INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2007; **17**:1002–1033
Published online 8 November 2006 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc.1147

Distributed multi-vehicle coordinated control *via* local information exchange

Wei Ren[1,*,†] and Ella Atkins[2]

[1] *Department of Electrical and Computer Engineering, Utah State University, Logan, UT 84322-4120, U.S.A.*
[2] *Department of Aerospace Engineering, University of Michigan, Ann Arbor MI 48109, U.S.A.*

SUMMARY

This paper describes a distributed coordination scheme with local information exchange for multiple vehicle systems. We introduce second-order consensus protocols that take into account motions of the information states and their derivatives, extending first-order protocols from the literature. We also derive necessary and sufficient conditions under which consensus can be reached in the context of unidirectional information exchange topologies. This work takes into account the general case where information flow may be unidirectional due to sensors with limited fields of view or vehicles with directed, power-constrained communication links. Unlike the first-order case, we show that having a (directed) spanning tree is a necessary rather than a sufficient condition for consensus seeking with second-order dynamics. This work focuses on a formal analysis of information exchange topologies that permit second-order consensus. Given its importance to the stability of the coordinated system, an analysis of the consensus term control gains is also presented, specifically the strength of the information states relative to their derivatives. As an illustrative example, consensus protocols are applied to coordinate the movements of multiple mobile robots.

Received 20 November 2005; Accepted 20 March 2006

KEY WORDS: consensus protocols; multi-vehicle systems; coordinated control; graph theory

1. INTRODUCTION

Cooperative control for multiple vehicle systems has been a topic of significant interest in recent years. For cooperative control strategies to be successful, numerous issues must be addressed. Among these issues, the study of shared information in a group of vehicles facilitates the coordination of these vehicles. As a result, a critical problem for cooperative control is to design appropriate protocols and algorithms so that the group of vehicles can converge to a consistent

*Correspondence to: Wei Ren, Department of Electrical and Computer Engineering, Utah State University, Logan, UT 84322-4120, U.S.A.
† E-mail: wren@engineering.usu.edu

view of the shared information in the presence of limited and unreliable information exchange and dynamically changing information exchange topologies.

Convergence to a common value is called the consensus or agreement problem in the literature [1]. Consensus problems have a history in computer science [2] and have recently been studied in the context of cooperative control of multiple vehicle systems [1, 3–8]. Consensus protocols have potential applications in formation control and cooperative timing and search problems for multiple robots, spacecraft, and unmanned air vehicles. For example, information consensus for dynamically evolving information was applied by Ren and Beard [9] to formation flying of multiple space-based interferometers.

One approach to consensus relies on algebraic graph theory, in which graph topologies are connected with the algebraic properties of the corresponding graph matrices. In [4], sufficient conditions are given for consensus of the heading angles of a group of agents under undirected switching information exchange topologies. Olfati-Saber and Murray [1] solved the average consensus problems for a network of integrators using directed graphs. Using directed graphs, Ren *et al.* [7] and Ren and Beard [8] show necessary and/or sufficient conditions for consensus of information under time-invariant and switching information exchange topologies, respectively.

Some other researchers make use of nonlinear mathematical tools to study consensus problems. A set-valued Lyapunov approach is used by Moreau [5] to consider consensus problems with unidirectional time-dependent communication links. Nonlinear contraction theory is used by Slotine and Wang [10] to study synchronization and schooling applications.

Optimality issues related to consensus problems are also studied in the literature. In [11], the fastest distributed linear averaging (FDLA) problem is addressed in the context of consensus-seeking among multiple autonomous agents. Bauso and Pesenti [12] consider distributed consensus protocols that minimize a team objective function.

All the previously mentioned references focus on consensus protocols that take the form of first-order dynamics. In reality, a broad class of vehicles require a second-order dynamic model. For example, some vehicle dynamics can be feedback linearized as double integrators, e.g. holonomic mobile robot dynamic models. In the case of first-order consensus protocols, the final consensus value is a constant. In contrast to the constant final consensus value, it might be proper to derive second-order consensus protocols such that some information states converge to a consistent value (e.g. position of the formation centre) while others converge to another consistent value (e.g. velocity of the formation centre). However, the extension of consensus protocols from first order to second order is non-trivial. In [9, 13–16], formation keeping algorithms taking the form of second-order dynamics are addressed to guarantee attitude alignment, agreement of position deviations and velocities, and/or collision avoidance in a group of vehicles. However, each algorithm mentioned above assumes an undirected information exchange topology. The case of directed information exchange topologies is much more challenging than that of undirected information exchange topologies due to the fact that the adjacency matrix for a directed graph is nonsymmetric.

In this paper, we assume a directed information exchange topology taking into account the general case where information flow may be unidirectional. For example, some vehicles may have transceivers, while other less capable team members only have receivers in heterogeneous teams. Also, vehicles in a team may have non-uniform communication power. In the case of information exchange through local sensing, vehicles may be equipped with sensors that only have a limited field of view, which may result in unidirectional information exchange topologies.

DOI: 10.1002/rnc

The main contributions of this paper are to introduce second-order consensus protocols and derive necessary and/or sufficient conditions under which consensus can be reached in the context of unidirectional information exchange topologies.

In [3, 17], formation stabilization and alignment problems are considered for multiple agents modelled by double integrator or more complicated linear dynamics. In [3] information exchange techniques are studied to improve stability margins and accuracy of vehicle formations. In addition, Roy *et al.* [17] show matrix theoretic conditions under which alignment can be achieved for multiple agents with double integrator dynamics in a general multi-observation setting. In contrast, the present paper applies graph theoretic tools to explore explicit graphical conditions of the information exchange topologies under which consensus can be achieved. This work focuses on analysing whether a given consensus protocol converges while [3, 17] consider whether a feedback gain exists to achieve formation stabilization or alignment.

2. BACKGROUND AND PRELIMINARIES

It is natural to model information exchange between vehicles by directed/undirected graphs. A digraph (directed graph) consists of a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite non-empty set of nodes and $\mathcal{E} \in \mathcal{N}^2$ is a set of ordered pairs of nodes, called edges. As a comparison, the pairs of nodes in an undirected graph are unordered. If there is a directed edge from node v_i to node v_j, then v_i is defined as the parent node and v_j is defined as the child node. A directed path is a sequence of ordered edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots$, where $v_{i_j} \in \mathcal{N}$, in a digraph. An undirected path in an undirected graph is defined analogously. A digraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node has exactly one parent except for one node, called the root, which has no parent, and the root has a directed path to every other node. A (directed) spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. We say that a graph has (or contains) a (directed) spanning tree if there exists a (directed) spanning tree that is a subset of the graph. Note that the condition that a digraph has a (directed) spanning tree is equivalent to the case that there exists at least one node having a directed path to all the other nodes. The union of a group of digraphs is a digraph with nodes given by the union of the node sets and edges given by the union of the edge sets of those digraphs.

The adjacency matrix $A = [a_{ij}]$ of a weighted digraph is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ where $i \neq j$. The adjacency matrix of a weighted undirected graph is defined analogously except that $a_{ij} = a_{ji}$, $\forall i \neq j$, since $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$. Let matrix $L = [\ell_{ij}]$ be defined as $\ell_{ii} = \sum_{j \neq i} a_{ij}$ and $\ell_{ij} = -a_{ij}$, where $i \neq j$. Matrix L satisfies the following conditions:

$$\ell_{ij} \leqslant 0, \quad i \neq j$$

$$\sum_{j=1}^{n} \ell_{ij} = 0, \quad i = 1, \ldots, n$$

For an undirected graph, L is called the Laplacian matrix [18], which has the property that it is symmetric positive semi-definite. However, matrix L for a digraph does not have

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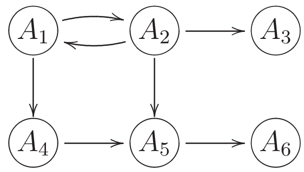

Figure 1. A digraph that has more than one possible (directed) spanning trees, but is not strongly connected.

this property. As an example of an adjacency matrix for a weighted digraph, the matrix

$$A = \begin{bmatrix} 0 & 1.5 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.1 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.2 & 0 \end{bmatrix}$$

can be a valid adjacency matrix corresponding to the digraph in Figure 1. Correspondingly, matrix L is defined as

$$L = \begin{bmatrix} 1.5 & -1.5 & 0 & 0 & 0 & 0 \\ -0.7 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & -1.1 & 1.1 & 0 & 0 & 0 \\ -0.8 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & -0.2 & 0 & -0.3 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -1.2 & 1.2 \end{bmatrix}$$

Let $\mathcal{I} = \{1, 2, \dots, n\}$. Let $\mathbf{1}$ and $\mathbf{0}$ denote the $n \times 1$ column vector of all ones and all zeros, respectively. Let I_n denote the $n \times n$ identity matrix and $0_{m\times n}$ denote the $m \times n$ matrix with all zero entries. Let $M_n(\mathbb{R})$ represent the set of all $n \times n$ real matrices. Given a matrix $A = [a_{ij}] \in M_n(\mathbb{R})$, the digraph of A, denoted by $\Gamma(A)$, is the digraph on n nodes v_i, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(A)$ from v_j to v_i if and only if $a_{ij} \neq 0$ (cf. [19]).

3. CONSENSUS PROTOCOLS

For systems modelled by

$$\dot{\xi}_i = u_i \tag{1}$$

where $\xi_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$, a first-order consensus protocol is proposed by Olfati-Saber and Murray [1], Jadbabaie *et al.* [4], Lin *et al.* [6], and Ren *et al.* [7] as

$$u_i = -\sum_{j=1}^{n} g_{ij} k_{ij} (\xi_i - \xi_j), \quad i \in \mathcal{I} \tag{2}$$

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where $k_{ij} > 0$ is uniformly bounded, $g_{ii} \triangleq 0$, and $g_{ij} = 1$ if information flows from vehicle j to vehicle i and 0 otherwise, $\forall i \neq j$. The adjacency matrix A of the information exchange topology is defined accordingly as $a_{ii} = 0$ and $a_{ij} = g_{ij}k_{ij}$, $\forall i \neq j$. Note that k_{ij} denotes the weight on information link (v_j, v_i).

By applying protocol (1) Equation (1) can be written in matrix form as

$$\dot{\xi} = -L\xi$$

where $\xi = [\xi_1, \ldots, \xi_n]^T$, and $L = [\ell_{ij}]$ is given as $\ell_{ii} = \sum_{j \neq i} g_{ij}k_{ij}$ and $\ell_{ij} = -g_{ij}k_{ij}$, $\forall i \neq j$.

The final consensus value using Equation (2) is given by $\xi^* = \sum_{i=1}^{n} \alpha_i \xi_i(0)$, where $\alpha = [\alpha_1, \ldots, \alpha_n]^T$ is a non-negative left eigenvector of $-L$ associated with eigenvalue 0 with $\alpha_i \geqslant 0$ and $\sum_{i=1}^{n} \alpha_i = 1$ [7].

Taking into account the second-order vehicle dynamics modelled by

$$\dot{\xi}_i = \zeta_i$$

$$\dot{\zeta}_i = u_i \tag{3}$$

where $\xi_i \in \mathbb{R}$, $\zeta_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$, we propose the following second-order consensus protocol:

$$u_i = -\sum_{j=1}^{n} g_{ij}k_{ij}[(\xi_i - \xi_j) + \gamma(\zeta_i - \zeta_j)], \quad i \in \mathcal{I} \tag{4}$$

where $k_{ij} > 0$ and $\gamma > 0$ are uniformly bounded, $g_{ii} \triangleq 0$, and $g_{ij} = 1$ if information flows from vehicle j to vehicle i and 0 otherwise, $\forall i \neq j$. Note that k_{ij} denotes the weight on information link (v_j, v_i) and γ denotes a scaling factor.

Note that consensus protocols (2) and (4) are distributed in the sense that each vehicle only needs information from its (possibly time-varying) local neighbours. The goal of consensus protocol (4) is to guarantee that $|\xi_i - \xi_j| \to 0$ and $|\zeta_i - \zeta_j| \to 0$ as $t \to \infty$. In the case that ξ_i and ζ_i denote the position and velocity of the ith vehicle, respectively, Equation (4) represents the acceleration of that vehicle.

Let $\xi = [\xi_1, \ldots, \xi_n]^T$ and $\zeta = [\zeta_1, \ldots, \zeta_n]^T$. By applying protocol (4), Equation (3) can be written in matrix form as

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} = \Gamma \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \tag{5}$$

where

$$\Gamma = \begin{bmatrix} 0_{n\times n} & I_n \\ -L & -\gamma L \end{bmatrix}$$

4. CONVERGENCE ANALYSIS UNDER TIME-INVARIANT INFORMATION EXCHANGE TOPOLOGIES

In this section, we focus on analysing consensus protocol (4) under time-invariant information exchange topologies.

Given a block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

it is known that $\det(M) = \det(AD - CB)$ if A and C commute, where $\det(\cdot)$ denotes the determinant of a matrix.

To find the eigenvalues of Γ, we can solve the equation $\det(\lambda I_{2n} - \Gamma) = 0$, where $\det(\lambda I_{2n} - \Gamma)$ is the characteristic polynomial of matrix Γ. Note that

$$\begin{aligned} \det(\lambda I_{2n} - \Gamma) &= \det\left(\begin{bmatrix} \lambda I_n & -I_n \\ L & \lambda I_n + \gamma L \end{bmatrix}\right) \\ &= \det(\lambda^2 I_n + (1 + \gamma\lambda)L) \end{aligned} \tag{6}$$

Also note that

$$\det(\lambda I_n + L) = \prod_{i=1}^{n} (\lambda - \mu_i) \tag{7}$$

where μ_i is the ith eigenvalue of $-L$.

By comparing Equations (6) and (7), we see that

$$\det(\lambda^2 I_n + (1 + \gamma\lambda)L) = \prod_{i=1}^{n} (\lambda^2 - (1 + \gamma\lambda)\mu_i)$$

which implies that the roots of Equation (6) can be obtained by solving $\lambda^2 = (1 + \gamma\lambda)\mu_i$. Therefore, it is straightforward to see that the eigenvalues of Γ are given by

$$\lambda_{i\pm} = \frac{\gamma\mu_i \pm \sqrt{\gamma^2\mu_i^2 + 4\mu_i}}{2} \tag{8}$$

where λ_{i+} and λ_{i-} are called eigenvalues of Γ that are associated with μ_i.

From Equation (8), we can see that Γ has $2m$ zero eigenvalues if and only if $-L$ has m zero eigenvalues. It is straightforward to see that $-L$ has at least one zero eigenvalue with an associated eigenvector $\mathbf{1}$ since all its row sums are equal to zero. Therefore, we know that Γ has at least two zero eigenvalues. Without loss of generality, we let $\lambda_{1+} = \lambda_{1-} = 0$. In addition, noting that $-L$ is diagonally dominant and has non-positive diagonal elements, we know that all non-zero eigenvalues of $-L$ have negative real parts from the Gersgorin disc theorem [19].

We have the following lemma regarding a necessary and sufficient condition for information consensus using consensus protocol (4).

Lemma 4.1
Consensus protocol (4) achieves consensus asymptotically if and only if matrix Γ has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. Specifically, $\xi \to \mathbf{1}p^{\mathrm{T}}\xi(0) + t\mathbf{1}p^{\mathrm{T}}\zeta(0)$ and $\zeta \to \mathbf{1}p^{\mathrm{T}}\zeta(0)$, for large t, where $p \in \mathbb{R}^n$ is a non-negative left eigenvector of $-L$ associated with eigenvalue 0 and $p^{\mathrm{T}}\mathbf{1} = 1$.

Proof (Sufficiency)
We first show that eigenvalue zero of Γ has geometric multiplicity equal to one in the case that Γ has exactly two zero eigenvalues. Letting $q = [q_a^{\mathrm{T}}, q_b^{\mathrm{T}}]^{\mathrm{T}}$, where $q_a, q_b \in \mathbb{R}^n$, be an eigenvector

DOI: 10.1002/rnc

of Γ associated with eigenvalue zero, then we know that

$$\Gamma q = \begin{bmatrix} 0_{n\times n} & I_n \\ -L & -\gamma L \end{bmatrix} \begin{bmatrix} q_a \\ q_b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

which implies that $q_b = \mathbf{0}$ and $-Lq_a = \mathbf{0}$. That is, q_a is an eigenvector of $-L$ associated with eigenvalue zero. Since Γ has exactly two zero eigenvalues, we know that $-L$ has exactly one zero eigenvalue. Therefore, we see that $-L$ has only one linearly independent eigenvector q_a associated with eigenvalue zero, which in turn implies that Γ has only one linearly independent eigenvector $q = [q_a^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$ associated with eigenvalue zero. That is, eigenvalue zero of Γ has geometric multiplicity equal to one.

Note that Γ can be written in Jordan canonical form as

$$\begin{aligned} \Gamma &= PJP^{-1} \\ &= [w_1, \ldots, w_{2n}] \begin{bmatrix} 0 & 1 & 0_{1\times(2n-2)} \\ 0 & 0 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & J' \end{bmatrix} \begin{bmatrix} v_1^{\mathrm{T}} \\ \vdots \\ v_{2n}^{\mathrm{T}} \end{bmatrix} \end{aligned} \tag{9}$$

where $w_j \in \mathbb{R}^{2n}$, $j = 1, \ldots, 2n$, can be chosen to be the right eigenvectors or generalized eigenvectors of Γ, $v_j \in \mathbb{R}^{2n}, j = 1, \ldots, 2n$, can be chosen to be the left eigenvectors or generalized eigenvectors of Γ, and J' is the Jordan upper diagonal block matrix corresponding to non-zero eigenvalues λ_{i+} and λ_{i-}, $i = 2, \ldots, n$.

Without loss of generality, we choose $w_1 = [\mathbf{1}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$ and $w_2 = [\mathbf{0}^{\mathrm{T}}, \mathbf{1}^{\mathrm{T}}]^{\mathrm{T}}$, where it can be verified that w_1 and w_2 are an eigenvector and generalized eigenvector of Γ associated with eigenvalue 0, respectively. Noting that Γ has exactly two zero eigenvalues, we know that $-L$ has a simple zero eigenvalue, which in turn implies that there exists a non-negative vector p such that $p^{\mathrm{T}}L = 0$ and $p^{\mathrm{T}}\mathbf{1} = 1$ as shown in Reference [7]. It can be verified that $v_1 = [p^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$ and $v_2 = [\mathbf{0}^{\mathrm{T}}, p^{\mathrm{T}}]^{\mathrm{T}}$ are a generalized left eigenvector and left eigenvector of Γ associated with eigenvalue 0, respectively, where $v_1^{\mathrm{T}} w_1 = 1$ and $v_2^{\mathrm{T}} w_2 = 1$. Noting that eigenvalues λ_{i+} and λ_{i-}, $i = 2, \ldots, n$, have negative real parts, we see that

$$\begin{aligned} \mathrm{e}^{\Gamma t} &= P\mathrm{e}^{Jt}P^{-1} \\ &= P \begin{bmatrix} 1 & t & 0_{1\times(2n-2)} \\ 0 & 1 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & \mathrm{e}^{J't} \end{bmatrix} P^{-1} \end{aligned}$$

which converges to $\begin{bmatrix} \mathbf{1}p^{\mathrm{T}} & t\mathbf{1}p^{\mathrm{T}} \\ 0_{n\times n} & \mathbf{1}p^{\mathrm{T}} \end{bmatrix}$ for large t.

Noting that for large t

$$\begin{bmatrix} \xi(t) \\ \zeta(t) \end{bmatrix} \to \begin{bmatrix} \mathbf{1}p^{\mathrm{T}} & t\mathbf{1}p^{\mathrm{T}} \\ 0_{n\times n} & \mathbf{1}p^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \xi(0) \\ \zeta(0) \end{bmatrix}$$

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we see that $\xi(t) \to \mathbf{1}p^{\mathrm{T}}\xi(0) + t\mathbf{1}p^{\mathrm{T}}\zeta(0)$ and $\zeta(t) \to \mathbf{1}p^{\mathrm{T}}\zeta(0)$ for large t. As a result, we know that $|\xi_i(t) - \xi_j(t)| \to 0$ and $|\zeta_i(t) - \zeta_j(t)| \to 0$ as $t \to \infty$. That is, consensus is achieved for the group of vehicles.

(Necessity) Suppose that the sufficient condition that Γ has exactly two zero eigenvalues and all the other eigenvalues have negative real parts does not hold. Noting that Γ has at least two zero eigenvalues, the fact that the sufficient condition does not hold implies that Γ has either more than two zero eigenvalues or it has two zero eigenvalues and at least one eigenvalue with positive real part. Without loss of generality, assume that $\varsigma_1 = \varsigma_2 = 0$ and $\mathrm{Re}(\varsigma_3) \geqslant 0$, where ς_k, $k = 1, \ldots, 2n$, denotes the kth eigenvalue of Γ and $\mathrm{Re}(\cdot)$ represents the real part of a number. Letting $J = [j_{k\ell}]$ be the Jordan canonical form of Γ, we know that $j_{kk} = \varsigma_k$, $k = 1, \ldots, 2n$. Then we see that $\lim_{t\to\infty} \mathrm{e}^{j_{kk}t} \neq 0$, $k = 1, 2, 3$, which in turn implies that the first three rows of $\lim_{t\to\infty} \mathrm{e}^{Jt}$ are linearly independent. Therefore, we know that the rank of $\lim_{t\to\infty} \mathrm{e}^{Jt}$ is at least three, which implies that the rank of $\lim_{t\to\infty} \mathrm{e}^{\Gamma t}$ is at least three. Note that consensus is reached asymptotically if and only if $\lim_{t\to\infty} \mathrm{e}^{\Gamma t} \to \begin{bmatrix} \mathbf{1}p^{\mathrm{T}} \\ \mathbf{1}q^{\mathrm{T}} \end{bmatrix}$, where p and q are $n \times 1$ vectors. As a result, the rank of $\lim_{t\to\infty} \mathrm{e}^{\Gamma t}$ cannot exceed two. This results in a contradiction. □

If all non-zero eigenvalues of $-L$ are real and therefore negative, it is straightforward to verify that all non-zero eigenvalues of Γ have negative real parts following Equation (8). In the general case, some non-zero eigenvalues of Γ may have positive real parts even if all non-zero eigenvalues of $-L$ have negative real parts as shown in the following examples.

We consider several cases as follows.

Case 1: *Information exchange topology with separated subgroups*. In the case that the information exchange topology has separated subgroups as shown in Figure 2, consensus cannot be achieved for the team of vehicles since the information states from different separated groups do not affect one another. In fact, we also know that $-L$ has at least two zero eigenvalues in this case [7], which in turn implies that Γ has at least four zero eigenvalues.

Hereafter, we assume that $k_{ij} = 1$ and $\gamma = 1$ in Equation (4) unless otherwise indicated. In addition, we let $\xi_i(0) = 0.2(i-1)$ and $\zeta_i(0) = 0.1(i-1)$, $i = 1, \ldots, 4$. Figure 3 shows the evolution of the information states ξ_i and ζ_i, $i = 1, \ldots, 4$, using consensus protocol (4) under the information exchange topology given by Figure 2. Note that A_1 and A_2 reach consensus, and A_3 and A_4 also reach consensus although the whole group cannot reach consensus.

Case 2: *Information exchange topology with multiple leaders*. If a vehicle only has outgoing information links but without incoming information links, we call that vehicle a leader. That is, a leader only transmits information but does not receive any information from other vehicles. In the case that the information exchange topology has multiple leaders as shown in Figure 4,

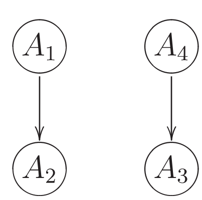

Figure 2. A digraph with separated subgroups.

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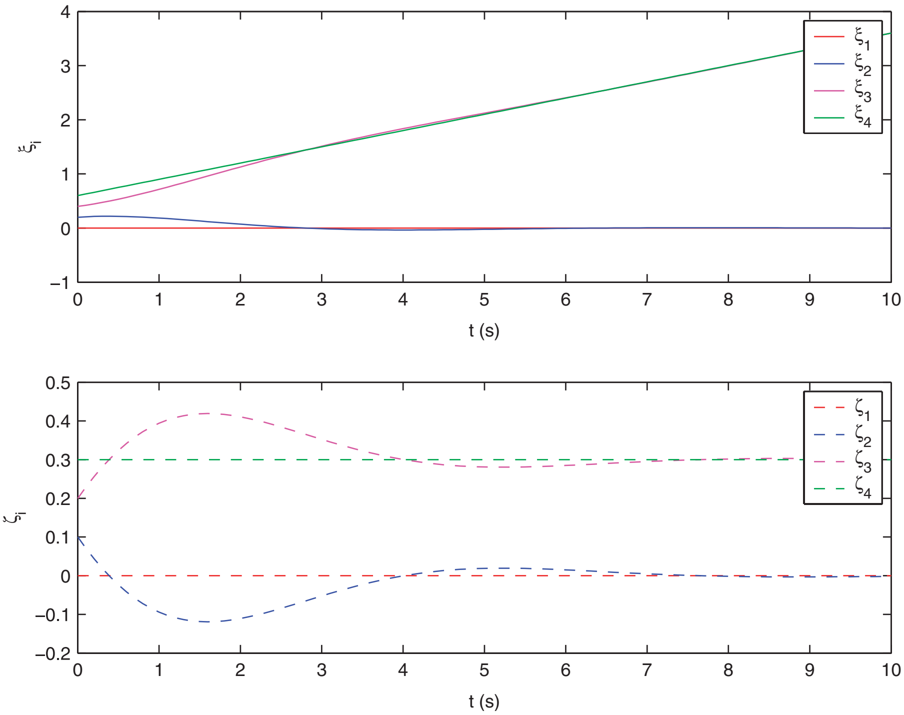

Figure 3. Evolution of the information states under the information exchange topology given by Figure 2.

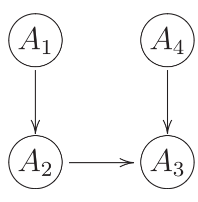

Figure 4. A digraph with multiple leaders.

where both A_1 and A_4 are leaders, consensus cannot be achieved for the team of vehicles since each leader's information state is not affected by any other vehicle's information state in the team. Noting that $-L$ has at least two rows with all zero entries in this case, we know that $-L$ has at least two zero eigenvalues, which in turn implies that Γ has at least four zero eigenvalues.

Figure 5 shows the evolution of the information states ξ_i and ζ_i, $i = 1, \ldots, 4$, using consensus protocol (4) under the information exchange topology given by Figure 4. Note that only A_1 and A_2 reach consensus.

Case 3: *Connected undirected information exchange topology*. If the information exchange topology is undirected as shown in Figure 6, we know that matrix L is symmetric positive semi-definite, which implies that all eigenvalues of L are real. Therefore, all non-zero eigenvalues of Γ have negative real parts.

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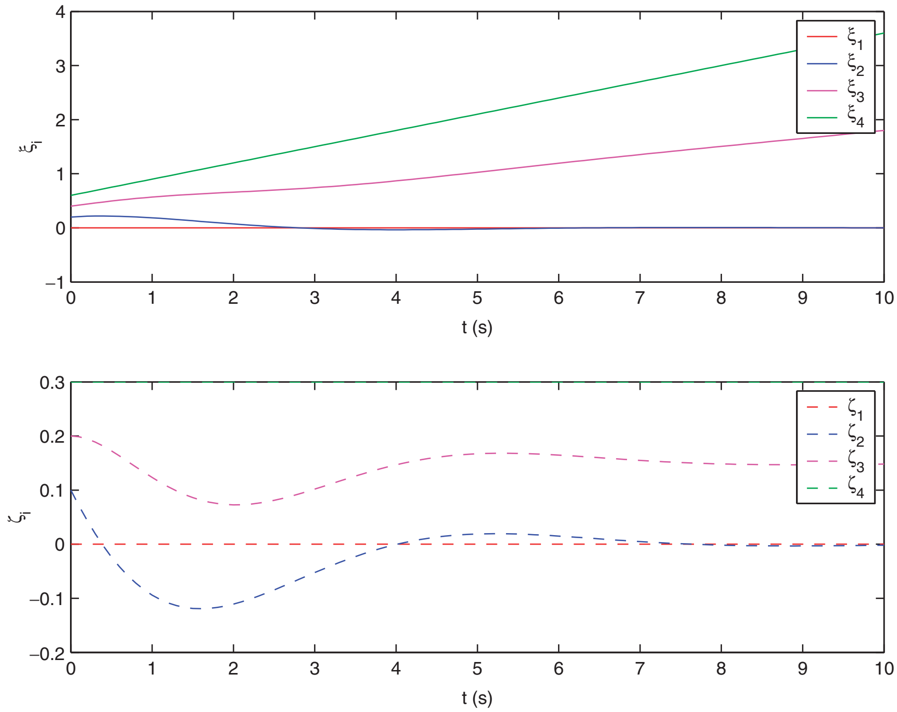

Figure 5. Evolution of the information states under the information exchange topology given by Figure 4.

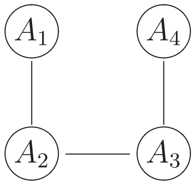

Figure 6. A connected undirected graph.

In the case of undirected graphs, matrix L has a simple zero eigenvalue if and only if the graph is connected. Therefore, we know that consensus is achieved asymptotically if and only if the undirected graph is connected.

Figure 7 shows the evolution of the information states ξ_i and ζ_i, $i = 1, \ldots, 4$, using consensus protocol (4) under the information exchange topology given by Figure 6.

Case 4: *Leader–follower information exchange topology*. In the case that the information exchange topology has a leader–follower structure as shown in Figure 8, it is straightforward to see that L can be written as an upper diagonal matrix by permutation transformations. As a result, we know that zero is a simple eigenvalue of L and all non-zero eigenvalues are real. Therefore, we know that consensus is achieved asymptotically in the case of leader–follower information exchange topologies.

DOI: 10.1002/rnc

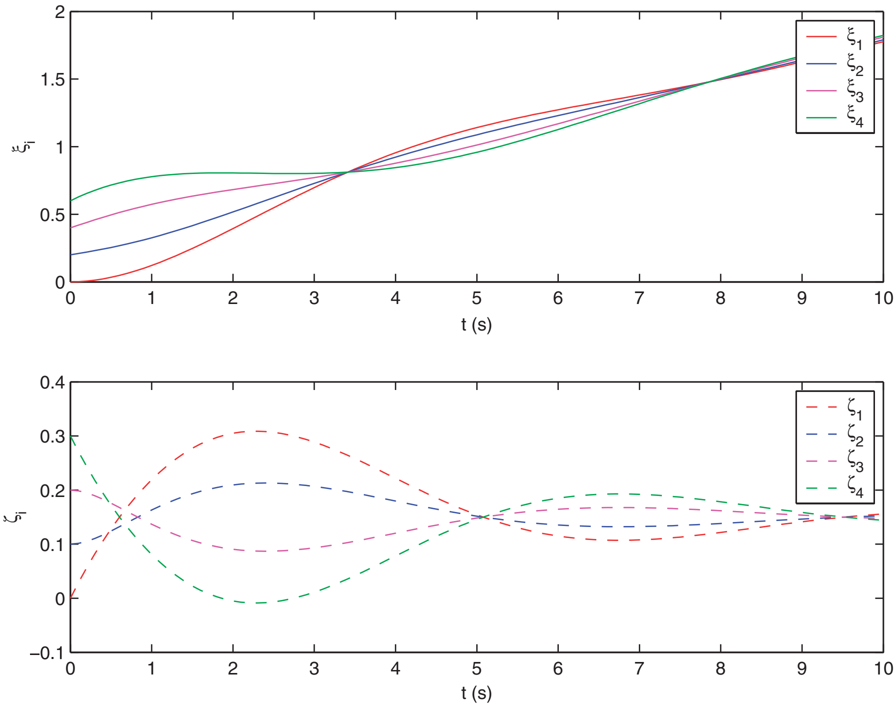

Figure 7. Evolution of the information states under the information exchange topology given by Figure 6.

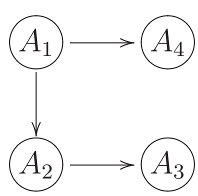

Figure 8. A digraph with a leader–follower topology.

Figure 9 shows the evolution of the information states ξ_i and ζ_i, $i = 1, \ldots, 4$, using consensus protocol (4) under the information exchange topology given by Figure 8.

Case 5: *Information exchange topology with a* (*directed*) *spanning tree*. Note that the connected undirected topology and the leader–follower topology can be thought of as special cases of an information exchange topology with a (directed) spanning tree.

In the case that the information exchange topology has a (directed) spanning tree as shown in Figure 10, consensus may not be achieved as in the case where the consensus protocol is given by Equation (2). However, having a (directed) spanning tree is a necessary condition for information consensus as will be shown below.

Figures 11 and 12 show the evolution of the information states ξ_i and ζ_i, $i = 1, \ldots, 4$, using consensus protocol (4) under the information exchange topology given by Figure 10 with $\gamma = 1$

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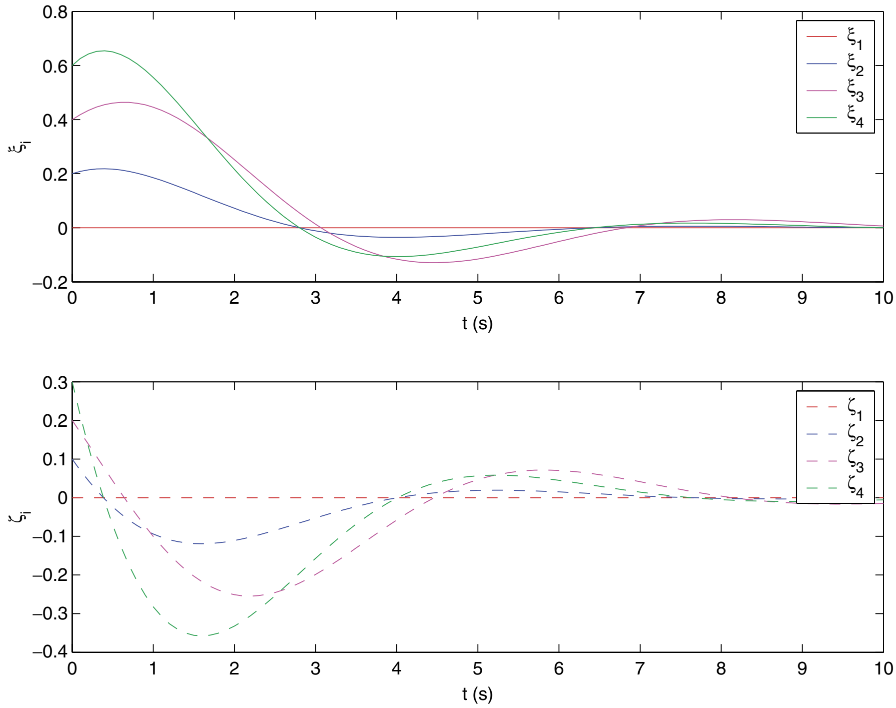

Figure 9. Evolution of the information states under the information exchange topology given by Figure 8.

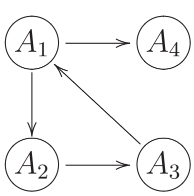

Figure 10. A digraph with a (directed) spanning tree.

and $\gamma = 0.4$, respectively. Note that consensus can be reached for $\gamma = 1$ but cannot be reached for $\gamma = 0.4$. Unlike the previous cases where convergence does not depend on γ, consensus may not generally be reached with small γ given information exchange topologies with a (directed) spanning tree other than those from Cases 3 and 4.

By comparing Figures 8 and 10, we see that more information exchange is involved in Figure 10 than in Figure 8 in the sense that A_3 sends information to A_1 in Figure 10. However, while the consensus protocol converges under the information exchange topology given by Figure 8 for any $\gamma > 0$, the consensus protocol cannot converge under the information exchange topology given by Figure 10 if γ is too small. This is somewhat contradictory to our intuition in the sense that more information exchange may lead to instability for the whole group.

In the special case that all eigenvalues of L are real, we have the following lemma.

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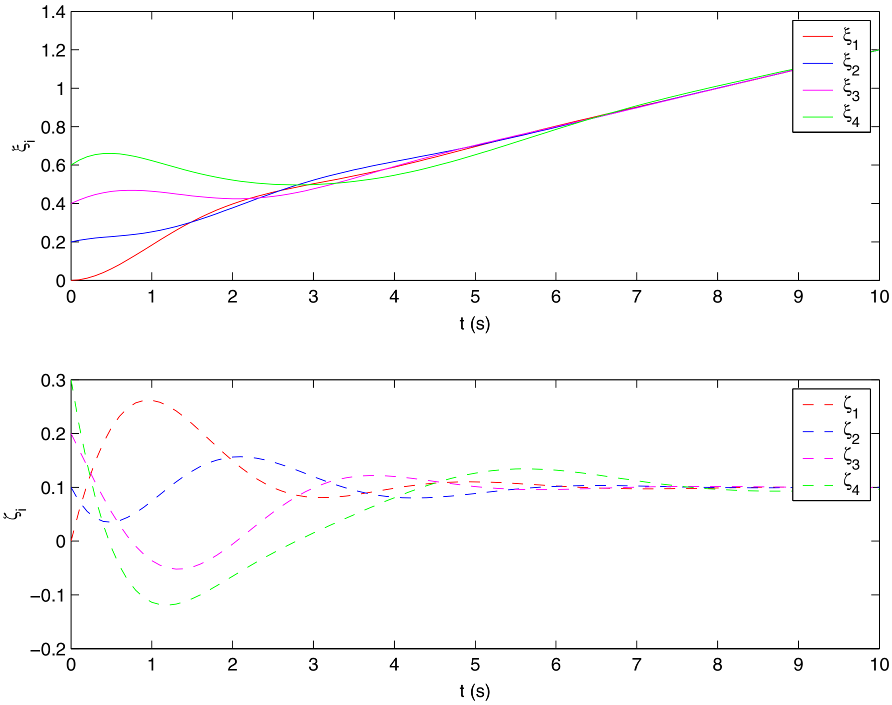

Figure 11. Evolution of the information states under the information exchange topology given by Figure 10 with $\gamma = 1$.

Lemma 4.2
If $-L$ has a simple zero eigenvalue and all the other eigenvalues are real, consensus protocol (4) achieves consensus for any $\gamma > 0$.

To show that having a (directed) spanning tree is a necessary condition for information consensus, we need the following lemma.

Lemma 4.3 (Ren et al. [7])
Matrix L of a directed weighted graph has a simple zero eigenvalue and all the other eigenvalues have positive real parts if and only if the graph has a (directed) spanning tree.

We have the following necessary condition for information consensus.

Theorem 4.1
Consensus protocol (4) achieves consensus asymptotically only if the information exchange topology has a (directed) spanning tree.[‡]

[‡] As a comparison, the first-order consensus protocol (2) achieves consensus asymptotically if and only if the information exchange topology has a (directed) spanning tree [7].

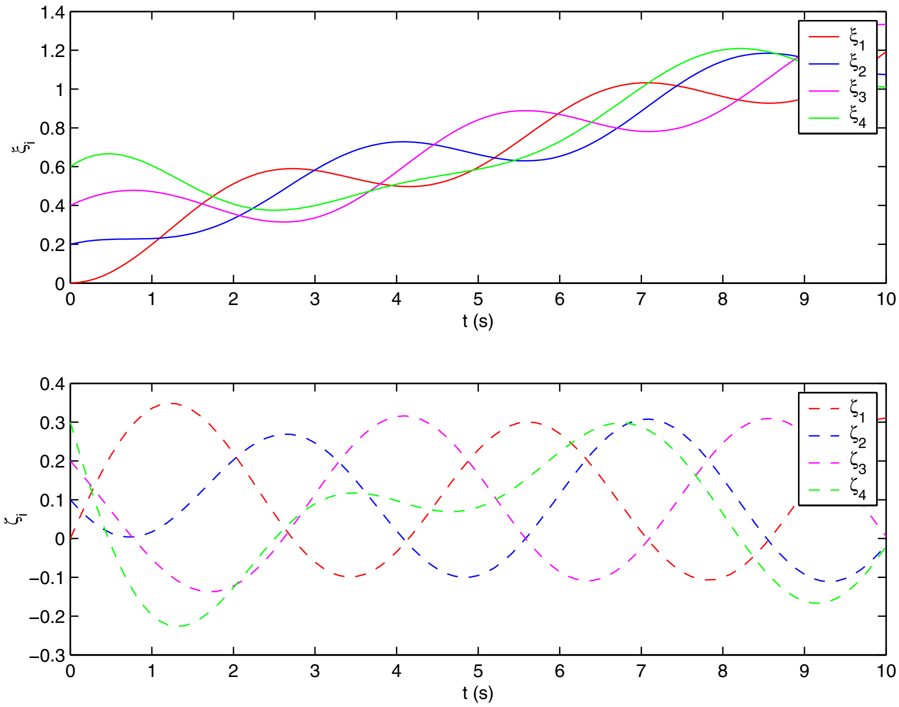

Figure 12. Evolution of the information states under the information exchange topology given by Figure 10 with $\gamma = 0.4$.

Proof

If consensus protocol (4) achieves consensus asymptotically, we know that Γ has exactly two zero eigenvalues following Lemma 4.1. Therefore, we see that matrix L has a simple zero eigenvalue, which in turn implies that the information exchange topology has a (directed) spanning tree following Lemma 4.3. □

Next, we show a sufficient condition for information consensus. To support this condition, we need the following lemma.

Lemma 4.4

Let

$$\rho_{\pm} = \frac{\gamma\mu - \alpha \pm \sqrt{(\gamma\mu - \alpha)^2 + 4\mu}}{2}$$

where $\rho, \mu \in \mathbb{C}$. If $\alpha \geqslant 0$, $\mathrm{Re}(\mu) < 0$, and

$$\gamma > \sqrt{\frac{2}{|\mu| \cos\left(\dfrac{\pi}{2} - \tan^{-1} \dfrac{-\mathrm{Re}(\mu)}{\mathrm{Im}(\mu)}\right)}} \tag{10}$$

then $\mathrm{Re}(\rho_{\pm}) < 0$, where $\mathrm{Re}(\cdot)$ represents the real part of a number.

Proof

Motivated by [16], we use Figure 13 to show the notations used in the proof. Let $a = \gamma\mu$, $b = \sqrt{(\gamma\mu)^2 + 4\mu}$, and $c = \gamma\mu - \alpha$. Also let $s_1 = (\gamma\mu)^2$, $s_2 = 4\mu$, and $s_3 = (\gamma\mu)^2 + 4\mu$. Furthermore we let $q_1 = (\gamma\mu - \alpha)^2$, $q_2 = 4\mu$, and $q_3 = (\gamma\mu - \alpha)^2 + 4\mu$.

In the case of $\alpha = 0$, the proof follows the argument of Theorem 6 in [17]. The basic idea is that inequality (10) guarantees that $|s_1| > |s_3|$, which in turn implies that $|a| > |b|$. Noting that the phase angle of b is smaller than a, we know that $\mathrm{Re}((a \pm b)/2) < 0$.

In the case of $\alpha > 0$, for the triangle composed of vectors s_1, s_2, and s_3, we know that $\eta_1 > \eta_3$ using the law of cosines, where η_i is the inner angle of the triangle that faces edge s_i. Represent a and c in polar coordinates as (r_1, θ_1) and (r_2, θ_2), respectively. We only need to consider μ located in the second quadrant of the complex plane since any μ located in the third quadrant is a complex conjugate of some μ located in the second quadrant. Let $\theta_i \in (\pi/2, \pi]$, where $i = 1, 2$. Then s_1 and q_1 can be represented in polar coordinates as $(r_1^2, 2\theta_1)$ and $(r_2^2, 2\theta_2)$, respectively. Consider another triangle composed of vectors q_1, q_2, and q_3 with inner angles given by ϕ_i, where ϕ_i faces edge q_i. Noting that $\theta_2 > \theta_1$, we know that $2\theta_2 > 2\theta_1$. We can then show that $\phi_1 > \eta_1$ and $\phi_3 < \eta_3$ by noting that $s_2 = q_2$ and comparing the triangles composed of q_i and s_i, respectively, which implies that $\phi_1 > \phi_3$. Using the law of cosines, we show that $|q_1| > |q_3|$. As a result, we know that $\mathrm{Re}(\rho_{\pm}) < 0$ by following an argument similar to that for $\alpha = 0$. □

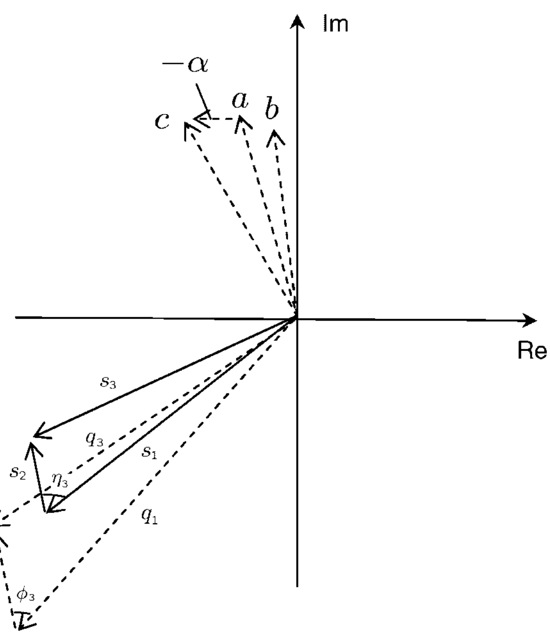

Figure 13. Notations used in the proof for Lemma 4.4.

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Theorem 4.2
Consensus protocol (4) achieves consensus asymptotically if the information exchange topology has a (directed) spanning tree and

$$\gamma > \max_{\mu_i \neq 0} \sqrt{\frac{2}{|\mu_i| \cos\left(\frac{\pi}{2} - \tan^{-1} \frac{-\mathrm{Re}(\mu_i)}{\mathrm{Im}(\mu_i)}\right)}} \tag{11}$$

where μ_i, $i = 1, \ldots, n$, are the eigenvalues of $-L$, and $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ represent the real and imaginary parts of a number, respectively.

Proof
If the information exchange topology has a (directed) spanning tree, we know that $-L$ has a simple zero eigenvalue and all the other eigenvalues have negative real parts from Lemma 4.3. Without loss of generality, we let $\mu_1 = 0$ and $\mathrm{Re}(\mu_i) < 0$, $i = 2, \ldots, n$. Then, we know that Γ has exactly two zero eigenvalues. It is left to show that non-zero eigenvalues of Γ have negative real parts. If inequality (11) is true, we know that $\mathrm{Re}(\lambda_{i\pm}) < 0$, $i = 2, \ldots, n$, following Lemma 4.4, where $\lambda_{i\pm}$ are eigenvalues of Γ associated with μ_i. As a result, we see that consensus can be achieved asymptotically from Lemma 4.1. □

Note that some of the results presented above for fixed information exchange topologies can be recovered using the matrix theoretic framework in [17]. However, in contrast to [17], the main purpose of this paper is to derive graphical conditions, specifically, the minimum connectivity between vehicles, under which consensus is achieved. In particular, this paper extends convergence results presented in [7, 8] for first-order consensus protocols to second-order consensus protocols under both fixed and switching directed information exchange topologies.

We also have the following lemma regarding the final consensus value.

Lemma 4.5
Suppose that Γ has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. If $\zeta_i(0) = 0$, $i \in \mathscr{I}$, then as $t \to \infty$, $\xi_i(t) \to \sum_{i=1}^n p_i \xi_i(0)$ and $\zeta_i(t) \to 0$, where $i \in \mathscr{I}$ and $p = [p_1, \ldots, p_n]^{\mathrm{T}}$ is a non-negative left eigenvector of $-L$ associated with eigenvalue 0 satisfying $p_i \geqslant 0$ and $\sum_{i=1}^n p_i = 1$. In addition, if $\zeta_i(0) = 0$, $i \in \mathscr{I}_L$, where $\mathscr{I}_L$ denotes the set of vehicles that have a directed path to all the other vehicles in the information exchange topology, then $\xi_i(t) \to \sum_{i \in \mathscr{I}_L} p_i \xi_i(0)$ and $\zeta_i(t) \to 0$, $i \in \mathscr{I}$, as $t \to \infty$.

Proof
The first part of the lemma follows directly from the fact that $\xi(t) \to \mathbf{1} p^{\mathrm{T}} \xi(0) + t \mathbf{1} p^{\mathrm{T}} \zeta(0)$ and $\zeta(t) \to \mathbf{1} p^{\mathrm{T}} \zeta(0)$ for large t.

For the second part of the lemma, we note that $p_i \neq 0$ if vehicle i has a directed path to all the other vehicles in the information exchange topology and $p_i = 0$ if there does not exist such a directed path [7]. As a result, we know that $\xi_i(t) \to \sum_{i \in \mathscr{I}_L} p_i \xi_i(0) + t \sum_{i \in \mathscr{I}_L} p_i \zeta_i(0)$ and $\zeta_i(t) \to \sum_{i \in \mathscr{I}_L} p_i \zeta_i(0)$ for large t and the second part of the lemma is proved. □

Note that $\xi \to \mathbf{1} p^{\mathrm{T}} \xi(0) + t \mathbf{1} p^{\mathrm{T}} \zeta(0)$ and $\zeta \to \mathbf{1} p^{\mathrm{T}} \zeta(0)$ for large t with consensus protocol (4). Under some circumstances, it might be desirable that $\xi \to \mathbf{1} q^{\mathrm{T}}$ and $\zeta \to \mathbf{0}$, where q is an $n \times 1$

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vector. For example, in formation stabilization applications, we want each vehicle to agree on their *a priori* unknown fixed formation centre, which has a constant position and zero velocity. In this case, we propose the following second-order consensus protocol:

$$u_i = -\alpha\zeta_i - \sum_{j=1}^{n} g_{ij}k_{ij}[(\xi_i - \xi_j) + \gamma(\zeta_i - \zeta_j)] \tag{12}$$

where $\alpha > 0$.

Equation (12) can be written in matrix form as

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} = \Sigma \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \tag{13}$$

where

$$\Sigma = \begin{bmatrix} 0_{n\times n} & I_n \\ -L & -\alpha I_n - \gamma L \end{bmatrix}$$

Similarly, we can solve the equation $\det(\lambda I_{2n} - \Sigma) = 0$ to find the eigenvalues of Σ. Note that

$$\begin{aligned} \det(\lambda I_{2n} - \Sigma) &= \det(\lambda^2 I_n + \gamma\lambda L + \alpha\lambda I_n + L) \\ &= \det((\lambda^2 + \alpha\lambda)I_n + (1 + \gamma\lambda)L) \end{aligned} \tag{14}$$

By comparing Equations (14) and (7), we see that the roots of Equation (14) can be obtained by solving $\lambda^2 + \alpha\lambda = \mu_i(1 + \gamma\lambda)$. Therefore, it is straightforward to see that the eigenvalues of Σ are given by

$$\rho_{i\pm} = \frac{\gamma\mu_i - \alpha \pm \sqrt{(\gamma\mu_i - \alpha)^2 + 4\mu_i}}{2} \tag{15}$$

where ρ_{i+} and ρ_{i-} are called eigenvalues of Σ that are associated with μ_i.

Unlike Γ, Σ has m zero eigenvalues if and only if $-L$ has m zero eigenvalues. Without loss of generality, we let $\mu_1 = 0$, which implies that $\rho_{1+} = 0$ and $\rho_{1-} = -\alpha$. We have the following lemma regarding a necessary and sufficient condition for information consensus using consensus protocol (12).

Lemma 4.6

Let p be a non-negative left eigenvector of $-L$ associated with eigenvalue 0 and $p^{\mathrm{T}}\mathbf{1} = 1$. With consensus protocol (12), $\xi(t) \to \mathbf{1}p^{\mathrm{T}}\xi(0) + (1/\alpha)\mathbf{1}p^{\mathrm{T}}\zeta(0)$ and $\zeta(t) \to \mathbf{0}$ asymptotically as $t \to \infty$ if and only if matrix Σ has a simple zero eigenvalue and all the other eigenvalues have negative real parts.

Proof (Sufficiency)

The proof follows a similar line to that of Lemma 4.1 by noting that $v_1 = [p^{\mathrm{T}}, (1/\alpha)p^{\mathrm{T}}]^{\mathrm{T}}$ and $w_1 = [\mathbf{1}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$ and

$$J = \begin{bmatrix} 0 & 0_{1\times(2n-1)} \\ 0_{(2n-1)\times 1} & J' \end{bmatrix}$$

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where J' is the Jordan upper diagonal block matrix corresponding to the $2n-1$ eigenvalues that have negative real parts.

(Necessity) If $\xi(t) \to \mathbf{1}p^{\mathrm{T}}\xi(0) + (1/\alpha)\mathbf{1}p^{\mathrm{T}}\zeta(0)$ and $\zeta(t) \to \mathbf{0}$ asymptotically as $t \to \infty$, we know that $\lim_{t\to\infty} P\,\mathrm{e}^{Jt}P^{-1}$ has a rank one, which in turn implies that $\lim_{t\to\infty} \mathrm{e}^{Jt}$ has a rank one. However, if the sufficient condition does not hold, we know that $\lim_{t\to\infty} \mathrm{e}^{Jt}$ has a rank larger than one by following a similar argument to the necessity proof of Lemma 4.1. This results in a contradiction. □

We also have the following sufficient condition for information consensus using consensus protocol (12).

Theorem 4.3

Let p be defined as in Lemma 4.6. Under the conditions in Theorem 4.2, consensus protocol (12) guarantees that $\xi \to \mathbf{1}p^{\mathrm{T}}\xi(0) + (1/\alpha)\mathbf{1}p^{\mathrm{T}}\zeta(0)$ and $\zeta \to \mathbf{0}$ asymptotically.

Proof

If the information exchange topology has a (directed) spanning tree, we know that $-L$ has a simple zero eigenvalue and all the other eigenvalues have negative real parts from Lemma 4.3. Without loss of generality, we let $\mu_1 = 0$ and $\mathrm{Re}(\mu_i) < 0$, $i = 2, \ldots, n$. It is straightforward to see that $\rho_{1+} = 0$ and $\rho_{1-} = -\alpha$. In addition, if

$$\gamma > \max_{\mu_i \neq 0} \sqrt{\frac{2}{|\mu_i| \cos\left(\dfrac{\pi}{2} - \tan^{-1}\dfrac{-\mathrm{Re}(\mu_i)}{\mathrm{Im}(\mu_i)}\right)}}$$

we know that $\mathrm{Re}(\rho_{i\pm}) < 0$, $i = 2, \ldots, n$, following Lemma 4.4. Therefore, we see that consensus can be achieved asymptotically from Lemma 4.6. □

5. CONVERGENCE ANALYSIS UNDER SWITCHING INFORMATION EXCHANGE TOPOLOGIES

In the case of switching information exchange topologies, the convergence analysis is more involved than that of a fixed information exchange topology. Next, we show several examples to provide a quantitative analysis. Here we focus on consensus protocol (4). The analysis for consensus protocol (12) is similar and is, therefore, omitted.

In the case of the first-order consensus protocol, we have shown that consensus can be achieved asymptotically under switching information exchange topologies if there exist infinitely many uniformly bounded consecutive time intervals such that the union of the information exchange topology across each such interval has a (directed) spanning tree [8]. However, as shown in the following example, this condition is generally not sufficient for information consensus in the case of the second-order protocol.

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Let

$$L_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Also let $\gamma_1 = \gamma_2 = \gamma_3 = 1$. Let Γ_i be defined as

$$\Gamma_i = \begin{bmatrix} 0_{n\times n} & I_n \\ -L_i & -\gamma_i L_i \end{bmatrix}$$

where $i = 1, 2, 3$. Note that the graphs of L_1 and L_3, denoted as $\mathscr{G}_1$ and $\mathscr{G}_3$, respectively, do not have a (directed) spanning tree while the graph of L_2, denoted as $\mathscr{G}_2$, does. Also note that Γ_2 has exactly two zero eigenvalues and all the other non-zero eigenvalues have negative real parts while both Γ_1 and Γ_3 have exactly four zero eigenvalues and all the other non-zero eigenvalues have negative real parts. At each time interval of 5 s, we let the information exchange topology be $\mathscr{G}_1$ during 90% of the time and be $\mathscr{G}_2$ during the rest of the time. Note that at each time interval of 5 s the union of the information exchange topologies ($\mathscr{G}_1 \cup \mathscr{G}_2$) has a (directed) spanning tree. Using the first-order consensus protocol, consensus can be achieved as shown in Figure 14. However, consensus cannot be achieved using the second-order consensus protocol as shown in Figure 15. In contrast, if we increase the gain γ_2 to be 10, consensus can be achieved asymptotically as shown in Figure 16. Alternatively, if we reduce the length of each time interval to be 1 s, consensus can be achieved asymptotically as shown in Figure 17. In addition, if we let the information exchange topology be $\mathscr{G}_1$ during 50% of the time and be $\mathscr{G}_2$ during the rest of the time, consensus can be achieved asymptotically as shown in Figure 18. Next, at each time interval of 5 s, we let the information exchange topology be $\mathscr{G}_1$ during 90% of the time and be $\mathscr{G}_3$ during the rest of the time. Note that at each time interval of 5 s the union of the information

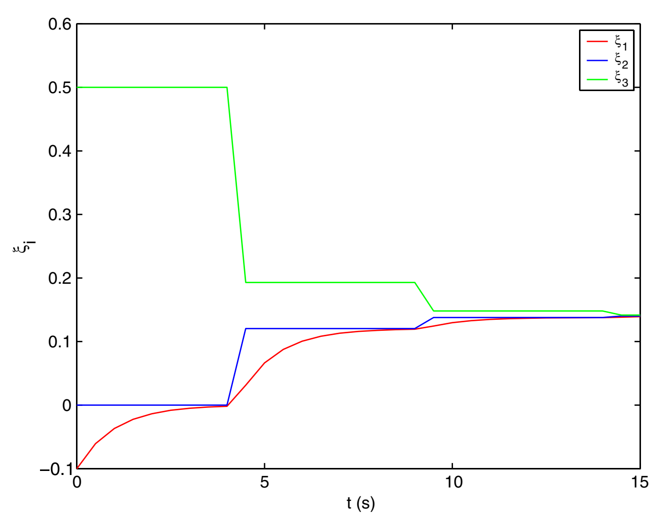

Figure 14. Consensus of information under switching topologies using the first-order consensus protocol ($\mathscr{G}_1$: 90%, $\mathscr{G}_2$: 10%).

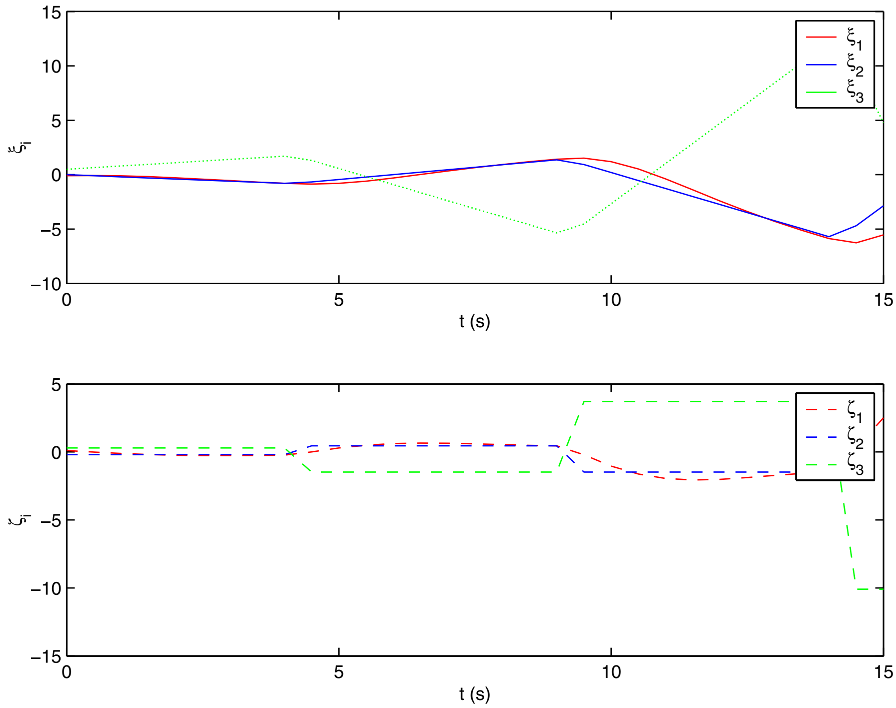

Figure 15. Consensus of information under switching topologies using the second-order consensus protocol ($\mathcal{G}_1$: 90%, $\mathcal{G}_2$: 10%).

exchange topologies ($\mathcal{G}_1 \cup \mathcal{G}_3$) has a (directed) spanning tree. Also note that graph $\mathcal{G}_3$ is only a subset of graph $\mathcal{G}_2$. Compared to Figure 15, Figure 19 shows that consensus can be achieved asymptotically even if graph $\mathcal{G}_3$ has less information exchange than graph $\mathcal{G}_2$.

In the simple case that the information exchange topology between vehicles is undirected and is based on their physical proximity, that is, there is information exchange between vehicle i and j if and only if the distance between them is below a certain threshold, we have the following lemma for information consensus motivated by [15].

Lemma 5.1

If the (time-varying) information exchange topology is undirected and connected at each time, consensus protocol (4) achieves consensus asymptotically.

Proof

Let $V_{ij} = \frac{1}{2}k_{ij}(\xi_i - \xi_j)^2$, where $k_{ij} > 0$ is defined in Equation (4). The second equation in Equation (3) can be rewritten as

$$\dot{\zeta}_i = -\sum_{j=1}^{n} g_{ij} \frac{\partial V_{ij}}{\partial \xi_i} - \sum_{j=1}^{n} g_{ij}\gamma k_{ij}(\zeta_i - \zeta_j) \tag{16}$$

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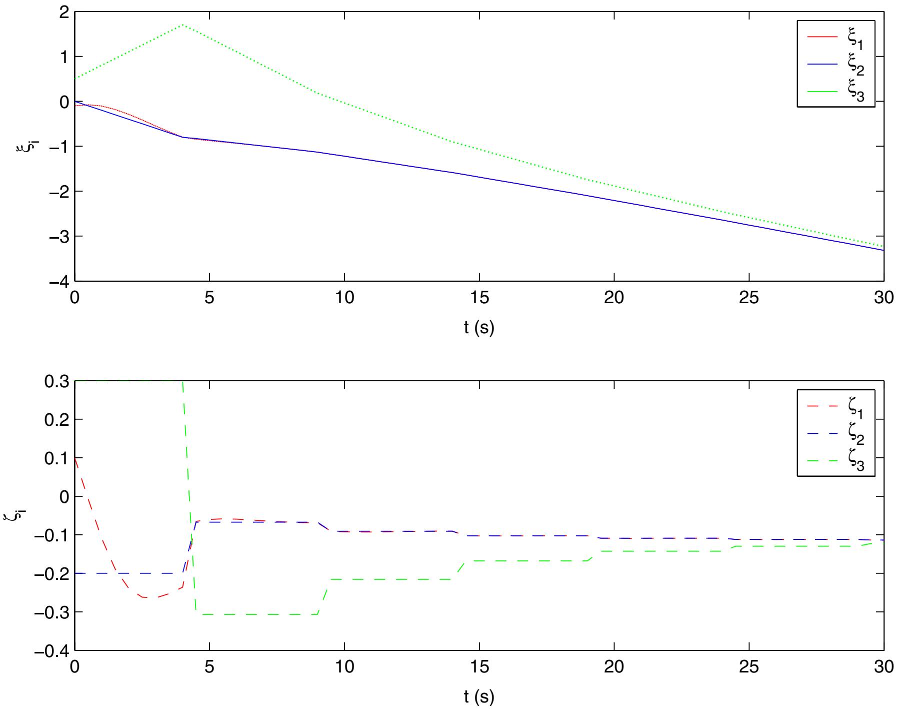

Figure 16. Consensus of information under switching topologies using the second-order consensus protocol with increased γ_2 ($\mathscr{G}_1 : 90\%$, $\mathscr{G}_2 : 10\%$).

Note that $k_{ij} = k_{ji}$ in the case of undirected information exchange. Also note that Equation (16) can be written in matrix form as $\dot{\zeta} = -L(t)\xi - \gamma L(t)\zeta$, where $L(t)$ is defined as in Section 3 corresponding to the undirected information exchange topology at time t. Noting that Equation (16) has the same form as Equation (4) in [15], we can follow a similar proof to that of Theorem VI.2 in [15] to show that $|\zeta_i - \zeta_j| \to 0$ and $\dot{\zeta}_i \to 0$. As a result, we know that $L\xi \to 0$, which implies that $|\xi_i - \xi_j| \to 0$ since the information exchange topology is connected. $\square$

In the general case that the information exchange topology between vehicles is directed and is switching randomly with time, we assume that Equation (5) can be written as

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} = \Gamma_\sigma \begin{bmatrix} \xi \\ \zeta \end{bmatrix}$$

where $\sigma : [0, \infty) \to \mathscr{P}$ is a piecewise constant switching signal with switching times $t_0, t_1, \ldots,$ and $\mathscr{P}$ denotes a set indexing the class of all possible directed information exchange topologies for the n vehicles that have a (directed) spanning tree. Here we assume that $\Gamma(t)$ is piecewise constant and satisfies $\Gamma(t) = \Gamma(t_i)$, $t \in [t_i, t_{i+1})$.

Let $\xi_{ij} = \xi_i - \xi_j$ and $\zeta_{ij} = \zeta_i - \zeta_j$ be the consensus error variables. Note that $\xi_{ij} = \xi_{1j} - \xi_{1i}$ and $\zeta_{ij} = \zeta_{1j} - \zeta_{1i}$. Defining the consensus error vector as $\tilde{\xi} = [\xi_{12}, \xi_{13}, \ldots, \xi_{1n}]^{\mathrm{T}}$ and

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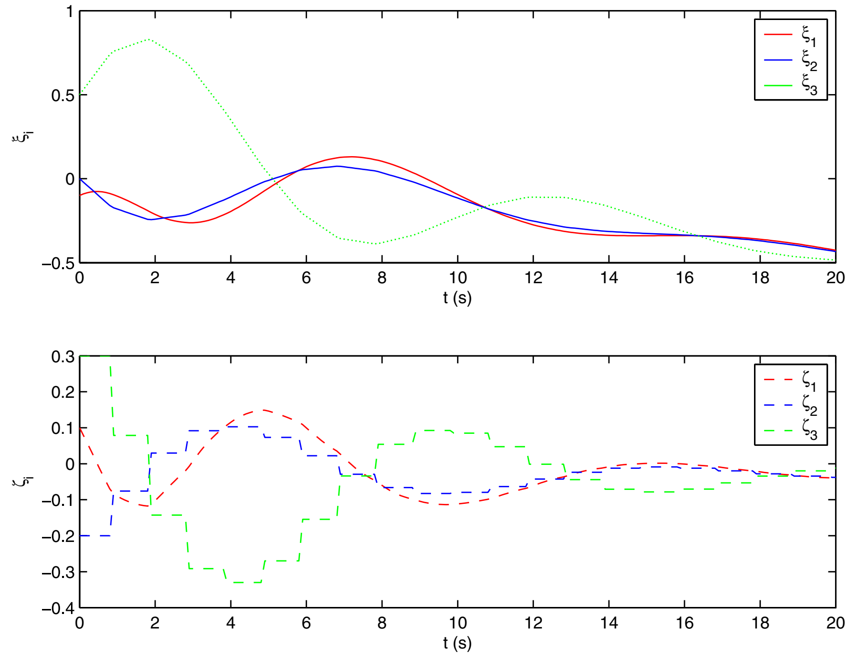

Figure 17. Consensus of information under switching topologies using the second-order consensus protocol with decreased interval length ($\mathscr{G}_1$: 90%, $\mathscr{G}_2$: 10%).

$\tilde{\zeta} = [\zeta_{12}, \zeta_{13}, \ldots, \zeta_{1n}]^{\mathrm{T}}$, we get the following equation:

$$\begin{bmatrix} \dot{\tilde{\xi}} \\ \dot{\tilde{\zeta}} \end{bmatrix} = \Delta_\sigma \begin{bmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{bmatrix} \tag{17}$$

where Δ_σ is a $2(n-1) \times 2(n-1)$ matrix that can be derived from Γ_σ. If Δ_σ is stable, we can find $a_\sigma \geqslant 0$ and $\chi_\sigma > 0$ such that $\|\mathrm{e}^{\Delta_\sigma t}\| \leqslant \mathrm{e}^{(a_\sigma - \chi_\sigma t)}$, $t \geqslant 0$.

We have the following theorem for information consensus under switching information exchange topologies.

Theorem 5.2

Let $t_0, t_1, \ldots$ be the times when the information exchange topology switches. Also, let τ be the dwell time such that $t_{i+1} - t_i \geqslant \tau$, $\forall i = 0, 1, \ldots$. If the information exchange topology has a (directed) spanning tree for each $t \in [t_i, t_{i+1})$, the condition for γ in Theorem 4.2 is satisfied for each Γ_σ, where $\sigma \in \mathscr{P}$, and the dwell time τ satisfies $\tau > \sup_{\sigma \in \mathscr{P}} \{a_\sigma / \chi_\sigma\}$, then consensus protocol (4) achieves consensus asymptotically and is robust to information exchange noise under switching information exchange topologies.

Proof

Given a certain $\sigma_\ell \in \mathscr{P}$, note that the information exchange topology has a (directed) spanning tree for $t \in [t_\ell, t_{\ell+1})$ and the condition for γ in Theorem 4.2 is satisfied for Γ_{σ_ℓ}. Then, we know that consensus is achieved asymptotically if $\sigma(t) \triangleq \sigma_\ell$, $\forall t \geqslant 0$, from Theorem 4.2. That is,

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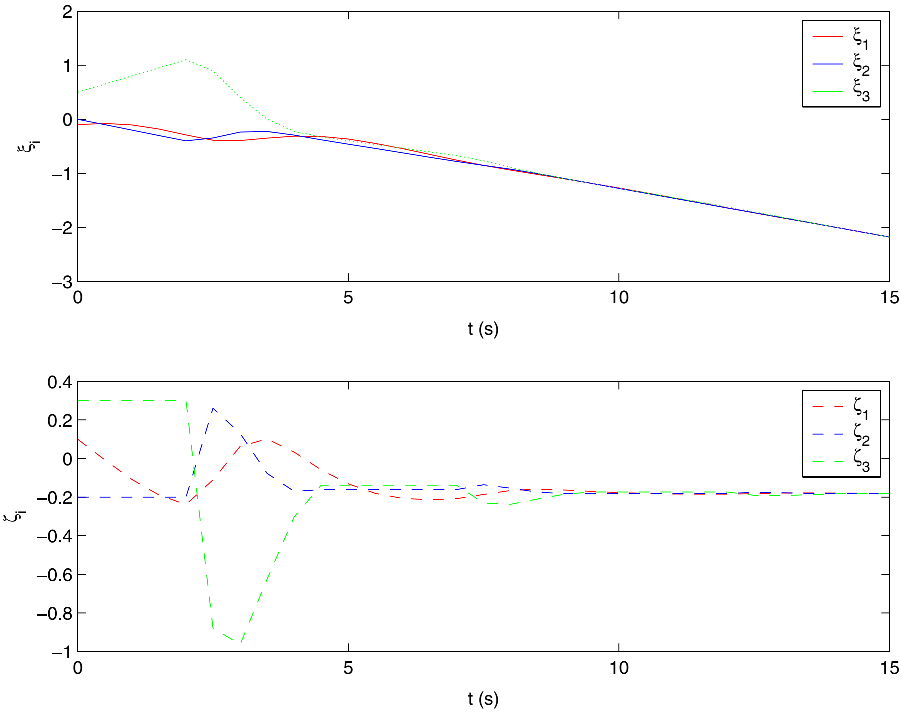

Figure 18. Consensus of information under switching topologies using the second-order consensus protocol ($\mathcal{G}_1$: 50%, $\mathcal{G}_2$: 50%).

$\xi_i \to \xi_j$ and $\zeta_i \to \zeta_j$, $\forall i \neq j$, if $\sigma(t) \triangleq \sigma_\ell$. Equivalently, we know that $\tilde{\xi} \to 0$ and $\tilde{\zeta} \to 0$ asymptotically if $\sigma(t) \triangleq \sigma_\ell$, which implies that switched system (17) is stable for each $\sigma \in \mathcal{P}$ under the conditions of the theorem. As a result, switched system (17) is globally exponentially stable if the dwell time τ satisfies $\tau > \sup_{\sigma \in \mathcal{P}} \{a_\sigma/\chi_\sigma\}$ [20]. The stability of switched system (17) implies that consensus can be achieved asymptotically. The robustness of the consensus protocol (4) to information exchange noise comes from the fact that Equation (17) is globally exponentially stable. □

6. ILLUSTRATIVE EXAMPLE

In this section, we apply the second-order consensus protocol to coordinate the movement of multiple mobile robots.

The equations of motion of a non-holonomic mobile robot are given by

$$\begin{aligned} \dot{x}_i &= v_i \cos(\theta_i) \\ \dot{y}_i &= v_i \sin(\theta_i) \\ \dot{\theta}_i &= \omega_i \\ m_i \dot{v}_i &= f_i \\ J_i \dot{\omega}_i &= \tau_i \end{aligned} \tag{18}$$

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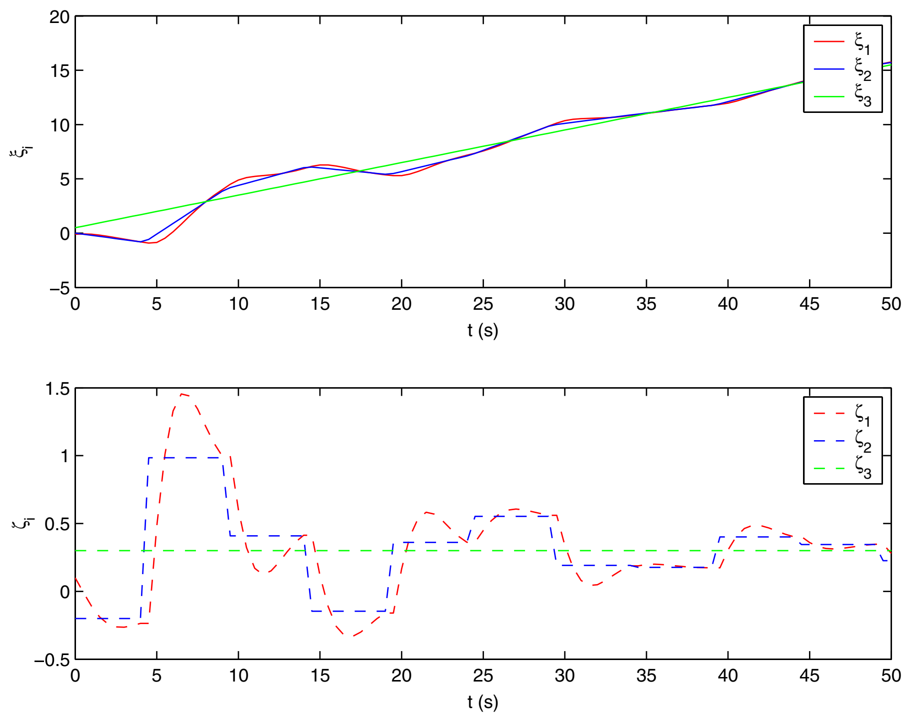

Figure 19. Consensus of information under switching topologies using the second-order consensus protocol ($\mathcal{G}_1$: 90%, $\mathcal{G}_3$: 10%).

where (x_i, y_i) is the Cartesian position of the robot centre, θ_i is the orientation, v_i is the linear velocity, ω_i is the angular velocity, m_i is the mass, J_i is the mass moment of inertia, f_i is the force, and τ_i is the torque applied to the robot. Here, friction effects have been neglected. In this paper, the specifications of each robot are given by $m_i = 10$ kg and $J_i = 0.15$ kg m^2.

To avoid the non-holonomic constraint introduced by Equation (18), we define

$$\begin{bmatrix} x_{hi} \\ y_{hi} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} + d_i \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix}$$

where (x_{hi}, y_{hi}) is a position off the wheel axis of the ith mobile robot by a distance d_i.

Motivated by [13], if we let

$$\begin{bmatrix} f_i \\ \tau_i \end{bmatrix} = \begin{bmatrix} \frac{1}{m_i}\cos(\theta_i) & -\frac{d_i}{J_i}\sin(\theta_i) \\ \frac{1}{m_i}\sin(\theta_i) & \frac{d_i}{J_i}\cos(\theta_i) \end{bmatrix}^{-1} \begin{bmatrix} v_{xi} + v_i\omega_i\sin(\theta_i) + d_i\omega_i^2\cos(\theta_i) \\ v_{yi} - v_i\omega_i\cos(\theta_i) + d_i\omega_i^2\sin(\theta_i) \end{bmatrix}$$

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we obtain the following equations of motion:

$$\begin{aligned}\dot{x}_{hi} &= v_{xi}\\ \dot{v}_{xi} &= v_{xi}\\ \dot{y}_{hi} &= v_{yi}\\ \dot{v}_{yi} &= v_{yi}\end{aligned} \tag{19}$$

In the following, we will focus on the design for control efforts v_{xi} and v_{yi}. Note that Equation (19) also represents equations of motion for a holonomic mobile robot.

We directed four mobile robots to move from their initial locations to pre-defined destinations. During the transition, these four mobile robots are required to preserve a square formation. The information exchange topology for the robots is given by Figure 10, where a directed edge from the ith robot to the jth robot means that the jth robot can obtain $x_{hi} - x^d_{hi}$, $y_{hi} - y^d_{hi}$, $\dot{x}_{hi}$, and $\dot{y}_{hi}$ from the ith robot. Note that Figure 10 has a (directed) spanning tree. We assume that the control effort is saturated and satisfies $|v_{xi}| \leqslant 1$ and $|v_{yi}| \leqslant 1$. In the following, we assume that no collision avoidance occurs between robots. Note that a collision avoidance behaviour may be added to Equation (20) by following [15, 16].

Let (x^d_{hi}, y^d_{hi}) be the desired destination for the ith robot. We propose the following control law for v_{xi} and v_{yi} as

$$\begin{aligned}v_{xi} =& -\alpha_x(x_{hi} - x^d_{hi}) - \gamma_x\alpha_x\dot{x}_{hi}\\ &- \sum_{j=1}^{n} g_{ij}k_{ij}[(x_{hi} - x^d_{hi}) - (x_{hj} - x^d_{hj})] - \sum_{j=1}^{n} g_{ij}\gamma_x k_{ij}(\dot{x}_{hi} - \dot{x}_{hj})\\ v_{yi} =& -\alpha_y(y_{hi} - y^d_{hi}) - \gamma_y\alpha_y\dot{y}_{hi}\\ &- \sum_{j=1}^{n} g_{ij}k_{ij}[(y_{hi} - y^d_{hi}) - (y_{hj} - y^d_{hj})] - \sum_{j=1}^{n} g_{ij}\gamma_y k_{ij}(\dot{y}_{hi} - \dot{y}_{hj})\end{aligned} \tag{20}$$

where $\alpha_* > 0$ and $\gamma_* > 0$. Note that the first two terms in Equation (20) are used to guarantee that each robot arrives at its destination (goal seeking) and the last two terms are used to guarantee that the desired formation shape between robots is preserved (formation keeping) during the transition.

Let $x_{ei} = x_{hi} - x^d_{hi}$ and $y_{ei} = y_{hi} - y^d_{hi}$. Also let $x_e = [x_{e1}, \ldots, x_{en}]^{\mathrm{T}}$ and $y_e = [y_{e1}, \ldots, y_{en}]^{\mathrm{T}}$. Equation (20) can be written in matrix form as

$$\begin{bmatrix}\dot{x}_e\\ \ddot{x}_e\end{bmatrix} = \underbrace{\begin{bmatrix}0_{n\times n} & I_n\\ -(L+\alpha_x I_n) & -\gamma_x(L+\alpha_x I_n)\end{bmatrix}}_{A_x}\begin{bmatrix}x_e\\ \dot{x}_e\end{bmatrix}$$

and

$$\begin{bmatrix}\dot{y}_e\\ \ddot{y}_e\end{bmatrix} = \underbrace{\begin{bmatrix}0_{n\times n} & I_n\\ -(L+\alpha_y I_n) & -\gamma_y(L+\alpha_y I_n)\end{bmatrix}}_{A_y}\begin{bmatrix}y_e\\ \dot{y}_e\end{bmatrix}$$

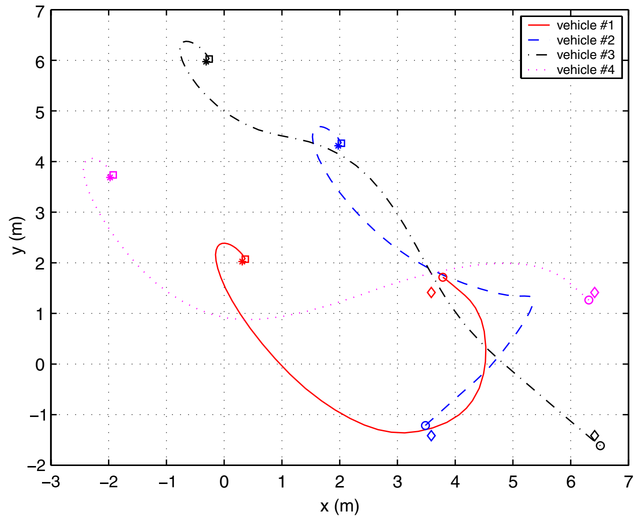

Figure 20. Trajectories of the four robots with $\gamma = 1$.

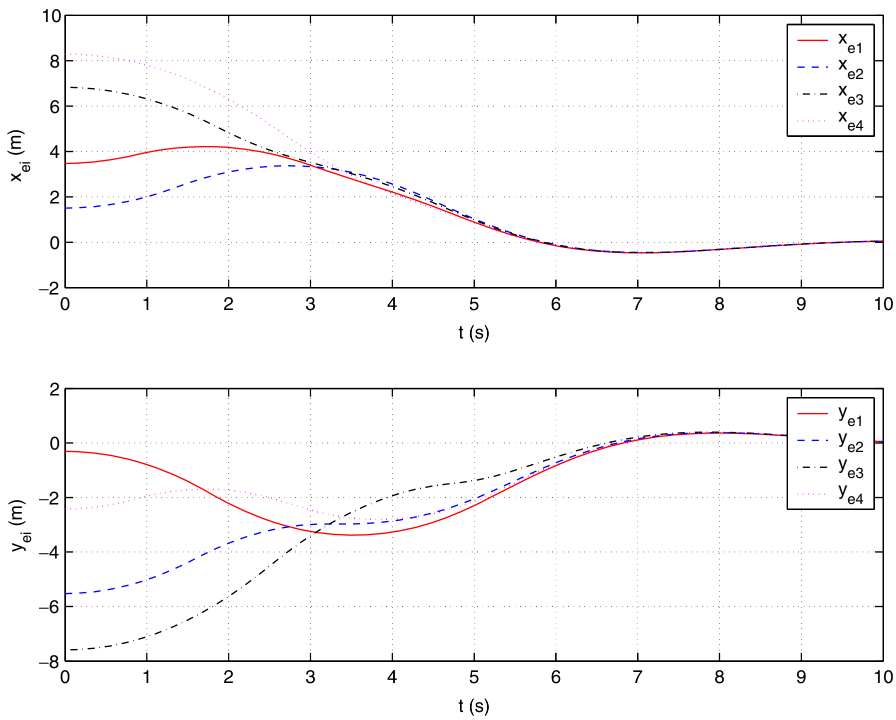

Figure 21. Goal-seeking errors of the four robots with $\gamma = 1$.

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In the first case, we let $k_{ij} = 5$, $\alpha_x = \alpha_y = 5$, and $\gamma_x = \gamma_y = 1$. Note that both

$$\Gamma_x = \begin{bmatrix} 0_{n\times n} & I_n \\ -L & -\gamma_x L \end{bmatrix}$$

and

$$\Gamma_y = \begin{bmatrix} 0_{n\times n} & I_n \\ -L & -\gamma_y L \end{bmatrix}$$

have two zero eigenvalues and all the other eigenvalues have negative real parts. Also note that A_x and A_y are stable matrices. If we let $\alpha_x = \alpha_y = 0$, the desired formation shape will be preserved but the robots are not guaranteed to reach their destinations. As a comparison, if we choose arbitrary stable matrices A_x and A_y, all mobile robots will reach their destinations but the desired formation shape is not guaranteed to be preserved during the transition.

Figure 20 shows the trajectories of the four robots, where circles and diamonds represent the actual and desired starting positions of each robot, respectively, and squares and stars represent the actual and desired ending positions of each robot, respectively. Figures 21 and 22 show the goal-seeking errors and formation-keeping errors of those robots, respectively. Note that the desired square formation is preserved well between robots. Figure 23 shows the control efforts of each robot.

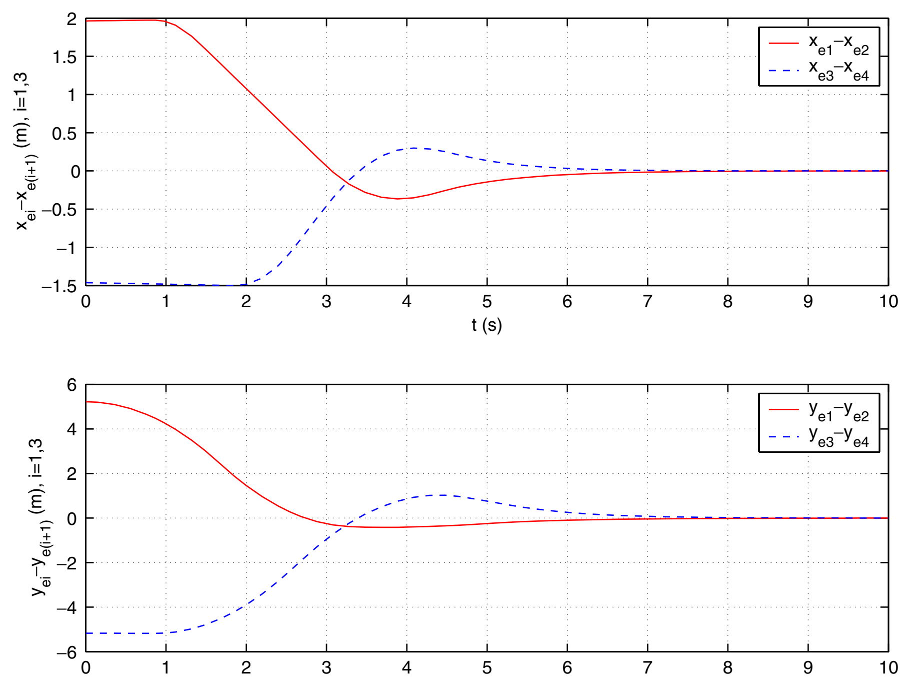

Figure 22. Formation-keeping errors of the four robots with $\gamma = 1$.

DOI: 10.1002/rnc

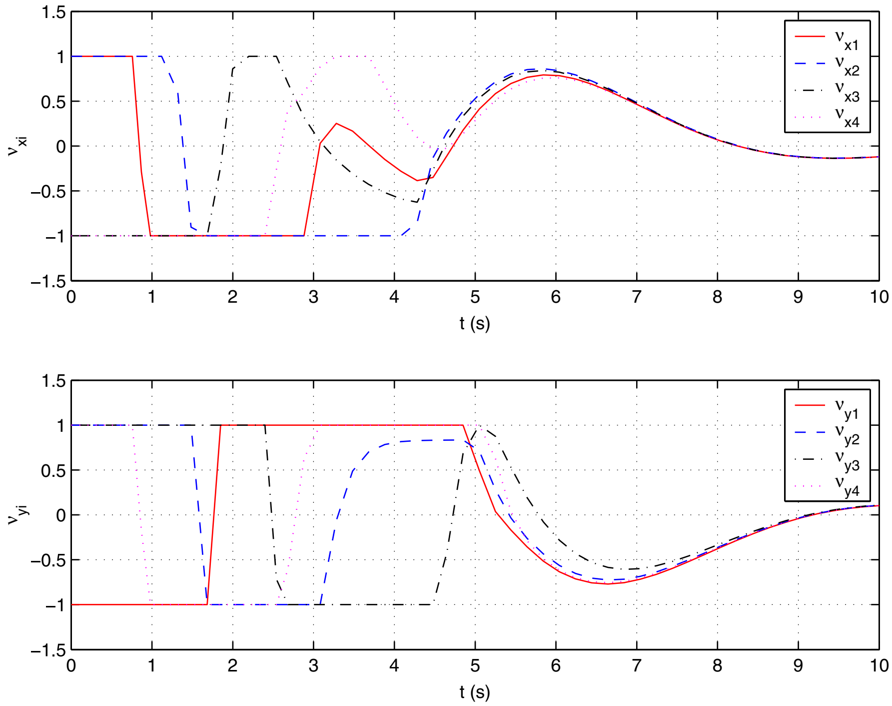

Figure 23. Control efforts of the four robots with $\gamma = 1$.

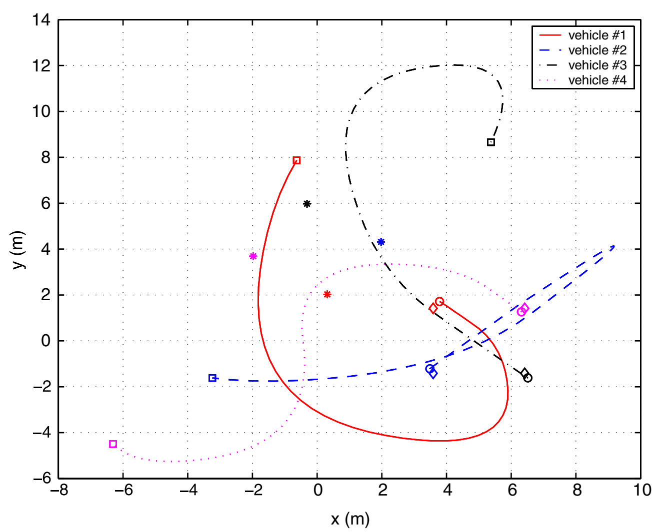

Figure 24. Trajectories of the four robots with $\gamma = 0.1$.

DOI: 10.1002/rnc

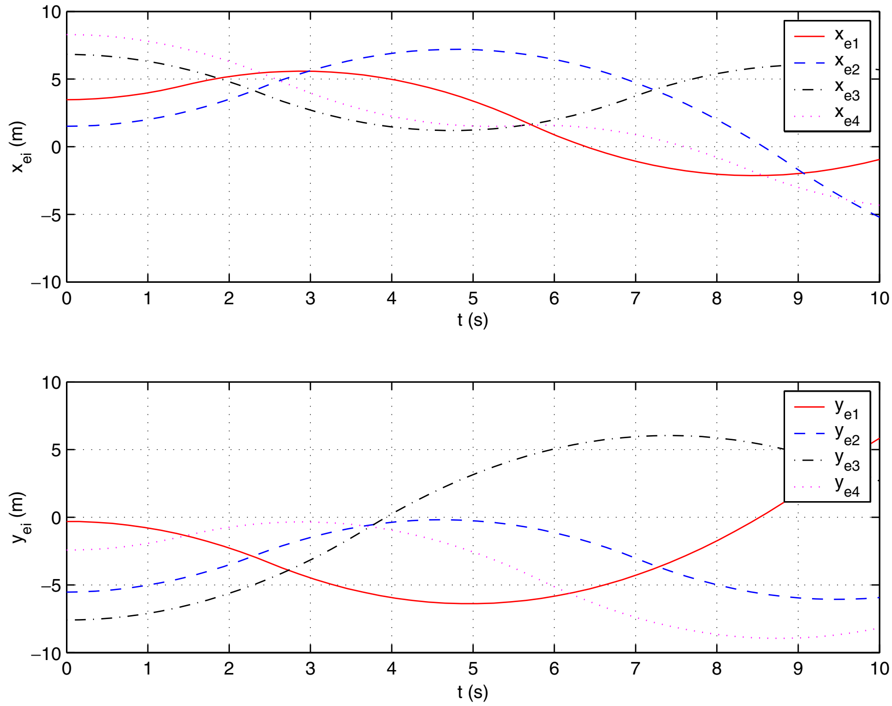

Figure 25. Goal-seeking errors of the four robots with $\gamma = 0.1$.

As a comparison, we let $\gamma_x = \gamma_y = 0.1$ in the second case. It can be shown that two eigenvalues of Γ_x and Γ_y have positive real parts, which implies that consensus cannot be achieved as shown in Case 5 in the previous section. Figure 24 shows the trajectories of the four robots. Figures 25 and 26 show the goal-seeking errors and formation-keeping errors of those robots, respectively. Note that the desired square formation cannot be preserved between robots. Figure 27 shows the control efforts of each robot.

In the third case, we let $\alpha_x = \alpha_y = 0$ and $\gamma_x = \gamma_y = 1$ to see how consensus can be achieved under switching information exchange topologies. At each $t_i = 0.05i$, $i = 0, 1, \ldots$, the information exchange topology is chosen randomly from the set of graphs given by Figures 6, 8, and 10. Figure 28 shows the trajectories of the four robots, and Figure 29 shows the formation-keeping errors of those robots, respectively. Since $\alpha_x = \alpha_y = 0$, those robots cannot reach their targets as expected. However, the desired formation between robots is preserved even if the information exchange topologies are switching with time.

In [14], a bi-directional ring information exchange topology is required for formation keeping between robots. We have shown that with a sufficiently large γ, any information exchange topology that has a (directed) spanning tree is sufficient for formation keeping. Under certain conditions, formation keeping between robots is also guaranteed in the case of switching information exchange topologies.

DOI: 10.1002/rnc

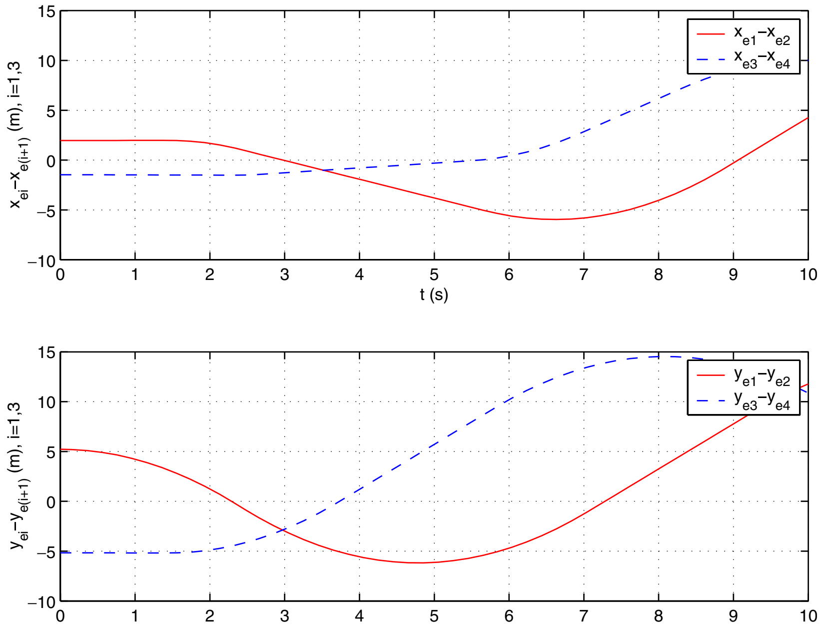

Figure 26. Formation-keeping errors of the four robots with $\gamma = 0.1$.

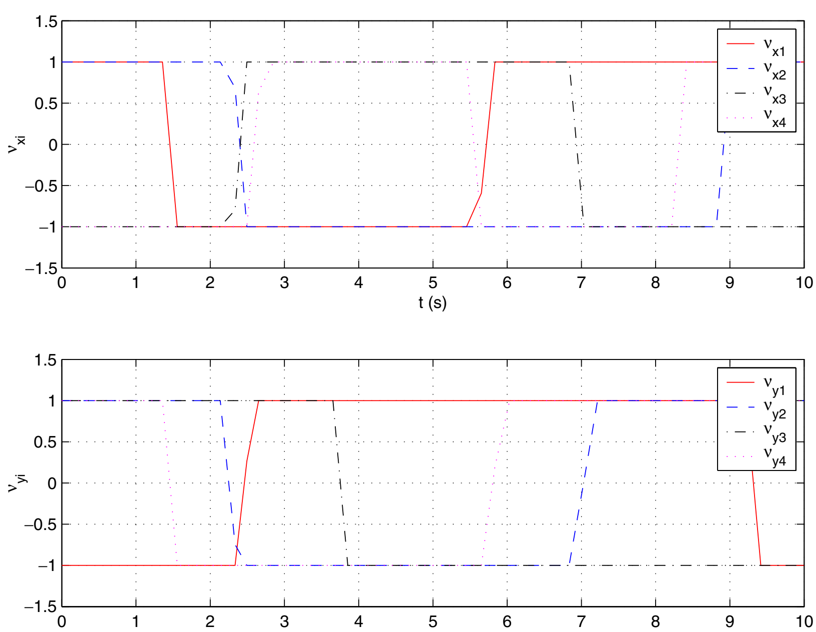

Figure 27. Control efforts of the four robots with $\gamma = 0.1$.

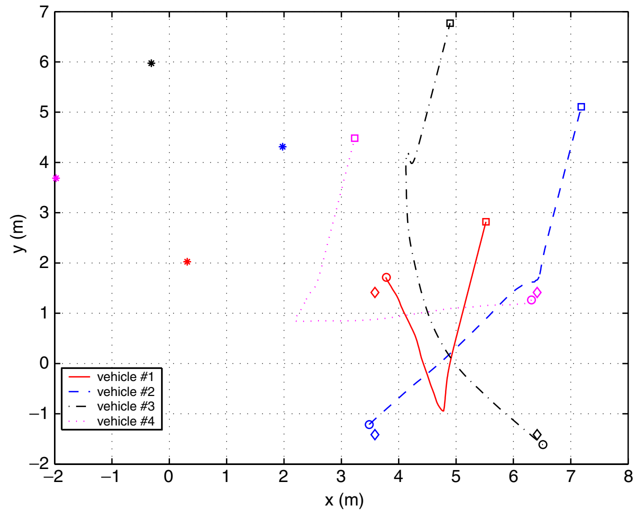

Figure 28. Trajectories of the four robots with $\gamma = 1$ under switching information exchange topologies.

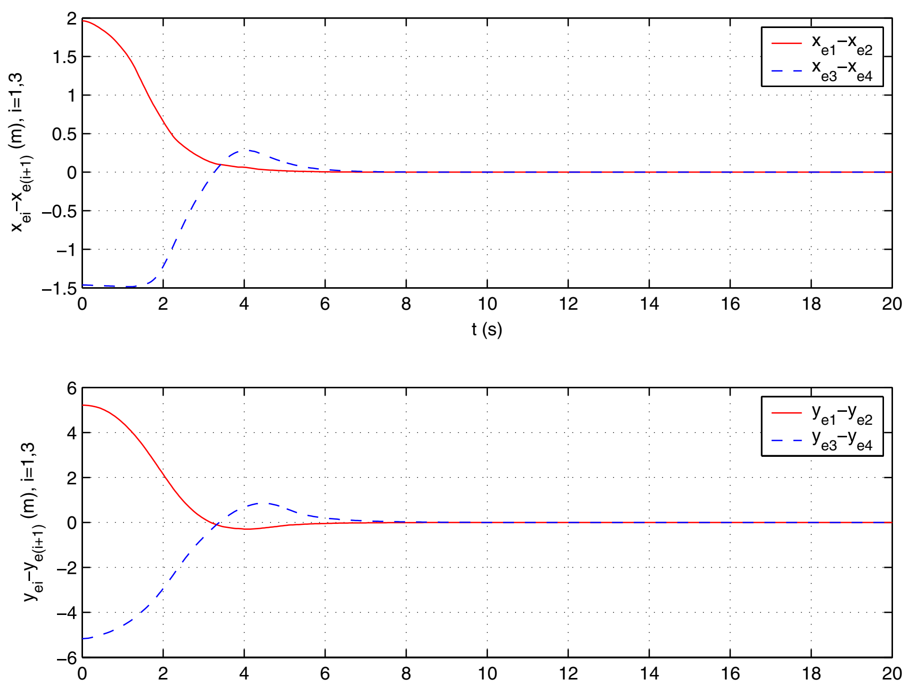

Figure 29. Formation-keeping errors of the four robots with $\gamma = 1$ under switching information exchange topologies.

DOI: 10.1002/rnc

7. CONCLUSION AND FUTURE WORK

We have proposed second-order protocols for information consensus among multiple vehicles. We have also shown necessary and/or sufficient conditions under which consensus can be achieved in the context of unidirectional information exchange topologies. The second-order consensus protocols have been applied to coordinate the movements of multiple mobile robots as a proof of concept.

It is worthwhile to mention that although double integrator dynamics are assumed in this paper, the protocols discussed in this paper may be extended to more complicated nonlinear vehicle dynamics. In addition, a practical enforcement of the lower bound on the dwell time will be a topic of future research.

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DOI: 10.1002/rnc