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SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY DIFFERENCE METHODS USING THE ELECTRONIC DIFFERENTIAL ANALYZER

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SUMMARY

This final report summarizes the investigation of the solution of both linear and nonlinear partial differential equations by difference techniques using the electronic differential analyzer. Complete details of the research are given in three technical reports. Two categories of physical problems are considered: (1) the lateral vibration of beams, and (2) heat flow. In both cases time is preserved as a continuous variable, while the spacial variable is broken into stations or cells. Thus spacial derivatives are approximated by finite differences, and the resulting set of simultaneous ordinary differential equations are solved by the electronic differential analyzer.

Theoretical accuracy of the difference method as a function of the number of cells along the spacial variable is investigated by comparing the eigenvalues (normal-mode frequencies for the beam, decay constants for heat flow) and normal-mode shapes with those for a continuous medium. Results show suprisingly few cells are required for representation of the first few modes to several percent accuracy. Theoretical as well as computer solutions are obtained for cantilever, hinged-hinged, free-free, and clamped-clamped beams, both uniform and non-uniform. Beams with viscous damping and time-varying boundary conditions were also solved, and the vibration of beams including deflection due to transverse shear are treated by the difference method. Cantilever beams with nonlinear damping terms such as velocity-squared damping and Coulomb damping are solved successfully.

Dynamic heat-flow problems treated by the difference method, both theoretically and with electronic differential analyzer solutions, include one, two, and three-dimensional flow in rectangular media as well as flow in cylindrical and spherical media. Change of independent variable to improve accuracy and to solve flow in semi-infinite media is demonstrated. The nonlinear heat equation for a medium having a conductivity

proportional to temperature is solved with the electronic differential analyzer and results are compared with a particular exact solution.

As a result of these investigations it seems clear that the electronic differential analyzer is an excellent tool for rapidly solving many partial differential equations, both linear and nonlinear. There is a one-to-one correspondence between resistor values in the circuit and physical parameters in the problem, and the desired dependent variables (e.g., beam displacement, velocity, bending moment, or temperature, heat flux, etc.) are all available as time-varying output voltages of the computer.

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SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY DIFFERENCE METHODS USING THE ELECTRONIC DIFFERENTIAL ANALYZER

INTRODUCTION

Whenever we wish to solve partial differential equations by means of the electronic differential analyzer, it is necessary first to convert the equations to ordinary differential equations, since the analyzer can integrate with respect to only one variable, namely time. If the partial differential equation is linear, this conversion to ordinary differential equations can often be done by separation of variables, which results in ordinary differential equations of the eigenvalue type. The normal modes, or eigenfunctions, can then be found, after which the complete solution is built up by combining the normal modes.

The above method of separating variables and obtaining a series type of solution can be carried out fairly efficiently on an electronic differential analyzer.[1-4] Certainly, for most problems the analyzer is much faster than any hand methods. But for the engineer who is interested in getting quantitative answers to specific problems even the analyzer approach might seem somewhat tedious. Then too, the control engineer would often like to have a real-time simulation of the system being controlled, and in many cases partial differential equations are required to describe this system. It therefore would be highly advantageous to solve the partial differential equations directly with the electronic differential analyzer. This can be done by replacing some of the partial derivatives by finite differences in order to convert the original partial differential equations into a system of ordinary differential equations.

Assume we are interested in solving a partial differential equation in which the dependent variable $y(x, t)$ is a function of both a distance variable x and a time variable t. Instead of measuring the variable y at all distances x, let us measure y only at certain stations along x; thus, let y_1

be the value of y at the first x station, y_2 be the value of y at the second x station, y_n be the value of y at the nth x station. Further, let the distance between stations be a constant Δx.

Now clearly a good approximation to $\partial y/\partial x|_{1/2}$ (i.e., the partial derivative of y with respect to x at the 1/2 station) is given by the difference

$$\left.\frac{\partial y}{\partial x}\right|_{1/2} = \frac{y_1 - y_o}{\Delta x} . \tag{1-1}$$

In fact the limit of equation (1-1) as $\Delta x \to 0$ is just the definition of the partial derivative at that point. Writing (1-1) in more general terms

$$\left.\frac{\partial y}{\partial x}\right|_{n-1/2} = \frac{y_n - y_{n-1}}{\Delta x} . \tag{1-2}$$

In the same way

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_{n} = \frac{1}{\Delta x}\left\{\left.\frac{\partial y}{\partial x}\right|_{n+1/2} - \left.\frac{\partial y}{\partial x}\right|_{n-1/2}\right\} \tag{1-3}$$

or from equation (2-2)

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_{n} = \frac{y_{n+1} - 2y_n + y_{n-1}}{(\Delta x)^2} . \tag{1-4}$$

Thus we have converted partial derivatives with respect to x into algebraic differences. The only differentiation needed now is with respect to the time variable t, so that we are left with a system of ordinary differential equations involving dependent variables $y_o(t)$, $y_1(t)$, . . . $y_n(t)$, . . .

SOLUTION OF THE HEAT EQUATION

2.1 Basic Equations for Heat Flow

The basic equation of heat flow is given by

$$c\,\delta \frac{\partial u}{\partial t} = \nabla \cdot K \nabla u + S \tag{2-1}$$

where

u = temperature and is a function of the spacial coordinates and time,

K = thermal conductivity, in general, a function of the spacial coordinates,

c = specific heat, a function of spacial coordinates,

δ = density, also a function of spacial coordinates,

$\bar{t}$ = time,

$\bar{S}$ = rate of heat supplied per unit volume by sources, in the medium, a function of spacial coordinates and time.

The left-hand side of equation (2-1) represents the rate at which heat is stored in a unit volume due to the heat capacity of the medium. The right-hand side represents the rate at which the unit volume receives heat, first due to heat conduction into the volume from the neighboring medium (the $\nabla \cdot K \nabla u$ term) and second, due to the heat flow into the volume from sources within the volume itself (the $\bar{S}$ term). The conductivity times the gradient of the temperature ($-K\nabla u$) is a vector representing the heat flux. The components of $-K\nabla u$ represent the heat flow through a unit surface normal to the direction along which the component is taken.

In a given heat-flow problem it is necessary to stipulate spacial boundary conditions either on the temperature u or the heat flux $-K\nabla u$, as well as the initial temperature distribution throughout the medium.

In technical report AIR-10[5] the form of Equation (2-1) and its solution by difference methods and separation of variables is discussed for Cartesian coordinates, cylindrical coordinates, and spherical coordinates. For example, in Cartesian coordinates for heat flow through a homogeneous slab we have the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{2-2}$$

where x is dimensionless distance through the slab and t is dimensionless time. For boundary conditions

$$u(0,t) = u_o(t) \tag{2-3}$$

$$\frac{\partial u}{\partial x}(1,t) = 0 \tag{2-4}$$

corresponding to a prescribed temperature $u_o(t)$ at the left boundary and zero heat flow at the right boundary, the difference equations become

$$\frac{du_1}{dt} = \frac{1}{(\Delta x)^2}\left[u_2 - 2u_1 + u_o(t)\right]$$

$$\frac{du_2}{dt} = \frac{1}{(\Delta x)^2}\left[u_3 - 2u_2 + u_1\right]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad (2\text{-}5)$$

$$\frac{du_{N-3/2}}{dt} = \frac{1}{(\Delta x)^2}\left[u_{N-1/2} - 2u_{N-3/2} + u_{N-5/2}\right]$$

$$\frac{du_{N-1/2}}{dt} = \frac{1}{(\Delta x)^2}\left[-u_{N-1/2} + u_{N-3/2}\right]$$

where Δx is the interval between stations along x and N-1/2 is the total number of x stations across the slab.

This set of N-1/2 simultaneous first-order differential equations can be set up and solved directly on the electronic differential analyzer. In the technical report[5] the time-dependent solutions for the temperature at each station are shown for a number of representative initial conditions.

3. SOLUTION OF THE BEAM EQUATION

3.1 Basic Equations for Lateral Beam Vibrations

The basic equation for lateral displacement y of a beam is given by

$$\frac{\partial^2}{\partial \bar{x}^2} EI(\bar{x}) \frac{\partial^2 y}{\partial \bar{x}^2} + \rho(\bar{x}) \frac{\partial^2 y}{\partial t^2} = \bar{f}(\bar{x}, \bar{t}) \qquad (3\text{-}1)$$

where

$\bar{x}$ = distance along the beam,

EI = flexual rigidity, in general u function of x,

ρ = mass per unit length, in general a function of x,

$\bar{t}$ = time

$\bar{f}(\bar{x}, \bar{t})$ = external applied force per unit length.

The bending moment M is given by

$$M(\bar{x}, \bar{t}) = EI(\bar{x}) \frac{\partial^2 y}{\partial \bar{x}^2} \qquad (3\text{-}2)$$

while the shear force V is

$$V(\bar{x}, \bar{t}) = \frac{\partial M(\bar{x}, \bar{t})}{\partial \bar{x}} \qquad (3\text{-}3)$$

Boundary conditions depend on the type of end fastening. For a cantilever (clamped end at $\bar{x} = L$

$$y(L, \bar{t}) = \frac{\partial y(L, \bar{t})}{\partial \bar{x}} = 0 . \qquad (3\text{-}4)$$

For a free end at $\bar{x} = L$

$$M(L, \bar{t}) = V(L, \bar{t}) = 0 \qquad (3\text{-}5)$$

while for a simple supported (hinged) end at $\bar{x} = L$

$$y(L, \bar{t}) = M(L, \bar{t}) = 0 . \qquad (3\text{-}6)$$

In technical report AIR-7 the normal-mode frequencies and shapes for Equation (3-1) are given for uniform free-free, cantilever, hinged-hinged, and clamped-clamped beams, along with theoretical solutions using the difference approximation. Electronic differential analyzer circuits and solutions using the difference technique are presented for the above cases and for nonuniform beams, beams with viscous damping, beams with time varying boundary conditions, and beams for which Equation (3-1) is modified to include transverse shear effects (this is necessary for beams whose thickness is not small compared with their length). In addition, technical report AIR-8 presents analyzer solutions for beams with nonlinear damping terms, both velocity-squared damping and coulomb (dry-friction) damping.

3.2 Difference Equations for Uniform Cantilever Beam

As a simple example, consider the equation for lateral displacement y of a uniform beam. It can be written as

$$\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = f(x, t) \qquad (3\text{-}7)$$

where x is dimensionless distance (the beam length in x is unity) and t is dimensionless time. To solve Equation (3-7) using the difference method we consider the transverse displacement y only at equally spaced stations approximately as

$$m_n = \left.\frac{\partial^2 y}{\partial x^2}\right|_n \cong \frac{1}{(\Delta x)^2}(y_{n+1} - 2y_n + y_{n-1}) \qquad (3\text{-}8)$$

where Δx is the distance between stations. In the same way the complete difference equation at the n-th station becomes

$$\frac{d^2 y_n}{dt^2} = -\frac{1}{(\Delta x)^2}(m_{n+1} - 2m_n + m_{n-1}) + f_n(t) \qquad (3\text{-}9)$$

where $f_n(t)$ is the applied force at the n-th station.

For a built-in end at the 1/2 station, the boundary conditions of Equation (3-4) imply that

$$y_o = y_1 = 0 \qquad (3\text{-}10)$$

while for a free end at the N + 1/2 station the boundary conditions of Equation (3-6) imply that

$$m_N = m_{N+1} = 0\,. \qquad (3\text{-}11)$$

Thus the complete set of difference equations for an N-cell cantilever beam becomes

$$\frac{d^2 y_2}{dt^2} = -\frac{1}{(\Delta x)^2}(m_3 - 2m_2 + m_1) + f_2(t)$$

$$\frac{d^2 y_3}{dt^2} = -\frac{1}{(\Delta x)^2}(m_4 - 2m_3 + m_2) + f_3(t)$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

. . .

$$\frac{d^2 y_{N-2}}{dt^2} = -\frac{1}{(\Delta x)^2}(m_{N-1} - 2m_{N-2} + m_{N-3}) + f_{N-2}(t)$$

$$\frac{d^2 y_{N-1}}{dt^2} = -\frac{1}{(\Delta x)^2}(- 2m_{N-1} + m_{N-2}) + f_{N-1}(t)$$

$$\frac{d^2 y_{N}}{dt^2} = -\frac{1}{(\Delta x)^2}(m_{N-1}) + f_{N}(t)$$

where

$$m_1 = \frac{1}{(\Delta x)^2}(y_2)$$

$$m_2 = \frac{1}{(\Delta x)^2}(y_3 - 2y_2)$$

$$m_3 = \frac{1}{(\Delta x)^2}(y_4 - 2y_3 + y_2)$$

. .
. .
. .

$$m_{N-2} = \frac{1}{(\Delta x)^2}(y_{N-1} - 2y_{N-2} + y_{N-3})$$

$$m_{N-1} = \frac{1}{(\Delta x)^2}(y_N - 2y_{N-1} + y_{N-2})$$

Thus we have converted the original partial differential equation given by (3-7) to a set of N - 1 simultaneous second-order ordinary differential equations which can readily be solved with the electronic differential analyzer. Computer output voltages represent time-dependent displacement, velocity, and bending-moment at each station, while the external forces at each station are represented by time-varying voltage inputs.

For computer circuits and recorded solutions of this and many other representative beam problems, the reader is referred to the technical reports.[6,7] For convenience the detailed list of topics and figures covered in these reports is given in Appendices II and III.

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Outline for Technical Report AIR-7 Entitled

APPLICATION OF DIFFERENCE TECHNIQUES TO THE LATERAL VIBRATION OF BEAMS USING THE ELECTRONIC DIFFERENTIAL ANALYZER

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