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SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY
DIFFERENCE METHODS USING THE
ELECTRONIC DIFFERENTIAL ANALYZER

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SUMMARY

This final report summarizes the investigation of the solution of both linear and nonlinear partial differential equations by difference techniques using the electronic differential analyzer. Complete details of the research are given in three technical reports. Two categories of physical problems are considered: (1) the lateral vibration of beams, and (2) heat flow. In both cases time is preserved as a continuous variable, while the spacial variable is broken into stations or cells. Thus spacial derivatives are approximated by finite differences, and the resulting set of simultaneous ordinary differential equations are solved by the electronic differential analyzer.

Theoretical accuracy of the difference method as a function of the number of cells along the spacial variable is investigated by comparing the eigenvalues (normal-mode frequencies for the beam, decay constants for heat flow) and normal-mode shapes with those for a continuous medium. Results show suprisingly few cells are required for representation of the first few modes to several percent accuracy. Theoretical as well as computer solutions are obtained for cantilever, hinged-hinged, free-free, and clamped-clamped beams, both uniform and non-uniform. Beams with viscous damping and time-varying boundary conditions were also solved, and the vibration of beams including deflection due to transverse shear are treated by the difference method. Cantilever beams with nonlinear damping terms such as velocity-squared damping and Coulomb damping are solved successfully.

Dynamic heat-flow problems treated by the difference method, both theoretically and with electronic differential analyzer solutions, include one, two, and three-dimensional flow in rectangular media as well as flow in cylindrical and spherical media. Change of independent variable to improve accuracy and to solve flow in semi-infinite media is demonstrated. The nonlinear heat equation for a medium having a conductivity

proportional to temperature is solved with the electronic differential analyzer and results are compared with a particular exact solution.

As a result of these investigations it seems clear that the electronic differential analyzer is an excellent tool for rapidly solving many partial differential equations, both linear and nonlinear. There is a one-to-one correspondence between resistor values in the circuit and physical parameters in the problem, and the desired dependent variables (e. g. , beam displacement, velocity, bending moment, or temperature, heat flux, etc.) are all available as time-varying output voltages of the computer.

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SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY
DIFFERENCE METHODS USING THE
ELECTRONIC DIFFERENTIAL ANALYZER

INTRODUCTION

Whenever we wish to solve partial differential equations by means of the electronic differential analyzer, it is necessary first to convert the equations to ordinary differential equations, since the analyzer can integrate with respect to only one variable, namely time. If the partial differential equation is linear, this conversion to ordinary differential equations can often be done by separation of variables, which results in ordinary differential equations of the eigenvalue type. The normal modes, or eigenfunctions, can then be found, after which the complete solution is built up by combining the normal modes.

The above method of separating variables and obtaining a series type of solution can be carried out fairly efficiently on an electronic differential analyzer.¹⁻⁴ Certainly, for most problems the analyzer is much faster than any hand methods. But for the engineer who is interested in getting quantitative answers to specific problems even the analyzer approach might seem somewhat tedious. Then too, the control engineer would often like to have a real-time simulation of the system being controlled, and in many cases partial differential equations are required to describe this system. It therefore would be highly advantageous to solve the partial differential equations directly with the electronic differential analyzer. This can be done by replacing some of the partial derivatives by finite differences in order to convert the original partial differential equations into a system of ordinary differential equations.

Assume we are interested in solving a partial differential equation in which the dependent variable $y(x, t)$ is a function of both a distance variable x and a time variable t . Instead of measuring the variable y at all distances x , let us measure y only at certain stations along x ; thus, let y_1

be the value of y at the first x station, y_2 be the value of y at the second x station, y_n be the value of y at the n th x station. Further, let the distance between stations be a constant Δx .

Now clearly a good approximation to $\partial y / \partial x \big|_{1/2}$ (i. e., the partial derivative of y with respect to x at the $1/2$ station) is given by the difference

$$\frac{\partial y}{\partial x} \bigg|_{1/2} = \frac{y_1 - y_0}{\Delta x} \quad (1-1)$$

In fact the limit of equation (1-1) as $\Delta x \rightarrow 0$ is just the definition of the partial derivative at that point. Writing (1-1) in more general terms

$$\frac{\partial y}{\partial x} \bigg|_{n-1/2} = \frac{y_n - y_{n-1}}{\Delta x} \quad (1-2)$$

In the same way

$$\frac{\partial^2 y}{\partial x^2} \bigg|_n = \frac{1}{\Delta x} \left\{ \frac{\partial y}{\partial x} \bigg|_{n+1/2} - \frac{\partial y}{\partial x} \bigg|_{n-1/2} \right\} \quad (1-3)$$

or from equation (2-2)

$$\frac{\partial^2 y}{\partial x^2} \bigg|_n = \frac{y_{n+1} - 2y_n + y_{n-1}}{(\Delta x)^2} \quad (1-4)$$

Thus we have converted partial derivatives with respect to x into algebraic differences. The only differentiation needed now is with respect to the time variable t , so that we are left with a system of ordinary differential equations involving dependent variables $y_0(t), y_1(t), \dots, y_n(t), \dots$

SOLUTION OF THE HEAT EQUATION

2.1 Basic Equations for Heat Flow

The basic equation of heat flow is given by

$$c \delta \frac{\partial u}{\partial t} = \nabla \cdot K \nabla u + S \quad (2-1)$$

where

u = temperature and is a function of the spacial coordinates and time,

- K = thermal conductivity, in general, a function of the spacial coordinates,
 c = specific heat, a function of spacial coordinates,
 δ = density, also a function of spacial coordinates,
 \bar{t} = time,
 \bar{S} = rate of heat supplied per unit volume by sources, in the medium, a function of spacial coordinates and time.

The left-hand side of equation (2-1) represents the rate at which heat is stored in a unit volume due to the heat capacity of the medium. The right-hand side represents the rate at which the unit volume receives heat, first due to heat conduction into the volume from the neighboring medium (the $\nabla \cdot K \nabla u$ term) and second, due to the heat flow into the volume from sources within the volume itself (the \bar{S} term). The conductivity times the gradient of the temperature ($-K \nabla u$) is a vector representing the heat flux. The components of $-K \nabla u$ represent the heat flow through a unit surface normal to the direction along which the component is taken.

In a given heat-flow problem it is necessary to stipulate spacial boundary conditions either on the temperature u or the heat flux $-K \nabla u$, as well as the initial temperature distribution throughout the medium.

In technical report AIR-10⁵ the form of Equation (2-1) and its solution by difference methods and separation of variables is discussed for Cartesian coordinates, cylindrical coordinates, and spherical coordinates. For example, in Cartesian coordinates for heat flow through a homogeneous slab we have the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (2-2)$$

where x is dimensionless distance through the slab and t is dimensionless time. For boundary conditions

$$u(0, t) = u_0(t) \quad (2-3)$$

$$\frac{\partial u}{\partial x}(1, t) = 0 \quad (2-4)$$

corresponding to a prescribed temperature $u_0(t)$ at the left boundary and zero heat flow at the right boundary, the difference equations become

$$\begin{aligned} \frac{du_1}{dt} &= \frac{1}{(\Delta x)^2} [u_2 - 2u_1 + u_0(t)] \\ \frac{du_2}{dt} &= \frac{1}{(\Delta x)^2} [u_3 - 2u_2 + u_1] \\ &\vdots \\ &\vdots \\ \frac{du_{N-3/2}}{dt} &= \frac{1}{(\Delta x)^2} [u_{N-1/2} - 2u_{N-3/2} + u_{N-5/2}] \\ \frac{du_{N-1/2}}{dt} &= \frac{1}{(\Delta x)^2} [-u_{N-1/2} + u_{N-3/2}] \end{aligned} \tag{2-5}$$

where Δx is the interval between stations along x and $N-1/2$ is the total number of x stations across the slab.

This set of $N-1/2$ simultaneous first-order differential equations can be set up and solved directly on the electronic differential analyzer. In the technical report⁵ the time-dependent solutions for the temperature at each station are shown for a number of representative initial conditions.

3. SOLUTION OF THE BEAM EQUATION

3.1 Basic Equations for Lateral Beam Vibrations

The basic equation for lateral displacement y of a beam is given by

$$\frac{\partial^2}{\partial \bar{x}^2} EI(\bar{x}) \frac{\partial^2 y}{\partial \bar{x}^2} + \rho(\bar{x}) \frac{\partial^2 y}{\partial t^2} = \bar{f}(\bar{x}, \bar{t}) \tag{3-1}$$

where

\bar{x} = distance along the beam,

EI = flexural rigidity, in general a function of x ,

ρ = mass per unit length, in general a function of x ,

\bar{t} = time

$\bar{f}(\bar{x}, \bar{t})$ = external applied force per unit length.

The bending moment M is given by

$$M(\bar{x}, \bar{t}) = EI(\bar{x}) \frac{\partial^2 y}{\partial \bar{x}^2} \quad (3-2)$$

while the shear force V is

$$V(\bar{x}, \bar{t}) = \frac{\partial M(\bar{x}, \bar{t})}{\partial \bar{x}} \quad (3-3)$$

Boundary conditions depend on the type of end fastening. For a cantilever (clamped end at $\bar{x} = L$

$$y(L, \bar{t}) = \frac{\partial y(L, \bar{t})}{\partial \bar{x}} = 0. \quad (3-4)$$

For a free end at $\bar{x} = L$

$$M(L, \bar{t}) = V(L, \bar{t}) = 0 \quad (3-5)$$

while for a simple supported (hinged) end at $\bar{x} = L$

$$y(L, \bar{t}) = M(L, \bar{t}) = 0. \quad (3-6)$$

In technical report AIR-7 the normal-mode frequencies and shapes for Equation (3-1) are given for uniform free-free, cantilever, hinged-hinged, and clamped-clamped beams, along with theoretical solutions using the difference approximation. Electronic differential analyzer circuits and solutions using the difference technique are presented for the above cases and for nonuniform beams, beams with viscous damping, beams with time varying boundary conditions, and beams for which Equation (3-1) is modified to include transverse shear effects (this is necessary for beams whose thickness is not small compared with their length). In addition, technical report AIR-8 presents analyzer solutions for beams with nonlinear damping terms, both velocity-squared damping and coulomb (dry-friction) damping.

3.2 Difference Equations for Uniform Cantilever Beam

As a simple example, consider the equation for lateral displacement y of a uniform beam. It can be written as

$$\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = f(x, t) \quad (3-7)$$

where x is dimensionless distance (the beam length in x is unity) and t is dimensionless time. To solve Equation (3-7) using the difference method we consider the transverse displacement y only at equally spaced stations approximately as

$$m_n = \frac{\partial^2 y}{\partial x^2} \Big|_n \cong \frac{1}{(\Delta x)^2} (y_{n+1} - 2y_n + y_{n-1}) \quad (3-8)$$

where Δx is the distance between stations. In the same way the complete difference equation at the n -th station becomes

$$\frac{d^2 y_n}{dt^2} = - \frac{1}{(\Delta x)^2} (m_{n+1} - 2m_n + m_{n-1}) + f_n(t) \quad (3-9)$$

where $f_n(t)$ is the applied force at the n -th station.

For a built-in end at the $1/2$ station, the boundary conditions of Equation (3-4) imply that

$$y_0 = y_1 = 0 \quad (3-10)$$

while for a free end at the $N + 1/2$ station the boundary conditions of Equation (3-6) imply that

$$m_N = m_{N+1} = 0. \quad (3-11)$$

Thus the complete set of difference equations for an N -cell cantilever beam becomes

$$\frac{d^2 y_2}{dt^2} = - \frac{1}{(\Delta x)^2} (m_3 - 2m_2 + m_1) + f_2(t)$$

$$\frac{d^2 y_3}{dt^2} = - \frac{1}{(\Delta x)^2} (m_4 - 2m_3 + m_2) + f_3(t)$$

. . .
 . . .
 . . .

$$\frac{d^2 y_{N-2}}{dt^2} = -\frac{1}{(\Delta x)^2} (m_{N-1} - 2m_{N-2} + m_{N-3}) + f_{N-2}(t)$$

$$\frac{d^2 y_{N-1}}{dt^2} = -\frac{1}{(\Delta x)^2} (-2m_{N-1} + m_{N-2}) + f_{N-1}(t)$$

$$\frac{d^2 y_N}{dt^2} = -\frac{1}{(\Delta x)^2} (m_{N-1}) + f_N(t)$$

where

$$m_1 = \frac{1}{(\Delta x)^2} (y_2)$$

$$m_2 = \frac{1}{(\Delta x)^2} (y_3 - 2y_2)$$

$$m_3 = \frac{1}{(\Delta x)^2} (y_4 - 2y_3 + y_2)$$

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$$m_{N-2} = \frac{1}{(\Delta x)^2} (y_{N-1} - 2y_{N-2} + y_{N-3})$$

$$m_{N-1} = \frac{1}{(\Delta x)^2} (y_N - 2y_{N-1} + y_{N-2})$$

Thus we have converted the original partial differential equation given by (3-7) to a set of N - 1 simultaneous second-order ordinary differential equations which can readily be solved with the electronic differential analyzer. Computer output voltages represent

time-dependent displacement, velocity, and bending-moment at each station, while the external forces at each station are represented by time-varying voltage inputs.

For computer circuits and recorded solutions of this and many other representative beam problems, the reader is referred to the technical reports.^{6,7} For convenience the detailed list of topics and figures covered in these reports is given in Appendices II and III.

APPENDIX I

Outline for Technical Report AIR-7 Entitled
**APPLICATION OF DIFFERENCE TECHNIQUES TO THE
 LATERAL VIBRATION OF BEAMS USING THE
 ELECTRONIC DIFFERENTIAL ANALYZER**

TABLE OF CONTENTS

<u>Chapter</u>	<u>Title</u>	<u>Page</u>
	PREFACE	i
	ILLUSTRATIONS	v
1	INTRODUCTION	1
	1.1 Basic Equations for a Thin Beam	2
	1.2 Solution by Separation of Variables	5
	1.3 Replacement of Partial Derivatives by Finite Differences	8
2	DERIVATION OF THE DIFFERENCE EQUATION FOR BEAMS	11
	2.1 Equation for the n-th Cell	11
	2.2 Boundary Conditions	13
	2.3 Initial Conditions	15
	2.4 Summary of Variable Transformations	15
3	ELECTRONIC DIFFERENTIAL ANALYZER CIRCUIT FOR SOLVING THE BEAM EQUATION BY THE DIF- FERENCE METHOD	17
	3.1 Linear Operations of the Electronic Differential Analyzer	17
	3.2 Analyzer Circuit for a Cantilever Beam	19
	3.3 Analyzer Circuit for a Hinged-Hinged Beam	19
4	THEORETICAL ACCURACY OF THE DIFFERENCE TECHNIQUE FOR UNIFORM BEAMS	24
	4.1 Uniform Hinged-Hinged Beam	24
	4.2 Uniform Cantilever Beam	26
	4.3 Uniform Free-Free Beam	31
	4.4 Summary of Theoretical Accuracy Calculations of the Difference Techniques for Beams	34

<u>Chapter</u>	<u>Title</u>	<u>Page</u>
5	APPLICATION TO CANTILEVER BEAMS	35
	5.1 Transient Response of a Uniform Cantilever Beam	35
	5.2 Static Deflection for a Uniform Load	35
	5.3 Determination of Normal Mode Frequencies	37
	5.4 Component Accuracy Requirements	40
	5.5 Effect of Voltage Transients	41
	5.6 Tapered Cantilever Beam	42
	5.7 Uniform Cantilever Beam with Concentrated Mass Load at the Free End	46
6	APPLICATION TO HINGED-HINGED BEAMS	51
	6.1 Analyzer Circuit for the Hinged-Hinged Beam	51
	6.2 Uniform Hinged-Hinged Beam	52
	6.3 Uniform Hinged-Hinged Beam with Concentrated Mass at the Center	53
7	APPLICATION TO FREE-FREE BEAMS	56
	7.1 Uniform Free-Free Beam	56
	7.2 Stability of the Free-Free Beam Circuit	58
8	BEAMS WITH VISCOUS DAMPING	61
	8.1 Beam Equations Including Viscous Damping	61
	8.2 Difference Equations Including Viscous Damping	62
	8.3 Computer Circuit for the Viscous-Damping Case	63
	8.4 Impulse Response of an 8-Cell Cantilever Beam with Viscous Damping	64
9	BEAMS WITH TIME-VARYING BOUNDARY CONDITIONS	68
	9.1 Method of Introducing Time-Varying Boundary Conditions	68
	9.2 Cantilever Beam with Specified Displacement at the Free End	68
	9.3 Beam on Elastic Foundations	69
10.	VIBRATION OF BEAMS INCLUDING DEFLECTION DUE TO TRANSVERSE SHEAR	71
	10.1 Equations for Transverse Beam Motion, Including Shear Displacements	71
	10.2 Difference Equations Including the Transverse Shear Effects	73
	APPENDIX I - CALCULATION OF MODE FREQUENCIES AND SHAPES FOR CELLULAR FREE-FREE BEAMS	A-1
	APPENDIX II - CALCULATION OF MODE FREQUENCIES FOR CELLULAR CANTILEVER BEAMS	A-12

APPENDIX III - MODE FREQUENCIES AND SHAPES FOR CELLULAR CLAMPED-CLAMPED BEAMS	<u>Page</u> A-15
BIBLIOGRAPHY	A-15

ILLUSTRATIONS

<u>Figure No.</u>	<u>Title</u>	<u>Page</u>
1-1	Cantilever Beam	3
1-2	Station Arrangement for Cantilever Beam	9
3-1	Operational Amplifiers	18
3-2	Analyzer Circuit for a Cantilever Beam	20
3-3a	Response of 8-cell Uniform Cantilever Beam to a Uniform Impulse; Stations 6, 7, and 8	21
3-3b	Response of 8-cell Uniform Cantilever Beam to a Uniform Impulse; Stations 2, 3, 4, and 5	21-a
3-4	Hinged-Hinged Beam	22
3-5	Analyzer Circuit for Hinged-Hinged Beam	23
4-1	Normal-Mode Frequency Deviation for a Uniform Hinged-Hinged Beam	27
4-2	Normal-Mode Frequency Deviation for a Uniform Cantilever Beam	29
4-3	Comparison of Mode Shapes for Cellular and Con- tinuous Uniform Cantilever Beams	30
4-4	Normal-Mode Frequency Deviation for a Uniform Free-Free Beam	32
4-5	Comparison of Mode Shapes for Cellular and Con- tinuous Free-Free Beams	33
5-1	Bending-Moment Response of 8-cell Cantilever Beam to a Uniform Impulse	36
5-2	Excitation of the Second-Mode of an 8-cell Cantilever Beam	39
5-3	Tapered Cantilever Beam with Concentrated Mass Load	43
5-4	Displacement at Wing-Tip Following a One-Second Uniform Impulse	44
5-5	Comparison of the Moments at Stations 1 and 4 of the Tapered Cantilever Beam with and without Concen- trated Mass Load	45
5-6	Uniform Cantilever Beam with Concentrated Mass Load at the Free End	46

<u>Figure No.</u>	<u>Title</u>	<u>Page</u>
5-7	$1/\beta_L^2$ versus M_L/M_B for an 8-cell End-Loaded Cantilever Beam	50
6-1	Uniform Hinged-Hinged Beam with Concentrated Mass Load at the Center	53
7-1	Cellular Free-Free Beam	56
7-2	Drift of an 8-Cell Free-Free Beam Following Release of Zero Initial Conditions	59
8-1	Analyzer Circuit at the nth Cell Including Viscous Damping	64
8-2	Unit Impulse Response of a Uniform Cantilever Beam with Viscous Damping (Input and Displacement at Stations 7 and 8)	65
8-3	Unit Impulse Response of a Uniform Cantilever Beam with Viscous Damping (Displacement at Stations 2, 3, 4, 5, and 6)	66
9-1	Hinged-Hinged Beam on Elastic Supports	69
10-1	Analyzer Circuit for Transverse Beam Displacement at the nth Station, Including Transverse Shear	75
10-2	Comparison of Transverse Shear Effect on Normal-Mode Frequency for a Uniform Free-Free Beam	76
A-1	Comparison of Mode Shapes for Cellular and Continuous Clamped-Clamped Beams	A-17

APPENDIX II

Outline for Technical Report AIR-8 Entitled
 ELECTRONIC DIFFERENTIAL ANALYZER SOLUTION
 OF BEAMS WITH NONLINEAR DAMPING

TABLE OF CONTENTS

<u>Chapter</u>	<u>Title</u>	<u>Page</u>
	PREFACE	ii
	ILLUSTRATIONS	v
1.	INTRODUCTION	1
	1.1 Equation for Lateral Vibration of Beams	1
	1.2 Finite Difference Method for Approximating Derivatives	5
	1.3 Principles of Operation of the Electronic Differential Analyzer	5
2	CANTILEVER BEAM WITH VELOCITY-SQUARED DAMPING	10
	2.1 Beam Equation Including Velocity-Squared Damping	10
	2.2 Equivalence of Damping-Coefficient Size and Amplitude of Vibration	10
	2.3 Difference Equations for the Cantilever Beam with Velocity-Squared Damping	11
	2.4 Analyzer Circuit for the Cantilever Beam with Velocity-Squared Damping	13
	2.5 Damped First-Mode Oscillation	14
	2.6 Approximate Theoretical Solution	14
	2.7 Impulse Response of the Cantilever Beam with Velocity-Squared Damping	17
3	CANTILEVER BEAM WITH COULOMB DAMPING	
	3.1 Beam Equation Including Coulomb Damping	22
	3.2 Difference Equations for the Cantilever Beam with Coulomb Damping	23
	3.3 Analyzer Circuit for the Cantilever Beam with Coulomb Damping	23
	3.4 Impulse Response of the Cantilever Beam with Coulomb Damping	24
	BIBLIOGRAPHY	

ILLUSTRATIONS

<u>Figure No.</u>	<u>Title</u>	<u>Page</u>
1-1	Cantilever Beam	3
1-2	Cantilever Beam Divided into Stations	6
1-3	Operational Amplifier	7
1-4	Servo Multiplier	8
2-1	Analyzer Circuit at the n th Station for the Cantilever Beam with Velocity-Squared Damping	13
2-2	Damped First-Mode Oscillations of Uniform Cantilever Beam with Velocity-Squared Damping	15
2-3	Variation of Logarithmic Decrement δ with Amplitude of Oscillation	18
2-4	Unit Impulse Response of 5-Cell Uniform Cantilever Beam with Velocity-Squared Damping; Displacements at Stations 2, 3, 4, and 5	20
2-5	Unit Impulse Response of 5-Cell Uniform Cantilever Beam with Velocity-Squared Damping; Bending-Moment at Stations 1, 2, 3, and 4	21
3-1	Analyzer Circuit at the n th Station for Cantilever Beam with Coulomb Damping	24
3-2	Impulse Response of a 5-Cell Uniform Cantilever Beam with Coulomb Damping	25

APPENDIX III

Outline for Technical Report AIR-10 Entitled
APPLICATION OF DIFFERENCE TECHNIQUES
TO HEAT FLOW PROBLEMS USING
THE ELECTRONIC DIFFERENTIAL ANALYZER

TABLE OF CONTENTS

<u>Chapter</u>	<u>Title</u>	<u>Page</u>
	PREFACE	
1	INTRODUCTION	
	1. 1 Basic Equations for Heat Flow	1
	1. 2 Equations for One-Dimensional Flow	2
	1. 3 Solution by Separation of Variables	5
	1. 4 Replacement of Partial Derivatives by Finite Differences	8
	1. 5 Principles of Operation of the Electronic Differential Analyzer	11
2	ONE-DIMENSIONAL HEAT FLOW	15
	2. 1 Equations to be Solved	15
	2. 2 Boundary Conditions	15
	2. 3 Initial Conditions	16
	2. 4 Complete Set of Difference Equations for One-Dimensional Heat Flow	16
	2. 5 Electronic Differential Analyzer Circuit for Solving One-Dimensional Heat Flow	17
	2. 6 Theoretical Accuracy of the Difference Method	19
	2. 7 Universal Curves for Obtaining Temperature Distributions for Arbitrary Initial Conditions	24
3	HEAT FLOW IN TWO AND THREE DIMENSIONS; CARTESIAN COORDINATES	34
	3. 1 Heat Equation in Three Dimensions using Cartesian Coordinates	34
	3. 2 Solution by Separation of Variables in a Homogeneous Rectangular Medium	35
	3. 3 Theoretical Accuracy of the Difference Method	37
	3. 4 Solution in a Two-Dimensional Homogeneous Medium	38

4	HEAT FLOW IN CYLINDERS	43
	4.1 The Heat Equation in Cylindrical Coordinates	43
	4.2 Solution by Separation of Variables	44
	4.3 Theoretical Accuracy of the Difference Method for Cylindrical Heat Flow	47
	4.4 Analyzer Solution of Axially-Symmetric Heat Flow with Initially Constant Temperature	50
5	HEAT FLOW IN SPHERES	62
	5.1 Heat Equation in Spherical Coordinates	62
	5.2 Difference Equation for Spherically-Symmetric Heat Flow	63
	5.3 Solution by Separation of Variables in the Spherically-Symmetric Case	64
	5.4 Theoretical Accuracy of the Difference Method for Spherically-Symmetric Heat Flow	65
	5.5 Analyzer Solution of Spherically-Symmetric Heat Flow Starting with Unit Temperature	65
	5.6 An Alternative Method of Writing the Difference Equations for Spherically Symmetric Heat Flow	71
6	CHANGE OF VARIABLE TO IMPROVE ACCURACY	74
	6.1 Regrouping of Station Locations	74
	6.2 Solution of Temperature Distributions in a Uniform Slab	74
7	SOLUTION OF HEAT-FLOW PROBLEMS IN SEMI- INFINITE MEDIA	80
	7.1 Equation to be Solved	80
	7.2 Exact Theoretical Solution	82
	7.3 Solution by the Difference Method Using a Change of Spacial Variable	82
8	NONLINEAR HEAT-FLOW PROBLEMS	87
	8.1 Introduction	87
	8.2 Nonlinear Problem to be Solved	87
	8.3 Formulation of the Difference Equations and Analyzer Circuit	88
	8.4 Analyzer Solution for Uniform Initial Temperature Distribution Across the Slab	90
	8.5 Exact Particular Solution by Separation of Variables; Comparison with the Difference- Equation Solution	90

APPENDIX I - CALCULATION OF DECAY CONSTANTS AND MODE SHAPES FOR THE DIFFERENCE APPROXIMATION TO AXIALLY-SYMMETRIC HEAT FLOW	97
APPENDIX II - CALCULATION OF DECAY CONSTANTS AND MODE SHAPES FOR THE DIFFERENCE APPROXIMATION TO SPHERICALLY-SYMMETRIC HEAT FLOW	104
BIBLIOGRAPHY	109

ILLUSTRATIONS

<u>Figure No.</u>	<u>Title</u>	<u>Page</u>
1-1	Temperature Distribution Between Two Slabs	3
1-2	Temperature Distributions Across Conducting Slab	9
1-3	Station Arrangement for $N = 10 - 1/2$	10
1-4	Operational Amplifiers	12
1-5	Schematic of Servo Multiplier	14
2-1	Analyzer Circuit for One-Dimensional Heat Flow	18
2-2	Analyzer Circuit for Heat Flow Through a Homogeneous Medium	20
2-3	Analyzer Solution for Heat Flow in a Uniform Slab with Unit Initial Temperature Distribution	21
2-4	Comparison of Analyzer and Theoretical Solution to Heat-Flow Problem	22
2-5	Percentage Deviation in Normal-Mode Constant $\bar{\beta}_n$ for One-Dimensional Heat Flow	25
2-6	Temperature versus Time in a Uniform Slab with Unit Initial Temperature at All Stations	26
2-7	Temperature versus Time in a Uniform Slab with Unit Initial Temperature at Station 1	28
2-8	Temperature versus Time in a Uniform Slab with Unit Initial Temperature at Station 2	29
2-9	Temperature versus Time in a Uniform Slab with Unit Initial Temperature at Station 3	30
2-10	Temperature versus Time in a Uniform Slab with Unit Initial Temperature at Station 4	31
2-11	Temperature versus Time in a Uniform Slab with Unit Initial Temperature at Station 5	32
2-12	Temperature versus Time in a Uniform Slab with Unit Initial Temperature at Station 6	33
3-1	Two-Dimensional Problem in Heat Flow	40
3-2	Analyzer Circuit for Two-Dimensional Heat Flow in an Isotropic Homogeneous Medium	41
3-3	Analyzer Solution for Two-Dimensional Heat Flow in a Square Isotropic Homogeneous Medium	42
4-1	Percentage Deviation of Decay Constant as a Function of Number of Cells for Radial-Dependent Modes of Cylindrical Heat Flow	51

4-2	Comparison of 4 - 1/2 - Cell Mode Shapes with J_0 , Modes 1 and 2	52
4-3	Comparison of 4 - 1/2-Cell Mode Shapes with J_0 , Modes 3 and 4	53
4-4	Axially-Symmetric Temperature Distributions in a Uniform Cylinder Following Unit Initial Temperature Distribution	55
4-5	Axially-Symmetric Temperature Distribution with $u_{1/2}^{(0)} = 1$	56
4-6	Axially-Symmetric Temperature Distribution with $u_{1-1/2}^{(0)} = 1$	57
4-7	Axially-Symmetric Temperature Distribution with $u_{2-1/2}^{(0)} = 1$	58
4-8	Axially-Symmetric Temperature Distribution with $u_{3-1/2}^{(0)} = 1$	59
4-9	Axially-Symmetric Temperature Distribution with $u_{4-1/2}^{(0)} = 1$	60
4-10	Axially-Symmetric Temperature Distribution with $u_{5-1/2}^{(0)} = 1$	61
5-1	Percentage Deviation of Decay-Constant for Spherically Symmetric Heat Flow as a Function of the Number of Cells	66
5-2	Comparison of 4-1/2-Cell and Continuous First and Second-Mode Shapes for Spherically Symmetric Heat Flow	67
5-3	Comparison of 4-1/2-Cell and Continuous Third and Fourth-Mode Shapes for Spherically-Symmetric Heat Flow	68
5-4	Comparison of 10-1/2-Cell and Continuous First and Second-Mode Shapes for Spherically-Cymmetric Heat Flow	69
5-5	Comparison of 10-1/2-Cell and Continuous Third and Fourth-Mode Shapes for Spherically-Symmetric Heat Flow	70
5-6	Analyzer Solution of Spherically-Symmetric Heat Flow with Unit Initial Temperature and Walls at Zero Temperature	72
6-1	Comparison of Station Locations for the Space-Variable Transformation $y = \sqrt{x}$	75
6-2	Analyzer Solution for Heat Flow in a Uniform Slab; Equal Stations along y , where $y = \sqrt{x}$	77

6-3	Comparison of Analyzer and Theoretical Temperature Distributions Across Uniform Slab with Unit Initial Temperature	79
7-1	Semi-Infinite Bar	80
7-2	Temperature Distributions along a Semi-Infinite Bar at Various Times	83
7-3	Analyzer Solution for Temperatures in a Semi-Infinite Bar	85
8-1	Analyzer Circuit for the Temperature at the nth Station; Conductivity Proportional to the Temperature	89
8-2	Analyzer Solution for Heat Flow with Conductivity Proportional to Temperature; Conducting Wall Held at Temperature $U_0 = 0$.	91
8-3	Analyzer Solution for Heat Flow with Conductivity Proportional to Temperature; Conducting Wall Held at Temperature $U_0 = 0.2$	92
8-4	Analyzer Solution for Heat Flow with Conductivity Proportional to Temperature; Conducting Wall Held at Temperature $U_0 = 0.5$	93
8-5	Analyzer Solution for Checking the Exact Solution by Separation of Variables	96

BIBLIOGRAPHY

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