FURTHER APPLICATION OF THE ELECTRONIC DIFFERENTIAL ANALYZER TO THE OSCILLATION OF BEAMS

by Carl E. Howe

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PREFACE

The investigations described in this report were carried out during the summer months of 1949 under the sponsorship of the Research Techniques Group of Project Wizard. This report forms a natural sequel to one aspect of the report UMM-28 mentioned in the Introduction. The investigations described in this previous report concerning the general utility of the electronic differential analyzer for engineering problems were carried out by the Research Techniques Group principally during the summer months of 1947 and 1948. At the time the work was started it was not generally clear what role the electronic differential analyzer was to play in the field of engineering research, development, and teaching.[1] Since that time it has been demonstrated in many quarters that application of this type of computer is simple and relatively inexpensive and that the range of its application is very wide.

Policy changes in Project Wizard have made it impossible to continue basic investigations in the field of structural dynamics beyond those described in this report. In recognition of this, the Aeronautical Research Center has made other funds available to continue this work during the summer of 1950. A complete report will be issued.

It has been deemed advisable to call the type of computer described in UMM-28 an electronic differential analyzer rather than an electronic analog computer as was done originally. There is another type of electric computer using direct circuit analogs which is better described by the term analog computer.[2]

The work described in this report was greatly aided by facilities and equipment of the Department of Aeronautical Engineering and by use of the

[1] M. H. Nichols and D. W. Hagelbarger, "A Simple Electronic Differential Analyzer as a Demonstration and Laboratory Aid to Instruction in Engineering," to be published in The Journal of Engineering Education.

[2] See, for example, McCann, Wilts, and Locanthi, IRE, <u>37</u> 954 (1949); McCann and MacNeal, ASME Jour. Appl. Mech., <u>17</u> 13 (1950) and references thereto.

constant temperature and humidity room of the Department of Chemical and Metallurgical Engineering.

L. L. Rauch

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INTRODUCTION

In a previous report[1] on the uses of an electronic differential analyzer one of the problems investigated was that of determining the normal modes of oscillations of beams. In the case of an oscillating beam with free ends it was assumed that the proper end conditions were fulfilled by setting equal to zero, at both ends of the beam, the second and third derivatives of the "displacement".

This assumption led to satisfactory solutions when the effects of shearing force and rotary inertia were neglected. However, when the effects were taken into consideration the solutions obtained by the differential analyzer showed mode shapes corresponding to oscillations of the center of gravity of the beam, a physical impossibility.

As explained on page 98 of the previous report this discrepancy was due to the use of incorrect end conditions. "The reason for this discrepancy is that end conditions . . . . are incorrect. The correct end conditions for a free-free beam are that the bending moment M and shearing force V are zero at both ends. When shear and rotary inertia are considered, these are no longer proportional to the second and third derivatives respectively."

The present report deals primarily with the determination of the normal modes of oscillations of uniform free-free beams, using as end conditions the equating to zero, at both ends, the expressions for bending moment and shearing force. Suggestions for further investigations of oscillating beams are made.

In addition there are described a number of modifications which have been made in the computing and recording equipment. Certain proposals for additional changes have been included.

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[1] Hagelbarger, Howe and Howe, University of Michigan Engineering Research Institute. External Memorandum on Investigation of the Utility of an Electronic Analog Computer in Engineering Problems (April 1, 1949), Air Force Contract W33-038-ac-14222 (Project MX-794).

It is assumed that the reader is familiar with the fundamental principles and techniques involved in the use of the electronic differential analyzer as described in the previous report to which reference has been made above.

CHAPTER I

NORMAL MODES OF OSCILLATIONS OF UNIFORM BEAMS

The equation of motion of a uniform beam may be deduced from a free body diagram by applying the laws of dynamics and elementary strength of materials. The five basic equations are[1]

$$\rho \frac{\partial^2 y}{\partial t^2} + \frac{\partial V}{\partial x} = 0 \tag{1}$$

$$-\frac{\partial M}{\partial x} + I' \frac{\partial^2 \beta}{\partial t^2} + V = 0 \tag{2}$$

$$V = -kAG\alpha \tag{3}$$

$$\frac{\partial \beta}{\partial x} = \frac{M}{EI} \tag{4}$$

$$\frac{\partial y}{\partial x} = \alpha + \beta \tag{5}$$

Notation:

x = horizontal distance from left end of beam

y = vertical deflection

α = neutral axis slope due to shear

β = neutral axis slope due to bending

ρ = mass per unit length of beam

[1] These equations, with slight changes in notation, are taken from - Ormondroyd, Hess, Hess and Edman; University of Michigan Engineering Research Institute, Second Progress Report (March 31, 1948), Office of Naval Research Contract N5Ori-116 (Univ. of Mich. No. M670-4). Title: <u>Theoretical Research on the Dynamics of a Ship's Structure</u>.

I = area moment of inertia

E = modulus of elasticity

A = cross sectional area

G = modulus of shear

k = ratio of average shear stress to shear stress at neutral axis

M = bending moment

V = vertical shear force

I' = mass moment of inertia per unit length of beam

ℓ = length of beam

t = time

ω = angular frequency of vibration

EI = flexural rigidity

kAG = shear rigidity

From equations (1) to (5) there may be obtained

$$EI\frac{\partial^4 y}{\partial x^4} - \left(\frac{EI}{kAG} + I'\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{I'\rho}{kAG}\frac{\partial^4 y}{\partial t^4} + \rho\frac{\partial^2 y}{\partial t^2} = 0 \qquad (6)$$

$$M = EI\frac{\partial^2 y}{\partial x^2} - \frac{EI}{kAG}\rho\frac{\partial^2 y}{\partial t^2} \qquad (7)$$

$$V + \frac{I'}{kAG}\frac{\partial^2 V}{\partial t^2} = EI\frac{\partial^3 y}{\partial x^3} - \left(\frac{EI}{kAG}\rho + I'\right)\frac{\partial^3 y}{\partial x \partial t^2} \qquad (8)$$

Equation (6) is the differential equation of motion of the uniform beam. Equations (7) and (8) are expressions for the bending moment, M, and the vertical shear force, V.

In obtaining a normal mode of oscillation it is assumed that the motion of the beam is simple harmonic. Let:

$$y(x,t) = X(x)\, e^{j\omega t}, \qquad (9)$$

$$V(x,t) = V(x)\, e^{j\omega t}, \qquad (10)$$

and

$$M(x,t) = M(x)\, e^{j\omega t}. \tag{11}$$

Equations (6), (7) and (8) may now be written

$$EI\frac{d^4X}{dx^4} + \left(\frac{EI}{kAG} + \frac{I'}{\rho}\right)\rho\omega^2\frac{d^2X}{dx^2} - \left(1 - \frac{I'\omega^2}{kAG}\right)\rho\omega^2 X = 0 \tag{12}$$

$$M = EI\frac{d^2X}{dx^2} + \frac{EI}{kAG}\rho\omega^2 X \tag{13}$$

$$V = \frac{1}{1 - \frac{I'\omega^2}{kAG}}\left[EI\frac{d^3X}{dx^3} + \left(\frac{EI}{kAG} + \frac{I'}{\rho}\right)\rho\omega^2\frac{dX}{dx}\right] \tag{14}$$

where X, M and V are now functions of x alone.

Equation (12) is to be solved by the electronic differential analyzer, subject to boundary conditions imposed by the restraints, or lack of them, placed on the beam. In order to solve the equation by using the computer it is necessary to change the independent variable.

The length of the uniform beam is designated by ℓ so that the range of the solution for the independent variable, x, is $0 \leq x \leq \ell$. In solving equation (12) by the electronic analog computer the independent variable, x, is proportional to the time in seconds (as shown on the oscillogram) elapsed from the time of starting the solution. Suppose the range of solution as shown on the oscillogram, is covered in L seconds. Then the independent variable, x, in the above equations should be changed to a new independent variable, τ, according to the relation

$$\tau = \frac{L}{\ell}x. \tag{15}$$

Then,

$$\frac{d}{dx} = \frac{L}{\ell}\frac{d}{d\tau},$$

$$\frac{d^2}{dx^2} = \frac{L^2}{\ell^2}\frac{d^2}{d\tau^2},$$

and, in general,

$$\frac{d^n}{dx^n} = \frac{L^n}{\ell^n}\frac{d^n}{d\tau^n} \tag{16}$$

In terms of the new independent variable, τ, equations (12), (13) and (14) may be written

$$\frac{EI}{\rho\omega^2\ell^4}L^4\frac{d^4X}{d\tau^4} + \left(\frac{EI}{kAG\ell^2} + \frac{I'}{\rho\ell^2}\right)L^2\frac{d^2X}{d\tau^2} - \left(1 - \frac{I'\omega^2}{kAG}\right)X = 0 \tag{17}$$

$$M = \rho\omega^2\frac{\ell^2}{L^2}\left(\frac{EI}{\rho\omega^2\ell^4}L^4\frac{d^2X}{d\tau^2} + \frac{EI}{kAG\ell^2}L^2X\right) \tag{18}$$

$$V = \frac{1}{1 - \frac{I'\omega^2}{kAG}}\rho\omega^2\frac{\ell}{L}\left[\frac{EI}{\rho\omega^2\ell^4}L^4\frac{d^3X}{d\tau^3} + \left(\frac{EI}{kAG\ell^2} + \frac{I'}{\rho\ell^2}\right)L^2\frac{dX}{d\tau}\right] \tag{19}$$

Let

$$R^2 = \frac{\rho\omega^2\ell^4}{EI} \tag{20}$$

$$S = \frac{EI}{kAG\ell^2} \tag{21}$$

and

$$U = \frac{I'}{\rho\ell^2}. \tag{22}$$

Then

$$R^2SU = \frac{I'\omega^2}{kAG} \tag{23}$$

Equations (17), (18) and (19) then become

$$\frac{L^4}{R^2}\frac{d^4X}{d\tau^4} + \left(SL^2 + UL^2\right)\frac{d^2X}{d\tau^2} - \left(1 - R^2SU\right) X = 0 \tag{24}$$

$$M = \frac{\rho\omega^2\ell^2}{L^2}\left(\frac{L^4}{R^2}\frac{d^2X}{d\tau^2} + SL^2X\right) \tag{25}$$

$$V = \frac{\rho\omega^2\ell}{I\left(1 - \frac{I'\omega^2}{kAG}\right)}\left[\frac{L^4}{R^2}\frac{d^3X}{d\tau^3} + \left(SL^2 + UL^2\right)\frac{dX}{d\tau}\right] \tag{26}$$

Now let

$$C = \frac{L^4}{R^2}, \tag{27}$$

$$D = SL^2, \tag{28}$$

and

$$F = UL^2. \tag{29}$$

Then

$$\frac{DF}{C} = R^2SU. \tag{30}$$

Equations (24), (25) and (26) may then be written using the further notation that

$$\frac{dX}{d\tau} = X', \quad \frac{d^2X}{d\tau^2} = X'', \text{ etc.}$$

$$CX^{IV} + (D + F)\, X'' - \left(1 - \frac{DF}{C}\right)X = 0 \tag{31}$$

$$M = \frac{\rho\omega^2\ell^2}{L^2}\,(CX'' + DX) \tag{32}$$

and

$$V = \frac{\rho\omega^2 \ell}{L\left(1 - \frac{I'\omega^2}{kAG}\right)}\left[CX''' + (D + F)\,X'\right] \tag{33}$$

These equations are now in a form suitable for solution by the computer. It will be shown later that they can be used to determine the modes of vibration of a non-uniform beam by causing the parameters C, D and F to change appropriately. For the solution of a uniform beam, however, the equations can be put into more useful form.

For a uniform beam

$$I' = \frac{\rho I}{A} \tag{34}$$

Equation (22) may then be written

$$U = \frac{I'}{\rho\ell^2} = \frac{\rho I}{\rho A\ell^2} = \frac{I}{A\ell^2} = \frac{I}{A\ell^2} \times \frac{kAG\ell^2}{EI} \times \frac{EI}{kAG\ell^2},$$

whence

$$U = \frac{EI}{kAG\ell^2} \times \frac{kG}{E} = S\,\frac{kG}{E} = SN, \tag{35}$$

where

$$N = \frac{kG}{E}. \tag{36}$$

Then

$$SL^2 + UL^2 = SL^2(1 + N) = D(1 + N), \tag{37}$$

$$F = UL^2 = SL^2N = DN, \tag{38}$$

and

$$\frac{DF}{C} = \frac{D^2N}{C} \tag{39}$$

Equations (31), (32) and (33) become, in terms of the parameters C, D and N,

$$CX^{IV} + D(1 + N)X'' - \left(1 - \frac{D^2N}{C}\right)X = 0 \tag{40}$$

$$M = \frac{\rho\omega^2\ell^2}{L^2}(CX'' + DX) \tag{41}$$

and

$$V = \frac{\rho\omega^2\ell}{L\left(1 - \frac{I'\omega^2}{kAG}\right)}\left[CX''' + D(1 + N)X'\right] \tag{42}$$

It should be noted that the new parameter N, as given by equation (36)

$$N = \frac{kG}{E} \tag{36}$$

is dependent solely upon the ratio, G/E, of the modulus of shear and the modulus of elasticity and upon the shape of the cross section of the beam, through the constant k.

A. Normal Modes of Oscillation of a Uniform Beam with Free Ends

For a beam with free ends the bending moment, M, and the vertical shear force, V, must be zero at each of the ends, i.e.,

$$M = V = 0, \; x = 0, \ell, \text{ or } \mathcal{T} = 0, L. \tag{43}$$

For this case the equation to be solved is equation (40)

$$CX^{IV} + D(1 + N)X'' - \left(1 - \frac{D^2N}{C}\right)X = 0 \tag{40}$$

subject to the boundary conditions obtained by setting equations (41) and (42) equal to zero at $\mathcal{T} = 0, L$,

$$CX'' + DX = 0 \quad \text{at } \mathcal{T} = 0, L \tag{44}$$

and

$$CX''' + D(1 + N)X' = 0 \qquad \text{at } \tau = 0,\ L. \tag{45}$$

(1) Neglecting the Effect of Rotary Inertia

Ormondroyd, Hess and Hess[1] have obtained a series of solutions for the first three normal modes of oscillation of a uniform beam. Their solutions include the effects of bending moment and vertical shear force, but exclude the effect of rotary inertia. The exclusion of rotary inertia makes the parameter N equal to zero. Equations (40), (44) and (45) then become

$$CX^{IV} + DX'' - X = 0 \tag{46}$$

with

$$CX'' + DX = 0 \qquad \text{at } \tau = 0,\ L. \tag{47}$$

and

$$CX''' + DX' = 0 \qquad \text{at } \tau = 0,\ L. \tag{48}$$

The computer circuit for obtaining the solution of equation (46) is shown in Figure 1-1. The first six amplifiers represent the equation itself. The last two amplifiers are used to observe the requisite end conditions as expressed by equations (47) and (48). These boundary conditions are met by the proper adjustment of the initial voltages (at time $\tau = 0$) V_3, V_4, V_5 and V_6 on the feed back capacitors of amplifiers A_3, A_4, A_5 and A_6. The switches S_3, S_4, S_5 and S_6 are relays with normally closed contacts. The solution is started by energizing all four relays simultaneously.

The solution of the equation is obtained by trial and error. The voltage V_6 is made a suitable fixed voltage, e.g., 6 or 24 volts. The potentiometer associated with amplifier A_4 is then adjusted until the output from

[1] Ormondroyd, Hess and Hess, University of Michigan Engineering Research Institute, Third Progress Report on Dynamics of a Ship's Structure (March 1, 1949), Office of Naval Research Contract N5ori - 116 (Univ. of Mich. No. M670-4).

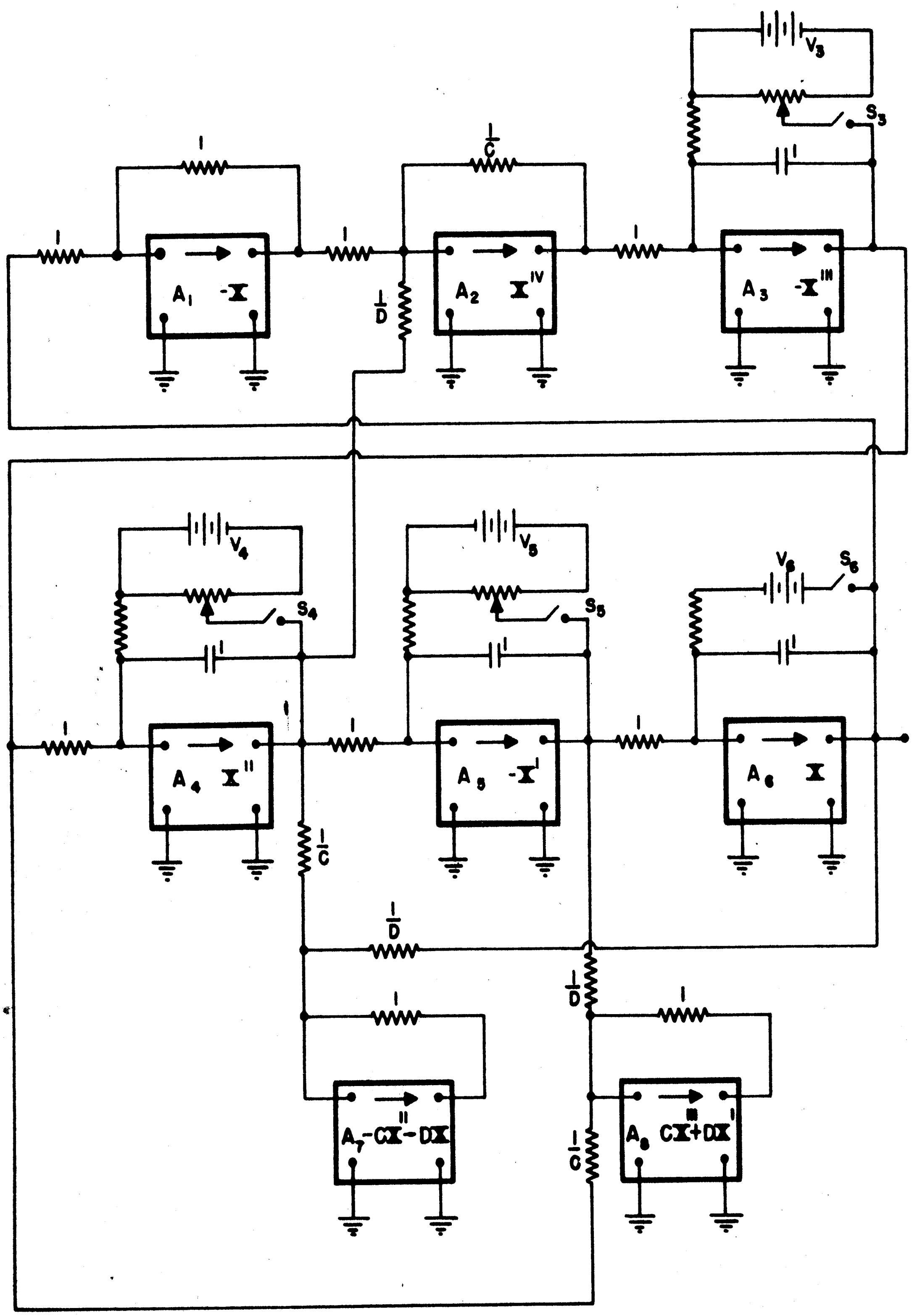

Figure 1-1. Computer circuit for obtaining normal mode solutions of uniform "floating" beam, excluding the effect of rotary inertia.

amplifier A_7 is zero, thereby satisfying equation (47) at the time $\tau = 0$. The voltage V_5 is then given an arbitrary value and subsequently the voltage V_3 is adjusted to such a value that the output of amplifier A_8 is zero. This satisfies equation (48) at the time $\tau = 0$.

With the initial boundary conditions having been set, a trial solution is obtained by energizing the initial condition relays. The output voltages from amplifiers A_7 and A_8 are recorded on a suitable recording device. A satisfactory solution is obtained when these output voltages are simultaneously equal to zero. It is highly improbable that a correct solution will be obtained on the first trial. The voltage V_5 is now given a new value and the voltage V_3 is again adjusted for zero output of amplifier A_8 and another trial solution obtained. These trial solutions are continued until the final boundary conditions are satisfied, i.e., until a solution is obtained where the outputs from amplifiers A_7 and A_8 are simultaneously zero.

Figure 1-2 shows a correct solution for a first normal mode of vibration. It should be noted that for the correct solution the curve of equation (47) is tangent to the zero axis at the end of the solution. At the same time the curve of equation (48) must cross the zero axis since it is the derivative of equation (47). Although the record of equation (48) is not necessary for observing the final end conditions it is needed for determining the time L during which the solution takes place. As described above, this time L is proportional to the length of the beam being studied and is necessary in interpreting the solution.

The procedure, details of which are given later, is to run a series of solutions with appropriate coefficients C and D. In each case the time, L, for the solution is determined. From the value of L, the quantities S and R are obtained by using equations (27) and (28). The several values of $R = \left(\omega \ell^2 \sqrt{\frac{\rho}{EI}}\right)$ are plotted against the corresponding values of $S \left(= \frac{EI}{kAG\ell^2}\right)$ as shown in Figure 1-5.

Since the values of R and S involve the square of L it is necessary to determine L as accurately as possible. To this end there are recorded on the oscillogram time pulses (in some cases four per second, in others ten per second) obtained from a synchronous motor contactor. The exact value of the time interval between successive pulses is obtained from the instantaneous line

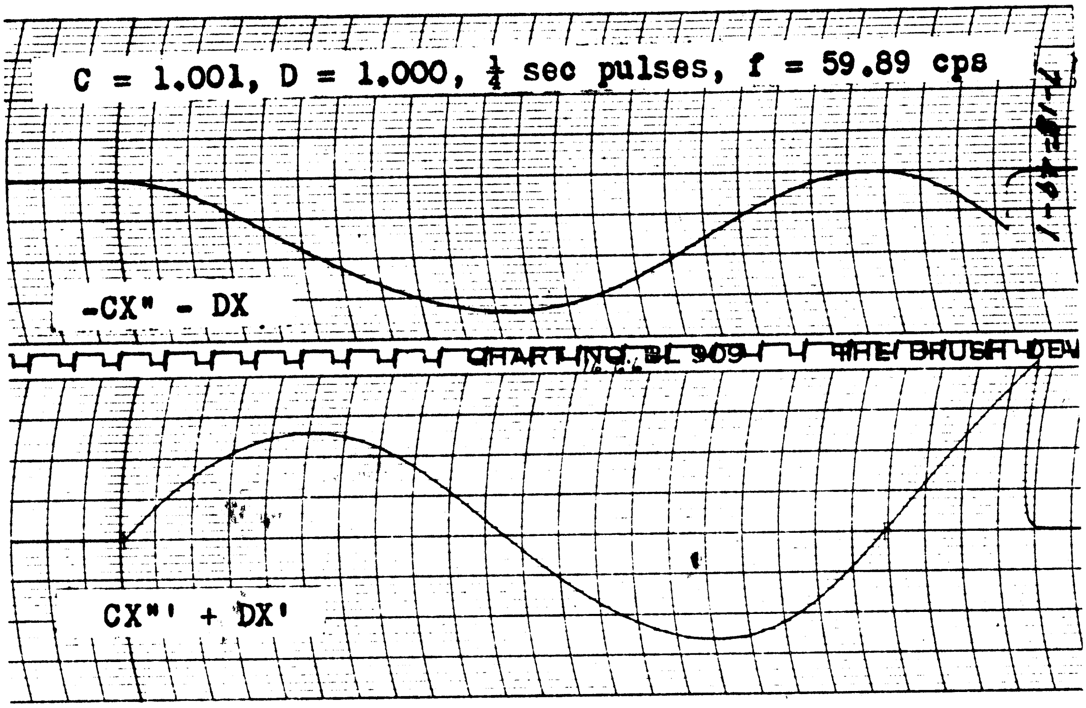

Figure 1-2. First mode solution for uniform "floating" beam excluding the effect of rotary inertia.

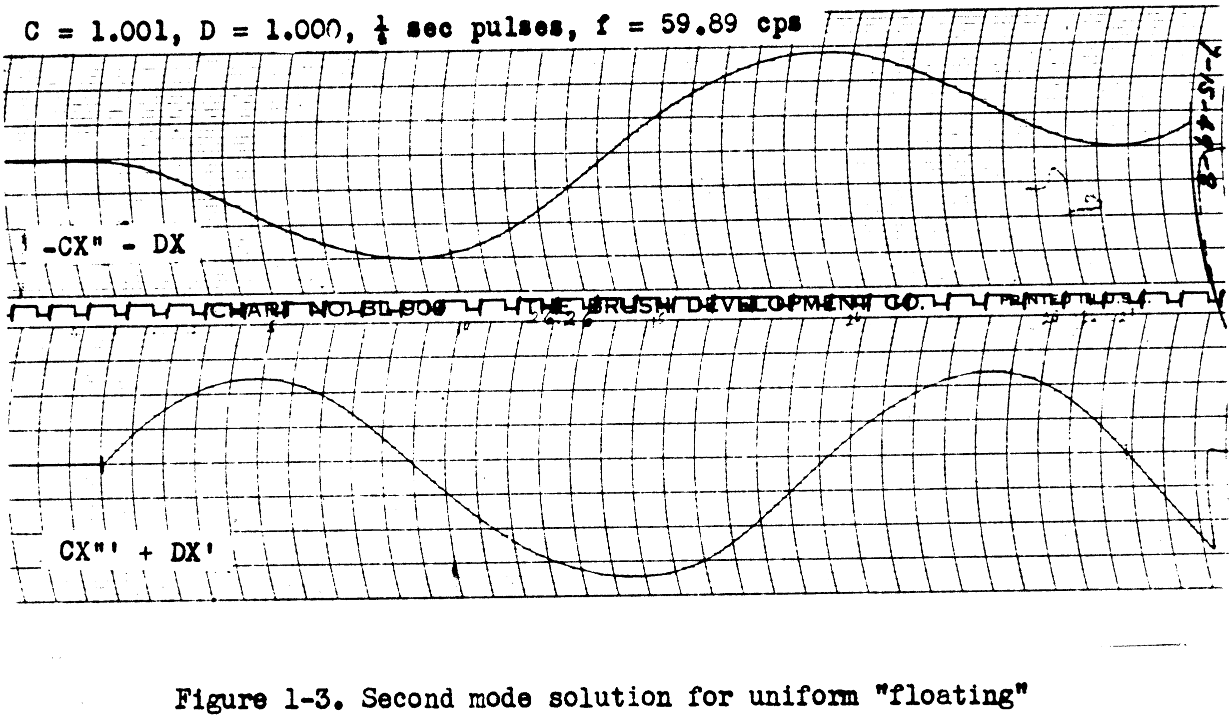

Figure 1-3. Second mode solution for uniform "floating" beam excluding the effect of rotary inertia.

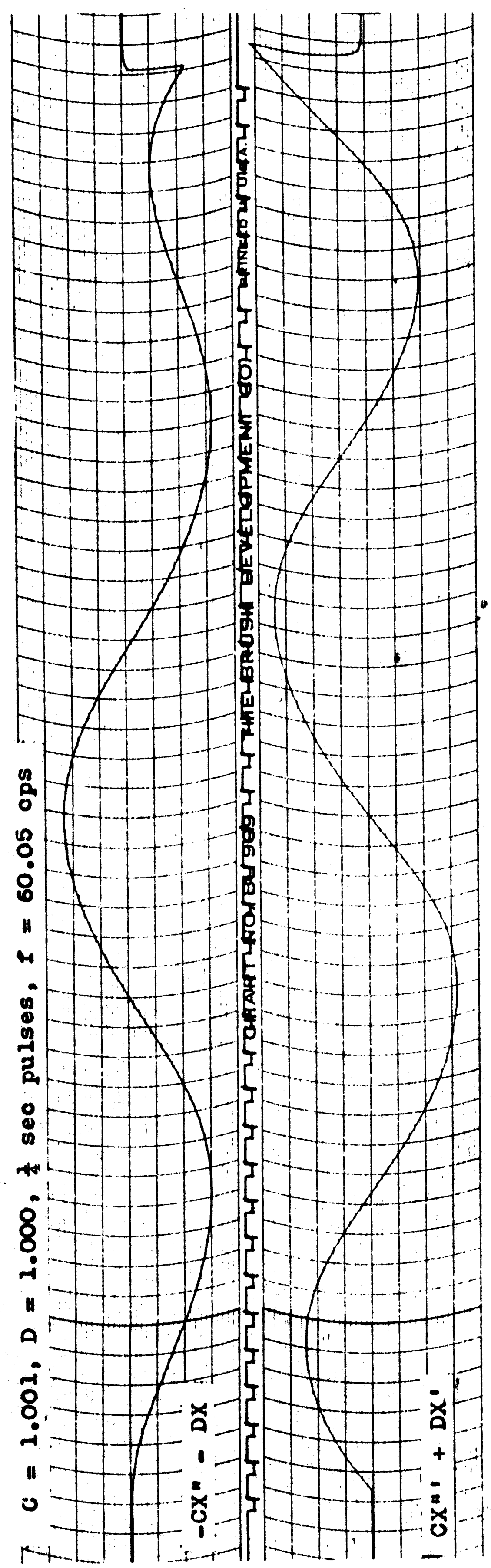

Figure 1-4. Third mode solution for uniform "floating" beam excluding the effect of rotary inertia.

frequency as indicated by a Leeds and Northrup recording frequency meter.

The value of L in seconds is obtained as follows. By comparative measurements the length of the solution on the oscillogram is expressed as L" in terms of the time pulses. This value of L" is then corrected for line frequency error and expressed as L'. The value of L' is then reduced to a value L in seconds. In Table I is an example of determining one value of L and the corresponding values of R and S. These values of R and S are recorded on the first line of Table II.

TABLE I

Oscillogram No.	Mode	L" (1/4 sec) Uncorrected	Frequency (cycles/sec)	Frequency Correction	L' (1/4 sec)
7/15/49/1	1	16.6	59.89	+ 0.03	16.69
7/15/49/2	1	16.70	59.94	+ 0.02	16.72
7/15/49/3	1	16.70	59.94	+ 0.02	16.72
7/15/49/4	1	16.68	59.92	+ 0.02	16.70
7/15/49/5	1	16.66	59.92	+ 0.02	16.68
				L' (avg)	16.70

$$L = 16.70 \times \frac{1}{4} = 4.175 \text{ sec.} \qquad 1/C = 1.000$$

$$L^2 = 17.43 \qquad 1/D = 1.001$$

$$R = \frac{L^2}{\sqrt{C}} = 17.4 \qquad S = \frac{D}{L^2} = 0.0573$$

Slightly different values of the initial voltage, V_5, (with V_3 adjusted for zero output from amplifier A_8, Figure 1) will produce solutions corresponding to higher normal modes. Figures 1-3 and 1-4 are the records of equations (47) and (48) obtained for the second and third normal modes with the coefficients C and D of equation (46) being, $\frac{1}{C} = 1.000$ and $\frac{1}{D} = 1.001$.

In Table II are listed the pairs of values of R and S obtained for the first three normal modes of vibration for various values of coefficients

TABLE II

Mode	1/C	1/D	R	S
1	1.000	1.001	17.4	.0573
1	1.001	1.399	18.8	.0381
1	1.001	2.000	20.0	.0251
1	1.001	4.008	21.3	.0117
2	1.001	0.3989	27.1	.0926
2	1.001	0.5011	31.3	.0639
2	1.000	1.001	43.5	.0230
2	1.001	1.399	48.1	.0149
2	1.001	2.000	52.1	.0096
2	1.001	4.008	57.4	.0044
3	1.001	0.3989	50.3	.0499
3	1.001	0.5011	58.6	.0341
3	1.000	1.001	82.8	.0121
3	1.001	1.399	91.8	.0078
3	1.001	2.000	100.4	.0050

C and D. These pairs of values of R and S appear as circles in Figure 1-5. The solid lines are curves obtained by Ormondroyd, Hess and Hess.[1]

The curves of Figure 1-5 may be used as follows. From the dimensions of the beam and the properties of the material of which it is made the value of S is calculated. The corresponding value of R is determined, for the first, second or third mode, as desired, and the value of the angular frequency, ω, calculated.

If desired the mode shape may be recorded by using the output, X, from amplifier A_6 (Figure 1-1). In general it is more satisfactory to take this record at the same time records are made for determining the proper end conditions. The conditions under which the accompanying results were obtained made it impracticable to make more than two recordings simultaneously. Consequently very few oscillograms are shown for mode shapes.

[1] Ibid

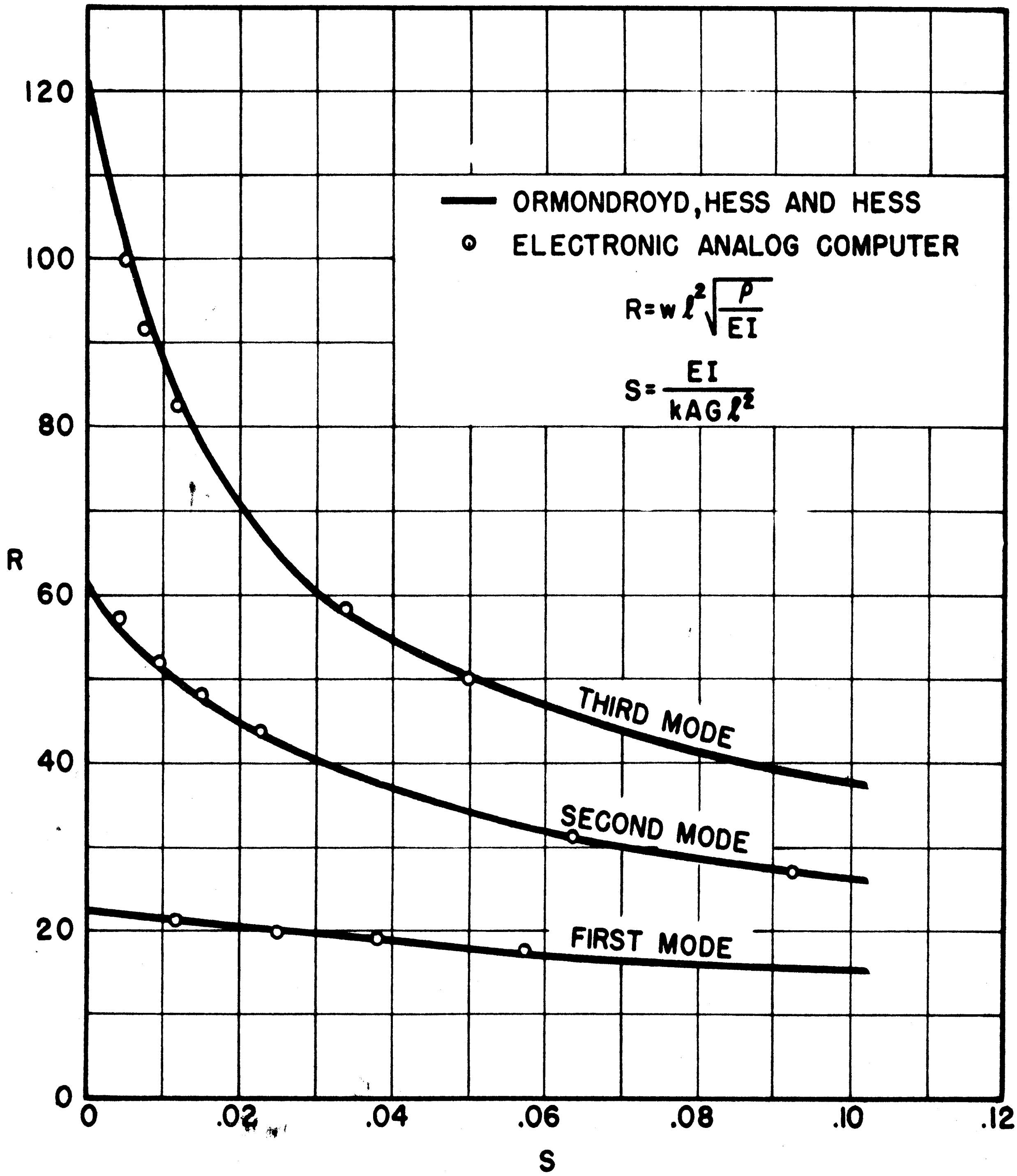

Figure 1-5. R vs S for uniform "floating" beam excluding the effect of rotary inertia.

(2) Including the Effect of Rotary Inertia

When the effect of rotary inertia is included the equation to be solved is equation (40),

$$CX^{iv} + D(1 + N)X'' - (1 - \frac{D^2N}{C})X = 0, \qquad (40)$$

subject to the boundary conditions

$$CX'' + DX = 0, \qquad \text{at } \tau = 0, L, \qquad (44)$$

and

$$CX''' + D(1 + N)X' = 0, \qquad \text{at } \tau = 0, L. \qquad (45)$$

In these equations the rotary inertia is introduced by means of the parameter N. Since N = kG/E, and solutions are being obtained for uniform beams this rotary inertia term depends only on the material of the beam and the shape of its cross section. For the solutions given in this report N was arbitrarily given the value 0.25.

The computer circuit for obtaining the solution of equation (40) is shown in Figure 1-6. Fundamentally this circuit is the same as that shown in Figure 1-1. The first six amplifiers (A_1 to A_6) represent the equation itself, while the last two amplifiers (A_7 and A_8) are used to observe the requisite end conditions as expressed by equations (44) and (45). The circuits differ to the extent that N was set equal to zero in the circuit of Figure 1-1.

The technique for obtaining a solution from the circuit of Figure 1-6 is the same as that previously described. Arbitrary initial conditions which satisfy equations (44) and (45) are set up and trial solutions made (with different initial conditions) until the final boundary conditions are satisfied.

In Figures 1-7, 1-8 and 1-9 are shown correct solutions for the first three normal modes of vibration. It should be noted that for the correct solution the curve of equation (44) is not tangent to the zero axis at the end of the solution, as was the case for the corresponding equation (47) when the effect of rotary inertia was not included. In the present case the criterion for a correct solution is that the curves for equations (44) and (45) cross the zero axis at the same time. When the effect of rotary inertia was not included a

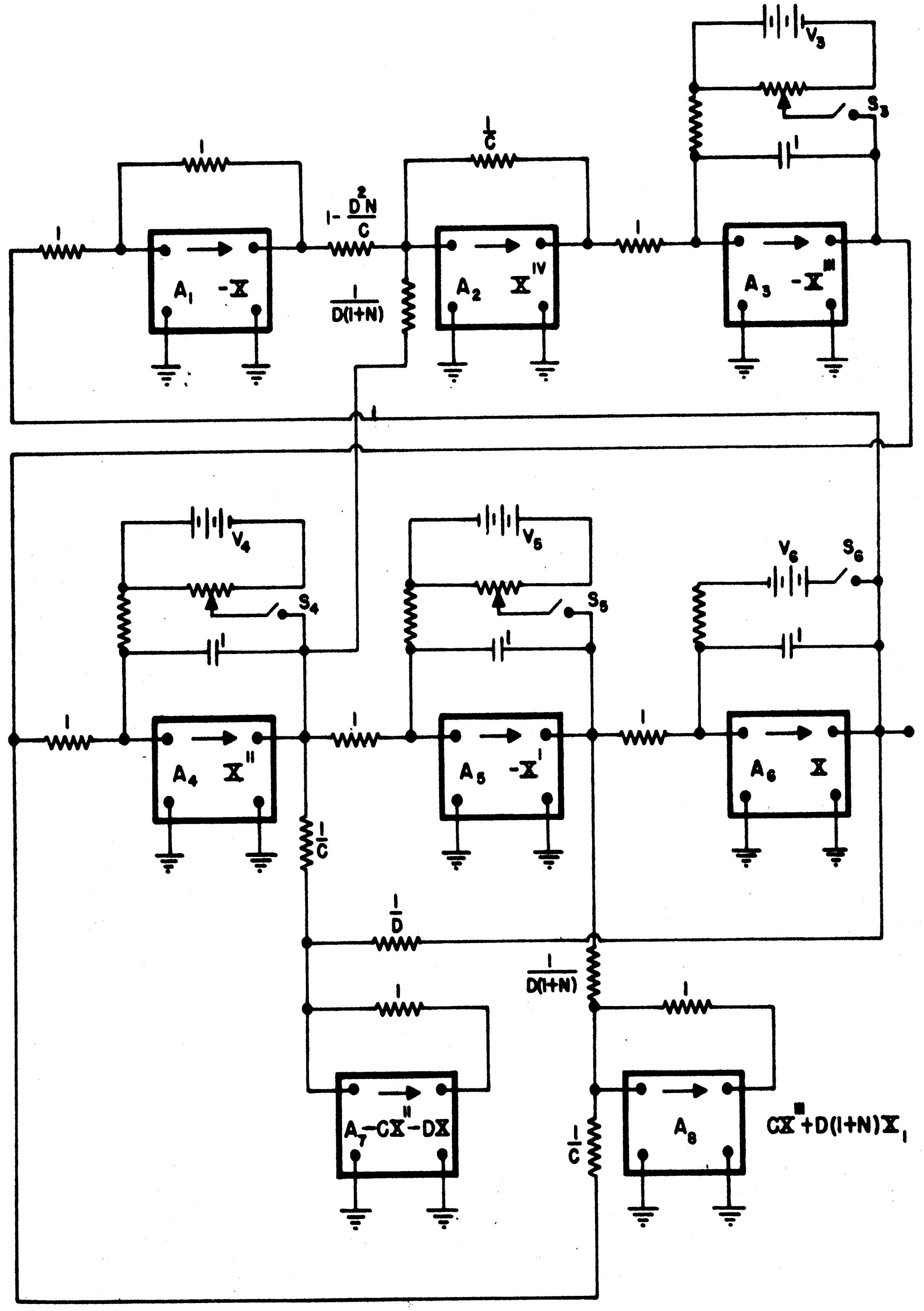

Figure 1-6. Computer circuit for obtaining normal mode solutions of uniform "floating" beam, including the effect of rotary inertia.

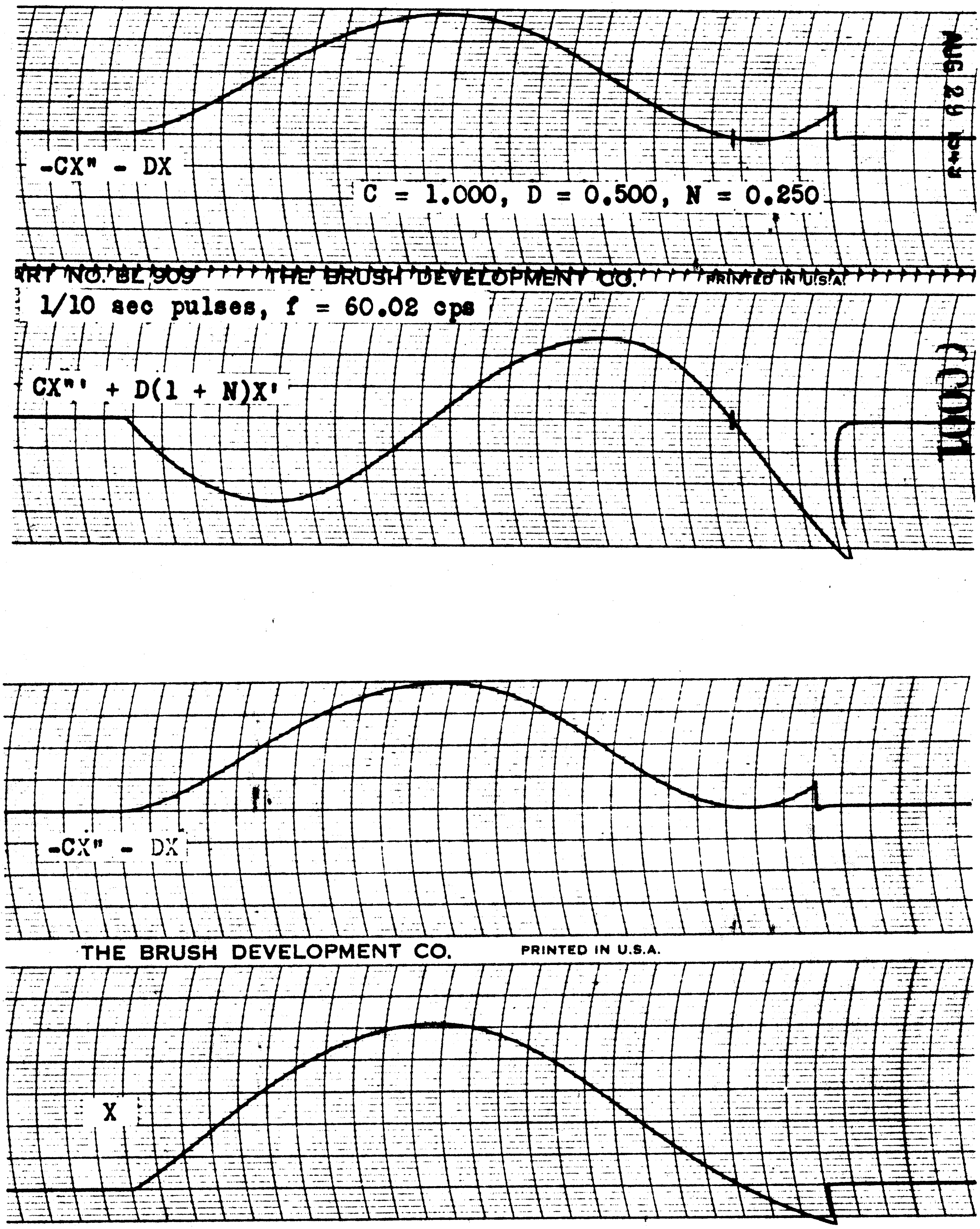

Figure 1-7. First mode solution for uniform "floating" beam including the effect of rotary inertia.

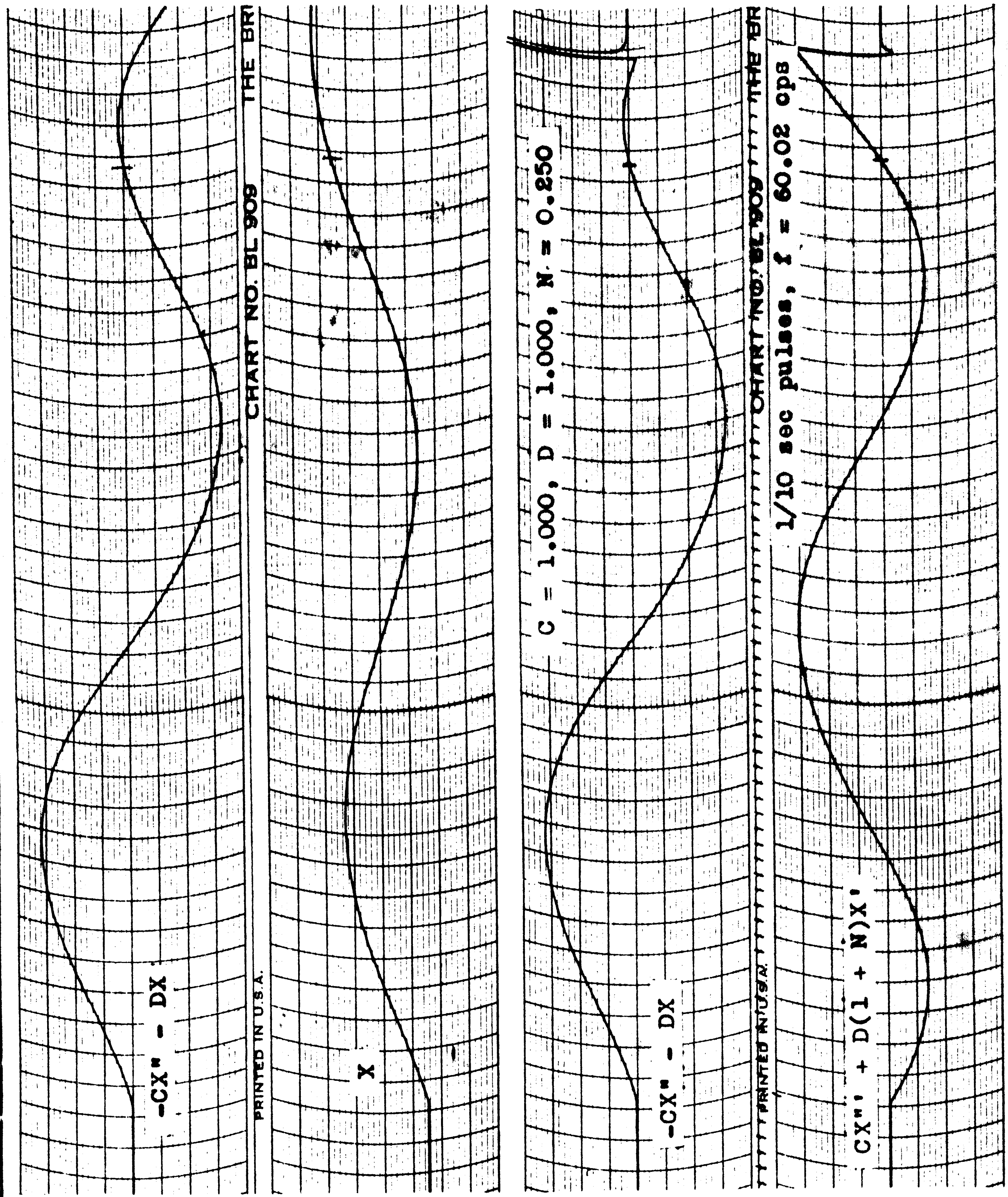

Figure 1-8. Second mode solution for uniform "floating" beam including the effect of rotary inertia.

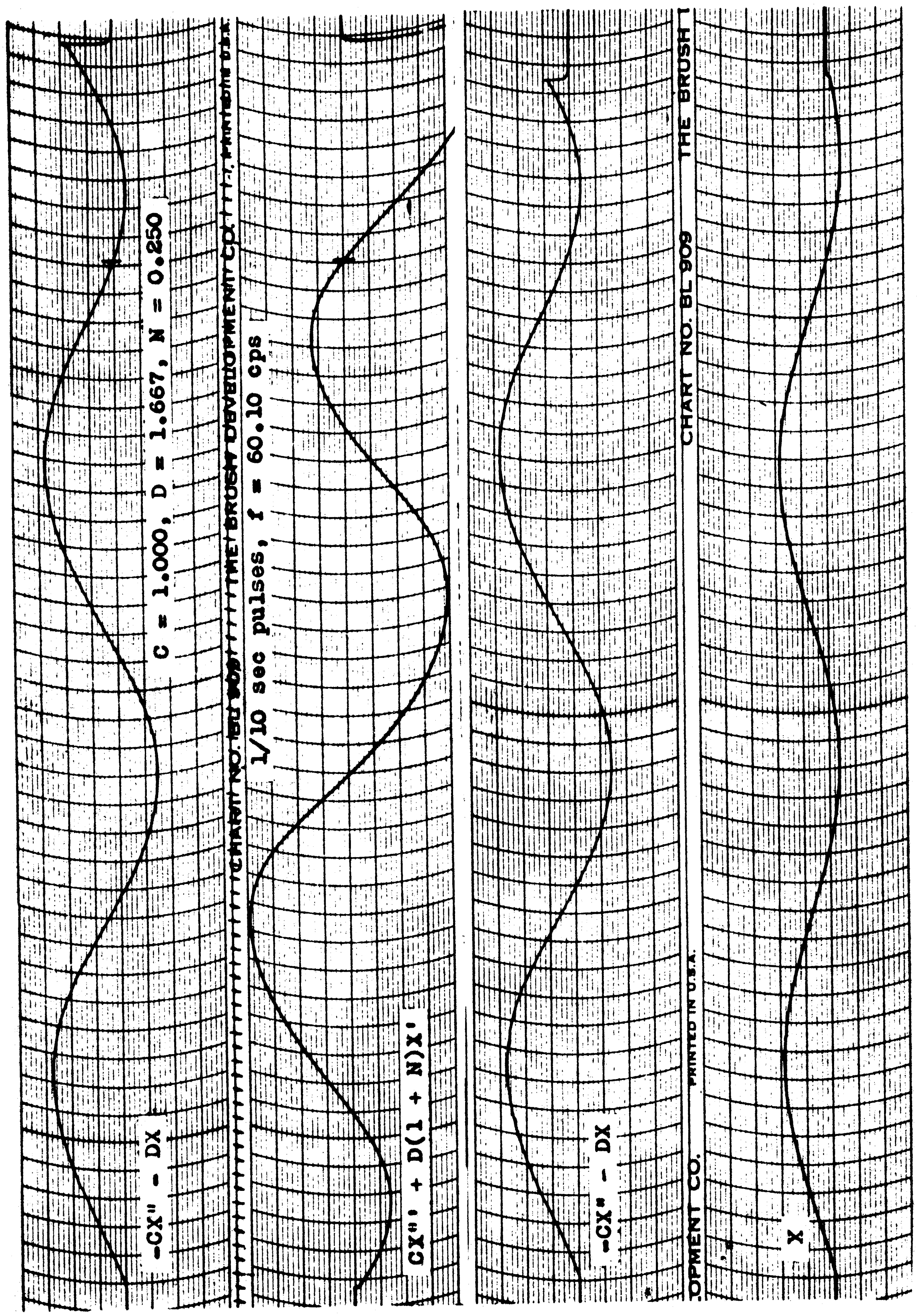

Figure 1-9. Third mode solution for uniform "floating" beam including the effect of rotary inertia.

correct solution could be obtained by observing the bending moment curve alone, since the vertical shear force curve is proportional to the derivative of the former. With the effect of rotary inertia included the vertical shear force equation is not proportional to the derivative of the bending moment equation and hence both curves must be recorded to obtain a correct solution.

Figures 1-7, 1-8 and 1-9 include mode shapes for the first three modes. Examination of the mode shape, X, for the third mode solution as shown in Figure 1-9 leaves some doubt as to whether the center of gravity of the curve is on the zero axis as it should be for a correct solution. It was just this situation, but more manifest, as discovered in previous work, that led to a correction in setting up the criteria for the proper boundary conditions. This problem should be investigated thoroughly to see if other modifications in the theory are necessary.

In Table III are given the data obtained for solutions including the effect of rotary inertia and the computed values for R, S and 1/S. In order to obtain curves which have a greater spread R is plotted against 1/S on semi-logarithmic paper as shown in Figure 1-10. The results in Table III are plotted as triangles in this figure. The other curves represent the results obtained when the effect of rotary inertia is not included in the solution. The large circles represent the results given in Table II, these results being obtained by using as end conditions the equating to zero of both the bending moment, M, and the vertical shear force, V. The small circles represent the results obtained in previous work where the end conditions were specified as being that the second and third derivatives of the displacement should be equal to zero. The data for these points are given in Tables IV, V, VI and VII. It is interesting to note that points obtained by both methods fall on the same curve for lower values of 1/S with a slight departure for higher values.

The curves shown in Figure 1-10 permit the determination of the frequency of vibration of a uniform "floating" beam of given size, shape and material. It would be of interest to investigate more thoroughly the effect of proper end conditions on the R vs 1/S solutions for normal modes when rotary inertia is neglected, particularly for large values of 1/S, which corresponds to long thin beams.

TABLE III

C = 1.000, N = 0.25

Mode	D	Avg L	L^2	$R = \frac{L^2}{\sqrt{C}}$	$S = \frac{D}{L^2}$	$\frac{1}{S}$
1	1.000	3.782	14.30	14.30	0.0699	14.30
1	1.000	3.784	14.32	14.32	.0698	14.32
1	0.667	4.052	16.42	16.42	.0406	24.63
1	0.500	4.221	17.82	17.82	.0280	35.64
1	0.333	4.369	19.09	19.09	.0175	57.27
1	0.250	4.466	19.94	19.94	.0125	79.78
1	0.200	4.541	20.62	20.62	.0097	103.1
2	1.667	4.927	24.27	24.27	.0687	16.18
2	1.333	5.573	31.06	31.06	.0429	23.30
2	1.000	6.088	37.06	37.06	.0270	37.06
2	0.667	6.638	44.06	44.06	.0151	66.09
2	0.500	6.941	48.18	48.18	.0104	96.36
2	0.250	7.405	54.83	54.83	.00456	219.3
3	1.667	7.264	52.77	52.77	.0316	35.18
3	1.333	7.875	62.02	62.02	.0215	46.52
3	1.000	8.520	72.59	72.59	.0138	72.59
3	0.667	9.282	86.15	86.15	.00774	129.2
3	0.500	9.653	93.18	93.18	.00537	186.4

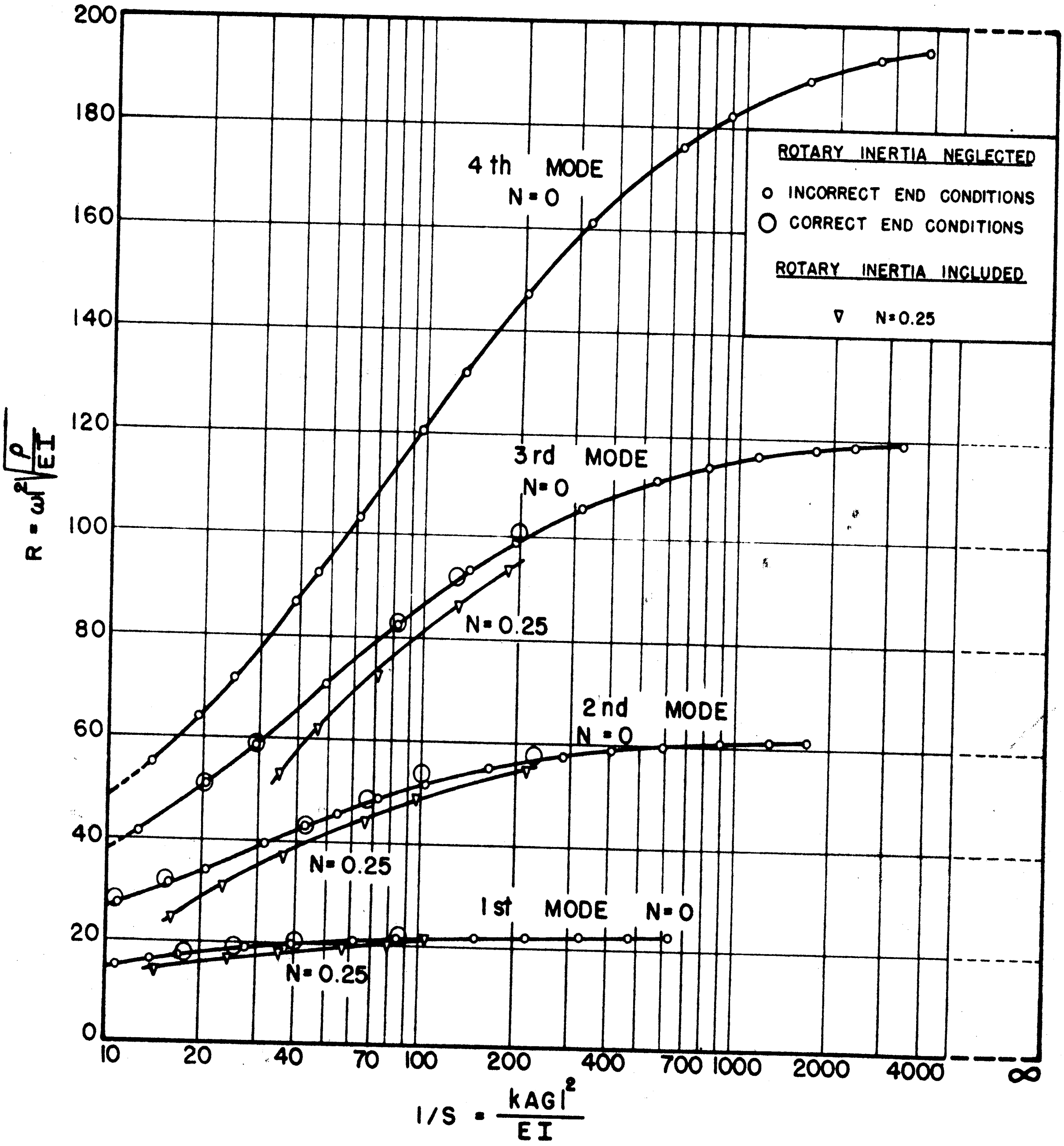

Figure 1-10. R vs 1/S for uniform "floating" beam.

TABLE IV

First Mode - Rotary Inertia Neglected

Incorrect End Conditions

C	D	L	L^2	$R = \frac{L^2}{\sqrt{C}}$	$\frac{1}{S} = \frac{L^2}{D}$
1.000	1.429	3.900	15.21	15.21	10.65
1.000	1.176	4.030	16.24	16.24	13.80
1.000	1.000	4.132	17.07	17.07	17.07
1.000	0.667	4.326	18.71	18.71	28.06
1.000	0.500	4.426	19.59	19.59	39.18
1.000	0.3333	4.521	20.44	20.44	61.32
1.000	0.2500	4.567	20.86	20.86	83.43
1.000	0.2004	4.614	21.29	21.29	106.2
1.000	0.1431	4.641	21.54	21.54	150.6
4.000	0.2013	6.625	43.89	21.95	218.1
1.000	0.0669	4.692	22.02	22.02	328.9
2.000	0.0669	5.608	31.45	22.24	469.9
2.000	0.0503	5.607	31.44	22.23	624.8
1.000	0	4.73	22.37	22.37	∞
2.000	0	5.63	31.70	22.41	∞

TABLE V

Second Mode - Rotary Inertia Neglected

Incorrect End Conditions

C	D	L	L^2	$R = \frac{L^2}{\sqrt{C}}$	$\frac{1}{S} = \frac{L^2}{D}$
1.000	2.500	5.192	26.96	26.96	10.78
1.000	2.000	5.588	31.23	31.23	15.61
1.000	1.818	5.700	32.49	32.49	17.87
1.000	1.667	5.838	34.08	34.08	20.45
1.000	1.250	6.263	39.23	39.23	31.38
1.000	1.000	6.541	42.78	42.78	42.78
1.000	0.833	6.736	45.37	45.37	54.40
1.000	0.667	6.947	48.26	48.26	72.40
1.000	0.500	7.177	51.51	51.51	103.0
1.000	0.3333	7.401	54.77	54.77	164.3
1.000	0.2004	7.560	57.15	57.15	285.2
1.000	0.1429	7.670	58.83	58.83	411.2
1.000	0.1005	7.735	59.83	59.83	595.3
1.000	0.0669	7.780	60.53	60.53	905
1.000	0.0503	7.790	60.68	60.68	1206
2.000	0.0503	9.290	86.30	61.02	1715
1.000	0	7.85	61.62	61.62	∞
2.000	0	9.33	87.05	61.55	∞

TABLE VI

Third Mode - Rotary Inertia Neglected

Incorrect End Conditions

C	D	L	L^2	$R = \frac{L^2}{\sqrt{C}}$	$\frac{1}{S} = \frac{L^2}{D}$
1.000	3.326	6.414	41.17	41.17	12.38
1.000	2.500	7.129	50.82	50.82	20.33
1.000	2.000	7.640	58.37	58.37	29.19
1.000	1.429	8.395	70.48	70.48	49.33
1.000	1.000	9.065	82.17	82.17	82.17
1.000	0.667	9.635	92.83	92.83	139.2
1.000	0.500	9.920	98.41	98.41	196.8
1.000	0.3333	10.27	105.5	105.5	316.4
1.000	0.2494	10.55	111.2	111.2	555.3
1.000	0.1430	10.71	114.6	114.6	802
1.000	0.1005	10.80	116.6	116.6	1161
1.000	0.0669	10.86	117.8	117.8	1760
1.000	0.0503	10.89	118.6	118.6	2357
2.000	0.0503	12.99	168.7	119.3	3353
1.000	0	11.00	121.00	121.00	∞
0.500	0	9.245	85.47	120.87	∞
0.250	0	7.780	60.53	121.06	∞

TABLE VII

Fourth Mode - Rotary Inertia Neglected

Incorrect End Conditions

C	D	L	L^2	$R = \frac{L^2}{\sqrt{C}}$	$\frac{1}{S} = \frac{L^2}{D}$
1.000	4.000	7.397	54.72	54.72	13.68
1.000	3.326	7.987	63.79	63.8	19.18
1.000	2.853	8.455	71.49	71.5	25.06
1.000	2.222	9.280	86.12	86.1	38.75
1.000	2.000	9.591	91.99	92.0	46.00
1.000	1.667	10.15	103.0	103.0	61.8
1.000	1.250	10.96	120.1	120.1	96.1
1.000	1.000	11.48	131.8	131.8	131.8
1.000	0.7133	12.14	147.3	147.3	206.5
1.000	0.5000	12.70	161.3	161.3	322.6
1.000	0.2857	13.28	176.4	176.4	617.3
1.000	0.2000	13.51	182.5	182.5	913
1.000	0.1176	13.78	189.9	189.9	1614
1.000	0.0717	13.94	194.2	194.2	2709
1.000	0.0503	13.96	194.9	194.9	3873
1.000	0	14.10	198.8	198.8	∞
0.5000	0	11.87	140.9	199.2	∞

B. Normal Modes of Oscillation of a Uniform Beam with Other End Conditions

In a previous report[1] solutions for normal modes were obtained (1) for a beam clamped on both ends, (2) for a beam hinged on both ends, and (3) for a cantilever beam.

The solutions obtained for these beams were made with the assumption that the effect of shear forces and rotary inertia could be neglected. The end conditions included the equating to zero the second and third derivatives of the deflection. In order to obtain completely accurate solutions the effects of shear forces and rotary inertia should be included as well as the correct end conditions of setting equal to zero the bending moment, M, as expressed by equation (44) and the shear force, V, as expressed by equation (45).

[1] Hagelbarger, Howe and Howe, University of Michigan Engineering Research Institute, External Memorandum on Investigation of the Utility of an Electronic Analog Computer in Engineering Problems (April 1, 1949), Air Force Contract W33-038-ac-14222 (Project MX-794).

CHAPTER II

NORMAL MODES OF OSCILLATIONS OF NON-UNIFORM BEAMS

In obtaining normal modes of oscillation of non-uniform beams it is convenient to consider the beam to be divided along its length into an equal number of parts. For each of these parts there is determined, or approximated, the average properties of that section of the beam. The beam is then considered to be made up of a discrete number of uniform parts. For each of these parts the equation to be solved is

$$CX^{iv} + D(1 + N)X'' - (1 - \frac{D^2N}{C})X = 0 . \qquad (40)$$

For each section of the beam the coefficients C, D and N take on different values. Resistors corresponding to the successive values of these coefficients can be introduced into the computing circuit by means of stepping relays, or their equivalent, as described in the previous report.

From equations (20) and (27), (21) and (28), and (36) it can be seen that the expressions for these coefficients are

$$C = \frac{EI}{\rho} \left(\frac{L}{\ell}\right)^2 \frac{1}{\omega^2} ,$$

$$D = \frac{EI}{kAG} \left(\frac{L}{\ell}\right)^2 ,$$

$$N = \frac{kG}{E} .$$

These relations involve, in addition to the physical properties of the section of the beam, the ratio L/ℓ and the frequency ω. The ratio L (the length of time of the solution) to ℓ (the length of the beam) is arbitrarily and conveniently chosen. The values of D and N can then be determined uniquely for each section of the beam and appropriate computer resistances found.

The value of C can not be determined uniquely since it involves the unknown quantity ω. However, since $1/\omega^2$ enters as a common constant factor for all the sections of the beam, the resistances involving C can be set up so that the part of C that changes from section to section will be handled by the stepping relays and the part involving $1/\omega^2$ will constitute other computer resistances to be changed from trial solution to trial solution. For example, the changing part of C could be put in the feedback resistance of an amplifier and the $1/\omega^2$ in the input resistance.

The details of the most satisfactory computer circuit have not yet been worked out. In general the computer would follow the plan of the circuit shown in Figure 1-6, with such modifications as would make a minimum the number of variable resistors required and make the solutions obtainable with a reasonable number of circuit changes from trial solution to trial solution. One very evident change is to arrange the two amplifiers (corresponding to A_7 and A_8 in Figure 1-6), for obtaining the proper end conditions, so that but one variable resistance is required for each amplifier.

In order to obtain a correct solution not only must the initial voltages on the integrating capacitors be adjusted so that the proper end conditions are observed, but also the variable part of C, i.e., $1/\omega^2$, must be adjusted so the solution is obtained in exactly the right amount of time, L.

CHAPTER III

PROPOSED WORK WITH COMPUTER

In view of information gained by the use of the computer while doing the experimental work included in this report, the following additional work is suggested.

1. The investigation of whether the mode shapes obtained with the revised end conditions are in agreement with theory.
2. The determination of the effect of the revised end conditions on the R vs 1/S curves.
3. The solutions for normal modes of vibration of uniform beams other than the free-free beam.
4. The solutions for normal modes of non-uniform beams.
5. The solution of the Bessel equation with variable resistances only in the feed back circuits of the amplifiers.

In the original solutions for normal modes of vibration of uniform beams it was discovered that the mode shapes obtained in some cases were such as to indicate a vibration of the center of gravity of the beam, a physical impossibility. As a consequence, the boundary conditions at either end of the beam were changed from, setting equal to zero the second and third derivations of the displacement, to, equating to zero the expressions for the bending moment, M and the vertical shear force, V. In connection with the present work it was found that these new end conditions improved the situation but left some doubt concerning whether the mode shapes obtained are in perfect agreement with the physical case. For small amplitudes of vibration of the bar the area under the mode shape curve, X, should be equal to zero for the center of gravity of the curve to remain on the axis. Consequently, the output of an amplifier, integrating the value of the displacement, X, for the duration of the solution, should give the required information. In case the value of this integral, at the end of the solution, is not equal to zero the matter of boundary conditions should be investigated further.

In Figure 1-10 are plotted curves of R vs 1/S on semi-logarithmic paper. For the cases where rotary inertia is neglected there are plotted data obtained by using (1) the original end conditions (second and third derivatives of the displacement equal to zero) and (2) the revised end conditions (M and V equal to zero). Since the points representing this data fall almost exactly on the same curves, it is suggested that further investigation be made to determine what effect, if any, the end conditions have on determining the frequency of vibration of the beam.

As implied earlier in the report it would be of interest to make studies of uniform beams other than the free-free beam. This would include the clamped-clamped, hinged-hinged and cantilever beams. The correct end conditions should be used with particular attention to how these end conditions affect the determination of the frequency. Mode shapes should be examined to see if their centers of gravity remain on the zero axis.

Since the ultimate objective of the project involving the study of vibrating beams by means of the electronic differential analyzer is to determine the normal modes of vibration of non-uniform structures, such as ships and aircraft, it is suggested that considerable attention be given to the development of a computing circuit which will handle this problem satisfactorily.

In connection with the solution of Bessel's equation as given in the previous report[1] it was pointed out that the step approximations of $1/x$ and $1/x^2$ are not equivalent to the reciprocals of the step approximations of x and x^2, respectively. Because of the difficulty involved in obtaining step approximations of the reciprocal of a variable in the neighborhood of zero, it is suggested that the equation be used in the form

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 .$$

In this form the step approximations for x and x^2 can be used satisfactorily. Some of the computer curves, such as those for $J_{1/4}$ and $J_{1/3}$ differed

[1] loc. cit.

materially from the theoretical curves. It would be of interest to determine how the proposed method of solution affects the disagreement.

CHAPTER IV

CHANGES IN THE COMPUTER

For the work associated with this report several changes and additions were made to the computer. These include an improvement in the technique of balancing individual amplifiers, a more exact method of measuring time intervals on the records, and a plug-in mounting for a number of amplifiers.

Balancing of Individual Amplifiers

In order to facilitate the balancing of individual amplifiers there was mounted on the chassis of each amplifier a double pole double throw switch. When this switch is closed in one direction (for balancing) the input of the amplifier is connected through a resistor R_i (approximately 160,000 ohms) to ground and simultaneously a feedback resistor R_f (approximately 10 megohms) is connected into the circuit (Figure 4-1). Since the values of R_i and R_f are selected to give large amplification, balancing is readily accomplished by adjusting one or both of the balancing controls of the amplifier until the output voltage as read on a DC vacuum tube voltmeter is reduced to zero. When the switch is closed to the other position the input and feedback impedances used for computing are properly connected. This procedure permits the operator to check the DC balance of the amplifier without disconnecting any of the impedances used in the computing operation.

Time Measurements

The results obtained in the experimental work depend on the square of L, the time, as measured on the oscillograph records, for a solution to be obtained. In order to obtain the desired accuracy in these measurements, a pen-marker was arranged to record time pulses on the oscillograph paper. These time pulses were obtained from a one revolution per second synchronous motor with a regular polygon cam operating a microswitch. For some of the records the time pulses were four per second; for others, ten per second. Simultaneously with the taking of the record there was observed and recorded the power line frequency

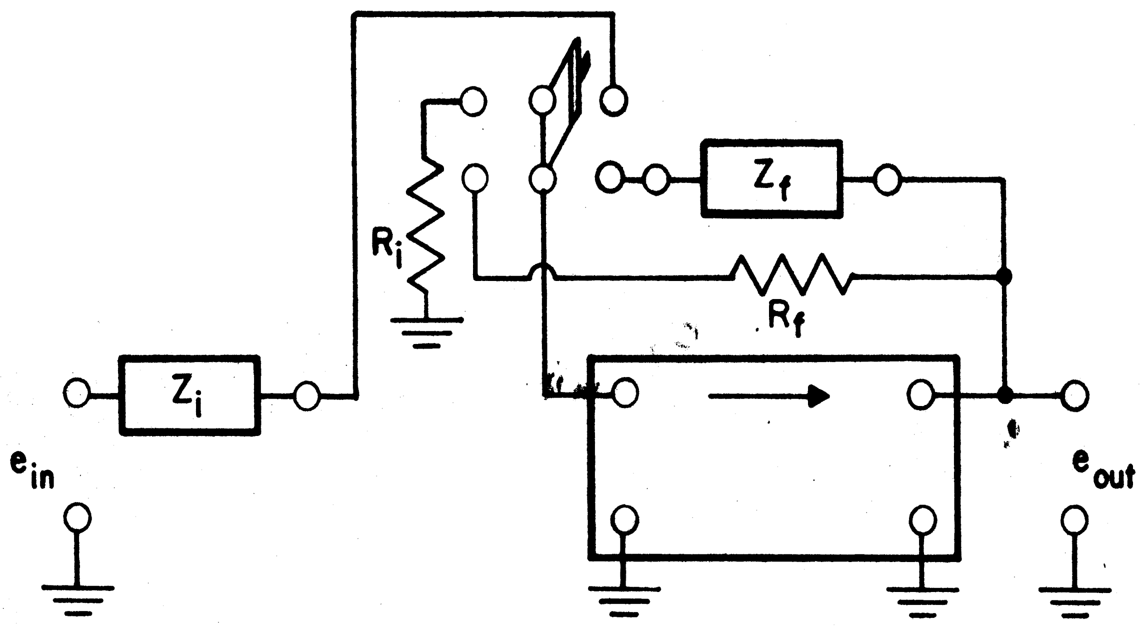

Figure 4-1. Balancing switch on amplifier chassis.

as indicated by a Leeds and Northrup frequency recorder. From these observations time measurements could be made to an accuracy of about one-tenth of one percent.

Plug-In Mounting

In order to arrange the amplifiers so that the computer should occupy a minimum amount of space, a plug-in mounting chassis was made to accommodate ten amplifiers. This chassis consisted of two shelves arranged step-wise, one above the other, each shelf supporting five amplifiers. The female power supply receptacles were mounted in vertical risers and spaced so that adjacent amplifiers were separated about one-eigth inch. All leads to these receptacles were shielded. In addition to giving a compact set-up, this arrangement dispenses with the clumsy power cable leading to each amplifier.

CHAPTER V

PROPOSED CHANGES AND MODIFICATIONS FOR THE COMPUTER

During the work with the computer it became evident that a number of modifications and additions should be considered. These proposed changes include automatic balancing of individual amplifiers, the application of initial conditions from a remote position, reconstruction of the basic amplifiers, different types of recording equipment, improvement in the making of accurate time measurements, etc.

These suggestions should be interpreted in the light of the work to be done with the computer. Some of them would be important for certain types of work and not for others.

Automatic Balancing of Individual Amplifiers

Since the individual amplifiers of the computer frequently require balancing it is suggested that some method of automatic balancing be considered. In the method here proposed the individual amplifiers are balanced consecutively. By means of auxiliary equipment each amplifier is automatically converted into a high gain DC amplifier, the output of which is reduced to zero by a suitable control mechanism. Each amplifier is caused to be balanced in turn by a stepping relay.

In Figure 5-1 are shown several additions to the individual amplifier chassis. One is a DPDT relay used for automatic balancing. In its normal, unenergized position this relay connects the input impedance Z_i and the feedback impedance Z_f to the amplifier for computing. When the relay is energized (by remote control) the amplifier is converted into a high gain DC amplifier using as input and feedback impedances the resistors R_i and R_f. When the balancing process is completed these two resistors are disconnected from the computer, except for a ground connection, so that any voltage picked up on the long balancing output lead is not introduced into the computer.

It should be noted that during the process of balancing the output terminal at e_o remains connected to the feedback impedance Z_f so that the initial

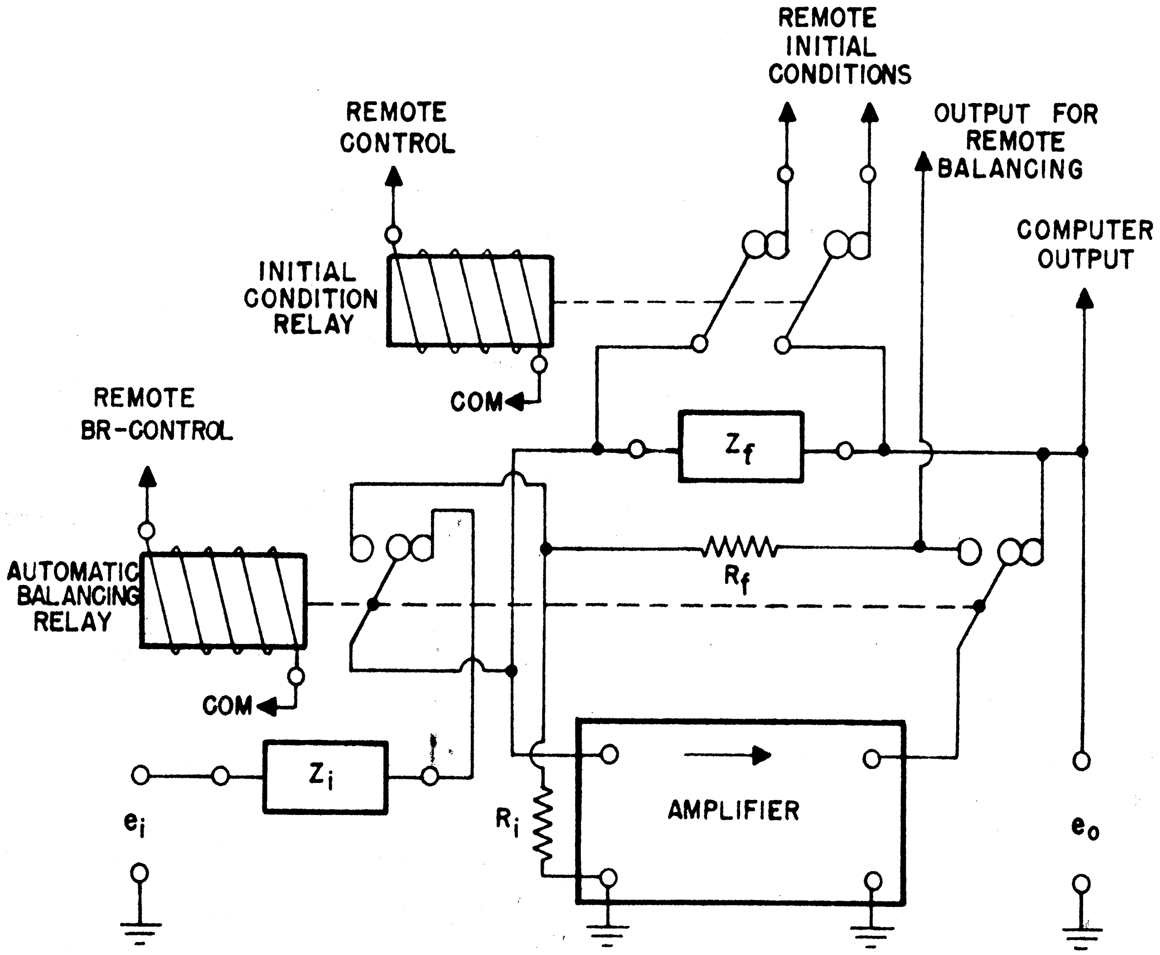

Figure 5-1. Partial wiring diagram of an individual amplifier chassis, showing the connections to the automatic balancing relay and the remote initial condition relay.

condition, if any, imposed on Z_f is maintained at the output terminals.

In Figure 5-2 are shown some of the details of the auxiliary automatic balancing equipment. The stepping relay has three banks of contacts, each bank having connections to the individual amplifiers. The terminals of bank I, labeled BA1, BA2, etc., are connected to the balancing output leads of the several amplifiers, which are connected consecutively to the input terminals of the special amplifier for balancing. The terminals of bank II, labeled BM1, BM2, etc., are connected to leads carrying power to small, reversible DC motors, one on each amplifier chassis for operating a balancing control. The terminals of bank III, labeled BR1, BR2, etc., are connected to leads which energize, at the proper time, the automatic balance relay on each amplifier chassis.

The stepping relay is stepped forward periodically by a contactor operated by a motor of suitable speed. As shown, the circuit provides for continuous automatic balancing. Provision could be made for one cycle of balancing, for push-button balance of any one amplifier, for automatic "homing" of the stepping relay when fewer than the maximum number of amplifiers are being used, etc.

As described above, during the process of balancing the output voltage of each amplifier is connected to the input of the special amplifier for balancing. This amplifier has input and feedback impedances, B_i and B_f, of such values as to give suitable gain to the amplifier. The output of this amplifier is fed into a load consisting of two resistors, R_1 and R_2, in series. These two resistors are given such values that the output voltage is limited for relatively large input voltages and that voltages of suitable magnitude are applied to the input of the controller. If necessary for the proper performance of the controller, attention should be paid to its input impedance.

The controller as shown in the circuit is indicated as performing a simple FORWARD-OFF-REVERSE operation, actuating either one of the two relays, electrically interlocked, to turn the balancing motors in the necessary direction. This method of balancing, if unmodified, would probably result in sluggish action (if the balancing motors are geared way down) or in excessive hunting (if the motors operate the balancing controls very rapidly). There are several methods by which this condition could be corrected. One is to use a controller which energizes the balancing motors for a fraction of a given time cycle, this fraction of time to be proportional to the error in balancing. This method is used

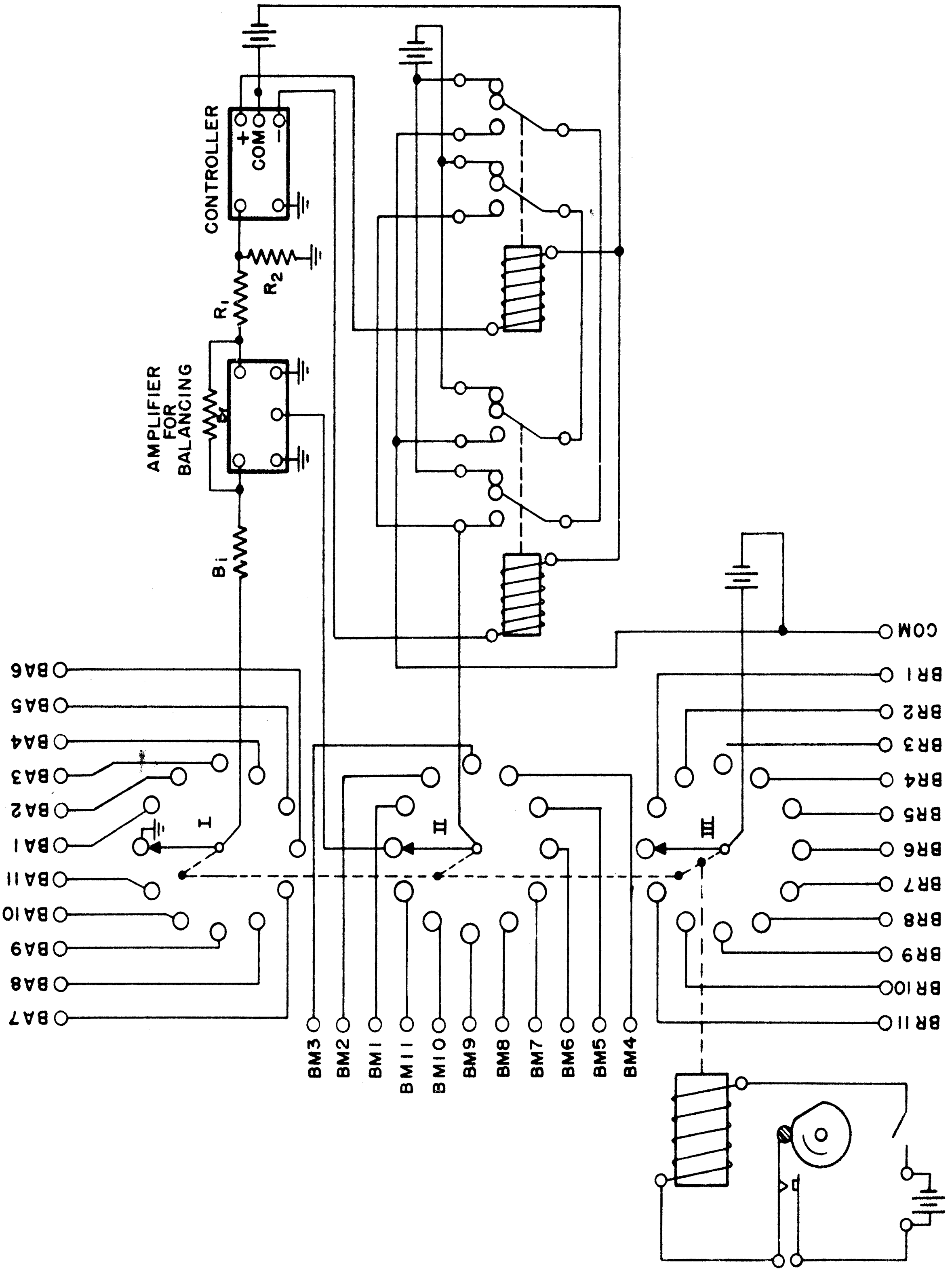

Fig. 5-2 Circuit for Controlling the Automatic Balancing of Individual Amplifiers.

by the Leeds and Northrup Company in their Micromax frequency controller. Another method would be to use a Speedomax (Leeds and Northrup) controller with a potentiometer or rheostat mounted on it in such a way as to cause the speed of the balancing motors to be proportional to the error in balance. A still further method would be to use electromagnetic damping on the balancing motors.

For this method of balancing it is suggested that the balancing motors be small 24 V DC permanent magnet motors of which there are many available from war surplus. Each motor should be geared down to operate one of the two DC amplifier balance controls. This control could advantageously be of the helipot type. The details of these motor operated controls are not described here, but in Figure 5-3 is shown a dial on the amplifier chassis to indicate the position of the automatic balancing control.

Reference to Figure 5-2 shows that the special amplifier for balancing is itself balanced automatically in its turn. This amplifier needs no balancing relay. The chassis relays for automatic balancing and the small DC motors have one lead in common. The battery furnishing power for the balancing motors is floating so that either one of its terminals is connected as required to the common line. If desired two batteries could be used, one with the positive lead connected to the common line and the other with the negative lead connected.

This method of balancing would probably result in some economy of equipment as compared with some other methods. However, since the amplifiers are balanced one after the other it entails some loss of time. How serious this loss would be depends upon the types of problems being solved as well as the frequency with which the balancing needs to be done.

A much more efficient method would be to have an automatic balancing mechanism for each individual amplifier[1]. Such a system might consist of a two-phase AC motor driving the balance control, this motor having one phase of its supply voltage furnished by a DC to AC converter-servo-amplifier such as is used in the Speedomax (Leeds and Northrup) of in the Electronik (Brown Instrument Co.) recorders. This method would permit the balancing of all amplifiers simultaneously in a few seconds.

A compromise between these two methods would be to use individual two-phase motors on each chassis with a single DC to AC servo-amplifier, the automatic balancing to be done consecutively but rapidly.

[1] See Appendix I

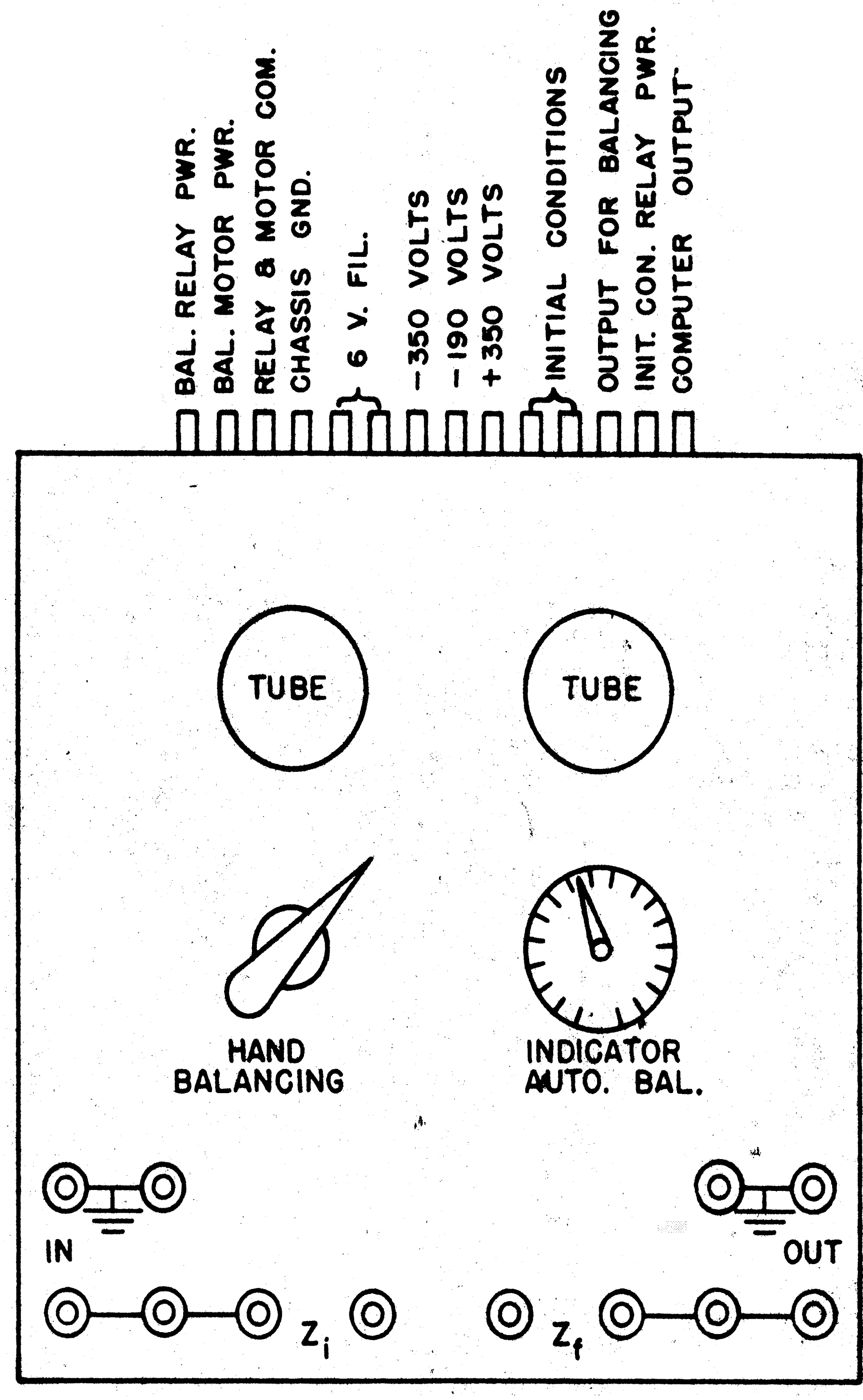

Figure 5-3. Amplifier chassis showing indicator for automatic balancing.

Initial Conditions

In order to keep the computing assembly more compact and convenient for operation it is proposed that an initial condition relay be mounted permanently on each chassis, the connections to it being as indicated in Figure 5-1. This scheme permits the initial condition voltage to be applied to the feedback capacitor from a remote position. It would be convenient to have the remote initial condition terminals for the several amplifiers mounted on a panel. On the same panel, or on an adjacent one, there should be available a suitable number of variable voltage sources for the initial condition voltages. Connections would be made by patch cords. In the cases where the initial condition voltage is zero, a shorting resistor of low value could be placed across the remote initial condition terminals.

The coils of all of the initial condition relays would be connected in parallel for simultaneous operation by a single starting switch. When these relays are energized the remote initial condition voltage supplies are completely disconnected from the computing system, as can be seen in Figure 5-1.

In case there is a situation where one initial condition voltage depends on another one, it could easily be arranged to have this relationship adjusted automatically by means of a servomechanism similar to that used in the automatic balancing of the amplifiers. This provision would have been particularly useful in the solution of the problem on the determination of the modes of vibration of oscillating beams where two such interrelations between initial conditions occur.

Amplifiers

In doing the experimental work contributing to the results summarized in the first part of this report considerable difficulty was experiencedwith the basic DC amplifiers. Without apparent cause an amplifier would change so that it could no longer be balanced with the two balancing controls provided. The trouble was found to be caused by large changes in the values of some of the 1 megohm and 2 megohm resistors in the amplifier circuit. There seemed to be some correlation with the hot, humid weather prevailing at the time. Consequently, the computer was moved to a room with constant temperature (70° F) and constant humidity (50%). The mortality rate of the amplifiers changed from one or two a day to almost none. In the six weeks during which the computer was used in this room only one amplifier had to be repaired.

While it was never determined just what caused the resistors to change values, the fact that the difficulty disappeared as soon as the amplifiers were put into a constant temperature and constant humidity atmosphere suggests high humidity as a large contributing factor. In order to avoid difficulties of this kind it is suggested that the ordinary carbon resistors used in the amplifiers be replaced by a more stable type, such as the Continental X-type. Alternately, or in addition, the resistors could be covered with a layer of suitable wax to prevent the absorption of moisture. The use of the more stable type of resistor is recommended.

On the other hand, if the trouble experienced was really due to atmospheric conditions it would be desirable to investigate the effect of these conditions on other electronic equipment. The results of such an investigation might indicate the desirability of housing the computing equipment in a room in which the atmosphere is maintained at suitable temperature and humidity.

Since the amplifiers used in this work were constructed other DC amplifiers with more satisfactory characteristics have been developed. This general field should be investigated with the idea of adopting new basic amplifiers for use in the computer.[1]

Recording Equipment

The recording equipment used in the experimental work consisted of a Brush[2], Model BL-202, double channel magnetic oscillograph and two Brush, Model BL-913, DC amplifiers designed to work into the magnetic oscillograph. Considerable difficulty was experienced with zero drift of these amplifiers, so much so that the work was made tedious and much time consumed in taking unusable records.

The Brush magnetic oscillograph has the advantage of speed in recording, being useful from DC up to frequencies above 60 cycles per second. For

[1] New and improved basic DC amplifiers are now under construction and should be available by June.

[2] Brush Development Company, Cleveland, Ohio.

moderate accuracy it is suggested that this oscillograph be used with a more satisfactory input amplifier.[1]

Where more accuracy is demanded, and speed is not so important, it is suggested that either Speedomax (Leeds and Northrup Company) or Electronik (Brown Instrument Company) recording potentiometers be tried. Both of these instruments have wide chart scales, ten and eleven inches, respectively, and record with an accuracy of the order of one-fifth of one percent. At least one of them (Speedomax) can be obtained with chart speeds up to six inches per minute.

Because of the high input impedance of the recording potentiometer no intermediate amplifier is needed between the computer and the recorder. The Speedomax recorder with a range of 0-10 millivolts has an effective input impedance, when off balance, of the order of magnitude of 7500 ohms. As a consequence it could be "driven" from a voltage divider connected directly across an output of the computer.

These recording potentiometers have the disadvantage of being, primarily, DC recorders with the result that they can satisfactorily record only very low frequency alternating voltages. Consequently their use would require the slowing down of the computer solution to such frequencies as the recorders would handle satisfactorily.

For general computer work it would be highly desirable to have both the high speed magnetic recorders with suitable amplifiers and the more accurate recording potentiometers.

Time Measurements

In using the computer time is the independent variable and if maximum use is to be made of the result obtained, the length (in time) of a solution should be known to a high degree of accuracy. While the method for obtaining accurate time pulses on the computer records as described above under Computer Changes is practicable, there is an alternative method of more convenience and accuracy.

Several manufacturers produce secondary frequency standards of high accuracy. The Hewlitt-Packard Company's Models 100A and 100B produce frequencies

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1 See Appendix II

of 100, 10, 1 and 0.1 kilocycles per second with accuracies of $\pm$ 0.01% and 0.001% respectively. A supplementary frequency divider is available to obtain a frequency of ten cycles per second. The General Radio Company offers for sale Type 1100-A primary and secondary frequency standards with a frequency drift of the order of one part in 10^7 or 10^8 per day. These standards give frequencies of 100, 10, 1 and 0.1 kilocycles per second.

The General Radio Company uses multivibrators for dividing the frequencies. No 100 to 10 cycles per second multivibrator is offered for sale by General Radio, but this multivibrator could be constructed easily. One disadvantage of multivibrator frequency division is, if any failure, or loss of synchronism, occurs in the system of frequency division, one or more of the multivibrators will become free running and give inaccurate frequencies. However, for well designed circuits the probability of this occurrence is small.

The Hewlitt-Packard Company uses a regenerative modulation frequency dividing circuit in its instrument. This circuit has the characteristic that it is non-regenerative in the sense that it can not function without a suitable input frequency and hence can not operate free running.

For records taken at high speed the tenth-second time pulses could be recorded directly on the recording paper by using a suitable pen. For slower speeds one-second pulses would probably be desirable. These could be obtained by another multivibrator or by any other suitable frequency dividing device. If desired, a synchronous motor could be driven by a power amplifier, using 50 or 100 cycles per second obtained from the standard frequency source. Gears and cams connected to the motor could be used to obtain any time pulses desired.

In any event the time pulses used would be so accurate that no corrections would be needed. In case variable input or feedback resistances are to be obtained by stepping relays, or their equivalent, pulses from the standard source should be used for operating these relays.

APPENDIX I

CLOSED-LOOP CONTINUOUS DRIFT COMPENSATION

For future work it is planned to use DC computer amplifiers equipped with an additional drift-free slow response feedback loop to compensate slow drifts in the basic DC computer amplifier. A slow response drift-free DC amplifier with an amplification of several hundred is connected with its input to the input junction of the high-gain computer amplifier. The output of the drift-free amplifier is connected to the other input of the computer amplifier which was normally used for manual balancing. The polarity is such that if the input junction of the computer amplifier is not at zero potential[1] then the drift-free amplifier applies a voltage to the balancing input of the computer amplifier resulting in an output voltage which acting through the feedback impedance returns the input junction to zero.

The slow response drift-free DC amplifier is of the type which uses a synchronous mechanical vibrator switch to modulate the input signal on a 60-cycle supressed carrier. The sidebands are then amplified by an AC amplifier and demodulated back to a relatively large DC signal by another synchronous vibrator switch. The effective drift at the input of such a slow response amplifier can be made negligibly small.

The application of this closed-loop continuous drift compensation scheme in no way changes the high frequency response of the computer amplifier.

L.L.R.

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[1] The condition for ideal operation of the computer amplifier (input grid current neglected) is that the potential of the input junction remains exactly at zero.

APPENDIX II

MODIFICATION OF THE BRUSH[1] DC AMPLIFIER BL-913 FOR USE WITH AN ELECTRONIC DIFFERENTIAL ANALYZER

by

L. L. Rauch and R. H. Dougherty

The problem was to make a simple modification in the Brush D.C. Amplifier Model BL-913 in order to reduce excessive zero-point drift. The Brush Company developed the amplifier primarily for use with their Magnetic Direct Inking Oscillographs such as Model BL-202. The amplifier and recorder together form a convenient means of recording the solutions of an electronic differential analyzer, however, the zero-point drift of the amplifiers over periods of several minutes was found to be many times greater than the maximum computer errors.

Since the weak link in the computer accuracy was the Brush amplifiers there were two alternatives, either design new driving amplifiers for the recorders, or modify the Brush instruments. For economy and convenience modification of the Brush instruments offered the best solution.

The electronic differential analyzer never requires a full scale recorder sensitivity greater than approximately one volt. The unmodified Brush equipment has approximately fifty times this sensitivity and the weak point is that the Brush amplifier sensitivity control is a voltage divider at the input. Thus the amplifier operates at full gain for all sensitivity adjustments and no reduction in the zero-point drift is obtained when operating at reduced sensitivity.

Most of the voltage drift comes from the first stage and rather than attempt to reduce the equivalent voltage drift at the input grid it appears more practical, in view of the lower sensitivity requirements of the computer,

[1] Brush Development Company, Cleveland, Ohio

to override the drift by operating at a larger equivalent input signal voltage. This is easily accomplished by an inverse feedback loop from the output of the amplifier to the input stage which reduces the voltage gain by a factor of 50 from the original 1000.

Herewith is presented the determination of the feedback loop, the design of a D.C. calibration circuit and the characteristics of the modified Brush D. C. Amplifier and Recorder.

SUMMARY

The modifications for the feedback loop are simple and inexpensive, two resistors and a condenser are required (Figure 1). If the original characteristics of the amplifier are desired again it is only necessary to short R_f and open R and the stabilizing condenser. The characteristics of the modified amplifier are as follows:

1. The drift is less than the width of the pen line over hours of operation as compared to as much as 5 mm on an unmodified amplifier over periods of several minutes.
2. The amplifier - recorder gain versus frequency response is flat to 25 cps (Figure 5) compared to 80 cps on an unmodified amplifier. This reduced bandwidth is more than enough for the computer application so no compensation was included to extend it.
3. The maximum voltage gain of the amplifier is 20, compared with approximately 1000 originally.
4. The balance control has a range of 5 mm each way from center compared to more than full scale each way on the unmodified amplifier.

A D.C. calibration voltage is more convenient than the 60 cycle A.C. employed originally because the A.C. required critical adjustments and had to be readjusted for the frequency response of different recorders. The D.C. calibration circuit is designed to put plus or minus one volt D.C. on the input

grid (Figure 2). The original milliammeter and 50 K potentiometer (R-46) are used. The extras required are a four pole - double throw switch (C-H 8888) which is off unless depressed, one resistor (R_p) and a 6 volt battery. The milliammeter is recalibrated to read 1 volt at half scale (.375 ma) and is a constant check on the calibration circuit. R-46 is adjusted for a half scale reading on the milliammeter.

To calibrate the amplifier remove any input voltage, turn the attenuator switch to "calibrate", throw the calibration switch to plus or minus one volt, observing that the meter is reading half scale, and vary the gain control for desired output.

THE FEEDBACK LOOP

To stabilize the amplifier against oscillation when employing the feedback loop the .01 mf condenser shunting R-14 was necessary. Bode's "Minimal Phase Shift Criterion" indicated that oscillation would occur upon application of 34 db of feedback ($\mu\beta = 50$) because of the 27 db per octave slope shown in Figure 4; by Bode's criterion the slope must not exceed 12 db per octave. The shunting condenser acts as part of an attenuation network to give a maximum slope of 8 db per octave in the first 34 db of attenuation without feedback.

The feedback loop was designed to leave the amplifier gain approximately 20, or a little less than one volt input for full scale deflection on the recorder. The values of R and R_f were determined as follows:

db gain at 20 cps without feedback (Figure 4) = 60

Voltage gain without feedback = 1000 = μ

Voltage gain with feedback = 20 = A

From negative feedback relationship,

$$A = \frac{\mu}{1-\mu\beta}\ ; \quad \beta = .049$$

Also

$$\beta = \frac{R_f}{R+R_f}\ ; \quad R_f = \frac{\beta R}{1-\beta}$$

R was chosen as 5000 ohms so R_f = 250 ohms.

THE CALIBRATION CIRCUIT

The D.C. calibration circuit for plus or minus one volt input is made up of circuit components already in the amplifier except for a 6 volt battery, a double throw - four pole switch and a resistor. When the switch is depressed R_p is placed in parallel with R-40 and the series network of R-45 and R-56 so that one volt will appear across the combination at .375 ma. The 50K potentiometer (R-46) is adjusted for a half scale reading on the milliammeter which is .375 ma, thus the meter is a check on the condition of the calibration circuit. The calibration switch does not open the input lead, thus it is necessary to disconnect any input voltage when calibrating.

There is space for a 6 volt "A" battery on top of the chassis; the battery was wrapped with asbestos paper to reduce voltage variation due to the heat from the tubes.

The attenuation switch was relabeled as shown in Figure 3.

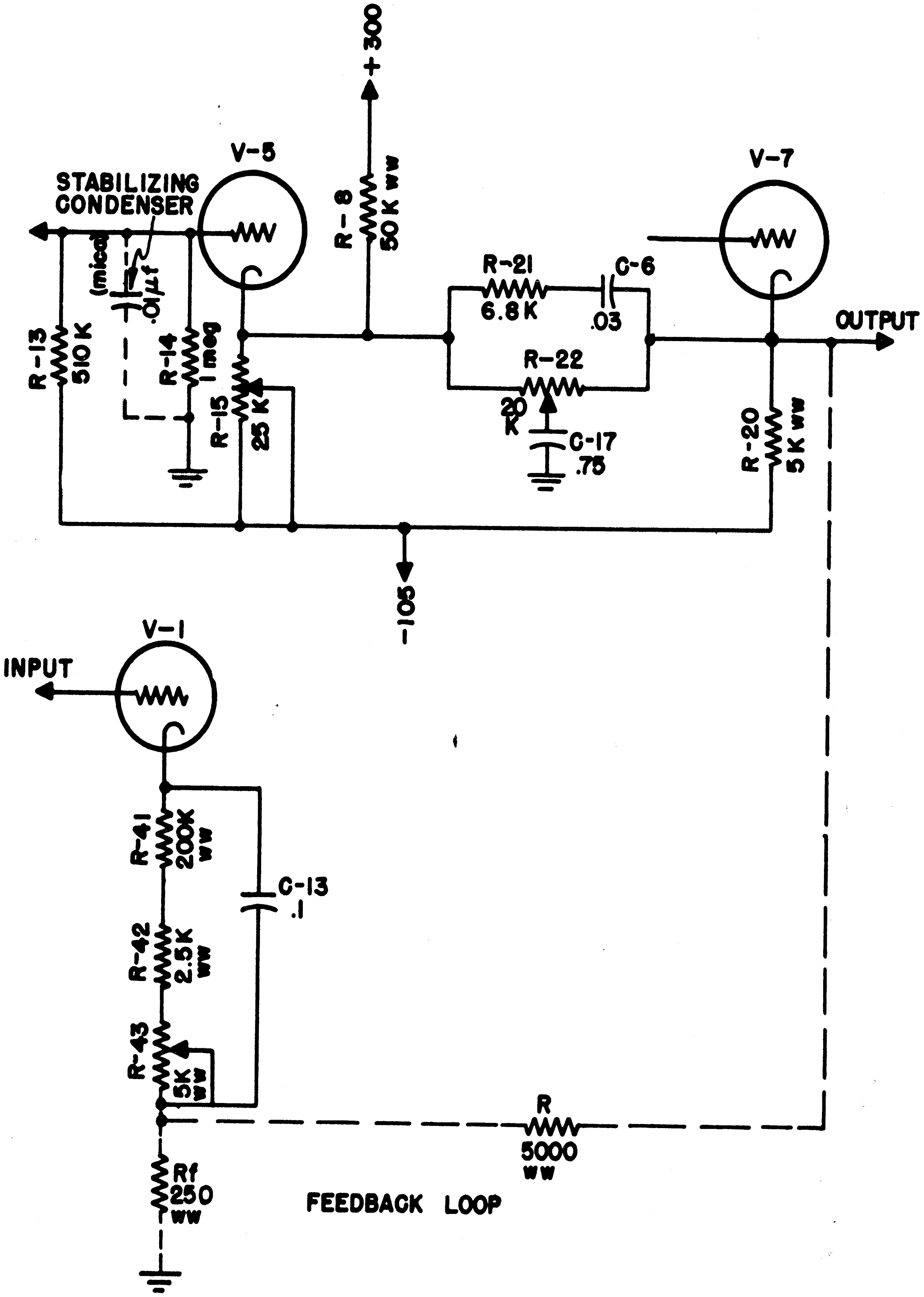

Figure 1. Modifications of Brush amplifier BL-913.

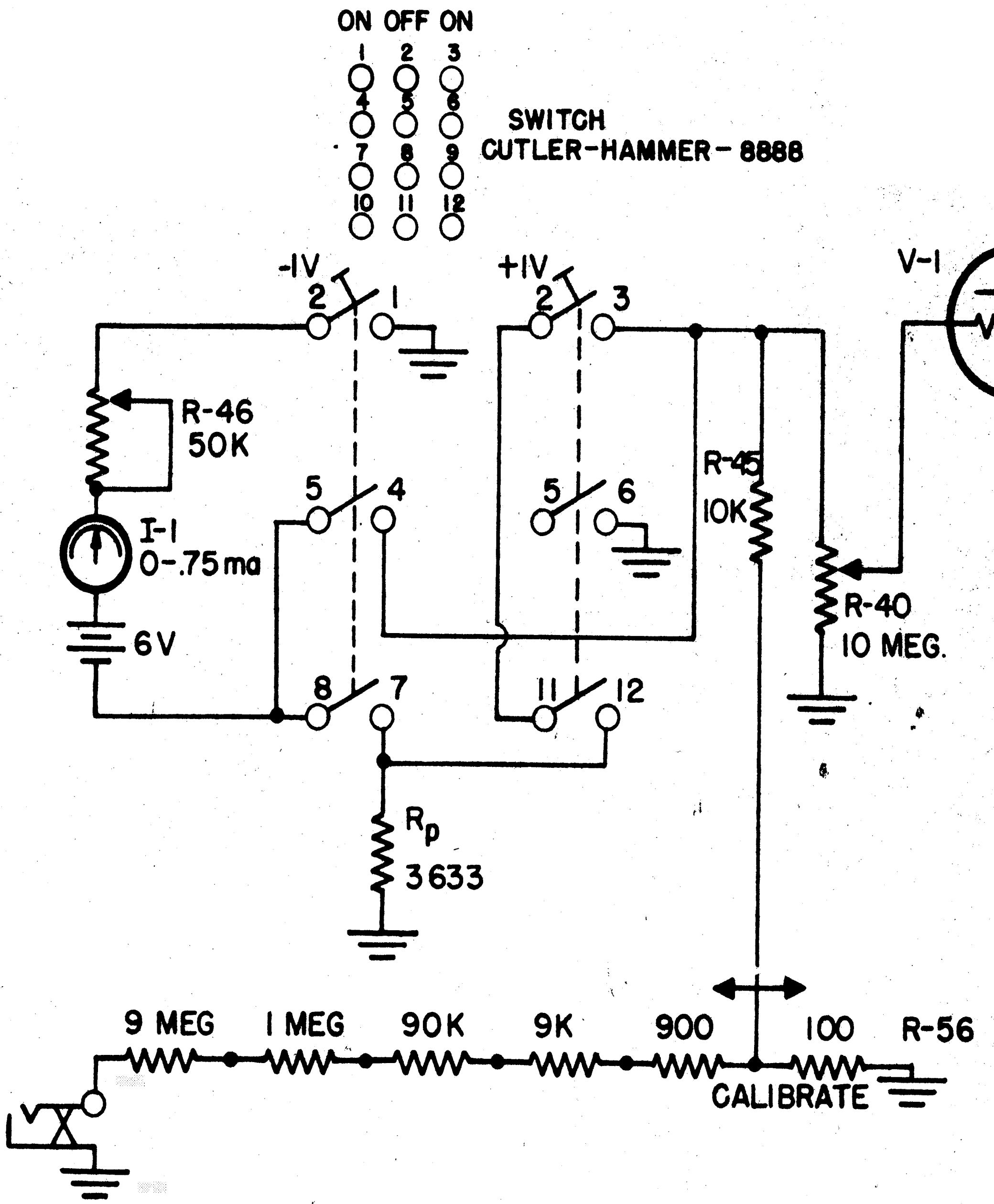

Figure 2 - Calibration Circuit (+) or (-) IV D-C Input.

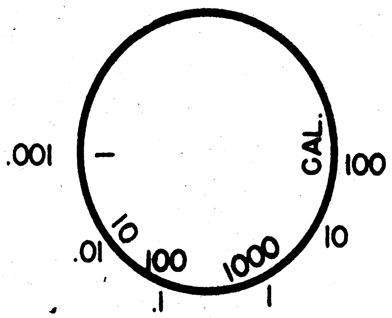

Figure 3 - Recalibration of Attenuator Control

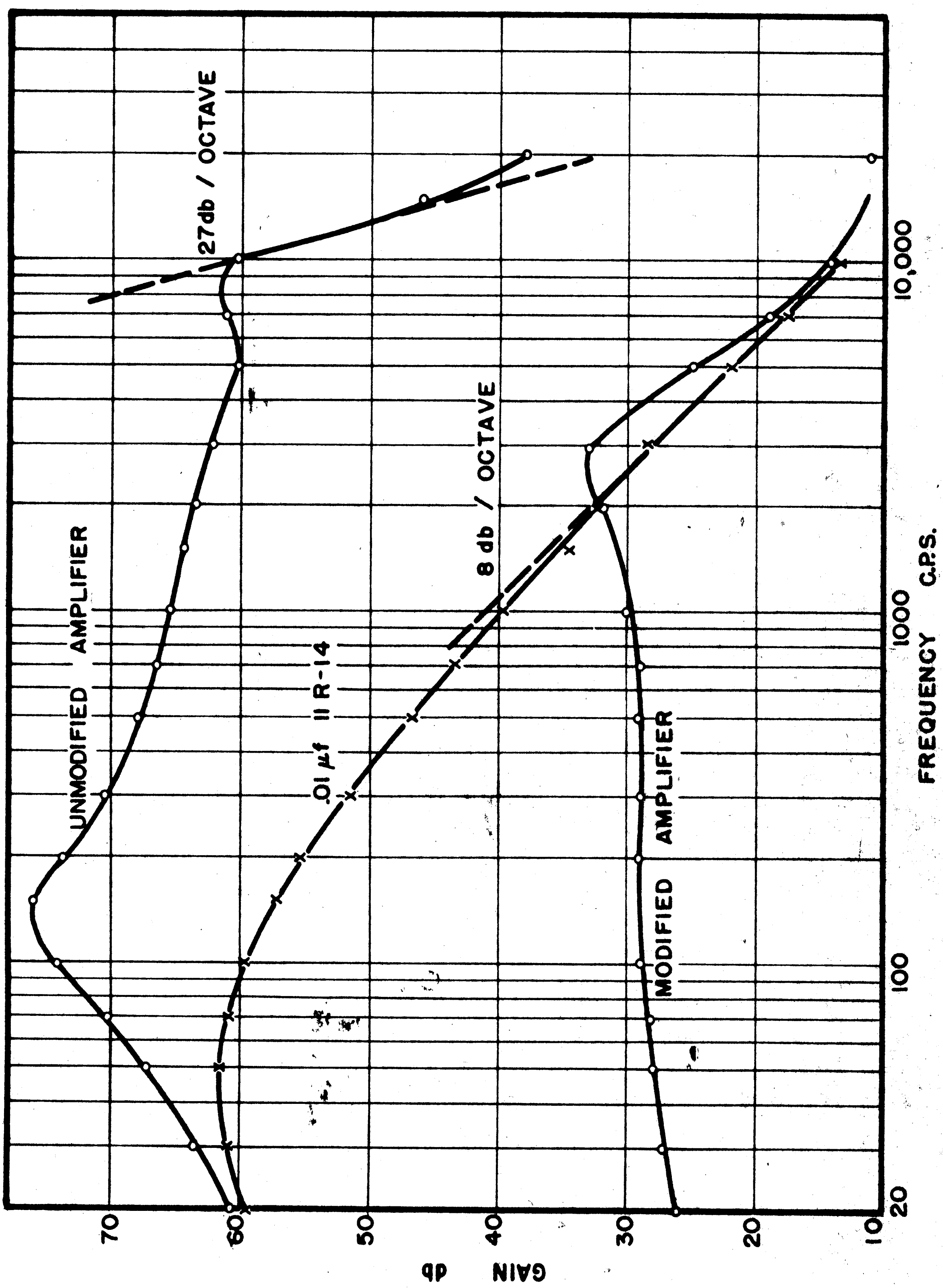

Figure 4. Frequency response of Brush d.c. amplifier BL-913.

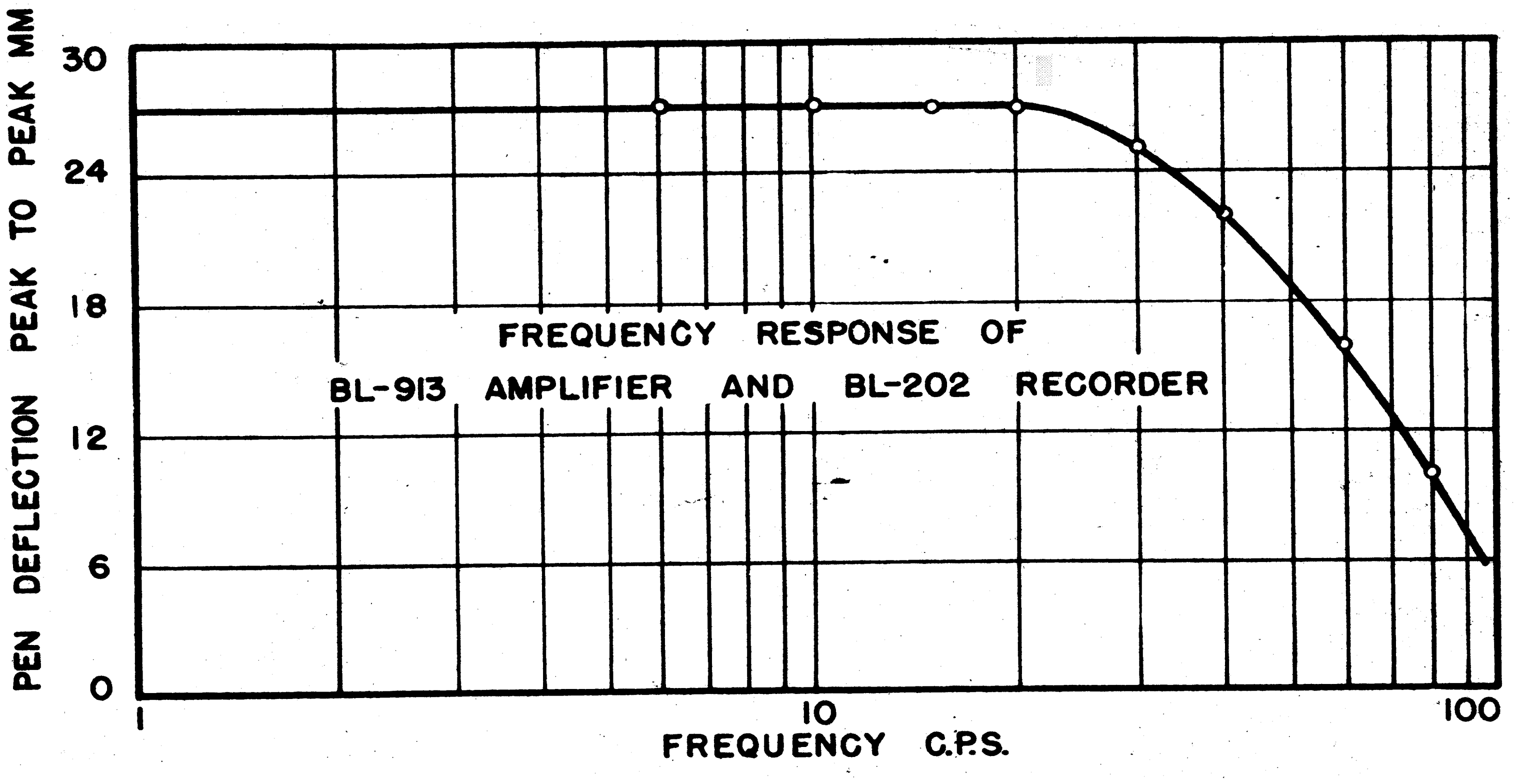

Figure 5 - Frequency response of BL-913 Amplifier and BL-202 Recorder