THE UNIVERSITY OF MICHIGAN
COLLEGE OF ENGINEERING
Department of Aeronautical and Astronautical Engineering

Technical Report

TRAJECTORY CALCULATIONS FOR SOUNDING ROCKETS

By
Robert M. Howe

UMRI Project 04003

under contract with:
DEPARTMENT OF THE ARMY
BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

administered by:
THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE   ANN ARBOR

July 1959

TRAJECTORY CALCULATIONS FOR SOUNDING ROCKETS

SUMMARY

This report describes the derivation of the equations used to calculate sounding-rocket peak altitude, range, and sensitivity to velocity error, flight path elevation error, and yaw angle at burnout. Curves which allow rapid computation of these quantities are shown for rockets with equivalent straight-up altitudes of 0-2000 miles and burnout flight path angles of 60-90 degrees.

1. Basic Free-Fall Equations

We will use as an inertial coordinate frame of reference a system with origin fixed at the center of the earth. The axes of this system will be non-rotating, so that the earth rotates with respect to the system. We will neglect perturbations due to the non-spherical earth corrections, so that any free-fall trajectory will be in a plane determined by its velocity vector at any point and the center of the earth. In general, we will use polar coordinates r, θ as shown in Fig. 1.1 to locate the rocket in this plane.

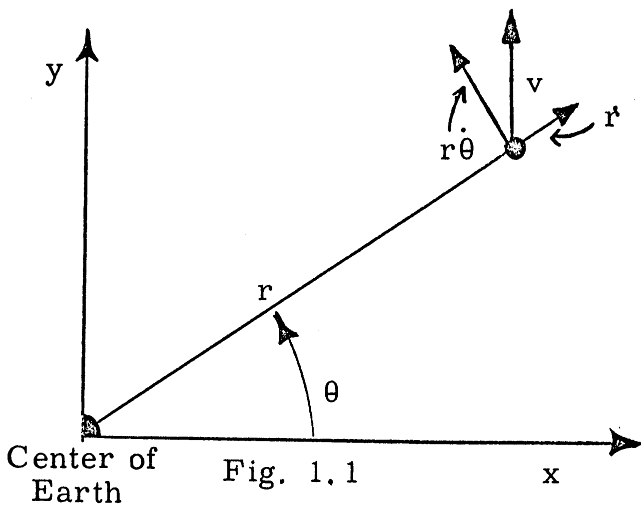


Fig. 1.1

Since we will find it convenient to use energy methods for many of the calculations, we will use Lagrange's equation to set up the usual equations of motion for a point mass m (the rocket mass = m) in a central force field. Thus the Lagrangian function L = T-V, where T = kinetic energy and V = potential energy*. If q_i is the ith general coordinate, then the equations of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (1.1)$$

For our polar-coordinate case there are two general coordinates, $q_1 = r$, $q_2 = \theta$. Hence

$$\text{Kinetic Energy} = T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) \qquad (1.2)$$

*Slater and Frank, Mechanics, McGraw-Hill, 1947, pp 72-74.

and

$$\text{Potential Energy} = V = -\frac{\gamma Mm}{r} \tag{1.3}$$

where γ is the gravitational constant and M is the mass of the earth. Eq. (1.3) can be written in more useful form by noting that at the earth's surface ($r = r_0$)

$$mg_0 = \frac{\gamma Mm}{r_0^2} \tag{1.4}$$

where g_0 is the acceleration of gravity at the earth's surface, not including the term $r_0\omega^2$ due to the angular rate of earth rotation ω. In subsequent calculations we will assume $g_0 = 32.2$ ft/sec^2. In terms of Eq. (1.4), Eq. (1.3) becomes

$$V = -mg_0 \frac{r_0^2}{r} \tag{1.5}$$

Thus L becomes

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + mg_0 \frac{r_0^2}{r} \tag{1.6}$$

Substituting Eq. (1.6) into (1.1) we have

$$\ddot{r} - r\dot{\theta}^2 + g_0\frac{r_0^2}{r^2} = 0 \tag{1.7}$$

and

$$\frac{d}{dt}\left[mr^2\dot{\theta}\right] = 0 \tag{1.8}$$

Eq. (1.7) states that the acceleration due to the trajectory balances the acceleration due to gravitational attraction. Eq. (1.8) is simply the equation for constant angular momentum, since after integration

$$mr^2\dot{\theta} = p = \text{angular momentum} = \text{constant.}$$

The time variable can be eliminated from Eqs. (1.7) and (1.8) to yield an equation describing the trajectory shape (i.e., involving r and θ only) by defining a new variable u given by

$$u = \frac{1}{r} \tag{1.9}$$

Then

$$\dot{r} = \frac{dr}{du}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -\frac{p}{m}\frac{du}{d\theta} \tag{1.10}$$

Similarly

$$\ddot{r} = -\left(\frac{p}{m}\right)^2 \frac{1}{r^2} \frac{d^2u}{d\theta^2} \qquad (1.11)$$

From Eqs. (1.10) and(1.11) Eq. (1.7) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{m^2 g_0 r_0^{\,2}}{p^2} \qquad (1.12)$$

But the solution to this equation can be written immediately as

$$\frac{1}{r} = u = C \cos(\theta - \theta_0) + \frac{m^2 g_0 r_0^{\,2}}{p^2} \qquad (1.13)$$

where θ_0 and C are arbitrary constants which depend on the initial conditions of the trajectory, or, in this case, specified values of r and θ at some given time and the angular momentum p.

Solving Eq. (1.13) for r, we have

$$r = \frac{\dfrac{p^2}{m^2 g_0 r_0^{\,2}}}{1 + C\dfrac{p^2}{m^2 g_0 r_0^{\,2}} \cos(\theta - \theta_0)} \qquad (1.14)$$

The equation for an ellipse of eccentricity ϵ has the form

$$r = \frac{a(1-\epsilon^2)}{1 + \epsilon \cos\theta} \qquad (1.15)$$

where a is the semimajor axis and where the origin of polar coordinates is at the right hand foci of the ellipse (see Fig. 1.2). This

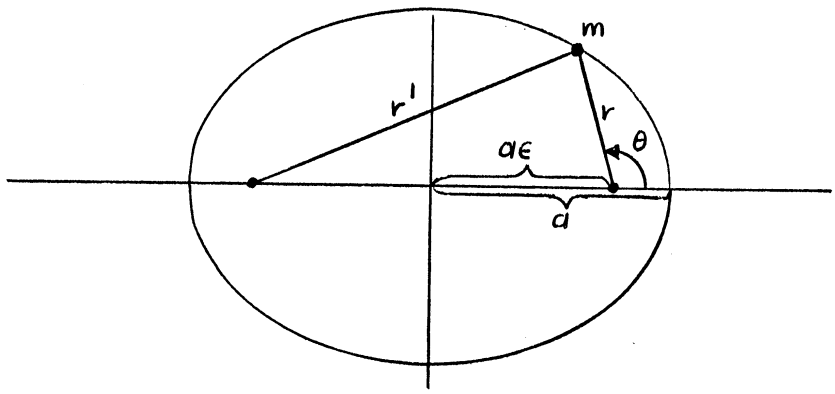


Fig. 1.2 Ellipse

can be seen from the law of cosines for the triangle formed by r and r' and from the equation $r + r' = 2a = \text{constant}$, which defines an ellipse.

$$r'^2 = r^2 + (2a\epsilon)^2 + 4r(a\epsilon)\cos\theta$$

Next we consider the determination of ϵ and a in terms of initial conditions for the trajectory. The solution to Eq. (1.12) can also be written as

$$\frac{1}{r} = u = A\cos\theta + B\sin\theta + \frac{m^2 g_0 r_0^2}{p^2} \qquad (1.16)$$

where A and B are constants determined by initial conditions. Differentiating Eq. (1.16) with respect to time

$$-\frac{1}{r^2}\dot{r} = \left[-A\sin\theta + B\cos\theta\right]\dot{\theta} \qquad (1.17)$$

At $r = r_1$, let $\dot{r} = \dot{r}_1$, $\theta = \theta_1$, $\dot{\theta} = \dot{\theta}_1$

Then Eqs. (1.16) and (1.17) can be written as

$$A\cos\theta_1 + B\sin\theta_1 = \frac{1}{r_1} - \frac{g_0 r_0^2}{r_1^4\dot{\theta}_1^2} \qquad (1.18)$$

and

$$-A\sin\theta_1 + B\cos\theta_1 = -\frac{1}{r_1^2}\frac{\dot{r}_1}{\dot{\theta}_1} \qquad (1.19)$$

Solving for A and B, we have

$$A = \cos\theta_1\left[\frac{1}{r_1} - \frac{g_0 r_0^2}{r_1^4\dot{\theta}_1^2}\right] + \sin\theta_1\left[\frac{\dot{r}_1}{r_1^2\dot{\theta}_1}\right] \qquad (1.20)$$

and

$$B = \sin\theta_1\left[\frac{1}{r_1} - \frac{g_0 r_0^2}{r_1^4\dot{\theta}_1^2}\right] + \cos\theta_1\left[-\frac{\dot{r}_1}{r_1^2\dot{\theta}_1}\right] \qquad (1.21)$$

Note that $r_1\dot{\theta}_1 = v_{\theta_1}$, where v_{θ_1} is the component of total velocity v parallel to the earth's surface at $r = r_1$.

Comparison of Eqs. (1.14) and(1.15) shows that $\epsilon = C\frac{p^2}{m^2 g_0 r_0^2}$. But from Eqs. (1.14) and (1.16) it is clear that $C = \sqrt{A^2 + B^2}$. Thus

$$\epsilon = \sqrt{A^2 + B^2}\ \frac{p^2}{m^2 g_0 r_0^2} = \sqrt{A^2 + B^2}\ \frac{r_1^4 \dot{\theta}_1^2}{g_0 r_0^2}$$

or

$$\epsilon = \sqrt{\left(\frac{r_1^3 \dot{\theta}_1^2}{g_0 r_0^2} - 1\right)^2 + \left(\frac{r_1^2 \dot{r}_1 \dot{\theta}_1}{g_0 r_0^2}\right)^2} \qquad (1.22)$$

For a circular satellite orbit, $\epsilon = 0$. For $\epsilon > 1$ the ellipse turns into a hyperbole and the rocket escapes from the earth's gravitational field. Thus to prevent escape, $\epsilon < 1$, and from Eq. (1.22) this implies that

$$(r_1 \dot{\theta}_1)^2 + (\dot{r}_1)^2 < 2\, r_1 g_0 \left(\frac{r_0}{r_1}\right)^2 \qquad (1.23)$$

But $r_1 \dot{\theta}_1 = v_{\theta_1}$ and $\dot{r}_1 = v_{r_1}$, where v_{θ_1} and v_{r_1} are the components of rocket velocity at $r = r_1$ parallel and perpendicular, respectively, to the earth's surface. Hence $(r_1 \dot{\theta}_1)^2 + \dot{r}_1^2 = v_{\theta_1}^2 + v_{r_1}^2 = v_1^2$, where v_1 = total rocket velocity. At the earth's surface $r_1 = r_0$ and $v_1^2 < 2 r_0 g_0$ to prevent escape. Hence we define the escape velocity v_E as

$$v_E = \sqrt{2 r_0 g_0} \qquad (1.24)$$

$$= \sqrt{2 \times 3950 \times 32.2 \times 5280} = 36{,}700 \text{ ft/sec}$$

In terms of the escape velocity v_E Eq. (1.22) for ϵ can be rewritten as

$$\epsilon = \sqrt{\left(2 \rho_1 \left(\frac{v_{\theta_1}}{v_E}\right)^2 - 1\right)^2 + \left[2\, \rho_1\ \frac{v_{r_1} v_{\theta_1}}{v_E^2}\right]^2} \qquad (1.25)$$

where $\rho_1 = \rho$ at $r = r_1$, and where ρ is a dimensionless radius equal to the ratio of actual radius r to earth's radius r_0. Thus

$$\rho = \frac{r}{r_0} \qquad (1.26)$$

If we know the radius ρ_1 and the velocity components v_{θ_1} and v_{r_1}, then Eq. (1.23) allows us to calculate ϵ. To calculate the semimajor axis a we note by comparing Eqs. (1.14) and (1.15) that

$$a(1-\epsilon^2) = \frac{p^2}{m^2 g_0 r_0^2} = 2\rho_1^2 \left(\frac{v_{\theta_1}}{v_E}\right)^2 r_0$$

or

$$a = \frac{2\rho_1^2 \left(\frac{v_{\theta_1}}{v_E}\right)^2 r_0}{1-\epsilon^2} \qquad (1.27)$$

Now that we have developed the basic equations describing trajectories of a rocket in free-flight in the gravitational field of the spherical earth, we turn to consideration of some specific trajectory problems.

2. Vertical Launch Free Flight Trajectory

In the case of a vertical launch the problem is complicated by the rotation of the launching platform, namely, the earth. Thus the velocity vector at burnout will be the sum of the rotating-earth component and the propulsion-unit contribution. To simplify the initial analysis, let us assume that this sum is such that at final-stage burnout, i.e., at the beginning of free-flight, the total velocity vector is perpendicular to the earth's surface. Since this will give maximum altitude, small deviations from this perpendicular direction will only cause second order effects in total altitude.

Under these conditions a calculation based on energy conservation is the easiest approach. Assume $r = r_1$ at burnout and r_2 at the apex of the trajectory. Then the change in potential energy $V_1 - V_2$ must equal the change in kinetic energy $-T_1$, or from Eq. (1.5)

$$m g_0 r_0^2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right) = -\frac{1}{2} m v_1^2 \qquad (2.1)$$

where v_1 is the total velocity at burnout. Replacing $g_0 r_0$ by $v_E^2/2$ in accordance with Eq. (1.25) and solving for v_1 we have

$$v_1 = v_E \sqrt{\frac{1}{\rho_1} - \frac{1}{\rho_2}} \qquad (2.2)$$

where $\rho_1 = r_1/r_0$ and $\rho_2 = r_2/r_0$. If we let h_1 equal the altitude at burnout and h_2 equal the peak altitude, then $\rho_1 = 1 + \frac{h_1}{r_0}$, $\rho_2 = 1 + \frac{h_2}{r_0}$. Since $h_1/r_0 \ll 1$ under normal circumstances, $1/\rho_1 \cong 1 - h_1/r_0$ and Eq. (2.2) becomes

$$v_1 \cong v_E \sqrt{1 - \frac{h_1}{r_0} - \frac{1}{1 + \frac{h_2}{r_0}}}, \quad h_1 << r_0 \tag{2.3}$$

Eq. (2.3) can be solved exactly for peak altitude h_2. Thus

$$h_2 \cong \frac{h_1 + r_0 \frac{v_1^2}{v_E^2}}{1 - \frac{h_1}{r_0} - \frac{v_1^2}{v_E^2}}, \quad \frac{h_1}{r_0} << 1 \tag{2.4}$$

In Fig. 2.1 the peak altitude h_2 in miles for various burnout velocities v_1 is plotted for $h_1 = 0$. For $h_1 > 0$, the maximum altitude is obtained approximately by adding

$h_1 \Big/ \left[1 - \left(\frac{v_1}{v_E}\right)^2\right]$ to the ordinate shown, providing $\frac{h_1}{r_0} << 1$.

3. Calculation of Peak Altitude for Non-Vertical Launch

Next we consider a calculation of the peak altitude reached when the velocity vector at burnout is not vertical but makes an angle γ with the plane parallel to the earth's surface directly below the burnout point. It is assumed that the velocity vector is measured with respect to the inertial (non-rotating) frame of reference with the origin at the center of the earth, as in Section 1. Let v_1 be the velocity at burnout

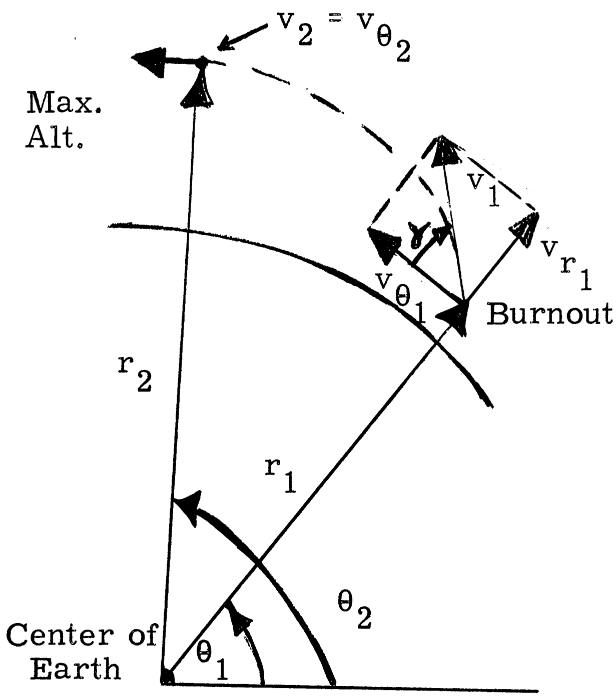
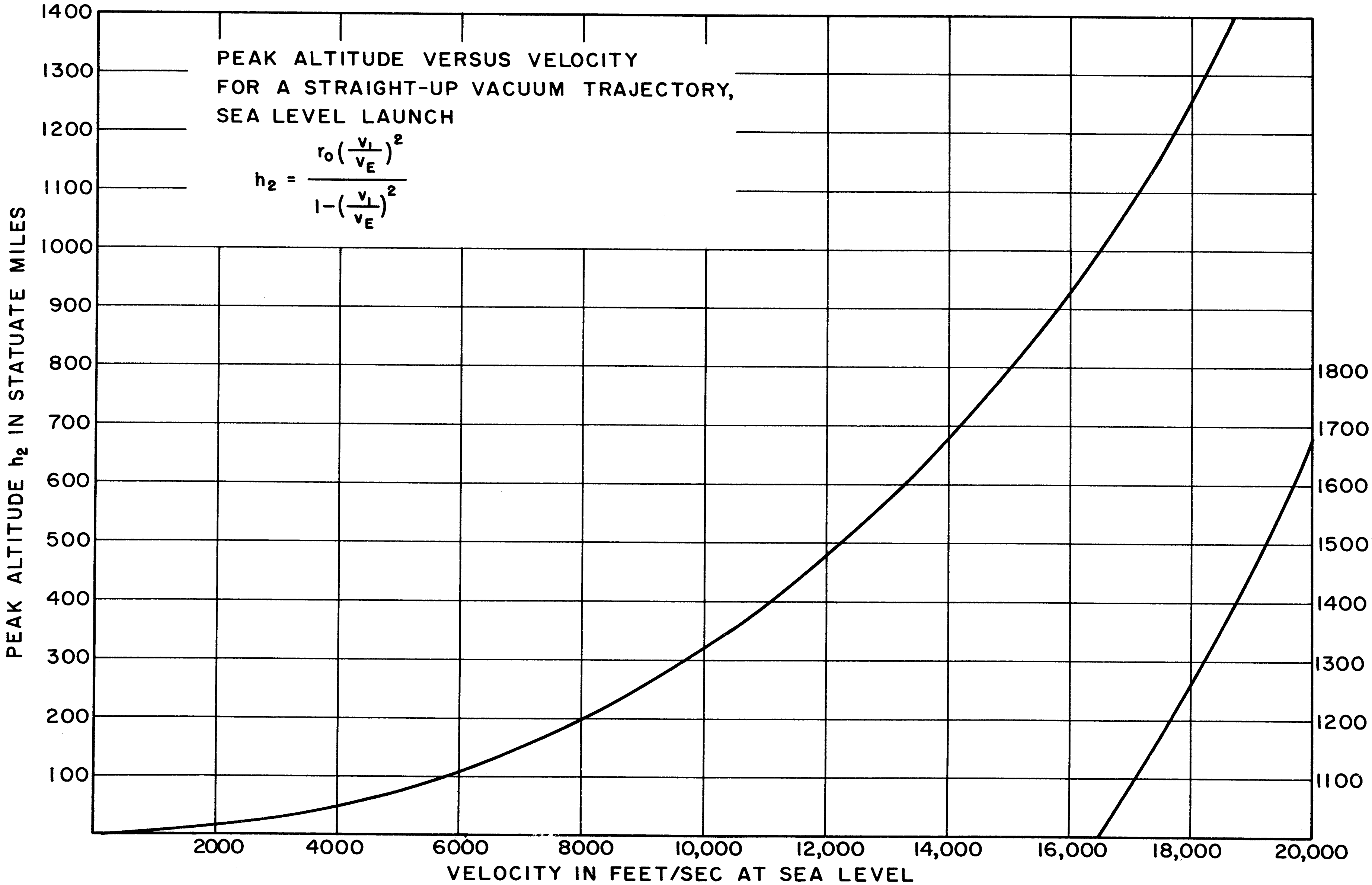


Fig. 3.1



PEAK ALTITUDE VERSUS VELOCITY
FOR A STRAIGHT-UP VACUUM TRAJECTORY,
SEA LEVEL LAUNCH
$h_2 = \dfrac{r_0\left(\frac{v_1}{v_E}\right)^2}{1-\left(\frac{v_1}{v_E}\right)^2}$
PEAK ALTITUDE h_2 IN STATUATE MILES
1400
1300
1200
1100
1000
900
800
700
600
500
400
300
200
100
1800
1700
1600
1500
1400
1300
1200
1100
2000
4000
6000
8000
10,000
12,000
14,000
16,000
18,000
20,000
VELOCITY IN FEET/SEC AT SEA LEVEL

Figure 3.1

with components v_{θ_1} and v_{r_1} parallel to and perpendicular to, respectively, the earth's surface, as shown in Fig. 3.1. Let r_1 be the polar radius of the rocket at this point. At the apex of the trajectory the velocity $v_2 = v_{\theta_2}$, and $r = r_2$. The most direct method to calculate r_2 and hence h_2, the maximum altitude, is to use equations for constant energy and constant angular momentum. From Eq. (1.5) we have for constant total energy

$$\frac{1}{2} m v_1^2 - mg_0 \frac{r_0^2}{r_1} = \frac{1}{2} m v_2^2 - mg_0 \frac{r_0^2}{r_2} \tag{3.1}$$

Replacing $g_0 r_0$ by $v_E^2/2$, where v_E is the escape velocity, it follows that

$$\left(\frac{v_1}{v_E}\right)^2 - \left(\frac{v_2}{v_E}\right)^2 - \frac{1}{\rho_1} + \frac{1}{\rho_2} = 0 \tag{3.2}$$

where $\rho_1 = r_1/r_0$, $\rho_2 = r_2/r_0$, r_0 = radius of earth.

Since the angular momentum is constant, we have

$$m r_1 v_1 \cos \gamma = m r_2 v_2 \tag{3.3}$$

Eliminating v_2 in Eq. (3.1) by means of (3.3), we obtain

$$\rho_2^2 \left[\frac{1}{\rho_1} - \left(\frac{v_1}{v_E}\right)^2\right] - \rho_2 + \left(\frac{v_1}{v_E}\right)^2 \cos^2 \gamma \, \rho_1^2 = 0 \tag{3.4}$$

Solving for ρ_2 which represents maximum ρ, we have

$$\rho_2 = \frac{1}{2\left[\frac{1}{\rho_1} - \left(\frac{v_1}{v_E}\right)^2\right]} \left[1 + \sqrt{1 - 4\left(\frac{v_1}{v_E}\right)^2 \rho_1^2 \cos^2\gamma \left[\frac{1}{\rho_1} - \left(\frac{v_1}{v_E}\right)^2\right]}\right] \tag{3.5}$$

The above expression reduces to $\rho_2 = \rho_1$ for $\gamma = 0$ and to Eq. (2.2) of the previous section for $\gamma = 90^\circ$. For the case where $4\left(\frac{v_1}{v_E}\right)^2 \rho_1^2 \cos^2\gamma \left[\frac{1}{\rho_1} - \left(\frac{v_1}{v_E}\right)^2\right] \ll 1$, which occurs when $\left(\frac{v_1}{v_E}\right) \ll 1$ or $\cos^2\gamma \ll 1$, Eq. (3.5) can be written approximately in terms of $h_2 = r_0(\rho_2 - 1)$. Thus

$$h_2 \cong h_{2_{\gamma = \pi/2}} \sin^2 \gamma \tag{3.6}$$

where $h_{2_{\gamma = \pi/2}}$ is the maximum altitude given in Eq. (2.4) for the case of a vertical launching. Thus the effect of burnout angle γ different from 90 degrees is, approximately, to reduce the altitude by the factor $\sin^2 \gamma$. The same result is obtained by considering the firing as one straight up with a velocity equal to the vertical component $v_1 \sin \gamma$ of the velocity v_1. The correction factor which must be applied to Eq. (3.6) for various altitudes and launch angles is shown in Fig. 3.2. The peak altitude reached is always somewhat higher than that given by Eq. (3.6) due to the influence of centrifugal acceleration.

4. Calculation of Sounding Rocket Range

The calculation of range is of interest not only because it is necessary for the prediction of impact point but also because it provides the equations necessary to calculate the sensitivity of range to burn-out velocity and flight-path angle.

The equation for the trajectory of free-flight after burnout is given by Eq. (1.16), which is repeated here for convenience

$$A \cos \theta + B \sin \theta = \frac{1}{r} - \frac{g_0 r_0^2}{r_1^4 \dot{\theta}_1^2} \tag{4.1}$$

where A and B are given in terms of the burnout variables, i.e., r_1, θ_1, $\dot{r}_1$, $\dot{\theta}_1$, in Eqs. (1.20) and (1.21). Let us arbitrarily assume that the impact occurs at $\theta = 0$. Also $r = r_0$ at impact. For these values of r and θ we have from Eqs. (4.1) and (1.20)

$$\left[\frac{1}{r_1} - \frac{g_0 r_0^2}{r_1^4 \dot{\theta}_1^2}\right] \cos \theta_1 + \left[\frac{\dot{r}_1}{r_1^2 \dot{\theta}_1}\right] \sin \theta_1 = \frac{1}{r_0} - \frac{g_0 r_0^2}{r_1^4 \dot{\theta}_1^2} \tag{4.2}$$

Replacing $g_0 r_0$ by $v_E^2/2$, $r_1 \dot{\theta}_1$ by v_{θ_1}, and $\dot{r}_1$ by v_{r_1}, we have

$$\left[\left(\frac{v_{\theta_1}}{v_E}\right)^2 - \frac{1}{2\rho_1}\right] \cos \theta_1 + \left(\frac{v_{r_1}}{v_E}\right)\left(\frac{v_{\theta_1}}{v_E}\right) \sin \theta_1 = \rho_1 \left(\frac{v_{\theta_1}}{v_E}\right)^2 - \frac{1}{2\rho_1} \tag{4.3}$$

where $\rho_1 = r_1/r_0$. Finally, $v_{r_1} = v_1 \sin \gamma$, $v_{\theta_1} = v_1 \cos \gamma$. Thus Eq. (4.3) becomes

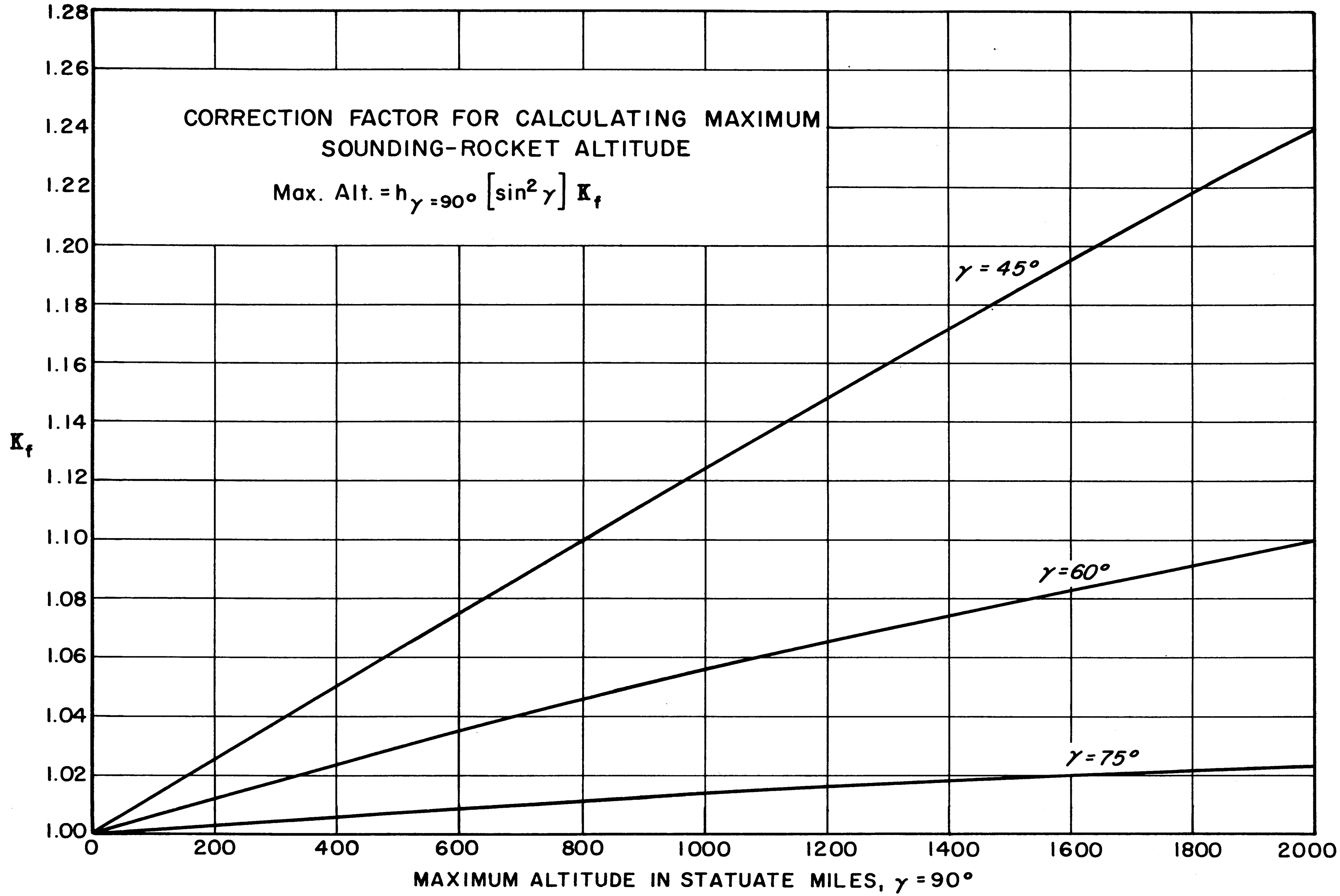

CORRECTION FACTOR FOR CALCULATING MAXIMUM
SOUNDING-ROCKET ALTITUDE
Max. Alt. = $h_{\gamma=90°} [\sin^2 \gamma] K_f$
K_f
1.28
1.26
1.24
1.22
1.20
1.18
1.16
1.14
1.12
1.10
1.08
1.06
1.04
1.02
1.00
0
200
400
600
800
1000
1200
1400
1600
1800
2000
$\gamma = 45°$
$\gamma = 60°$
$\gamma = 75°$
MAXIMUM ALTITUDE IN STATUATE MILES, $\gamma = 90°$

Figure 3.2

$$\left[\left(\frac{v_1}{v_E}\right)^2 \cos^2\gamma - \frac{1}{2\rho_1}\right] \cos\theta_1 + \left[\frac{1}{2}\left(\frac{v_1}{v_E}\right)^2 \sin 2\gamma\right] \sin\theta_1 =$$

$$\rho_1\left(\frac{v_1}{v_E}\right)^2 \cos^2\gamma - \frac{1}{2\rho_1} \qquad (4.4)$$

For a given velocity v_1, flight-path angle γ, and radius ρ_1 at burnout, the angle θ_1 given by Eq. (4.4) represents the burnout θ coordinate necessary to give impact at $\theta = 0$. The range of the rocket is, then simply $-r_0\theta_1$.

Eq. (4.4) has the general form

$$C_1 \cos\theta_1 + C_2 \sin\theta_1 = D \qquad (4.5)$$

But this can be rewritten as

$$\sqrt{C_1^2 + C_2^2} \cos(\theta_1 + \eta) = D \qquad (4.6)$$

where

$$\eta = -\tan^{-1} \frac{C_2}{C_1} \qquad (4.7)$$

Hence

$$\theta_1 = \cos^{-1} \frac{D}{\sqrt{C_1^2 + C_2^2}} + \tan^{-1} \frac{C_2}{C_1} \qquad (4.8)$$

By analogy between (4.4) and (4.5)

$$\theta_1 = \cos^{-1} \frac{\rho_1\left(\frac{v_1}{v_E}\right)^2 \cos^2\gamma - \frac{1}{2\rho_1}}{\sqrt{\left[\left(\frac{v_1}{v_E}\right)^2 \cos^2\gamma - \frac{1}{2\rho_1}\right]^2 + \frac{1}{4}\left(\frac{v_1}{v_E}\right)^4 \sin^2 2\gamma}}$$

$$+ \tan^{-1} \frac{\frac{1}{2}\left(\frac{v_1}{v_E}\right)^2 \sin 2\gamma}{\left(\frac{v_1}{v_E}\right)^2 \cos^2\gamma - \frac{1}{2\rho_1}} \qquad (4.9)$$

For the special case where $\rho_1 = 1$, i.e., where the burnout is assumed to take place at sealevel, the two angles on the right side of Eq. (4.9) are equal and

$$\theta_1 = 2 \tan^{-1} \frac{\frac{1}{2}\left(\frac{v_1}{v_E}\right)^2 \sin 2\gamma}{\left(\frac{v_1}{v_E}\right)^2 \cos^2\gamma - \frac{1}{2}}, \quad \rho_1 = 1 \tag{4.10}$$

Since the burnout altitude for most sounding rockets is low (perhaps 10-30 miles), Eq. (4.10) will give a reasonably accurate estimate of sounding-rocket range. For more accurate computation Eq. (4.9) can always be used. Eq. (4.10) should, however, be quite adequate for establishing the dependence of range on v_1 and γ. Note that we have also ignored the reentry trajectory here.

If a missile with velocity v_1 and flight path angle γ were launched in a constant-gravity field of acceleration g_0, then the total time of flight would be $(2v_1 \sin\gamma)/g_0$ and the horizontal distance traveled would be $\left[(2v_1 \sin\gamma)/g_0\right] \left[v_1 \cos\gamma\right] = (v_1^2 \sin 2\gamma)/g_0$. This makes a convenient norm to compare the actual distance traveled along the earth's surface as computed from Eq. (4.10). The required correction factor K_γ is shown in Fig. 4.1 as a function of rocket altitude for vertical burnout for various burnout angles γ. Thus

$$\text{Range} = \frac{v_1^2 \sin 2\gamma}{g_0} K_\gamma \tag{4.11}$$

This allows a quick calculation of range for a wide variety of burnout conditions.

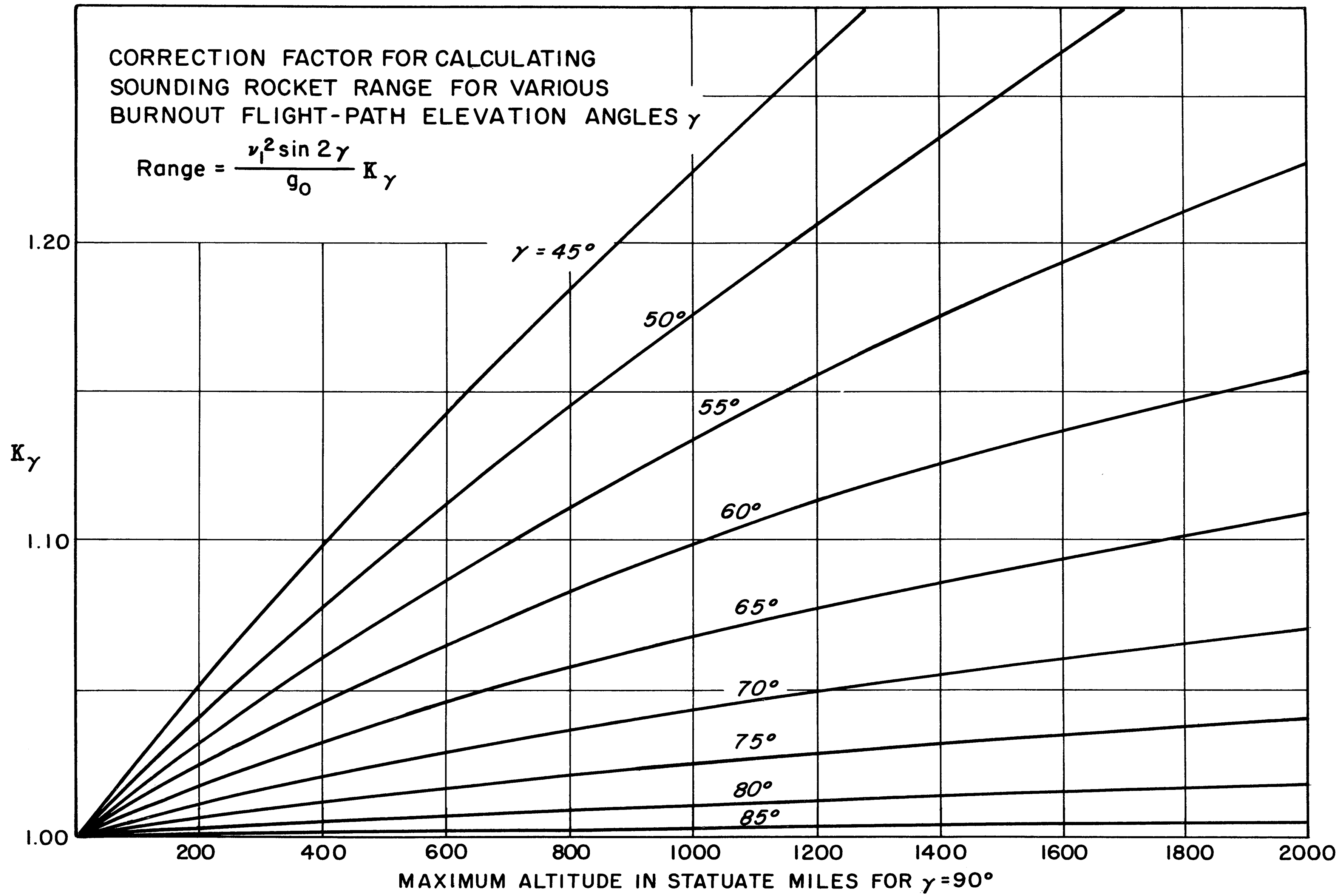
In Fig. 4.2 are plots of actual range R for various γ versus peak altitude for straight-up launch.

5. Effect of Errors in Burnout Velocity and Angle on Impact Point

5.1 Effect of Velocity Error

In Eq. (4.9) we derived the launch polar angle θ_1 for an impact at $\theta = 0^\circ$ with velocity v_1 and flight-path angle γ at launch. Eq. (4.10) is an excellent approximate representation of θ_1 (it is exact for the case where launch, or more correctly, burnout occurs at $\rho_1 = 1$, the earth's surface). Eq. (4.10) can be rewritten as

$$\theta_1 = -2 \tan^{-1} \frac{\sin 2\gamma}{\frac{1}{\rho_1}\left(\frac{v_E}{v_1}\right)^2 - 2\cos^2\gamma} \tag{5.1}$$



CORRECTION FACTOR FOR CALCULATING
SOUNDING ROCKET RANGE FOR VARIOUS
BURNOUT FLIGHT-PATH ELEVATION ANGLES γ
Range = $\frac{v_1^2 \sin 2\gamma}{g_0} K_\gamma$
K_γ
1.20
1.10
1.00
γ = 45°
50°
55°
60°
65°
70°
75°
80°
85°
200
400
600
800
1000
1200
1400
1600
1800
2000
MAXIMUM ALTITUDE IN STATUATE MILES FOR γ = 90°

Figure 4.1

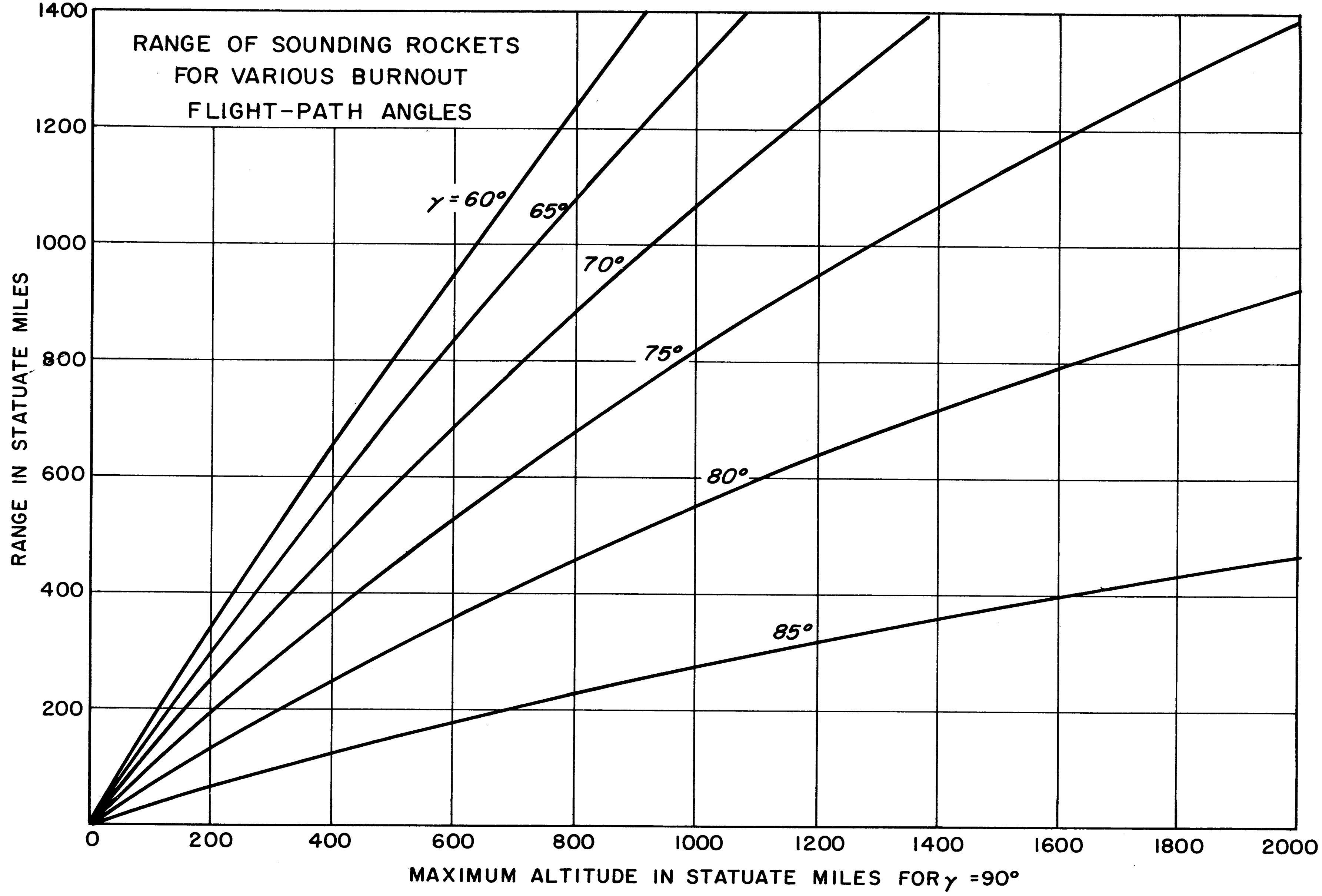

RANGE OF SOUNDING ROCKETS
FOR VARIOUS BURNOUT
FLIGHT-PATH ANGLES
γ = 60°
65°
70°
75°
80°
85°
RANGE IN STATUATE MILES
1400
1200
1000
800
600
400
200
0
200
400
600
800
1000
1200
1400
1600
1800
2000
MAXIMUM ALTITUDE IN STATUATE MILES FOR γ = 90°

Figure 4.2

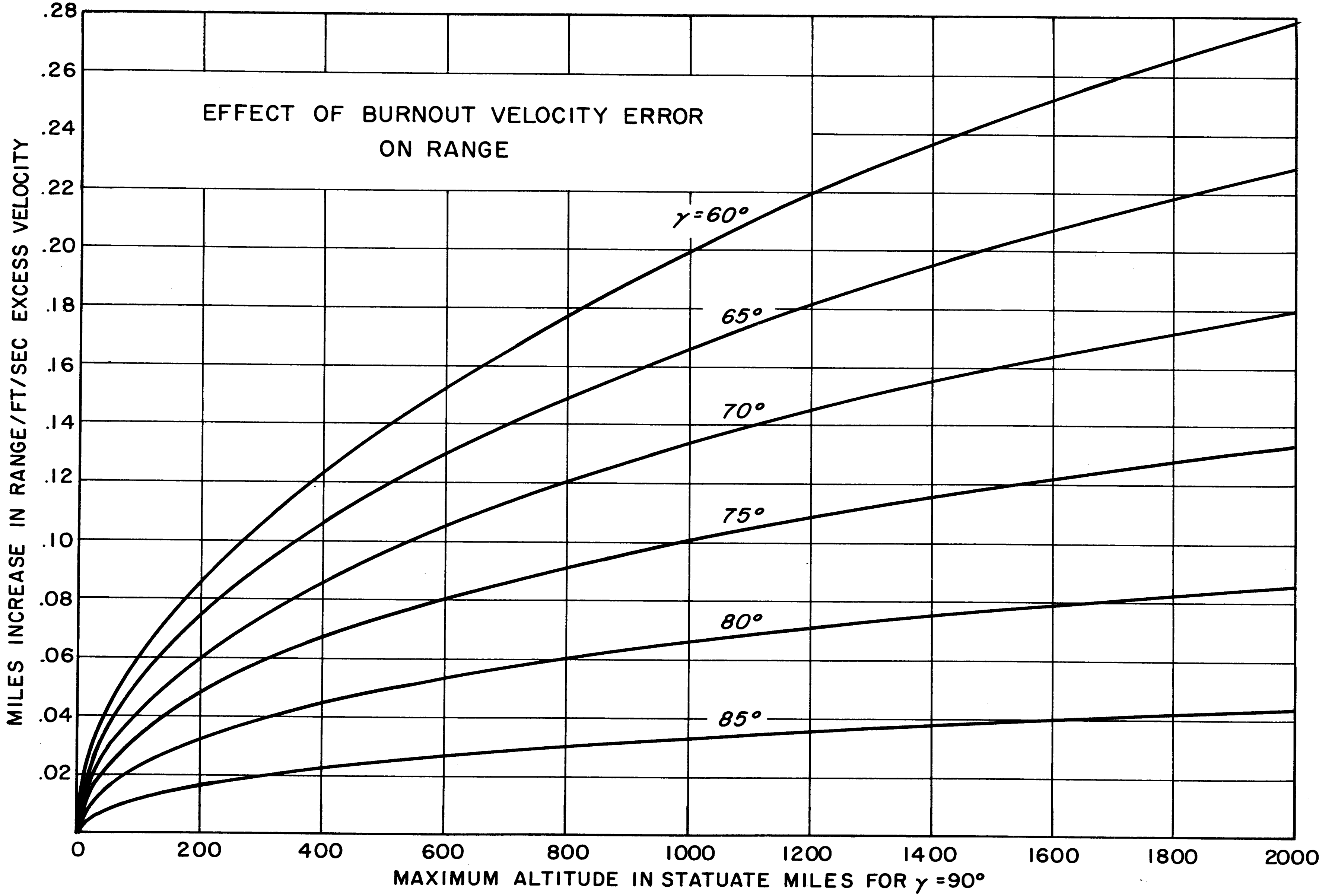

EFFECT OF BURNOUT VELOCITY ERROR
ON RANGE
γ=60°
65°
70°
75°
80°
85°
MILES INCREASE IN RANGE/FT/SEC EXCESS VELOCITY
.28
.26
.24
.22
.20
.18
.16
.14
.12
.10
.08
.06
.04
.02
0
200
400
600
800
1000
1200
1400
1600
1800
2000
MAXIMUM ALTITUDE IN STATUATE MILES FOR γ =90°

Figure 5.1

Taking the partial derivative with respect to v_1, we have

$$\frac{\partial \theta_1}{\partial v_1} = - \frac{\frac{4 \sin 2\gamma}{\rho_1 v_E}\left(\frac{v_E}{v_1}\right)^3}{\left[\frac{1}{\rho_1}\left(\frac{v_E}{v_1}\right)^2 - 2\cos^2\gamma\right]^2 + \sin^2 2\gamma} \qquad (5.2)$$

The change in range R due to changes in v_1 is simply $r_0 \; \partial\theta_1/\partial v_1$. Thus the error in range ΔR is

$$\Delta R = -r_0 \frac{\partial \theta_1}{\partial v_1} \Delta v_1 \qquad (5.3)$$

where Δv_1 is the change in burnout velocity from the norm value.

A good approximation to Eq. (5.2) can be written for Eq. (5.2) for near-vertical sounding rockets. Thus if $(v_E/v_1)^2 \gg 2\cos^2\gamma$, and $(v_E/v_1)^4 \gg \sin^2 2\gamma$,

$$\frac{\partial \theta_1}{\partial v_1} \cong - \frac{4\rho_1 \sin 2\gamma}{v_E}\left(\frac{v_1}{v_E}\right) \qquad (5.4)$$

For a 1000 mile sounding rocket with a burnout angle of 75^o the approximate expression (5.4) gives $r_0 \partial\theta_1/\partial v_1$ = -0.0968 mi/ft/sec compared with the exact value of -0.099 from Eq. (5.2). The exact value corresponds to 16.4 miles error in range for a 1% error in velocity v_1. In Fig. 5.1 are shown plots of $r_0 \partial\theta_1/\partial v_1$ for burnout angles between 60^o and 90^o.

5.2 Effect of Elevation Angle Error

Next we calculate the effect of errors in flight-path elevation angle γ at burnout. Taking the partial derivative of Eq. (5.2) with respect to γ, we have

$$\frac{\partial \theta_1}{\partial \gamma} = \frac{4}{\left[\frac{\frac{1}{\rho_1}\left(\frac{v_E}{v_1}\right)^2 - 2\cos^2\gamma}{\sin 2\gamma}\right]^2 + 1}\left\{1 - \text{ctn}\, 2\gamma\left[\frac{\frac{1}{\rho_1}\left(\frac{v_E}{v_1}\right)^2 - 2\cos^2\gamma}{\sin 2\gamma}\right]\right\} \qquad (5.5)$$

Again a fair approximation is obtained for sounding rockets when $(v_E/v_1)^2 \gg 2\cos^2\gamma$, $(v_E/v_1)^4 \gg \sin^2 2\gamma$, and $\left|(v_E/v_1)^2 \cos 2\gamma\right| \gg \sin^2 2\gamma$

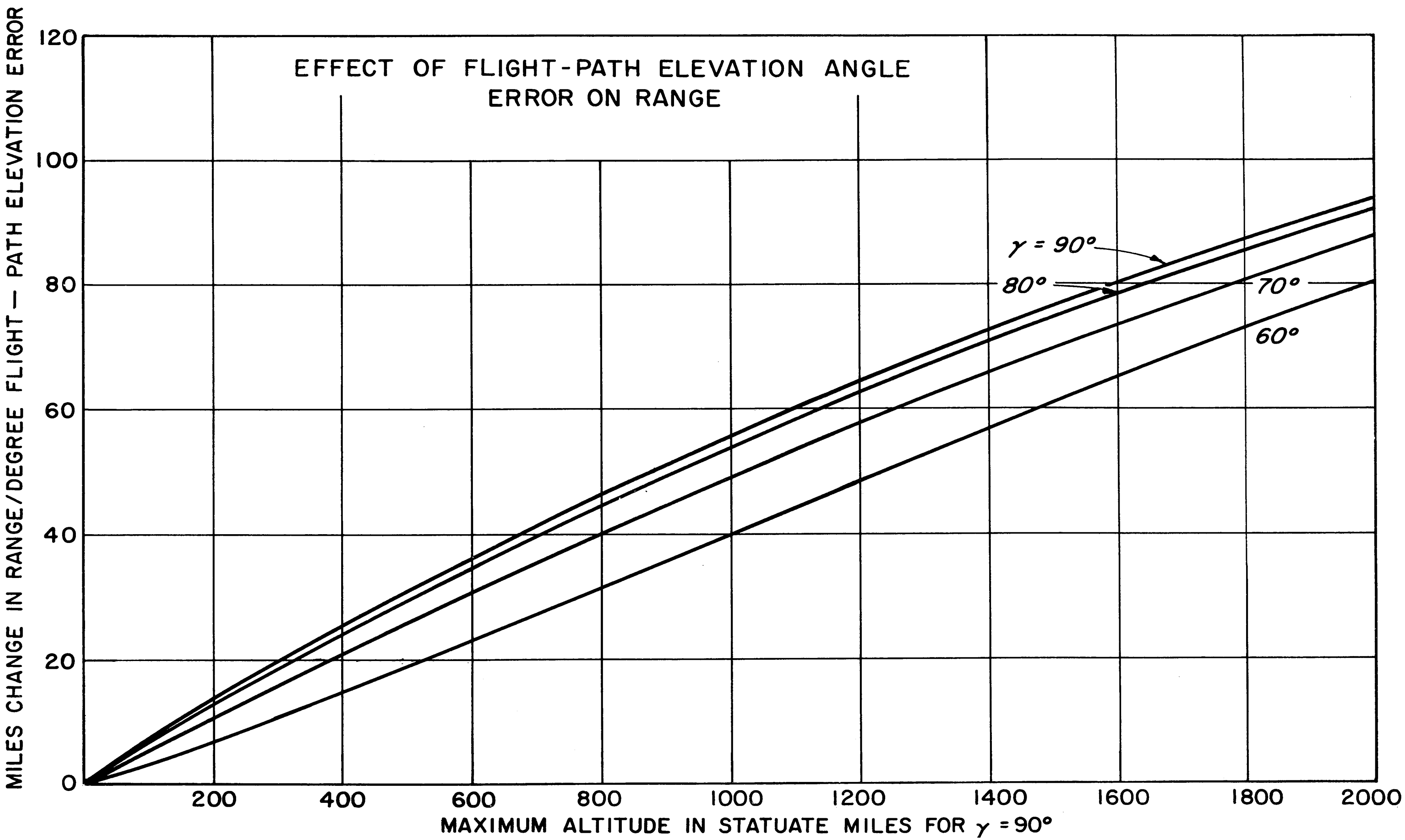

EFFECT OF FLIGHT-PATH ELEVATION ANGLE
ERROR ON RANGE
MILES CHANGE IN RANGE/DEGREE FLIGHT — PATH ELEVATION ERROR
0
20
40
60
80
100
120
200
400
600
800
1000
1200
1400
1600
1800
2000
MAXIMUM ALTITUDE IN STATUATE MILES FOR γ = 90°
γ = 90°
80°
70°
60°

Figure 5.2

$$\frac{\partial \theta_1}{\partial \gamma} \cong -4 \rho_1 \left(\frac{v_1}{v_E} \right)^2 \cos 2\gamma \qquad (5.6)$$

For a 1000 mile sounding rocket with $\gamma = 75^o$, $\partial \theta_1 / \partial \gamma = 0.706$ from the approximate Eq. (5.6) and 0.763 from the exact expression (5.5) the error in range ΔR is given by

$$\Delta R = r_0 \frac{\partial \theta_1}{\partial \gamma} \Delta \gamma \qquad (5.7)$$

where $\Delta \gamma$ is the error in γ at burnout. For the rocket example earlier this turns out to be 48.7 miles/degree.

In Fig. 5.2 are shown plots of $r_0 \; \partial\theta_1 / \partial \gamma$ for burnout angles between 60^o and 90^o.

5.3 Effect of Yaw Angle Error

Let us define a yaw angle η as measured in the plane through the desired velocity vector at burnout but normal to the desired plane of motion, as shown in Fig. 5.3.

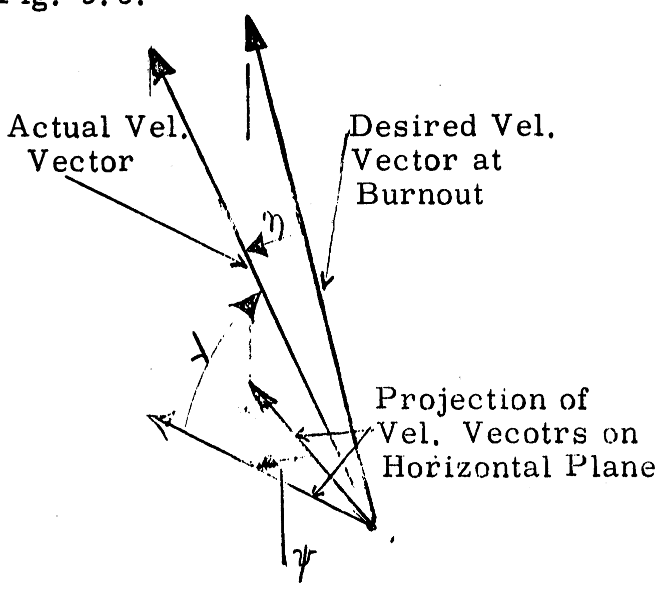


Fig. 5.3

This defines a new plane of free-flight motion which makes an angle $\psi = \eta / \cos \gamma$ with the original plane, assuming $\psi << 1$. Hence as long as the range R is not so long that spherical-surface effects predominate, the error in impact position d at right angles to the plane of the motion is given by

$$d = R \frac{\eta}{\cos \gamma} \qquad (5.8)$$

For the 1000 mile rocket with $\gamma = 75^o$, R = 820 miles and d = 55.3 miles/deg.

6. Effect of the Earth's Rotation on Sounding Rocket Trajectories

6.1 Calculation of the Burnout Flight-Path Angle and Velocity in Inertial Coordinates

All of the trajectory computations up to now have been made with respect to a non-rotating coordinate system with its origin at the center of the earth. Actually, the rocket is launched from the surface of the earth, which is moving eastward at a velocity $v_e = \Omega r_0 \cos\theta' = 1515 \cos\theta'$ ft/sec, where Ω is the angular velocity of rotation of the earth, r_0 is the earth radius, and θ' is the latitude of the launch point. All of the rocket performance calculations are made with respect to the moving-earth's surface at the launch point. To the burnout velocity v_1' calculated with respect to this system we must add vectorally the eastward velocity component v_e to obtain the burnout velocity v_1 with respect to the inertial coordinate system. The simplest case is the one where we fire the rocket directly east or directly west, i.e., where v_1' and v_e lie in the same plane, as shown in Fig. 6.1.

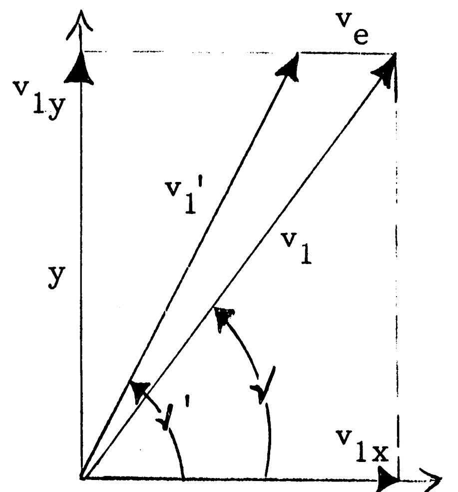


Fig. 6.1

Here the horizontal velocity component $v_{1x} = v_1'\cos\gamma' \pm v_e$ and the vertical component $v_{1y} = v_1'\sin\gamma'$, where γ' is is the burnout angle as calculated with respect to the rotating earth and γ is the burnout angle as calculated with respect to the inertial coordinates. For an eastward launch we have $+v_e$ in the formula for v_{1x}; for a westward launch we have $-v_e$. The total velocity v_1 in the inertial coordinate system is given by

$$v_1 = \sqrt{v_{1x}^2 + v_{1y}^2} = v_1'\sqrt{1 \pm \frac{2v_e}{v_1'}\cos\gamma' + \left(\frac{v_e}{v_1'}\right)^2} \qquad (6.1)$$

If we assume that $\left| \pm 2v_e/v_1' \cos\gamma' + (v_e/v_1')^2 \right| << 1$, we can write approximately

$$v_1 \cong v_1' \left[1 \pm \frac{v_e}{v_1'} \cos\gamma' + \frac{1}{2}\left(\frac{v_e}{v_1'}\right)^2 \right] \qquad (6.2)$$

This approximation formula should be quite accurate for near vertical launches of rockets to several hundred miles or higher.

From Fig. 6.1 it is apparent that

$$\gamma = \tan^{-1} \frac{v_{1y}}{v_{1x}} = \tan^{-1}\left[\frac{v_1' \sin\gamma'}{v_1' \cos\gamma' \pm v_e}\right] \qquad (6.3)$$

If $\left| \gamma' - \gamma \right| << 1$, then we can write the approximate formula

$$\gamma = \gamma' \mp \frac{v_e}{v_1'} \sin \gamma' \qquad (6.4)$$

As a specific example, consider a nominal 1000 mile sounding rocket fired eastward at a latitude of 40°N. Let $v_1' = 16,500$ ft/sec and $\gamma' = 75^{\circ}$. Then v_e = 1160 ft/sec, v_1 = 16840 ft/sec, and $\gamma = 71.1^{\circ}$. Thus the effect of launching the rocket east has been to raise the burnout velocity by 340 ft/sec and lower the flight path angle at burnout by 3.9°. If we had neglected this effect the peak altitude for the original 16500 ft/sec velocity and 75° burnout angle would have yielded a calculated peak altitude of 951 miles. With the earth's rotational velocity taken into account the peak altitude is 970 miles.

For a westward launch with v_1' = 16500 ft/sec and $\gamma' = 75^{\circ}$ as before, v_1 = 16240 ft/sec and $\gamma = 78.9^{\circ}$. The correct peak altitude is then 935 miles. Thus an eastward launch yields a slight advantage in peak altitude.

For azimuth angles at launch which are approximately eastward or westward, the above results will apply with reasonable accuracy. For other azimuth angles the analysis is more complicated but is equally straightforward.