ENGINEERING RESEARCH INSTITUTE
UNIVERSITY OF MICHIGAN
ANN ARBOR

THEORETICAL STUDY OF THE INITIATION OF OSCILLATIONS IN ELECTRON STREAMS THROUGH CROSSED FIELDS

Technical Report No. 16
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Project 2009
CONTRACT NO. DA-36-039 sc-15450
SIGNAL CORPS, DEPARTMENT OF THE ARMY
DEPARTMENT OF ARMY PROJECT NO. 3-99-13-022
SIGNAL CORPS PROJECT NO. 27-112B-0
November 1953

ABSTRACT

A stream of electrons is considered, partly filling the space between two plane, parallel admittance sheets of different d-c potential and focussed by a uniform magnetic field parallel to the sheets. The natural modes of propagation of small sinusoidal perturbations are calculated for various boundary admittances, and the conditions are investigated under which these modes represent growing waves.

Comparisons are made between propagation parallel with and perpendicular to the magnetic field.

For applications to magnetrons the cases are considered where the low-potential sheet (analagous to the cathode) has infinite admittance or is a pure conductance, respectively. In the latter case the calculations show that growing waves may exist even when the high-potential sheet (anode) is a capacitive admittance.

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Introduction

The purpose of this study is to find the general conditions under which spontaneous, approximately sinusoidal oscillations will build up in an idealized system consisting of a stream of electrons flowing between two admittance sheets in crossed electric and magnetic fields. The motivation for a small-signal, greatly oversimplified theory is as usual the hope that the results after comparison with experimental data may make it possible to reach a better at least qualitative understanding of the behavior of realizable systems and to show the way toward further improvement of the performance of these systems. The independent variables that are given particular attention are the admittances of the boundary surfaces, i.e., the "circuit" characteristics of the oscillator.

The very general statement of the problem makes it possible to make some comparisons between the mechanisms of oscillation in traveling-wave and magnetron oscillators.

It is hoped that some of the results will make it possible to look beyond the present frontiers, for instance, in the field of voltage-tunable operation of magnetrons where the understanding of the mechanism of oscillation and the relations between the various design parameters is very incomplete.

Statement and Solution of the Idealized Problem in Mathematical Form

Figure 1 relates a Cartesian coordinate system to the geometry of the system to be studied. Between two plane parallel admittance sheets $y = 0$ and $y = d$ a stream of electrons occupies the volume from $y = 0$ to $y = h$. It has an average velocity component v_{ox} in the direction of the x-axis and one $v_{oz} = u$ in the direction of the z-axis. A uniform magnetic flux density B is parallel to the z-axis, and a potential difference V_d is maintained between the admittance sheets $y = d$ and $y = 0$.

For the electric field E and the magnetic flux density B we introduce the variables

$$\vec{e} = \frac{e\,\vec{E}}{m} \tag{1}$$

$$\omega_c = \left| \frac{e\,\vec{B}}{m} \right| , \tag{2}$$

Translating the components of the acceleration into partial derivatives by means of the "hydrodynamic operator" $\partial/\partial t + \vec{v} \cdot \nabla$ we obtain Newton's equations of motion for an electron in the following form:

$$-(e_x + v_y \omega_c) = \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right) v_x \tag{3}$$

$$-(e_y - v_x \omega_c) = \frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right) v_y \tag{4}$$

$$-\,e_z = \frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right) v_z\,. \tag{5}$$

We shall assume that the unperturbed state of the system is the ideal Brillouin state of the cut-off linear magnetron and that the zero-order quantities (d-c) will remain invariant under small sinusoidal perturbations. Thus we have

$$\frac{\partial v_{ox}}{\partial x} = \frac{\partial v_{oz}}{\partial z} = e_{ox} = e_{oz} = v_{oy} = 0 \tag{6}$$

$$v_{ox} = - \omega_c y \tag{7}$$

$$v_{oz} = u \tag{8}$$

$$e_{oy} = - \omega_c^2 y . \tag{9}$$

For the first-order perturbations the equations (3)-(5) assume the following appearance

$$e_{1x} = - q v_{1x} \tag{10}$$

$$e_{1y} = v_{1x} \omega_c - q v_{1y} \tag{11}$$

$$e_{1z} = - q v_{1z} . \tag{12}$$

where the operator notation

$$q = \frac{\partial}{\partial t} - \omega_c y \frac{\partial}{\partial x} + u \frac{\partial}{\partial z} \tag{13}$$

has been introduced.

The analysis to be presented here is nonrelativistic, i. e., it is assumed throughout that the squares of the ratios of the electron velocities and the phase velocities of all wave motions to the velocity of light are negligible compared to unity. This means that the first-order electric field inside the electron stream satisfies Poisson's equation and outside Laplace's equation rather than the vector wave equation.

Inside the stream then

$$\text{div } e_o = \rho_o = - \omega_c^2 \tag{14}$$

$$\operatorname{div} e_1 = \rho_1 . \tag{15}$$

Furthermore, conservation of charge requires that

$$-\frac{\partial \rho}{\partial t} = \operatorname{div} (\vec{v}\,\rho) . \tag{16}$$

Note that the space-charge density ρ has equally arbitrary dimensions and units as the electric field e; the choice is justified by the resulting compact form of the equations.

Expansion of (16) gives

$$q \operatorname{div} \vec{e}_1 = {\omega_c}^2 \operatorname{div} \vec{v}_1 . \tag{17}$$

Making use of the fact that there will be no vortices inside the electron stream

$$\operatorname{curl} v_1 = 0 \tag{18}$$

we can eliminate the field components and all but one velocity component between the equations (10)-(12), (17) and (18). The result is a four-dimensional partial differential equation. For sinusoidal perturbations a separation of the variables is easily achieved. Let

$$v_{1x}(t,x,y,z) = v_{1x}(y) \exp\left\{ j\omega t - j\alpha x - j\gamma z \right\} . \tag{19}$$

Then the operator

$$q = j\omega + j\alpha\omega_c y - ju\gamma$$

$$= j\omega_c p = j\omega_c \left\{ \frac{\omega}{\omega_c} + \alpha y - \frac{u\gamma}{\omega_c} \right\} . \tag{20}$$

It is found convenient to express the solution as a function of the new variable p rather than of y. Evidently $\delta/\delta p$ is equal to $1/\alpha \cdot \delta/\delta y$.

The resulting differential equation is

$$\frac{\delta^2 v_{1x}}{\delta p^2} + \frac{2p}{p^2-1} \frac{\delta v_{1x}}{\delta p} - \frac{\alpha^2 + \gamma^2}{\alpha^2} v_{1x} = 0 . \qquad (21)$$

The same equation is obtained for the component v_{1z}.

The variable p is zero for an electron that has an average velocity equal to the phase velocity of the perturbation. The origin of p thus falls inside the stream only if such a condition of sychronism exists at any plane in the stream.

The solution of this equation, the electromagnetic field equations in the space between y = h and y = d and the appropriate boundary relations at y = 0, y = h, and y = d results in a number of natural modes of propagation along the stream. In the x and z dimensions we shall assume "cyclic" boundary conditions, i.e., that waves leaving the system in one direction, immediately reenter the system from the opposite direction. The components α and γ of the propagation constant then have a discrete set of real values, but the radian frequency of any particular natural mode of the system may be either real or complex. In the latter case the amplitude of the mode decays or grows exponentially with time. The existence of growing waves is a necessary condition for the system to be a self-excited oscillator. The sufficient set of conditions includes in addition some relations given by the initial state of the system. By the use of the device of cyclic boundary conditions we have eliminated the growth and decay in space that is typical of the traveling-wave amplifier and limited our study to reentrant oscillators like the magnetron, without having to deal with the additional complication of a cylindrical geometry. Since time has only one direction, complex roots in frequency are somewhat less confusing than complex roots in wave number.

The assumption of real values of α and γ is realistic when one of the admittance sheets is an interdigital structure connected to a single-mode resonator, such as a lumped RLC circuit. A perturbation in the resonator will then produce instantaneously an identical potential difference at all pairs of fingers. For other configurations this assumption may not be justifiable. In order to simplify the analysis, however, we shall here consider α and γ real quantities determined by the structures producing the wave admittances Y_h at $y = h$ and Y_k at $y = 0$.

We shall discuss the solution of (21) and corresponding natural modes of the system for two extreme conditions. The first is obtained by letting α go to zero, so that the wave propagation takes place parallel to the magnetic field. We shall refer to this case as that of a pure traveling-wave oscillator, since the system can be interpreted as an idealization of a hollow-beam coaxial-circuit traveling-wave tube with magnetic focusing.

The equation (21) in this case becomes

$$\frac{\delta^2 v_{1z}}{\delta y^2} - \gamma^2 v_{1z} = 0 \qquad \left\{\text{TW:} \quad \alpha = 0\right\} \qquad (22)$$

with the solutions

$$v_{1z} = v_1 \exp\left\{ j\omega t - j\gamma z \pm \gamma y \right\}. \qquad (23)$$

The opposite extreme is reached by setting $\gamma = 0$, so that the wave propagation is perpendicular to the direction of the magnetic field. We shall call this case that of the pure magnetron, since the system now obviously is an ideal linear magnetron. Then we have from (21) and (20)

$$\frac{\delta^2 v_{1x}}{\delta p^2} + \frac{2p}{p^2-1}\frac{\delta v_{1x}}{\delta p} - v_{1x} = 0, \quad (M: \gamma = 0) \qquad (24)$$

where

$$p = \frac{\omega}{\omega_c} + \alpha y, \qquad (25)$$

which is the differential equation studied by Macfarlane and Hay[2] and in slightly different form by Brewer.[3]

The solutions are one odd and one even function of p, which for real values of p are real and monotonic between the singular points $p = \pm 1$. These functions will be discussed further later on; for the time being we simply write the solutions in the form

$$v_{1x} = a_o f_o (p) + a_1 f_1 (p). \qquad (26)$$

The next step in the analysis is the evaluation of the "ripple" in the boundary of the electron stream. According to Hahn's [1] procedure the equivalent surface charge or discontinuity in the normal component of the electric field is found to be

$$\Delta e_{1y} = \frac{j\omega_c v_{1y}}{p}. \qquad (27)$$

At first sight it is tempting to set $v_{1x} = v_{1y} = v_{1z} = 0$ at $y = 0$, since these relations may seem the natural boundary conditions at a solid surface smoothly covered by the electron stream. However, we prefer to assume that the boundary conditions are given by the continuity of the tangential electric and magnetic field components and the admittance of the

1) Ref. 3, page 93.

sheet. For simplicity we also assume that the sheet will permit an infinitesimal ripple of the stream boundary and values of v_{1x} or v_{1z} and v_{1y} different from zero. Admittedly this assumption is unrealistic; we may not be justified in neglecting the power loss produced by the fact that the electrons impinging on the cathode have finite kinetic energy while those emitted have zero energy on the average.

The boundary conditions at y = 0 can then be written

$$Y_k = -\frac{H_x}{E_z} = \frac{\omega \epsilon_0 E_y}{\gamma E_z} = \frac{e_y}{e_z} \cdot \frac{\omega\epsilon_0}{\gamma} \qquad (\alpha = 0) \qquad (28)$$

for the first case, since

$$(\text{curl } H)_y = \frac{\delta H_x}{\delta z} = \epsilon_0 \frac{\delta E_y}{\delta t} \qquad (29)$$

and for the second case

$$Y_k = \frac{H_z}{E_x} = \frac{\omega \epsilon_0 E_y}{\alpha E_x} = \frac{e_y}{e_x} \cdot \frac{\omega \epsilon_0}{\alpha} \qquad (\gamma = 0). \qquad (30)$$

At y = h the boundary conditions will be expressed by the same relations with reversed signs, since the admittance then is measured in the direction of positive rather than negative y.

We shall here assume that Y_k is a pure conductance $1/R_k$ and introduce the dimensionless parameters

$$\left(\frac{e_y}{e_z}\right)_{y=0} = \frac{\gamma}{\omega \epsilon_0 R_k} = \frac{1}{\eta} \qquad (\alpha = 0) \qquad (31)$$

$$\left(\frac{e_y}{e_x}\right)_{y=0} = \frac{\alpha}{\omega \epsilon_0 R_k} = \frac{1}{\eta} \qquad (\gamma = 0) \qquad (32)$$

$$-\left(\frac{e_y}{e_z}\right)_{y=h} = \frac{\gamma}{\omega\epsilon_o} Y_h = j\, B_h \qquad (\alpha = 0) \qquad (33)$$

$$-\left(\frac{e_y}{e_x}\right)_{y=h} = \frac{\alpha}{\omega\epsilon_o} Y_h = j\, B_h \qquad (\gamma = 0). \qquad (34)$$

B_h is a dimensionless susceptance parameter that is complex when Y_h has a real component. It can be expressed in terms of the admittance Y_d at y = d and the distance d-h.

$$B_h = \frac{\tanh \gamma(d - h) + B_d}{1 + B_d \tanh \gamma(d-h)} \qquad (\alpha = 0) \qquad (35)$$

where

$$j\, B_d = \frac{\gamma}{\omega\epsilon_o} Y_d \,. \qquad (36)$$

For $\gamma = 0$ the corresponding expressions are obtained by replacing γ by α.

For large values of γ (d - h), B_h of course approaches unity, i.e., the normalized capacitive susceptance of free space.

In order to obtain a necessary condition for oscillation in the first case (traveling-wave oscillator, $\alpha = 0$), we combine the boundary conditions (31) and (33) with the wave solutions (23) and examine the complex roots of ω. Introducing (11), (12), and (27) into (31) we obtain the differential equation

$$\frac{1}{\eta} = \left[\frac{e_{1y} + \Delta e_{1y}}{e_{1z}}\right]_{y=0} = \left[\frac{\left[- j\, \omega_c\, p + \frac{j\omega_c}{p}\right] v_{1y}}{- j\, \omega_c\, p\, v_{1z}}\right]_{y=0}$$

$$= \left[\frac{\left(1 - \frac{1}{p^2}\right) \frac{\delta v_{1z}}{\delta y}}{- j \, v_{1z}} \right]_{y=0} \tag{37}$$

$$\frac{\delta v_{1z}}{\delta y} = - \frac{j \gamma}{\eta \phi} v_{1z} \qquad \text{for} \quad y = 0 \tag{38}$$

$$\phi = 1 - \frac{1}{p^2} . \tag{39}$$

In the same way (33) leads to the equation

$$\frac{\delta v_{1z}}{\delta y} = - \frac{\gamma B_h}{\phi} v_{1z} \qquad \text{for} \quad y = h. \tag{40}$$

Let us write the complete wave solution inside the space charge

(23)

$$v_{1z} = a_o \cosh \gamma y + a_1 \sinh \gamma y . \tag{41}$$

This expression can now be substituted for v_{1z} either in (38) and (40) or in the integrals of these equations. In the present case the forme alternative leads to the simpler result.

The compatibility relation of the two resulting equations determines the eigen-values of ω.

$$\frac{\tanh \gamma h + \frac{B_h}{\phi}}{1 + \frac{B_h}{\phi} \tanh \gamma h} = \frac{1}{\eta \phi} . \tag{42}$$

Here both ϕ and B_h may be functions of ω; consequently the general solution for the eigen-values of ω can hardly be written in simpler form.

When B_h is independent of ω a quadratic equation in ϕ is obtained

$$\phi^2 + \phi \left[\frac{B_h}{s} - \frac{j}{\eta s^2}\right] - j\,\frac{B_h}{\eta s} = 0 \tag{43}$$

where for short

$$\tanh \gamma h = s \, . \tag{44}$$

When also $\eta = 0$, we get simply

$$p^2 = \frac{1}{1 + B_h \tanh \gamma h} = \left[\frac{\omega - \gamma u}{\omega_c}\right]^2 . \tag{45}$$

At first sight, (45) may seem to be in conflict with the conventional small-signal theory of the traveling-wave tube, since it is of the second rather than of the third degree. However, if ω be considered real and $\tanh \gamma h$ replaced by γh, it is seen that the equation will be of the third degree in γ, as usual.

Incidentally, it is interesting to note that the electric field components calculated from (23) and (10)-(12) satisfy Laplace's equation, so that the first-order space-charge perturbation ρ_1 is zero. No bunching actually takes place; the effect of space-charge waves is produced entirely by the ripple along the surface of the electron stream.

In the magnetron case ($\gamma = 0$) the boundary conditions at $y = 0$ (32) and $y = h$ (34) in the same way give the differential equations

$$\left[\left(1 - \frac{1}{p^2}\right) \frac{1}{v_{1x}} \frac{\delta v_{1x}}{\delta p} + \frac{1}{p}\right]_{y=0} = \frac{1}{j\eta} \tag{46}$$

$$\left[\left(1 - \frac{1}{p^2}\right) \frac{1}{v_{1x}} \frac{\delta v_{1x}}{\delta p} + \frac{1}{p}\right]_{y=h} = -B_h \, . \tag{47}$$

The general wave solution inside the space charge is

$$v_{1x} = a_o f_o(p) + a_1 f_1(p) \ . \tag{26}$$

Expansion about the origin defines the functions $f_o(p)$ and $f_1(p)$ from (24) in the following way

$$f_o(p) = 1 + \frac{1}{2} p^2 + \frac{5}{24} p^4 + \frac{93}{720} p^6 + \frac{3848}{40320} p^8 + \ldots . \tag{48}$$

$$f_1(p) = p + \frac{1}{2} p^3 + \frac{11}{40} p^5 + \frac{321}{1680} p^7 + \ldots\ldots \tag{49}$$

The integrals of (46) and (47) consistent with (26) and valid in the interval $-1 < p < 1$ are

$$v_{1x} = \left\{ a_o \left[\left(\frac{1-p}{1+p} \right)^{1/2} e^p \right]^{1/j\eta} \right\}_{y=0}$$

$$= \left\{ a_o \psi \left(p, \frac{1}{j\eta} \right) \right\}_{y=0} \tag{50}$$

$$v_{1x} = \left\{ a_o \left[\left(\frac{1-p}{1+p} \right)^{1/2} e^p \right]^{-B_h} \right\}_{y=h}$$

$$= \left\{ a_o \psi \left(p, -B_h \right) \right\}_{y=h} \ . \tag{51}$$

If $\eta = 0$, then the tangential electric field e_{1x} and also v_{1x} should be zero. The right-hand side of (50), however, behaves quite differently depending on the direction in the complex plane from which η approaches zero. When the resistivity of the conductor is > 0, the magnetic field according to the skin-effect theory always lags the electric field, so that

$$\eta = r + jx \qquad (x > 0 \text{ if } r > 0). \tag{52}$$

Consequently

$$v_{1x} = a_o\left[\left(\frac{1 - p}{1 + p}\right)^{1/2} e^p\right]^{-\frac{x + jr}{x^2 + r^2}} \qquad (y = 0). \tag{53}$$

When x and r simultaneously approach zero, this expression also becomes zero, as it should.

The alternative forms of the equation of compatibility obtained by substituting (26) for v_{1x} in (46)-(47) and (50)-(51) respectively, are

$$\frac{f'_o\left(\frac{\omega}{\omega_c}\right)\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right] - f_o\left(\frac{\omega}{\omega_c}\right)\left[\frac{1}{j\eta} - \frac{\omega_c}{\omega}\right]}{f'_o\left(p_h\right)\left[1 - \frac{1}{p_h^{\ 2}}\right] + f_o\left(p_h\right)\left[B_h + \frac{1}{p_h}\right]} =$$

$$= \frac{f'_1\left(\frac{\omega}{\omega_c}\right)\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right] - f_1\left(\frac{\omega}{\omega_c}\right)\left[\frac{1}{j\eta} - \frac{\omega_c}{\omega}\right]}{f'_1\left(p_h\right)\left[1 - \frac{1}{p_h^{\ 2}}\right] + f_1\left(p_h\right)\left[B_h + \frac{1}{p_h}\right]} \tag{55}$$

$$\frac{f_1\left(\frac{\omega}{\omega_c}\right)}{f_1\left(p_h\right)} - \frac{\psi\left(\frac{\omega}{\omega_c}, \frac{1}{j\eta}\right) - f_o\left(\frac{\omega}{\omega_c}\right)}{\psi\left(p_h, -B_h\right) - f_o\left(p_h\right)} = 0. \tag{56}$$

The eigen-values of ω representing the natural modes of the system can be solved from either one of these equations. The first one is more convenient for an approximate solution based on the assumption

$$p^4 << 1 \tag{57}$$

$$\eta^2 << 1. \tag{58}$$

The result is

$$B_h \approx \frac{1 - \frac{1}{2} p_h^{\,2}}{p_h^{\,2}\left\{ \alpha h - j\eta \left[\left(\frac{\omega_c}{\omega}\right)^2 - \frac{1}{2}\right] \right\}} \tag{59}$$

Since it is known that magnetrons operate close to synchronism between the electron velocity at the edge of the space charge (y = h) and the phase velocity of the circuit wave ω/α, we may neglect $\frac{1}{2} p_h^{\,2}$ in comparison with unity and write

$$p_h^{\,2} = \left(\frac{\omega}{\omega_c} + \alpha h\right)^2 \approx \frac{1}{B_h\left\{ \alpha h - j\eta \left[\left(\frac{\omega_c}{\omega}\right)^2 - \frac{1}{2}\right]\right\}} \tag{60}$$

The value p_o at $y = 0$ is of course still subject to the condition (57) only.

Capacitive Boundary at y = d

We shall first discuss the conditions prevailing when the upper boundary is an admittance sheet of pure, capacitive susceptance. Then the admittance seen by the electrons at the edge of the beam (y = h) is also a capacitive susceptance. The normalized quantity B_h (33)-(34) is a positive real number independent of frequency.

For $\alpha = 0$ (i.e., TWO) the frequencies of the natural modes are determined by the equation (45),

$$\frac{\omega - \gamma u}{\omega_c} = \frac{1}{\left[1 + B_h \tanh \gamma h\right]^{1/2}} \tag{61}$$

if the lower boundary (y = 0) is an ideal conductor ($\eta = 0$). Evidently the roots of ω are real and no growing waves can exist.

On the other hand, if the conductivity of the lower boundary is finite ($\eta > 0$) and the frequency dependence of η is neglected, (42) leads to

$$\phi = 1 - \left(\frac{\omega_c}{\omega - \gamma u}\right)^2$$

$$= \frac{j}{2\eta s^2}\left\{ (1 + ja) \pm \sqrt{(1 + ja)^2 - 4\, jas^2} \right\} \qquad (62)$$

where

$$a = s\eta B_h \,. \qquad (63)$$

If a is a first-order small quantity and the minus sign is chosen, a second-order approximation of one root is

$$\phi = \frac{j}{\eta s^2}\left\{ja\, s^2 + a^2 s^2 (1 - s^2)\right\} \qquad (64)$$

$$= -\frac{a}{\eta} + \frac{j\, a^2 (1 - s^2)}{\eta}, \qquad (65)$$

and

$$\frac{1}{p^2} = \left(\frac{\omega - \gamma u}{\omega_c}\right)^{-2} = 1 + sB_h - j\eta\, s^2 B_h^{\,2} (1 - s^2), \qquad (66)$$

$$\omega = \gamma u \pm \frac{\omega_c}{\left\{1 + sB_h - j\eta\, s^2 B_h^{\,2} (1 - s^2)\right\}^{1/2}} \qquad (67)$$

Here the minus sign places the second term in the third quadrant, and the corresponding value of ω represents a growing wave with slightly less than synchronous frequency. The second root of ϕ must be discarded, since the result does not approach the correct limit as η approaches zero.

The result (67) will of course not be quantitatively correct when a is not small. The roots will always be complex, however, and placed in opposite quadrants, except in the limits $a = 0$, $a = \infty$, and $s = 1$; one root will thus in all practical cases give a growing wave.

Since the beam velocity u occurs only in the first term of (67), the frequency of oscillation will vary linearly with this velocity, while the rate of growth is independent of u. The oscillation thus appears to be perfectly "voltage-tunable," since u is proportional to the square root of the beam voltage.

Usually there is a whole set of discrete wave numbers γ possible. The question of "mode selection" is then important. With which one of these values is the oscillation most likely to take place?

The exponent of growth is the negative imaginary component of (67); for small η it is a monotonic, increasing function of η, which is inversely proportional to γ (31).

For given B_d also B_h is a function of γ (35). So is $s = \tanh \gamma h$. When $\tanh \gamma h$ and $\tanh \gamma (d-h)$ are small the exponent of growth increases rapidly with γ, so that the higher space harmonics are favored. When the hyperbolic tangents approach unity, on the other hand, this exponent is a decreasing function of γ, favoring the lower space harmonics. In the latter case, however, (67) is not very accurate; the second term is no longer small and it is not a good approximation to consider η independent of frequency.

It is understood, of course, that in most practical structures the variation of B_d with γ is not as simple as assumed in (36). The above discussion is an illustration only.

If the skin-effect inductance of the lower boundary ($y = 0$) is taken into account, the roots of (62) will be somewhat modified; there will still be two complex eigen values of ω, however, one of which will represent a growing wave.

If the upper boundary is a lossy capacitance, we have

$$B_h = \frac{\tanh \gamma (d-h) + B_d + \frac{\gamma G_d}{j\omega \epsilon_o}}{1 + \tanh \gamma (d-h) \left\{ B_d + \frac{\gamma G_d}{j\omega \epsilon_o} \right\}} \tag{68}$$

or in shorter notation

$$B_h = \frac{g + B_d - j \frac{\omega_g}{\omega}}{1 + g \left(B_d - j \frac{\omega_g}{\omega} \right)}$$

$$= \frac{(g + B_d)\ (\gamma u + \omega_c p) - j\omega_g}{g\left[\left(\frac{1}{g} + B_d\right)\ (\gamma u + \omega_c p)\ -\ j\omega_g\right]}, \tag{69}$$

and if $\eta = 0$

$$p^2 = \frac{1}{1 + s\ \dfrac{(g + B_d)\ (\gamma u + \omega_c p) - j\omega_g}{g\left[\left(\frac{1}{g} + B_d\right)\ (\gamma u + \omega_c p) - j\omega_g\right]}}$$

$$= \frac{A(1 + kp) - j\omega_g}{B(1 + kp) - j\beta\omega_g} \tag{70}$$

where

$$A = \left(\frac{1}{g} + B_d\right) \gamma u \tag{71}$$

$$B = \left\{\left(\frac{1}{g} + B_d\right) + s\left(1 + \frac{B_d}{g}\right)\right\} \gamma u \tag{72}$$

$$\beta = 1 + \frac{s}{g} \tag{73}$$

$$k = \frac{\omega_c}{\gamma u}\ . \tag{74}$$

The resulting characteristic equation is

$$p^3 + p^2 \frac{B - j\beta\omega_g}{kB} - p\frac{A}{B} - \frac{A}{kB} + \frac{j\omega_g}{kB} = 0 . \tag{75}$$

For comparison we set $\omega_g = 0$ and note that then

$$p^3 + p^2 \frac{1}{kB} - p\frac{A}{B} - \frac{A}{kB} = \left(p^2 - \frac{A}{B}\right)\left(p + \frac{1}{k}\right) = 0 . \tag{76}$$

The last root $p = -1/k$ is not a root of the original equation (70) for $\omega_g = 0$. Consequently we examine the roots of (75) that correspond to $p = \pm (A/B)^{1/2}$ for $\omega_g = 0$ and discard the one that corresponds to $-1/k$.

When ω_g and the imaginary components $-\delta_1$ and $-\delta_2$ of the first two roots are small it is found that

$$-\delta_1 \cong \frac{\beta\omega_g C}{2B\ (Ck + 1)} \tag{77}$$

$$-\delta_2 \cong \frac{\beta\omega_g C}{2B\ (Ck - 1)} \tag{78}$$

where

$$C^2 = A/B . \tag{79}$$

Since g and s are smaller than or equal to one

$$\frac{1}{1 + B_d} < C^2 < 1 . \tag{80}$$

The ratio k of the cyclotron or plasma frequency to the synchronous frequency γu can assume a wide range of values. When Ck is smaller than one a positive value of δ_2 and a growing wave are possible. The exponent of growth is particularly large close to the critical point $Ck = 1$. The mode selection in this case favors the wave numbers γ that approach the critical value γ_c which makes Ck = 1.

When γ = 0, i.e., for a magnetron, the characteristic equation was found to be

$$p_n^2 = \left(\frac{\omega}{\omega_c} + \alpha h\right)^2 \approx \frac{1}{\alpha h\ B_h}\ , \tag{81}$$

if the lower boundary were a perfect conductor (η = 0). Evidently no complex roots and growing waves can exist as long as B_h is real, positive, and independent of frequency.

If $\eta > 0$ we have

$$p_h^2 = \left(\frac{\omega}{\omega_c} + \alpha h\right)^2 \approx \frac{1}{\alpha h B_h \left\{1 - \frac{j\eta}{\alpha h}\left[\left(\frac{\omega_c}{\omega}\right)^2 - \frac{1}{2}\right]\right\}}\ , \tag{82}$$

It must be remembered that this equation is valid for small p_h only; since αh is in general smaller than one, B_h must be large. Otherwise more accurate solutions of (55) and (56) must be used.

If small roots of p_h exist, their approximate values are

$$p_h \approx \pm\left[\frac{1}{\alpha h\ B_h}\left\{1 + \frac{j\eta\ B_h}{(\alpha h)^2}\left(1 - \frac{1}{2}\,(\alpha h)^2\right)\right\}\right]^{1/2}, \tag{83}$$

one of which lies in the first quadrant, the other in the third quadrant. The latter represents a growing wave of the following radian frequency

$$\omega = \omega_c\left\{-\alpha h - \left[\frac{1}{\alpha h\ B_h}\left\{1 + \frac{j\eta\ B_h}{(\alpha h)^2}\left(1 - \frac{1}{2}\,(\alpha h)^2\right)\right\}\right]^{1/2}\right. . \tag{84}$$

Note that for synchronous or nearly synchronous waves α is numerically negative. Only the first term is affected by this sign since both η and B_h are normalized in such a way that they change sign with α [see (32) and (33)] .

Also here the real component of the natural frequency is slightly smaller than the synchronous frequency $\omega_s = - \alpha h \, \omega_c$, and for a given magnetic field the thickness h of the beam and the radian wave number α determine the frequency of oscillation.

Since under these conditions h depends only on the d-c potential difference between the planes y = d and y = 0, the oscillation is again "voltage-tunable."

The mode selection when the radian wave number α can have a number of discrete values can be discussed in the same way as for the traveling-wave oscillator. Because of the limited range of validity of the approximate solution given here, we can only consider large values of B_h; i.e., small values of tanh $\alpha(d-h)$. Under such conditions the exponent of growth is a rapidly decreasing function of α, indicating that the oscillation is most likely to occur with the lowest or fundamental wave number.

When no small roots of p_h exist it may be inferred from experience that no spontaneous oscillations are possible.

Also in this case it is interesting to investigate the effect of losses in the upper capacitive boundary. From (81) and a revised (69) we obtain

$$p^2 = \frac{\left\{\left(\frac{1}{g} + B_d\right)(\omega_c p - \alpha h) - j\omega_g\right\}}{\frac{\alpha h}{g}\left\{(g + B_d)(\omega_c p - \alpha h) - j\omega_g\right\}}$$

$$= \frac{A\,(1 + kp) - j\omega_g}{B\,(1 + kp) - j\beta\omega_g}\,, \tag{85}$$

which is exactly the same as the final form of (70), although the constants (defined so as to be positive numbers) have a different interpretation:

$$A = \alpha h \left(\frac{1}{g} + B_d \right) , \tag{86}$$

$$B = \frac{(\alpha h)^2}{g} (g + B_d) , \tag{87}$$

$$\beta = \frac{\alpha h}{g} , \tag{88}$$

$$k = - \frac{1}{\alpha h} . \tag{89}$$

The decay constants for the acceptable roots (77), (78) are both positive, if Ck is larger than one. Since in most practical cases $\alpha h < 1$,

$$C^2 k^2 \geq \frac{1}{(\alpha h)^3} > 1 \qquad (\text{if } B_d > 1), \tag{90}$$

and no growing waves can exist.

For very high space harmonics this inequality may not be true; in that case however the right side of (85) is not small, and the approximate theory does not hold. The analysis permits only a qualified conclusion that within the range of the present analysis no growing waves are in evidence in a magnetron with a lossy capacitance as anode boundary, as long as the cathode is a perfect conductor.

The solutions for a lossy capacitance admittance at y = d and $\eta > 0$ have not been computed. Qualitatively, however, it is obvious that they must be interpolated between the solutions of the two cases treated above. The losses in the anode circuit tend to prevent or restrict the oscillations that would be caused by the resistivity of the cathode, if the anode were a pure capacitance.

Inductive Boundary at y = d

When the admittance sheet at y = d acts as a pure inductance L, the dimensionless quantity B_h according to (35) is (for $\alpha = 0$)

$$B_h = \frac{\tanh \gamma (d-h) + \frac{\gamma}{j\omega \epsilon_o} \frac{1}{j\omega L}}{1 + \frac{\gamma}{j\omega \epsilon_o} \frac{1}{j\omega L} \tanh \gamma (d-h)}$$

$$= \frac{1 - g^2 \left(\frac{\omega}{\omega_o}\right)^2}{1 - \left(\frac{\omega}{\omega_o}\right)^2} , \qquad (91)$$

where the following shorter notation has been introduced

$$g = \tanh \gamma (d-h) \qquad (92)$$

$$\omega_o^2 = \frac{\gamma}{\epsilon_o L} \tanh \gamma (d-h) . \qquad (93)$$

In the magnetron case the same notation will be used, γ being replaced by α.

Because of the capacitive nature of the space between y = h and y = d, the admittance at y = h has a series-resonance frequency ω_o and a parallel-resonance frequency ω_o/g.

Let us again begin with the case $\alpha = 0$ (TWO) and perfect conductivity at y = 0 ($\eta = 0$). Then

$$\frac{1}{p^2} = \left(\frac{\omega_c}{\omega - \gamma u}\right)^2 = 1 + s \, \frac{1 - g^2 \left(\frac{\omega}{\omega_o}\right)^2}{1 - \left(\frac{\omega}{\omega_o}\right)^2} \qquad (94)$$

$$p^2 = \left(\frac{\omega - \gamma u}{\omega_c}\right)^2 = \frac{1 - \left(\frac{\omega}{\omega_o}\right)^2}{(1 + sg^2)\left\{\frac{1 + s}{1 + sg^2} - \left(\frac{\omega}{\omega_o}\right)^2\right\}} . \tag{95}$$

Since it is empirically known that traveling-wave tubes operate close to synchronism at least under small-signal conditions, it is expedient to look for solutions for which both members of (95) are very small compared to unity. It is then convenient to write

$$\frac{\omega}{\omega_o} = 1 + \Delta_1 - j\delta_1 \tag{96}$$

$$\frac{\omega}{\omega_c} = \frac{\gamma u}{\omega_c} + \Delta_2 - j\delta_2 \tag{97}$$

and to treat Δ_1 Δ_2 δ_1 δ_2 as small quantities.

The result is

$$\Delta_2 \approx - \frac{\omega_c}{\omega_o} \frac{g}{s(1 - g^2)} , \tag{98}$$

provided this is a small quantity ($\Delta_2^2 << 1$).

$$\delta_2^2 \approx \frac{2g}{s(1 - g^2)}\left\{\frac{\gamma u}{\omega_o} - 1\right\} - \frac{g^2}{s^2(1 - g^2)^2}\left(\frac{\omega_c}{\omega_o}\right)^2 \tag{99}$$

so that two real values of δ_2 exist, one positive and one negative, if

$$\frac{\gamma u}{\omega_o} - 1 > \frac{g}{s(1 - g^2)}\left(\frac{\omega_c}{\omega_o}\right)^2 . \tag{100}$$

If this inequality is satisfied, two normal modes exist with a phase velocity very close to the electron velocity u and a real frequency component

$$\omega_r = \gamma u - \frac{\omega_c^2}{\omega_o} \frac{g}{s(1 - g^2)} \tag{101}$$

one of them growing and the other decaying with time.

The right member of (100) is always positive; consequently the beam velocity u must be large enough for the synchronous frequency γu to exceed the series-resonance frequency ω_o at the edge of the electron stream $(y = h)$, in order to satisfy the necessary conditions for the existence of a growing wave.

As far as mode selection is concerned, it can be seen that for small values of s and g the right side (100) is nearly independent of γ while the left side grows very rapidly with γ, so that the high space harmonics are favored. When the factor $(1 - g^2)$ is small, however, also the right side grows rapidly with γ and the answer is more doubtful.

Since (95) is an equation of the fourth degree in ω, two more natural modes must exist. Their frequencies are far from the synchronous frequency γ_u, however, and from physical considerations they must be expected to show neither growth nor decay with time.

For finite conductivity of the lower boundary ($\eta > 0$), an explicit determination of Δ_2 and δ_2 becomes difficult. For a qualitative discussion, however, these quantities can be written in an implicit form that can be compared with (98)-(99). It is then possible to calculate a necessary condition for achieving a specified rate of growth δ_2 rather than just $\delta_2 > 0$.

If $\phi \approx - 1/p^2$ the equation (41) can be reduced to

$$(\Delta_2 - j\delta_2)^2 \approx \frac{1 + \dfrac{j\eta\, s}{(\Delta_2 - j\delta_2)^2}}{1 + \dfrac{j\eta}{s(\Delta_2 - j\delta_2)^2}} \; a(\Delta_1 - j\delta_1) \tag{102}$$

where

$$a(\Delta_1 - j\delta_1) = \frac{1}{sB_h} \tag{103}$$

$$a = \frac{2g}{s(1 - g^2)} \quad . \tag{104}$$

In order to find out how a finite η tends to change the previously determined solution for $\eta = 0$, let us arbitrarily choose η so small that approximate methods can be applied. Since $s < 1$, (102) becomes

$$(\Delta_2 - j\delta_2)^2 = - a' (\Delta_1 - j\delta_1) (1 - j\beta) \tag{105}$$

where a' is slightly different from a and where β is a small positive quantity monotonically increasing with η .

The relative frequency increment Δ_2 of the two relevant modes is found to be

$$\Delta_2 = - \frac{a'}{2} \frac{\omega_c}{\omega_o} \frac{\delta_2 + \beta b \dfrac{\omega_o}{\omega_c}}{\delta_2 + \beta \dfrac{a'\omega_c}{2\omega_o}} = - \frac{a'}{2} \frac{\omega_c}{\omega_o} F(\beta) \tag{106}$$

where

$$b = \frac{\gamma u}{\omega_o} - 1 \; . \tag{107}$$

If the condition (100) is satisfied, the second term in the numerator is larger than the second term in the denominator, so that the absolute value of Δ_2 increases with β.

In the shorter notation presently used the relation (99) above has the following form

$$\delta_2^{\,2} \approx a\left\{b - \frac{a}{4}\left(\frac{\omega_c}{\omega}\right)^2\right\}. \qquad (\eta = 0) \tag{108}$$

The corresponding equation when $\eta > 0$ is

$$\delta_2^{\,2} \approx a'\left\{b - \frac{a'}{4}\left(\frac{\omega_c}{\omega_0}\right)^2 \left[1 - (F(\beta) - 1)^2\right] - \beta\,\frac{\omega_c}{\omega_0}\,\delta_2\right\}. \tag{109}$$

If (108) is used as a first approximation to calculate $F(\beta)$ under the assumption that the second terms are at least one order of magnitude smaller than δ_2, the following expression is obtained:

$$\delta_2^{\,2} \approx a'\left\{b - \frac{a'}{4}\left(\frac{\omega_c}{\omega_0}\right)^2 \left[1 - \left(\beta\,\frac{\delta_2}{a}\right)^2\right] - \beta\,\frac{\omega_c}{\omega_0}\,\delta_2\right\}. \tag{110}$$

The main difference between (108) and (110) is the addition of the last term. Consequently, it takes a larger b or a higher beam velocity u to produce a specified rate of growth δ_2 when $\eta > 0$.

In the magnetron case ($\gamma = 0$) an inductive upper boundary and a zero-impedance lower boundary lead to a characteristic equation

$$p_h^{\,2} = (\Delta_2 - j\delta_2)^2 \approx \frac{1}{\alpha h\, B_h} = \frac{1 - \left(\frac{\omega}{\omega_0}\right)^2}{\alpha h\left[1 - g^2\left(\frac{\omega}{\omega_0}\right)^2\right]}$$

$$\approx -\,\frac{2\,(\Delta_1 - j\delta_1)}{\alpha h\,(1 - g^2)}\,. \tag{111}$$

Just as in the traveling-wave oscillator case the result can be written

$$\Delta_2 = -\,\frac{a}{2}\left(\frac{\omega_c}{\omega}\right) \tag{112}$$

$$\delta_2^{\,2} = ab - \frac{a^2}{4}\left(\frac{\omega_c}{\omega_0}\right)^2 \tag{113}$$

although the quantities a and b now are defined differently:

$$a = \frac{2g}{\alpha h\ (1 - g^2)} \tag{114}$$

$$b = -\alpha h - 1 \,. \tag{115}$$

A necessary condition for the existence of a real positive root of δ_2 and a corresponding growing wave is

$$b > \frac{a}{2}\left(\frac{\omega_c}{\omega_o}\right)^2 \tag{116}$$

or

$$-\alpha h - 1 > \left(\frac{\omega_c}{\omega_o}\right)^2 \frac{g}{\alpha h\ (1 - g^2)} \,. \tag{117}$$

It should be remembered that α is numerically negative, while g/α is always positive.

The discussion of mode selection is the same as for the traveling-wave oscillator.

When $\eta > 0$ the characteristic equation is

$$p_h^{\ 2} = \left(\frac{\omega}{\omega_c} + \alpha h\right)^2 = \frac{1 - \left(\frac{\omega}{\omega_o}\right)^2}{\left[1 - g^2\left(\frac{\omega}{\omega_o}\right)^2\right]\left\{\alpha h - j\eta\left[\left(\frac{\omega_c}{\omega}\right)^2 - \frac{1}{2}\right]\right\}} \,. \tag{118}$$

Again we compute the necessary condition for obtaining a specified positive value of δ_2. The corresponding frequency increment is

$$\Delta_2 \approx -\frac{a}{2}\left(\frac{\omega_c}{\omega_o}\right) \frac{\delta_2 - kb\left(\frac{\omega_o}{\omega_c}\right)}{\delta_2(1 + k^2) - k\,\frac{a}{2}\left(\frac{\omega_c}{\omega_o}\right)}$$

$$= -\frac{a}{2}\left(\frac{\omega_c}{\omega_o}\right) F(k) \tag{119}$$

where

$$k = \frac{\eta}{(\alpha h)^3} \left[1 - \frac{1}{2} (\alpha h)^2\right] . \tag{120}$$

If the second term of numerator and denominator of F(k) are an order of magnitude smaller than δ_2 and if (113) is used as a first approximation in evaluating these small quantities,

$$F(k) \approx 1 - \frac{k}{\delta_2}\left(\frac{\omega}{\omega_c}\right)\left[b - \frac{a}{4}\left(\frac{\omega_c}{\omega_o}\right)^2\right]$$

$$\approx 1 - \frac{k\delta_2}{a}\left(\frac{\omega}{\omega_c}\right) . \tag{121}$$

Then

$$\delta_2^{\,2} \approx a\left\{b - \frac{a}{4}\left(\frac{\omega_c}{\omega_o}\right)^2 \left[1 - (F(k) - 1)^2\right] + k\,\delta_2\left(\frac{\omega_c}{\omega_o}\right)\right\} \tag{122}$$

$$\approx a\left\{b - \frac{a}{4}\left(\frac{\omega_c}{\omega_o}\right)^2 + k\,\delta_2\left(\frac{\omega_c}{\omega_o}\right)\right\} . \tag{123}$$

In this case the required values of αh and of the beam voltage are somewhat lower when $\eta > 0$ ($k > 0$) than when $\eta = 0$ for the same specified rate of growth δ_2.

The corresponding calculations for a lossy inductive boundary have not been carried out but are not likely to offer any essential new results.

Resistive Boundary at y = h

If the upper boundary is moved down so that d = h, we can draw some general conclusions about the existence of spontaneous oscillations but it is difficult to get explicit solutions of the characteristic equation for the oscillations.

When $\alpha = 0$ and $\eta = 0$ we have from (45)

$$\left(\frac{\omega - \gamma u}{\omega_c} \right)^2 = \frac{1}{1 + \frac{\gamma s}{j\omega \epsilon_o R_h}} , \qquad (124)$$

where R_h is the resistance of the boundary sheet. This relation can be written

$$p^2 = \frac{p + ps}{p + ps - ja} \qquad (125)$$

or

$$p^3 + (ps - ja) p^2 - p - ps = 0 . \qquad (126)$$

By inspection it is clear that at least two roots are complex or imaginary, since the sum of all three roots is complex and their product real and positive. Then at least one root must lie in the third or fourth quadrant, i.e. have a positive rate of growth δ_2. We have here an oscillator corresponding to the resistance-wall amplifier of Birdsall et al.

When $\eta > 0$ and p is not small the solution becomes rather complicated and would not be likely to contribute to the understanding of the general problem under study. As long as η is small it can be expected to affect the result only under marginal conditions, i.e. when the right member of (124) differs very little from a real positive number.

In the magnetron case ($\gamma = 0$) the solution (60) is valid only for small p_h. It is therefore possible to study the influence of a resistive boundary only for very low resistivity so that

$$p_h^2 = \left(\frac{\omega}{\omega_c} + \alpha h\right)^2 \approx \frac{1}{\frac{\alpha^2 \, sh}{j\omega \;\; {}_oR_h} \left\{1 - \frac{j\eta}{(\alpha h)^3}\left[1 - \frac{1}{2}(\alpha h)^2\right]\right\}} << 1 \, . \quad (127)$$

Then for $\eta = 0$

$$p_h^2 = ja \, (p_h - \alpha h) \, , \quad (128)$$

where

$$a = \frac{\omega_c \, \epsilon_o \, R_h}{\alpha^2 sh} \, , \quad (129)$$

The solution is one growing and one decaying wave:

$$\omega \approx \omega_c \left\{ - \alpha h \pm \sqrt{1/2 \ a \ \alpha h} \ - j\left(\frac{a}{2} \pm \sqrt{1/2 \ a \ \alpha h}\right)\right\} \quad (130)$$

where the quantity $(1/2 \ a \ \alpha h)$ is independent of the sign of α.

From (129) and (130) it is easily seen that the mode selection favors the low wave numbers.

It is tempting to extrapolate the existence of a growing wave beyond the validity of (127), but no assurance that such an extrapolation is justified can be obtained without a study of the mapping of the function $f_o(p)$ and $f_1(p)$ in (26) over the complex p-plane.

Also in the magnetron case the modification of the solution by a small but finite resistivity at the lower boundary $y = 0$ (i.e., $\eta > 0$) is clearly small. The effect of η is to give the quantity a a small positive phase angle. In the solution of the quadratic equation (128) the radical is the more important term, and a small change from its phase angle -45° for $\eta = 0$ is of very little importance.

If the resistive wall is placed at $y = d > h$, the admittance at $y = h$ is a lossy capacitance and the discussion above for such a termination applies.

Discussion of Results

The theoretical investigation presented above studies the modes of oscillation in a reentrant system formed by an ideal laminar stream of electrons partly filling the space between two plane boundaries of specified wave admittances. Comparisons are made between propagation parallel and perpendicular to the constant magnetic field. The particular purpose of the study is to survey the conditions under which modes exist that in the small-signal range show exponential growth of amplitude with time, in the hope that the result provide some better understanding of the limits for spontaneous initiation of oscillations in traveling-wave oscillators and magnetrons.

When at least one of the boundary admittances contain inductance components, the wave guide formed by the admittance walls and the space between them is a propagating structure, and the classical small-signal traveling-wave-amplifier theory applies. More interesting are the cases where this structure is operated under cut-off conditions and attenuates only, in the absence of the beam. The wall admittances are then resistive and capacitive, and the analysis indicates that within certain ranges of the parameters spontaneous oscillations can still occur.

Each root to the characteristic equation obtained for a specific case of boundary admittances leads to a relation between the radian wave numbers α and γ, on the one hand, and the complex frequency on the other hand, with beam velocity and dimensions as parameters. A study of this relation reveals the range of the parameters for which self-excited oscillations can be obtained, the electronic tuning obtainable by varying the parameters, primarily the beam voltage, and the mode selection rules when a set of different wave numbers are permitted.

In this report only those natural modes have been considered for which either α or γ is zero, because a comparison of these modes appears particularly instructive. Actually any cavity, limited in all three dimensions, with or without a stream of electrons, has an infinite number of natural modes with non-vanishing values of both α and γ. In the presence of an electron stream and under appropriate conditions of approximate synchronism between electron and phase velocity in one dimension natural modes may exist that are growing waves. A closer study may reveal that the complex frequencies of these modes differ very little from each other, so that an oscillator may switch from one mode to another for very small variations of the operating conditions. In many space-harmonic slow-wave structures, however, modes of this kind are more or less effectively suppressed. This is in general true about traveling-wave tubes but not about magnetrons.

The significance of the lower boundary ($y = 0$) and of its admittance is apparent in a magnetron, where the cathode surface forms a boundary which may have considerable wave resistance and, in case of a helical shape, also appreciable wave reactance. When a potential minimum exists, it is a rather complicated problem to state the boundary conditions realistically. The analysis shows definitely, however, that even for very small perturbations the space charge is by no means a perfect shield around the cathode, and that the finite cathode admittance has an appreciable effect on the natural modes of oscillation of the system.

The wide-range operation of voltage-tunable magnetrons may possibly be explained on the basis of this analysis. It is too early to present such a detailed interpretation here; only a general outline may be in order. When the anode admittance is large, the resistivity of the cathode may provide the necessary condition for the existence of growing waves even at frequencies where the anode is primarily capacitive, thus making the oscillation only to a minor extent dependent on the anode circuit. The mode selection and the variation of output power with frequency is still likely to be determined by the anode circuit and its matching network, even if the frequency range over which oscillations occur is not to any appreciable degree.

Some attempts have been made to show that viscous losses in the space charge, produced, for instance, by the presence of a small amount of gas, may also create conditions favorable to the existence of growing waves. Such losses could be represented by a constant term in the operator q (See equations (10)-(13)), leading to the same differential equation (21) with a modified variable p and the expected conclusions. However, the operator q appears also in the continuity equation (17), where the added term cannot be justified. The differential equation (21) does consequently not apply, and no safe conclusions can be reached without working out the solution of the modified differential equation with appropriate boundary conditions.

More important than the viscous losses in the space charge itself are the r-f losses due to "backheating," and it is reasonable to assume that the latter can be taken into account as increased resistance losses in the cathode surface.

BIBLIOGRAPHY

1. Bunemann, O., "Generation and Amplification of Waves in Dense Charged Beams under Crossed Fields," Nature V. 165, p. 474, March 1950.

2. MacFarlane, G. C., Hay, H. G., "Wave Propagation in a Slipping Stream of Electrons: Small-Amplitude Theory," Proc. Phys. Soc., London, B LXIII, p. 409, 1950.

3. Brewer, G. R., "The Propagation of Electromagnetic Waves in a Magnetron Space Charge," Technical Report No. 8, Electron Tube Laboratory, Department of Electrical Engineering, University of Michigan, July 1951.

4. Hok, Gunnar, "Dependence of Initial Space-Charge Waves in a Magnetron on Tank-Circuit Impedance," paper read at the Conference on Electron Tube Research, Ottawa, June 1952.

5. Birdsall, C. K., Whinnery, J. R., "Waves in an Electron Stream with General Admittance Walls," Jour. Appl. Phys., Vol. 24, p. 314, March 1953.

6. Hok, Gunnar, "Circuit Considerations for Initiation of Oscillations in Space-Charge Flow through Crossed Fields," paper read at the Conference on Electron Tube Research, Stanford, June 1953.

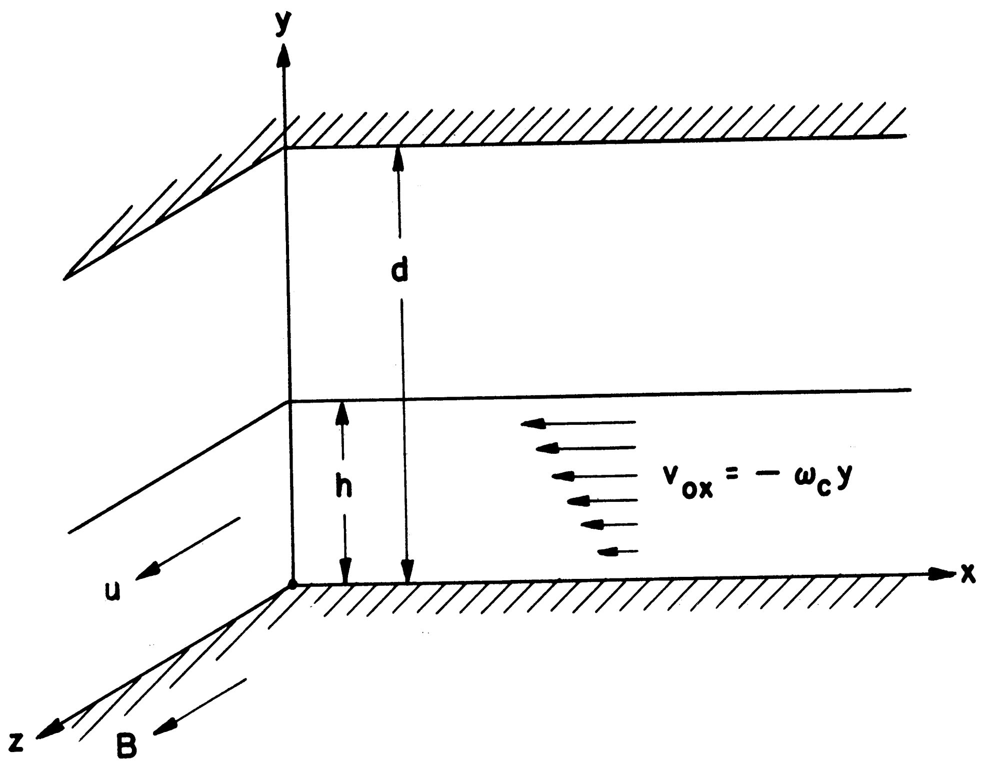

FIG. 1

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