U(t) \frac{\partial}{\partial t} \left\{ U(-t) \right\} U(-t), \]  

(3.12a)

\[ \Lambda^{(2)}(t) = U(t) \left\{ \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right) U(-t) \right\} - \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right) V_1(t) \right\} U(-t), \]  

(3.12b)

\[ \Lambda^{(3)}(t) = U(t) \left\{ \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right) U(-t) \right\} - \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right) V_1(t) - \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right) V_2(t) \right\} U(-t), \]  

(3.12c)

\[ \Lambda^{(m)}(t) = U(t) \left\{ \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right) U(-t) \right\} - \sum_{i=1}^{m-1} \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right) V_i(t) \right\} U(-t). \]  

(3.12m)

We may express the m-scatterer collision integral, \( \Lambda^{(m)}(t) \), purely in terms of the m-scatterer evolution operators, \( \Gamma^{(m)}(t) \), by using the inverse of Eqs. (3.3),

\[ V_1(t) = \Gamma^{(1)}(t), \]  

(3.13a)

\[ V_2(t) = \Gamma^{(2)}(t) - \Gamma^{(1)}(t)^2, \]  

(3.13b)

\[ V_m(t) = \sum_{\pi \in m} (-1)^{\nu(\pi)-1} \prod_{b \in \pi} \Gamma^{(b)}(t), \]  

(3.13m)
where $\Gamma^{(l)}(t)$ denotes $U(-t)\Gamma^{(l)}(t)$ and $v(\pi)$ denotes the number of summands in the partition $\pi$. This inverse Husimi-like expansion may be obtained by successive inversion of Eqs. (3.3). Insertion of Eqs. (3.13) into Eq. (3.12) gives

$$\Lambda^{(m)}(t) = U(t) \sum_{b_1 + \cdots + b_2 = m} \left( - \right)^{b_1} \frac{\partial}{\partial t} \prod_{i=1}^{b_1} \prod_{j=1}^{b_2} U(-t), \quad (3.14)$$

where the sum is over all partitions of $m$ into summands, and the left most operator is the partial time derivative of the evolution operator $\prod^{(b_1)}(t)$. We may thus express the $m$-scatterer collision operator $\Lambda^{(m)}(t)$ as

$$\Lambda^{(m)}(t) = U(t) \prod_{m} d \xi \cdots d \xi \ A(1 \cdots m; t) \ U(-t), \quad (3.15)$$

where the symmetric collision operator $A(1 \cdots m; t)$ is the inverse Husimi-like expansion of the script cluster operators,

$$A(1; t) = P^{(1)}(1) \frac{\partial}{\partial t} U(1; t), \quad (3.16a)$$

$$A(12; t) = P^{(2)}(12) \frac{\partial}{\partial t} U(12; t) - \sum_{\rho \in \rho} P^{(1)}(1) \frac{\partial}{\partial t} P^{(2)}(2) U(2; t), \quad (3.16b)$$

$$A(1 \cdots m; t) = \sum_{\pi^{(m)}} \left( - \right)^{v(\pi)} \prod_{\rho \in \rho} P(\rho) \frac{\partial}{\partial t} \prod_{\xi(1 \cdots m)} \ U(\xi; t), \quad (3.16m)$$

where this result is obtained from Eq. (3.14) by reversing the steps leading from Eq. (3.1) to Eq. (3.3).

We thus find in Eqs. (3.14) or (3.16) that the general or m-scatterer collision integral $\Lambda^{(m)}(t)$ is the inverse Husimi-like expansion of the
cluster operators with the restriction that the left-most operator in a product must be the partial time derivative of the cluster operator. We will demonstrate later that the inverse Husimi-like expansion of cluster operators is an irreducible expansion corresponding to the fact that only irreducible collision events contribute to the generalized collision integral. An irreducible collision event is, in the multiple scattering problem, a many-scatterer collision event which cannot be decomposed into a series of successive collision events among disjoint groups of scatterers.

B. THE CONVOLUTION EQUATION

We now apply the Husimi-like expansion to the Laplace-transformed cluster operators, \( R(12\ldots m) \), obtaining an equation with a convolution type of collision integral. We introduce the new operator \( K_s^{(m)}(12\ldots m) \), obtained from the Husimi-like expansion of the cluster operator, \( R(12\ldots m) \), of Eq. (2.36),

\[
\begin{align*}
\hat{\rho}^{(n)}(1) R(1) &= K_s^{(n)}(1) , \\
\hat{\rho}^{(l2)}(12) R(12) &= K_s^{(l2)}(12) + \sum_{\rho \in S} \hat{\rho}_{s}^{(1)} \hat{\rho}_{s}^{(2)} , \\
\hat{\rho}^{(l\cdot m)}(1\cdots m) R(1\cdots m) &= \sum_{\kappa^{\text{ord}} \in (1\cdots m)} \sum_{\rho \in S} \hat{\kappa}_{s}^{\text{ord}} K_s(B) ,
\end{align*}
\]

where the general term is the sum over order-preserving partitions of the sum over symmetrizing permutations of ordered products of \( K_s \) operators.

If we carry out the integral over scatterer positions, the Husimi-like expansion of Eqs. (3.17) becomes the sum over the partitions of a number
into summands,

\[ S_s^{-1} \Gamma_s^{(m)} = \sum_{\kappa \in m} \prod_{\ell \in \kappa} K_s^{(\ell)} , \]

where the m-scatterer collision integral, \( K_s^{(m)} \), is given by,

\[ K_s^{(m)} = \frac{1}{m!} \int \cdots \int d\vec{s}_1 \cdots d\vec{s}_m \ K_s^{(m)}(1 \cdots m) . \]

(3.19)

Taking the thermodynamic limit on Eq. (2.40), we obtain the operator relation,

\[ \Gamma_s = S_s + \sum_{m=1}^{\infty} \Gamma_s^{(m)} = S_s \left( 1 - \sum_{m=1}^{\infty} K_s^{(m)} \right)^{-1} , \]

(3.20)

and this result is formally identical with Eq. (3.5) for the Husimi-like expansion of \( \Gamma(t) \). If we now insert Eq. (3.20) into the expression for \( f_s(\mathbf{\hat{r}}, \mathbf{\hat{p}}) \), Eq. (2.39), we obtain the Laplace transform of the convolution equation,

\[ S_s^{-1} f_s(\mathbf{\hat{r}}, \mathbf{\hat{p}}) = f(\mathbf{\hat{r}}, \mathbf{\hat{p}}; 0) = \mathcal{L}_s f_s(\mathbf{\hat{r}}, \mathbf{\hat{p}}) , \]

(3.21)

where

\[ \mathcal{L}_s = \sum_{m=1}^{\infty} K_s^{(m)} S_s^{-1} . \]

(3.22)

Taking the inverse Laplace transform gives the convolution equation,

\[ \frac{\partial}{\partial t} f(\mathbf{\hat{r}}, \mathbf{\hat{p}}; t) + \frac{\vec{\mathbf{\hat{v}}}}{m} \cdot \frac{\partial}{\partial \mathbf{\hat{r}}} f(\mathbf{\hat{r}}, \mathbf{\hat{p}}; t) = \int_0^t dt' \mathcal{L}^{-1}(t-t') f(\mathbf{\hat{r}}, \mathbf{\hat{p}}; t') , \]

(3.23)

where
\[ \int_0^\infty e^{-st} \zeta(t) = \sum_{m=1}^{28} \kappa_s^{(m)} S_s^{-1}. \] (3.24)

C. IRREDUCIBLE FACTORIZATION

The motivation for studying the irreducible factorization* of the binary collision expansion is to reduce the expression for \( f_s(T,E) \), involving the infinity of complex collision events, to as simple and tractable a formula as possible; it is an outgrowth of this decomposition process that we are able to discover a dynamical equation. It must be emphasized that the primary goal of irreducible factorization is not to invert series or develop and equation in the manner of sections A and B; it is to understand the myriad of collision events.

We consider the concept of irreducible factorization applied to the sequences of labels corresponding to the products of binary collision operators in the evolution operator \( f_s^{(m)} \). A given sequence is termed irreducible if it cannot be factored into a product of disjoint subsequences (sequences with no labels in common). The process of irreducible factorization consists of factoring all sequences from the binary collision expansion into their unique irreducible products without altering the order of the labels, and then collecting together all products having the same structure. The structure of a product of irreducible factors refers to the number of factors, the number of distinct labels in each factor, and the order of the factors.

*We are concerned with factorization because the basic units of the binary collision expansion are products.
An example will illustrate the general process. Consider the sequence 12134374: it can be uniquely factorized into two irreducible sequences, 121 and 34374; its structure is characterized as, (12)(347), indicating that it has two factors, that the factors have the distinct labels 12 and 347, and that they occur in the indicated order from left to right. Other sequences having the same irreducible factorization, (12)(347), are 12123473, 1213473, etc. Thus we see that there is an infinity of sequences having the same factorization structure.

We consider the three problems of irreducible factorization: (1) to determine the general structure; (2) to collect together all sequences with a given structure; and (3) to apply this to the decomposition of the binary collision expansion in Eq. (2.36). We first give a prescription for the general factorization structure. Given a sequence we obtain its structure by partitioning its elements into disjoint blocks preserving the order of the original sequence among and within the blocks. Repeated labels in a factor are not repeated in the corresponding block. Hence the general structure is that of an order preserving partition. For example, the sequence 12134374 factors into the subsequences 121 and 34374, and the factorization corresponds to the order-preserving partition (12)(347) of the ordered set 12347. The repeated numbers, the second 1, 3, and 4, are left out of the structure notation.

We may simplify the collection of all irreducible factorizations corresponding to a given order-preserving partition using the following lemma:
The collection of all irreducible factorizations corresponding to a given order-preserving partition may be factored into the ordered product of the collections of all irreducible sequences whose elements are the labels of the blocks of the partition.

This lemma follows from the completeness of the expansion being irreducibly factored, the binary collision expansion. We illustrate the lemma with the following example. Consider the collection of all irreducible factorizations corresponding to the partition \((12)(43):\) \(121434, 1212434, 1214343, 12124343, 12121434, 121214343, \ldots.\) The lemma states that this collection may be written as the product of two factors. The left-most factor is the collection of all irreducible sequences whose elements are the labels \(12;\) viz., \(121, 1212, 12121, \ldots.\) The factor on the right is the collection of all irreducible sequences whose elements are the labels \(43;\) viz., \(434, 4343, 43434, \ldots.\) Note that the basic order \(1, 2, 4, 3\) is always preserved.

The lemma may be proved by using the disjointness of the irreducible factors as shown in Appendix B.

We now apply irreducible factorization to the cluster operators \(R(12\ldots m)\) of Eq. (2.32) which contain all sequence of labels from \(12\ldots m,\) each sequence containing all \(m\) labels, with no label occurring more than once in succession. Since the sum in \(R(12\ldots m)\) is complete we may apply the lemma, denoting the collection of all irreducible sequences having the labels \(12\ldots k\) in any order as \(C(12\ldots k),\)

\[
R(1) = C(1),
\]

\[
R(12) = C(12) + \sum_{\text{perm}} C(1)C(2),
\]

\((3.25a)\)

\((3.25b)\)
\[ R(1 \cdot m) = \sum_{\pi^{\text{ord}} \in (1 \cdot m)} \sum_{B \in \pi^{\text{ord}}} \prod_{B^{\text{ord}}} C(B), \quad (3.25\text{m}) \]

and this is the same sum that first appeared as the Husimi-like expansion in Eq. (3.1). Thus we find that the irreducible factorization of the binary collision expansion is formally identical with the Husimi-like expansion only because the structure of the factorization is that of the order-preserving partitions.

The general prescription for \( C(12 \cdot \cdot m) \) is,

\[ C(12 \cdot \cdot m) = \text{sum over all irreducible sequences of labels from } 12 \cdot \cdot m \text{ such that each sequence contains all } m \text{ labels in any order and not occurring more than once in succession.} \quad (3.26) \]

This is illustrated with a few examples:

\[ C(1) = B(1), \quad (3.27\text{a}) \]

\[ C(12) = B(1)B(2)B(1) + B^2 \sum_{\text{perm}} B(1)B(1)B(2)B(3) + \cdots, \quad (3.27\text{b}) \]

\[ C(122) = \sum_{\text{perm}} B(1)B(2)B(3)B(1) + \sum_{\text{perm}} B(1)B(2)B(3)B(1)B(1)B(3) + \cdots, \quad (3.27\text{c}) \]

where the sum on permutations is over the \( 3! \) symmetrizing permutations.

We next apply the method of irreducible factorization to the product \( p(m)(12 \cdot \cdot m) R(12 \cdot \cdot m) \). The preceding development can be carried over if the product is expressible as some complete expansion, so that the lemma can be applied. This is achieved by expanding the scatterer distribution \( p(m)(12 \cdot \cdot m) \) in an Ursell series\(^{17} \) of correlation functions,
\[ P^{(l)} = W^{(l)}, \quad (3.28a) \]
\[ P^{(l2)} = W^{(l2)} + W^{(l)} W^{(2)}, \quad (3.28b) \]
\[ P^{(l23)} = W^{(l23)} + W^{(l)} W^{(23)} + W^{(2)} W^{(3)} + W^{(l2)} W^{(3)} + W^{(l3)} W^{(2)}, \quad (3.28c) \]

\[ P^{(m)}(l2...m) = \text{sum over all partitions of the set } l2...m \text{ into disjoint subsets (blocks), of the product of correlation functions } W \text{ with the blocks as arguments}. \quad (3.28m) \]

The correlation functions are by definition irreducible. This means that the argument of a correlation function \( W \) is considered to be an irreducible sequence because \( W \) cannot be further decomposed. Note that the Ursell series is different from the Husimi-like series; the latter may be obtained from the former by symmetrizing all the Ursell products, keeping account of the order of the noncommuting factors. However, the Ursell series, Eqs. (3.28), is already symmetric since the order of the correlation functions is unimportant. Another difference between the Ursell and Husimi-like expansions is manifested in the inversion factor (the numerical factor in the inverted Ursell or Husimi-like series) which is \((\nu-1)!\) \((-)^{\nu-1}\) for the Ursell series and \((-)^{\nu-1}\) for the Husimi-like series, where \( \nu \) is the number of blocks in the partition.

The irreducible factorization of the product \( P^{(m)}(l2...m) R(l2...m) \) is obtained from the direct product of the cluster expansion of \( P^{(m)}(l2...m) \) and the Husimi-like expansion of \( R(l2...m) \). This direct product is a complete expansion in the sense of the lemma; therefore, after collecting to-
gether terms having the same structure, we obtain the expansion of Eqs. (3.17) in the same manner as we obtained Eqs. (3.25). The K operators of the Husimi-like expansion are therefore irreducible collision operators where irreducibility means that the sequence of labels cannot be factored because the labels are repeated, or the labels appear in the argument of a correlation function, or both.

The irreducible collision operators may be expressed in terms of the correlation functions and the irreducible C(12...m) operators as follows:

\[ K^{(1)}_{11} = P^{(1)}_{11} C^{(1)} \]

\[ K^{(2)}_{12} = P^{(2)}_{12} C^{(12)} + \sum_{\text{perm}} W^{(12)} C^{(1)}C^{(2)} \]

\[ K^{(3)}_{123} = P^{(3)}_{123} C^{(123)} + \sum_{\text{perm}} \left[ W^{(123)} + W^{(12)}W^{(3)} + W^{(13)}W^{(2)} \right] C^{(1)}C^{(2)}C^{(3)} + \]

\[ + \sum_{\text{perm}} \left[ W^{(123)} + W^{(13)}W^{(2)} + W^{(12)}W^{(3)} \right] C^{(12)}C^{(3)} + \]

\[ + \sum_{\text{perm}} W^{(123)} C^{(1)}C^{(2)}C^{(3)} \]

\[ K^{(m)}_{1...m} = \sum_{\pi^{\text{ord}}} \sum_{\text{perm}} \left[ \sum_{\sigma \in \pi^{\text{ord}}} \prod_{A \in \sigma} \frac{TT}{W(A)} \right] \prod_{B \in \pi^{\text{ord}}} C^{(B)} \]

where \( \sum_{\text{perm}} \) is the sum over all permutations among the blocks B,

\( \sum_{\pi^{\text{ord}} \in (12...m)} \) is the sum over order-preserving partitions \( \pi^{\text{ord}} \) of the set (12...m), and \( \sum_{\sigma \in \pi^{\text{ord}}} \) is the sum over all ordinary partition of the set (12...m) excluding those that coincide with or are less than (are derived from by further partitioning) \( \pi^{\text{ord}} \). This insures that the product of W's and C's is irreducible.

An approach similar to the method of irreducible factorization, the
connected diagram expansion, has been studied by Fujita.\textsuperscript{31} This method considers the connectivity of the terms in the interaction expansion of the system streaming operator, grouping together terms corresponding to certain connected diagrams. Since, by definition, a connected diagram corresponds to an irreducible sequence, the two expansions (the connected diagram expansion and the Husimi-like or irreducible expansion) are the same except in the way they are developed and in the fact that Fujita uses the interaction expansion instead of the binary collision expansion. However, the latter difference is trivial since one can transform from one expansion to the other by including or excluding identical adjacent indices in the expansion sequences. Fujita makes an unnecessary and incorrect approximation to his connected diagram expansion, enabling him to write the transport coefficient of a quantum Lorentz gas as an inverse density series, whereas the connected diagram expansion or the Husimi-like expansion, results in an inverted density series (see Eq. (3.5)) without invoking any approximations. The entire development of this section is equally valid for the interaction expansion.

D. COMMENTS ON THE TWO DYNAMICAL EQUATIONS

We have developed two equations for the particle distribution using the Husimi-like expansion. The product equation Eq. (3.9), expresses the time rate of change of \( f(\overrightarrow{r},\overrightarrow{p};t) \) due to collisions, denoted \( \frac{\partial}{\partial t} f(\overrightarrow{r},\overrightarrow{p};t) \), as a linear operator, the generalized collision operator \( \Lambda(t) \), acting on the particle distribution,
\[
\left[ \frac{\partial}{\partial t} f(\vec{r}, \vec{p}; t) \right]_{\text{coll}} = \Lambda(t) f(\vec{r}, \vec{p}; t).
\]  
(3.30)

The convolution equation, Eq. (3.23), expresses \( \left[ \frac{\partial}{\partial t} \right]_{\text{coll}} \) as the convolution integral of the generalized collision operator \( K(t) \) acting on the particle distribution,

\[
\left[ \frac{\partial}{\partial t} f(\vec{r}, \vec{p}; t) \right]_{\text{coll}} = \int_0^t dt' \Lambda(t-t') f(\vec{r}, \vec{p}; t'),
\]  
(3.31)

or in Laplace transform form it appears as a linear operator,

\[
\int_0^\infty dt e^{-st} \left[ \frac{\partial}{\partial t} f(\vec{r}, \vec{p}; t) \right]_{\text{coll}} = K_s \mathcal{F}_s (f(\vec{r}, \vec{p})),
\]  
(3.32)

where \( K_s = K \mathcal{S}_s^{-1} \).

The two expressions for \( \left[ \frac{\partial}{\partial t} f(\vec{r}, \vec{p}; t) \right]_{\text{coll}} \) are different by virtue of the fact that the product form, Eq. (3.30), is a time-dependent collision operator acting on \( f(\vec{r}, \vec{p}; t) \) and the convolution form, Eq. (3.31), is a Laplace-transformed operator acting on \( f_s(\vec{r}, \vec{p}) \), the Laplace transform of the particle distribution; \( K_s \) is not the Laplace transform of \( \Lambda(t) \). The relation between \( \Lambda(t) \) and \( K_s \) is given in Appendix C where it is also demonstrated that a given collision integral from Eq. (3.30) and Eq. (3.31) for scattering with \( m \) scatterers differ in that the former is an integral operator acting on the particle distribution evaluated at time \( t \) for collision events that are not irreducible, and the latter is an integral operator acting on the particle distribution evaluated at the time \( t - t_{\text{coll}} \), where \( t_{\text{coll}} \) is the duration of the irreducible collision event.

The dynamical equation is best expressed in the following form, obtained
in Chapter IV,
\[
\frac{2}{\hbar^2} f(\vec{r}, \vec{p}; t) + \sum_{m=1}^{\infty} \sum_{j_1 \cdots j_k} \sum_{\text{real or hyp.}} \int \mathcal{I}_1 \, \nu \, d\Omega \cdots \int \mathcal{I}_m \, \nu \, d\Omega_m \cdot \\
\int \cdots \int d\zeta_1 \cdots d\zeta_m \sum_{\text{real or hyp.}} (-i)^v \Theta(t - \zeta_{i,m}) f(\vec{r}', \vec{p}'; t - \zeta_{i,m})
\]

(3.33)

where \( \sum_{j_1 \cdots j_k} \) is the sum over all irreducible sequences of the \( m \) labels \( 12 \cdots m \);
the scatterers are assumed to be randomly distributed so that \( P(12 \cdots m) = n^m \);
\( \mathcal{I}_i \, d\Omega_i \) is the differential cross-section for the first encounter (following
the motion backward in time) of the particle with the \( i \)th distinct scatterer
in the sequence \( j_1 j_2 \cdots j_k \); \( \tau_i \) is the time the particle moves freely just prior
(following the motion backward in time) to the first encounter with the \( i \)th
distinct scatterer; \( \tau_{\text{coll}} \) is the duration of the collision; \( \sum_{\text{real or hyp.}} \) is the sum
over real or hypothetical-collisions for each of the \( l \) encounters; \( \nu \) is the
number of hypothetical-collisions in the sequence; and \( \vec{r}' \) and \( \vec{p}' \) are the
restituting position and momentum, i.e., those values of position and momentum
that evolve into \( \vec{r} \) and \( \vec{p} \) in the course of the collision. Where a hypothetical-
collision is an encounter with a scatterer which does not rotate the particle
momentum and which has an overall minus sign; it is similar to the direct col-
losion term of the Lorentz-Boltzmann collision integral.
IV. THE FORM OF THE COLLISION INTEGRALS FOR HARD SPHERES

A. INTRODUCTION

The nontriviality of the calculation of the collision integrals is attested to by the discrepancy in recent publications\(^{35-41}\) as to the correct form the Boltzmann collision integral should take for spatially inhomogeneous distributions and hard sphere interaction. This discrepancy affects the Lorentz-Boltzmann collision integral as well as the generalization of the collision integrals to higher densities. Enskog\(^3\) was the first to propose the form the Boltzmann collision integral should take when spatial dependence of the particle distribution is taken into account. The Enskog-Ansatz which is an improvement over Boltzmann's Stosszahlansatz takes account of the finite distance separating the particles at the moment of collision. The Enskog generalization for the Lorentz gas of hard spheres gives the collision integral,

\[
\mathcal{V} \int b_1 db_1 d\phi \left[ P_u(\hat{r}, \hat{p}; t) \hat{f}(\hat{r}, \hat{p}; t) - P_u(\hat{r}+\hat{\rho}; \hat{f}(\hat{r}, \hat{p}; t)) \right],
\]

(4.1)

where the scatterer distribution \(P^{(1)}\) is evaluated at \(\hat{r}+\hat{\rho}\) for the restituting collision and at \(\hat{r}+\hat{\rho}\) for the direct collision; \(\hat{\rho} = \hat{p} + \sqrt{\sigma^2 - b^2} \hat{v}\) is a vector of length \(\sigma\) (the hard core radius); \(\hat{p}\) is the impact vector perpendicular to \(\hat{v}\) and lying in the collision plane; and \(db = b_1 db_1 d\phi\), where \(b\) is the impact parameter and \(\phi\) is the azimuthal angle determining the orientation of the collision plane about the fixed particle velocity \(\hat{v}\).
If it is correct, the Enskog generalization of the binary collision integral should be obtained in a mechanical derivation starting from the Liouville equation. Bogoliubov\textsuperscript{3} mentions that it is not difficult to see that his generalized binary collision integral leads to the Enskog result; however, he does not give details of the transition to the Enskog form. Choh and Uhlenbeck\textsuperscript{13} acknowledge the statement of Bogoliubov that the latter's theory reduces to the Enskog result for hard spheres, but they present no verification.

A number of recent papers\textsuperscript{25-38} derive a result differing from the Enskog form. Rice, Kirkwood, Ross, and Zwanzig\textsuperscript{36} obtain a collision integral (using the Kirkwood theory) for which the distributions are evaluated at the same point for both the restituting and the direct collisions. They consider the origin of this disparity as due to Enskog's distinction between the variables before a collision and the variables after a collision. They claim there can be no such distinction.

There have also appeared a number of papers\textsuperscript{39-41} that do obtain the Enskog result. Sengers and Cohen\textsuperscript{39} present derivations of the generalized binary collision integral for hard spheres based on the Bogoliubov and the Kirkwood theories. They correctly point out an omission in the paper of Rice, Kirkwood, Ross, and Zwanzig which they claim gives the Enskog result when included.

In a more recent series of papers\textsuperscript{25,30} derivations of the generalized collision integral are given using the Fourier transform technique first employed by Zwanzig\textsuperscript{19}. It would appear that this is a step in the direction
of a more rigorous derivation compared with the earlier derivations\textsuperscript{36,39} which invoke many assumptions. However, in the papers of Haines, Dorfman, and Ernst\textsuperscript{25} and of van Leeuwen and Weijland\textsuperscript{30} it is assumed that the contribution to the collision integral from initial positions in which the hard spheres overlap can be neglected. As a result they obtain a general-
ized binary collision integral differing from the Enskog formula.

There is clearly a need for a careful derivation of the binary colli-
sion integral and of the generalized collision integrals for hard spheres. The philosophy of the recent non-equilibrium theories of dense gases and of the Lorentz gas is to replace the Stosszahlansatz and its accompanying intuitive derivation of the collision integral by a derivation based only on the mechanics of the motion and the Liouville equation. However, most derivations seem at one point or another to make use of an intuitive argu-
ment rather than a rigorous mathematical discussion. We shall derive the form of the collision integral for the Lorentz gas of hard sphere scatterers using only our knowledge of the single-particle streaming operator, $S(1;t)$, as the operator that carries the motion backward a time $t$ under the influence of the interaction with a scatterer.

B. THE STREAMING OPERATOR $S(1;t)$

We consider how the single-particle streaming operator acts on a func-
tion of the particle position and momentum $F(\vec{r},\vec{p})$ for the hard sphere po-
tential $V(r)$,

$$V(r) = \begin{cases} V_0 & t < \sigma \\ 0 & t > \sigma \end{cases} \quad (4.2)$$
where the limit $V_0 \to \infty$ is taken after performing other manipulations.

A natural description of the collision between particle and scatterer is in terms of the impact parameter $b$, the azimuthal angle $\phi$, and the initial distance a separating the particle and scatterer measured along the initial velocity $\hat{v}$. The angular momentum of the particle, is given in terms of the relative separation between the particle and fixed scattering center, $\vec{r} - \vec{x}_1$, by,

$$\vec{L} = (\vec{r} - \vec{x}_1) \wedge \vec{p} .$$

(4.3)

The impact parameter is then defined as the magnitude of the angular momentum divided by the particle momentum,

$$b = \left| \frac{|\vec{L}|}{|\vec{p}|} \right| = \left| (\vec{r} - \vec{x}_1) \wedge \vec{v} \right| .$$

(4.4)

Denoting the direction of $\vec{L}$ by the unit vector $\hat{l}$, the impact vector $\vec{b}$ is given as,

$$\vec{b} = b \, \hat{v} \wedge \hat{l} ,$$

(4.5)

and it is the projection of the relative separation, $\vec{r} - \vec{x}_1$, in the plane perpendicular to the particle velocity $\hat{v}$ ($\vec{b}$ is in the collision plane since it is perpendicular to $\hat{L}$). The orientation of $\vec{b}$ is given by the angle $\phi$.

We note that,

$$d\vec{b} = b \, db \, d\phi .$$

(4.6)
The component of the relative separation, $\vec{r} - \vec{x}$, along $\hat{\nu}$ is given by,

$$a = (\vec{r} - \vec{x}) \cdot \hat{\nu}.$$  \hspace{1cm} (4.7)

We may thus write the relative separation in the form,

$$\vec{r} - \vec{x} = \vec{b} + a \hat{\nu}.$$  \hspace{1cm} (4.8)

We may now consider the effect of the streaming operator for various values of $a$ and $\vec{b}$. In those regions of $a, \vec{b}$ for which the particle does not encounter the hard core by carrying the motion backward a time $t$, the single-particle streaming operator, $S(1;t)$, acts as the free streaming operator $S(t)$. Therefore we may write $S(1;t) = [S(1;t) - S(t)] + S(t)$, and consider the difference $[S(1;t) - S(t)]$ in those regions where a collision takes place between the initial time and a time $t$ in the past.

Clearly a collision can only take place if the particle aims to collide at some time in the past; this is the region,

$$|\vec{b}| < \sigma, \hspace{1cm} (4.9)$$

$$-\sqrt{\sigma^2 - \rho^2} < a.$$  \hspace{1cm} (4.10)

The first condition, Eq. (4.9) means that the particle aims to collide with the scatterer at some time, and the second condition, Eq. (4.10), means that it can only collide at a time in the past. We must further impose the condition that the particle collides at least by a time $t$ in the past. When the particle is initially outside the hard core, $+\sqrt{\sigma^2 - \rho^2} < a$, this is given by,
\[ a < \sqrt{\sigma^2 - b^2}, \quad (4.11) \]

which states that the initial distance measured along \( \mathbf{v} \) between the particle and the hard core surface, \( a - \sqrt{\sigma^2 - b^2} \), must be less than the distance the particle travels in time \( t, \mathbf{v} \). When the particle is initially inside the hard core, \( -\sqrt{\sigma^2 - b^2} < a < +\sqrt{\sigma^2 - b^2} \), \( [S(1;t) - S(t)] \) is nonzero only if the particle leaves the hard core in a time \( t \); this condition is given by Eq. (4.11).

By following the motion as the particle moves backwards we may evaluate
\[ S(1;t)F(\mathbf{r}, \mathbf{p}) \]
for the different values of \( a \) and \( \mathbf{p} \) described above. We first write down the answer as a sum of terms and then explain and verify each term.

\[ S(1;t)F(\mathbf{r}, \mathbf{p}) = T_1 + T_2 + T_3 + T_4 + F(\mathbf{r} - \mathbf{v}t, \mathbf{p}). \quad (4.12) \]

\[ T_1 = \int d\mathbf{b} \Theta \left( \frac{\sqrt{\mathbf{b}^2 \mathbf{V}_o^2}}{\sqrt{1 - \frac{b^2}{\sigma^2}}} - \mathbf{p} \right) \int_0^t \mathbf{v} d\tau \cdot \mathbf{S} \left( \mathbf{r} - \left( \mathbf{x} - \mathbf{p} \right) - \mathbf{v} \tau \right) \cdot \left[ F(\mathbf{x} - \mathbf{p} - \mathbf{v} (t-\tau), \mathbf{p}) - F(\mathbf{r} - \mathbf{p} - \mathbf{v} \tau, \mathbf{p}) \right]. \quad (4.13) \]

Equation (4.13) is the contribution when the particle is initially outside the hard core and does not have sufficient momentum to penetrate the hard core interior. The \( \Theta \)-function is the ordinary step function which is unity when its argument is positive and zero otherwise. It expresses the condition that the initial momentum \( \mathbf{p} \) is not sufficient for the particle to penetrate the effective potential barrier, \( V_o + \frac{L^2}{2mb^2} \). The Dirac delta expresses the condition that the particle is located at the position,
\[ \dot{\tau} = \dot{\tau}_1 + \dot{\rho} + \dot{\nu} \tau, \]

where,

\[ \dot{\rho} = \dot{b} + \sqrt{\frac{v}{c - 6}} \dot{\nu}, \]  \hspace{1cm} (4.14)

and \( \dot{\tau}_1 + \dot{\rho} \) is the location of the collision on the hard sphere, and that it collides at a time \( \tau \) in the past. The particle distribution is evaluated at the position and momentum which are obtained by carrying the motion backward a time \( t \) in the past from the initial values \( \dot{\tau} \) and \( \dot{\rho} \). In the first term, \( F \) is evaluated at, \( \dot{\tau}_1 + \dot{\rho} - \dot{\nu}(t-\tau) \), obtained by carrying the motion backwards the remaining time \( t-\tau \) along the restituting velocity \( \dot{\nu} \) starting at the point of collision, \( \dot{\tau}_1 + \dot{\rho} \). The restituting momentum, \( \dot{\rho}' \), is the momentum that evolves into \( \dot{\rho} \) in the course of the collision (time running forward) and may be expressed as a rotation operator \( R_{\dot{\rho}} \) acting on \( \dot{\rho} \),

\[ \dot{\rho}' = R_{\dot{\rho}} \dot{\rho} = \dot{\rho} - \frac{\dot{\rho} \times \dot{\rho}}{6c}. \] \hspace{1cm} (4.15)

In the second term of \( T_2 \), \( \dot{\tau}_1 + \dot{\rho} - \dot{\nu}(t-\tau) \) is the position obtained by carrying the motion backward a time \( t \) without interaction, viz., \( \dot{\tau} - \dot{\nu} t \).

\[ T_2 = \int d\tau \int d\rho \Theta \left( \rho - \frac{\rho_0}{\rho - \rho_0} \right) \delta \left( \dot{\tau} - \dot{\tau}_2 \right) \delta \left( \dot{\rho} - \dot{\rho}_2 - \dot{\nu}(t-\tau) \right) \cdot \left[ F(\dot{\tau}_2, \dot{\rho}_2 - \dot{\nu}_2(t-\tau_2), \dot{\rho}_2) - F(\dot{\tau}_2, \dot{\rho}_2, \dot{\nu}(t-\tau_2), \dot{\rho}_2) \right]. \] \hspace{1cm} (4.16)

The first term of \( T_2 \) is the contribution from the trajectory that begins outside the hard core, penetrates for a time \( \tau_2 \),

\[ \tau_2 = \frac{2c}{\sqrt{\rho^2 - \rho^2_0} - 2mV_0} \] \hspace{1cm} (4.17)
and then moves for a time \( t - \tau_1 - \tau_2 \) outside the hard core with momentum \( \vec{p}_2 \),
\[
\vec{p}_2 = \vec{p} - \frac{2(\vec{\sigma} - \hat{\Delta}) \cdot \vec{p}_2 (\vec{\sigma} - \hat{\Delta})}{|\vec{\sigma} - \hat{\Delta}|^2},
\]
where,
\[
\hat{\Delta} = m \hat{\tau}_2 \left( \sqrt{\rho \left( 1 - \frac{\beta^2}{c^2} \right) - 2} m \frac{\vec{p}}{\sqrt{1 - \beta^2}} + \rho \frac{\beta}{\c} \hat{\lambda} \hat{\rho} \right),
\]
is the distance the particle travels inside the hard core. The second term of \( T_2 \) is the contribution from the free streaming.

\[
T_3 = \iint d\hat{\tau} \Theta \left( \rho - \frac{2m \frac{\sqrt{\tau}}{\sqrt{1 - \frac{\beta^2}{c^2}}}}{\sqrt{1 - \frac{\beta^2}{c^2}}} \right) \int_{\tau - \tau}^{\tau} dt d\tau \delta (\hat{\tau} - \left( \hat{\tau}_1 \hat{p}_1 \right) - \hat{\tau}_2) \cdot \nonumber
\]
\[
\left[ F \left( \hat{\tau}_1 \hat{p}_1 - \vec{\Omega}(t-\tau), \vec{p}_2 \right) - F \left( \hat{\tau}_1 \hat{p}_1 - \vec{\Omega}(t-\tau), \vec{p} \right) \right].
\]
\[
T_3
\]
is the term whose interaction part corresponds to the trajectory that begins outside the hard core and terminates inside the hard core. The momentum inside the hard core is given by,
\[
\vec{p}_3 = \sqrt{\rho \left( 1 - \frac{\beta^2}{c^2} \right) - 2m \frac{\vec{p}}{\sqrt{1 - \beta^2}}} + \rho \frac{\beta}{\c} \hat{\lambda} \hat{\rho}.
\]
The second part of \( T_3 \) corresponds to the free streaming for the same integration domain.

\[
T_4 = \iint d\hat{\tau} \Theta \left( t - \frac{2\sqrt{\tau}}{\c} \right) \int_{-\sqrt{\tau} d^2}^{+\sqrt{\tau} d^2} da + \Theta \left( \frac{2\sqrt{\tau} d^2}{\c} - t \right) \int_{-\sqrt{\tau} d^2}^{-\sqrt{\tau} d^2} da \nonumber
\]
\[
\cdot \delta (\hat{\tau} - \left( \hat{\tau}_1 \hat{p}_1 + d \hat{\Omega} \right)) \left[ F \left( \hat{\tau}_1 \hat{p}_1 - \sqrt{\tau} d^2 \hat{\Omega} - \vec{\Omega}(t - \frac{a+\sqrt{\tau} d^2}{\c}), \vec{p}_2 \right) - \right.
\]
\[
\left. - F \left( \hat{\tau}_1 \hat{p}_1 + d \hat{\Omega} - \vec{\Omega}(t, \vec{p}) \right) \right].
\]
$T_4$ is the contribution when the particle is initially inside the hard core and in a time $t$ moves outside the hard core. The restituting momentum is,

$$
\vec{p}_4 = -\sqrt{\mathcal{P}^2(1 - \frac{\vec{b} \cdot \vec{v}}{c^2})} + \mathcal{P} \frac{\vec{b} \cdot \vec{v}}{c} \mathcal{P}_L \left( \frac{\vec{b} - \sqrt{\mathcal{P}^2 - \mathcal{P}^2 \vec{v}^2}}{c} \right),
$$

where $\mathcal{P}_L + \vec{b} - \sqrt{\mathcal{P}^2 - \mathcal{P}^2 \vec{v}^2}$ is the position at which the particle leaves the hard core.

**C. THE BOGOLIUBOV DERIVATION OF THE BINARY COLLISION INTEGRAL**

We present a discussion of how the streaming operator as given by Eq. (4.12) may be used to evaluate the binary collision integral for the Lorentz gas of hard spheres starting with the Bogoliubov formula,

$$
\Omega = \int \frac{d^3 \xi}{2\pi} \lim_{\tau \to \infty} \mathcal{P} \left( \frac{\partial}{\partial \xi} \right) \frac{\partial V}{\partial \vec{v}} \frac{\partial}{\partial \vec{v}} S(\xi; \tau) S(-\tau) f(\xi, \vec{v}; \tau) .
$$

This form differs from the collision integrals of Chapter III in that the limit of infinite time is taken and the exclusion of particle-scatterer overlap is not imposed. We may remove the explicit appearance of the potential with the transform,

$$
\frac{\partial V}{\partial \vec{f}} \cdot \frac{\partial}{\partial \vec{p}} S(\xi; \tau) = S(\tau) \frac{\partial S(-\tau)}{\partial \tau} S(\xi; \tau),
$$

and further transform the right-hand side of (4.25) in the following manner,

$$
\vec{u} \cdot \frac{\partial}{\partial \vec{f}} S(\xi; \tau) + S(\xi; \tau) \left[ \frac{\partial V}{\partial \vec{f}} \cdot \frac{\partial}{\partial \vec{p}} - \vec{u} \cdot \frac{\partial}{\partial \vec{f}} \right].
$$

Bogoliubov assumes that in the long time limit,

$$
S(\xi; \tau) \sqrt{(1 \vec{f} - \vec{v})},
$$

(4.27)
is zero so that the collision operator may be expressed as,

\[ \Omega = \int \frac{dS}{\tau} \lim_{\tau \to \infty} \left[ \nabla \cdot \left( \frac{2}{\sigma^2} S_{1r} \tau \right) - \nabla \cdot \left( \frac{2}{\sigma^2} \right) S_{1r} \tau \right] \frac{f}{F(r, \vec{v}, \tau)} \]  

(4.28)

It is this form, Eq. (4.28), that Bogoliubov claims can be reduced to the Enskog form, Eq. (4.1). We first examine the validity of Eq. (4.28) and then demonstrate the form to which it reduces.

Let us consider the function \( F(r, \vec{v}) \) of Eq. (4.12) to be the potential which is finite only when \(|\vec{r} - \vec{r}_1| < \sigma\). The only contributions from Eq. (4.12) are,

\[ S(1, \tau) \nabla \left( |\vec{r} - \vec{r}_1| \right) = \int \frac{d\vec{b}}{\tau} \Theta \left( \rho - \frac{\sqrt{2m\nu}}{\sqrt{1 - \frac{\nu}{\sigma^2}}} \right) \frac{\tau}{\tau - \tau} \delta \left( \vec{r} - \left( \vec{r}_1 + \vec{b} \right) \right) \cdot \nabla \left( |\vec{r} - \vec{r}_1| \right) + \]

\[ + \int \frac{d\vec{b}}{\tau} \left( \delta \left( \vec{r} - \left( \vec{r}_1 + \vec{b} + a \vec{v}_1 \right) \right) \nabla \left( |\vec{r} - \vec{v}_1| \right) \right) \]

(4.29)

and the second term on the right-hand side is zero for times greater than \( 2\sigma/\nu \). The first term of Eq. (4.29) gives a contribution only for momenta greater than the effective potential. If we take the limit of Eq. (4.29) as \( \tau \to \infty \) before performing the integral over \( \vec{r}_1 \) then the argument of the \( \delta \)-function of the first term can never be zero and the integral over \( \vec{r}_1 \) is zero. Therefore for finite \( \vec{r}_1 \) and \( V_1 \) the long time limit of Eq. (4.27) is strictly zero, and Bogoliubov's assumption is correct.

We next examine the integrand of Eq. (4.28) for the \( T_1 \) contribution of \( S(1, t) \),
\[
\left[ \hat{v} \cdot \frac{\partial}{\partial \tau} S(\tau) - S(\tau) \hat{v} \cdot \frac{\partial}{\partial \rho} \right] F(\tau, \rho).
\] (4.30)

We note that the left-most \( \hat{v} \cdot \frac{\partial}{\partial \tau} \) acts only on the \( S \)-function with the argument \( \tau - (\frac{t_1 + \rho}{\tau_1}) \), \( \tau_1 \) and may therefore be transformed to \( -\frac{\partial}{\partial \tau_1} \) acting only on the \( S \)-function. The right-most \( \hat{v} \cdot \frac{\partial}{\partial \rho} \) (which is to the right of \( S(1; t) \)) acts only on the argument of the function \( F \) and may be transformed to \( \frac{\partial}{\partial \tau_1} \) acting only on this function. Therefore the sum of the two terms yields \( -\frac{\partial}{\partial \tau_1} \) acting on the integrand; hence we simply evaluate the integrand at the limits of the integration domain of \( \tau_1 \). The \( T_1 \) contribution to (4.30) is thus given as,

\[
V \int d^6 \theta \left\{ \Theta \left( \frac{2mV_0}{\sqrt{1 - \frac{V_0^2}{c^2}}} - \rho \right) \left[ S \left( \tau - (\frac{t_1 + \rho}{\tau_1}) \right) \left[ F \left( \frac{t_1 + \rho}{\tau_1}, \frac{\rho}{\tau_1} \right) - F \left( \frac{t_1 + \rho}{\tau_1}, \frac{\rho}{\tau_1} \right) \right] - 
- S \left( \tau - (\frac{t_1 + \rho}{\tau_1}) \right) \left[ F \left( \frac{t_1 + \rho}{\tau_1}, \frac{\rho}{\tau_1} \right) - F \left( \frac{t_1 + \rho}{\tau_1}, \frac{\rho}{\tau_1} \right) \right] \right\}. \tag{4.31}
\]

The same argument applied to the other contributions, i.e., we evaluate \( T_2, T_3, \) and \( T_4 \) at the lower limit of the \( \tau \) or \( \int \) and subtract the value at the upper limit. The \( T_2 \) contribution is given as,

\[
V \int d^6 \theta \left( \theta - \frac{\sqrt{mV_0}}{\sqrt{1 - \frac{V_0^2}{c^2}}} \right) \Theta (\tau - \rho) \left[ S \left( \tau - (\frac{t_1 + \rho}{\tau_1}) \right) \cdot 
\cdot \left[ F \left( \frac{t_1 + \rho}{\tau_1}, \frac{\rho}{\tau_1} \right) - F \left( \frac{t_1 + \rho}{\tau_1}, \frac{\rho}{\tau_1} \right) \right] \right]. \tag{4.32}
\]

The \( T_5 \) contribution is given as,
\[ \mathcal{V} \left\{ d\mathbf{b} \left\{ \Theta \left( \mathbf{p} - \frac{2mV_0}{\sqrt{1-\mu^2}} \right) \left\{ \delta(t-(\mathbf{p}^- + \mathbf{e}^- - \mathbf{V} \mathbf{e}^-)) \left[ F(\mathbf{p}^+, \mathbf{p}^-, \mathbf{\bar{\zeta}}^+ , \mathbf{\bar{\zeta}}^-) - F(\mathbf{\bar{p}}^+, \mathbf{\bar{p}}^-, \mathbf{\bar{\zeta}}^+ , \mathbf{\bar{\zeta}}^-) \right] - \delta(t-(\mathbf{p}^+ + \mathbf{e}^+ - \mathbf{V} \mathbf{e}^-)) \left[ F(\mathbf{p}^-, \mathbf{\bar{p}}^+, \mathbf{\bar{\zeta}}^+ , \mathbf{\bar{\zeta}}^-) - F(\mathbf{\bar{p}}^-, \mathbf{p}^+, \mathbf{\bar{\zeta}}^+ , \mathbf{\bar{\zeta}}^-) \right] \right\} \right\} \right\}. \] (4.33)

The $T_4$ contribution is given as,
\[ \mathcal{V} \left\{ d\mathbf{b} \left\{ \delta(t-(\mathbf{p}^+ + \mathbf{e}^+ - \mathbf{V} \mathbf{e}^-)) \left[ F(\mathbf{p}^+, \mathbf{p}^+, \mathbf{\bar{\zeta}}^+ , \mathbf{\bar{\zeta}}^-) - \delta(t-(\mathbf{p}^- + \mathbf{e}^- - \mathbf{V} \mathbf{e}^-)) \left[ F(\mathbf{p}^-, \mathbf{\bar{p}}^+, \mathbf{\bar{\zeta}}^+ , \mathbf{\bar{\zeta}}^-) - \delta(t-(\mathbf{p}^+ + \mathbf{e}^+ - \mathbf{V} \mathbf{e}^-)) \left[ F(\mathbf{p}^+, \mathbf{\bar{p}}^-, \mathbf{\bar{\zeta}}^+ , \mathbf{\bar{\zeta}}^-) \right] \right] \right] \right\} \right\}. \] (4.34)

Adding the four terms we find that formula (4.30) is given by,
\[ \mathcal{V} \left\{ d\mathbf{b} \left\{ \delta(t-(\mathbf{p}^+ + \mathbf{e}^+ - \mathbf{V} \mathbf{e}^-)) \left[ \Theta \left( \frac{2mV_0}{\sqrt{1-\mu^2}} \right) F(\mathbf{p}^+, \mathbf{\bar{\zeta}}^+ , \mathbf{\bar{\zeta}}^-) + \Theta \left( \frac{2mV_0}{\sqrt{1-\mu^2}} \right) F(\mathbf{\bar{p}}^+, \mathbf{\bar{\zeta}}^+ , \mathbf{\bar{\zeta}}^-) \right] \right\} \right\}. \] (4.35)
The last $\delta$-function in Eq. (4.35) is zero in the limit as $\tau \to \infty$ since its argument can never be zero for finite $\bar{r}$, $\bar{\xi}_1$.

If we now take the limit of expression (4.35) as $\tau \to \infty$ noting that $F = S(-\tau) f(\bar{r}, \bar{\xi}_1; t)$ then we find that the collision integral is given by,

$$L = \int d \bar{r} \int d \bar{\xi}_1 \left\{ \int d \bar{b} \left\{ \delta(\bar{r} - (\bar{r}, \bar{\xi}_1)) \left( \bar{b} - \frac{m V_0}{1 - \bar{v}_1} \hat{e} \right) f(\bar{r}, \bar{\xi}_1, \bar{b}; t) + \right. \\
+ \Theta(p - \frac{m V_0}{1 - \bar{v}_1}) f(\bar{r}, \bar{\xi}_1, \bar{b} - \bar{v}_1 \hat{e}, \bar{b}; t) - \\
\left. - f(\bar{r} + \bar{b} - \frac{m V_0}{1 - \bar{v}_1} \hat{e} + \bar{v}_1 \hat{e}, \bar{b}; t) \right\} + \\
+ \delta(\bar{r} - (\bar{r} + \bar{b} - \frac{m V_0}{1 - \bar{v}_1} \hat{e})) \left[ f(\bar{r}, \bar{b} - \frac{m V_0}{1 - \bar{v}_1} \hat{e}, \bar{b}; t) - \\
- f(\bar{r}, \bar{b} - \frac{m V_0}{1 - \bar{v}_1} \hat{e}, \bar{b}; t) \right] \right\}.
$$

(4.36)

To obtain the Enskog result from this formula we now take the limit $V_0 \to \infty$ holding $p$ finite. We find that Eq. (4.36) is equal to the Enskog collision integral, Eq. (4.1), (we make the transform $\bar{b} \to -\bar{b}$ so that $b - \sqrt{\sigma^2 - b^2} \hat{e} \to -\bar{\rho})$ plus the contribution,

$$\int d \bar{r} \int d \bar{\xi}_1 \left\{ \int d \bar{b} \left[ \delta(\bar{r} - (\bar{r}, \bar{\xi}_1, \bar{b} - \frac{m V_0}{1 - \bar{v}_1} \hat{e})) f(\bar{r}, \bar{b} - \frac{m V_0}{1 - \bar{v}_1} \hat{e}, \bar{b}; t) - \\
- \delta(\bar{r} - (\bar{r}, \bar{\xi}_1, \bar{b})) f(\bar{r}, \infty, \bar{b}; t) \right] \right\}.
$$

(4.37)

We discuss in the next section the conditions that lead to the vanishing of this term.

D. EVALUATION OF THE BINARY COLLISION OPERATOR

Before considering the evaluation of the binary collision operator $B(1)$...
acting on some function \( F(\vec{r}, \vec{p}) \), we demonstrate its connection with the binary collision integral. From the development of Chapter III we found the binary collision integral has the form,

\[
\sum d\xi \rho(\xi) \int S(t) \frac{\partial}{\partial \xi} \left( \frac{S(t; \xi) - S(t)}{\Theta(1 - \xi - \sigma)} \right) \Theta(1 - \xi - \sigma) \cdot S(-t) f(\vec{r}, \vec{p}; t).
\]

(4.38)

We note that the operator in curly brackets may be transformed to the expression

\[
\frac{\partial}{\partial \vec{r}} \cdot \frac{\partial}{\partial \vec{p}} S(\vec{r}; t),
\]

(4.39)

by taking the time derivative; and this gives the Bogoliubov collision integral, Eq. (4.24), without the time limit. A more useful transformation yields the operator,

\[
\left( \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} \right) \left[ S(\vec{r}; t) - S(t) \right],
\]

(4.40)

where the time derivative acts only on the streaming operators \( S(\vec{r}; t) \) and \( S(t) \), and the spatial derivative acts on the streaming operators plus any function of \( \vec{r} \) to their right. The Laplace transform of expression (4.40) is the binary collision operator \( B(\vec{r}) \), Eq. (2.24),

\[
B(\vec{r}) = \int_{0}^{\infty} dt e^{-s t} \left( \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} \right) \left[ S(\vec{r}; t) - S(t) \right],
\]

(4.41)

since the initial value of \( S(\vec{r}; t) - S(t) \) is zero. Thus we find that the binary collision operator is the Laplace transform of that part of the binary collision integral which acts on \( S(-t) f(\vec{r}, \vec{p}; t) \).
We compute expression (4.40) operating on $F(\bar{\mathbf{r}}, \bar{\mathbf{p}})$ for the four contributions of Eq. (4.12).

\[
\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \frac{\partial}{\partial \mathbf{r}}\right) T_1 = \mathcal{U} \int d\mathbf{b} \cdot \Theta \left(\frac{\sqrt{2mV_0}}{\sqrt{1 - \mathbf{u}^2}} - \gamma\right) \delta(\mathbf{r} - (\bar{\mathbf{r}} + \mathbf{u} t, \mathbf{p})) \cdot \\
\left[ F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}') - F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}) \right], \tag{4.42}
\]

\[
\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \frac{\partial}{\partial \mathbf{r}}\right) T_2 = \mathcal{U} \int d\mathbf{b} \cdot \Theta \left(\frac{\sqrt{2mV_0}}{\sqrt{1 - \mathbf{u}^2}}\right) \Theta(\mathbf{r} - \mathbf{r}_0) \delta(\mathbf{r} - (\bar{\mathbf{r}} + \mathbf{u} t, \mathbf{p})) \cdot \\
\left[ F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}_0) - F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}) \right], \tag{4.43}
\]

\[
\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \frac{\partial}{\partial \mathbf{r}}\right) T_3 = \mathcal{U} \int d\mathbf{b} \cdot \Theta \left(\frac{\sqrt{2mV_0}}{\sqrt{1 - \mathbf{u}^2}}\right) \Theta(\mathbf{r} - \mathbf{r}_0) \delta(\mathbf{r} - (\bar{\mathbf{r}} + \mathbf{u} t, \mathbf{p})) \cdot \\
\left[ F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}_0) - F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}) \right], \tag{4.44}
\]

\[
\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \frac{\partial}{\partial \mathbf{r}}\right) T_4 = \mathcal{U} \int d\mathbf{b} \cdot \delta(\mathbf{r} - (\bar{\mathbf{r}} + \mathbf{u} - \sqrt{\frac{2 \mu V_0}{\mathbf{u}^2}} \cdot \mathbf{b})) \cdot \\
\left[ F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}_0) - F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}) \right] - \\
\mathcal{U} \int d\mathbf{b} \cdot \Theta \left(\frac{\sqrt{2mV_0}}{\sqrt{1 - \mathbf{u}^2}}\right) \delta(\mathbf{r} - (\bar{\mathbf{r}} + \mathbf{u} t, \mathbf{p})) \cdot \\
\left[ F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}_0) - F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}) \right], \tag{4.45}
\]

Adding all the contributions, we find that expression (4.40) may be written as the sum of two terms: (1) the Enskog-Lorentz-Boltzmann term,

\[
\mathcal{U} \int d\mathbf{b} \left[ \delta(\mathbf{r} - (\bar{\mathbf{r}}, \mathbf{p})) \Theta \left(\frac{\sqrt{2mV_0}}{\sqrt{1 - \mathbf{u}^2}}\right) F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}') - \delta(\mathbf{r} - (\bar{\mathbf{r}}, \mathbf{p})) F(\bar{\mathbf{r}}, \bar{\mathbf{p}} - \mathbf{u} \cdot \mathbf{b}, \mathbf{p}) \right], \tag{4.46}
\]
where we have introduced the transform $\mathfrak{F} \rightarrow -\mathfrak{F}$, so that $\mathfrak{F} - \sqrt{\sigma^2 - \mathfrak{F}^2} \rightarrow -\mathfrak{F}$; and (2) the remainder term,

$$
\begin{align*}
\mathcal{V} \int d\mathfrak{F} \left\{ \mathcal{S} \left[ \mathfrak{F}^2 - (\mathfrak{F}^2 + \mathfrak{P}^2) \right] & \left( \mathcal{E} \left( t - \frac{2mV_0}{\sqrt{t^2 - \mathfrak{F}^2}} \right) \right) \left( \mathcal{T} \left( t - \frac{2mV_0}{\sqrt{t^2 - \mathfrak{F}^2}} \right) \right) \right. \\
+ \mathcal{E} \left( t - \frac{2mV_0}{\sqrt{t^2 - \mathfrak{F}^2}} \right) & \left( \mathcal{T} \left( t - \frac{2mV_0}{\sqrt{t^2 - \mathfrak{F}^2}} \right) \right) - \left( \mathcal{T} \left( \frac{2mV_0}{\sqrt{t^2 - \mathfrak{F}^2}} \right) \right) \\
- \mathcal{F} \left( \mathfrak{F}, \mathfrak{P} - \sqrt{\mathfrak{F}^2 - \mathfrak{P}^2} \mathfrak{F} - \mathfrak{P} \left( t - \frac{2mV_0}{\sqrt{t^2 - \mathfrak{F}^2}} \right), \mathfrak{P} \right) & \left( \mathcal{T} \left( \frac{2mV_0}{\sqrt{t^2 - \mathfrak{F}^2}} \right) \right) \\
+ \mathcal{F} \left( \mathfrak{F}, \mathfrak{P} - \sqrt{\mathfrak{F}^2 - \mathfrak{P}^2} \mathfrak{F} - \mathfrak{P} \left( t - \frac{2mV_0}{\sqrt{t^2 - \mathfrak{F}^2}} \right), \mathfrak{P} \right) & \left( \mathcal{T} \left( \frac{2mV_0}{\sqrt{t^2 - \mathfrak{F}^2}} \right) \right) \\
& \left. \mathcal{F} \left( \mathfrak{F}, \mathfrak{P} - \sqrt{\mathfrak{F}^2 - \mathfrak{P}^2} \mathfrak{F} - \mathfrak{P} \left( t - \frac{2mV_0}{\sqrt{t^2 - \mathfrak{F}^2}} \right), \mathfrak{P} \right) \right\} \right.
\end{align*}
$$

(4.47)

We now take the limit as $V_0 \rightarrow \infty$. We assume that the Lorentz gas has finite total energy. Since the particle is initially excluded from the hard cores, the initial potential energy is zero; and we may write the total energy as

$$
\iint d\mathfrak{F} d\mathfrak{P} \frac{\mathfrak{P}^2}{2m} f(\mathfrak{F}, \mathfrak{P}; 0) = \text{finite} \qquad (4.48)
$$

Expression (4.48) gives a condition on the volume in phase space $(\mathfrak{F}, \mathfrak{P})$ for large momentum $\mathfrak{P}$. Similarly the kinetic energy at any time $t$ must be finite so that we also have the condition,

$$
\iint d\mathfrak{F} d\mathfrak{P} \frac{\mathfrak{P}^2}{2m} f(\mathfrak{F}, \mathfrak{P}; t) = \text{finite} \qquad (4.49)
$$

Both conditions, (4.48) and (4.49), may be satisfied by supposing that $f(\mathfrak{F}, \mathfrak{P}; t)$ behaves as

$$
-\alpha \mathfrak{P} \mathfrak{P} \qquad (4.50)
$$
as \( p \to \infty \), where \( \alpha > 0 \). This is valid for an appropriate choice of the initial distribution \( f(r, \vec{v}; 0) \). Therefore in the limit \( V_o \to \infty \),

\[
\Theta(\frac{\sqrt{2} m v_o}{\sqrt{1 - \epsilon^2}} - p) \to 0 \tag{4.51}
\]

\[
\Theta(\frac{\sqrt{2} m v_o}{\sqrt{1 - \epsilon^2}}, -p) \to 1 \tag{4.52}
\]

\[
|\vec{v}_o| \to \infty \tag{4.53}
\]

and we may thus write expression (4.40) as follows,

\[
\left( \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} \right) \left[ S(1; t) - S(t) \right] F(\vec{r}, \vec{p}) =
\]

\[
= u \int d\vec{r}' \left[ S(\vec{r} - (\vec{r}'\vec{p})) F(\vec{r}' - \vec{v}t - \vec{v}t', \vec{p}') - S(\vec{r} - \vec{r}'\vec{p}) F(\vec{r}', \vec{p}' - \vec{v}t, \vec{p}) \right] -
\]

\[
- S(\vec{r} - (\vec{r}'\vec{p})) \Theta(\frac{\sqrt{2} m v_o}{\sqrt{1 - \epsilon^2}}) F(\vec{r}' - \vec{v}t, \vec{p}) \right) \right] \tag{4.54}
\]

The first two terms on the right-hand side of Eq. (4.54) correspond to the Enskog-Lorentz-Boltzmann collision integral, and the third term is a short time contribution, \( t < \frac{2\sqrt{2} \sigma^2 - D^2}{v} \), in which the spatial argument of \( F \) remains inside the hard core.

If we substitute expression (4.54) in the curly brackets of Eq. (4.38) taking,

\[
F(\vec{r}, \vec{p}) = \Theta(|\vec{r} - \vec{r}'| - \sigma) f(\vec{r} + \vec{v}t, \vec{p}; t), \tag{4.55}
\]

then the third term of (4.54) is zero because the step function excludes
contributions from inside the hard core; and we obtain the Enskog-Lorentz-
Boltzmann binary collision integral,
\[
\begin{align*}
\int d\frac{\bar{\sigma}}{2} \rho^{(n)}(\frac{\bar{\sigma}}{2}) \nu \int d\frac{\bar{\sigma}}{2} \left[ \delta \left( \frac{\bar{\sigma}}{2} \cdot \frac{\bar{\sigma}}{2} \right) f \left( \frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2} ; t \right) - \\
- \delta \left( \frac{\bar{\sigma}}{2} \cdot \frac{\bar{\sigma}}{2} \right) f \left( \frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2} ; t \right) \right] = \\
= \nu \int d\frac{\bar{\sigma}}{2} \left[ \rho^{(n)}(\frac{\bar{\sigma}}{2}) f(\frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2} ; t) - \rho^{(n)}(\frac{\bar{\sigma}}{2}) f(\frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2} ; t) \right]
\end{align*}
\tag{4.56}
\]

We noted previously that the binary collision integral is the Laplace
transform of the operator in curly brackets of Eq. (4.38); therefore we
obtain the result,
\[
\begin{align*}
\beta(n) F(\frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2}) = \nu \int d\frac{\bar{\sigma}}{2} \left[ \delta \left( \frac{\bar{\sigma}}{2} \cdot \frac{\bar{\sigma}}{2} \right) \left( s + \hat{\mathbf{u}}' \cdot \frac{\partial}{\partial \frac{\bar{\sigma}}{2}} \right) F(\frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2}) - \\
- \delta \left( \frac{\bar{\sigma}}{2} \cdot \frac{\bar{\sigma}}{2} \right) \left( s + \hat{\mathbf{u}}' \cdot \frac{\partial}{\partial \frac{\bar{\sigma}}{2}} \right) \right] = \\
- \nu \int d\frac{\bar{\sigma}}{2} \delta \left( \frac{\bar{\sigma}}{2} \cdot \frac{\bar{\sigma}}{2} \right) \int dt e^{-st} F(\frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2} - \hat{\mathbf{u}}' t, \frac{\bar{\sigma}}{2} )
\end{align*}
\tag{4.57}
\]

If the binary collision operator acts on
\[
\Theta \left( 1 - \frac{\bar{\sigma}}{2} / \frac{\bar{\sigma}}{2} \right) S^{-1} \quad f_s \left( \frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2} \right)
\tag{4.58}
\]
where
\[
S^{-1} = s \hat{\mathbf{u}}' \frac{\partial}{\partial \frac{\bar{\sigma}}{2}}
\]
for the binary collision operator, Eq. (3.22), the result is the Laplace
transform of Eq. (4.56),
\[
\beta(n) \Theta \left( 1 - \frac{\bar{\sigma}}{2} / \frac{\bar{\sigma}}{2} \right) S^{-1} \quad f_s \left( \frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2} \right) = \\
= \nu \int d\frac{\bar{\sigma}}{2} \left[ \delta \left( \frac{\bar{\sigma}}{2} \cdot \frac{\bar{\sigma}}{2} \right) f_s \left( \frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2} ; t \right) - \delta \left( \frac{\bar{\sigma}}{2} \cdot \frac{\bar{\sigma}}{2} \right) f_s \left( \frac{\bar{\sigma}}{2}, \frac{\bar{\sigma}}{2} ; t \right) \right]
\tag{4.59}
\]
thus we have verified the first equality between the product equation collision integrals and the convolution equation collision integrals, Eq. (3.33). We find that this equality is due to the time independence of the operator,

$$\frac{\partial V}{\partial t} \cdot \frac{\partial}{\partial \hat{p}} \: \mathcal{S}(1 \mid t) \: \Theta(|\hat{r} - \hat{r}| - \sigma) \: \mathcal{S}(-t),$$  \hspace{1cm} (4.60)

for hard spheres.

E. EVALUATION OF THE MANY-SCATTERER COLLISION INTEGRALS

We shall use the binary collision operator as given by Eq. (4.57) to obtain an explicit form for the many-scatterer collision integrals which involve products of the binary collision operator. We first consider the product of two binary collision operators

$$B(1) B(2) \: F(\hat{r}, \hat{p}) \: \Theta(|\hat{r} - \hat{r}^\prime| - \sigma) \: \Theta(|\hat{r} - \hat{r}^\prime| - \sigma)$$  \hspace{1cm} (4.61)

We use the rotation operator $R^*_{\hat{p}}$ which acts in the manner of Eq. (4.15) on all functions of $\hat{p}$ to its right except $\hat{p}$, and the hypothetical collision operator* $h^*_{\hat{p}}$ which has the phase (-1) and operates only on $\hat{p}$ transforming it to $-\hat{p}$. We adopt the convention of numbering the $\hat{b}$ and $\hat{p}$ consecutively with the subscripts $1, 2, ...$ corresponding to the order in which they appear from left to right, e.g., and $l$th binary operator has the corresponding quantities $\hat{b}_l, \hat{p}_l, R_{\hat{p}l},$ and $h_l$. In writing out expression (4.61) we first note that the contribution from $B(2)$ (the third term of (4.57)) with the

---

*We use the terminology hypothetical collision, following Green,* to denote a collision event where the particle does not scatter-dropping the terminology, direct collision.
particle inside the hard core vanishes because of the \( \Theta \)-functions, but the same term from \( B(1) \) is not excluded by the \( \Theta \)-functions. Expression \((4.61)\) may now be written as follows,

\[
\nu\int[d\vec{b}_1]\left[\mathcal{P}_{\vec{a}_1} + h_1\right] \mathcal{S}\left(i - (\vec{f}_1 + \vec{f})\right) \left(s + \vec{v} \cdot \frac{\partial}{\partial (\vec{f}_1, \vec{f})}\right)^{-1} \nu\int[d\vec{b}_2] \cdot \\
\cdot \mathcal{S}\left((\vec{f}_1 + \vec{f}) - (\vec{f}_2 + \vec{f})\right) \left(s + \vec{v} \cdot \frac{\partial}{\partial (\vec{f}_2, \vec{f})}\right)^{-1} \Theta\left(1, \vec{f}_1 - \vec{f}_2 - \vec{v}t\right) \cdot \\
\cdot F\left(\vec{f}_1, \vec{f}_2, \vec{f}\right) - \nu\int[d\vec{b}_1] \mathcal{S}\left(i - (\vec{f}_1 + \vec{f})\right) \int_0^{\nu\int[d\vec{b}_2]} dt \nu\int[d\vec{b}_2]\left[\mathcal{P}_{\vec{a}_2} + h_2\right] \cdot \\
\cdot \mathcal{S}\left((\vec{f}_1 + \vec{v}t) - (\vec{f}_2 + \vec{v}t)\right) \left(s + \vec{v} \cdot \frac{\partial}{\partial (\vec{f}_2, \vec{f})}\right)^{-1} \Theta\left(1, \vec{f}_1 + \vec{v}t - \vec{f}_2 - \vec{v}t\right) F\left(\vec{f}_1 + \vec{v}t, \vec{f}\right).
\]

The second term in \((4.62)\) is finite only if the scatterers overlap since its second \( \mathcal{S} \)-function gives the condition,

\[
\left|\vec{f}_1 - \vec{f}_2\right| = \left|\vec{f}_1 - \left(\vec{f}_2 - \vec{v}t\right)\right|.
\]

and this is less than \(2\sigma\) since \(t < \frac{2\sqrt{\sigma^2 - b^2}}{\nu}\). Therefore if we add the condition on the scatterers that they cannot overlap, the extra contribution to the binary collision operator from particle positions inside the hard core is completely removed. Thus with both restrictions, exclusion of particle-scatterer overlap and exclusion of scatterer-scatterer overlap, the binary collision operator reduces to the simple Enskog form, which we write as,

\[
B(1)F\left(\vec{f}, \vec{f}\right) = \nu\int[d\vec{b}]\left[\mathcal{P}_{\vec{a}} + h_2\right] \mathcal{S}\left(i - (\vec{f}_1, \vec{f})\right) \left(s + \vec{v} \cdot \frac{\partial}{\partial (\vec{f}_1, \vec{f})}\right)^{-1} \cdot \\
\cdot F\left(\vec{f}_1, \vec{f}_2, \vec{f}\right).
\]

\((4.64)\)
This result holds for any number of products of binary collision operators in the evolution operator, $r_s^{(m)}$, and the collision operator, $\rho_s^{(m)}$.

The meaning of the first term of Eq. (4.62) may be clarified by examining the product of the Laplace-transformed streaming operator and the $\delta$-function upon which it acts,

$$
(\sigma + \vec{v} \cdot \frac{\partial}{\partial (\vec{r}_1, \vec{r}_2)})^{-1} \delta\left(\left(\vec{r}_1, \vec{r}_2\right) - \left(\vec{r}_1, \vec{r}_2\right)\right)
$$

(4.65)

If we make use of the vector identity,

$$
\vec{a} = \vec{a} \cdot \hat{\vec{n}} \hat{\vec{n}} + \vec{a} \cdot (\vec{a} \cdot \hat{\vec{n}})
$$

(4.66)

expression (4.65) may be written in the form,

$$
\frac{1}{\nu} \left(\begin{array}{c}
1 \\
1 \\
2
\end{array}\right) \mid D \mid \left(\begin{array}{c}
1 \\
1 \\
2
\end{array}\right)
$$

(4.67)

where $(\frac{i}{k} | \frac{j}{l})$ is the condition on the variables,

$$
\delta\left(\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} \right) \Theta\left(\left|\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{c}\right|\right)
$$

(4.68)

and $D (\frac{i}{k} | \frac{j}{l})$ is the distance,

$$
\left|\vec{r} - \vec{r}\right|
$$

(4.69)

An inspection of Eqs. (4.68) and (4.69) gives us an immediate physical intrepretation for formula (4.67). The $\delta$-function in $(\frac{i}{k} | \frac{j}{l})$, Eq. (4.68), is the condition that the relative separation between the point of collision with $\vec{r}_i$ and the point of collision with $\vec{r}_j$ in a plane perpendicular to the particle velocity $\vec{v}$ is zero; this means the momentum $\hat{p}$ aims to collide at
\[ \dot{\xi}_1 + \dot{\rho}_k \text{ and at } \dot{\xi}_j + \dot{\rho}_l \]. The \( \Theta \)-function in \( \left( \frac{1}{k} \| \frac{1}{l} \right) \) gives condition that the momentum aims to collide with \( \dot{\xi}_j \) before \( \dot{\xi}_1 \); if we follow the motion backward in time the particle encounters \( \dot{\xi}_1 \) before encountering \( \dot{\xi}_j \). The immediate generalization of the latter condition is that the collision sequence occurs physically in reverse order to the order in which the binary collision operators are written from left to right. \( D \left( \frac{1}{k} \| \frac{1}{l} \right) \) is the distance the particle travels freely between the collision at \( \dot{\xi}_1 + \dot{\rho}_k \) and at \( \dot{\xi}_j + \dot{\rho}_l \).

\[ \frac{1}{v} D \left( \frac{1}{k} \| \frac{1}{l} \right) \] is the time the particle travels freely between collisions (real or hypothetical).

We may thus write the product of binary collision operators of expression (4.61) in the compact form,

\[ v \left[ \int d \bar{b}_1 \left[ \mathcal{R}_{\bar{\beta}_1} + \mathcal{R}_1 \right] \delta \left( \bar{r} - \left( \bar{\xi}_1 + \bar{\rho} \right) \right) \right] v \left[ \int d \bar{b}_2 \left[ \mathcal{R}_{\bar{\beta}_2} + \mathcal{R}_2 \right] \frac{1}{v} \left( \frac{1}{v} \right) \right] \cdot \]

\[ - \frac{v}{v} D \left( \frac{1}{k} \| \frac{1}{l} \right) \left( s + v \cdot \frac{\partial}{\partial \left( \bar{\xi}_1 + \bar{\rho} \right)} \right)^{-1} F \left( \bar{\xi}_1, \bar{\beta}_1, \bar{\rho} \right) \cdot \]

This term has the interpretation that the particle initially on the collision hemisphere of \( \dot{\xi}_1, \bar{r} = \dot{\xi}_1 + \dot{\rho}_1 \), suffers a collision (real or hypothetical) moves backwards a time \( \frac{1}{v} D \left( \frac{1}{l} \| \frac{1}{k} \right) \) and suffers another collision on the collision hemisphere of \( \dot{\xi}_2, \dot{\xi}_2 + \dot{\rho}_2 \).

If we operate on expression (4.70) from the left with the free streaming operator \( S_b = \left( s + v \cdot \frac{\partial}{\partial \bar{r}} \right)^{-1} \) it transforms to the following form,

\[ v \left[ \int d \bar{b}_1 \left[ \mathcal{R}_{\bar{\beta}_1} + \mathcal{R}_1 \right] \frac{1}{v} \left( \frac{1}{v} \right) \right] e^{-\frac{v}{v} D \left( \frac{1}{k} \| \frac{1}{l} \right)} v \left[ \int d \bar{b}_2 \left[ \mathcal{R}_{\bar{\beta}_2} + \mathcal{R}_2 \right] \right] \cdot \]

\[ \frac{1}{v} \left( \frac{1}{v} \right) e^{-\frac{v}{v} D \left( \frac{1}{k} \| \frac{1}{l} \right)} \left( s + v \cdot \frac{\partial}{\partial \left( \bar{\xi}_1 + \bar{\rho} \right)} \right)^{-1} F \left( \bar{\xi}_1, \bar{\beta}_1, \bar{\rho} \right) \cdot \]

(4.71)
where \( \mathbf{r}_{1} \) is obtained from \( \left( ^{1} \kappa_{i} \right) \) by substituting \( \mathbf{r}_{1} \) + \( \mathbf{p}_{R} \rightarrow \mathbf{r} \). This term, Eq. (4.71), has the interpretation that the particle is initially at a distance \( D \left( \mathbf{r}_{1} \right) \) from \( \mathbf{r}_{1} \) and that it travels freely, backwards a time \( \frac{1}{v} D \left( \mathbf{r}_{1} \right) \) when it suffers a collision with \( \mathbf{r}_{1} \); it then moves freely a time \( \frac{1}{v} D \left( \mathbf{r}_{2} \right) \) and suffers a collision with \( \mathbf{r}_{2} \) at \( \mathbf{r}_{2} + \mathbf{p}_{2} \). Expression (4.70), without the free streaming operator, is representative of terms in the collision integral \( \eta_{s}^{(m)} \), and expression (4.71) is representative of the terms in the evolution operator \( \Gamma_{s}^{(m)} \).

We now evaluate a term from the three scatterer evolution operator,

\[
\frac{1}{\pi^{2}} \left\{|R_{1}^{2} | \mathbf{d}_{x}^{2} \mathbf{d}_{y}^{2} \left| \left\{ \mathcal{P} \left( \mathbf{r}_{1}, \mathbf{r}_{2}, \mathbf{r}_{3}, \mathbf{r}_{4} \right) \right\} \right. \right. \\
\left. \left. \mathcal{B}_{3} \mathcal{B}_{2} \mathcal{B}_{1} \mathcal{B}_{0} \right\} \right\} \mathbf{f} \left( \mathbf{r}, \mathbf{\hat{r}} ; \theta \right) \right. \\
(4.72)
\]

Inserting the form of the binary collision operator given by Eq. (4.64), and using expression (4.67) this term may be written as follows,

\[
\frac{1}{2} \left\{|R_{1}^{2} | \mathbf{d}_{x}^{2} \mathbf{d}_{y}^{2} \left| \left\{ \mathcal{P} \left( \mathbf{r}_{1}, \mathbf{r}_{2}, \mathbf{r}_{3}, \mathbf{r}_{4} \right) \right\} \right. \right. \\
\left. \left. \mathcal{B}_{3} \mathcal{B}_{2} \mathcal{B}_{1} \mathcal{B}_{0} \right\} \right\} \mathbf{f} \left( \mathbf{r}, \mathbf{\hat{r}} ; \theta \right) \\
(4.73)
\]

This term has the physical interpretation, following the motion backwards in time, that the particle begins at \( \mathbf{r} \) moves freely a time \( \frac{1}{v} D \left( \mathbf{r}_{1} \right) \) collides at \( \mathbf{r}_{1} + \mathbf{p}_{1} \), moves freely a time \( \frac{1}{v} D \left( \mathbf{r}_{1} \right) \), collides at \( \mathbf{r}_{2} + \mathbf{p}_{2} \), moves freely a time \( \frac{1}{v} D \left( \mathbf{r}_{3} \right) \), collides at \( \mathbf{r}_{3} + \mathbf{p}_{3} \), moves freely a time \( \frac{1}{v} D \left( \mathbf{r}_{4} \right) \), and finally recollides at \( \mathbf{r}_{2} + \mathbf{p}_{4} \). Each collision may be real or hypothetical depending on which of the operators \( R_{i}, h_{i} \) is taken.
Next we perform the integral over the dependent variables, thereby removing the \( \delta \)-functions. Each \( \delta \)-function imposes a condition on two variables, and we shall use the rules: (1) that we integrate over the two components of \( \mathbf{\xi}_j \) that are perpendicular to \( \mathbf{v} \) (\( \mathbf{\Phi} \wedge (\mathbf{\xi}_j \wedge \mathbf{v}) \)) for each \( (k_{\frac{1}{k}}j_{\frac{1}{k}}) \) in which \( \mathbf{\xi}_j \) appears for the first time; and (2) that we integrate over the two components of \( \mathbf{b}_k \) for each \( (k_{\frac{1}{k}}j_{\frac{1}{k}}) \) in which \( \mathbf{\xi}_j \) has already appeared in a condition to its left. The integrals over the independent \( b_{\frac{1}{k}} \)'s (there are \( m \) independent \( b_{\frac{1}{k}} \)'s for a sequence with \( m \) distinct scatterer indices) are left standing, and the integrals over the components of \( \mathbf{\xi}_j \) parallel to the particle velocity between collisions is transformed to the integral over the time \( \frac{1}{v} D(k_{\frac{1}{k}} j_{\frac{1}{k}}) \) where \( (k_{\frac{1}{k}}j_{\frac{1}{k}}) \) is the left-most condition containing the scatterer label \( j \).

Using these rules we may write expression (4.73) in the simpler form,

\[
\frac{(nu)^3}{3!} \int \cdots \int \prod_{i=1}^{v} \left[ \mathbf{R} \mathbf{\omega}_{i} + \mathbf{h}_{i} \right] \cdot \left( \sum_{s} \mathcal{T}_{\text{coll}} \cdot \left( s + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{v}} \right) \right)^{-1} f(\mathbf{r}', \mathbf{p}', c) ,
\]

(4.74)

where \( \mathcal{T}_{\text{coll}} \) is the total duration of the collision event,

\[
\mathcal{T}_{\text{coll}} = \frac{4}{v} \left[ D(1/1) + D(1/2) + D(2/3) + D(3/4) \right] ,
\]

(4.75)

the product,

\[
\prod_{i=1}^{v} \left[ \mathbf{R} \mathbf{\omega}_{i} + \mathbf{h}_{i} \right] ,
\]

(4.76)

represents the \( 8 \) possible sequences of real or hypothetical collision events,
and \( \vec{r}', \vec{p}' \) are the restituting position and momentum, i.e., they are the position and momentum which in the course of the collision sequence give rise to \( \vec{r} \) and \( \vec{p} \). If we follow the motion backward through the collision sequence corresponding to a product of operators in (4.76), starting with \( \vec{r}, \vec{p} \) the trajectory terminates on scatterer \( \vec{s}_2 \) at the position \( \vec{r}' \) and with momentum \( \vec{p}' \). The domain of integration over the \( 3 \times 3 = 9 \) integration variables in Eq. (4.74) is determined by the physics of the collision event and the limitation that the scatterers cannot overlap. We note that the integration domain, and the values of \( \tau_{\text{coll}}, \vec{r}', \text{ and } \vec{p}' \) all depend on which of the possible sequences of real or hypothetical collision is chosen from Eq. (4.76).

We are thus able to write the collision integral of Eq. (4.72) in a simple form, Eq. (4.74), where the variables are directly related to a physical collision event (a succession of single encounters). The variables and the integration domain are most easily obtained by drawing a diagram of the collision event and using the geometry of the collision sequences. We should remind ourselves of the interpretation of the integration variables in Eq. (4.74). The time \( \tau_1 \) is the time the particle travels freely (following the motion backwards in time) prior to its first encounter with the \( i \)th distinct scatterer in the collision sequence, and \( \vec{b}_i \) is the impact vector for that encounter.

With the above technique we may directly write down the simple form a typical term of the irreducible collision operator \( \Omega' \) takes. Let us consider the term,
This may be reduced to an integral over $3 \times 3 - 1 = 8$ variables since the first $8$-function, viz., $8(\mathbf{r} - (\mathbf{r}_1 + \mathbf{r}_2))$, is not operated on by the streaming operator; thus the integral over $\mathbf{r}_1$ gives the condition that the particle is initially located on the collision hemisphere of $\mathbf{r}_1$ (following the motion backwards in time). The collision integral, Eq. (4.77), may be written in the simple form,

$$
\frac{(nu)^3}{3!} \int \cdots \int d\tau_1 d\tau_3 \frac{d^3 \vec{b}_1}{dB_1} \frac{d^3 \vec{b}_2}{dB_2} \prod_{i=1}^{4} \left[ R_{\vec{r}_i} + h_i \right] 
\cdot e^{-s \mathbf{\tau}_{coll}} \mathbf{f}_5(\mathbf{r}', \mathbf{p}') \cdot \cdot \cdot
$$

where the variables have the same interpretation as in Eq. (4.74). We remind ourselves that the integral over $\tau_1$ is missing in the irreducible collision integral, Eq. (4.78), because $\tau_1 = 0$ or the particle is initially located on the collision hemisphere of the first scatterer encountered (following the motion backwards in time).

We denote the special encounters whose variables $\tau_1$, $\vec{b}_1$ appear in the simplified collision integrals, as the starred encounters. The starred encounter with the label $i$ corresponds to the $i$th distinct scatterer encountered in the sequence following the motion backwards in time or examining the sequences of labels of the binary collision operators from left to right. We recall that $\tau_1$ is the time the particle moves freely prior to the $i$th starred encounter (following the motion backwards in time), and $\vec{b}_1$ is the impact vector for the $i$th starred encounter.
We may now write down the generalized evolution operator, the generalized collision operator, and a generalized Stosszahlansatz. Consider the terms for m scatterers; a given sequence of encounters (a given sequence of labels) which must be irreducible for the collision operators and the Stosszahlansatz; and a given selection of real or hypothetical collisions. We denote the sequence of encounters by the set of labels,

\[ \tilde{j}_1, \tilde{j}_2, \tilde{j}_3, \cdots, \tilde{j}_m, \]  

and the selection of real or hypothetical collisions by the product,

\[ \prod_{i=1}^{l} \mathcal{O}_i \quad , \quad \mathcal{O}_i = \mathcal{R}_{2i} \quad \text{or} \quad \mathcal{L}_{2i}. \]  

The generalized evolution operator for the sequence of \((4.79)\) and the selection of \((4.80)\) is, in Laplace transform, given by,

\[
\left(\frac{nu}{m!}\right)^m \int \cdots \int d\tau_1 \cdots d\tau_m \int \cdots \int d\bar{\tau}_1 \cdots d\bar{\tau}_m \prod_{i=1}^{l} \mathcal{O}_i \quad e^{-s\tau_{coll}} \cdot \quad \left(s + \tilde{u} \cdot \frac{\partial}{\partial \tilde{r}}\right)^{-1} f(\tilde{r}', \tilde{p}'; 0),
\]

and appears as,

\[
\left(\frac{nu}{m!}\right)^m \int \cdots \int d\tau_1 \cdots d\tau_m \int \cdots \int d\bar{\tau}_1 \cdots d\bar{\tau}_m \prod_{i=1}^{l} \mathcal{O}_i \quad \Theta(t - \tau_{coll}) \cdot \quad f(\tilde{r}' - \tilde{u}(t - \tau_{coll}), \tilde{p}'; 0),
\]

in the time domain. \(\Gamma_m^{(m)}(t) f(\tilde{r}', \tilde{p}; 0)\) is the sum of expression \((4.82)\) over all sequences \((4.79)\) and selections \((4.80)\). We note that the phase of \((4.82)\) is \((-\nu)\), where \(\nu\) is the number of hypothetical collisions.
The generalized collision operator for sequence (4.79) and selection (4.80) is given in the Laplace transform domain as,

\[
\left(\frac{n_\nu}{m!}\right)^m \int \cdots \int d\tau_1 \cdots d\tau_m \; db_1 \cdots db_m \; \prod_{i=1}^{l} \mathcal{O}_i \; e^{-s \tau_{\text{coll}}} 
\cdot f_s(\vec{r}', \vec{p}') \]

(4.83)

and in the time domain by,

\[
\left(\frac{n_\nu}{m!}\right)^m \int \cdots \int d\tau_1 \cdots d\tau_m \; db_1 \cdots db_m \; \prod_{i=1}^{l} \mathcal{O}_i \; \Theta(t - \tau_{\text{coll}}) 
\cdot f(\vec{r}', \vec{p}'; t - \tau_{\text{coll}}) \]

(4.84)

\(f_s^{(m)}(\vec{r}, \vec{p})\) is the sum of expression (4.83) over all irreducible sequences (4.79) and selections (4.80). \(A^{(m)}(t) f(\vec{r}, \vec{p}; t)\) is the corresponding sum of expressions (4.84).

The generalized Stosszahlansatz states that the probability of a collision corresponding to the sequence of (4.79) and the selection (4.80) is given by,

\[
\left(\frac{n_\nu}{m!}\right)^m \int \cdots \int d\tau_1 \cdots d\tau_m \; db_1 \cdots db_m \; \prod_{i=1}^{l} \mathcal{O}_i \; \Theta(t - \tau_{\text{coll}}) f(\vec{r}', \vec{p}'; t - \tau_{\text{coll}}) \]

(4.85)

We find from expressions (4.82) and (4.85) that in addition to evaluating the particle distribution at the restituting position and momentum, it must also be evaluated at a time \(\tau_{\text{coll}}\) in the past.
V. SOME MODEL CALCULATIONS

A. INTRODUCTION

The main purpose of this chapter is to demonstrate how one may directly compute the terms of the evolution operator, $\Gamma^{(m)}_S$ and the collision operator, $\Omega^{(m)}_S$, from formulas (4.57) and (4.58). A complete evaluation for any model other than the one-dimensional hard core model presents considerable difficulty both in enumerating the various collision events with m-scatterers and in carrying out the calculations. We shall therefore only sketch the steps involved in calculating $\Gamma^{(m)}_S$ and $\Omega^{(m)}_S$, and demonstrate the divergence of the leading terms. The one-dimensional model is completely evaluated.

The procedure to be followed in computing a given m-scatterer term is first to draw a diagram of the physical collision event corresponding to the sequences of Eqs. (4.57) and (4.58), second to determine the independent variables ($dm$ for $\Gamma^{(m)}_S$ and $dm$ for $\Omega^{(m)}_S$ where $d$ is the dimension) from the diagrams, third to evaluate the dependent scattering angles and the total collision time as functions of the independent variables, fourth to determine the integration domain from the geometry of the complex collision event, and fifth to carry out the integration over as many variables as possible.

B. ONE-DIMENSIONAL POINT SCATTERERS MODEL

This model is perhaps the simplest tractable model and may even be
computed in a direct manner. It consists of randomly distributed point hard core scatterers on the line and assumes spatial uniformity of the particle distribution. We first compute the general evolution operator, \( \Gamma_{s}^{(m)} \), given by the basic formula obtained from Eq. (4,58),

\[
\Gamma_{s}^{(m)} \equiv \frac{1}{m!} \sum_{j_{1} \cdots j_{m}} (n \theta)^{m} \int \cdots \int \prod_{i=1}^{m} \delta_{j_{i}, j_{i+1}} \sum_{\nu} \left( -\frac{\Omega_{\nu}}{s} \right)^{\nu-1} \int f(p; o) \, dp', \quad (5.1)
\]

where: with the assumption of random scatterers we replace the m-scatterer distribution \( P(12 \ldots m) \) by \( n^{m} \) where \( n \) is the density of scatterers on the line; we replace the translation operator of Eq. (4,58) by \( 1/s; \) the sum \( \sum_{j_{1} \cdots j_{m}} \) is over all sequences containing the m labels 12 \ldots m excluding sequences with identical adjacent labels; and in one dimension there are no collision parameters \( \Omega \), since the particle always aims to collide with the scatterers. We recall that \( \tau_{1} \) is the time the particle moves freely (between encounters) prior (following the motion backward in time) to its first encounter with the i-th distinct scatterer encountered, \( \tau_{\text{coll}} \) is the total time of the collision event, \( \nu \) is the number of hypothetical collisions in the sequence, and \( p' \) is the restituting momentum which in one dimension can only take the values \( \pm p \).

We first evaluate the single-scatterer term \( \Gamma_{s}^{(1)} \). The sum over sequences is the single term \( j_{1} = 1, \tau_{\text{coll}} = \tau_{1} \), and \( \sum_{\text{real or hypt}} \) has only the two terms \( (-)^{0} f(-p; o) \) and \( (-)^{1} f(p; o) \) corresponding respectively to a real- or a hypothetical-collision. Therefore we may write \( \Gamma_{s}^{(1)} \) as,
\[
\int_0^{\infty} \eta(s) e^{-\frac{ts}{s^2}} [\mathcal{F}(\rho;0) - \mathcal{F}(\rho;\infty)] = \int_0^{\infty} \eta(s) e^{-\frac{ts}{s^2}} [\mathcal{F}(\rho;0) - \mathcal{F}(\rho;\infty)] .
\]

(5.2)

For the \(m\)-scatterer events we consider a schematic collision history with time plotted on the ordinate and position on the abscissa. We plot the trajectory (following the motion backward in time) according to the sequence \(r, j_1, j_2, \ldots, j_k\) so that it begins at \(r\) (the particle position) and moves downward in zig-zags ending on some scatterer, and the fixed scatterers are plotted as vertical lines extending over all time. We relabel the sequences in \(\sum_{m \leq \xi_m < \xi_{m-1} < \ldots < \xi_1}\) so that the particle momentum is to the right and \(\xi_m < \xi_{m-1} < \ldots < \xi_1\).

We may simplify the collision diagrams by noting that whenever the trajectory passes through a scatterer this may correspond either to a hypothetical collision with phase -1 or to a no encounter (no corresponding binary operator in the sequence) with phase +1; in both cases \(\tau_{\text{coll}}\) and \(p'\) are the same, and the Jacobian transforming from one set of independent times \(\tau_i\) to another is unity so that the contributions have equal magnitude. Both possibilities are not always realized but whenever they are, their sum is zero. They are both realized in \(p^{(m)}_s\) whenever a given scatter has a real collision and the trajectory also passes through it. This follows because when a scatterer suffers a real collision it has at least one binary operator in the sequence so that each passing through has both possibilities of no encounter or hypothetical collision. Thus we do not consider those diagrams
where a scatterer has both a passing through and either ends the trajectory or reflects the trajectory. (The end of the trajectory must correspond to a binary operator, and each reflection corresponds to a real collision.) Hence the only diagrams we consider are shown is Figure 1(a,b).

\[ \begin{array}{cc}
\text{(a)} & \text{(b)} \\
\end{array} \]

Figure 1. One-dimensional collision history.

In Figure 1(a) each passing through can only correspond to a hypothetical collision since each scatterer must have at least one binary operator in the sequence \( j_1 \ldots j_l \). The collision duration is the sum of the times \( \tau_i \),

\[ \tau_{coll} = \tau_1 + \tau_2 + \cdots + \tau_m, \quad (5.3) \]

the first \( m-1 \) scatterers encountered contribute the phase \( (-)^{\nu} = (-)^{m-1} \)
and the last scatterer encountered may have either a real- or a hypothetical-collision. Therefore the contribution to \( \Gamma_{s}^{(m)} \) from Figure 1(a) is,

\[ \Gamma_{s}^{(m)}(\rho; \omega) = \frac{\nu m^m}{\nu} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \cdot \cdot \cdot \int_{0}^{\infty} \cdot \cdot \cdot \int_{0}^{\infty} (-)^{m} \frac{f(\rho; \omega) - f(\rho; \omega)}{\tau_{coll}} = \]

\[ \frac{(m)^m}{m^m} \cdot (-)^{m-1} \left[ f(\rho; \omega) - f(\rho; \omega) \right]. \quad (5.4) \]
In Figure 1(b), there are all possibilities of no encounter or hypothetical collision for each passing through of the \( m-2 \) scatterers \( i_2, i_3, \ldots, i_{m-1} \) providing each has at least one hypothetical collision. The phase and counting of these possibilities is obtained as follows. Consider one of these scatterers. Suppose the trajectory passes through this scatterer \( j \) times so that there are in all \( 2^j \) combinations of no encounter and hypothetical collision. The sum of these \( 2^j \) combinations is zero since the phases cancel in pairs. Since the combination consisting of all no encounters is forbidden and has the phase \( +1 \) the sum of the remaining \( 2^j - 1 \) combinations must contribute only the phase \( -1 \). Thus each scatterer for which the trajectory only passes through contributes the phase \( -1 \). If \( l \) denotes the number of full paths between \( i_1 \) and \( i_m \) then scatterers \( i_1 \) and \( i_m \) contribute the product of operators \( P^l \), where in one dimension the rotation operator is the parity operator \( P \), and the last scatterer contributes both \( P \) and \( -1 \).

For convenience of calculation we may suppose the first encounter to be with \( i_m \), the next with \( i_{m-1} \), \( i_{m-2} \), \ldots. Therefore the collision duration is given as,

\[
\tau_{\text{coll}} = \tau_1 + \frac{1}{a} \left( \tau_2 + \cdots + \tau_m \right) \tag{5.5}
\]

and the domain of integration is \( 0 < \tau_1 < \tau_2 + \tau_3 + \cdots + \tau_m \) (this is the condition that \( \xi_1 < r < \xi_m \)), and \( 0 < \tau_1 < \infty \) for \( i = 2, 3, \ldots, m \). We may now write down the contribution to \( \Gamma_\xi^{(m)} \) from Figure 1(b),
\[ \Gamma_s^{(m)} f(p;0) = \frac{(n\nu)^m}{s} \sum_{l=1}^{\infty} \ln \left[ \sum_{l=1}^{\infty} d\tau_1 \cdots d\tau_m \right] \rho_l. \]

where \( P \) acting on \( [f(-p;0) - f(p;0)] = [P\, \lambda] f(p;0) \) reduces to the phase -1.

Adding the two contributions to \( \Gamma_s^{(m)} \), we find,

\[ \Gamma_s^{(m)} f(p;0) = \frac{2}{s} \frac{(n\nu)^m}{s} \sum_{l=1}^{\infty} \ln \left[ \sum_{l=1}^{\infty} \frac{(-)^l}{l^{m-1}} \right]. \] (5.7)

This result was first obtained by Ford. The complete evolution operator \( \Gamma_s \) gives the Laplace-transformed particle distribution,

\[ f_s(p) = \Gamma_s \ln f(p;0) = \frac{1}{s} f(p;0) + \frac{n\nu}{s} \ln \left[ f(-p;0) - f(p;0) \right] + \]

\[ + \frac{2}{s} \sum_{m=2}^{\infty} \frac{(n\nu)^m}{s} \sum_{l=1}^{\infty} \frac{(-)^l}{l^{m-1}} \ln \left[ f(-p;0) - f(p;0) \right]. \] (5.8)

Taking the inverse Laplace transform, we find the particle distribution \( f(p;t) \) is given by,

\[ f(p;t) = f(p;0) + \left\{ n\nu t + 2 \sum_{m=2}^{\infty} \frac{(n\nu)^m}{m!} \sum_{l=1}^{\infty} \frac{(-)^l}{l^{m-1}} \ln \left[ f(-p;0) - f(p;0) \right] \right\}. \] (5.9)

and we find that the \( m \)th density term diverges as \( t^m \).
We may write either of these expressions as a single sum by interchanging the summations, e.g., Eq. (5.8) is reduced to,

\[
\begin{align*}
\hat{f}_s(p) &= \frac{1}{s} \hat{f}(p;0) + \frac{\nu}{s^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{s^{2k}} \left[ \hat{f}(p;0) - \hat{f}(p;\nu) \right].
\end{align*}
\] (5.10)

This form may be expanded in powers of \( s \) starting with \( 1/s \) and used to demonstrate that the long time limit of the particle distribution is given by,

\[
\lim_{t \to \infty} \hat{f}(p;t) = \lim_{s \to 0^+} s \hat{f}_s(p) = \hat{f}(p;0) + \frac{1}{2} \left[ \hat{f}(p;0) - \hat{f}(p;\nu) \right] =
\]

\[
\frac{1}{2} \left[ \hat{f}(p;0) + \hat{f}(p;\nu) \right],
\] (5.11)

which is the result we expect physically, i.e., the asymptotic distribution is isotropic.

We will not compute the irreducible collision operator for this model because the task of enumerating the irreducible diagrams is not worthwhile. It is more expeditious to use the Husimi-like expansion relating \( \hat{\gamma}_s \) and \( \Gamma_s \). The functional relation between the complete evolution and collision operators is for this model given by,

\[
\Gamma_s = \left( s - \hat{\gamma}_s \right)^{-1}.
\] (5.12)

Solving for \( \hat{\gamma}_s \) we find,

\[
\hat{\gamma}_s = s - \Gamma_s = s - \left( \frac{1}{s} + \sum_{m=1}^{\infty} \Gamma_s^{(m)} \right)^{-1}.
\] (5.13)

From Eq. (5.13) we find that the m-scatterer collision operator diverges
as \( s^{1-m} \). By expanding \( \Gamma_s \) in powers of \( s \) using the evolution operator in the form given by Eq. (5.10), we find the following expansion for the collision operator,

\[
\Gamma^m_s = \left[ \frac{(mv)^2}{s} + \frac{3}{2} s + \cdots \right] \left[ \rho - 1 \right].
\]

(5.14)

The significance of this formula is discussed in Chapter VIII, where we demonstrate that there is zero transport for the point scatterers model.

We next drop the restriction to spatially uniform distribution and consider the complete evolution operator, \( \Gamma_s \), acting on \( f_s(x, p) \). The m-scatterer operator is given by Eq. (5.1) with \( 1/s \) replaced by \( (s + p/m \frac{\partial}{\partial \xi_{j,l}})^{-1} \) and \( f(x, p) \) evaluated at \( x = \xi_{j,l} \) where \( \xi_{j,l} \) is the position of the last scatterer encountered on the trajectory. The same collision diagrams will be used as in the previous calculation, Figure 1(a,b), except that we must distinguish diagrams that have a different terminal point. Considering Figure 1(a) first, the contribution to the m-scatterer evolution operator is,

\[
\Gamma^m_s f \equiv (nv)^m \int_0^\infty \cdots \int_0^\infty e^{-s \tau_{col}} \left[ \left( s + p/m \frac{2}{\partial x} \right)^{-1} f(x, p) \right]_{x = \tau_{col} - v \tau},
\]

(5.15)

where \( \tau_{col} \) is given by Eq. (5.3), and we let \( p \) denote the particle momentum which changes sign under \( P \), while \( v \) is the magnitude of the particle velocity. We introduce the change of variables \( \tau = \tau_{col} \), and integrate over the \( m-1 \) times \( \tau_2, \tau_3, \ldots, \tau_m \) giving,
\[ n^m \int_5 f(r, p) = (nU)^m (-)^m \int_0^\infty \int_0^r \int_0^{\tau_{coll}} \left[ \sum_{\ell = 1}^{\infty} \left\{ -S \tau_{coll} \right\} \right] \]

\[ \left[ \frac{2}{\alpha x} \right]^{-1} f(x, \eta) \right|_{x = r - \tau_{coll}} \left. \sum_{\ell = 1}^{\infty} \left[ \left[ \frac{2}{\alpha x} \right]^{-1} f(x, \eta) \right] \right|_{x = r - \tau_{coll}} \left[ \left[ \frac{2}{\alpha x} \right]^{-1} f(x, \eta) \right] \right|_{x = r - \tau_{coll}} \]

(5.17)

Next consider the contribution from Figure 1(b). We adopt the convention that \( \tau_1 \) is the time it takes the particle to travel from \( r \) to \( \xi_m \) and that \( \tau_2 + \ldots + \tau_m \) is the time the particle travels from \( \xi_m \) to \( \xi_1 \), so that \( \tau_{coll} = \tau_1 + \ell(\tau_2 + \ldots + \tau_m) \). When the particle makes an even number of trips between \( \xi_m \) and \( \xi_1 \), \( \ell = \text{even} \), then the particle distribution is evaluated at \( x = r - \tau_{coll} \), and when the number of trips is odd, \( \ell = \text{odd} \), the distribution is evaluated at \( x = r - \tau_{coll} + \ell(\tau_2 + \ldots + \tau_m) \). Therefore we obtain the contribution of Figure 1(b),

\[ \int_5 f(r, p) = (nU)^m (-)^m \int_0^\infty \int_0^r \int_0^{\tau_{coll}} \left[ \sum_{\ell = 1}^{\infty} \left\{ -S \tau_{coll} \right\} \right] \]

\[ \left[ \frac{2}{\alpha x} \right]^{-1} f(x, \eta) \right|_{x = r - \tau_{coll}} \left. \sum_{\ell = 1}^{\infty} \left[ \left[ \frac{2}{\alpha x} \right]^{-1} f(x, \eta) \right] \right|_{x = r - \tau_{coll}} \left[ \left[ \frac{2}{\alpha x} \right]^{-1} f(x, \eta) \right] \right|_{x = r - \tau_{coll}} \]

(5.17)

If we introduce the change of variables \( \tau_2 \to \tau_\alpha = \tau_2 + \tau_3 + \ldots + \tau_m \), then the integral over the \( m - 2 \) times \( \tau_3, \tau_4, \ldots, \tau_m \) is performed in the same manner as the above, giving,
\[ (n\nu)^m \left( \frac{\nu}{m-2} \right) \sum_{0}^{\infty} \int_{0}^{\infty} \frac{\tau_{d}}{(m-2)!} \left\{ \sum_{k=1}^{\infty} e^{-s[\tau_{r} + k \tau_{d}]} \right\} [p-1]. \]

\[ \cdot \left[ \left( s + \frac{\nu}{2} \frac{\partial}{\partial s} \right)^{-1} f(r, p) \right]_{x = 1 - \nu \tau_{r} + \nu \tau_{d}} - \sum_{\nu = 1}^{\infty} e^{-s[\tau_{r} + \nu \tau_{d}]} \left[ 1 \right]. \]

\[ \cdot \left[ \left( s + \frac{\nu}{2} \frac{\partial}{\partial s} \right)^{-1} f(r, p) \right]_{x = 1 - \nu \tau_{r} + \nu \tau_{d}} \right\}, \] (5.18)

and this may be reduced to,

\[ (n\nu)^m \left( \frac{\nu}{m-2} \right) \sum_{0}^{\infty} \int_{0}^{\infty} \frac{\tau_{d}}{(m-2)!} \left( \int_{0}^{\infty} \frac{\partial}{\partial s} \right)_{k=1}^{\infty} e^{-s[\tau_{r} + 2k \tau_{d}]} \left. \left[ \left( s - \frac{\nu}{2} \frac{\partial}{\partial s} (1 - \nu \tau_{r}) \right)^{-1} f(r, p) \right]_{x = 1 - \nu \tau_{r} + \nu \tau_{d}} \right. \]

Thus the m-scatterer evolution operator acting on \( f(r, p; o) \) is the sum of Eqs. (5.16) and (5.19).

C. ONE-DIMENSIONAL FINITE SCATTERERS MODEL

We discuss the one-dimensional model of finite hard core scatterers with diameter 2\( \sigma \). The m-scatterer distribution, \( p^{(m)}(12...m) \), is approximated by \( n^m \) multiplied by the condition that the scatterers do not overlap, and the particle has zero extent. Consider first the single-scatterer evolution operator, \( l_s^{(1)} \), which has the two contributions corresponding to a real and a hypothetical collision. In both, the time the particle travels freely before the first encounter, \( \tau_1 \), has the limits 0 to \( \infty \). Therefore the evolution operator is given by,
\[
\left(\frac{n(0)}{(-)}\right)_m^m \int_0^\infty \int_0^{\tau_1} \cdots \left(\tau_{-1} \cdot \frac{2 \sigma}{m-1} \right)^{m-1} e^{-s \tau_{-1}} \left(5 + \frac{2 \sigma}{2(1 - v \tau_{-1})}\right)^{-1} \cdot \left(1 - \frac{2 \sigma}{2(1 - v \tau_{-1})}\right) \cdot \int \left(1 - u \tau_{-1} \cdot \rho; 0\right) d\tau_1 \cdots d\tau_m.
\]

(5.23)

We next consider the contribution of Figure 1(b) when the number of complete trips between \(\xi_1\) and \(\xi_m\) is even so that the restituting position is at \(\xi_{m+1}\), for a last, real encounter. For convenience we choose the starred encounters so that \(\tau_1\) is the time the particle travels between \(r\) and its first encounter with \(\xi_1\), \(\tau_2\) is the time it travels to \(\xi_2\), etc. We perform the integral over \(\tau_2, \tau_3, \ldots, \tau_m\), denoting their sum as \(\tau\), using the technique that led to Eq. (5.22). The result is given by the following,

\[
\left(\frac{n(0)}{(-)}\right)_m^m \int_0^\infty \left(\tau - \frac{2 \sigma}{m-2}\right)^{m-2} e^{-s \tau} \left(5 - \frac{2 \sigma}{2(1 - v \tau)}\right)^{-1} \cdot \int \left(1 - u \tau \cdot \rho; 0\right) d\tau_1 \cdots d\tau_m.
\]

(5.24)

The contribution for the last, hypothetical collision and even \(l\) is obtained from Eq. (5.24) by adding \(2 \sigma / v\) to the collision duration and \(-2 \sigma\) to the restituting position,

\[
\left(\frac{n(0)}{(-)}\right)_m^{m-1} \int_0^\infty \left(\tau - \frac{2 \sigma}{m-2}\right)^{m-2} e^{-s \tau} \left(5 + \frac{2 \sigma}{2(1 - v \tau)}\right)^{-1} \cdot \int \left(1 - u \tau \cdot \rho; 0\right) d\tau_1 \cdots d\tau_m.
\]

(5.25)

The contribution for odd \(l\) when the trajectory terminates on the rightmost scatterer, \(\xi_1\), is taken from Eq. (5.24) with the same integrals over \(\tau\) and \(\tau_1\), but with the restituting position at \(r + v(\tau - \tau_1)\) for the last,
real collision and at \( r + v(\tau - \tau^1) + 26 \) for the last, hypothetical collision. Therefore if we make the change of variable, \( \tau - \tau^1 = \tau' \), we obtain the following contribution for the last, real collision,

\[
(n \nu)^m (-)^m \int_0^\infty d\tau \left( \frac{\tau - (m-2) \frac{2\xi}{v}}{(m-2)!} \right)^{m-1} \tau \left[ \xi (m+1) \tau - \tau' \right] \nonumber \]

\[
\left( 5 + \frac{v}{2 (1 + v \tau')} \right)^{-1} f(k + v \tau' + 26, -\rho; \sigma) \]  \hspace{1cm} (5.26)

and the following contribution for the last, hypothetical collision,

\[
(n \nu)^m (-)^m \int_0^\infty d\tau \left( \frac{\tau - (m-2) \frac{2\xi}{v}}{(m-2)!} \right)^{m-1} \tau \left[ \xi (m+1) \tau - \tau' + \frac{2\xi}{v} \right] \nonumber \]

\[
\left( 5 - \frac{v}{2 (1 + v \tau' + 26)} \right)^{-1} f(k + v \tau' + 26, -\rho; \sigma) \]  \hspace{1cm} (5.27)

A comparison of Eqs. (5.22) through (5.27) with Eqs. (5.16) and (5.18) shows that these results correctly reduce to the point scatterers calculation in the limit as \( \sigma \to 0 \).

If we impose the restriction of spatial uniformity and add all contributions to \( \Gamma^{(m)}_s \) we find,

\[
\prod_{s} \sum_{m=1}^{\infty} \frac{(n \nu)^m (-)^m}{(m+1)!} \frac{\xi}{\xi} \frac{-\xi (m+1)}{(m+2)\frac{2\xi}{v}} \left[ 1 + \frac{-\xi}{\xi} \right]. \hspace{1cm} (5.28)
\]

The complete evolution operator may be obtained by adding the m-scatterer operators, \( \Gamma^{(m)}_s \), and interchanging the sum over m and \( \xi \) thereby expressing the result as a single sum,
\[
\Gamma_s \int \rho; \omega) = \frac{1}{2} \int \rho; \omega) + \frac{nu}{\Gamma_s} \left[ \sum_{\ell=1}^{\infty} \left( \frac{-1}{\ell + \frac{1}{\Gamma_s}} \right) \left[ \frac{5 \sigma}{nv} + \frac{-2 \Gamma_s}{v} \right] \right] \cdot \left[ \rho - 1 \right] \int \rho; \omega) .
\]

(5.29)

Expanding Eq. (5.29) in powers of \( s \) and inserting it in Eq. (5.13) gives an expression for the irreducible collision operator valid in the long time limit,

\[
\mathcal{N}_s \int \rho; \omega) = \left[ \frac{1}{2} \frac{v}{\frac{3}{2} n \epsilon} - \left( 2 - \frac{3}{2} n \epsilon + \frac{1}{4 (n \epsilon)^2} \right) s + \cdots \right] \cdot \left[ \rho - 1 \right] \int \rho; \omega) .
\]

(5.30)

Thus \( \mathcal{N}_s \) is well-behaved in the long time limit for the one-dimensional finite scatterers model in contrast to its behavior for the point scatterers model where it diverges.

D. WIND-TREE MODEL

The behavior of the long time divergence of the collision integrals is discussed for the Ehrenfest wind-tree model. This two-dimensional model consists of a large number of parallel squares (the scatterers) with diagonal \( 2a \), and a point particle that moves parallel to the scatterer diagonals. When the particle suffers a real collision it scatters to the right or left by \( \pi/2 \) radians; the rotation operator, \( R \), rotates the particle velocity \( \rho' \), so that \( \rho' \), given by

\[
\rho' = R \rho
\]

is the restituting momentum that evolves into \( \rho \) with a rotation of \( +\pi/2 \).
The integral over the impact vector in the collision integrals of Chapter IV reduces to the integral over the impact parameter $b$ which runs from $-\sigma$ to $+\sigma$. The rotation operator, $R_p^b$, of Chapter IV is denoted by $R(b)$ and is expressed as,

$$R(b) = \begin{cases} R & b > 0 \\ Rp & b < 0 \end{cases}, \quad (5.32)$$

where $P$ is the parity operator. We note that the great simplicity of the wind-tree model over the hard spheres model is that all possible scattering events (rotations) in the former form a finite group of order 4 whereas in the latter the scattering events form a continuous group. The elements of the group of rotations in the wind-tree model are,

$$1, \; P, \; R, \; RP. \quad (5.33)$$

The vector describing the locus of collisions on a square, $\hat{\beta}$, is expressed as,

$$\dot{\beta} = \left(e^{-ib/\hbar}\right) \dot{\nu} + b RP \dot{\nu}. \quad (5.34)$$

We shall assume spatial uniformity throughout the wind-tree calculation.

Let us consider first the single-scatterer evolution operator, $R_{s}^{(1)}$, operating on $f(\beta;0)$. From Chapter IV this is written as an integral over the time $\tau_1$ and the impact parameter $b_1$,

$$\int_{s}^{(1)} f(\beta;0) = \frac{m\nu}{\hbar} \int d\tau_1 \int db_1 e^{-s\tau_1} \left[ R(b) - 1 \right] f(\beta;0), \quad (5.35)$$
where the domain of $\tau_1$ is $0$ to $\infty$ for both the real and the hypothetical collisions. We again note that as in the one-dimensional case, the initial particle position, $\vec{r}$, may be inside a scatterer and the trajectory may pass through a scatterer; it is only the restituting position, $\vec{r}'$, that must be outside the hard core. Performing the integrals in (5.35) we obtain the formula for the single-scatterer evolution operator,

$$\Pi_1^{(1)} \hat{f}(\vec{r}'; \phi) = \frac{\hbar \nu \sigma}{S^2} \left[ R(1+\rho) - 2 \right] \hat{f}(\vec{r}; \phi). \quad (5.36)$$

In evaluating the two scatterer evolution operator, we must specify the sequence of labels and the selection of real or hypothetical collisions, (r) or (h), that we are computing. Let us consider first the sequence 12, the simplest and most divergent part of $\Gamma_s^{(2)}$. The factor $1/m!$ of Eq. (4.81) may be removed by summing over the $m!$ permutations of the labels in $\Gamma_s^{(m)}$. There are four possible selections of real or hypothetical collisions; for some of the selections the integration domain is complicated while for others it is quite simple. Since we are mainly interested in the long time behavior of the collision integrals we integrate over the domains $-\sigma < b_1 < \sigma$, $0 < \tau_1 < \infty$, and this amounts to relaxing the exclusion of scatterer-scatterer overlap. Thus for all selections, the dominant part of the 12 contribution to $\Gamma_s^{(2)}$ is given by,

$$\left( \frac{\hbar \nu \sigma}{S^2} \right)^2 \left[ R(1+\rho) - 2 \right] \hat{f}(\vec{r}; \phi). \quad (5.37)$$

where the exact 12 contribution adds higher powers of $\sigma/\nu$ to this result. Applying this to the m-scatterer evolution operator we similarly find that
the dominant part of \( \Gamma_s^{(m)} \) is given by,
\[
\frac{(n \nu \sigma)^m}{S^{m+1}} \left[ R(1+\rho) - 2 \right]^m \hat{f}(\vec{p}; \omega). \tag{5.38}
\]

If we sum these most dominant parts for all \( m \), we find they give the following contribution to the complete evolution operator in the time domain, \( \Gamma(t) \),
\[
\frac{n \nu \sigma \frac{d}{dt} \left[ R(1+\rho) - 2 \right]}{S} \hat{f}(\vec{p}; \omega). \tag{5.39}
\]

This term is equivalent to the well-known result of considering only binary collisions. We see that it is obtained by calculating with only the simplest sequences in each \( \Gamma_s^{(m)} \) (the completely reducible sequences) and by approximating their value with a simplification of the integration domain (allowing the scatterers to overlap). Any attempt at improving Eq. (5.39) must carefully re-evaluate the contribution of the completely reducible sequences as well as take account of the more complex sequences.

As an example we next write down the contribution to \( \Gamma_s^{(2)} \) from the sequence 121 for the collision history of Figure 2(a),
\[
\frac{(n \nu \sigma)^2}{S} \int_0^\infty \frac{d\tau_1}{\tau_1} \int_0^\infty \frac{d\tau_2}{\tau_2} \int_0^{\infty} \frac{d\tau_3}{\tau_3} \hat{f}(\vec{p}; \omega) \left[ R-1 \right]^2, \tag{5.40}
\]

where there are the two possibilities of a real or hypothetical collision for the recollision. Carrying out the integration we find the contribution,
\[
\frac{(n \nu \sigma)^2}{S} \left[ \frac{1}{2 \tau} - \frac{1 - \frac{\omega}{2 \tau}}{2 \tau} \right] \left[ R-1 \right] \hat{f}(\vec{p}; \omega). \tag{5.41}
\]
Expanding the result in powers of $\sigma/\nu$ we obtain,

$$\left(\frac{\nu u_0}{s}\right)^{1/2} \frac{1}{s} \left[ \frac{1}{2} - \frac{25\pi^2}{3} \frac{s}{3} + \frac{(25\pi^2)^2}{4} \right] \cdots \{r-1\} f(\bar{\rho}, \gamma). \quad (5.42)$$

However if we add all contributions for the 121 sequence we find that all the terms cancel. This comes about because the single-scatterer contribution was only approximately evaluated. There is a contribution from the sequence 12 of the same form as Eq. (5.42) that prevents the cancellation when properly included. This points out a feature of the wind-tree model and all other models, that is identical to the feature used in simplifying the one-dimensional calculations, viz., the contributions from collision histories in which the trajectory passes through a scatterer with which the particle also suffers a real collision are zero (this rule applies to the terms of the evolution operators and must be modified for the irreducible collision operators). Thus the collision history of Figure 2(b) does not contribute to $\Gamma^{(2)}$ because the contribution from sequence 121 and 21 differ only in phase and therefore exactly cancel.

![Diagram](image)

Figure 2. Two-scatterer wind-tree collision histories.
Let us for a moment examine the nature of the contributions from the 12 and the 121 sequences: in the 12 sequence the dominant term varies as,
\[
\left( \frac{n\nu 6}{s^3} \right)^2 \tag{5.43}
\]
and it contains powers of \( s\sigma /\nu \); in the 121 sequence the dominant term varies as,
\[
\left( \frac{n\nu 6}{s^3} \right)^2 \frac{s\sigma}{\nu} \tag{5.44}
\]
and it contains higher powers of \( s\sigma /\nu \). Thus the two scatterer evolution operator is a power series of the form,
\[
\Gamma^{(2)}_S \begin{pmatrix} \vec{p} ; \sigma \end{pmatrix} = \frac{(n\nu 6)}{s^3} \sum_{\ell=0}^{\infty} A^{(2)}_{\ell} \left( \frac{s\sigma}{\nu} \right)^{\ell} f(\vec{p} ; \sigma) \tag{5.45}
\]
where \( A^{(2)}_{\ell} \), the coefficient of the \( \ell \)th power of \( s\sigma /\nu \) is a sum of the operators \( 1, P, R, R' \) with certain numerical coefficients. The calculation of \( A^{(2)}_{0} \) is obtained by considering only the sequence 12; the calculation of \( A^{(2)}_{1} \) requires an analysis of both the 121 and the 12 sequences; the calculation of \( A^{(2)}_{2} \) is obtained from the sequences 12, 121, and 1212, and so forth. Generalizing expression (5.45) it is easily shown that the dominant part of \( \Gamma^{(m)}_S \) varies as \( (n\nu 6)^m / s^{m+1} \) and that it contains powers of \( s\sigma /\nu \) so that its form appears as,
\[
\Gamma^{(m)}_S \begin{pmatrix} \vec{p} ; \sigma \end{pmatrix} = \frac{(n\nu 6)}{s^{m+1}} \sum_{\ell=0}^{\infty} A^{(m)}_{\ell} \left( \frac{s\sigma}{\nu} \right)^{\ell} f(\vec{p} ; \sigma) \tag{5.46}
\]
The complete evolution operator, \( \Gamma_S \), may then be expressed as follows:
\[
\Gamma_S \begin{pmatrix} \vec{p} ; \sigma \end{pmatrix} = \frac{1}{s} f(\vec{p} ; \sigma) + \frac{n\nu 6}{s} A^{(1)} \begin{pmatrix} \vec{p} ; \sigma \end{pmatrix} + \frac{1}{s} \sum_{m=2}^{\infty} \frac{(n\nu 6)^m}{s^m} \sum_{\ell=0}^{\infty} A^{(m)}_{\ell} \left( \frac{s\sigma}{\nu} \right)^{\ell} f(\vec{p} ; \sigma) \tag{5.47}
\]
Thus the cluster series for \( \Gamma_S(\vec{p}) \), expression (5.47), is a power series.
in $\frac{NV_0}{s}$ or $NV_0 t$ in the time domain, and is not a series in powers of the small parameter $no^2$. We may regroup the terms of expansion (5.47), assuming it is a valid operation, and write the sum as a series in powers of the small parameter, $no^2$,

$$\int_s f(x_j0) = \frac{1}{s} \left[ 1 + \frac{NV_0}{s} a^{(1)} + \sum_{k=0}^{\infty} \left( \frac{NV_0}{s} \right)^k \sum_{l=0}^{\infty} a^{(k+l)} \right] f(x_j0).$$  \hspace{1cm} (5.48)

The transition from expression (5.47) to the expansion in powers of $no^2$, Eq. (5.48), has not been justified. It is in the form that one usually seeks in a perturbation treatment of the many-body problem. Yet from an observation of the recently found divergence in the transport coefficients and in the collision integrals one might doubt the existence of the coefficients of the higher powers of $no^2$, i.e., there is no assurance that the coefficients,

$$\sum_{k=2}^{\infty} \left( \frac{NV_0}{s} \right)^k a^{(k)} \sum_{l=0}^{\infty} a^{(l)}$$  \hspace{1cm} (5.49)

exist in the long time limit (the limit as $s \rightarrow 0^+$).

The term-by-term divergence of the cluster series may be dismissed by saying it is not a good physical decomposition of the many-body problem since it does not expand in powers of the small parameter, $no^2$, rather it expands in powers of $NV_0 t$ which may be arbitrarily large. In answer to the question of how one does expand in powers of $no^2$ without recourse to the brute-force rearrangement of Eq. (5.47) and the question
of whether such an expansion exists in the long time limit, we will demonstrate in Chapter VII that the renormalized expansion of Chapter VI is a natural physical expansion since it expands in dressed collision events rather than the bare collision events of the cluster series, that it amounts to an expansion in powers of $n_0^2$, and that it is strictly term-by-term convergent. The latter statement makes no claims on the convergence of the complete renormalized series for arbitrary $n_0^2$; we are guaranteed that the complete evolution operator $\Gamma_s$ is finite in the long time limit and therefore that the infinite sum, (5.47) or (5.48), must exist because $f_s(\vec{p}) = \Gamma_s f(\vec{p}; o)$ exists. However there is no guarantee that the collision operator, $\mathcal{Q}_s$, exists in the limit $s \to o^+$ for arbitrary $n_0^2$.

We consider a brief demonstration of the assumed form of the m-scatterer evolution operator, (5.46), which tells us the order of the divergence of the various m-scatterer collision events. We study the terms of the irreducible collision operator $\mathcal{Q}_s^{(m)}$ which are related to the terms of the evolution operator $\Gamma_s^{(m)}$ by multiplying the irreducible terms of $\Gamma_s^{(m)}$ by $s^2$.

The m-scatterer irreducible collision operator, $\mathcal{Q}_s^{(m)}$, contains an infinite series of sequences of increasing complexity. The complexity of a collision event refers to the number of recollisions or in the language of sequences, the number of repeated labels in a sequence. In Chapter VI a classification scheme is given whereby the sequences are catalogued according to their k-number which is the difference between the total number of labels and the number of distinct labels in a sequence. The k-
number serves to denumerate the number of repeats of all the labels of a given sequence or to measure the complexity of a collision event, since it is the sum of the number of times each label is repeated. For example, for the sequence, 12312113, the k-number is 8-3=5.

Using the formula of Chapter IV for the m-scatterer collision integral, Eq. (4.83), we examine the order of the divergence of the terms with a given k-number. We demonstrate that our interpretation of the k-number is correct, i.e., the least complex collision event is the most divergent term and has k-number equal to unity. If we imagine the collision histories for m-scatterers for the unit k-number sequences, we observe that the trajectory begins on some scatterer, suffers successive encounters (one each) with the other m-1 scatterers, and then recollides with the first scatterer. Consider the integration domain \((\tau_\ell, b_\ell)\) of the last distinct scatterer representing a real collision, encountered before the recollision. With the other integration variables fixed, the position of the scatterer with which the particle recollides is fixed, and the ray making up the possible collision loci of the last real collision prior to the recollision is fixed. Since real collisions rotate the particle by \(\pm \pi/2\), the domains of \(\tau_\ell\) and \(b_\ell\) are independent. The domain of \(\tau_\ell\) along the fixed ray is \(\tau_\beta\) to \(\tau_\beta + \frac{2\sigma}{V}\) so that the trajectory aims to recollide upon rotation by \(\pm \pi/2\). The domain of \(b_\ell\) ranges over \(0\) to \(\sigma\). Thus neglecting the dependence of the integrand on \(\tau_\ell\) and \(b_\ell\), the integral over \(\tau_\ell\) and \(b_\ell\) yields \(\sigma \frac{2\sigma}{V}\). Thus the integral over the time preceding the collision that rotates the momentum so that the particle aims to
recoUide does not yield a factor \( \frac{1}{s} \). The integral over each of the remaining m-2 times, \( \tau_i \), yields \( \frac{1}{s} \) since the domain of each is not restricted to a range of order \( \sigma/v \). Hence the dominant part of the unit k-number part of the m-scatterer collision operator, \( \Omega^{(m)}_s \), varies as,

\[
(n\nu\sigma)^m \frac{1}{s^{m-2}} \frac{1}{s^j} \tag{5.50}
\]

and it contains less divergent parts in the form of a power series in \( \frac{\sigma}{v} \).

We may apply the same argument to the terms of \( \Omega^{(m)}_s \) with arbitrary k-number by examining those times the particle travels prior to the last real collision with a starred scatterer before a recollision. For each of these times the integral contributes in lowest order the factor \( \sigma/v \), and since there are k such times there is a net factor of \( (\sigma/v)^k \). The remaining m-1-k integration times contribute the factor \( (1/s)^{m-1-k} \) since each has a macroscopic range. Therefore the most dominant part of the m-scatterer collision integral with k recollisions varies as,

\[
(n\nu\sigma)^m \frac{(\sigma/v)^k}{s^{m-1-k}} = (n\nu\sigma)^{m-k} \frac{(\sigma/v)^k}{s^{m-1-k}} \tag{5.51}
\]

The same term has less divergent contributions which contains higher powers of \( s\sigma/v \).

Therefore the m-scatterer irreducible collision operator for the wind-tree model may be written as a sum over contributions having k-number from 1 to \( \infty \),

\[
\Omega^{(m)}_s = \sum_{k=1}^{\infty} (n\nu\sigma)^{m-k} \left( \frac{\sigma}{v} \right)^k I^{(m),k} \left( \frac{s\sigma}{v} \right) \tag{5.52}
\]
where $I^{(m),k}$ is an operator which may be written as a power series in $\alpha/v$,

$$I^{(m),k} \left( \frac{\alpha}{v} \right) = \sum_{x=0}^{\infty} \left( \frac{\alpha}{v} \right)^x J^{(m),k}_x,$$

(5.53)

where $J^{(m),k}_x$ is a linear combination of $l, P, R, RP$. This result confirms the assumed form of Eq. (5.46).*

E. HARD DISKS MODEL

The hard disks model consists of a large number of two-dimensional hard disk scatterers of diameter $2\sigma$ and a point particle. The single-scatterer evolution operator for a spatially uniform particle distribution is given by the following,

$$\Gamma^{(1)} f(\bar{r};0) = \frac{\mu v}{\pi} \int_0^{\infty} ds \int d\beta \epsilon^{-5/2} \left[ \mathcal{R}(b) - 1 \right] f(\bar{r};0),$$

(5.54)

where we recall that the rotation operator is given by,

$$\mathcal{R}(b) \bar{r} = \bar{r} - 2 \bar{r} \frac{\bar{r} \cdot \hat{b}}{b^2} =$$

$$= -\cos 2\theta \hat{b} + \sin 2\theta \hat{b} \cdot \hat{b},$$

(5.55)

where $b = \sigma \sin \theta_1$, and $\hat{b}$ is a unit vector in the direction of the angular momentum. Performing the integral over $\tau_1$ in Eq. (5.11) we find,

$$\Gamma^{(1)} f(\bar{r};0) = \frac{\mu v}{\pi} \int_0^{\infty} d\theta \cos \theta \left[ \mathcal{R}(b) - 1 \right] f(\bar{r};0),$$

(5.56)

The single-particle collision integral $\mathcal{K}^{(1)}_{s}$ is, according to Chapter IV, given by,

*The rules giving rise to Eq. (5.52) breakdown for certain collision sequences, therefore $k$ is not strictly the number of recollisions, see Appendix G.
\begin{align}
K_s^{(1)} \ f_5(\vec{\rho}) &= n\nu \int_{-\pi}^{+\pi} d\theta \left[ R(\theta) - 1 \right] f_5(\vec{\rho}) \\
&= n\nu \int_{-\pi}^{+\pi} d\theta \cos \theta \left[ R(\theta) - 1 \right] f_5(\vec{\rho}). \tag{5.57}
\end{align}

In the remainder of this section we shall demonstrate the types of divergent terms that appear in the collision integrals, \( H_s^{(m)} \). We write down the first logarithmically divergent term in the collision integral, \( H_s^{(2)} \). This term corresponds to the simplest irreducible sequence, 121.

Using the prescription of Chapter IV, the integral may be written in the form,

\begin{equation}
(n\nu)^2 \int d\epsilon_1 \int d\epsilon_2 \int d\epsilon_3 \left[ R(\epsilon_1) - 1 \right] \left[ R(\epsilon_2) - 1 \right] \left[ R(\epsilon_3) - 1 \right] \frac{1}{\epsilon_1^2 \epsilon_2^2} f_5(\vec{\rho}). \tag{5.58}
\end{equation}

where we have included all selections of real and hypothetical collisions.

The dependence of \( \beta_3 \) and \( \tau_{\text{coll}} \) on the 3 independent variables may be found either by going back a step to the form of the integrals containing the conditions \( \left( \frac{1}{k}, \frac{1}{l} \right) \) or from an analysis of the geometry of the collision sequence, and the same applies to the determination of the integration domain. The collision history for the simplest selection, hhrh, is shown in Figure 3. This collision integral (5.58) cannot be evaluated exactly, although it can be written in detail. However we are primarily interested in the divergent character of this integral, particularly in how the logarithmic divergence arises. Therefore we shall only compute the dominant part of Eq. (5.58), coming from large values of \( \tau_2 \), the time the particle travels freely before encountering scatterer 2. The integration is carried out over the domain, \( T < \tau_2 < \infty \), where \( T \) is some finite time.
Figure 3. Collision history of first logarithmic divergence for hard disks.

longer than $\sigma/v, T > \sigma/v$. The dependent impact parameter, $b_3$, is approximately given by the relation,

$$\frac{b_1 + b_3}{v^2 \tau} = 2 \theta_2,$$

(5.59)

since $2\theta_2 < 1$. It is convenient to introduce a change of variables from $b_1, b_2, \tau$ to $b_1, b_3, \tau$. The Jacobian of this transform is, $\sigma/2v\tau_2$.

Thus the dominant part of (5.58) may be written in the form,

$$(\Pi v)^2 \int_0^{\infty} \frac{1}{2 \nu \tau^2} \int db_1 \int db_3 \left[ \mathcal{R}(b_1) - 1 \right] \mathcal{R}(b_3) - 1 \right] \cdot \varepsilon^{-25 \tau} \int \mathcal{R}(\bar{\rho})$$

(5.60)

where the collision time is approximately $2\tau_2$, and $\mathcal{R}(b_2) = \mathcal{R}(0)$ since
\[ \theta_2 < 1, \text{ i.e., the collision with scatterer 2 rotates the trajectory by } \pi. \]

Clearly the domain of \( b_1 \) is \(-\sigma\) to \(+\sigma\) since \( b_1 \) merely fixes the location of \( x_1 \) with respect to the initial particle momentum \( \vec{p} \), and the domain of \( b_3 \) runs from \(-\sigma\) to \(+\sigma\) since this is equivalent to sweeping \( \theta_2 \) over its possible values, i.e., over all angles that subtend scatterer 1. We may now carry out the integral over \( \tau_2 \), giving,

\[ \frac{(-\tau)^2}{2 \sigma^2} E_x \left( -25\tau^2 \right) \int_{-\sigma}^{+\sigma} \int_{-\sigma}^{+\sigma} \left[ \mathcal{R}(b_1) - 1 \right] \mathcal{R}(b_2) \mathcal{R}(b_3) - 1 \mathcal{R}(\vec{\vec{p}}) \right] \]

where \( E_x \) is the logarithmic integral which has the following expansion for small \( x \),

\[ E_x (-x) = \ln |x| - x + \frac{x^2}{2 \cdot 2!} - \frac{x^3}{3 \cdot 3!} + \cdots \]

where \( \gamma \) is Euler's constant. Thus in the limit of small \( s \) the irreducible collision integral for the the 121 sequence has a logarithmic divergence which appears, because the range of the angle that subtends the recollision is of the order \( \sigma \sqrt{\tau} \), and this term appears in the integral of \( \tau_2 \). The divergence does not appear for those parts of the integral containing higher powers of \( \frac{s\sigma}{\sqrt{\tau}} \); for integrals that contain powers of \( \tau_2 \) in addition to the \( \frac{1}{\tau_2} \) contribution, the logarithmic divergence is replaced by a power divergence.

Consider the generalization of the result for the 121 sequence of the collision operator, \( \mathcal{K}^{(2)}_s \), to an arbitrary sequence in \( \mathcal{K}^{(m)}_s \). Each recollision places a restriction on the integration domain of one of the independent impact parameters reducing it from \( \sigma \) to \( \sigma \sqrt{\tau} \), where \( \sqrt{\tau} > \sigma \).
and $\tau$ is one of the times the particle travels freely between collisions (it may or may not be one of the independent times). Contrast this behavior with that of the wind-tree model wherein each recollision reduces the integration domain by the factor $s\sigma/v$. The reduction in both cases involves the ratio $\sigma/v$, but this is multiplied by $s$ for the wind-tree model and by $1/\tau$ for the hard disks model. Thus the latter model will have logarithmic terms whenever the integrand $d\tau/\tau$ occurs (no higher or lower powers of $\tau$ appear), and the former model can never have logarithmic terms.

Thus for the hard disks model, the sequence with $k$-number equal to $k$ has a factor,

\[
\frac{k}{\tau_1^{\frac{k}{\tau}}} = \frac{6}{\tau_1^{\frac{k}{\tau}}}
\]  

(5.63)

in the integrand where all, some, or none of the $\tau_{j_1}$ are equal, and the $\tau_{j_1}$ may or may not be the independent times, $\tau_i$. If there are only $k$ independent times $\tau_i$ which is the case when $m-l=k$, then the behavior of the divergence can immediately be obtained from an examination of (5.63).

In this case the integral over the $\tau_i$ yields a factor of the order of,

\[
(k \ln s\tau)^{\ell} = \frac{1}{\tau} k^{k-1} \quad \ell < k-1
\]

(5.64)

\[
(k \ln s\tau)^{k} = \ell = k
\]

where $\ell$ is the number of times in (5.63) that appear singly. Thus all powers of the logarithm occur. For example, the sequence 12131 contributes $(\ln s\tau)^2$, the sequence 1213141 contributes $(\ln s\tau)^3$, etc. It should not
be considered unusual when $ln sT$ or powers of it appear in the terms of the $\Psi_s^{(m)}$ since it comes from integrands containing a given time, $\tau$, to the minus one power.

When the number of integration times, $m-1$, is greater than $k$, then the integrand contains powers of the $\tau_1$ multiplying expression \( (5.63) \). In this case each of the integration times $\tau_1$ may appear in the integrand with a positive or zero exponent, with the exponent $-1$, or with exponent less than $-1$. For the first possibility there appears a power divergence ($1/s$ raised to the exponent plus one), for the second there appears a logarithmic divergence, $ln sT$, and for the third there is no divergence. The net term which is the product of these three possibilities thus contains in general a power divergence multiplied by a power of the logarithmic divergence.

It is more difficult to write a formula giving the general behavior of the collision integrals in terms of just the $k$-number and the number of scatterer labels, $m$, for the hard disks model, than for the wind-tree model. However using the above arguments one can write down the divergent behavior for any given collision history.

F. HARD SPHERES MODEL

The hard spheres model consists of a large number of hard sphere scatterers and a point particle. This model is obtained from the hard disks model by allowing the collision plane to vary, i.e. we add the integral over azimuthal angles $\delta \phi$. From the formulas of Chapter IV, the
single-particle collision integral is as follows,

\[ \mathcal{R}_s^{(1)} f_s(\vec{p}) = \nu \sigma \int d\theta d\phi \int_{R_0^2} \frac{\partial}{\partial \rho_1} f_s(\vec{p}) \]

(5.65)

where the rotation operator, \( R_\phi \), is given by Eq. (5.55) with the orientation of \( \hat{r} \) determined by the azimuthal angle \( \phi \).

A discussion of the divergent behavior of the dominant terms of \( \mathcal{M}_S^{(m)} \) follows from the hard disks discussion by replacing each factor \( \sigma/\nu \tau \), obtained from the integral over angles subtending a recollision in the hard disks model by the factor \( (\sigma/\nu \tau)^2 \), obtained from the integral over the solid angle subtending a recollision in the hard spheres model. It follows at once that the hard spheres model contains terms with power divergence, logarithmic divergence, and in general products of powers of both. Using the discussion of Section 5 we can specify the nature of the divergence of any given collision history in the hard spheres model.

We discuss the terms of the two-scatterer collision integral. There is only one integration time \( \tau_2 \) and no powers of \( \tau_2 \) occur in the integrand since \( m-1=1 \) cannot be greater than the k-number. Thus we immediately find that \( \mathcal{M}_S^{(2)} \) contains no power divergence and can contain no logarithmic divergence since only even powers of \( \frac{1}{\tau} \) occur.

Considering the three-scatterer collision integral, \( \mathcal{M}_m^{(3)} \), we see that \( m-1=2 \) is greater than the k-number for \( k=1 \); therefore the integrand contains a factor \( \tau \) from the integral over one of the times \( \tau_1 \), and the factor \( 1/\tau^2 \) from the integral over the solid angle subtending the recollision. Thus the unit k-number term is logarithmic divergent. The terms
of $\gamma_s^{(3)}$ for k-number greater than unity all converge because no powers of $\tau$ appear, $m-1 > k$, and each recollision adds the factor $\frac{1}{\tau^2}$.

Examining the four-scatterer collision integrals, $\gamma_s^{(4)}$, we find that $m-1=2$ for the unit k-number contribution, and thus it diverges linearly since $\tau_2$ appears in the numerator canceling the $1/\tau^2$ factor coming from the integral over the solid angle subtending the recollision. The term for k-number of 2 contains a logarithmic divergence since the numerator of the integrand contains $\tau, m-1-k=1$, which cancels with one of the $\tau_1$ of the denominator. The higher k-number contributions to $\gamma_s^{(4)}$ converge.

The five-scatterer collision integral diverges as $1/s^2$ for the k=1 term; as $1/s$ (linearly) and as $(\ln Ts)^2$ for the k=2 term; and so forth.
VI. THE RENORMALIZATION PROCEDURE

A. THE RING DIAGRAM TECHNIQUE

The first attempt at removing the divergence from the nonequilibrium gas theory was made by Kawasaki and Oppenheim\textsuperscript{27} using the ring diagram technique of Montroll and Mayer\textsuperscript{28}. Other authors using the same technique include van Leeuwen and Weijland\textsuperscript{30} who applied it to transport coefficient calculations for the classical Lorentz gas and Fujita\textsuperscript{31} who applied it to the conductivity coefficient of an electron impurity system.

Montroll and Mayer first developed the ring diagram technique to simplify the enormous task of computing the higher virial coefficients of imperfect gases. Their basic idea was: to extract the simplest parts (of the integrand) from each virial coefficient; to simplify their form (using a Fourier representation); to sum these simplest parts into closed form; to compute approximately these closed form expressions; and then to repeat this procedure for the next simplest parts. Thus the first ring diagram expansion was introduced to simplify and organize the computation of an infinite series of complicated integrals. The second application of the ring diagram technique was introduced by Mayer\textsuperscript{44} in an effort to eliminate the divergence arising from the long range coulomb potential in ionic solutions. Thus what was first introduced as a device for simplifying the calculation of complicated but convergent manyfold integrals is now applied to the renormalization of divergent manyfold integrals.

Later Gell-Mann and Brueckner\textsuperscript{45} applied the ring diagram method to
the calculation of the correlation energy of an electron gas at high den...
terms, this is done by going to the Fourier representation or an equivalent representation (it may be necessary to simplify the terms before classifying them); (3) sum all terms of the infinite series having a given classification (topology or combination of indices) into a simple or closed form; and (4) approximately compute the closed form or renormalized expressions. The functional behavior of the divergence and of the renormalized expressions depends on both the specific nature of the problem (just what we are computing, what the form of the integrand is), and on the topology of the various terms. In a complete theory one should be able to verify that the first order ring diagram term is the most dominant one and that the higher terms form a convergent series which may be truncated in some order of approximation. In the next section we demonstrate the application of the ring diagram technique to the classical Lorentz gas.

B. THE RING DIAGRAM TECHNIQUE APPLIED TO THE LORENTZ GAS WITHOUT CORRELATIONS

The basic idea of the ring diagram technique is to classify terms in the virial expansions of the evolution operator, $\Gamma_s$, or the collision operator, $\mathcal{H}_s$, and to collect together all terms having the same classification. Since the terms for the Lorentz gas are most easily represented as sequences of labels (from the binary collision expansion), we shall classify the sequences without recourse to diagrams. If we examine any sequence of labels from the binary collision expansion (which contains all sequences of labels excluding those with identical adjacent labels),
e.g., 1213516427, we notice that we may distinguish two types of labels: those that occur only once in a sequence called free labels, and those that occur more than once called bound labels (repeated labels). The classification scheme is via the combinations of bound labels disregarding the number of free labels. The general ring diagram sum therefore is the sum over all free labels for a given combination of bound labels. For example, for the combination of bound labels, 11, the first order ring diagram sum is over the sequences 121, 1231, 12341, ....

We may formally sum the terms of a given classification before obtaining a simple representation for them. We first consider the resummation without correlations for the terms of \( \rho_S^{(m)} \), the m-scatterer irreducible collision operator. Recall that in this case the operator is given as,

\[
\rho_S^{(m)} = \frac{n^m}{m!} \int \cdots \int \frac{d^3 \vec{S}_1 \cdots d^3 \vec{S}_m}{s_{12} \cdots s_{m}} C(12 \cdots m) \ S_{S}^{-1} \tag{6.1}
\]

where \( C(12 \cdots m) \) is the sum of all irreducible sequences of the m labels 12...m excluding terms with identical adjacent labels. Suppose that we have selected a particular combination of bound labels. If we integrate over the scatterer coordinates of the free labels we may replace the free binary operators by,

\[
\Lambda = n \int d^3 \vec{S}_i \ B \tag{6.2}
\]

We also relabel all sequences so that the indices first appear from left to right in the order 1,2,3,...,m thereby removing the factor 1/m! of Eq. (6.1).
If we now sum over all terms of $\gamma$ with a given combination of bound indices, that is over all possibilities of placing free labels between the bound labels, then we obtain the sum $\sum_{r=0}^{\infty} \Lambda^r$ between every pair of distinct bound labels and the sum $\sum_{r=1}^{\infty} \Lambda^r$ between every pair of identical bound labels. The coefficient is unity in both sums since the binary collision expansion, Eq. (2.28), contains only unit coefficients (each sequence occurs once and only once). Thus the ring diagram sum gives the factors: 

$$(1-\Lambda)^{-1}$$ between every pair of distinct bound labels, and $\Lambda(1-\Lambda)^{-1}$ between every pair of identical bound labels.

We may catalog the combinations of bound indices in the following manner which according to Chapter V orders them in a series of decreasing contributions. Our way of cataloging the terms is to specify the $k$-number which is the difference between the total number of bound labels $l$, and the number of distinct bound labels, $i$,

$$k = l - i.$$  

(6.3)

We may then construct a table of all combinations for a given $k$-number.

**k-NUMBER CLASSIFICATION SCHEME**

<table>
<thead>
<tr>
<th>$k$</th>
<th>combinations of bound labels</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>111, 1212, 1221</td>
</tr>
<tr>
<td>3</td>
<td>1111, 11212, 12212, 12112, 11222, 11221, 12221, 12211, 12121, 12132, 12132, 12312, 12312, 122313, 123213, 123213, 123213, 123312, 123231, 123321, 123231</td>
</tr>
</tbody>
</table>
We may write the complete irreducible collision operator, \( \mathcal{Q}_s \), as a sum over the renormalized terms as follows

\[
\mathcal{Q}_s = \Lambda \mathcal{S}_s^{-1} + \sum_{m=1}^{\infty} \mathcal{K}_s^{(m)} \mathcal{S}_s^{-1},
\]

where \( \mathcal{K}_s^{(m)} \) denotes the sum of all renormalized sequences with \( m \) distinct labels,

\[
\mathcal{K}_s^{(1)} = \hbar \left[ \left( d \frac{d}{d \vec{p}} \sum_{\text{perm}} B(1)(1-A)^{-1} B(1)(1-A)^{-1} B(1)(1-A)^{-1} \right) \cdot B(1) + \cdots \right],
\]

\[
\mathcal{K}_s^{(2)} = \frac{\hbar^2}{2!} \left[ \left( d \frac{d}{d \vec{p}} \sum_{\text{perm}} B(1)(1-A)^{-1} B(2)(1-A)^{-1} B(2)(1-A)^{-1} \right) \cdot B(1) + \cdots \right]
\]

\[
+ \frac{\hbar^2}{2!} \left[ \left( d \frac{d}{d \vec{p}} \sum_{\text{perm}} B(1)(1-A)^{-1} B(2)(1-A)^{-1} B(3)(1-A)^{-1} \right) \cdot B(3) + \cdots \right],
\]

\[
\mathcal{K}_s^{(3)} = \frac{\hbar^3}{3!} \left[ \left( d \frac{d}{d \vec{p}} \sum_{\text{perm}} B(1)(1-A)^{-1} B(2)(1-A)^{-1} B(3)(1-A)^{-1} \right) \cdot B(3) + \cdots \right].
\]

where \( \sum_{\text{perm}} \) is the sum over all symmetrizing permutations of the labels. The general term is the sum over all irreducible sequences of labels such that each label occurs at least twice with the factor \( (1-A)^{-1} \) between distinct labels and the factor \( \Lambda(1-A)^{-1} \) between identical labels. We note that the first ring diagram term, the one considered by van Leeuwen-Weijland\(^30\) and Kawasaki-Oppenheim\(^27\), is the first term on the right-hand side of Eq. (6.5a).
Next we apply the ring diagram technique to the complete evolution operator for randomly distributed scatterers,

$$\Pi_s = S_s^{-1} \left[ 1 + \sum_{m=1}^{\infty} \frac{\Lambda^m}{m!} \int \cdots \int d^{2m} R_{12} \cdots R_{m} \right]^{-1},$$  \hspace{1cm} (6.6)$$

where $R_{12} \cdots m$ is the sum over all sequences such that each sequence contains the $m$ labels $12 \cdots m$ with no label occurring twice in succession.

We may obtain the renormalized form of Eq. (6.6) from that of the collision operator using the relation between the operators,

$$\Pi_s = \left( S_s^{-1} - \Lambda S_s \right)^{-1}. \hspace{1cm} (6.7)$$

Insertion of Eq. (6.4) into Eq. (6.7) yields,

$$\Pi_s = \left( S_s^{-1} - \Lambda S_s^{-1} - \sum_{m=1}^{\infty} \frac{\Lambda^m}{m!} S_s^{-1} \right)^{-1}$$

$$= S_s (1 - \Lambda)^{-1} \left( 1 - \sum_{m=1}^{\infty} \frac{\Lambda^m}{m!} S_s^{-1} (1 - \Lambda)^{-1} \right)^{-1} \hspace{1cm} (6.8)$$

And expanding this result out we find,

$$\Pi_s = S_s \left[ (1 - \Lambda)^{-1} + (1 - \Lambda)^{-1} \sum_{i=1}^{\infty} \frac{\Lambda^m}{m!} (1 - \Lambda)^{-1} \right.$$  

$$+ \left. (1 - \Lambda) \sum_{i=1}^{\infty} \frac{\Lambda^m}{m!} (1 - \Lambda)^{-1} \right] \hspace{1cm} (6.9)$$

This result could also be found directly by summing over all free labels and using the same rule of placing the factor $(1 - \Lambda)^{-1}$ between distinct bound labels and the factor $\Lambda(1 - \Lambda)^{-1}$ between identical bound labels, adding the rule of multiplying on the left and right of each renormalized operator by $(1 - \Lambda)^{-1}$, since the terms are in general reducible and may
contain any number of free labels on the left and right. The first term of Eq. (6.9) comes from the sum over all terms containing no bound labels.

We may introduce a notation for the renormalized terms of $\Gamma_s$,

$$\Gamma_s = \sum_s \left[ (1-\Lambda)^{-1} + (1-\Lambda)^{-1} \sum \gamma_s^{(m)} (1-\Lambda)^{-1} \right]$$

(6.10)

where,

$$\gamma_s^{(1)} = \Gamma_s^{(1)}$$

(6.11a)

$$\gamma_s^{(2)} = \Gamma_s^{(2)} + \Gamma_s^{(1)} (1-\Lambda)^{-1} \Gamma_s^{(1)}$$

(6.11b)

$$\gamma_s^{(3)} = \Gamma_s^{(3)} + \Gamma_s^{(1)} (1-\Lambda)^{-1} \Gamma_s^{(1)} + \Gamma_s^{(1)} (1-\Lambda)^{-1} \Gamma_s^{(1)}$$

$$+ \Gamma_s^{(1)} (1-\Lambda)^{-1} \Gamma_s^{(1)} (1-\Lambda)^{-1} \Gamma_s^{(1)}$$

(6.11c)

where the general term is a Husimi-like sum of products of the $K_s^{(m)}$. We may also write $\gamma_s^{(m)}$ as,

$$\gamma_s^{(m)} = \frac{1}{m!} \int \cdots \int d\bar{s}_2 \cdots d\bar{s}_m \gamma_s(12 \cdots m)$$

(6.12)

where

$$\gamma_s(12 \cdots m) = \text{sum of all sequences of labels such that each label occurs at least twice, with the factor } (1-\Lambda)^{-1} \text{ inserted between } \Lambda(1-\Lambda)^{-1} \text{ inserted between identical labels.}$$

(6.13)

C. THE RING DIAGRAM TECHNIQUE WITH CORRELATIONS

We next consider the sum over free labels when the scatterer correlations are included. Free labels now means only labels that are argu-
ments of the single scatterer correlation function \( W(i) = P^{(1)}(i) \) and which are free in the previous sense of not occurring more than once in a sequence of binary operators. We shall now denote by \( \Lambda \) the integral,

\[
\Lambda = \int d^3 \vec{r} \, P^{(1)}(\vec{r}) \left( i \right) \ . 
\]

(6.14)

Since we sum over those labels that occur only once in a sequence and which only appear in the single scatterer distribution all other labels are considered bound and are included in the classification scheme. Following the same procedure as the above, the renormalized terms are obtained by removing all free labels and inserting the factor \((1-\Lambda)^{-1}\) between distinct bound labels and the factor \(\Lambda(1-\Lambda)^{-1}\) between identical bound labels.

We may classify the sequences with correlations in the same manner as the classification without correlations since the effect of the correlations is equivalent to making the correlated indices identical, thereby transforming them into repeated or bound indices. We write the first ring diagram term (for the classification 11) as follows,

\[
\int d^3 \vec{r} \, P^{(1)}(\vec{r}) B(1) \Lambda(1-\Lambda)^{-1} B(1) + \\
+ \frac{1}{2!} \int \int d^3 \vec{r} d^3 \vec{r}' \sum_{\text{perm}} W(12) B(1) (1-\Lambda)^{-1} B(2) \ , 
\]

(6.15)

and the classification 1212 includes the terms,
\[
\frac{1}{2!} \sum_{\text{perm}} B_{(1)}(1) (1) B_{(2)}(1) (1) B_{(3)}(1) (1) B_{(4)}(1) + \\
+ \frac{1}{3!} \sum_{\text{perm}} \left[ W_{(1)} W_{(2)} B_{(1)}(1) (1) B_{(2)}(1) (1) B_{(3)}(1) (1) B_{(4)}(1) \right] + \\
+ \frac{1}{4!} \sum_{\text{perm}} \left[ W_{(1)} W_{(2)} W_{(3)} B_{(1)}(1) (1) B_{(2)}(1) (1) B_{(3)}(1) (1) B_{(4)}(1) \right] 
\]

where the sum on permutations is the sum over the symmetrizing permutations. The renormalized expressions for \(\Gamma_s^{(m)}\) and \(\Gamma_s\) will have the same form as in Eqs. (6.4) and (6.9) except that \(\Gamma_s^{(m)}\) is generalized in the manner of Eqs. (6.15) and (6.16).

D. THE RENORMIALIZED COLLISION INTEGRALS

We demonstrate how the evolution operator, \(\Gamma_s^{(m)}\), and the collision operator, \(\Omega_s^{(m)}\), can be written in terms of dressed collision events such that each collision sequence gives a finite contribution in the long time limit. The transition from bare collision events (events that do not account for the presence of the other scatterers) to the dressed collision events (events that properly take account of the influence of the other scatterers, i.e., the fact that the frequency of a collision event with \(m\)-scatterers must diminish as the spatial dimensions of the event increases) is obtained directly from the factor \((1-\Lambda)^{-1}\).

We insert the explicit form of the binary collision operator, \(B(i)\), in \(\Lambda\), giving,
where the contribution to the binary collision operator from the third

term of Eq. (4.57) vanishes because of the exclusion of scatterer-
scatterer and particle-scatterer overlap. The prime on \( \vec{v} \) and \( \vec{p} \) denotes the restituting values obtained by the rotation operator \( R^\rho_\rho \) of Eq. (4.15),

\[ \vec{p}' = R^\rho_\rho \vec{p}. \]

Carrying out the integral over \( s_1 \) we find,

\[
\Lambda F(\vec{r}, \vec{p}) = \frac{1}{2} d\vec{\delta} \int d\vec{\delta} \left\{ \rho^{(1)}(\vec{r}, \vec{p}) \, R^\rho_\rho - \rho^{(1)}(\vec{r} + \vec{p}) \right\} \cdot \left( s + \vec{v} \cdot \frac{\partial}{\partial \vec{p}} \right)^{-1} F(\vec{r}, \vec{p})
\]

\[ (6.18) \]

We may now treat \( \Lambda \), acting on any function of \( \vec{r} \) and \( \vec{p} \), as the operator on the right-hand side of Eq. (6.18) that acts on \( F(\vec{r}, \vec{p}) \). Inserting this form of the \( \Lambda \) operator in \((1 - \Lambda)^{-1}\), we obtain the following,

\[
(1 - \Lambda)^{-1} = \left( 1 - \frac{1}{2} \int d\vec{\delta} \left\{ \rho^{(1)}(\vec{r}, \vec{p}) \, R^\rho_\rho - \rho^{(1)}(\vec{r} + \vec{p}) \right\} \right)^{-1} S_5
\]

\[ = S_5^{-1} \left( S_5^{-1} - \int d\vec{\delta} \left\{ \rho^{(1)}(\vec{r}, \vec{p}) \, R^\rho_\rho - \rho^{(1)}(\vec{r} + \vec{p}) \right\} \right)^{-1} \]

\[ (6.19) \]

where \( S_5 \) is the free streaming operator, \((s + \vec{v} \cdot \frac{\partial}{\partial \vec{p}})^{-1}\).

Next we collect the inverse of the free streaming operator and the hypothetical collision part of the binary collision operator into the
dressed free streaming operator, $S_p$,
\[
S_p = \left( \rho + \vec{v} \cdot \frac{2}{\partial r} \rho^{(1)}(\vec{r}+\vec{\rho}) \right)^{-1},
\]
(6.20)

where the strictly nonzero parameter $p$ is given as,
\[
p = s + \int \int d\vec{\rho} \rho^{(1)}(\vec{r}+\vec{\rho}) \rho^{(1)}(\vec{r}+\vec{\rho})
\]
(6.21)

Hence Eq. (6.19) may be written in dressed form,
\[
\left( -\Lambda \right)^{-1} = S_s^{-1} \left( S_p^{-1} - \int \int d\vec{\rho} \rho^{(1)}(\vec{r}+\vec{\rho}) \rho^{(1)}(\vec{r}+\vec{\rho}) S_p \right)^{-1}
\]
\[
= S_s^{-1} S_p \left( 1 - \int \int d\vec{\rho} \rho^{(1)}(\vec{r}+\vec{\rho}) \rho^{(1)}(\vec{r}+\vec{\rho}) S_p \right)^{-1},
\]
(6.22)

where the transition from $s$ to $p$ in the free streaming operator is obtained by dropping the hypothetical collision from $\Lambda$. We note that the positive definite part of $p$ is the inverse of the mean free time, $T$, of the particle in the neighborhood of the position $\vec{r}$,
\[
\frac{1}{T'} = \int \int d\vec{\rho} \rho^{(1)}(\vec{r}+\vec{\rho})
\]
(6.23)

We obtain a physical picture of the renormalization, or the transition to the dressed form of Eq. (6.22), by noting that the long time divergence, the limit as $s \to 0^+$, of the various collision integrals comes from letting the particle travel freely between encounters for a long time; in the dressed form the bare, free streaming operator $S_s$ is replaced by the renormalized free streaming operator (for almost all travels between encounters) $S_p$, and in the long time limit, $p \to 1/T$, so that the particle
is effectively not allowed to travel a greater distance than the mean free path, $\nu T$, between successive encounters. The particle may travel a very large distance but the collision integrals are damped exponentially with damping constant, $1/\nu T$, the inverse mean free path.

We may also understand the loss of certain hypothetical collisions by noting that in the bare form the series of collision events that successively add on more and more hypothetical collisions differ in magnitude but not in the angles through which the trajectory scatters. Such successive series form a divergent series (higher powers of the integration times, $\tau_i$, are brought into the integrand) in the long time limit. Summing this series of hypothetical collisions gives a single convergent collision event. In contrast, the sum of successive real collision events (none of which are bound) does not reduce to a single equivalent collision event as witnessed by the presence of the real collision term in Eq. (6.22).

Let us now consider the effect of inserting the renormalized form of $(1-\Lambda)^{-1}$ back into the expressions for $f_s^{(m)}$ and $h_s^{(m)}$. First we consider the factor $(1-\Lambda)^{-1}$ placed between two distinct bound labels,

$$\mathcal{B}(i) (1-\Lambda)^{-1} \mathcal{B}(j) .$$

(6.24)

The renormalized form according to Eq. (6.22) is,

$$\mathcal{B}_p(i) \left(1 - \frac{1}{\Lambda_p^{NH}}\right)^{-1} \mathcal{B}(j) ,$$

(6.25)

where $\mathcal{B}_p(i)$ is the ordinary binary collision operator with $s$ replaced by $p$, and $\Lambda_p^{NH}$ is the integral over the scatterer positions of the real col-
collision operator,

\[
\Lambda_{p}^{nh} \mathcal{F}(\vec{r}, \vec{p}) = \int d\vec{r}' P^{(1)}(\vec{x}, \vec{r}) \nabla \nabla \cdot \vec{\Sigma}(\vec{r} - (\vec{r} + \vec{p})) \cdot \left( \vec{p} + \vec{u}', \frac{\partial}{\partial (\vec{r} + \vec{p})} \right)^{-1} \mathcal{F}(\vec{r} + \vec{p}, \vec{p}') \right). \tag{6.26}
\]

If we expand expression (6.25) in powers of \( \Lambda_{p}^{nh} \), eventually integrating only over the dependent collision variables, and transforming to the form of Eqs. (4.81) through (4.85), we obtain the following rules for writing \( \Gamma_{s}^{(m)} \) and \( \mathcal{H}_{s}^{(m)} \) in renormalized form: (1) between successive distinct bound encounters there are no hypothetical collisions but only real dressed collisions (dressed meaning \( s \) is replaced by \( p \)); and (2) each bound encounter followed by a distinct bound encounter has dressed real and hypothetical collisions.

Next we consider the factor \( \Lambda(1-\Lambda)^{-1} \) placed between two identical bound labels,

\[
B(\iota) \Lambda(1-\Lambda)^{-1} B(\iota'). \tag{6.27}
\]

The renormalized form of (6.27) is given as,

\[
B(\iota) \Lambda_{p} (1 - \Lambda_{p}^{nh})^{-1} B(\iota'). \tag{6.28}
\]

This gives us the following set of rules: (1) each bound operator which is followed by the same bound operator is bare (\( s \) is not replaced by \( p \)); and (2) between each pair of like bound binary collision operators, the left-most free encounter may be a dressed, real or hypothetical collision.
and the remaining free encounters are dressed, real collisions.

In the collision integral, \( \gamma_s^{(m)} \), the right-most binary collision operator which is a bound operator is acted on to the right only by \( S_s^{-1} \), therefore it is unchanged by the renormalization. In the evolution operator, \( \gamma_s^{(m)} \), the right-most bound operator is acted on to the right by \( (1-\Lambda)^{-1} \) and is therefore a dressed operator; also the left-most bound operator is operated on to the left by \( S_s (1-\Lambda)^{-1} \), therefore the left-most free streaming operator is dressed.

We next examine how these rules may be applied in writing the renormalized equivalents of Eqs. (4.81) through (4.85). The free streaming operators contained in each binary collision operator and the left-most free streaming operator in the evolution operator \( \gamma_s^{(m)} \) when acting to the right on a binary collision operator, \( B(i) \), have the interpretation that the particle moves freely prior to the encounter with scatterer \( \tau_i \), e.g., \( S_s B(i) \) means that the particle moves freely from \( \tau_i \) to the collision hemisphere of \( \tau_i \) prior to the encounter with \( \tau_i \), and \( B(j)B(i) \) means the particle moves freely from the collision hemisphere of \( \tau_j \) to the collision hemisphere of \( \tau_i \) prior to the collision with \( \tau_i \) (as usual we are following the motion backwards in time). Therefore \( S_s B(i) \) and \( B_p(j)B(i) \) mean the particle moves freely prior to its encounter with \( \tau_i \) for a time \( \tau \) and they contain the factor,

\[
\mathcal{C} - (s + \frac{\tau}{\nu}) \tau,
\]

(6.29)
where $T$ is the mean free time of Eq. (6.23). Thus each dressed free streaming operator adds a damping factor to the collision sequence. The times that are damped, according to the above rules, are all times the particle moves freely between collisions except those times it moves immediately after a bound encounter which is followed by the same bound encounter (there may be any number of free encounters between the two bound encounters).

We write the renormalized form of formula (4.81) for a general term in the evolution operator, $\gamma_s^{(m)}$, as follows,

$$\frac{H^m}{m!} \int d\gamma \cdots d\gamma d\gamma \cdots d\gamma \left[ \begin{array}{c} \gamma \\ \gamma \end{array} \right] \frac{s^2}{T_{\text{coll}}} \cdot \epsilon = \cdots \frac{s^2}{T} \left( 1 + \frac{\epsilon}{\frac{1}{2}} \cdot \frac{2}{T} \right) \frac{s^2}{T} \left( \frac{s^2}{T} \left( \frac{s^2}{T} \right) \right) \cdots$$

where the restriction on the product over real or hypothetical collisions excludes all free, hypothetical collisions except the one that immediately follows a bound encounter which is followed by the same bound encounter; the sum, $\sum_T$, is over all times the particle travels freely between encounters except the times immediately following a bound encounter which is followed by the same bound encounter; $T$ is the mean free time; and the other variables are as defined in Eq. (4.81). Let us recall that a bound encounter is an encounter with a scatterer which the particle encounters more than once in a given collision sequence, and a free encounter is with a scatterer that is only encountered once during the collision sequence.

Taking the inverse Laplace transform of Eq. (6.30) gives the renor-
malized form of Eq. (4.82),
\[
\frac{(n\nu)^m}{m!} \int \cdots \int \frac{d^3 \vec{r}_1 \cdots d^3 \vec{r}_m d^3 \vec{p}_1 \cdots d^3 \vec{p}_m}{(2\pi)^3} \left[ \Pi Q \right]_{\text{restricted}} \Theta(t - \tau_{\text{coll}}) \cdot \\
- \frac{\epsilon}{\tau} \cdot f_{\tau}(\vec{r}_1 \cdots \vec{r}_m, \vec{p}_1 \cdots \vec{p}_m). \tag{6.31}
\]

The renormalized form of Eq. (4.83) for a term of the generalized collision operator, \( \Theta_\tau^{(m)} \), is given as,
\[
\frac{(n\nu)^m}{m!} \int \cdots \int \frac{d^3 \vec{r}_1 \cdots d^3 \vec{r}_m d^3 \vec{p}_1 \cdots d^3 \vec{p}_m}{(2\pi)^3} \left[ \Pi Q \right]_{\text{restricted}} \Theta(t - \tau_{\text{coll}}) \cdot \\
- \frac{\epsilon}{\tau} \cdot f_{\tau}(\vec{r}_1 \cdots \vec{r}_m, \vec{p}_1 \cdots \vec{p}_m). \tag{6.32}
\]

Taking the inverse Laplace transform of Eq. (6.32), we obtain the renormalized form of Eq. (4.84),
\[
\frac{(n\nu)^m}{m!} \int \cdots \int \frac{d^3 \vec{r}_1 \cdots d^3 \vec{r}_m d^3 \vec{p}_1 \cdots d^3 \vec{p}_m}{(2\pi)^3} \left[ \Pi Q \right]_{\text{restricted}} \Theta(t - \tau_{\text{coll}}) \cdot \\
- \frac{\epsilon}{\tau} \cdot f(t', \vec{p}'; t - \tau_{\text{coll}}). \tag{6.33}
\]

Finally we write the renormalized form of the generalized Stosszahlansatz, Eq. (4.85), as follows,
\[
\frac{(n\nu)^m}{m!} \int \cdots \int \frac{d^3 \vec{r}_1 \cdots d^3 \vec{r}_m d^3 \vec{p}_1 \cdots d^3 \vec{p}_m}{(2\pi)^3} \left[ \Pi Q \right]_{\text{restricted}} \Theta(t - \tau_{\text{coll}}) \cdot \\
- \frac{\epsilon}{\tau} \cdot f(t', \vec{p}'; t - \tau_{\text{coll}}). \tag{6.34}
\]

We will demonstrate in Chapter VII that the damping factor,
\[
- \frac{\epsilon}{\tau}, \tag{6.35}
\]
guarantees the convergence of expressions (6.30) through (6.34) in the long time limit. We note that the damping constant $T$, the mean free time, appears naturally in the theory.
VII. SOME RENORMALIZED CALCULATIONS

A. CONVERGENCE OF THE RENORMALIZED EXPRESSIONS

The term-by-term convergence of the renormalized terms of the evolution operator \( R_s^{(m)} \) and of the collision operator \( \Phi_s^{(m)} \) in the limit as \( s \to 0^+ \) is guaranteed by the damping factor (6.35). It serves to damp exponentially the collision sequences for a time of flight between successive single encounters greater than the mean free time \( T \). As was pointed out at the end of Chapter VI, the renormalization process brings the mean free time into the theory in a natural way as a result of taking account of the presence of the \( N \) scatterers for collision events among \( m \)-scatterers, \( m < N \). The presence of the \( N \) scatterers is manifested in the exponential decay \( e^{-\tau/T} \) of the legs of the \( m \)-scatterer collision event where the particle travels freely between collisions for a time \( T \). Note that the exponential decay is present in the collision integrals, the evolution operator, and the Stosszahlansatz and is unchanged under a transition from the Laplace-transform domain to the time domain.

We recall that the sum, \( \sum \tau_i \), in the damping factor is over all times \( \tau \) excluding the integration times immediately following a bound operator followed by the same bound operator. All the integration times, \( \tau_i \), are still damped however. This may be seen in either of two ways. First, since the starred time that is omitted from \( \sum \tau \) is part of a loop that forms a recollision, there is at least one time in this loop that ap-
proaches the excluded one in magnitude as the latter grows indefinitely (all other times in the loop have the damping factor). Second, we may transform to a new set of integration variables, \( \tau' \)'s and \( b' \)'s, obtained by examining the collision event in reverse. The Jacobian of this transformation is unity since the succession of single encounters may be parameterized either following the motion forward or backward in time. Using the new set of integration variables and the same integrand (unchanged), each of the times that is excluded from the damping constant is now no longer an integration variable since it is a dependent time; thus each new integration time is exponentially damped.

Note that the one-dimensional model is not considered in this chapter since it may be solved for exactly. The general diagrams are difficult to enumerate, and there seems to be no connection between the first few renormalized terms of the one-dimensional model and the asymptotic behavior of the complete solution.

B. WIND-TREE MODEL

We first calculate some of the terms of the renormalized evolution operators for the spatially homogeneous wind-tree model using formula (6.30). From the exclusion of most free, hypothetical collisions, the single-scatterer evolution operator has only a real collision contribution. This is given by

\[
\Gamma_{\lessgtr}^{(1)} f(\vec{p}; 0) = \frac{n V}{\rho} \int_{0}^{\infty} d\tau_1 \int_{b_1}^{+\infty} db_1 R(b_1) e^{-b_1 \frac{\tau_1}{\rho}} f(\vec{p}; 0) =
\]
\[ = \frac{n\nu s}{p^2} R(1+p) \tilde{f}(\tilde{p}; 0), \tag{7.1} \]

where we recall that \( p = s + \frac{1}{T} \), and the mean free time, \( T \), is equal to \((2\nu\sigma)^{-1}\). Similarly, the contribution to \( r_s^{(m)} \) from the completely reducible sequence is found to be,

\[ \frac{(n\nu s)^m}{p^{m+1}} \left[ R(1+p) \right]^m \tilde{f}(\tilde{p}; 0), \tag{7.2} \]

where we simplify the calculation by allowing the scatterers to overlap. This renormalized term, Eq. (7.2), is to be contrasted with the unrenormalized term of Eq. (5.28). If we add all the contributions to the evolution operator from the completely reducible sequences (the most divergent terms) and take the inverse Laplace transform, we obtain the following renormalized result,

\[ \left[ \int_s f(\tilde{p}; 0) \right]_{\text{reducible}} = e^{-\frac{T}{2} \left[ R(1+p) + \frac{1}{2} \left[ \frac{1}{R(1+p)} \right]^2 \right] \tilde{f}(\tilde{p}; 0)}, \tag{7.3} \]

where the first term in brackets corresponds to the free streaming operator, the second term results from the single-scatterer operator, and so forth. We observe that each term in Eq. (7.3) is well-behaved for long times because of the damping factor \( e^{-t/T} \). However, if we perform the infinite sum, expression (7.3) reduces to,

\[ \left[ \int_s f(\tilde{p}; 0) \right]_{\text{reducible}} = e^{\frac{t}{2T} \left[ R(1+p) - \frac{2}{2} \right] \tilde{f}(\tilde{p}; 0)}, \tag{7.4} \]
and this is precisely the unrenormalized result, Eq. (5.39). This is not unexpected since the renormalization is merely a method of expanding the exact result in a term-by-term convergent manner. The renormalization comes about by removing the hypothetical part of the operator, \([R+RP^{-2}]\), and placing it as the convergent factor multiplying the terms of expression (7.3), and this is equivalent to removing the hypothetical collision part of \(\Lambda\) in the derivation of Eq. (6.22).

Equation (7.3) points out what we already know about the renormalization, viz., that it is a different way of expanding the complete \(\Gamma_s\) or \(\mathcal{K}_s^{(m)}\). It also shows that the renormalization is meaningful only if a small parameter appears in the theory with which we can truncate the infinite series, otherwise it is necessary to obtain a complete solution. We shall search for a small expansion parameter by analyzing the parametric dependence of the renormalized terms of \(\mathcal{K}_s^{(m)}\).

Considering the irreducible collision operators, we note that the renormalization does not alter the single-scatterer collision integral \(\mathcal{K}_s^{(1)}\)

\[
\mathcal{K}_s^{(1)} = \frac{1}{2} \left[ \mathcal{R}(1+\mathcal{R}) - 2 \right] \mathcal{S}(\mathcal{R}^2).
\]  

(7.5)

Thus the Lorentz-Boltzmann collision integral is the first term of the renormalized expansion. Let us consider the general behavior of the renormalized \(m\)-scatterer collision operator, \(\mathcal{K}_s^{(m)}\), for a given \(k\)-number. The discussion parallels that of the unrenormalized collision operators; thus we may use formulas (5.50) and (5.51) replacing \(s\) by \(p\) almost everywhere. The exclusion of almost all hypothetical collisions does not affect the general form of Eqs. (5.50) and (5.51). Thus the dominant part (lowest order in \(\sigma/v\)) of the unit \(k\)-number contribution to the \(m\)-scatterer collision operator, \(\mathcal{K}_s^{(m)}\), varies as,
\[(n\nu\sigma)^m \frac{1}{\rho^{m-2}} \frac{\mathcal{S}_s}{\mathcal{V}} \]  

(7.6)

In the limit as \( s \to 0^+ \), this term varies as,
\[
\frac{1}{\tau} n\nu^2 .
\]  

(7.7)

Thus to lowest order in \( \sigma/\nu \) all unit k-number contributions in the limit \( s \to 0^+ \) behave as the first correction to the Lorentz-Boltzmann collision integral in an expansion in powers of the parameter \( n\nu^2 \). This is an initial demonstration that in the limit \( s \to 0^+ \), the renormalized expansion is an expansion in powers of the parameter \( n\nu^2 \).

From formula (5.51), the m-scatterer collision integral with k recollisions contributes a renormalized term of the form,
\[
(n\nu\sigma)^m \left( \frac{\mathcal{S}_s}{\mathcal{V}} \right)^k \frac{1}{\rho^{m-1-k}}.
\]  

(7.8)

In the limit as \( s \to 0^+ \) this term varies as,
\[
\frac{1}{\tau} (n\nu^2)^k ,
\]  

(7.9)

thus demonstrating that the dominant part of the contribution to \( \mathcal{K}_{s}^{(m)} \) with k-number equal to k varies as the k-th power of \( n\nu^2 \).*

\( n\nu^2 \) is the small expansion parameter with which we may truncate the renormalized series, since it is dimensionless (the ratio of the volume per particle to the specific volume) and small (we are dealing with only a moderately dense Lorentz gas). Through \( \sigma/\nu \) may be a small time, it is not dimensionless.

We may write the following expression for the general behavior of the renormalized series for \( \mathcal{K}_{s}^{(m)} \) obtained from Eq. (5.52) by replacing \( s \) by \( p \) and by modifying slightly the operator \( I^{(m),k} \).

*The k-number is equal to the number of uncorrelated recollisions (Appendix G).
\[ \alpha_s^{(m)} \bigg|_{\text{renormalized}} = \sum_{k=1}^{\infty} \left( \frac{n v s}{p} \right)^{m-k} \int_{\text{ten.}}^{(m),k} \int_{(s/\rho, \rho, s/\rho)}^{(m),k} \binom{m-k}{k} \frac{1}{2^{m-1} \Gamma^2} \left( 1 + \frac{T_s}{T} \right)^{m-k} \int \left( 2 n s^2 + \frac{s s}{\rho} \right). \] (7.10)

and this may be rewritten in terms form,

\[ \sum_{k=1}^{\infty} \frac{(s+\frac{T}{T_s}) (2 n s^2)^k}{2^m (1+\frac{T_s}{T})^{m-k}} \int_{\text{ten.}}^{(m),k} \int_{(s/\rho, \rho, s/\rho)}^{(m),k} \left( 2 n s^2 + \frac{s s}{\rho} \right). \] (7.11)

For values of \( s \) less than \( 1/T \) the collision operator becomes independent of \( s \) and the decay to equilibrium becomes exponential. We will show in Chapter VIII that the transport coefficient depends on \( \alpha_s^{(m)} \), and therefore for its computation we may consider the Lorentz gas to be described by a kinetic equation with a time-independent collision operator. Thus on the hydrodynamic scale the Lorentz gas decays exponentially. On the kinetic scale, the renormalized collision operator (7.11) depends on \( s \) and the system does not decay exponentially.

Hence the renormalized expansion amounts to an expansion in the powers of \( n s^2 \) in the limit as \( s \to 0^+ \) where the coefficient of the \( m \)th power of \( n s^2 \) comes from sequences with all values of \( m \) (the number of distinct scatterer labels) and with \( k \)-number not greater than \( k \). We emphasize that for the two-dimensional wind-tree model the contribution to the complete collision operator \( \alpha_s^{(m)} \) to order \( (n s^2)^k \) may be computed by dropping all terms with \( k \)-number greater than \( k \), and by employing terms having all values of \( m \). To obtain even the first correction to the Lorentz-Boltzmann collision integral one must compute collision integrals for collision histories with an infinite number of scatterers.

We compute the first correction to the Lorentz-Boltzmann collision
integral, Eq. (7.5), in an expansion in powers of \( n\sigma^2 \) for the wind-tree model. This involves the calculation of all renormalized collision integrals (all values of \( m \), the number of scatterers from 2 to \( \infty \)) with unit k-number—the complete first ring diagram term, \( T_1 \),

\[
T_1 = n \int \frac{d^3 \hat{r}}{8\pi} B(u) \Lambda (1-\Lambda)^{-1} B(v) S_s^{-1} f_s(\hat{r}). \tag{7.12}
\]

In dressed form this is written as,

\[
T_1 = n \int \frac{d^3 \hat{r}}{8\pi} B(u) \Lambda_p (1-\Lambda_p^{nh})^{-1} B(v) S_s^{-1} f_s(\hat{r}). \tag{7.13}
\]

This term can be evaluated by expanding the operator \((1-\Lambda_p^{nh})^{-1}\) in powers of \( \Lambda_p^{nh} \). However, this presents considerable difficulty because of the task of enumerating an infinite series of collision diagrams. We shall therefore develop a technique for transforming (7.13) to a form whose evaluation includes only a few diagrams. The simple form of the rotation operator, \( R \) or \( RP \), allows us to re-express (7.13) in a form in which the rotation operators are removed from the denominator.

Introduce the notation,

\[
F(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4) = (1-\Lambda_p^{nh})^{-1} B(u) S_s^{-1} f_s(\hat{r}). \tag{7.14}
\]

then \( F(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4) \) is a solution of the following equation,

\[
B(u) S_s^{-1} f_s(\hat{r}) = F(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4) - \Lambda_p^{nh} F(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4). \tag{7.15}
\]

This equation is valid for all models and may be exactly solved for in

*There are other contributions of order \( n\sigma^2 \) which have not been included.
the wind-tree model, and the hard spheres model. If we introduce the function \( H_p(\xi^+_1, \xi^+_2, \xi^+_3) \) given by,

\[
H_p(\xi^+_1, \xi^+_2, \xi^+_3) = S_p F(\xi^+_1, \xi^+_2, \xi^+_3),
\]

then \( H_p(\xi^+_1, \xi^+_2, \xi^+_3) \) when inserted in Eq. (7.15) satisfies the following equation,

\[
B(1) S_5^{-1} f_5(\xi) = S_p^{-1} H_p(\xi^+_1, \xi^+_2, \xi^+_3) - n N \int d\theta \int_0^{2\pi} H_p(\xi^+_1, \xi^+_2, \xi^+_3),
\]

and this result is still independent of the model. The method of solving this equation depends on the model; in Section D we will demonstrate the technique used for the hard spheres model. For the wind-tree model the integral over impact parameters of the rotation operator acting on \( H \) is,

\[
\frac{1}{2T} R(1+\rho) H_p(\xi^+_1, \xi^+_2, \xi^+_3).
\]

Therefore, the formal solution of \( H_p \) is as follows,

\[
H_p(\xi^+_1, \xi^+_2, \xi^+_3) = \left( S_p^{-1} - \frac{1}{2T} R(1+\rho) \right)^{-1} B(1) S_5^{-1} f_5(\xi).
\]

After some algebra this expression reduces to the following explicit solution for \( H_p(\xi^+_1, \xi^+_2, \xi^+_3) \),

\[
H_p(\xi^+_1, \xi^+_2, \xi^+_3) = \frac{1-\rho}{2} B(1) S_5^{-1} f_5(\xi) + \\
+ \left[ \frac{1}{2p} \left( S_{x1}^{-1} - S_{x1}^{-1} \right) \right] D \left( S_{x1}^{-1} S_{x1}^{-1} \right) R \left( \frac{1-\rho}{2} S_{x1}^{-1} B(1) S_5^{-1} f_5(\xi) \right). \]
where,

\[ p = \frac{1}{T} \quad (7.21) \]

\[ S_{1s} = \left( I + \frac{\vec{u} \cdot \partial}{\partial r} \right)^{-1} \quad (7.22) \]

\[ S_{2s} = \left( I - \frac{\vec{u} \cdot \partial}{\partial r} \right)^{-1} \quad (7.23) \]

\[ S_{3s} = \left( I + \frac{\vec{v}_R \cdot \partial}{\partial r} \right)^{-1} \quad (7.24) \]

\[ S_{4s} = \left( I - \frac{\vec{v}_R \cdot \partial}{\partial r} \right)^{-1} \quad (7.25) \]

where \( P_{1s} = S_{2s} P, \; R_{1s} = S_{3s} R, \; R_{PS_{1s}} = S_{4s} R P, \; R = \Phi_R R, \)

and the denominator \( D \) is given by,

\[ D = S_{1p}^{-1} S_{2p}^{-1} S_{3p}^{-1} S_{4p}^{-1} - p^4. \quad (7.26) \]

It is shown in Appendix D how this form of \( H_p, \) Eq. (7.20), when inserted in \( T_1, \) Eq. (7.13), allows us to compute the complete first renormalized correction to the Lorentz-Boltzmann collision integral using only a few collision histories. The result of the calculation is as follows,

\[ \lim_{s \to 0^+} T_1 = n u v e \sum_{n} \left\{ \left[ -\frac{n}{v} - 2 \right] (R-1) (1+\rho) + \right. \]

\[ + \left. \frac{2}{v} (1-\rho) \right\} . \quad (7.27) \]

With a little more effort it should be possible to compute the coefficient of \((n \sigma^2)^2\) using the same technique.
C. HARD DISKS MODEL

The immediate question of interest to the hard disks model is the density dependence of the renormalized terms of $Y_s^{(m)}$. To this end we may carry over the discussion of Section E, Chapter V by replacing $s$ almost everywhere by $p = s + \frac{1}{T}$, where we recall that $\frac{1}{T} = 2n\sigma$. If in the renormalized integral, $Y_s^{(m)}$, we replace each integration time $\tau_i$ by the dimensionless variable $y_i$, $y_i = \tau_i/T$, then a renormalized term from $\tilde{Y}_s^{(m)}$ with a given sequence of single encounters may be written as,

$$\frac{1}{T^{2m}} \int \cdots \int d\mathbf{r}_1 \cdots d\mathbf{r}_m \cos \theta_1 \cdots \cos \theta_m \exp \left[ \frac{i}{\hbar} \int_{t_0}^{t_f} \sum_{i,j} \phi_{ij}(\mathbf{r}_i, \mathbf{r}_j) \right] \cdot$$

$$\cdot \left( -sT \gamma_{\text{coll}} - \sum_{j} \epsilon \gamma_{j} \right) \left( \mathfrak{f}_{s} \left( \frac{r_j^0}{\mathbf{r}_j}, \mathbf{r}_j' \right) \right),$$

(7.28)

where this general form is taken from Eq. (6.32). The boundary of the integration domain of (7.28) which was formerly expressed in the form of equations involving the distance $\sigma$, $\nu \tau_i$, and functions of the angles $\theta_i$, is now dependent on the dimensionless parameter $2n\sigma^2 = \sigma/\nu T$ and the dimensionless integration variables, $y_i$ and $\theta_i$. Therefore in the limit as $s \to \sigma^+$ the integral of (7.28) depends only on the small parameter $2n\sigma^2$. From the argument of Chapter V, those terms having $k$ recollisions, $k$-number of $k$, contain the product,

$$(N \sigma^2)^k \frac{1}{Y_{i_1}} \frac{1}{Y_{i_2}} \cdots \frac{1}{Y_{i_k}},$$

(7.29)

when transforming from the impact parameter of the collision that aims the trajectory to recollide, to the impact parameter on the scatterer.
with which the trajectory recollides. Any number of the \( y \)'s in (7.29) may be the same, depending on the specific collision sequence. For the sequence 121, \( k = 1 \) and expression (7.29) reduces to

\[ n \sigma^2 \frac{1}{\gamma}. \]  

(7.30)

As shown in Appendix E, the integral of expression (7.28) for the sequence 121 is decomposed into an integral over the two regions \( y < n \sigma^2 \) and \( y > n \sigma^2 \). The latter integral contains the logarithm integral, equation (5.62), which behaves as \( \ln(\gamma n \sigma^2) \) for small values of the parameter \( n \sigma^2 \). Therefore the first renormalized sequence, 121, contributes a term varying as,

\[ \frac{1}{T} n \sigma^2 \ln \left( n \sigma^2 \right). \]  

(7.31)

and varies as the logarithm of the small parameter \( n \sigma^2 \). Note that this term approaches zero in the limit as \( n \sigma^2 \to 0 \), so that it is well-behaved for a very dilute Lorentz gas, \( n \sigma^2 \ll 1 \), but it cannot be expressed mathematically as a power series in \( n \sigma^2 \) about \( n \sigma^2 = 0 \), i.e., it is nonanalytic about \( n \sigma^2 = 0 \). It is very difficult to prove that there are not other contributions of the same form as (7.31) from other sequences. There will be other terms in \( \ln(n \sigma^2) \) having higher powers of \( n \sigma^2 \) as coefficients, for example, the sequence 121341. There also will occur terms varying as \( \ln(n \sigma^2) \) raised to some power, e.g., from the sequence 12131 we find a term varying as
\[ \frac{1}{T} \left( n \sigma^2 \ln(n \sigma^2) \right)^2. \] (7.32)

Thus the density dependence of the renormalized terms of the hard disks model is very complex.

One systematic way of approaching the hard disks calculation (which avoids the search for the logarithmic terms) is to adopt the method of the wind-tree calculation by computing the coefficients of the various powers of \( n \sigma^2 \), treating \( \ln(n \sigma^2) \) (or a power of it) as a number making up part of the coefficient. In this way we may be able to compute the coefficient of \( (n \sigma^2)^l \) by considering collision sequences with k-number not greater than \( l \). In carrying out the program, we would compute the complete first renormalized term to first order in \( n \sigma^2 \) using Eq. (7.13) and consider this as the first density correction to the Lorentz-Boltzmann collision integral, treating \( \ln(n \sigma^2) \) as a number. The second density correction would be obtained from the first renormalized term computed to second order in \( n \sigma^2 \) and the second renormalized term computed to first order in \( n \sigma^2 \). The result would be a power series in \( n \sigma^2 \) and \( \ln(n \sigma^2) \).

D. HARD SPHERES MODEL

The density dependence of the hard spheres model may be found upon replacing \( s \) by \( s + \frac{1}{T} \) almost everywhere in the unrenormalized collision integral where,

\[ \frac{1}{T} = n \sum \pi \sigma^2. \] (7.33)

Following Section C, if we replace \( \tau \) by \( T \tau \), then a renormalized term
from \( \mathcal{R}_s^{(m)} \) for a given sequence of encounters may be written in the form,

\[
\frac{1}{T} \frac{1}{\mathcal{N}^m_m!} \left[ \cdots \int_{x'_{m-1}} d\vec{y}_{m-1} \int_{x_{m-1}} d\vec{y}_{m-1} \cdots \int_{x_{1}} d\vec{y}_{1} \right]_{\text{restricted}} \\
\cdot e^{-sT \gamma_{\text{coll}}} e^{-\Sigma y} f_{s}(\vec{r}', \vec{\bar{r}}'),
\]  

(7.34)

where \( I_{i} d\Omega_{i} = \sin\theta_{i} \cos\theta_{i} d\theta_{i} d\phi_{i} \). Again the boundaries of the domain of integration are described by equations in \( y_{1}, \Theta_{1}, \phi_{1} \) and the dimensionless parameter \( \pi n \sigma^3 \), so that in the limit as \( s \rightarrow 0^+ \) Eq. (7.34) depends functionally on the small parameter \( \pi n \sigma^3 \). The argument of Chapter V states that upon transforming from the \( d\Omega \) of the collision that aims to recollide to the \( d\Omega \) of the recollision, the integrand acquires the factor,

\[
\left( n \sigma^3 \right)^{2-k} \frac{1}{y_1^2} \frac{1}{y_2^2} \cdots \frac{1}{y_k^2}.
\]

(7.35)

As in the hard disks model there will appear terms in \( \ln(n \sigma^3) \) and terms in \( \ln(n \sigma^3) \) raised to some power. The most systematic procedure therefore is probably to compute the coefficient of the various powers of the small parameter \( n \sigma^3 \), treating \( \ln(n \sigma^3) \) as a numerical factor. The first correction to the Lorentz-Boltzmann collision integral would be obtained from the first complete renormalized term, \( T_1 \), and varies parametrically as,

\[
\frac{1}{T} n \sigma^3 \left[ a + b n \sigma^3 \ln(n \sigma^3) \right],
\]

(7.36)

where \( a \) and \( b \) are numerical coefficients multiplying integrals over the rotation operators.
Let us consider the technique for evaluating the first renormalized term, $T_1'$, of Eq. (7.13). We note that the denominator in (7.13) contains the operator $\Lambda^{nh}_p$ which when acting on a function of the particle momentum is given by,

$$\Lambda^{nh}_p F(\mathbf{p}) = \nu \int d^6 \mathbf{p} \sum_{\sigma} F(\mathbf{p}) . \quad (7.37)$$

It has been pointed out by van Leeuwen-Weijland\textsuperscript{30} that expression (7.37) is independent of the orientation of $\mathbf{p}'$ since upon the transformation from the integral over the impact vector, $\mathbf{v}$, to the integral over the angles of $\mathbf{v}' = R_{\mathbf{p}} \mathbf{v}$ the expression reduces to,

$$\Lambda^{nh}_p F'(\mathbf{p}) = \frac{\nu \sigma}{\mathcal{A}} \int d^{2} v' \left( \mathbf{v} + \frac{\mathbf{p} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{p}} \mathbf{v} \right)^{-1} F(\mathbf{p}'), \quad (7.38)$$

where $d\mathcal{A}$ is the infinitesimal solid angle describing the orientation of $\mathbf{v}$. Since $\Lambda^{nh}_p$ acting on any function of $\mathbf{p}$ depends only on the magnitude of $\mathbf{p}$, when two of the operators $\Lambda^{nh}_p$ act in succession, the two may be factored,

$$\Lambda^{nh}_p F(\mathbf{p}) \Lambda^{nh}_p G(\mathbf{p}) = \left( \Lambda^{nh}_p F(\mathbf{p}) \right) \left( \Lambda^{nh}_p G(\mathbf{p}) \right) . \quad (7.39)$$

Thus the denominator in (7.13) acts as follows,

$$\left( I - \Lambda^{nh}_p \right)^{-1} \sum_{\sigma} \mathcal{S}^{-1}_s f_s(\mathbf{p}) = \sum_{\sigma} \mathcal{S}^{-1}_s f_s(\mathbf{p}) +$$

$$+ \sum_{\mathcal{K}} \left( \frac{\nu \sigma}{\mathcal{A}} \int d^{2} v' \left( \mathbf{v} + \frac{\mathbf{p} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{p}} \mathbf{v} \right)^{-1} \right) \cdot$$

$$\Lambda^{nh}_p \sum_{\sigma} \mathcal{S}^{-1}_s f_s(\mathbf{p}) . \quad (7.40)$$
It is difficult to see how expression (7.40) could be evaluated in the manner of the wind-tree calculation using only a few collision diagrams. van Leeuwen and Weijland\textsuperscript{30} have shown how it can be evaluated by transforming to the Fourier representation.
VIII. THE LORENTZ GAS CONDUCTIVITY

A. INTRODUCTION

There are two procedures for calculating transport coefficients, the distribution function method, and the correlation function method, which have been shown to be equivalent to lowest and to first order in the density. The former method has its origins in the Chapman-Enskog theory and was further developed for dense gases by Choh-Uhlenbeck. The latter method has been developed in recent years by several authors.

The distribution function method considers the behavior of the system not in equilibrium. One solves the Boltzmann or Lorentz-Boltzmann equation with the disturbance turned on, to some order in a uniformity parameter, inserts the solution into a phenomenological relation defining the transport coefficient, and computes its long time behavior. The correlation function method analyses the long time response of the system to an initial disturbance expressing the transport coefficient as the equilibrium average of the time correlation between microscopic variables with the disturbance turned off. By the disturbance we here mean the electric field.

We shall present two derivations of the Lorentz gas conductivity that are similar to the distribution function method and the correlation function method, denoting the results, respectively, as $\sigma_L$ and $\sigma_K$. We will demonstrate, using the generalized Lorentz-Boltzmann equation as developed in the preceding chapters, that the two conductivities are identical.
B. THE CONDUCTIVITY FORMULAS

The conductivity, $\sigma$, is obtained from the phenomenological relation between the macroscopic current, $\vec{J}$, and the electric field, $\vec{E}$. It is taken as the coefficient of the portion of the current that is linear in the field,

$$\vec{J} = \sigma \vec{E}.$$  \hspace{1cm} (8.1)

The current $\vec{J}$ is computed from the particle distribution by the following relation,

$$\vec{J} = e \int d\vec{p} \ \vec{v} \ f(\vec{v}; t),$$ \hspace{1cm} (8.2)

where for simplicity we assume spatial uniformity. Solving Eqs. (8.1) and (8.2) for $\sigma$, we find,

$$\sigma = \frac{e}{E} \int d\vec{p} \ \vec{v} \cdot \vec{E} \ f(\vec{v}; t).$$ \hspace{1cm} (8.3)

This prescription is not yet complete for what we mean by the conductivity, $\sigma$, is that phenomenological coefficient that relates $\vec{J}$ and $\vec{E}$ under steady-state conditions a long time after the disturbing field, $E$, has been turned on, and in the limit of a weak field. We therefore define the Lorentz conductivity $\sigma_L$ to be given by the following equation,

$$\sigma_L = \lim_{t \to \infty} \lim_{E \to 0} \frac{e}{E} \int d\vec{p} \ \vec{v} \cdot \vec{E} \ f(\vec{v}; t),$$ \hspace{1cm} (8.4)

with the time limit replaced by the time average when the former does not exist, and where the initial distribution is given as the equilibrium distribution,
\[ f(p; \varrho) = f_{eq} = \left( \frac{2\pi mkT}{\varrho} \right)^{\frac{3}{2}} e^{-\frac{\varrho^2}{2mkT}} \quad (8.5) \]

Note that for the multiple scattering system the equilibrium distribution can be any well-behaved function of the magnitude of the momentum, \( p \); we choose the Maxwell distribution to gain similarity with the dense gas problem. To complete the specification of \( \sigma \) we must state also that \( f(p; t) \) is the initial value solution of the Lorentz-Boltzmann equation,

\[ \frac{\partial f(p; t)}{\partial t} + e \mathbf{E} \cdot \frac{\partial}{\partial p} f(p; t) = \left[ \frac{\partial f(p; t)}{\partial t} \right]_{\text{coll}} \quad (8.6) \]

with \( [\partial f/\partial t]_{\text{coll}} \) given by the development of the preceding chapters, and with the initial value given by Eq. (8.5). The specification of the transport coefficient is not unique, e.g., we could have chosen a different initial distribution, yet it is in deepening with other transport coefficient derivations that we set the distribution equal to the equilibrium distribution at some long time in the past.

The correlation function derivation of the conductivity formula consists simply of a perturbation solution of the Liouville equation (with the unperturbed solution taken as the equilibrium distribution), calculating the linear response of the current, \( \mathbf{j} \), to the electric field, \( \mathbf{E} \). Starting with the Liouville equation, satisfied by the particle distribution, \( D(12\ldots N, \mathbf{r}, \mathbf{p}; t) \), we expand the latter into the unperturbed equilibrium distribution and the first perturbation,

\[ D = D_{\text{eq.}} + D^{(1)} \quad (8.7) \]
and we write the Hamiltonian as the sum of an unperturbed part $H^{(0)}$ given by Eq. (2.1), and the perturbation due to the electric field, $-eE \cdot \vec{r}$,

$$
H = H^{(0)} - e\vec{E} \cdot \vec{r}.
$$

(8.8)

Substitution into the Liouville equation then gives the equation satisfied by $p^{(1)}$,

$$
\frac{\partial D^{(1)}}{\partial t} = \left\{ H^{(0)}, D^{(1)} \right\} - e\vec{E} \cdot \frac{\partial}{\partial \vec{p}} D_{eq}. \quad (8.9)
$$

As with the previous derivation, we take the distribution, $D$, to be equal to the equilibrium distribution in the infinite past, obtaining the solution to Eq. (8.9) in the following steps,

$$
\frac{\partial}{\partial t} S(12; N; -t) D^{(1)}(12; N; \vec{r}, \vec{p}; t) = -S(12; N; -t) e\vec{E} \cdot \frac{\partial}{\partial \vec{p}} D_{eq}, \quad (8.10)
$$

where $S(12...N;t)$ is the system streaming operator without the electric field.

Solving Eq. (8.10) we find the result,

$$
D^{(1)}(12; N; \vec{r}, \vec{p}; t) = -\int_{-\infty}^{t} dt' S(12; N; t'; t) e\vec{E} \cdot \frac{\partial}{\partial \vec{p}} D_{eq}. \quad (8.11)
$$

using the assumption that $D^{(1)}(1...N,t,\vec{r},\vec{p};-\infty) = 0$. Simplification of Eq. (8.11) gives,

$$
D^{(1)}(12; N; \vec{r}, \vec{p}; t) = -\int_{0}^{\infty} dt' S(12; N; t') e\vec{E} \cdot \frac{\partial}{\partial \vec{p}} D_{eq}, \quad (8.12)
$$

and thus we find that $D^{(1)}$ is independent of the time $t$. With the perturbed distribution, $D^{(1)}$, we compute the perturbed macroscopic current which is strictly linear in the electric field,
\[ \mathbf{J} = e \int d\mathbf{p} \sum d\mathbf{\hat{p}}_i \cdots d\mathbf{\hat{p}}_N \mathcal{D}^{(\text{df})}_{1(2 \cdots N)} \mathcal{P}_{1(2 \cdots N)} = -e^2 \int d\mathbf{\hat{p}} \mathcal{M} \int_0^\infty dt \int d\mathbf{\hat{p}}_1 \cdots d\mathbf{\hat{p}}_N \mathcal{P}_{1(2 \cdots N)} \cdot S_{1(2 \cdots N; t')} \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{p}} \text{Eq.} \]  

Since the equilibrium distribution is taken to be the Maxwell distribution, \( f_{eq} \), we may transform Eq. (8.13) into the following form,  

\[ \mathbf{J} = \frac{e^2}{m^2 kT} \int d\mathbf{\hat{p}} \mathcal{P} \mathcal{M} \int_0^\infty dt \mathbf{\hat{p}}(t) \cdot \mathbf{E} \cdot f_{eq}, \]  

where  

\[ \mathcal{P}(t) = \int \cdots \int d\mathbf{\hat{p}}_1 \cdots d\mathbf{\hat{p}}_N \mathcal{P}_{1(2 \cdots N)} \mathbf{\hat{p}}_t, \]  

is the average over all scatterer positions of the momentum \( \mathbf{\hat{p}}_t \) which in time \( t \) evolves into \( \mathbf{\hat{p}} \) under the influence of the scatterers, but in the absence of an external force.

Let \( \mathbf{\hat{p}}(t) \) denote the vector,  

\[ \mathbf{\hat{p}}(t) = f_{eq}(\mathbf{p}) \mathbf{\hat{p}}(t). \]  

This vector satisfies the following equation,  

\[ \frac{\partial \mathbf{\hat{p}}(\mathbf{p}; t)}{\partial t} = \left[ \frac{\partial \mathbf{\hat{p}}(\mathbf{p}; t)}{\partial t} \right]_{\text{coll}}, \]  

with \( \frac{\partial \mathbf{\hat{p}}/\partial t}_{\text{coll}} \) given by the theory of the preceding chapters, and having the initial value,  

\[ \mathbf{\hat{p}}(0) = \mathbf{\hat{p}} f_{eq}(\mathbf{p}). \]
We now demonstrate the equality of the two conductivities, Eqs. (8.4) and (8.14). We obtain an explicit expression for \( \sigma_K \) that involves the irreducible collision operator from the Laplace transformed solution of Eq. (8.17),

\[
\hat{\varphi}_s = \left( s - \hat{\varphi}_s \right)^{-1} \hat{\varphi}(0),
\]

(8.19)

where \( \hat{\varphi}_s \) is the irreducible collision integral for spatial uniformity and in the absence of an external force. If we insert an integrating factor in Eq. (8.14), \( e^{-st} \), taking the limit \( s \to 0^+ \) outside the integral, we obtain the following formula for the current,

\[
\dot{J} = \lim_{s \to 0^+} \frac{e^{-s}}{mkT} \int d\vec{p} \dot{\vec{p}} \int d\vec{E} \left( s - \hat{\varphi}_s \right)^{-1} \hat{\varphi} \cdot \vec{E}
\]

(8.20)

If the system is isotropic, \( \dot{\hat{p}}(t) \) is equal to \( \dot{\vec{p}} \) multiplied by a scalar function of the time, and in the integrand of Eq. (8.14), \( \dot{\vec{p}} \cdot \vec{E} \) may be replaced by \( \dot{\vec{p}} \cdot \dot{\vec{p}}(t)/3 \vec{E} \), so that the resulting scalar coefficient of \( \vec{E} \) is the conductivity. The conventional form of the time correlation formulas contains the integral of \( \dot{\vec{p}} \cdot \dot{\vec{p}}(t)/3 \), in three dimensions and therefore assumes isotropy. All the models we have discussed are isotropic, although an interesting anisotropic system is obtained from the wind-tree model by allowing the particle to scatter only to the right when it suffers a real collision. In the most general case we may introduce the matrix, \( k_{ij}(s) \), defined by,

\[
\hat{\varphi}_s \hat{\varphi}_i = k_{ij}(s) \hat{\varphi}_j,
\]

(8.21)
where $k_{ij}(s)$ is just a function (no longer an operator). Writing, $A_{ij}(s) = (s-K_s)^{-1} p_i$, we find in the limit $s \to \infty$, $A_{ij}(o) = -\delta_{ij}$ or, $A_{ij}(o) = -(k(o))^{-1}_{ij}$, and thus we obtain,

$$J_i = -\frac{e}{3m^2kT} \int d\vec{p} \frac{p_i}{f_{eq}(\vec{p})} \left( k(o) \right)_i^{ij} E_j,$$  \hspace{1cm} (8.22)

where $k(o)$ is the inverse matrix of the limit as $s \to \infty$ of the matrix of Eq. (8.21). For an isotropic system this reduces to,

$$J = -\frac{e}{3m^2kT} \int d\vec{p} \frac{p^2}{f_{eq}(\vec{p})} \frac{1}{k(o)} \vec{E},$$  \hspace{1cm} (8.23)

so that the correlation function conductivity is found to be,

$$\sigma_k = -\frac{e}{3m^2kT} \int d\vec{p} \frac{p^2}{f_{eq}(\vec{p})} \frac{1}{k(o)}.$$  \hspace{1cm} (8.24)

We next obtain the Lorentz conductivity by solving Eq. (8.6) for the particle distribution to terms linear in the electric field. In the Laplace transform domain, the equation is,

$$s f_s(\vec{p}) + e E \cdot \frac{\partial}{\partial \vec{p}} f_s(\vec{p}) = f_s(\vec{p}; 0) + \chi_s^{E} f_s(\vec{p}),$$  \hspace{1cm} (8.25)

where the superscript on the irreducible collision integral indicates the presence of the electric field. To terms linear in $E$, the particle distribution may be written in the form,

$$f_s(\vec{p}) = f_{eq} \left[ \frac{1}{s} + \vec{u} \cdot \vec{E} \chi_s \right],$$  \hspace{1cm} (8.26)

where $\chi_s$ is in general some function of the magnitude of the particle momentum. Substitution of Eq. (8.26) into Eq. (8.25) gives the equation for $\chi_s$. 

\[ s X_s - \frac{e}{kT} \frac{1}{s} = (\vec{v} \cdot \vec{E})^{-1} \kappa_s^E \vec{v} \cdot \vec{E} X_s. \] (8.27)

The solution is given by,
\[ X_s = \frac{e/kT \frac{1}{s}}{s - \kappa_s^E}, \] (8.28)
where \( \kappa_s^E \) is given by,
\[ \kappa_s^E \frac{d}{d} = \vec{v} \cdot \vec{E} \kappa_s^E, \] (8.29)
and is dependent on the electric field.

Substitution of Eq. (8.26) into the Laplace transform of the current yields,
\[ \bar{J_s} = \frac{e}{s} \int d\vec{p} \bar{v} f_{eq} + \frac{e^2}{kT s} \int d\vec{p} \bar{v} \cdot \vec{E} (s - \kappa_s^E)^{-1}. \] (8.30)

The first term of Eq. (8.30) is zero, and the second may be written as,
\[ \frac{e^2}{3m^2kT} \frac{1}{s} \int d\vec{p} \bar{v} f_{eq}(\vec{p}) (s - \kappa_s^E)^{-1}. \] (8.31)

The Lorentz conductivity may now be found by substituting the Laplace-transformed current into Eq. (8.4) with the time limit replaced by the limit as \( s \to 0^+ \) of \( s \) multiplied by the Laplace transform. The result is as follows,
\[ \sigma_L = \lim_{s \to 0^+} \lim_{E \to 0} \frac{e^2}{3m^2kT} \int d\vec{p} \bar{v} f_{eq}(\vec{p}) (s - \kappa_s^E)^{-1}, \] (8.32)
and this reduces to the correlation function conductivity given by Eq. (8.24) since,
\[ \lim_{s \to 0^+} \kappa_s^E(s) = \kappa(s), \] (8.33)
where \( k(s) \) is the value obtained with the electric field turned off.

Since \( k(0) \) is independent of \( p \) we may perform the integral over the momentum in Eq. (8.24) using the form of the equilibrium distribution as given by Eq. (8.5). The result though computed in three dimensions has the same form in one and two dimensions. We shall therefore take the Lorentz gas conductivity to be given by the following formula,

\[
\sigma = -\frac{e^2}{m} k(0^+) \tag{8.34}
\]

where \( k(0^+) \) is given by,

\[
k(0^+) = \lim_{s \to 0^+} \frac{1}{p^2} \hat{p} \cdot \mathbf{K}_s \hat{p} \tag{8.35}
\]

It should be pointed out that the equivalence of the two conductivities \( \sigma_L \) and \( \sigma_K \) follows readily with the use of the convolution dynamical equation. If we instead use the product equation and are not careful to include all terms linear in the electric field, the two conductivities would not be equal. This follows because in the product dynamical equation the term,

\[
\int_{-\infty}^{\infty} E(t) f_{eq}(p) \tag{8.36}
\]

contains a finite contribution linear in \( E(\mathbf{K}_s E f_{eq}(p) \) is strictly zero).

Consider the derivation of the Lorentz conductivity using the product dynamical equation,

\[
\frac{\partial f(p; t)}{\partial t} + e\mathbf{E} \cdot \frac{\partial f(p; t)}{\partial p} = \int_{-\infty}^{\infty} f(p; t) \tag{8.37}
\]

with,
\[ f(A; t) = \int_{\text{eq}}(p) \left[ 1 + \nabla \cdot E \right] \text{d}E \text{d}A(t) \]  

(8.38)

Recall that in the derivation of both the Lorentz and the correlation formula conductivity we used the fact that,

\[ \mathcal{K}_S^E f_{\text{eq}}(p) = 0 \]  

(8.39)

this is easily proved by recalling that the right-most binary collision operator in each term of \( \mathcal{K}_S^E \) is multiplied on its right by \( S_s^{-1} \), therefore the right most operator is \( \left[ R_p^{-1} \right] \) which is identically zero when operating on \( f_{\text{eq}}(p) \).

However it is not true that \( \Lambda^E f_{\text{eq}}(p) \) is zero as we shall demonstrate below.

We may express \( \Lambda^E(t) \) in terms of \( k(s) \) in the following manner.

\[ \Lambda^E(t) = \left( \frac{\partial}{\partial t} + \beta \cdot \nabla \right) \Gamma^E \cdot \Gamma^E^{-1} \]  

(8.40)

where \( \Gamma^E(t) \) is the complete evolution operator. We may write,

\[ \Gamma(t) = \frac{1}{2\pi i} \int ds e^{+is} \int \sum \]  

(8.41)

and use the relation between \( \Gamma^E \) and \( \mathcal{K}_S^E \) obtained from the Husimi-like expansion,

\[ \sum = \left( S_s^{-1} \right)^{-1} \]  

(8.42)

We now construct the solution of the product equation to first order in \( \nabla \cdot v \) with \( \Lambda(t) \) and \( \Lambda^E(t) \) respectively. Use of Eq. (8.42) yields,

\[ \Gamma(t) f_{\text{eq}} \left[ a + b \nabla \cdot v \right] = \int_{\text{eq}} \left[ a + b \gamma(t) \nabla \cdot v \right] \]  

(8.43)

where \( \gamma(t) \) is related to \( k(s) \) by,
\[
\int_0^\infty e^{-st} \dot{\gamma}(t) = (s - \kappa(s))^{-1},
\]  
(8.44)

where
\[
\kappa(s) = \frac{1}{\rho^2} \vec{\rho} \cdot \vec{\kappa}_s \vec{\rho}.
\]  
(8.45)

Use of Eqs. (8.43) and (8.40) gives,
\[
\Delta(t) \int_{\text{eq}} \left[ 1 + \vec{E} \cdot \vec{\nabla} \right] = \int_{\text{eq}} \vec{\nabla} \cdot \vec{E} \chi (x) \dot{\gamma}(t) \dot{\gamma}(t),
\]  
(8.46)

thus the solution to Eq. (8.38) is,
\[
\chi(t) = \frac{\hbar}{\kappa T} \dot{\gamma}(t) \int_0^t \dot{\gamma}(s) \dot{\gamma}(s) \text{d}s.
\]  
(8.47)

We contrast this with the operation of the evolution operator with the field turned on,
\[
\Gamma^E(t) \int_{\text{eq}} \left[ a + b \vec{\rho} \cdot \vec{E} \right] = \int_{\text{eq}} \left[ a + b \dot{\gamma} + \frac{\alpha E}{\kappa T} \dot{\gamma} \right] \vec{E} \cdot \vec{\nabla},
\]  
(8.48)

so that,
\[
\Delta^E(t) \int_{\text{eq}} \left[ 1 + \vec{E} \cdot \vec{\nabla} \right] = \int_{\text{eq}} \vec{E} \cdot \vec{\nabla} \left[ -\frac{\kappa T}{\delta} \dot{\gamma} + \frac{\delta}{\delta} (X - \frac{\Theta}{\kappa T}) + \frac{\Theta}{\kappa T} \dot{\gamma} \right],
\]  
(8.49)

and solving the product dynamical equation for \( \chi \) we find,
\[
\chi^E(t) = \frac{\hbar}{\kappa T} \dot{\gamma}(t) + \frac{\Theta}{\kappa T} \chi(t).
\]  
(8.50)

Inserting in Eq. (8.48) the value \( t=0 \) we find, \( \gamma(0) = 0 \), \( \dot{\gamma}(0) = 1 \), therefore the constant must be zero since \( \chi(0) = 0 \), thus
\[
\chi^E(t) = \frac{\hbar}{\kappa T} \dot{\gamma}(t).
\]  
(8.51)

Thus the solutions are quite different in the two cases when the field
is present or absent in the evolution operator, whereas in the convolution form, to first order in $E, \hat{\chi}_s^E$ may be replaced by $\hat{\chi}_s$, the irreducible collision operator with the field turned off. Taking the Laplace transform of $\chi(t)$ from Eq. (8.51) we find the same solution as Eq. (8.28) obtained from the convolution equation.

C. ONE-DIMENSIONAL MODEL CONDUCTIVITY

Using Eq. (5.14) for the irreducible collision operator of the one-dimensional point scatterers model we find,

$$\kappa(s) = -2 \left( \frac{nu}{2} \right)^2 - 3s + \cdots \quad (8.52)$$

therefore $k(0^+)$ diverges linearly. This simply means that there is no transport for the one-dimensional point scatterers,

$$\sigma = \frac{-e^2}{m \kappa(0^+)} = 0 \quad (8.53)$$

However for the finite point scatterers, there is finite transport since,

$$\kappa(s) = -\frac{v_p}{\epsilon} + O(s) \quad (8.54)$$

and therefore,

$$\sigma = \frac{e^2}{m v_p} = \frac{6 e^2}{m v} \quad (8.55)$$

a result which is independent of the density. Note that if the one-dimensional Lorentz gas may be described by the binary collision operator, then we would have found that $k(0^+) = -2nv$ for both the finite and the point scatterers models, and therefore that the conductivity is given by $\sigma = e^2/(2mnv)$ which is
inversely proportional to the density.

D. CONDUCTIVITY COEFFICIENT IN THE WIND-TREE MODEL

The conductivity coefficient, \( \sigma \), may be obtained to first order in the expansion in the small parameter \( \epsilon = 2n a^2 \) by using the result of Appendix D. It is found that to first order in \( \epsilon \),

\[
\kappa (0) = - \frac{T}{T} - \frac{\epsilon}{T} \left( \frac{1}{n_\perp} + \frac{1}{n_\parallel} \right) + \cdots + O(\epsilon^2) \tag{8.56}
\]

Therefore the conductivity is given by the formula,

\[
\sigma = \frac{\epsilon^2 T}{m (1 + \epsilon (\frac{1}{n_\perp} + \frac{1}{n_\parallel}) \epsilon^2)} \tag{8.57}
\]

This result is obtained from a complete evaluation of the first ring diagram term (an infinite sum of collision integrals) to lowest order in \( \sigma/v \).

E. CONDUCTIVITY COEFFICIENT IN THE HARD DISKS MODEL

The density dependence of the collision operator, \( \mathcal{M}_s \), and therefore of the number \( k(0) \) is shown in Appendix E to be nonanalytic since it is the Laplace transform of a function \( F(\xi) \) with \( \epsilon = 2n a^2 \) as the Laplace transform variable. For large values of \( \xi \), \( (\xi > 1) \) \( F(\xi) \) may be expanded in powers of \( 1/\xi \) and the term linear in \( 1/\xi \) behaves as \( \epsilon \ln 1/\xi \). The coefficient of \( \epsilon \) has contributions from all terms in the first ring diagram and may be computed only numerically.

The Lorentz-Boltzmann contribution to \( k(0) \) is,

\[
k_{L-B} = - \frac{1}{T} \frac{\epsilon}{T} \tag{8.58}
\]
and the first correction (first ring diagram term to lowest order in $\epsilon$),

$$k^{(1)}(0) = -\frac{\epsilon}{8T} \left(\frac{2}{3}\right)^2 \ln \frac{T}{\epsilon} - \frac{\epsilon}{8T} C_1,$$  \hfill (8.59)

where $C_1$ is the underdetermined coefficient of $\epsilon$. The conductivity is then given by the formula,

$$\sigma = \frac{\frac{3}{4} \frac{\epsilon^2 T}{m}}{1 + \frac{3}{2} \ln \frac{T}{\epsilon} + \frac{3}{3} C_1 \epsilon + \ldots}.$$  \hfill (8.60)

There are terms containing powers of $\epsilon \ln 1/\gamma \epsilon$ in $k(0)$, some of which contribute terms varying as,

$$-\frac{3}{4} T \left(\frac{2}{3} \ln \frac{T}{\epsilon}\right)^m.$$  \hfill (8.61)

This is highly suggestive that the terms in $k(0)$ may factor into a geometric series so that it could be written in the form,

$$k(0) = \frac{-\frac{1}{T} \frac{3}{4}}{1 - \frac{3}{5} \frac{\epsilon \ln \frac{T}{\epsilon} - \epsilon C'}{C' + \ldots}}.$$  \hfill (8.62)

therefore the conductivity would be proportional to the denominator of $k(0)$ and would thus not contain powers $\ln \epsilon$.

The conductivity coefficient in the hard spheres model may also be expressed as the Laplace transform of a function with $\epsilon = \pi n \sigma^2$ as the Laplace transform variable. For the 121 sequence (first part of the first ring diagram term) the function varies as, $a_1/\sigma^2 + a_2/\sigma^3 + \ldots$, therefore it contri-
butes the terms $\epsilon$, $\epsilon^2 \ln 1/\gamma\epsilon$, plus higher orders of $\epsilon$. The 1231 sequence contributes terms varying as $\epsilon^2 \ln 1/\gamma\epsilon$, $\epsilon^2$, plus higher order terms. Van Leeuwen and Weijland\textsuperscript{20} find a $\epsilon^2 \ln 1/\gamma\epsilon$ contribution only from the 1231 sequence, with the 121 sequence contributing only a term in $\epsilon$. The fact that $\epsilon \ln 1/\gamma\epsilon$ is small compared with unity may resolve this difference. The calculation of the hard sphere collision integrals is quite tedious in the physical representation used herein and is therefore not discussed.
APPENDIX A

FORM OF THE BINARY COLLISION EXPANSION

The functional form of the binary collision expansion is given as,

\[ F(\bar{a}) = \left(1 + \sum_{\alpha} \left\{ \frac{1}{1+\bar{a}} \right\}^{-1} \right)^{-1}, \]  

(A.1)

where \( \bar{a} \) denotes \( B_{\alpha} \), and the streaming operator of Eq. (2.27) has been placed on the left-hand side. This may be expressed in the alternate form,

\[ F(\bar{a}) = 1 + \sum_{\alpha} \left( \bar{a} - \bar{a}^2 + \bar{a}^3 + \ldots \right) F(\bar{a}) \].  

(A.2)

It is clear that \( F(\bar{a}) \) may be written in the form,

\[ F(\bar{a}) = a_0 + a_1 + a_2 + \cdots \]  

(A.3)

where \( a_n \) denotes the sum of all terms in \( F(\bar{a}) \) with \( n \) and only \( n \) \( \bar{a} \). Insertion of Eq. (A.3) into Eq. (A.2) yields the formula,

\[ a_n = \sum_{\alpha} \bar{a}^{n+1} a_n + \sum_{\alpha} \bar{a}^n a_{n-1} + \cdots + (-1)^{n+1} \sum_{\alpha} \bar{a}^n \]  

(A.4)

Define \( b_n \) to be the sum over all products of \( n \) \( \bar{a} \)'s such that no adjacent \( \bar{a} \)'s are identical (the labels occur only once in succession).

As a consequence of their definition the \( b_n \) satisfy the relation,

\[ \sum_{\alpha, left \ most} \bar{a}^k b_n = \sum_{\alpha, \ not \ left \ most} \bar{a}^{k+1} b_{n-1} \]  

(A.5)

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for any $k$, which follows from,

$$
b_n = \sum_{\alpha \neq 1, \text{ leftmost of } b_{n-1}} \alpha \cdot b_{n-1} \quad \quad \text{(A.6)}$$

Suppose now that $a_k = b_k$ for $k = 0, 1, 2, \ldots, n-1$, then, rearranging terms in A.4 we find,

$$a_n = \sum_{\alpha \neq 1} \alpha \cdot b_{n-1} + \left( \sum_{\alpha = 2} \alpha \cdot b_{n-1} - \sum_{\alpha = 2} \alpha^2 \cdot b_{n-2} \right) - \cdots$$

$$+ (-1)^n \left( \sum_{\alpha = 3} \alpha^{n-1} \cdot b_1 - \sum \alpha^n \right) =$$

$$= \sum_{\alpha \neq 1} \alpha \cdot b_{n-1} + 0 \quad \quad \text{(A.7)}$$

hence $a_n = b_n$; and thus by induction this is valid for all $n$. Therefore it has been proved that,

$$F(\alpha) = b_0 + b_1 + b_2 + \cdots \quad \quad \text{(A.8)}$$

or the binary collision expansion is the sum over all sequences of labels occurring only once in succession.
APPENDIX B

PROOF OF THE IRREDUCIBLE FACTORIZATION LEMMA

The lemma reads as follows: the collection of all irreducible factorizations corresponding to a given order-preserving partition may be factored into the ordered product of the collections of all irreducible sequences whose elements are the labels of the blocks of the partition.

Let $C(\pi^{\text{ord}})$ denote the collection of all irreducible factorizations that correspond to a given order-preserving partition $\pi^{\text{ord}}$. A typical member appears as $(B)(B) \ldots (B)$. This is the collection of all sequences of labels such that each disjoint subsequence, $(B)$, contains the labels of the corresponding block, $B$, of $\pi^{\text{ord}}$ such that the order of the labels among and within the blocks $B$ is the same as the order of the labels among and within the subsequences.

In each of the sequences in the collection $C(\pi^{\text{ord}})$, imagine that all subsequences are fixed except the $l$th one; the latter corresponds to the $l$th block $B_l$ of $\pi^{\text{ord}}$. The collection of all subsequences corresponding to $B_l$ are irreducible, and may be denoted by, $C(B_l)$. $C(B_l)$ is the collection of all irreducible sequences whose elements are the labels of the block $B_l$. This latter statement follows from the completeness of the sequences of labels that are contained in the binary collision expansion; meaning that there is no sequence of labels that does not occur.

If all members of $C(\pi^{\text{ord}})$ having all but the $l$th block fixed, are added, the result is,
\[ \sum_{(B_i, B_j) \in C(\tau^{\text{ord}})} (B_i)(B_j) \cdots C(B_k) \cdots = (B_i)(B_j) \cdots C(B_k) \cdots, \quad (B.1) \]

since the blocks and their corresponding subsequences are disjoint, and the terms of the binary collision expansion occur once and only once. Repeating this process for the other blocks, it follows that,

\[ \sum (B)(B) \cdots (B) = \prod_{B \in \tau^{\text{ord}}} \sum (B) = \prod_{B \in \tau^{\text{ord}}} C(B). \quad (B.2) \]

Thus the collection of sequences having the irreducible factorization characterized by \( \pi^{\text{ord}} \) is the ordered product of the collections of sequences characterized by the blocks of \( \pi^{\text{ord}} \). This is a consequence of the uniqueness and completeness of the binary collision expansion, and the disjointness of the irreducible factors.
APPENDIX C

A COMPARISON OF THE PRODUCT AND THE CONVOLUTION COLLISION OPERATORS

The complete product collision operator, $\Lambda(t)$, and the complete convolution collision operator, $^{(2)}\mathcal{K}_s$, are related by the following operator identity,

$$\Lambda(t) = \frac{1}{2\pi i} \int d\xi e^{-it\mathcal{S}} (\mathcal{R} \mathcal{S})^{-1} = \frac{1}{2\pi i} \int d\xi e^{-it\mathcal{S}} (\mathcal{S}^{-1} - \mathcal{R}^{-1})^{-1}.$$  \hspace{1cm} (C.1)

This relation may be used to evaluate $\Lambda(t)$ when $^{(2)}\mathcal{K}_s$ is known.

To gain an understanding of the physical collision events that constitute the product collision operator, it is sufficient to examine the collision histories of $\Lambda^{(3)}(t)$, the three scatterer collision operator. We consider $\Lambda^{(3)}(t)$ acting on $f(x, p; t)$ where according to Eqs. (3.15) and (3.16) $\Lambda^{(3)}(t)$ is given by,

$$\Lambda^{(3)}(t) = \frac{U(t)}{3!} \prod_{\mathcal{S}_i} \int d^3 \mathcal{S}_i \mathcal{F}_i \mathcal{F}_i \mathcal{F}_i \mathcal{A}(123; t) U(-t),$$  \hspace{1cm} (C.2)

where

$$\mathcal{A}(123; t) = \sum_{\text{perm}} P(13) \frac{\partial U(t) U(12, 3; t)}{\partial t} - \sum_{\text{perm}} P(12) \frac{\partial U(t) U(1, 2, 3; t)}{\partial t} U(t) U(23, 1; t) - \sum_{\text{perm}} P(12) \frac{\partial U(t) U(1, 2, 3; t)}{\partial t} U(-t) U(3; t) + \sum_{\text{perm}} P(1) \frac{\partial U(t) U(1; t)}{\partial t} U(-t) U(2, 1; t) U(-t) U(3; t).$$  \hspace{1cm} (C.3)

Each operator in (C.3) may be evaluated by taking the Laplace transform followed by the inverse Laplace transform. In this way we find,
\[
\frac{1}{3!} \int \left[ \frac{d^3 \vec{r}_1}{d \tau_1} \frac{d^3 \vec{r}_2}{d \tau_2} P(123) U(t) \frac{\partial}{\partial \vec{r}} U(-t) U(123; t) F(\vec{r}, \vec{p}) \right] = \\
= \frac{1}{2\pi^2} \int ds \epsilon_s \frac{e^{i s}}{3!} \int \left[ \frac{d^3 \vec{r}_1}{d \tau_1} \frac{d^3 \vec{r}_2}{d \tau_2} P(123) R(123) F(\vec{r}, \vec{p}) \right],
\]

(C.4)

where \( R(123) \) is the sum of products of binary collision operators with three labels. From the results of chapter four, this may be written as the integral over integration times and impact variables,

\[
\sum_{\text{collision histories}} \frac{(Mv)^3}{2\pi^2} \int \left[ \frac{d^3 \vec{r}_1}{d \tau_1} \frac{d^3 \vec{r}_2}{d \tau_2} \frac{d^3 \vec{r}_3}{d \tau_3} \Theta(t - \tau_1) \Theta(t - \tau_2) \Theta(t - \tau_3) \right] F(\vec{r}, \vec{p}),
\]

(C.5)

where the collision histories consist of all sequences of single encounters among three scatterers in which the trajectory begins on one scatterer and ends on a scatterer. The collision histories are not restricted to be irreducible.

Using the same technique, we find that the term \( U(12;t) \) may be written as,

\[
\frac{1}{2!} \int \left[ \frac{d^3 \vec{r}_1}{d \tau_1} \right] U(12; t) F(\vec{r}, \vec{p}) = \sum_{\text{collision histories}} \frac{(Mv)^3}{2\pi^2} \int \left[ \frac{d^3 \vec{r}_1}{d \tau_1} \frac{d^3 \vec{r}_2}{d \tau_2} \frac{d^3 \vec{r}_3}{d \tau_3} \Theta(t - \tau_1) \Theta(t - \tau_2) \Theta(t - \tau_3) \right] \cdot F(\vec{r}, \vec{p})
\]

\[
\cdot \Theta(t - \tau_{\text{coll}}) F(\vec{r} - \vec{v} (t - \tau_{\text{coll}}), \vec{p}'),
\]

(C.6)

where in contrast with (C.5), the collision histories consist of all sequences of single encounters among two scatterers in which the trajectory begins at \( \vec{r} \) which is at a distance \( v \tau_{\perp} \) from the first encounter with a scatterer, and it ends on a scatterer. The sequences are not restricted to be irreducible. Using relations (C.5) and (C.6) we may write the terms of \( \Lambda^{(3)}(t) \) noting that the presence of the free streaming operator \( U(-t) \) after each operator effects the transformation,
\[ F(\mathbf{r}' - \mathbf{v}(t - \tau_{\text{coll}}'), \mathbf{p}') \rightarrow F(\mathbf{r}' + \mathbf{v}(t - \tau_{\text{coll}}), \mathbf{p}') \]  

(2.7)

in (C.5) and (C.6). Therefore the contribution of the first term of (C.3) is given as,

\[ \sum_{\text{collision histories}} \frac{(\nu \omega)^3}{3!} \int d\tau_1 d\tau_2 d\tau_3 d\mathbf{b}_1 d\mathbf{b}_2 d\mathbf{b}_3 \Theta(t - \tau_{\text{coll}}) f(\mathbf{r}' + \mathbf{v}(t - \tau_{\text{coll}}), \mathbf{p}'; t) \]  

(2.8)

where the sum over collision histories is over all collisions among three scatterers. The second term has the contribution,

\[ -\sum_{\text{collision histories}} \frac{(\nu \omega)^3}{3!} \int d\tau_1 d\tau_2 d\tau_3 d\mathbf{b}_1 d\mathbf{b}_2 d\mathbf{b}_3 \Theta(t - \tau_{\text{coll}}) f(\mathbf{r}' + \mathbf{v}(t - \tau_{\text{coll}}), \mathbf{p}'; t) \]  

(2.9)

where the sum over collision histories is over all collisions in which the particle encounters one scatterer and then scatters among the remaining two scatterers, never returning to the first scatterer encountered (following the motion backwards in time). The third term of (C.3) has the contribution,

\[ -\sum_{\text{collision histories}} \frac{(\nu \omega)^3}{3!} \int d\tau_1 d\tau_2 d\tau_3 d\mathbf{b}_1 d\mathbf{b}_2 d\mathbf{b}_3 \Theta(t - \tau_{\text{coll}}) \int d\tau_1 d\tau_2 d\mathbf{b}_3 \Theta(t - \tau_3) f(\mathbf{r}'' + \mathbf{v}(t - \tau_{\text{coll}}), \mathbf{p}''; t) \]  

(2.10)

where the sum over collision histories is over all encounters among two particles; the duration of these sequences is \( \tau_{\text{coll}} \). The net collision event is as follows. The particle interacts in any manner with two particles and ends up on the collision hemisphere of one of them at the position \( \mathbf{r}' \) and having momentum \( \mathbf{p}' \), after traveling backwards for a time \( \tau_{\text{coll}} \). The particle then jumps to the position, \( \mathbf{r}' + \mathbf{v}(t - \tau_{\text{coll}}) \). Starting from the latter point, and with momentum \( \mathbf{p}' \), the particle moves backwards a time \( \tau_3 \) until
it encounters the third scatterer, where it scatters once at the position \( \mathbf{r}'' \) and acquires the momentum, \( \mathbf{p}'' \).

The fourth term of (C.3) has the contribution,

\[
\left( \frac{n_0 \nu}{2} \right)^3 \int d\mathbf{r}_1 \, d\tau_1 \, d\mathbf{v}_1 \, \Theta(t-\tau_1) \int d\mathbf{r}_3 \, d\mathbf{v}_3 \, \Theta(t-\tau_3) \cdot \left( \mathbf{v}_1'' + \mathbf{v}_3'' \right) \cdot (\mathbf{r}_1'' + \mathbf{r}_3'') \cdot (\mathbf{p}_1'' + \mathbf{p}_3'') \cdot t \right) .
\]

(C.11)

The collision history consists of the following events. The particle begins on a scatterer and suffers a collision giving rise to the position \( \mathbf{r}' = r \) and the momentum \( \mathbf{p}' \). Starting at this point it moves backwards a time \( \tau_2 \) with momentum \( \mathbf{p}' \) until a collision occurs with a second scatterer at the position \( \mathbf{r}'' \) and giving rise to the momentum \( \mathbf{p}'' \). It then jumps to the position \( \mathbf{r}'' + \mathbf{v}''\tau_2 \). Starting at the latter point and with the momentum \( \mathbf{p}'' \) it moves backwards a time \( \tau_3 \) until it suffers a collision with the third scatterer at the position \( \mathbf{r}''' \) and gives rise to the momentum \( \mathbf{p}''' \).

The collision events of each of the four contributions do not cancel one another so as to yield an irreducible collision sequence that is characteristic of the convolution collision operator \( \mathcal{H}_S^{(3)} \). We find that by evaluating the particle distribution at time \( t \) for the product collision operator instead of at the time \( t-\tau_{\text{coll}} \), as for the convolution collision operator, the contributing collision sequences are quite complicated, viz., they are not irreducible and they consist of sequences of reducible events that are connected by free streaming trajectories. Thus from the standpoint of the physical interpretation of the collision sequences in \( \Lambda(t) \) versus \( \mathcal{H}_S \), the convolution collision operator is the more natural one.
APPENDIX D

FIRST CORRECTION TO THE LORENTZ GAS CONDUCTIVITY
FOR THE WIND-TREE MODEL

The complete first ring diagram term, $T_1$, in the renormalized expression for the collision operator $\mathbf{K}_s$ is computed to lowest order in the expansion parameter, $\epsilon = 2n_0^2$, in the limit as $s \to 0^+$. This yields the first correction to the conductivity, i.e., the coefficient of the small parameter $\epsilon$ in the expansion of $\sigma$,

$$\sigma = -\frac{\epsilon^2}{m \left(k^{(0)}_{LB} + k^{(1)}_{LB} + \cdots \right)} , \quad (D.1)$$

where $k^{(0)}_{LB}$ is the zeroth order coefficient obtained from the Lorentz-Boltzmann equation,

$$k^{(0)}_{LB} = -\frac{1}{T} \quad (D.2)$$

where $1/T = 2n_0^2$ is the mean free time, and $k^{(1)}(0)$, the first correction term, is found to be given by,

$$k^{(1)}(0) = + \frac{\epsilon T}{\kappa} \quad (D.3)$$

The first ring diagram term is written as,

$$T_1 = \mathbf{K}_s \mathbf{B}(1) \Lambda_p^{-1} \left(1 - \Lambda_p^{nh} \right)^{-1} B(1) \mathbf{S}_s^{-1} f_s(\vec{z}) \quad (D.4)$$

The operator, $(1 - \Lambda_p)^{-1}$, when solved for exactly, has the form,
\[
(1 - \Lambda_p^{nh})^{-1} = S_p^{-1} \left[ 1 - \frac{i}{2p} (S_{2p}^{-1} - S_{2p}') \right] \mathcal{D}^{-1} (S_{2p}^{-1} S_{4p}') \cdot \left( \frac{1 + p}{2} S_{2p}^{-1} + \frac{1 - p}{2} \right) .
\]

(D.5)

where the denominator, \(\mathcal{D}\), is given by,

\[
\mathcal{D}^{-1} = \sum_{\ell=0}^{\infty} (p^y)^\ell \left( S_{2p} S_{2p} S_{2p} S_{4p} \right) \frac{1}{(2\ell)!}.
\]

(D.6)

To avoid evaluating an infinite series of diagrams corresponding to the sum in (D.6), we express \(\mathcal{D}^{-1}\) as an operator acting on only four streaming operators,

\[
\mathcal{D}^{-1} = J_{op} S_{1x} S_{2x} S_{3y} S_{4y} .
\]

(D.7)

where \(J_{op}\) acting on some function \(F(x,y)\) is given by,

\[
J_{op} F(x,y) = \lim_{\beta \to 0} \sum_{\ell=0}^{\infty} (p^y)^\ell \left( \frac{d}{dx} \frac{d}{dy} \right)^\ell F(x,y) .
\]

(D.8)

Thus by writing \((1 - \Lambda_p^{nh})^{-1}\) in the form of a geometric series of streaming operators the terms of which do not contain the rotation operators, we are able to compute the first ring diagram term using only a few diagrams. The use of the operator of (D.8) is a device which sums the infinite series of diagrams.

Insertion of relations (D.5) - (D.8) into the first ring diagram term gives the following expression,

\[
T_1 = \frac{n^2 \beta}{2} J_{op} \int d^2 \vec{p} \int d^2 \vec{q} \left\{ B(1) B(2) S_{2p}^{-1} S_{1x} S_{2x} S_{3y} S_{4y} \right. \cdot \left[ 2 S_{2p}^{-1} S_{2p}' S_{4p}' - \frac{i}{p} S_{2p}' S_{2p} S_{4y} (1 - \rho) + \frac{p}{2} S_{2p}^{-1} S_{4y} R + p S_{2p}' R p \right] \left\{ B(1) S_{5}^{-1} f_5(\vec{p}) \right\} .
\]

(D.9)
The contribution from the term of (D.5) not containing the denominator, D, is zero because it corresponds to the collision diagram where scatterers 1 and 2 are touching.

Rearrangement of the terms and use of the transformation,

\[ S_{1\rho}^{-1} = S_{1\chi}^{-1} + \rho^{-x} \]  

allows us to write \( T_1 \) in the form,

\[
T_1 = \frac{\hbar}{2m} \sum_{\rho} \int dS_{\rho} dS_{\rho} B(1)B(2)B(1)S_{\rho}^{-1} \cdot \\
\left[ S_{1x}S_{2x}S_{3y}S_{4y} \left( 2\rho (\rho^{-y})^2 (\rho^{-x}) + (\rho^{-x})^2 (\rho^{-y})^2 (\rho^{-1}) + \rho^2 (\rho^{-x})(\rho^{-y}) R(1+P) \right) + \\
+ S_{1x}S_{2x}S_{3y}S_{4y} \rho^2 (\rho^{-x}) R + S_{1x}S_{2x}S_{4y} \rho^2 (\rho^{-x}) R P + \\
+ S_{1x}S_{3y}S_{4y} \left( 2\rho (\rho^{-y})^2 + (\rho^{-x})(\rho^{-y}) R(1+P) \right) + \\
+ S_{1x}S_{2x} \left( 2\rho (\rho^{-x}) + (\rho^{-x})^2 (\rho^{-1}) \right) + \\
+ S_{3y}S_{4y} \rho^2 (\rho^{-y}) (\rho^{-1}) + S_{1x}S_{3y} \rho^2 R + \\
+ S_{1x}S_{3y} \rho^2 R P + 2\rho S_{1x} \int B(1) S_{s}^{-1} f_{\tilde{S}}(\tilde{S}) \right].
\]

Each of the nine terms in (D.11) is evaluated by drawing a diagram corresponding to the particular collision history represented by the given product of operators and then integrating over the independent collision variables, each free streaming operator corresponds physically to the free streaming of
the particle along one of the four directions without a rotation. Thus for example, the product of operators,

\[ B(1) B(2) S_p^{-1} S_{ix} S_{2x} B(1) S_f^{-1} \]

(D.12)
corresponds to the collision history where the particle scatters with \( \xi_1 \), moves for a time \( \tau_2 \) to scatterer 2, scatters with \( \xi_2 \), then moves for a time \( \tau_3 \) in the direction \(-R_1 R_2 \hat{p} \), where \( R_1 \) and \( R_2 \) are the rotation operators for the encounters with \( \xi_1 \) and \( \xi_2 \) respectively. Then it moves for a time \( \tau_4 \) in the direction \(+R_1 R_2 \hat{p} \), and ends on the collision hemisphere of \( \xi_1 \) where it scatters again. Each free streaming operator may be treated as a binary collision operator without the integral over impact parameters, and which does not rotate the momentum. The subscripts on the streaming operators have the interpretation that the number indicates the direction of the free streaming and the letter is the Laplace transform variable.

Without giving the details of the evaluations, the results of integrating over the impact parameters and streaming times are listed below,

\[ S_{ix} S_{2x} S_{iy} S_{fy} : \quad \frac{\nu \sigma \epsilon_r}{2} \left( 1 - \frac{x^2}{y^2} \right) \left( \frac{1}{x^2} + \frac{1}{y^2} \right) \left(R - 1\right) \left(1 + \nu \right) \]

(D.13)

\[ + \frac{\nu \sigma \epsilon_r}{2} \left( 1 - \frac{x^2}{y^2} \right) \left(1 - \nu \right) \]

\[ S_{ix} S_{2x} S_{iy} : \quad \frac{\nu \sigma \epsilon_r}{2} \left( 1 - \frac{x^2}{y^2} \right) \left(-2 \left(R - 1\right) \left(1 + \nu \right) + \nu - 1 \right) \]

(D.14)
\[ S_{1x} S_{2x} S_{xy} : -\frac{\nu \sigma \Delta}{\lambda} \times \gamma \left[ -2(R-1)(1+\rho) + \rho - 1 \right], \] (D.15)

\[ S_{1x} S_{3y} S_{xy}, S_{1x} S_{2x}, S_{3y} S_{xy} : 0 \] (D.16)

\[ S_{1x} S_{3y} : -\frac{\nu \sigma \Delta}{\lambda} \times \gamma \left[ 3\rho - R\rho - 3R + 1 \right], \] (D.17)

\[ S_{1x} S_{4y} : -\frac{\nu \sigma \Delta}{\lambda} \times \gamma \left[ 3\rho - R\rho - 3R + 1 \right], \] (D.18)

\[ S_{1x} : -\nu \sigma \Delta \times \gamma \left[ (R-1)(1+\rho) - (1-\rho) \right]. \] (D.19)

The contributions from (D.16) are zero when operated on by \( J_{op} \) since the latter reduces to the operation of taking the limit as \( x \) or \( y \rightarrow p \).

The limit \( s \rightarrow 0^+ \) has been taken in all terms (D.13) - (D.19), whereby \( p = 1/T \).

Adding together all the contributions from (D.13) - (D.19) gives the result,

\[ \lim_{s \rightarrow 0} T_1 = \nu \sigma \Delta \times \gamma \left( \frac{\nu \sigma \Delta}{\lambda} \times \gamma \left[ \frac{2}{\lambda} \times (-1)(R-1)(1+\rho) - \frac{\nu \sigma \Delta}{\lambda} \times \gamma \left( \frac{\nu \sigma \Delta}{\lambda} \times (1+\rho) \right) \right] \right), \] (D.20)

where the effect of \( J_{op} \) operating on the various terms is given by the following,

\[ J_{op} \times \gamma = \lim_{y \rightarrow p} \gamma = 1, \] (D.21)
\[
\frac{J_0}{x^2} = \lim_{x \to \infty} \sum_{\ell=0}^{\infty} (2\ell)! (\frac{1}{2\ell})^\ell \gamma \frac{e^{-\frac{x^2}{4\ell}}}{(4\ell)^{3/2}}.
\]

Note that,
\[
\left(1 + \frac{x^2}{4\ell}\right)^{\ell} x^{-n} = \binom{\frac{n}{2}}{\ell} x^{n-\ell},
\]
therefore,
\[
J_0 \frac{y}{x^2} = \sum_{\ell=0}^{\infty} \binom{\frac{n}{2}}{\ell} (-1)^{\ell} = \sqrt{-1} = 0.
\]

\[
J_0 \frac{y}{x} = \sum_{\ell=0}^{\infty} \frac{(-\ell)!}{(\ell)!} \frac{\gamma}{\ell!} = \sum_{\ell=0}^{\infty} \frac{(-\ell)!}{(\ell)!} \frac{\gamma}{\ell!} = \Gamma \left(-\ell, \ell; 1; 1\right).
\]

The hypergeometric function can be written in the form,
\[
\Gamma \left(-\ell, \ell; 1; 1\right) = \frac{\prod(c) \prod(c-a-b)}{\sqrt{\Gamma(c-a) \Gamma(c-b)}},
\]
for \(c-a-b > 0\). Hence we find that,
\[
J_0 \frac{y}{x} = \frac{\prod(1) \prod(1)}{\sqrt{\prod(1) \prod(1)}} = \frac{2}{\pi}.
\]

Thus the first ring diagram term is given by,
\[
\lim_{s \to 0} T_s = n v e n s^2 \left(\frac{4}{\pi} - 1)(R - 1)(1 + \rho) - n v e n s^2 \left(\frac{4}{\pi} + 1)(1 - \rho)\right).
\]

The last ring is,
\[
\kappa^\text{1st ring} = - n v e n s^2 \left(\frac{4}{\pi} + 2\right).
\]
Only the operator \( l^\dagger P \) contributes when operating on \( \hat{p} \). Expressing the result with \( 1/T = 2\pi \nu \sigma \) and \( \epsilon = 2\pi \sigma^2 \) we find the result,
\[
\kappa^{1\text{st ring}} = -\frac{\epsilon\pi}{T} \left( \frac{1}{\bar{a}} + \frac{1}{a} \right) \tag{D.30}
\]

Note that this is the contribution only from the first ring diagram; there are other contributions of order \( \epsilon \) which have not been computed. Appendix G gives some examples of these extra contributions.
APPENDIX E

FIRST RENORMALIZED TERM IN THE HARD DISKS MODEL

The dependence on the small parameter $\epsilon = 2n\sigma^2$ of the first sequence, \( \ell_2 \), of the first ring diagram term is discussed for the hard disks model. The coefficient of the term varying as $\epsilon \ln 1/\gamma \epsilon$ is computed for this sequence and the coefficient of $\epsilon$ is given by quadrature.

The $\ell_2$ contribution is given by the formula,

$$K_{\ell_2}^{(0)} = \lim_{\rho \to 0^+} \rho \cdot \int d^2 \rho' R'_{11} B_{11} R_{11} \rho' \cdot \left( \int \frac{d^2 \rho}{\rho} \right)^{-1}. \tag{E.1}$$

This reduces to the integral over one time, $\tau'$, and the two angles, $\theta_2$, $\theta_3$, where $b_1 = \sigma \sin \theta_1$ is the impact parameter.

$$K_{\ell_2}^{(0)} = -\left( \frac{\pi \rho b_1}{4} \right)^2 \int \cos \theta_3 d \theta_3 \cos \theta_2 d \theta_2 d \tau' \cdot \left[ \cos(2\theta_1 + 2\theta_2) + \cos(2\theta_2 + 2\theta_3) + \cos(2\theta_1 + 2\theta_3) \right] \in \frac{-2\epsilon \cos \theta_3}{-2\epsilon \cos \theta_3}. \tag{E.2}$$

The integration variables are related according to the formulas,

$$\epsilon \cos(2\theta_1 + \theta_2) + y' \cos 2\theta_2 - \gamma = \epsilon \cos \theta_1, \tag{E.3}$$

$$\epsilon \sin(2\theta_1 + \theta_2) + y' \sin 2\theta_2 = -\epsilon \sin \theta_1, \tag{E.4}$$

where $y_2$ is the dimensionless time the particle travels (backwards) from 1 to 2 and $y'$ is the dimensionless time it travels from 2 to 1. Note that these
relations contain only the small parameter \( \epsilon \). It is readily verified that,

\[
dy_2 \cos \theta_1 \, d\theta_1 \, \cos \theta_2 \, d\theta_2 = dy' \cos \theta_3 \, d\theta_3 \, \cos \theta_4 \, d\theta_4, \tag{E.5}
\]

so that the integral, (E.2), may be expressed as the integral over \( y_2, \theta_1, \theta_2 \) or the integral over \( y', \theta_2, \theta_3 \), as mentioned in Chapter VII.

It is expedient to transform to the three variables, \( y', \theta_1, \theta_3 \). The Jacobian of the transform is,

\[
\frac{1}{2} \epsilon \cos \theta_1 \left( y' \cos 2\theta_2 + \epsilon \cos (2\theta_1 + \theta_3) \right) \tag{E.6}
\]

therefore \( k_{121}^{(0)} \) may be written as,

\[
k_{121}^{(0)} = -\frac{\epsilon}{4\pi} \int_C -\frac{dy \cos \theta_1 \cos \theta_2 \cos \theta_3}{\sin \theta_2} \left[ \frac{1}{2} \cos \theta_1 \right] \tag{E.7}
\]

using the substitution \( y' = \epsilon \xi \). The advantage of this set of variables lies in the simple integration domain. Using (E.3) and (E.4), we may solve for \( \theta_2, \theta_3 \),

\[
\sin 2\theta_2 = \frac{-\xi \cos \theta_2 + \sin \theta_3 \sqrt{(\xi + \cos \theta_2)^2 + \sin^2 \theta_3 - \sin^2 \theta_1}}{(\xi + \cos \theta_2)^2 + \sin^2 \theta_3}, \tag{E.8}
\]

\[
\cos 2\theta_2 = \frac{-\sin \theta_3 \sin \theta_1 + (\xi + \cos \theta_2) \sqrt{(\xi + \cos \theta_2)^2 + \sin^2 \theta_3 - \sin^2 \theta_1}}{(\xi + \cos \theta_2)^2 + \sin^2 \theta_3}. \tag{E.9}
\]

Thus we may write \( k_{121}^{(0)} \) as the Laplace transform of a function \( f(\xi) \) with \( \epsilon \) as
the Laplace transform variable,

\[ k_{121}^{(0)} = \frac{-\xi}{\eta} \int_0^\infty e^{-\xi s} f(s) \, ds \]  

(E.10)

where,

\[ f(s) = \frac{1}{\pi^2} \left[ \sin(2\theta_1 + \theta_2) \sin(2\theta_1 - \theta_2) \right] + \frac{\lambda}{\pi^2} \left[ \cos(2\theta_1 + \theta_2) + \cos(2\theta_1 - \theta_2) \right] \left[ \cos(2\theta_1 + \theta_2) - \cos(2\theta_1 - \theta_2) \right] \]

\[ + \frac{\lambda}{\pi^2} \left[ \cos(2\theta_1 + \theta_2) + \cos(2\theta_1 - \theta_2) \right] \left[ \cos(2\theta_1 + \theta_2) - \cos(2\theta_1 - \theta_2) \right] \]

\[ + \frac{\lambda}{\pi^2} \left[ \cos(2\theta_1 + \theta_2) + \cos(2\theta_1 - \theta_2) \right] \left[ \cos(2\theta_1 + \theta_2) - \cos(2\theta_1 - \theta_2) \right] \]

\[ f(s) \text{ has the expansions,} \]

\[ f(s) = a_1 s + a_2 s^2 + \cdots , \quad s > 1 \]  

(E.12)

and,

\[ f(s) = f^{(0)} + \frac{\xi}{1} f^{(1)} + \cdots , \quad s < 1 \]  

(E.13)

Since the integrals in (E.11) cannot be carried out, \( f(\xi) \) must be expanded in series (E.12) or (E.13). The use of a power series over the entire range of \( \xi \) would yield an expansion for \( k_{121}^{(0)} \) in powers of \( 1/\xi \). Since we are interested in the expansion of \( k_{121}^{(0)} \) for small \( \xi \), both expansions, (E.12) and (E.13),
must be employed. Thus we write,

$$ k_{12}^{(o)} = -\frac{\xi}{g} \int_0^1 d\xi \, e^{-\xi f(\xi)} - \frac{\xi}{g^2} \int_0^\infty d\xi \, e^{-\xi f(\xi)} , \quad (E.14) $$

and expand \( f(\xi) \) in the first term on the right-hand side of \((E.14)\) in powers of \( \xi \) and in the second term, in powers of \( 1/\xi \). Expanding both terms and retaining only the lowest order in \( \epsilon \), we have,

$$ \int_0^1 d\xi \, e^{-\xi f(\xi)} = \int_0^1 d\xi \, f(\xi) + O(\epsilon) , \quad (E.15) $$

$$ \int_0^\infty d\xi \, e^{-\xi f(\xi)} = a \int_0^\infty d\xi \, e^{-\xi \frac{a}{\xi}} + \int_0^\infty d\xi \left[ f(\xi) - \frac{a}{\xi} \right] + O(\epsilon) , \quad (E.16) $$

where,

$$ \int_0^\infty d\xi \, e^{-\xi f(\xi)} = -E_1(-\epsilon) = \ln \frac{1}{\xi} + O(\epsilon) . \quad (E.17) $$

Thus to lowest order in the density, \( k_{12}^{(o)} \) is given by,

$$ k_{12}^{(o)} = -\frac{\xi}{g^2} \left\{ a \ln \frac{1}{\xi} + \int_0^1 d\xi \, f(\xi) + \int_0^\infty d\xi \left[ f(\xi) - \frac{a}{\xi} \right] \right\} . \quad (E.18) $$

The calculation of the constant term in \((E.18)\) can only be carried out numerically. The coefficient of the logarithmic term is given as,

$$ a_1 = \int_0^\infty d\theta_1 d\theta_2 \ln 3 \left[ \ln(2\theta_1 \theta_2) + \ln 2\theta_1 + \ln 2\theta_2 + 1 \right] = \left( \frac{\theta_1}{\theta_2} \right)^2 . \quad (E.19) $$

The contributions from the higher sequences may be written down by generalizing the preceding treatment. Doing this it is found that there are no further contributions to the \( \epsilon \) line term, that the higher terms contain contributions of order \( \epsilon \) and higher, including terms \( \epsilon^2 \) line, \( \epsilon^3 \) line, etc. Since even the coefficient of \( \epsilon \) cannot be evaluated for the simplest sequence of the first
ring diagram term, we avoid writing the contributions from the more complex sequences, which are even more difficult to evaluate.
APPENDIX F

EXTENSION OF THE THEORY TO THE DENSE GAS

We consider briefly how the preceding method may be applied to the dense gas of hard sphere molecules. As in the Lorentz gas, the complex collision events among m hard sphere molecules are sequences of binary encounters; therefore a description in terms of sequences of binary collision operators appears natural. The terms in the binary collision expansion of the system streaming operator correspond to sequences of pairs of labels since each binary collision operator contains the two labels of the molecules undergoing the binary collision.

One thus observes that the generalization of the techniques of Chapters II-VIII amounts to the problem of extending the manipulations with sequences of single labels to manipulations with sequences of pairs of labels. In this extension some of the ideas can be directly carried over, such as irreducible factorization and renormalization, although in a slightly different form.

One difference between the sequences of labels and sequences of pairs of labels is that the irreducible factorization of the latter is a factorization into products of subsequences that are not-more-than singly connected, thus which are not in general disjoint; whereas the former is a factorization into disjoint subsequences. The irreducible factorization is similar in form to the Husimi expansion and may be obtained from the latter by symmetrization of the noncommuting operators.

The irreducible factorization gives rise to an inversion of the series
in the manner of the relation between the Lorentz gas evolution operator $\Gamma_s$, and collision operator, $\mathcal{H}_s$, given by Eq. (8.42). However an important difference lies in the fact that the inverted operator in the dense gas operates on the initial value distributions in the manner of a differential operator so that the inversion is not strictly analogica inversion of operators acting in the space of $\mathcal{F}$ and $\mathcal{P}$.

The renormalization procedure is analogous to that of the Lorentz gas with the sum over free labels replaced by the sum over singly connected pairs. The renormalization may be thought of as the process of summing over all hypothetical collisions that do not alter a physical scattering sequence; this takes into account the presence of all the molecules and their effect on the $m$-body collision sequence by the introduction of a convergence factor that essentially cuts off the collision integrals for times between collisions greater than the mean free time.
APPENDIX G

THE ORDER OF TERMS IN THE WIND-TREE MODEL

It has pointed out by E. H. Hauge that there are several contributions of order $\epsilon$ in the wind-tree model that do not follow the classification scheme according to which the sequences with $k$ recollisions are of order $\epsilon^k$. Thus the first ring diagram term is not the only contribution of order $\epsilon$.

Let us consider the rule for determining the order of a term (power of $\epsilon$). From a naive viewpoint each recollision, of which there are $k$ (= number of indices minus number of distinct indices in a sequence), restricts the integration time of the scatterer that aims the trajectory to recollide and therefore replaces $1/p$ by $\sigma/v$ giving rise to the factor $\epsilon$. However more than one recollision can correspond to the same aiming to recollide; in fact all recollisions in a collision sequence may correspond to the same aiming to recollide.

Therefore the $k$-number used in ordering the terms in powers of the small parameter, $\epsilon$, is equal to the number of scattering events that first aim the particle to experience a recollision, rather than the number of recollisions. For example the sequence, 123412341, corresponds to an event having 5 recollisions but only one aiming to recollide and thus contributes a term of order $\epsilon$. Similarly the sequence, 12345321, corresponds to an event with 3 recollisions but with only one aiming to recollide and therefore is of order $\epsilon$.  

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50. Ford, G. W., private communication.
ERRATA

Page 18  Eq. (2.41) should read:

$${}_{m}^{m_{s}} = \frac{1}{m!} \int \ldots \int d\xi_{1} \ldots d\xi_{m} \, f^{(m)}(12 \ldots m) \, S_{s} \, R(12 \ldots m).$$

Page 23  Line 8 should read, "U(-t) f(\hat{r}, \hat{p}; t)."

Page 27  Line 3 should read, "limit of Eq. (2.40) . . ."

Page 28  Line 7 should read, "series or develop an equation . . ."

Page 28  Line 11 should read, "evolution operator, \( {}_{m}^{m_{s}} \), Eq. (2.41)."

Page 34  Line 4 should read, "The connected diagram of Fujita is a graphical representation of sequences of single encounters; the connection refers to the elements or factors of a sequence having the same labels in common, therefore connected factors are irreducible factors and the two expansions (the connected diagram expansion)."

Page 34  Lines 11 and 12 should read, "Fujita writes the disjoint irreducible factors in a non-disjoint manner allowing him to write the transport coef . . ."

Page 46  Line 8 should read, "gives a contribution only for values of the momentum"
ERRATA (Concluded)

Page 129  Line 11 should read, "The correlation function method analyzes"

Page 131  Line 9 should read, "yet it is in keeping with..."

Page 143  Add the comment, "E. H. Hauge informed the author that in their
           most recent preprint, van Leeuwen and Weijland obtain the $e^2 \ln \frac{1}{\gamma \varepsilon}$
           contribution from both the 121 and the 1231 sequences."

Page 152  Line 8 should read, "and $k^{1st}$ ring, the first ring diagram"

Page 152  Eq. (D.3) should read, "$k^{1st}$ ring $= - \frac{e^2}{T} \left( \frac{1}{\pi} + \frac{1}{2} \right)"

Page 167  Add the comment to Ref. 6, "This paper treats the problem of
           obtaining a time-independent collision integral, or a kinetic
           equation from the time-dependent dynamical equation."

Page 167  Ref. 11 should read, "translator: M. J. Moravcsik"

Page 168  Ref. 12 should read, "(The Hague, Holland"

Page 168  Ref. 19 should read, "Zwanzig, R."

Page 168  Ref. 17 should read, "Cambridge Phil."