THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

THE TORSION OF SHAFTS
OF VARYING CIRCULAR CROSS SECTIONS

Hubert L. Hunzeker

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1958

September, 1958

IP-320

Doctoral Committee:

Professor George E. Hay, Chairman
Assistant Professor A. Bruce Clarke
Associate Professor Robert L. Hess
Professor Wilfred Kaplan
Professor Gail S. Young

ACKNOWLEDGEMENTS

This research was supported in whole or in part by the United States Air Force under Contract No AF 49(638)-104 monitored by the AF Office of Scientific Research of the Air Research and Developement Command.

The author thanks Professor Wilfred Kaplan and others of the faculty who have generously entered into discussions on the problem. The author is grateful to Professor George E. Hay for the inspiration and many courtesies extended.

TABLE OF CONTENTS

CHAPTER Page

LIST OF ILLUSTRATIONS

CHAPTER I

THE PROBLEM

1. Introduction. An elastic body is defined as one which deforms when it is subjected to an external load, the deformation disappearing when the load is removed. When such a body is loaded, the various particles of the body are displaced and exert forces on neighboring particles. The displacement of a general particle is a vector described by its three scalar components which are functions of position and are called the displacements of the body. The deformation is described by six functions called the strain components which are related to the displacements by six strain-displacement relations. Furthermore, the internal reactions in the body are described by six functions called the stress components which must satisfy three equations of equilibrium which are first order linear partial differential equations. The strain components must satisfy six second order linear partial differential equations called the equations of compatibility. Collectively, the equilibrium conditions, the stress-strain relations the strain-displacement equations and the equations of compatibility are known as the fundamental equations of elasticity.

In this thesis we consider the torsion of a shaft of varying circular cross sections, which problem is described more precisely as follows: The elastic body under consideration is a solid of revolution with plane ends perpendicular to the axis of revolution, and the body is twisted by twisting couples applied to the ends. The problem is to find the components of stress, strain, and displacement throughout the body. It is further assumed that the body is elastically isotropic and homogeneous, which means that the elastic properties of the material are independent of direction and position, respectively.

According to the above formulation, the solution to the torsion problem should provide for twisting couples which are of arbitrary magnitude and which are distributed arbitrarily over the ends of the shaft. However, gross difficulty is usually encountered when solutions of this type are sought and it is standard practice to specify only the magnitude of the twisting couple, and not its distribution over the ends of the shart. Solutions obtained in this way require that the twisting couples be distributed over the ends in a particular way. However, the Principle of Saint Venant [1] permits application of such solutions to cases of general distribution of the twisitng couples over the ends, for according to this principle, if the force system acting on a portion P of the surface of a body is replaced by a second force system statically equivalent to the first, there will be no significant change in the distribution of stress at points sufficiently far from the region P.

A treatment of the question of the distribution of twisting couple, which is somewhat equivalent to the above, is referred to by Synge [4] as the exponential condition, whereby the difference between any two distributions of twisting couples with equal magnitude produces effects in a shaft which decrease exponentially with distance from the end. In this thesis we shall follow the standard practice, and shall not specify the distribution of twisting couple over the ends.

Historical Survey. The torsion of a shaft of uniform but general cross section has an extensive history. This problem reduces to a Neumann boundary value problem in two dimensions, that is, to the determination of a function harmonic throughout a cross section D and with a prescribed normal derivative on the boundary of D. An equivalent representation involves a Dirichlet boundary value problem in two dimensions, that is, the determination of a function harmonic in D with prescribed values on the boundary of D. The torsion problem for the shaft of varying circular cross sections resembles in some respects the torsion of the shaft of uniform but general cross section; it also reduces to two-dimensional Neumann and Dirichlet boundary value problems, but these problems are now generalized, that is, the differential equations are not Laplace's equations; further, the two coordinates involved are in the meridian section of the shaft rather than the cross section.

In his historical review of the torsion problem for a solid of revolution, Higgins [5] attributes the first investigations to Michell [6] and Föppl [7]. The generalized Neumann and Dirichlet problem approach is due to Willers [8] who used numerical methods to investigate the stresses in particular shafts and to verify the interesting result of Larmor [9] that, on a circumferential scratch around a shaft of uniform cross sections, the maximum shearing stress is double that at the same spot of the uniform shaft. The development of the fundamental equations for the torsion of a solid of revolution in terms of orthogonal curvilinear coordinates is due to Timpe [10] who has more recently used the biharmonic equation [11] to obtain solutions in terms of spherical coordinates for shafts bounded by spherical and conical surfaces [12] [13]. Quite recently Hay [14] formulated the fundamental equations for the torsion of a solid of revolution in terms of general curvilinear coordinates.

While no general solution is available for the torsion of a general shaft of varying circular cross sections, solutions in closed form are available for the torsion of a few shafts of simple geometrical form.

Mellan [15] has used Timpe's analysis in conjunction with infinite integrals of Bessel function products to obtain a solution for the torsion of the shaft bounded by a portion of an arbitrary hyperboloid of revolution of one sheet, one end of the shaft being that cross section of the hyperboloid which has the smallest radius. Pöschl [16] used Timpe's analysis to obtain solutions for the torsion of an arbitrary hyperboloid of a revolution of one sheet, an arbitrary cone of revolution, an arbitrary sphere, an arbitrary oblate ellipsoid and an arbitrary paraboloid of revolution. In his study of notch stresses, Neuber utilized his three function theorem and curvilinear coordinates to obtain, among other things, the solution to the torsion problem for a hyperboloid of revolution of one sheet [17] and for a shaft of nearly uniform cross sections which has a symmetrically located oblate ellipsoidal cavity [18]. Sonntag [19] has given an exact solution for the torsion of a uniform circular shaft which has an arbitrary semi-circular circumferential groove; he used polar coordinates in a meridian section and developed simple formulae for the maximum shearing stress in shafts which are of common use in machine construction.

Many other solutions of this torsion problem have been obtained indirectly. Among the more recent investigations are those of Sokolow [20] who used orthogonal curvilinear coordinates to obtain solutions for the torsion of a few shafts with boundary surfaces which have algebraic curves in a meridian section as generators. Also, Abbassi [21] used spherical coordinates to obtain Legendre function solutions to the fundamental partial differential equation from which solutions are chosen for a few simple cases when the boundary of the shaft is generated by an algebraic curve in a meridian section. In a later paper, Abassi [22] also used Hermite polynomials in a similar manner. Recently, Chattajari [23] found two infinite sets of polynomials in terms of rectangular cartesian coordinates in a meridian section;

and linear combinations of these polynomials were used to construct simple curves which generate the boundary of the shaft.

Also, Reissner and Wennagil [24] used Bessel functions multiplied by exponential or trigonometric functions to build solutions which determine a generator for the boundary of a shaft; numerical analysis then led to solutions for the torsion of certain shafts of technical interest. Brousse [25] considered the torsion problem for a uniform shaft with a circumferential groove by choosing a sequence of domains, in the meridian section of the shaft, which converged to the domain of the shaft; a solution of the generalized Dirichlet problem for each domain then led to a sequence of solutions which converged to the desired solution. Ling [26] has given the solution for the torsion of a shaft with a symmetrically located spherical cavity in terms of an infinite sum of Lengendre functions, and Das [27] has also obtained a solution to this problem in terms of dipolar coordinates. In addition, Poritsky [28], has shown that solutions of the differential equation in the generalized Dirichlet problem are identical with harmonic functions in a five-dimensional space, and has used these harmonic functions to construct solutions for certain shafts which approximate some cases of technical interest. Weiss and Payne [29] have given the solution to the torsion problem for a shaft with an arbitrary toroidal cavity in an n-dimensional axial symmetry body.

Summary. In Chapter I we consider the fundamental equations of elasticity, and reduce these for the case of the torsion of a shaft of varying circular cross sections, using a tensorial approach following Hay [14]. In chapter II we consider the torsion of a hollow prolate ellipsoidal shaft, and of a shaft of nearly uniform cross sections with an arbitrary prolate ellipsoidal cavity. Non-orthogonal coordinates are used. The solutions are apparently new, and the solutions are also obtained in terms of classical prolate elliptic coordinates.

In Chapter III we apply Hay's analysis to obtain the solution to the Saint Venant torsion problem for a shaft of nearly uniform cross sections with an arbitrary symmetrically located oblate ellipsoidal cavity. Non-orthogonal coordinates are again used and the solutions are compared with a known solution due to Neuber [18]. Chapter IV contains a result, which is apparently new, for the torsion of a shaft bounded by a portion of an arbitrary hyperboloid of revolution of two sheets, in terms of a classical elliptic coordinates. A similar solution for the torsion of a hollow shaft bounded by a portion of arbitrary hyperboloids of revolution of one sheet is obtained in terms of non-orthogonal coordinates, and in the special case when the shaft is soild, the result is compared with those of Pöschl [16] and Neuber [17].

In Chapter V we consider the torsion of a general shaft of varying circular cross section. We use rectangular cartesian coordinates in a meridian section of the shaft, and introduce known solutions to the basic partial differential equation of the related generalized Dirichlet problem. From these we construct an orthonormal system of functions which depends on only one variable along a general boundary of the merician section of the shaft. The boundary conditions on this boundary are then satisfied by the use of a Fourier type series of orthogonal functions. The method and results seem to be new.

2. The Fundamental Equations of Elasticity. Tensor analysis with the usual summation convention is used. Latin suffixes have the range 0, 1, 2, and Greek suffixes have the range 1, 2.

Let us consider a homogeneous isotropic elastic body occupying a region V bounded by a surface S. Let x^i $(i = 0, 1, 2)$ be curvilinear coordinates of a general point P in V and let (x, y, z) be the rectangular cartesian coordinates of P. Then we have the relations

$$x^i = x^i(x, y, z). \tag{2.1}$$

We consider only coordinate systems x^i for which the transformation (2.1) has a finite non-vanishing Jacobian, except possibly at certain singular points.

The differential element of arc length is given by the relation

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{2.2}$$

where g_{ij} denotes the covariant metric tensor. The contravariant metric tensor is denoted by g^{ij} and we have the identities

$$g_{ij}\, g^{jk} = \delta^k_j , \tag{2.3}$$

where δ^k_j is the Kronecker delta.

Christoffel symbols of the first and second kinds are defined respectively by the equations

$$[ij, k] \equiv \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right), \tag{2.4}$$

$$\Gamma^k_{ij} \equiv g^{ks}\, [ij, s]. \tag{2.5}$$

We denote the covariant derivative of a tensor with respect to the metric g_{ij} of the space by a subscript following a double bar. Thus, for example, if A^{ij} is a second order, contravariant tensor, its covariant derivative is

$$A^{ij}{}_{\|k} = \frac{\partial A^{ij}}{\partial x^k} + \Gamma^i_{nk}\, A^{nj} + \Gamma^j_{nk}\, A^{in}.$$

Let $\Delta\bar{S}$ denote an oriented element of area containing a point P. Let λ^i denote the unit vector at P normal to $\Delta\bar{S}$ on the positive side. Let T^i denote the stress vector at P corresponding to the direction λ^i, that is, T^i is a contravariant vector describing the force per unit area at P exerted by particles on the positive side of $\Delta\bar{S}$ on particles on the negative side of $\Delta\bar{S}$. Let T^{ij} denote the contravariant stress components at P. We then have the usual relations

(2.6) $$T^i = T^{ij} \lambda_j .$$

Furthermore, the stress components must satisfy the equations of equilibrium which are

(2.7) $$T^{ij}{}_{\|j} = 0.$$

Let u_i denote the covariant component of displacement. Then the strain components e_{ij} are defined by the relations

(2.8) $$e_{ij} = \frac{1}{2} (u_{i\|j} + u_{j\|i}).$$

For the isotropic body under consideration, the stress-strain relations are

(2.9) $$T^{ij} = \lambda \Delta g^{ij} + 2\mu e^{ij},$$

where

(2.10) $$\Delta \equiv u^k{}_{\|k},$$

and λ, μ are elastic constants sometimes called the Lamé constants; μ is also referred to as the shear modulus. The Lamé constants are related to Young's modulus E and Poisson's ratio σ by the equations

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)} .$$

If the body were not isotropic, that is, if all directions at a point were not elastically equivalent, the number of elastic constants could range up to 21; the precise number of such constants depends on the type of elastic symmetry, and is of course two for isotropic bodies.

To obtain the three well-known differential equations for the three components of displacement, we first substitute in equations (2.9) for the strain components from equations (2.8) obtaining the stress-displacement equations

$$(2.11) \qquad T^{ij} = \lambda \Delta g^{ij} + \mu (g^{jk} u^{i}{}_{\|k} + g^{ik} u^{j}{}_{\|k}).$$

Substitution for T^{ij} from these relations in the equations of equilibrium (2.7) gives rise to the desired equations

$$(2.12) \qquad (\lambda + \mu)\Delta_{,i} + \mu \nabla_3^2 u_i = 0,$$

where ∇_3^2 is the three dimensional Laplacian operator, that is,

$$(2.13) \qquad \nabla_3^2 u_i \equiv g^{jk} u_{i\|jk}.$$

Equations (2.12) are known as the generalized Navier equations, and must be satisfied by the displacements u^i throughout V.

Now T_i is the stress vector at P corresponding to a direction λ_i, and represents the force per unit area acting across a surface $\Delta\bar{S}$ at P perpendicular to λ^i. The component of T_i in the direction of λ_i is the normal stress N at P. It is given by the relation

$$(2.14) \qquad N = T_i \lambda^i.$$

The component of T_i which is tangent to the element of surface $\Delta\bar{S}$ at P is called the shearing stress S, and is given by the equation

(2.15) $S^2 = T_i T^i - N^2.$

The maximum shearing stress S_{max} at P can readily be found in the following way. We first consider the equation

(2.16) $|T_{ij} - \sigma g_{ij}| = 0$

which is a cubic in σ. We denote the roots by σ_1, σ_2, σ_3 with $\sigma_1 \leqq \sigma_2 \leqq \sigma_3$; these are called the principal stresses at P and it can be shown that

(2.17) $S_{max} = \frac{1}{2}(\sigma_3 - \sigma_1).$

The quanities N, S, S_{max}, σ_1, σ_2, and σ_3 are invariant under coordinate transformations.

3. Fundamental Equations in the Case of Torsion; Neumann Form. Let us consider a shaft of non-uniform, circular, cross section with plane ends, as shown in Figure 1, acted upon by equal and opposite twisting couples applied to the ends. The shaft is then in torsion. The intersection of the shaft and a half-plane emanating from the axis of revolution of the shaft is called a meridian section of the shaft.

Let (x, y, z) be rectangular cartesian coordinates, the z-axis being coincident with the axis of the shaft, as shown in Figure 1. Let x^i (i = 0, 1, 2) be general curvilinear coordinates, where x^0 is the angle between the xz-plane and a general meridian section D, and x^α (α = 1, 2) are general curvilinear coordinates in D. We note that D is a two dimensional subspace of the three dimensional space V.

The parametric lines of x^0 are circles normal to the meridian sections, so that

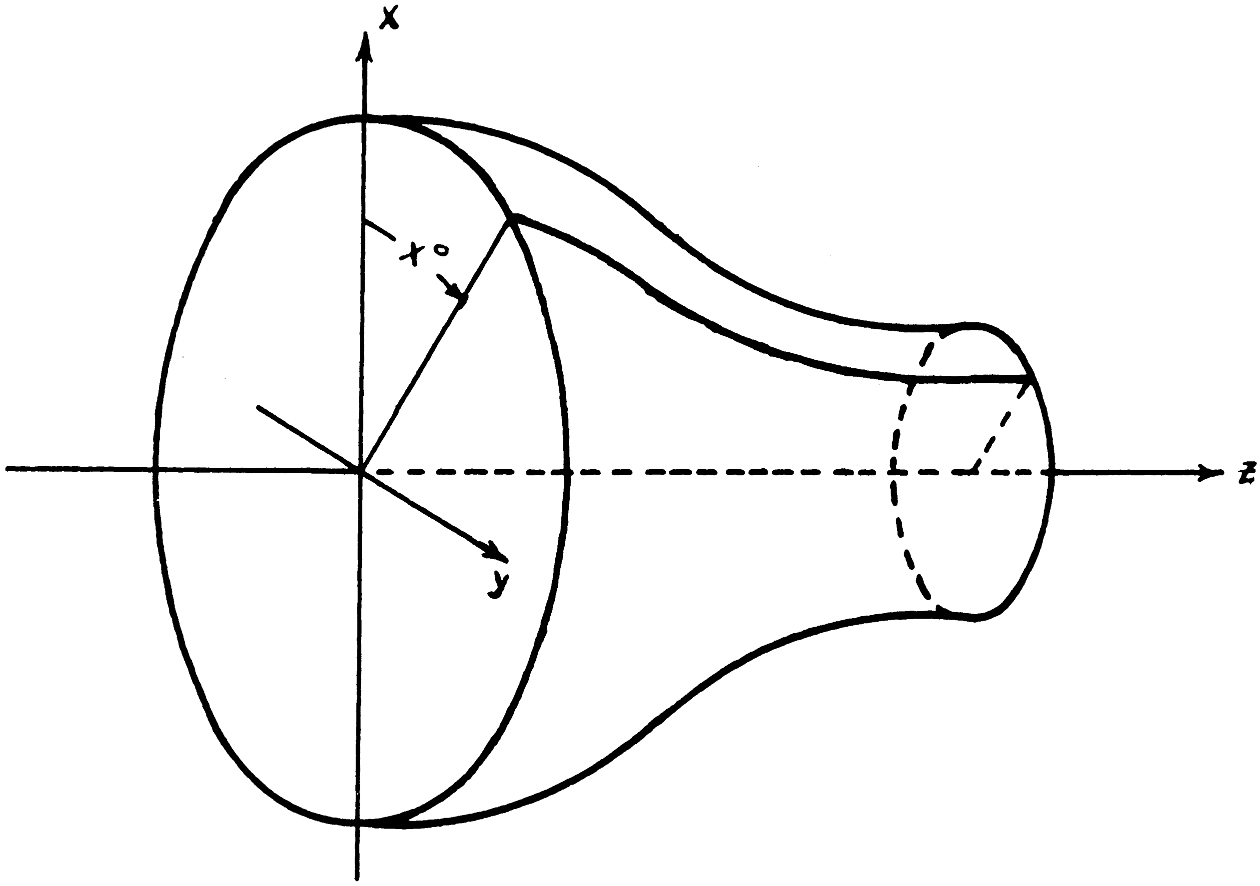

Figure 1. A Shaft

(3.1) $$g_{0\alpha} = g^{0\alpha} = 0.$$

Hence, the expression for arc length (2.2) reduces to

(3.2) $$ds^2 = g_{00} dx^0 dx^0 + g_{\alpha\beta} dx^\alpha dx^\beta.$$

It is immediately apparent that

(3.3) $$\sqrt{g_{00}} = r,$$

where r is the distance from the axis of revolution to a general point in D. Also, from equations (2.3), we have

$$g^{00} = 1/g_{00} = 1/r^2.$$

For the two dimensional space D, we have from (3.2),

(3.4) $$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta ,$$

so that $g_{\alpha\beta}$ is the metric tensor of D. In Section 2 of this chapter, we denoted covariant differentiation with respect to the metric g_{ij} of the three dimensional space V by a latin subscript following a vertical double bar. To denote covariant differentiation with respect to the metric $g_{\alpha\beta}$ of D, we use a Greek subscript following a single vertical bar. Also, partial differentiation will be denoted by a subscript following a comma. For the Christoffel symbols we have

(3.5)
$$[\alpha\beta, \gamma] = \frac{1}{2}(g_{\beta\gamma,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}),$$
$$[\alpha\beta, 0] = [\alpha 0, \beta] = [00, 0] = 0,$$
$$-[00, \alpha) = [0\alpha, 0] = \frac{1}{2} g_{00,\alpha} ,$$

$$(3.6)\quad \begin{cases} F^{\gamma}_{\alpha\beta} = g^{\delta\gamma}\,[\alpha\beta,\delta], \\ F^{o}_{\alpha\beta} = F^{\alpha}_{o\beta} = F^{o}_{oo} = 0, \\ F^{\alpha}_{oo} = -\frac{1}{2}\, g^{\alpha\beta}\, g_{oo,\beta}\,, \quad F^{o}_{o\alpha} = \frac{1}{2}(\ln g_{oo})_{,\alpha}. \end{cases}$$

We make the usual assumption, based on symmetry, that each particle is displaced along the parametric line of x^{o}. Hence, it should follow that $u^{\alpha} = 0$. To verify this, let $\lambda^{i}_{(j)}$ denote the unit vector tangent to the parametric line of x^{j}. Then u^{i} is normal to $\lambda^{i}_{(1)}$ and $\lambda^{i}_{(2)}$ so that $u^{i}\lambda_{(\alpha)i} = 0$, $(\alpha = 1, 2)$. Now it can be readily deduced that

$$(3.7)\qquad \lambda^{i}_{(o)} = (1/r,\, 0,\, 0), \quad \lambda^{i}_{(1)} = (0,\, 1/\sqrt{g_{11}}\,,\, 0), \quad \lambda^{i}_{(2)} = (0, 0, 1/\sqrt{g_{22}}),$$

so the above condition yields

$$u^{1}\, g_{11} + u^{2}\, g_{22} = 0,$$

$$u^{1}\, g_{12} + u^{2}\, g_{22} = 0.$$

Since the determinant $|g_{\alpha\beta}|$ of this system vanishes only at those points in D which are singular points of the coordinate system, we have the desired result

$$(3.8)\qquad u^{\alpha} = 0.$$

Furthermore, if u is the magnitude of displacement, we have

$$(3.9)\qquad u_{o} = ru = r^{2}\, u^{o},$$

and, from axial symmetry, we have also

$$(3.10)\qquad u^{o} = u^{o}(x^{1},\, x^{2}).$$

Now in general

$$u^{i}\|_{j} = u^{i}{}_{,j} + F^{i}_{jk}\, u^{k}.$$

Thus, in view of equations (3.1), (3.8) and (3.10), we have

$$(3.11)\quad \begin{cases} u^{o}\|_{o} = 0 = u^{\alpha}\|_{\beta} = u^{\alpha}|_{\beta}, \\ u^{\alpha}\|_{o} = -\frac{1}{2} g^{\alpha\beta} g_{oo,\beta} u^{o}, \\ u^{o}\|_{\alpha} = u^{o}{}_{,\alpha} + \frac{1}{2} g^{oo} g_{oo,\alpha} u^{o} . \end{cases}$$

A substitution from these equations into equation (2.10) yields

$$(3.12)\qquad \Delta = u^{i}\|_{i} = 0.$$

Furthermore,

$$u^{i}\|_{jk} = (u^{i}\|_{j})_{,k} + F^{i}_{mk} (u^{m}\|_{j}) - F^{m}_{jk} (u^{i}\|_{m}),$$

so that, in the present case, we have

$$(3.13)\quad \begin{cases} u^{o}\|_{oo} = r g^{\alpha\beta} r_{,\alpha} u^{o}{}_{,\beta} , \\ u^{o}\|_{\alpha\beta} = u^{o}|_{\alpha\beta} + (\frac{r_{,\alpha} u^{o}}{r})_{,\beta} + \frac{r_{,\alpha}}{r} u^{o}\|_{\beta} . \end{cases}$$

We now turn to the generalized Navier equations (2.12), which, because of (3.8), (3.10) and (3.12), reduce to the single equation

$$(3.14)\qquad 0 = \nabla^{2}_{3} u_{o} = g^{ik} u_{o}\|_{jk} .$$

Now $u_{o}\|_{jk} = g_{oo} u^{o}\|_{jk}$ so that this basic partial differential equation is equivalent to

$$0 = g^{jk} u^{o}\|_{jk} = g^{\alpha\beta} u^{o}\|_{\alpha\beta} + g^{oo} u^{o}\|_{oo} .$$

When we substitute from equations (3.13) into this last expression and simplify, we obtain

$$(3.15)\qquad g^{\alpha\beta} u^{o}|_{\alpha\beta} + g^{\alpha\beta} \underset{1}{r}_{\alpha\beta} + \frac{3}{r} g^{\alpha\beta} r_{,\alpha} u^{o}{}_{,\beta} = 0.$$

Now $g^{\alpha\beta} r_{|\alpha\beta} = \nabla^2 r$, where ∇^2 is the Laplacian operator for the coordinates x^α in D. If we choose for x^α the rectangular cartesian coordinates (z, r), we find that $\nabla^2 r = 0$. Now $\nabla^2 r$ is an invariant with respect to transformation of the coordinates x^α and so $\nabla^2 r = 0$ for all coordinates. Thus, (3.15) reduces to

$$g^{\alpha\beta}(u^o{}_{|\alpha\beta} + \frac{3}{r} r_{,\alpha} u^o_{,\beta}) = 0,$$

or

$$\nabla^2 u^o + \frac{3}{r} \nabla r \cdot \nabla u^o = 0. \tag{3.16}$$

The stress-displacement equations (2.11) relate the stress components T^{ij} to the displacement u^o. Since Δ is zero, these equations simplify to

$$\frac{1}{\mu} T^{ij} = g^{jk} u^i{}_{\|k} + g^{ik} u^j{}_{\|s} .$$

Because of equations (3.11), we now have

$$(3.17) \quad \begin{cases} T_{o\alpha} = \mu r^2 u^o{}_{,\alpha} , \\ T_{ij} = 0 \quad \text{otherwise.} \end{cases}$$

The boundary conditions on u^o in the meridian section D are obtained from the condition that the surface of the shaft is free of load. We let C denote a boundary curve of the meridian section of the shaft. We let n^j denote the outward unit normal to the shaft at a point P on C. Then the stress vector T^i at P corresponding to n^j vanishes and, since $n^o = 0$, equation (2.6) then yields

$$T_{ij} n^j = 0.$$

A substitution here from equations (3.17) gives the boundary conditions in the form

$$(3.18) \qquad n^{\alpha} u^{o}_{,\alpha} = 0.$$

A displacement u^{o} satisfying the generalized Neumann problem, which is defined by the partial differential equation (3.16) and the boundary conditions (3.18), completely characterizes the problem of torsion of a shaft by assigned twisting couples.

We remark that the function u^{o} is called the twist function. No doubt this stems from the physical interpretation of u^{o}, which was first recognized by Willers [6] in terms of rectangular cartesian coordinates in the meridian section. To consider the physical meaning of u^{o}, as before we let u denote the magnitude of the displacement; then u is also a physical component of the displacement vector u^{o} in the direction of the parametric line of x^{o} at a general particle P in the body, as shown in Figure 2. From equation (3.9), we have $u^{o} = u/r$. Now in the linear theory of elasticity, the displacements considered must be small in comparison with the dimensions of the body so that u^{o} is the angle of rotation or twist angle of the point P about the center of the cross section containing P.

4. Fundamental Equations in the Case of Torsion; Dirichlet Form. An obvious disadvantage of the above formulation of the problem in terms of the displacement u^{o} lies with the Neumann type boundary conditions (3.18). A more desirable form in this respect is as a generalized Dirichlet problem involving a new dependent variable $F(x^{1}, x^{2})$ prescribed on the boundary C of the meridian section. When the coordinates are rectangular cartesian, it is known that F can be found so that it is constant on C. We shall now extend this to the case of general coordinates x^{α}.

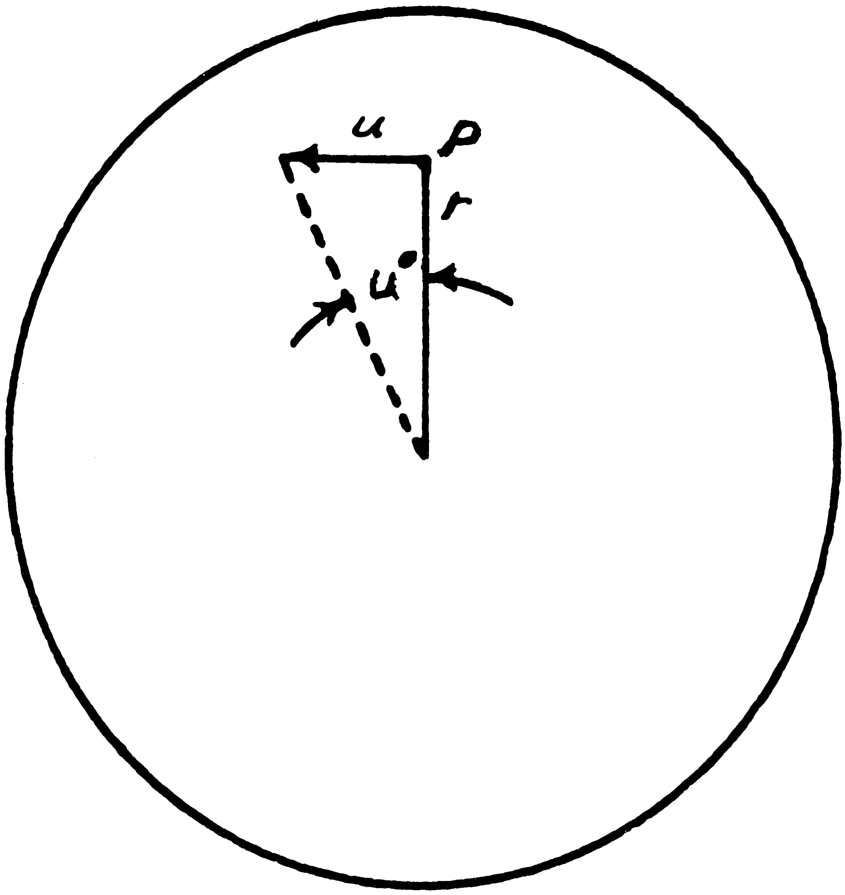

Figure 2. Twist Angle u°

The basic generalized Neumann problem of equations (3.16) and (3.18) has been expressed in terms of quanities and operations pertaining only to the two-dimensional meridian section D. Thus, in the remainder of this thesis, it is necessary to work only with the metric $g_{\alpha\beta}$ in D.

Now n^{α} is the outer normal to the boundary curve C of the meridian section D, as shown in Figure 3. Let t^{α} be the unit tangent vector to C oriented so that $t^{\alpha} = \eta^{\alpha\beta} n_{\beta}$, where $\eta_{\alpha\beta}$ is the absolute tensorial permutation symbol, that is,

$$\eta_{\alpha\beta} = g^{-1/2} \epsilon^{\alpha\beta} ,$$

$$g = \det(g_{\alpha\beta}),$$

$$\epsilon^{12} = 1, \quad \epsilon^{21} = -1, \quad \epsilon^{11} = \epsilon^{22} = 0.$$

It then follows that $n_{\alpha} = \eta_{\beta\alpha} t^{\beta}$, so the boundary condition (3.18) becomes

$$(4.1) \qquad \eta^{\alpha\beta} u^{o}{}_{,\alpha} g_{\beta\gamma} t^{\gamma} = 0.$$

Now let s be the arc length of C, the direction of s increasing being such that $t^{\gamma} = \frac{dx^{\gamma}}{ds}$, so the above boundary condition becomes

$$(4.2) \qquad \eta^{\alpha\beta} u^{o}{}_{,\alpha} g_{\gamma\beta} \frac{dx^{\gamma}}{ds} = 0.$$

This condition reduces to the condition $F(x^1, x^2)$ = constant on C if there exists functions $F(x^1, x^2)$ and $G(x^1, x^2)$ such that

$$(4.3) \qquad \eta^{\alpha\beta} u^{o}{}_{,\alpha} g_{\beta\gamma} = G F_{,\gamma} .$$

We now eliminate F from the two equations (4.3) to obtain information about G.

The function $u^{o}(x^1, x^2)$ is invariant under coordinate transformations on x^{α} in the meridian section D. Hence, we may assume that F and G

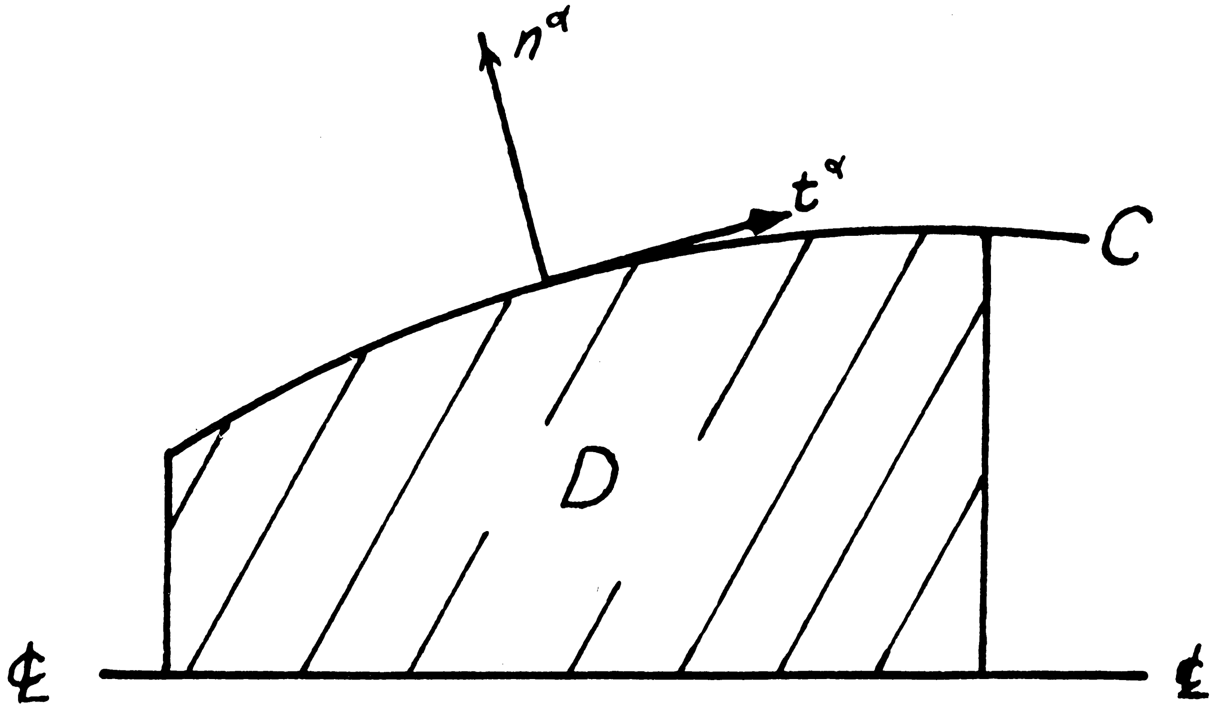

Figure 3. A Meridian Curve

are invariants. Thus, $F_{,\gamma}$ is a covariant vector, and its covariant derivative is $(F_{,\gamma})_{|\delta} = F_{|\gamma\delta}$. Since we are dealing with a flat space, we have $F_{|\gamma\delta} = F_{|\delta\gamma}$. We substitute into this equation for $F_{|\delta\gamma}$ and $F_{|\gamma\delta}$ as determined from (4.3) and obtain the equation $G\nabla^2 u^o - \nabla u^o \cdot \nabla G = 0$. But u^o must satisfy the fundamental equation (3.16), so that $\frac{1}{G}\nabla G = -\frac{3}{r}\nabla r$, or $\ln G = \ln r^{-3}$. Thus, we may choose $G = r^{-3}$, and equation (4.3) becomes $r^3 \eta^{\alpha\beta} g_{\beta\gamma} u^o{}_{,\alpha} = F_{,\gamma}$; this may be solved for $u^o{}_{,\alpha}$ to yield

$$(4.4) \qquad u^o{}_{,\alpha} = r^{-3} \eta_{\alpha\beta} g^{\beta\gamma} F_{,\gamma} \ .$$

A substitution of $u^o{}_{,\alpha}$ from equation (4.4) into the differential equation (3.16) for u^o, followed by some direct calculation, gives the fundamental differential equation for $F(x^1, x^2)$ in the form

$$(4.5) \qquad \nabla^2 F - \frac{3}{r}\nabla r \cdot \nabla F = 0.$$

The boundary condition on F is now

$$(4.6) \qquad F(x^1, x^2) = \text{constant on } C.$$

We remark that a meridian curve, on which this boundary condition is satisfied, is sometimes called a stress line since it generates a stress free boundary of a shaft.

The generalized Dirichlet problem of (4.5) and (4.6) also completely characterizes the torsion problem under consideration. Once F has been determined, the displacement u^o can readily be determined from (4.4). It is desirable to express the stress components directly in terms of F so that they too can be readily determined. These relations, which are obtained by substitution for $u^o{}_{,\alpha}$ from equations (4.4) into (3.17), are

$$(4.7) \quad \begin{cases} T_{o\alpha} = \frac{\mu}{r} \eta_{\alpha\beta}\, g^{\beta\gamma}\; F_{,\gamma}\, , \\ T_{ij} = 0 \quad \text{otherwise.} \end{cases}$$

It is worth noting that the curves $u^o(x^1, x^2) = \text{constant}$ and $F(x^1, x^2) = \text{constant}$ are orthogonal, since $g^{\alpha\beta}\, u^o{}_{,\alpha}\, F_{,\beta} = 0$.

5. <u>The Twisting Couple</u>. Let us consider a hollow shaft with a meridian section, as shown in Figure 4, C_1 denoting a curve on the inner surface of the shaft and C_2 a curve on the outer surface of the shaft. In view of the boundary condition (4.6), F must be constant on both C_1 and C_2. We write

$$(5.1) \qquad [F]_{C_1} = \text{constant} = k_1, \quad [F]_{C_2} = \text{constant} = k_2.$$

To obtain an expression for the twisting couple M, let the curvilinear coordinates x^α be the two cylindrical coordinates (z, r) which are rectangular cartesian coordinates in the meridian section, as shown in Figure 4. Since load is applied only on the ends of the shaft, the couple acting across a general right section $z = z_o$ is M, and we have

$$(5.2) \qquad M = \int_a^b \int_o^{2\pi} r^2\, \tau_{o1}\, (z_o, r)\, dr\, dx^o,$$

where τ_{o1} is a physical component of stress for the cylindrical co-ordinates (x^o, z, r). Now $\tau_{o1} = \frac{1}{r} T_{o1}$, where T_{o1} is a covariant component of stress for the above coordinate system. From (4.7) we have

$$T_{o1} = \frac{\mu}{r} F_{,r}$$

so $\tau_{o1} = \frac{\mu}{r^2} F_{,r}$. and hence substitution into (5.2) followed by an elementary integration gives

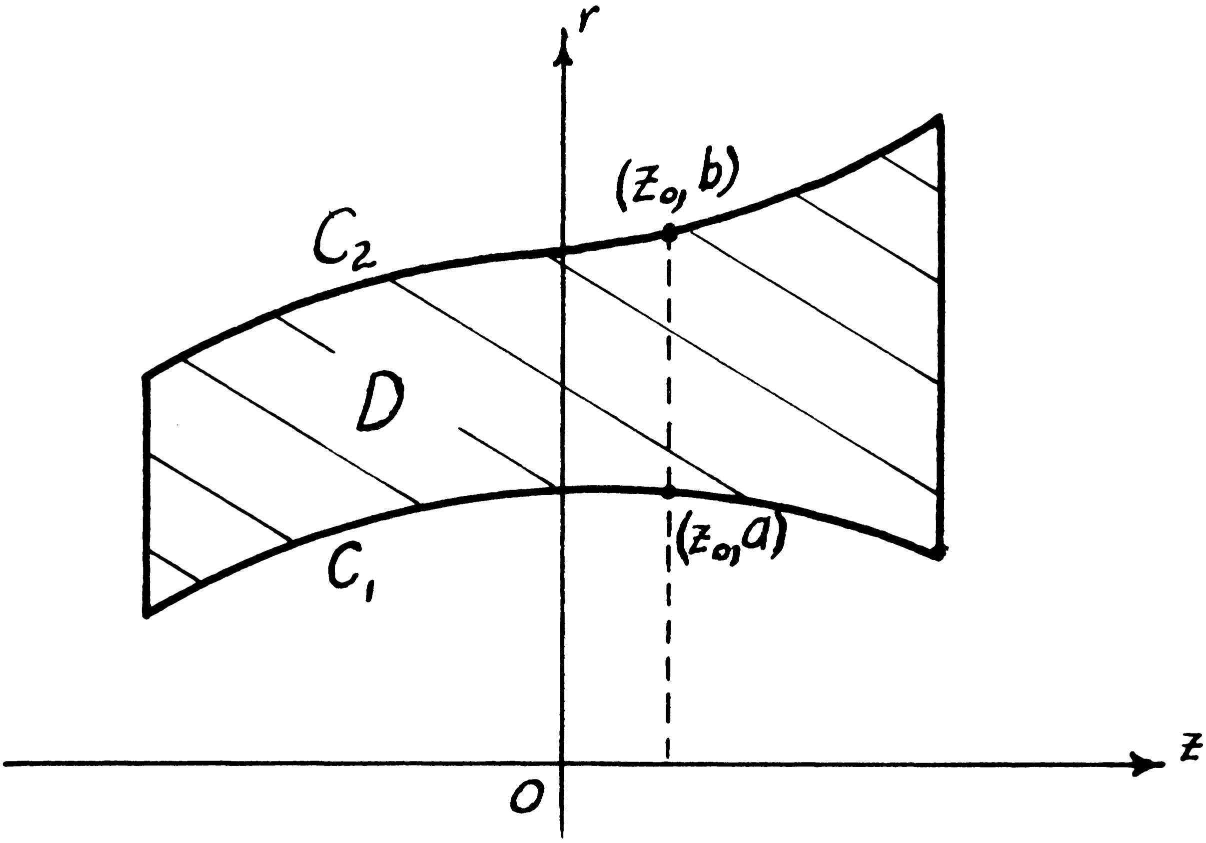

Figure 4. A General Meridian Section

$$(5.3) \qquad M = 2\pi\mu\left[[F]_{C_2} - [F]_{C_1}\right] ,$$

or

$$(5.4) \qquad M = 2\pi\mu(k_2 - k_1) .$$

Now F is an invariant, and hence (5.3) is an invariant relation. Therefore, (5.3) holds for all coordinate systems.

If the shaft is not hollow, then C_1 is the axis of revolution of the shaft and equation (5.3) still holds. But M must have the same constant value for all cross sections, and $[F]_{C_2}$ = constant. Thus, euqation (5.3) requires that $[F]_{C_1}$ = constant so we conclude that: _if the shaft is not hollow,_ F = constant = k_1 _on the axis of revolution and (5.4) still holds._

When the shaft has a cavity which is symmetrical about the axis of the shaft, some cross sections of the shaft are simply connected while others are not. In this case, C_1 consists of the boundary of the cavity in a meridian section, plus that part of the axis of the shaft which is outside of the cavity. Thus, F = constant = k_1 on the surface of the cavity and on the axis of the shaft outside of the cavity; also (5.4) still holds.

CHAPTER II

THE SHAFT WITH A PROLATE ELLIPSOIDAL CAVITY

6. The Fundamental Equations in Non-orthogonal Coordinaties. We shall use the general theory of Sections 4 and 5 to discuss the torsion of two shafts which have symmetrically located prolate ellipsoidal cavities. The first shaft is bounded by confocal ellipsoids and the second has nearly uniform cross sections. The first analysis for both discussions stems from the same set of fundamental equations which follow.

Let us consider rectangular cartesian coordinates (z, r) in a meridian section, where as usual r is the distance from the axis of revolution and z is the distance from some right cross section, as shown in Figure 5. Let us define a variable ξ by the relation

$$(6.1) \qquad z = \epsilon\sqrt{d^2 \cosh^2\xi - r^2 \coth^2\xi}$$

where d is a constant and ϵ is an indicator which is defined by the relations

$$(6.2) \qquad \begin{cases} \epsilon = +1 & \text{in quadrant I of the } (z, r) \text{ system,} \\ \epsilon = -1 & \text{in quadrant II of the } (z, r) \text{ system.} \end{cases}$$

Let (ξ, r) be the coordinates (x^1, x^2) in the meridian section. The parametric lines of ξ are straight lines parallel to the z-axis. The parametric lines of r are confocal semi-ellipses with centers at the origin O and foci at the points A and B, as shown in Figure 5.

It should be noted that the r-axis in Figure 5 is a singular line of the coordinates (ξ, r) for the parametric lines of these coordinates

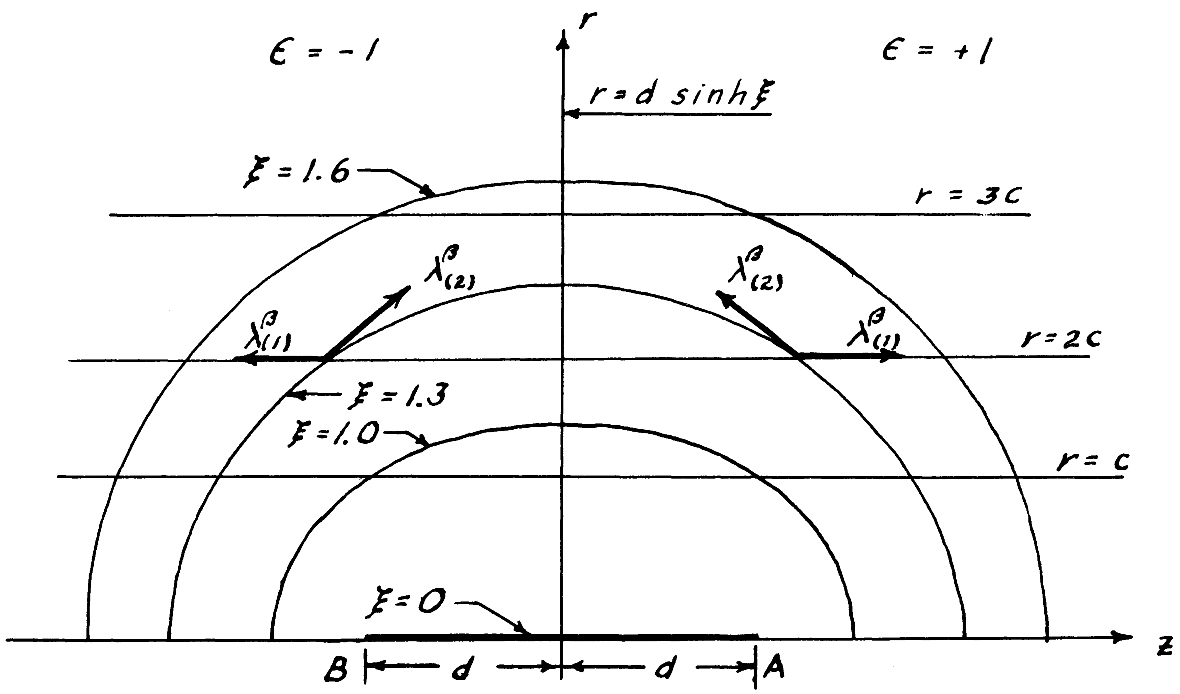

Figure 5. The ξ, r Coordinates

have common tangents on this line. In fact, if $\lambda^{\beta}_{(\alpha)}$ denotes the unit vector tangent to a parametric line of x^{α} in the direction of the parametric line as shown in Figure 5, then at points on the singular line the angle between $\lambda^{\beta}_{(1)}$ and $\lambda^{\beta}_{(2)}$ is π radians. Furthermore, the coordinates (x^{o}, ξ, r) form a right handed system when z is positive and a left handed system when z is negative. In order to account analytically for this change in orientation of the coordinates (ξ, r), we make extensive use of the indicator ϵ defined by equations (6.2).

The differential element of arc length in the meridian section is given by

$$ds^2 = dz^2 + dr^2. \tag{6.3}$$

From equations (6.1) we see that

$$dz = \epsilon\, D^{-1/2}(E d\xi - r \coth\xi\, dr),$$

where

$$\begin{aligned} D = D(\xi, r) &\equiv d^2 \sinh^2\xi - r^2, \\ E = E(\xi, r) &\equiv d^2 \sinh^2\xi + r^2 \operatorname{csch}^2\xi. \end{aligned} \tag{6.4}$$

We substitute for dz in equation (6.3) and pick up the tensor notation to get

$$ds = g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}$$

where

$$g_{11} = E^2/D, \quad g_{12} = g_{21} = -\frac{rE}{D} \coth\xi, \quad g_{22} = E/D. \tag{6.5}$$

Furthermore,

$$\det g_{\alpha\beta} \equiv g = E^2/D,$$

$$g^{11} = 1/E, \quad g^{12} = g^{21} = \frac{r}{E} \coth\xi, \quad g^{22} = 1.$$

The Jacobian of the transformation from rectangular cartesian coordinates (z, r) to curvilinear coordinates (ξ, r) is $J\left(\frac{z,\ r}{\xi,\ r}\right) = \epsilon\, g^{1/2}$.

The fundamental differential equation is (4.5). To obtain this equation in terms of the (ξ, r) coordinates under consideration, we use the familiar identities:

$$(6.6) \qquad \nabla^2 F = g^{-1/2} (g^{1/2} g^{\alpha\beta} F_{,\alpha})_{,\beta} ,$$

$$\frac{1}{r} \nabla r \cdot \nabla F = g^{\alpha\beta} (\ln r)_{,\alpha} F_{,\beta} .$$

After some direct calculations we find the required differential equation in the form

$$(6.7) \qquad F_{,\xi\xi} - 3 \coth \xi \, F_{,\xi} + 2r \coth \xi \, F_{,\xi r} + E(F_{,rr} - \frac{3}{r} F_{,r}) = 0.$$

The boundary conditions are, from equations (5.1),

$$(6.8) \qquad \begin{cases} F(\xi, r) = \text{constant} = k_1 \quad \text{on} \quad C_1, \\ F(\xi, r) = \text{constant} = k_2 \quad \text{on} \quad C_2. \end{cases}$$

The differential equation (6.7) is of second order and is linear with variable coefficients, but the coefficient $E(\xi, r)$, which is given by equation (6.4) makes it impossible to obtain solutions by separation of variables. However, since F is invariant it is easy to get the solution for a shaft of uniform cross sections. In rectangular cartesian coordinates, the well-known solution in this case is

$$(6.9) \qquad F(z, r) = a_1 r^4 + a_2$$

where a_1 and a_2 are arbitrary constants. To obtain the solution for a uniform circular shaft in terms of the (ξ, r) coordinates, it is only necessary to shift from coordinates (z, r) to coordinates (ξ, r) by substitution for z from equation (6.1). When this transformation is applied to (6.9), we obtain $F(\xi, r) = a_1 r^4 + a_2$ which is precisely the form of equation (6.9).

Let us seek a solution to the fundamental equation (6.7) in the form $F(\xi, r) = (b_1 r^4 + b_2)[a_1 f(\xi) + a_2 g(\xi)]$ where $f(\xi)$ and $g(\xi)$ are functions of ξ alone and the constants a_1, a_2, b_1, and b_2 are arbitrary. We readily obtain the particular integral

$$(6.10) \qquad F(\xi, r) = r^4[a_1 f(\xi) + a_2] + a_3 g(\xi) + a_4$$

where

$$(6.11) \qquad f(\xi) = \int \operatorname{csch}^5 \xi \, d\xi$$

$$= \operatorname{csch}^3 \xi \coth \xi - \frac{3}{2} \operatorname{csch} \xi \coth \xi - \frac{3}{2} \ln(\coth \xi - \operatorname{csch} \xi),$$

$$(6.12) \qquad g(\xi) = \int \sinh^3 \xi \, d\xi = \frac{1}{3} \cosh^3 \xi - \cosh \xi$$

and the constants a_1, a_2, a_3, and a_4 are arbitrary.

7. The Hollow Ellipsoidal Shaft. Let us consider the torsion of a shaft whose sides are confocal ellipsoids of revolution, so the meridian section of the shaft is as shown in Figure 6. Then the curves C_1 and C_2 which bound D in part have the equations

$$(7.1) \qquad \xi = \text{constant} = \xi_1, \qquad \xi = \text{constant} = \xi_2,$$

respectively. We must have $\xi_1 < \xi_2$, with ξ_1 and ξ_2 both non-negative.

We introduce the particular integral (6.10) and the boundary conditions (6.8) in the form

$$(7.2) \qquad F(\xi_1, r) = 0, \quad F(\xi_2, r) = \text{constant} = k_2,$$

since we lose no generality in setting $k_1 = 0$. These boundary conditions require that $a_1 = a_2 = 0$, and that $a_3 g(\xi_1) + a_4 = 0$, $a_3 g(\xi_2) + a_4 = k_2$, so that

$$a_3 = \frac{k_2}{g(\xi_2) - g(\xi_1)}, \quad a_4 = - a_3 g(\xi_1).$$

We then have

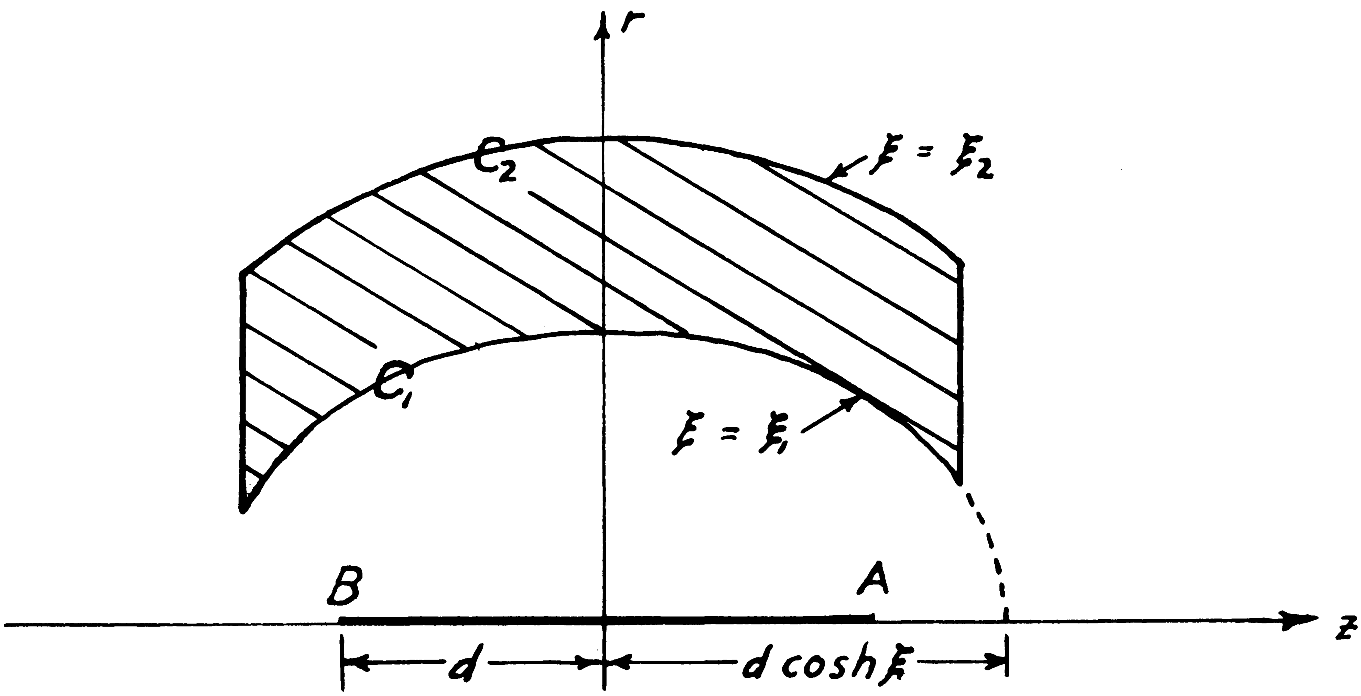

Figure 6. Meridian Section of a Hollow Ellipsoidal Shaft

$$(7.3) \qquad F(\xi, r) = k_2 \frac{g(\xi) - g(\xi_1)}{g(\xi_2) - g(\xi_1)} .$$

From equation (5.4), we recall that $M = 2\pi\mu k_2$, so that the solution to the problem of torsion for the hollow elliptic shaft is

$$(7.4) \qquad F(\xi, r) = \frac{M}{2\pi\mu} \frac{g(\xi) - g(\xi_1)}{g(\xi_2) - g(\xi_1)}$$

where M is the twisting couple, and $g(\xi)$ is defined in (6.12).

From equations (4.4), we obtain for the twist function u^o, the expression

$$(7.5) \qquad u^o = - \frac{\epsilon M}{4 d^3 \pi\mu [g(\xi_2) - g(\xi_1)]} \left[\ln(d \sinh\xi - D^{1/2}) - \ln r - \frac{D^{1/2} d \sinh\xi}{r^2} \right] + a_5,$$

where a_5 is an arbitrary constant. From Section 3, we recall that the physical component of displacement in the direction of the parametric line of x^o is $u = r u^o$.

By the use of equations (4.7), we obtain the stress components in the form

$$T_{o1} = \frac{\epsilon M \sinh^2\xi \cosh\xi}{2\pi D^{1/2} [g(\xi_2) - g(\xi_1)]} ,$$

$$T_{o2} = \frac{-\epsilon M \sinh^3\xi}{2\pi r D^{1/2} [g(\xi_2) - g(\xi_1)]} ,$$

$$T_{ij} = 0 \text{ otherwise.}$$

The normal stress N and the shearing stress S, corresponding to a direction specified by the unit vector ζ^α, are, by (2.14) and (2.15) respectively,

$$(7.6) \qquad N = 0, \qquad S(\xi, r) = \frac{M \sinh^2\xi \, [\zeta^1 \cosh\xi - r^{-1}\zeta^2 \sinh\xi]}{2\pi r D^{1/2} \, [g(\xi_1) - g(\xi_2)]} .$$

The expression for $F(\xi, r)$ deduced above is analytic in the meridian section of the shaft. However, the function u^o, T^{ij} and S which are obtained from $F(\xi, r)$, have discontinuities which will be considered in the next section.

8. <u>Discontinuities of the Solution for the Torsion of the Hollow Ellipsoidal Shaft</u>. The functions u^o, T_{ij} and S deduced in the previous section have discontinuities which occur, in whole or in part, when $r = 0$ and $d = 0$.

The second discontinuity occurs when $D = 0$. From equation (6.4) we see that this implies $r = d \sinh \xi$, which is equivalent to $z = 0$. Now the coordinate system (ξ, r) is singular when $z = 0$, and it is thus suspected that the singularities at $z = 0$ are removable, which is indeed the case. To see this, we note that $F(\xi, r)$, as given by (7.4) with (6.12), is a function of $\cosh \xi$ only. Now from (6.1) we have

$$(8.1) \qquad \sinh \xi = \frac{\sqrt{2}}{2d} \left[d^2 - z^2 - r^2 + \sqrt{(d^2 - z^2 - r^2)^2 + 4r^2 d^2} \right]^{1/2},$$

$$(8.2) \qquad \cosh \xi = \frac{\sqrt{2}}{2d} \left[d^2 + z^2 + r^2 + \sqrt{(d^2 + z^2 + r^2)^2 - 4z^2 d^2} \right]^{1/2}.$$

Since $\cosh \xi$ is an analytic function of z, r on $z = 0$, then F is an analytic function of z, r on $z = 0$, and so the singularity which appeared earlier for $z = 0$ is removable.

The twist function u^o at the singular points on $z = 0$ can be obtained readily from (7.5) above by a limit operation. However, the determination of a quantity involving stress components corresponding to a certain direction at a point $z = 0$ is more complex. This is due to the fact that the representation of a direction by the tensorial components of a unit vector is not possible at a point on $z = 0$. Again we use a limit operation.

Let us suppose we wish the shearing stress at the point P on the singular line $z = 0$, corresponding to the direction of a unit vector $\vec{v}$ which makes an angle θ with the line r = constant through P, as shown in Figure 7. We introduce a field of unit vectors ζ^{α} on this line, all with the same direction as $\vec{v}$. Figure 7 shows the vector ζ^{α} at a general point Q. We proceed to find ζ^{α}.

Now $\lambda^{\alpha}_{(1)}$ is the unit vector at Q in the direction of the parametric line of ξ. We have accordingly

$$(8.3) \qquad \cos\theta = g_{\alpha\beta}\lambda^{\alpha}_{(1)}\zeta^{\beta}, \quad 1 = g_{\alpha\beta}\zeta^{\alpha}\zeta^{\beta}.$$

Now from equations (3.7), $\lambda^{\alpha}_{(1)} = (1/\sqrt{g_{11}}, 0)$ so equations (8.3) become

$$g_{11}\zeta^1 + g_{12}\zeta^2 = \sqrt{g_{11}} \cos\theta,$$
$$g_{11}(\zeta^1)^2 + 2g_{12}\zeta^1\zeta^2 + g_{22}(\zeta^2)^2 = 1.$$

From these two equations in ζ^1, ζ^2, the solution is found to be

$$\zeta^1 = \frac{\cos\theta}{\sqrt{g_{11}}} - \frac{g_{12}}{\sqrt{g g_{11}}}\sin\theta,$$
$$\zeta^2 = \sqrt{g_{11}/g}\,\sin\theta.$$

From (6.5) and following equations, we have then

$$(8.4) \qquad \begin{aligned} \zeta^1 &= \frac{\sqrt{D}}{E}\cos\theta + \frac{r}{E}\coth\xi\,\sin\theta, \\ \zeta^2 &= \sin\theta. \end{aligned}$$

We use the relations (8.4) in (7.6) to compute the shearing stress S corresponding to the direction of the unit vector ζ^{α}. This yields

$$S(\xi, r) = \frac{M \sinh^2 \xi}{2\pi r\, E[g(\xi_1)-g(\xi_2)]} [\cosh \xi \cos \theta - \frac{\sqrt{D}}{r} \sinh \xi \sin \theta].$$

We now pass to the limit as Q approaches P. This requires that $r \to d \sinh \xi$, $D \to 0$, $E \to d^2 \cosh^2 \xi$. Hence, we get for the shearing stress at P corresponding to the direction of the vector $\vec{v}$

$$[S]_Q = \frac{M \sinh \xi \cos \theta}{2\pi d^3 \cosh \xi\, [g(\xi_1)-g(\xi_2)]};$$

here, ξ for the point P is given in terms of the distance r from P to the axis of revolution by the relation $r = d \sinh \xi$.

9. <u>The Shaft of Almost Uniform Cross Sections</u>. Let us consider a long shaft with a symmetrically located prolate ellipsoidal cavity, as indicated in Figure 8, the exact form of the outer surface being unspecified for the moment. We use the coordinate system (ξ, r) of Section 6, so that the boundary surface of the cavity is generated by a parametric line of r in the meridian section. The inner boundary C_1 of the meridian section of the shaft is then specified by the relations

$$(9.1) \qquad \begin{cases} \xi = \xi_1 & |z| \leqq d \cosh \xi_1, \\ r = 0 & |z| \leqq d \cosh \xi_1. \end{cases}$$

The outer boundary C_2 remains unspecified for the moment.

We introduce the particular integral (6.10), and the boundary conditions (6.8) in the form

$$(9.2) \qquad F(\xi, r) = 0 \text{ on } C_1, \qquad F(\xi, r) = k_2 \text{ on } C_2$$

where the constant k_2 is specified by equation (5.4) with $k_1 = 0$. Now $F(\xi, r)$ must be bounded throughout the meridian section of the shaft, so

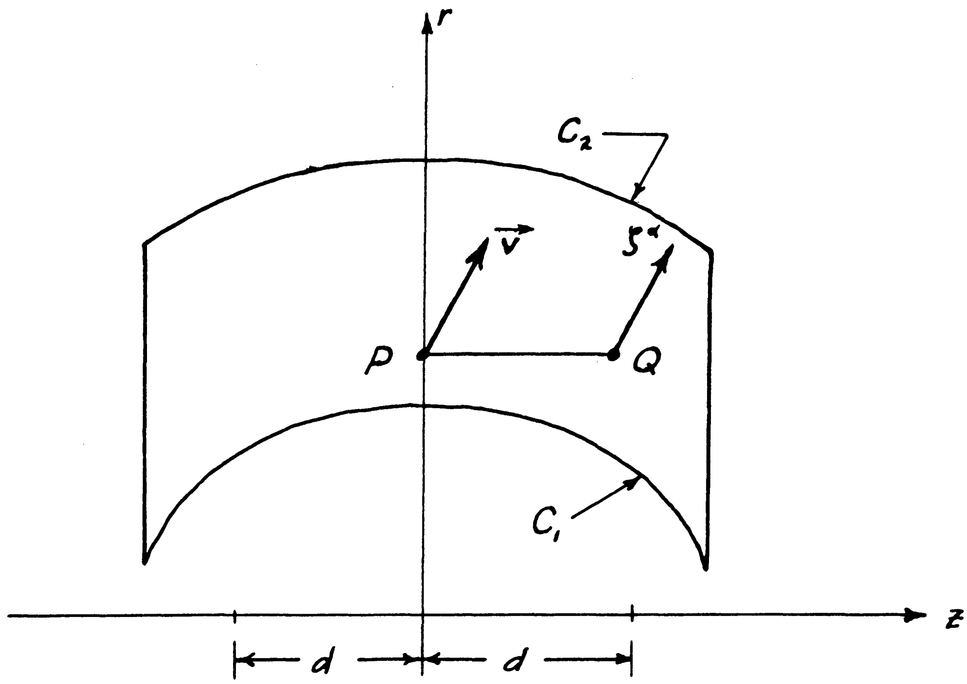

Figure 7. The Unit Vector Field ζ^{α}

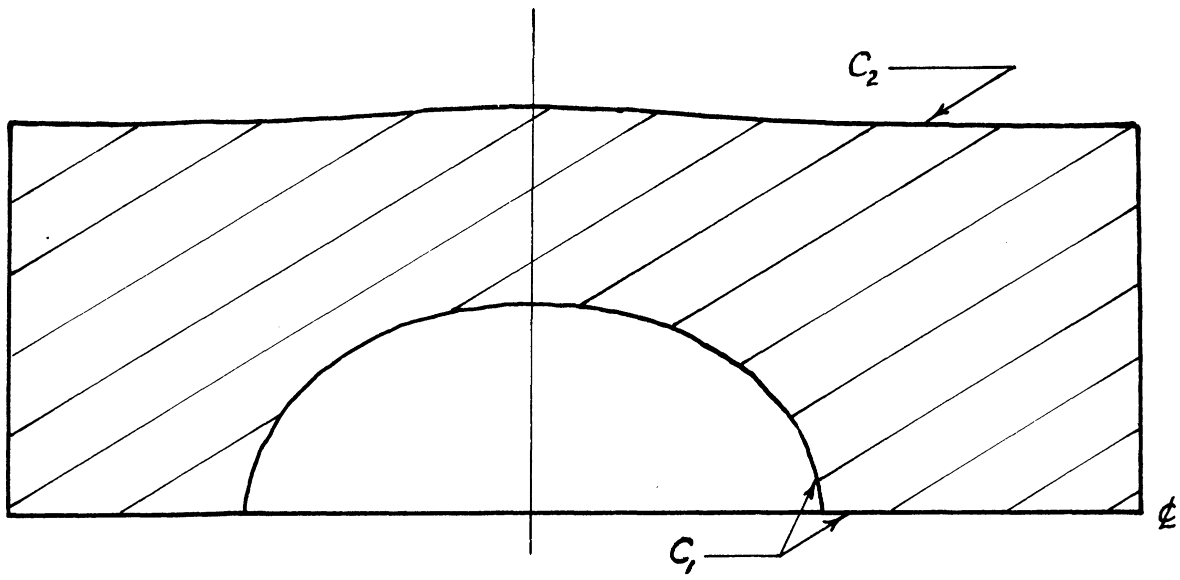

Figure 8. Meridian Section of an Almost Uniform Shaft with a Prolate Ellipsoidal Cavity

that in the particular integral (6.10) we have $a_3 = 0$. The first condition of equation (9.2) requires that $a_4 = 0$, and $a_2 = -a_1 f(\xi_1)$. Hence the particular integral (6.10) reduces to

$$(9.3) \qquad F(\xi, r) = a_1 r^4 [f(\xi) - f(\xi_1)],$$

where $f(\xi)$ is a known function defined by equations (6.11). We note that as ξ approaches infinity, $F(\xi, r)$ asymptotically approaches $r^4 \times$ constant, which is the well known solution for the uniform circular shaft.

The remaining boundary condition of equation (9.2) is satisfied on a curve C_2 with the equation

$$(9.4) \qquad r^4 [1 - \frac{f(\xi)}{f(\xi_1)}] = - \frac{k_2}{a_1 f(\xi_1)} = \text{constant} = K,$$

for all non-negative K. Any value of K determines an outer boundary C_2 for the shaft in the meridian section. Representatives of the family are illustrated in Figure 9, for $d = 1$, $\xi_1 = \text{arcsinh } 1$ and selected values of K. It should be noted that the radius of the outer surface of the shaft is almost uniform when the size of the cavity is not too large in comparison with the radius of the outer surface.

Except for the unspecified constant a_1, the function $F(\xi, r)$ of (9.3) is then the solution of the torsion problem for a long shaft with a symmetrically located prolate ellipsoidal cavity. The arbitrary constant a_1 is determined by the twisitng couple M from equation (5.4). In the present case we have

$$M = 2\pi\mu k_2 = -2\pi\mu a_1 K f(\xi_1),$$

from which it follows that the desired solution is

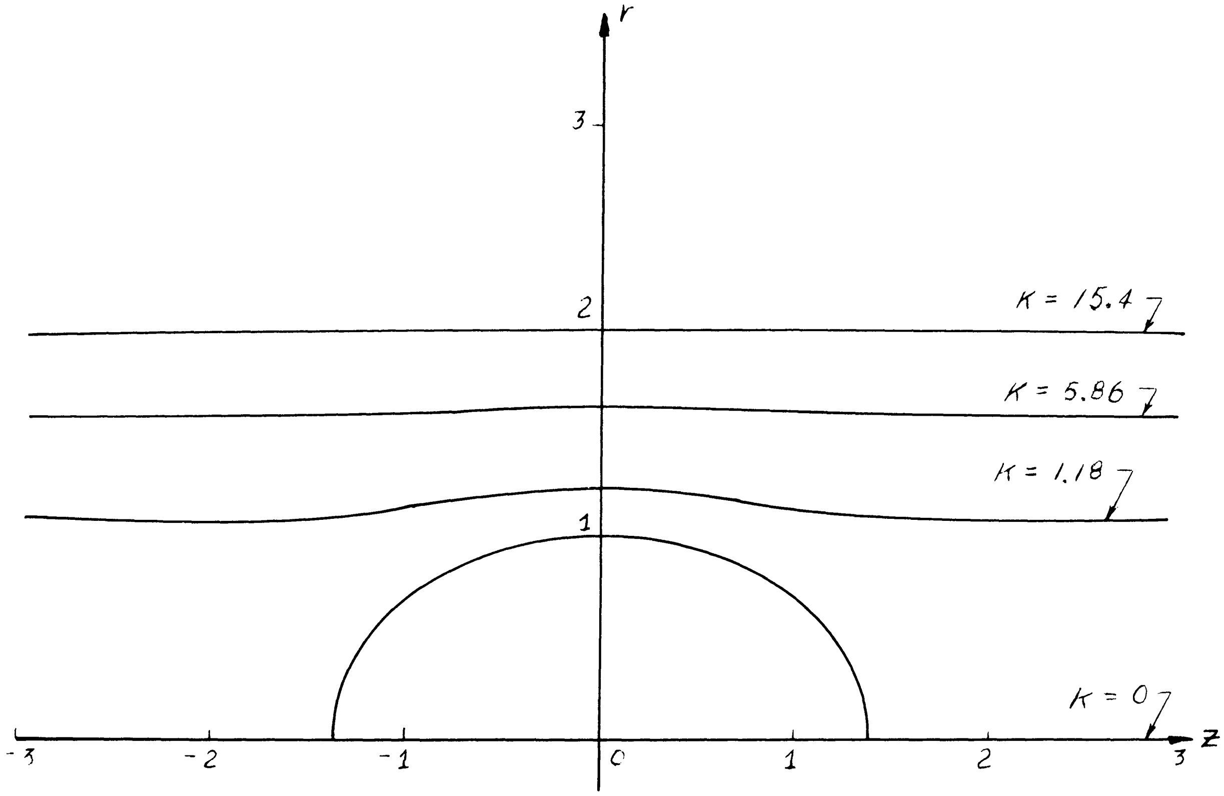

Figure 9. Graph of $r^4(1 - \frac{f(\xi)}{f(\xi_1)}) = K$, for $d = 1$ and ξ_o = arc sinh 1

$$(9.5) \qquad F(\xi, r) = -\frac{M r^4 [f(\xi) - f(\xi_1)]}{2 \varkappa \mu K f(\xi_1)} ,$$

where $f(\xi)$ is defined in equation (6.11).

The single non-vanishing component of displacement may be determined by the use of equation (4.4). A straight forward calculation shows that

$$u = -\epsilon \frac{M r D^{1/2}}{2 \varkappa \mu K f(\xi_1)} \left[\operatorname{csch}^5 \xi + 4 \coth \xi \, [f(\xi) - f(\xi_1)]\right] + a_5 r$$

where $u = r u^0$ is the physical component of displacement in the direction of the parametric line of x^0. The presence of the term $a_5 r$ is due to the fact that u is unspecified to within a rigid body rotation about the z-axis. It may be evaluated by assigning the displacement u at any one point in the shaft.

The only non-vanishing components of stress are obtained directly from equations (4.7). In this case we have

$$(9.6) \qquad \begin{aligned} T_{01} &= -\frac{\epsilon M r^2}{2 \varkappa K f(\xi_1) D^{1/2}} \left[r^2 \operatorname{csch}^5 \xi \coth \xi + E[f(\xi) - f(\xi_1)]\right] , \\ T_{02} &= +\frac{\epsilon M r^3}{2 \varkappa K f(\xi_1) D^{1/2}} \left[\operatorname{csch}^5 \xi + 4 \coth \xi \, [f(\xi) - f(\xi_1)]\right] . \end{aligned}$$

When these equations are used in equations (2.14) to compute the normal stress associated with an arbitrary direction ζ^a, we find that $N = 0$. Similarly, for the shearing stress S from equation (2.5), we have

$$S = \pm \frac{M r}{2 \varkappa K f(\xi_1) D^{1/2}} \Big[\left[r^2 \operatorname{csch}^5 \xi \coth \xi + 4E[f(\xi) - f(\xi_1)] \right] \zeta^1 - r^2 \left[\operatorname{csch}^5 \xi + 4 \coth \xi \, [f(\xi) - f(\xi_1)]\right] \zeta^2 \Big] .$$

Here, as in the case of the hollow ellipsoidal shaft, singularities occur due to the singular line in the coordinate system. These may be removed, and the various pertinent quanities may be computed for points on the singular line as was done in the previous section.

It is of interest to see that other known solutions are obtained when certain limit operations are applied to the solution (9.3). Let us take this equation in the form

$$(9.7) \qquad F(\xi, r) = r^4[a\, f(\xi) - b],$$

where the constants a and b are chosen so that $F(\xi, r) = 0$ on C_1.

If the ellipsoidal cavity collapses, we should have the case of a solid circular shaft. This implies that $\xi_1 = 0$. From the definition of $f(\xi)$ given by equation (6.11), it is not difficult to show that $\lim\limits_{\xi \to 0} f(\xi) = \infty$. Hence, we must take $a = 0$ which leaves $F = r^4 \times$ constant as the desired solution.

The solution of a hollow shaft of uniform circular cross sections is also readily obtained. In this case, we keep the semi-minor axis of the ellipsoidal cavity fixed and let the distance between the foci become infinite. This means that

$$\lim_{\substack{d \to \infty \\ \xi \to 0}} d \sinh \xi = d \sinh \xi_1$$

when $\xi = \xi_1$ defines the boundary of the ellipsoidal cavity. Now $f(\xi)$ does not depend on d so that

$$\lim_{\substack{d \to \infty \\ \xi \to 0}} f(\xi) = \lim_{\xi \to 0} f(\xi) = \infty .$$

Hence, we must take $a = 0$ and again $F = r^4 \times$ constant.

The solution for a shaft of almost uniform circular cross sections with a symmetrically located spherical cavity may be obtained from the solution (9.3). Here, we map the semi-ellipses given by the equation $\xi = \xi_0$ in the coordinates (ξ, r) onto the semi-circles given by $\rho = \rho_0$ in terms of plane polar coordinates in the meridian section. This requires that ξ_0 become infinite as the interfocal distance 2d approaches zero in such a way that

$$\lim_{\substack{d\to 0\\ \xi_0\to\infty}} \frac{d}{2}\,(e^{\xi_0} \pm e^{-\xi_0}) = \rho_0.$$

After some calculations with this limit on equation (9.3), we find that

$$\lim_{\substack{d\to 0\\ \xi\to\infty}} F(\xi, r) = r^4\,(\frac{9\,a_1}{20}\,\rho^{-5} + a_2),$$

which solution has been obtained by Hay [14] using the general theory of Sections 4 and 5, with r, ρ as non-orthogonal coordinates in the meridian section.

10. The Second Determination of the Solution for the Torsion of the Shaft of Almost Uniform Cross Sections by the use of Prolate Elliptic Coordinates. The results of Sections 7 and 9 are easily determined again by the use of the classical prolate elliptic coordinates x^1 and x^2 in the meridian section. The coordinates (x^1, x^2) are connected with rectangular cartesian coordinates (z, r) by the relations

$$(10.1)\qquad z = d\cosh x^1 \cos x^2, \qquad r = d \sinh x^1 \sin x^2,$$

where $0 \leq x^1$, $0 \leq x^2 \leq \pi$, and d is a constant. The parametric lines of x^1 are confocal hyperbolas given by the equations

$$(10.2)\qquad \frac{z^2}{(\cos x^2)^2} - \frac{r^2}{(\sin x^2)^2} = d^2.$$

The parametric lines of x^2 are the confocal ellipses

$$(10.3)\qquad \frac{z^2}{(\cosh x^1)^2} + \frac{r^2}{(\sinh x^1)^2} = d^2,$$

as shown in Figure 10. The constant d is one-half of the distance between foci of the parametric lines.

For the meridian section we have $ds^2 = dz^2 + dr^2$, which becomes, by equations (10.1)

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

where

$$g_{11} = g_{22} = g^{1/2}, \qquad g_{12} = g_{21} = 0,$$

$$g = \det g_{\alpha\beta} = d^4(\sinh^2 x^1 + \sin^2 x^2)^2,$$

$$g^{11} = g^{22} = g^{-1/2}, \quad g^{12} = g^{21} = 0.$$

The basic partial differential equation for the torsion problem is (4.5). The identities (6.6) again simplify the calculations which yield this differential equation, in prolate elliptic coordinates (x^1, x^2), in the form

$$F_{,11} + F_{,22} - 3 \coth x^1\, F_{,1} - 3 \cot x^2\, F_{,2} = 0\,. \tag{10.4}$$

The boundary conditions are, as usual, by equations (5.1),

$$\left\{ \begin{aligned} F(x^1, x^2) &= k_1 \quad \text{on} \quad C_1, \\ F(x^1, x^2) &= k_2 \quad \text{on} \quad C_2. \end{aligned} \right. \tag{10.5}$$

Let us now consider the hollow ellipsoidal shaft of Section 7 which is bounded by confocal prolate ellipsoids. The boundaries of the meridian section have the equations $x^1 = \text{constant} = \xi_1$, $x^1 = \text{constant} = \xi_2$, and, in view of the boundary conditions (10.5), we look for a solution of (10.4) which is a function of x^1 alone. It readily follows by the usual elementary techniques that such a solution is

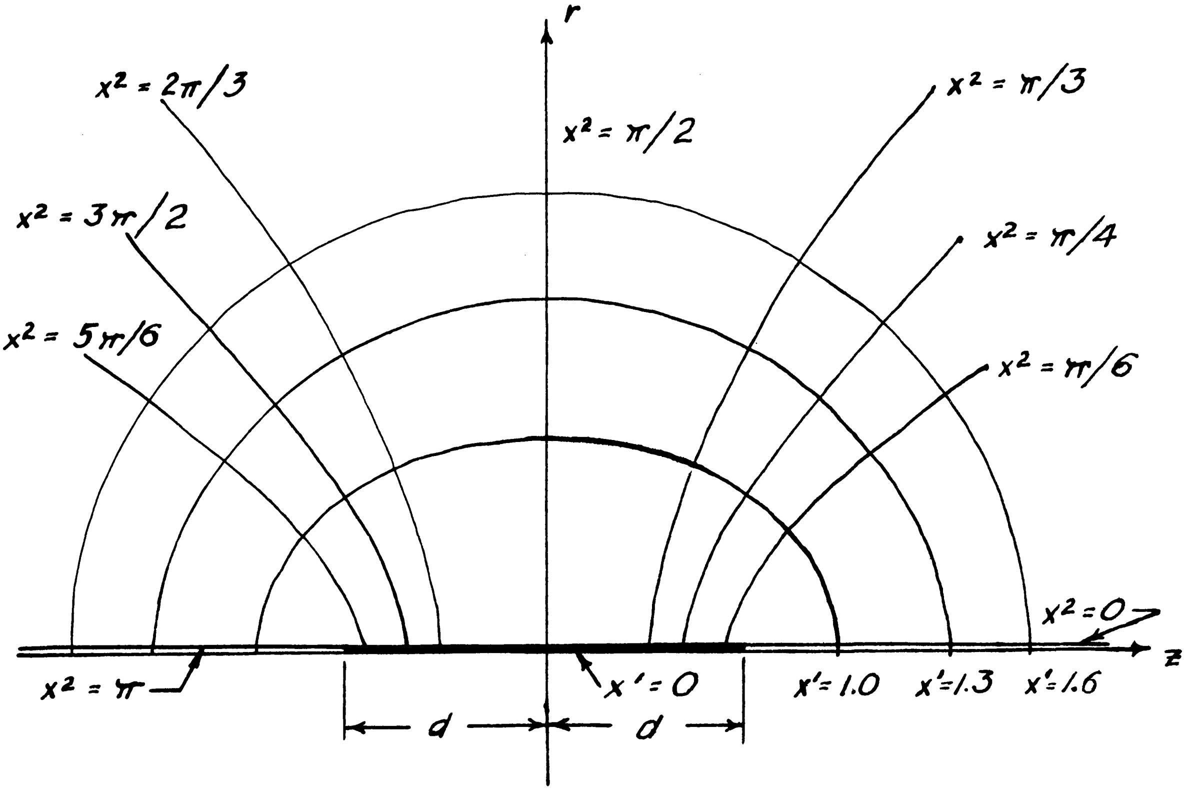

Figure 10. Prolate Elliptic Coordinates in a Meridian Section

$$F(x^1, x^2) = a(\frac{1}{3} \cosh^3 x^1 - \cosh x^1) + b$$

where a and b are arbitrary constants. The boundary conditions (10.5) permits the determination of a and b, all of which yields the solution for F as in Section 7.

For the nearly uniform shaft of Section 9, a determination of the solution by the use of the prolate elliptic coordinate is quite interesting. In this case the boundary conditions are

$$(10.6) \quad \begin{cases} F(x^1, x^2) = 0 & \text{on } C_1, \\ F(x^1, x^2) = k_2 & \text{on } C_2. \end{cases}$$

Here C_1 is the boundary of the cavity and is defined by the relations

$$(10.7) \quad \begin{cases} x^1 = \xi_1, \quad (0 < x^2 < \pi) \\ x^2 = 0, \quad x^2 = \pi, \quad (x^1 > \xi_1) \end{cases}$$

while C_2 is unspecified for the present.

The classical separation of variables is employed. We look for solutions of the form $F(x^1, x^2) = X_1(x^1)\, X_2(x^2)$, and find the two ordinary differential equations

$$(10.8) \quad \begin{cases} X_1'' - 3 \coth x^1 X' - \quad X_1 = 0, \\ X_2'' - 3 \cot x^2 X_2' + \quad X_2 = 0, \end{cases}$$

where λ is a separation constant. The change of variables $x = \cos x^2$ and $X_2(x^2) = \sin^2 x^2\, y(x)$ transforms the second differential equation into

$$(1 - x^2)\, y'' - 2x\, y' + (2 - \lambda - \frac{4}{1 - x^2})y = 0.$$

If we let $\lambda = 2 - n(n + 1)$ for $n = 1, 2, \ldots$, we obtain the associated Legendre differential equation

$$(1 - x^2)\, y'' - 2\, x\, y' + [n(n + 1) - \frac{4}{1 - x^2}]\, y = 0.$$

Well known linearly independent solutions of this equation are the associated Legendre functions $P_n^2(x)$ and $Q_n^2(x)$ of the first and second kind respectively for $n = 1, 2, \ldots$. Similarly, the change of variables $z = \operatorname{sech} x^1$ and $X_1(x^1) = w(z)$ transforms the first differential equation of (10.8) into

$$w'' - \frac{2(2 - z^2)}{z(z-1)(z+1)}\, w' + \frac{(n-1)(n-2)}{z^2(z-1)(z+1)}\, w = 0. \tag{10.9}$$

This differential equation has the three regular singular points, $z = 0$ and $z = \pm 1$. If we follow the usual technique for obtaining power series solutions near a singular point, we find the two solutions

$$\left.\begin{array}{l} z^{n-1}\ {}_2F_1(\frac{n}{2}, \frac{n-1}{2};\ n + \frac{3}{2}\ ;\ z^2), \\ z^{-n-2}\ {}_2F_1(-\frac{n}{2} - 1, -\frac{n+1}{2}\ ;\ -n + \frac{1}{2};\ z^2) \end{array}\right\},\ (n = 1, 2, \ldots).$$

When we return to the original variables (x^1, x^2) and collect results, we get for $F(x^1, x^2)$, the expression

$$\left\{\begin{array}{l} \operatorname{sech}^{n-1} x^1\ {}_2F_1(\frac{n}{2}, \frac{n-1}{2};\ n + \frac{3}{2};\ \operatorname{sech}^2 x^1) \\ \operatorname{sech}^{-n-2} x^1\ {}_2F_1(-\frac{n}{2} - 1, -\frac{n+1}{2};\ -n + \frac{1}{2};\ \operatorname{sech}^2 x^1) \end{array}\right\} \times \left\{\begin{array}{l} (\sin x^2)^2 P_n^2(\cos x^2) \\ (\sin x^2)^2\, Q_n^2(\cos x^2) \end{array}\right\}. \tag{10.10}$$

The boundary conditions (10.6) require that $F(x^1, x^2)$ be symmetric about the line $z = 0$ where $x^1 = \pi/2$. Thus, we must take $n = 2m$, $(m = 1, 2, \ldots)$, since $P_n^2(\cos x^2)$ is a poloynomial of degree n in $\cos x^2$. Also, from the theory of Legendre functions [33], if we write $u = \cos x^2$ we see that

$$Q_{2n}^2(u) = \frac{1 - u^2}{2}[P_{2n}''(u) \ln(\frac{u + 1}{u - 1}) - \frac{4}{u^2 - 1}\, P_{2n}'(u)$$

$$+ \frac{4u}{u^2 - 1}\, P_{2n}(u) + P_{2n-1}'(u)],\ (n = 1, 2, \ldots).$$

Hence, $Q_{2m}^{2}(u)$ are not symmetric about $u = 0$ so they must be discarded. The restriction that $F(x^1, x^2)$ be bounded for all x^1 further reduces the set of available solutions to linear combinations of the following:

$$(10.11)\quad \begin{cases} \Big[a \operatorname{sech} x^1 {}_2F_1(1, \tfrac{1}{2}; \tfrac{7}{2}; \operatorname{sech}^2 x^1) \\ \qquad + b \sinh^4 x^1 {}_2F_1(-2, -\tfrac{3}{2}; -\tfrac{3}{2}; \operatorname{sech}^2 x^1)\Big] \sin^2 x^2\, P_2^2(\cos x^2), \end{cases}$$

$$\operatorname{sech}^{2m-1} x^1 \sin^2 x^2 {}_2F_1(m, m-\tfrac{1}{2}; 2m-\tfrac{3}{2}; \operatorname{sech}^2 x^1)\, P_{2m}^2(\cos x^2),$$

$$(m = 2, 3, \ldots),$$

where a and b are arbitrary constants. However, the only function of this set which vanishes on the elliptic boundary $x^1 = \xi_1$ is (10.11), and this occurs only for appropriate choices for a and b. Because of the identities

$${}_2F_1(-2, 1; 1; \operatorname{sech}^2 x^1) = \tanh^4 x^1, \quad P_2^2(\cos x^2) = 3 \sin^2 x^2,$$

and, since $r = d \sinh x^1 \sin x^2$ from equations (10.1), we find that our solution must be of the form

$$(10.12)\quad F(x^1, x^2) = \frac{3r^4}{d^4}\left[a \operatorname{csch}^4 x^1 \operatorname{sech} x^1\, {}_2F_1(1, 1; 7; \operatorname{sech}^2 x^1) + b\right].$$

Let us compare this expression for F with the expression determined earlier and given in equation (9.3). These agree if

$$\cosh^4 x^1 \operatorname{sech} x^1\, {}_2F_1(1, 1; 7; \operatorname{sech}^2 x^1) = a \int \operatorname{csch}^5 x^1\, dx^1 + b$$

for some constants a and b. Since either member of this relation, when multiplied by r^4 from equations (10.1), satisfies the differential equation (10.4), we have

$$F(x^1, x^2) = r^4\left[a_0 \int \operatorname{csch}^5 x^1\, dx^1 + b_0\right],$$

where a_o and b_o are arbitrary constants. We recall that the parametric lines of r in the coordinates (ξ, r) are semi-ellipses defined by equation (6.11) and that the parametric lines of x^2 in the coordinates (x^1, x^2) are also semi-ellipses defined by equation (10.3). Thus, the equations $\xi = x^1 = \xi_1$ = constant define the boundary of the ellipsoidal cavity in either coordinate system and the solution above is equivalent to that given by equation (9.3).

CHAPTER III

THE TORSION OF A SHAFT WITH AN OBLATE ELLIPSOIDAL CAVITY

11. <u>The Fundamental Equations in Non-orthogonal Coordinates</u>. Neuber [17] has considered the torsion problem for a shaft of nearly uniform cross sections with an oblate ellipsoidal cavity as indicated in Figure 12. He used his three function theorem with an oblate ellipsoidal coordinate system. His method of solution is rather involved and we shall here develop a simpler and direct method of solution using the general theory of Sections 4 and 5 with a non-orthogonal coordinate system. The coordinate system is developed in this section and the solution is obtained in Section 12. In Section 13, we discuss an oblate ellipsoidal coordinate system so that, in Section 14, we may compare our solution with that of Neuber. The discussion, which now follows, is analogous to that for the torsion of the shaft with a prolate ellipsoidal cavity.

Let us consider retangular cartesian coordinates (z, r) in a meridian section of the shaft. We introduce a variable ζ by the relations

$$(11.1) \qquad z = \epsilon\sqrt{d^2 \sinh^2\zeta - r^2 \tanh^2\zeta}\,, \quad (\zeta \geq 0),$$

where the indicator ϵ is defined by the equations

$$(11.2) \qquad \begin{cases} \epsilon = 1 & \text{in quadrant I of the } (z,\ r) \text{ system,} \\ \epsilon = -1 & \text{in quadrant II of the } (z,\ r) \text{ system.} \end{cases}$$

We regard (ζ, r) as non-orthogonal curvilinear coordinates in the meridian section of a shaft, as indicated in Figure 11, and denote them by x^α $(\alpha = 1, 2)$ upon occasion. The parametric lines of ζ are straight lines parallel to

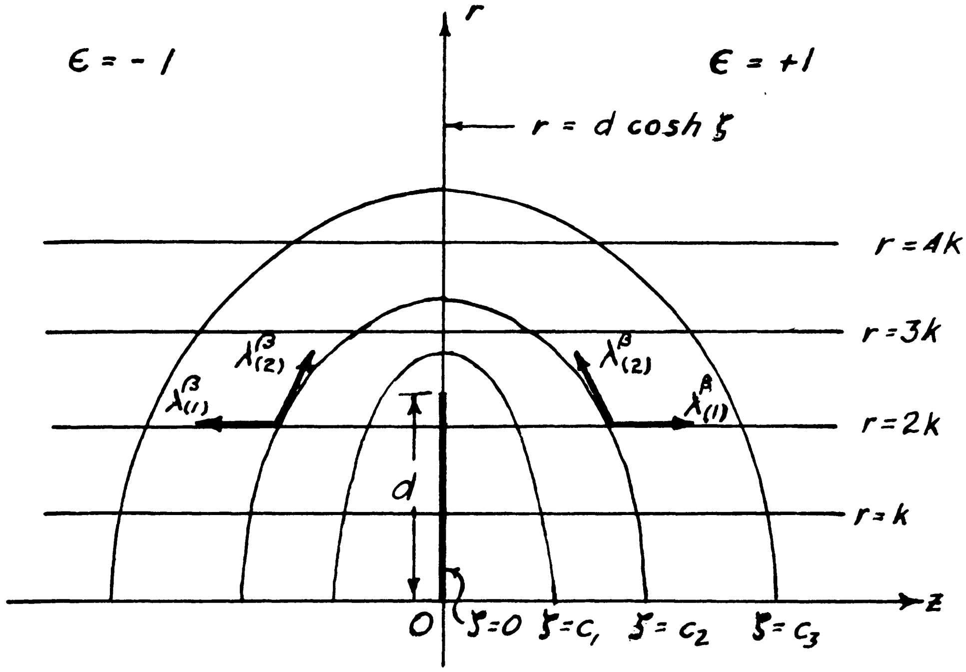

Figure 11. The (ζ, r) Coordinate System

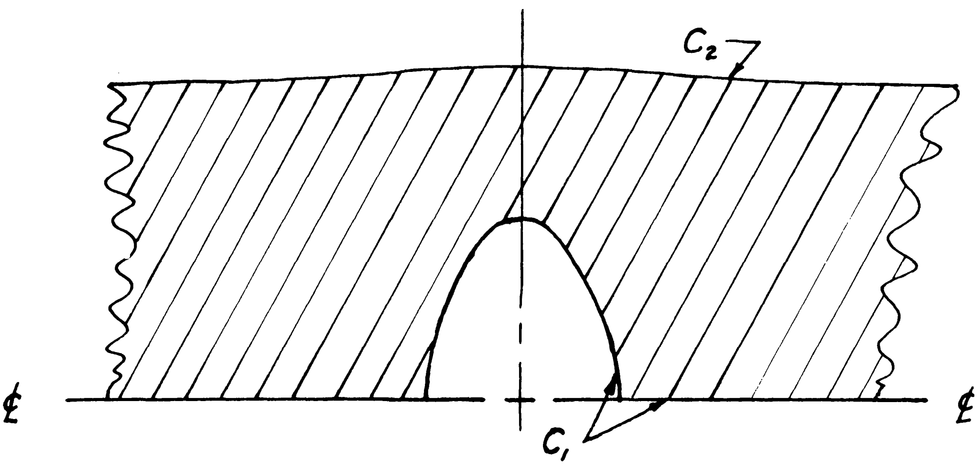

Figure 12. Meridian Section of a Nearly Uniform Shaft with an Oblate Ellipsoidal Cavity

the z-axis, and the parametric lines of r are confocal semi-ellipses whose foci have the rectangular cartesian coordinates (0, d). For every non-negative constant ζ_1, the equation

$$\zeta = \zeta_1 \tag{11.3}$$

defines a semi-ellipse of the family

$$\frac{z^2}{\sinh^2\zeta} + \frac{r^2}{\cosh^2\zeta} = d^2 \tag{11.4}$$

in the meridian section, which generates an oblate ellipsoid. Thus, the boundary of an ellipsoidal cavity has the simple equation (11.3).

Let $\lambda^i_{(j)}$ be unit vectors in the direction of the parametric lines of x^j ($j = 0, 1, 2$). It will be noted that the angle between $\lambda^\beta_{(1)}$ and $\lambda^\beta_{(2)}$ is π at points on that part of the r-axis which is above the focus, so that this line segment is a singular line of the (ζ, r) coordinate system. The equation of this singular line segement is $r = d\cosh\zeta$.

Furthermore, it will be observed in Figure 11 that the orientation of the coordinates $x^\alpha = (\zeta, r)$ changes from right handed to left handed in passage from quadrant one to two of the (z, r) coordinate system. In order to take this change into account analytically with convenience, we make use of the indicator ϵ defined by equations (11.2).

For the arc length ds in the meridian plane, we have

$$ds^2 = dz^2 + dr^2. \tag{11.5}$$

From equation (11.1) we obtain

$$dz = \epsilon D^{-1/2} \tanh\zeta\,(E\,d\zeta - r\tanh\zeta\,dr),$$

where

$$\begin{aligned} D = D(\zeta, r) &\equiv d^2\cosh^2\zeta - r^2, \\ E = E(\zeta, r) &\equiv d^2\cosh^2\zeta - r^2\operatorname{sech}^2\zeta\,. \end{aligned} \tag{11.6}$$

Substitution in (11.5) for dz yields

$$ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta,$$

where

(11.7) $g_{11} = E^2/D, \; g_{12} = g_{21} = -rED^{-1} \tanh\zeta \, , \; g_{22} = E/D.$

It then follows that

(11.8) $\det g_{\alpha\beta} \equiv g = E^2/D,$

(11.9) $g^{11} = E^{-1}, \; g^{12} = g^{21} = rE^{-1} \tanh\zeta \, , \; g^{22} = 1,$

and that the Jacobian of the transformation from rectangular cartesian coordinates (z, r) to the non-orthogonal curvilinear coordinates $(\zeta, r) = x^\alpha$ is $J(\frac{z, r}{x^1, x^2}) = \epsilon E/D^{1/2}$.

With the aid of the tensorial expressions for $\nabla^2 F$ and $\frac{1}{r}\nabla r \cdot \nabla F$, the fundamental differential equation (4.5) for F is readily expressed in terms of the coordinates (ζ, r) in the form

(11.10)
$$F_{,\zeta\zeta} - 3 \tanh\zeta \; F_{,\zeta} + 2r \tanh\zeta \; F_{,\zeta r} + E(F_{,rr} - \frac{3}{r} F_{,r}) = 0.$$

The boundary conditions are given by equations (5.1) which we may take in the form

(11.11) $F(\zeta, r) = 0$ on C_1, $F(\zeta, r) = K$ on C_2,

where the constant K determines the magnitude of the twisting couple. In fact, from equation (5.4), we have

(11.12) $K = M/2\pi\mu.$

12. <u>The Nearly Uniform Shaft With an Oblate Ellipsoidal Cavity</u>. Let us consider a nearly uniform shaft with an oblate ellipsoidal cavity as indicated in Figure 12. The inner boundary C_1 of the meridian section is prescribed by the relations

(12.1)
$$\begin{cases} \zeta = \text{constant} = \zeta_1, & |z| \leqq d \sinh \zeta_1, \\ r = 0, & |z| \geqq d \sinh \zeta_1, \end{cases}$$

and the outer boundary C_2 is left unspecified for the moment.

The differential equation (11.10) is linear and of second order. The coefficient E prevents a classical separation of variables. However, if we seek a solution of the form $F(\zeta, r) = (r^4 + b_1)[a_1 f(\zeta) + a_2 g(\zeta)]$ where the constants a_1, a_2 and b_1 are arbitrary, we obtain the particular integral

$$(12.2) \qquad F(\zeta, r) = r^4[a_1 f(\zeta) + a_2] + a_3 g(\zeta) + a_4$$

where a_1, a_2, a_3, a_4 are arbitrary constants, and

$$(12.3) \qquad f(\zeta) = \int \operatorname{sech}^5\zeta \, d\zeta = \operatorname{sech}^3\zeta \tanh\zeta - \frac{3}{2}\operatorname{sech}\zeta \tanh\zeta - \frac{3}{2}\operatorname{arc\,cot}\sinh\zeta ,$$

$$(12.4) \qquad g(\zeta) = \int \cosh^3\zeta \, d\zeta = \frac{1}{3}\sinh^3\zeta + \sinh\zeta .$$

The first boundary condition (11.11) together with (12.1) requires that $a_2 = - a_1 f(\zeta_1)$, $a_3 = a_2 = 0$. Thus, we have

$$(12.5) \qquad F(\zeta , r) = a_1 r^4[f(\zeta) - f(\zeta_1)],$$

with $f(\zeta)$ defined by equation (12.3).

The second boundary condition is satisfied for any outer boundary C_2 with the equation

$$(12.6) \qquad r^4[f(\zeta) - f(\zeta_1)] = K_1,$$

where K_1 is a non-negative constant. This outer boundary C_2 can be varied by choice of the parameter K_1.

The arbitrary constant a_1 of the solution (12.5) is determined by the second of the boundary conditions (11.11). We use this in conjunction with equations (12.5), (12.6), (11.12) and some direct calculation to find that

$$(12.7)\qquad a_1 = K/K_1 = M(2\pi\mu K_1)^{-1} ,$$

where M is the twisting couple. Thus, the particular solution (12.5) becomes

$$(12.8)\qquad F(\zeta, r) = \frac{Mr^4}{2\pi\mu K_1}[f(\zeta) - f(\zeta_1)].$$

We shall show, in Section 14, that the curves (12.6) on which F = constant are the stress lines obtained by Neuber, who gives, in addition, expressions for the stress components.

We add the physical component of displacement $u = ru^0$ in the direction of the parametric line of x^0 which is obtained from equations (4.4). Some direct calculation yields

$$u = \epsilon \frac{rMD^{1/2}}{2\pi K_1\mu}\left[\operatorname{sech}^5\zeta + 4\tanh\zeta\,[f(\zeta) - f(\zeta_1)]\right] + cr$$

where the term cr corresponds to an arbitrary rigid body rotation about the axis of the shaft.

13. Oblate Ellipsoidal Coordinates. In order to compare in Section 14 the results of Section 11 above with those of Neuber, we now introduce an oblate ellipsoidal coordinate system which is three dimensional and contains that of Neuber as a special case. These coordinates reduce to elliptic coordinates in a meridian section, and will be referred to in Section 16.

Let us consider rectangular cartesian coordinates (x, y, z) with the x-axis on the axis of a shaft. Then, let us consider the ellipsoidal coordinates (u, v, w) which are related to the rectangular cartesian coordinates (x, y, z) by the equations

$$(13.1)\qquad \left\{\begin{aligned} x &= d\sinh u\cos v,\\ y &= d\cosh u\sin v\cos w,\\ z &= d\cosh u\sin v\sin w.\end{aligned}\right.$$

The planes defined by w = constant are meridian sections of the shaft. Thus, without loss of generality, we may consider the coordinate system in the meridian section $w = 0$.

Now we have used (z, r) to denote rectangular cartesian coordinates in the meridian section. To avoid confusion in that which follows, we write (z, r) in place of (x, y). The coordinates (u, v) above are elliptic coordinates in a meridian section. They are related to our rectangular cartesian coordinates (z, r) by the equations

$$(13.2) \qquad z = d \sinh u \cos v, \quad r = d \cosh u \sin v,$$

where d is a constant. The parametric lines of v are defined by the equation

$$(13.2) \qquad \frac{z^2}{\sinh^2 u} + \frac{r^2}{\cosh^2 u} = d^2,$$

which prescribes confocal ellipses. The foci are at the points $(0, \pm d)$ in coordinates (z, r), as shown in Figure 13. Similarly, the parametric lines of u are defined by the equation

$$(13.4) \qquad \frac{r^2}{\sin^2 v} - \frac{z^2}{\cos^2 v} = d^2,$$

which represents confocal hyperbolas with foci at $(0, \pm d)$ in the coordinates (z, r).

We remark that the range of u and v must be restricted if continuity of particular parametric lines is desired and if the transformation (13.2) is to be one-valued. We shall impose such restrictions as the occasion arises.

14. <u>Comparison of Results from Section 12 with Neuber's Solution for the Torsion of the Nearly Uniform Shaft with an Oblate Ellipsoidal Cavity</u>. A meridian curve which generates a stress free boundary for a

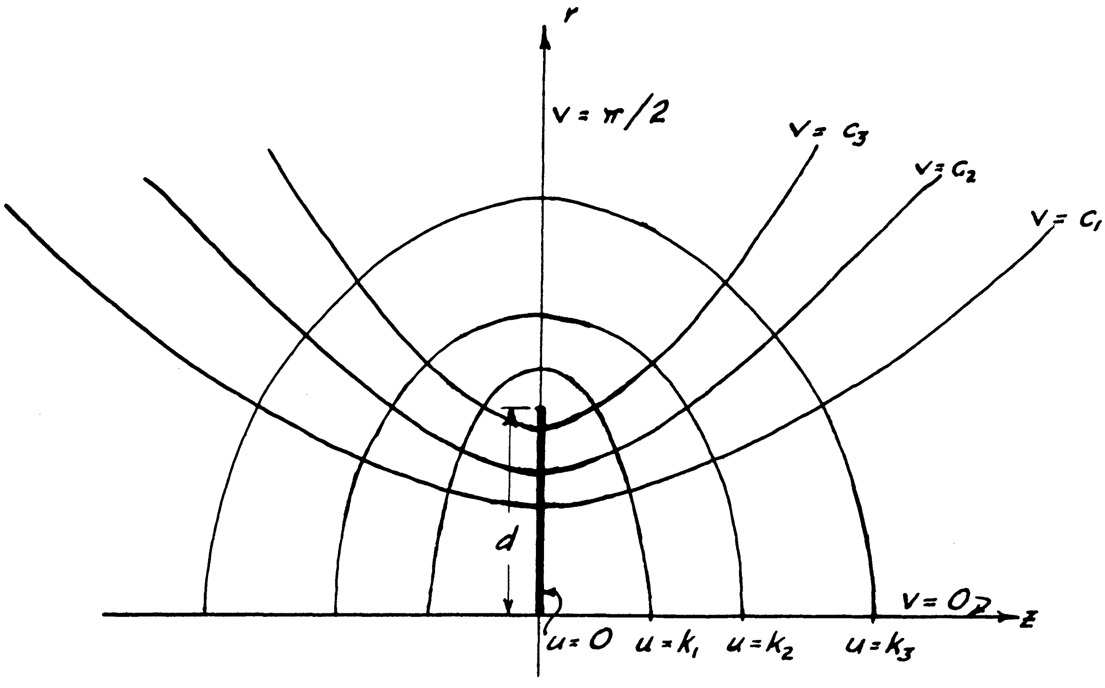

Figure 13. Oblate Elliptic Coordinates

shaft is referred to as a stress line by Neuber [17]. We now compare our solution (12.8), or equivalently equation (12.6), with Neuber's stress lines. He uses the oblate ellipsoidal coordinates of our Section 13 with $d \equiv 1$. We remark that the inner boundary C_1 of the shaft is defined in this coordinate system by the relations

$$(14.1) \quad \begin{cases} u = \text{constant} = u_o, & (0 < v < \pi), \\ v = 0,\ v = \pi, & (0 < u_o \leqq u), \end{cases}$$

where the inequalities are necessary if the first equation is to completely prescribe the elliptic portion of the boundary and, at the same time, if the transformations (13.2) are to be one valued.

The equations of the stress lines, as given by Neuber [17] for the shaft under consideration, are

$$(14.2) \quad \left\{\cosh^4 u + A[\cosh^4 u\ T - \cosh^2 u \sinh u - \tfrac{2}{3} \sinh u]\right\} \sin^4 v = \text{constant}.$$

where

$$\frac{1}{A} = - T(u_o) + \frac{\sinh u_o}{\cosh^2 u_o} + \frac{2 \sinh u_o}{3 \cosh^4 u_o},$$

$$t = \text{arc} \cot(\sinh u).$$

If we factor a $\cosh^4 u$ from the bracketed quanity in the left hand member of (14.2) and recall that $\cosh^4 u \sin^4 v = r^4$ from equations (13.2), we obtain.

$$r^4 \left[1 - \frac{T - \operatorname{sech} u \tanh u - \frac{2}{3} \operatorname{sech}^3 u \tanh u}{T(u_o) - \operatorname{sech} u_o \tanh u_o - \frac{2}{3} \operatorname{sech}^3 u_o \tanh u_o} \right] = \text{constant}.$$

We refer to equations (12.3) for the definition of $f(\xi)$, in terms of which this last equation becomes

$$(14.3) \quad r^4[f(u) - f(u_o)] = \text{constant}.$$

This is precisely the form of our equation (12.6) with the variable u in place of ζ. When we compare equations (13.3) and (11.4) with $d = 1$, we see that

$$u = c\zeta$$

for some constant c. Hence, our solution (12.8) is equivalent to that obtained by Neuber.

CHAPTER IV

TORSION OF HYPERBOLOIDS OF REVOLUTION

15. The Case of the Hyperboloid of Two Sheets. Let us consider a hollow shaft bounded inside and outside by portions of confocal hyperboloids of two sheets. A meridian section for such a shaft is shown in Figure 14, the foci being at A and B. Let (x^1, x^2) be elliptic coordinates in a meridian section as defined in Section 10. If C_1 and C_2 denote the inner and outer boundaries of the meridian section respectively, we then have

$$(15.1) \qquad \begin{aligned} x^2 &= \text{constant} = b_1 \quad \text{on} \quad C_1, \\ x^2 &= \text{constant} = b_2 \quad \text{on} \quad C_2. \end{aligned}$$

As we shall see later, C_2 could be a portion of the axis of revolution, so that the shaft is solid, provided however that no points on the line segment AB joining the foci shall lie on C_2.

The fundamental differential equation in terms of the elliptic coordinates (x^1, x^2) is equation (10.4). The boundary conditions are prescribed, in general, by equation (5.1). In view of equations (15.1), these boundary conditions become

$$(15.2) \qquad F(x^1, b_1) = 0, \qquad F(x^1, b_2) = K$$

where the constant K is determined by the twisting couple, from equation (5.4). We have in the present case

$$(15.3) \qquad M = 2\pi\mu K.$$

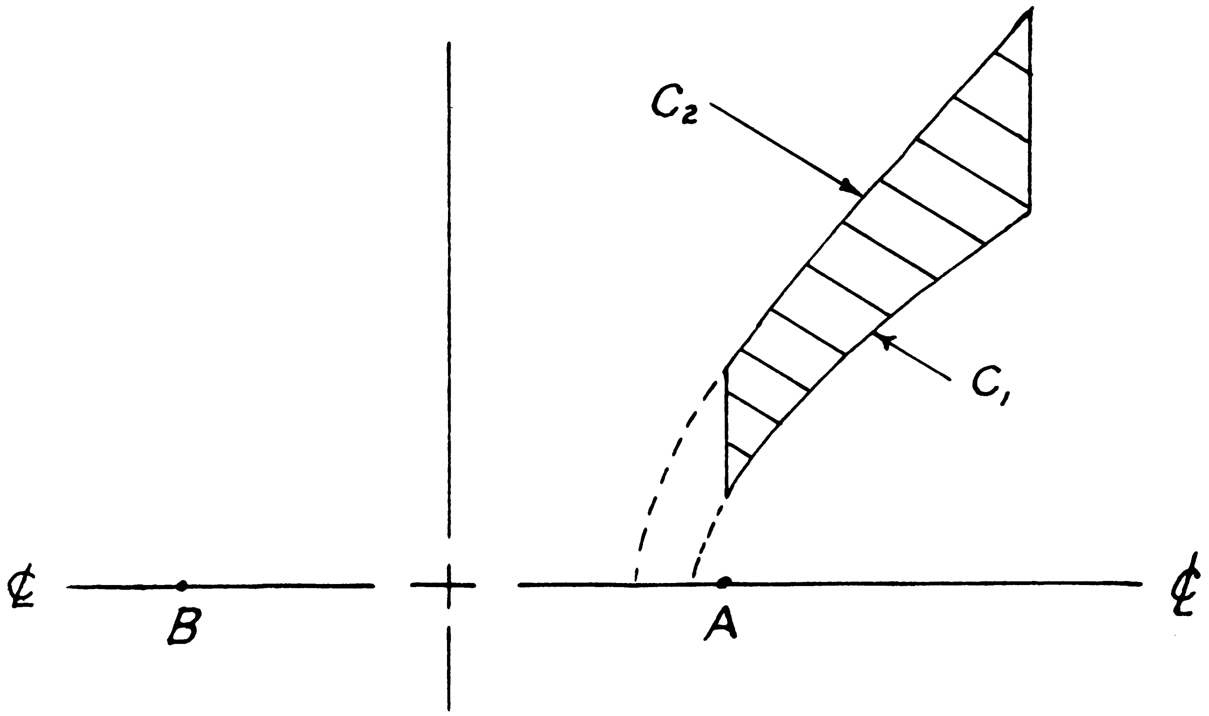

Figure 14. A Meridian Section of a Shaft with Boundary Surfaces Which Are Portions of One Sheet of Confocal Hyperboloids of Two Sheets

In view of the boundary conditions (15.2), we look for solutions of the fundamental differential equation (10.4) which involve x^2 only. We then find that F satisfies an elementary ordinary differential equation whose general solution is

$$F = a_1(\cos^3 x^2 - 3\cos x^2) + a_2, \tag{15.4}$$

where a_1 and a_2 are arbitrary constants.

The boundary conditions (15.2) yield

$$\begin{cases} a_2 = -a_1(\cos^3 b_1 - 3\cos b_1), \\ a_1 = K(\cos^3 b_2 - 3\cos b_2 - \cos^3 b_1 + \cos b_1)^{-1}. \end{cases} \tag{15.5}$$

Thus, with K from (15.3) above, we have

$$F = \frac{M}{2\kappa\mu}\left[\frac{\cos^3 x^2 - 3\cos x^2 - \cos^3 b_1 + 3\cos b_1}{\cos^3 b_2 - 3\cos b_2 - \cos^3 b_1 + 3\cos b_1}\right]. \tag{15.6}$$

The physical component of displacement $u = r\,u^0$ in the direction of the parametric line of x^0 can be found from F by the use of equation (4.4). A straight forward calculation yields

$$u = -\frac{Mr}{2\kappa\mu d^3}\left[\frac{\operatorname{csch} x^1 \coth x^1 + \ln(\operatorname{csch} x^1 - \coth x^1)}{\cos^3 b_2 - 3\cos b_2 - \cos^3 b_1 + 3\cos b_1}\right] + cr$$

where the term cr corresponds to an arbitrary rigid body rotation of the shaft about its axis.

Equations (4.7) give the stress components in terms of the function F. Hence, we have in the present case

$$\begin{cases} T_{01} = \dfrac{M}{2\kappa d}\operatorname{csch} x^1 \sin^2 x^2\,[\cos^3 b_2 - 3\cos b_2 - \cos^3 b_1 + 3\cos b_1]^{-1}, \\ T_{ij} = 0 \text{ otherwise.} \end{cases} \tag{15.7}$$

The principal stresses σ_1, σ_2, σ_3, are the roots of equation (2.16); in the present case we obtain $\sigma_1 = -\sigma_3 = r^{-1}g^{-1/2}\,T_{01}$, $\sigma_2 = 0$. Hence, the maximum shearing stress S_{max} is, by (2.17)

$$(15.8) \qquad S_{max} = r^{-1} g^{-1/4} T_{01}$$
$$= \frac{M \alpha \operatorname{csch}^2 x^1 \sin x^2}{2\pi d^3 \sqrt{\sinh^2 x^1 + \sin^2 x^2}} [\cos^3 b_2 - 3 \cos b_2 - \cos^3 b_1 + 3 \cos b_1].$$

Now, csch x^1 is unbounded as x^1 approaches zero, so that the solution (15.4) is valid only for $x^1 > 0$. Since x^1 vanishes on the line segment AB joining the foci of the bounding hyperbolas, Figure 14, it is only necessary that this line segement be exterior to the shaft. Thus, C_2 could be a portion of the axis of revolution and the shaft would then be solid, provided that C_2 does not contain any points on the line segement AB.

16. <u>The Case of a Hollow Shaft Bounded by Hyperboloids of Revolution of One Sheet</u>. The torsion problem for a solid hyperboloid of revolution of one sheet has been considered by Pöschl [16] and Neuber [18]. The latter refers to it as the torsion of a body with a deep circumferential groove. We shall consider the torsion of a hollow shaft bounded by two confocal hyperboloids of revolution of one sheet, whose meridian section is shown in Figure 16, by the use of the theory of Sections 4 and 5. We shall use non-orthogonal coordinates and shall show that the solution which we obtain is equivalent to those obtained by the two writers mentioned above in the special case when the shaft is solid.

Let us consider the usual retangular cartesian coordinates (z, r) in the meridian section. We introduce a variable ϕ by the relations

$$(16.1) \qquad z = \epsilon\sqrt{r^2 \cot^2 \phi - d^2 \cos^2 \phi}, \qquad (0 \leq \phi \leq \pi/2),$$

where ϵ is the indicator defined, as before, by the equations

$$(16.2) \qquad \begin{cases} \epsilon = 1 & \text{in quadrant I of the } (z, r) \text{ system,} \\ \epsilon = -1 & \text{in quadrant II of the } (z, r) \text{ system.} \end{cases}$$

We regard $(r, \phi) = (x^1, x^2)$ as curvilinear coordinates in a meridian section. The parametric lines of ϕ are straight lines parallel to the z-axis, as shown in Figure 15. The parametric lines of r are one branch of a family of confocal hyperbolas with foci whose (z, r) coordinates are $(0, d)$. This branch of the hyperbola generates a hyperboloid of revolution of one sheet. Thus, the boundary surface of the shaft is given by the simple equations

$$(16.3) \quad \begin{cases} \phi = \text{constant} = \phi_1, \ (0 \leqq \phi_1 < \pi/2), \\ \phi = \text{constant} = \phi_2, \ (0 \leqq \phi_1 < \phi_2 \leqq \pi/2). \end{cases}$$

The line described by the equations $r = d \cos \phi$ is a singular line of the coordinate system. If we use the now familiar $\lambda^i_{(j)}$ to denote a unit vector tangent to the parametric line of x^j, then the angle between $\lambda^\beta_{(1)}$ and $\lambda^\beta_{(2)}$ is π at points on this singular line. Furthermore, the orientation of the coordinates (x^0, r, ϕ) changes from right handed to left handed upon passage from quadrant I to quadrant II respectively. Here, as in the previous cases, we use the indicator ϵ of equations (16.2), to account for this in analytical work.

The square of distance between adjacent points, in terms of rectangular cartesian coordinates, is

$$(16.4) \quad ds^2 = dz^2 + dr^2.$$

From equation (16.1), we see that

$$dz = \epsilon D^{-1/2}(r \cot \phi \, dr - E d\phi),$$

where

$$(16.5) \quad \begin{cases} D = D(r, \phi) = r^2 - d^2 \sin^2 \phi, \\ E = E(r, \phi) = r^2 \csc^2 \phi - d^2 \sin^2 \phi. \end{cases}$$

When this expression for dz is substituted into equations (16.4), we have

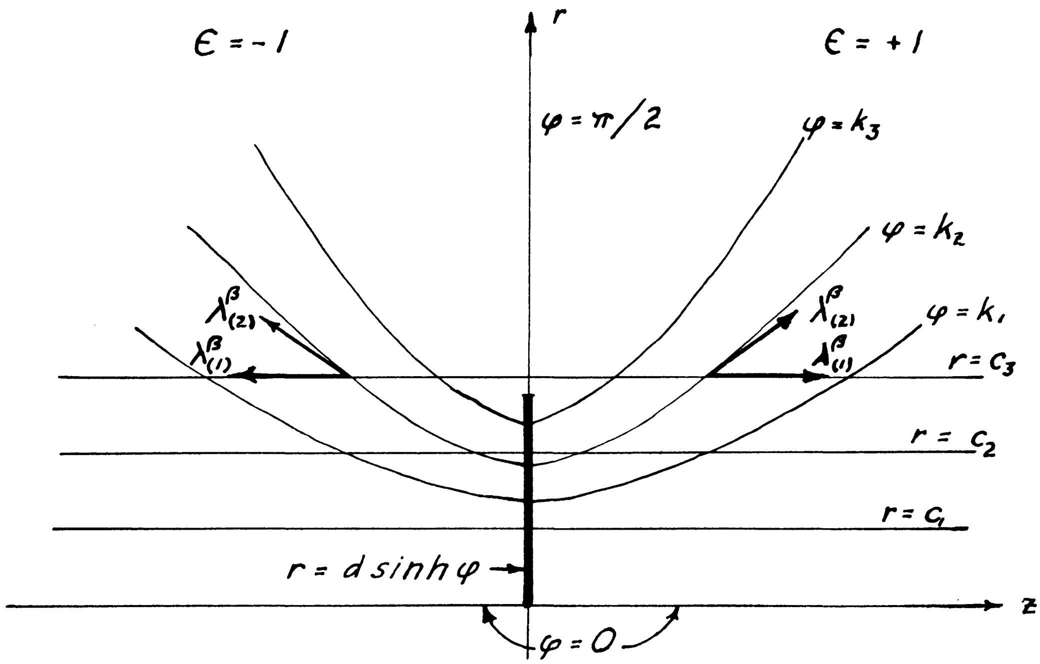

Figure 15. The (r, ϕ) Coordinate System

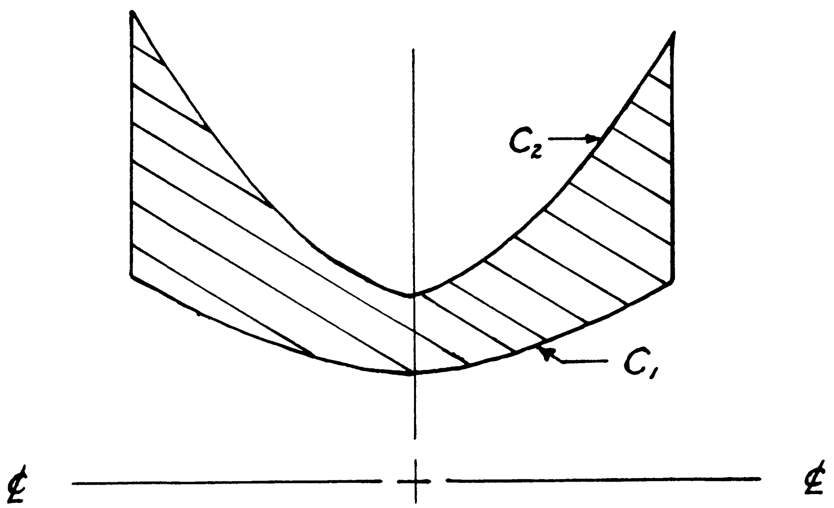

Figure 16. Meridian Section of a Hyperboloid of Revolution

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

where

(16.6) $g_{11} = E/D,\ g_{12} = g_{21} = -\ r\ E\ D^{-1} \cot \phi,\ g_{22} = E^2/D.$

A further calculation yields

(16.7) $\det g_{\alpha\beta} = g = E^2/D,$

(16.8) $g^{11} = 1,\ g^{12} = g^{21} = \ \ r\ E^{-1} \cot \phi,\ g^{22} = E^{-1},$

$$J\left(\frac{z,\ r}{x^1,\ x^2}\right) = -\epsilon E\ D^{-1/2}\ .$$

The fundamental differential equation (4.5) for F is readily expressed in terms of the coordinates (r, ϕ), with the aid of the tensorial expressions for $\nabla^2 F$ and $\frac{1}{r}\nabla r \cdot \nabla F$, to yield

(16.9) $E(F,_{rr} - \frac{3}{r} F,_r) + 2r \cot \phi\ F,_{r\phi} + F,_{\phi\phi} - 3 \cot \phi\ F,_\phi = 0.$

The boundary conditions for a general shaft are given by equations (5.2). Thus, in view of equations (16.3), the boundary conditions take the form

(16.10) $F(r, \phi_1) = k_1, \quad F(r, \phi_2) = k_2$

where k_1, k_2 are constants whose difference is determined by the twisting couple M from the familiar equation

(16.11) $M = 2\pi\mu(k_2 - k_1).$

The basic differential equation is linear with variable coefficients. If we apply the usual technique of separation of variables, we obtain finally,

(16.12) $F(r, \phi) = a_1 r^4[2 \csc^3 \phi \cot \phi + 3 \csc \phi \cot \phi + 3\ \text{arc tanh} (\cos \phi)]$

$$+\ a_2(\cos^3 \phi - 3 \cos \phi) + a_3$$

where a_1, a_2, a_3 are arbitrary constants. Now $\csc \phi$ is unbounded as ϕ approaches zero. Also, on any line L given in terms of rectangular cartesian coordinates (z, r) by the equation $z/a + r/b = 1$, we have, with the aid of equations (16.1),

$$\lim_{\substack{r \to 0 \\ \text{on } L}} F(r, \phi) = 8a_1(a^2 + d^2)^2 - \frac{2}{3}a_2 + a_3$$

where it is noted that a is the intercept of line L on the axis of the shaft. Thus, from the boundary conditions (16.10), $a_1 = 0$. Further application of the boundary conditions (16.10) to $F(r, \phi)$ as given by (16.12) yields

$$\begin{aligned} a_3 &= k_2 - a_2 (\cos^3 \phi_1 - 3 \cos \phi_1), \\ a_2 &= (k_2 - k_1)(\cos^3 \phi_2 - 3 \cos \phi_2 - \cos^3 \phi_1 + 3 \cos \phi_1)^{-1}. \end{aligned} \tag{16.13}$$

Hence, the solution is

$$F(r, \phi) = \frac{M}{2\kappa} \left[\frac{\cos^3\phi - 3\cos \phi - \cos^3\phi_1 + 3\cos \phi_1}{\cos^3\phi_2 - 3\cos \phi_2 - \cos^3\phi_1 + 3\cos \phi_1}\right]. \tag{16.14}$$

If we let ϕ_1 approach zero, the inner boundary C_1 becomes the axis of the shaft and the shaft becomes solid. In this particular case, the solution (16.14) is that which Pöschl [16] obtained through the theory of the generalized Dirichlet problem, using the elliptic coordinates, which are defined by equations (13.2), in a meridian section. The above solution is also equivalent to that obtained by Neuber [18] with the use of his three function theorem and the ellipsoidal coordinates which are defined by equations (13.1) when $d = 1$.

CHAPTER V

TORSION OF A SEMI-INFINITE SHAFT WITH A GENERAL MONOTONE MERIDIAN SECTION

17. <u>The Fundamental Equations</u>. Let us consider a semi-infinite shaft with a meridian section as shown in Figure 17. As one moves from the end of the shaft, the radius increases monotonically for a finite distance and is uniform thereafter. The analysis which follows is independent of the length of the uniform portion of the shaft so that the results obtained are equally valid for a shaft of finite length with a general monotone increasing meridian section. Let the radius of the uniform portion be b. We introduce rectangular cartesian coordinates (z, r) with the positive z-axis on the axis of the shaft as usual and the origin at one end of the shaft as shown in Figure 17. The curves C_1 and C_2, forming the inner and outer boundaries of the meridian section, have the following equations:

$$\text{(17.1)} \qquad \text{for } C_1, \quad r = 0 \qquad (0 \leqq z);$$

$$\text{(17.2)} \qquad \text{for } C_2, \begin{cases} z = f(r) & (0 \leqq z \leqq z_0), \\ r = b & (z_0 \leqq z), \end{cases}$$

where z_0 is any positive constant and $f(r)$ is a monotone increasing function of r on $a \leqq r \leqq b$ or, equivalently on $0 \leqq z \leqq z_0$. In terms of the rectangular cartesian coordinates (z, r), the fundamental differential equation and associated boundary conditions are

$$\text{(17.3)} \qquad \begin{cases} F,_{zz} + F,_{rr} - \frac{3}{r} F,_r = 0, \\ F(z, 0) = 0, \quad F(z, r) = k_2 \text{ on } C_2, \end{cases}$$

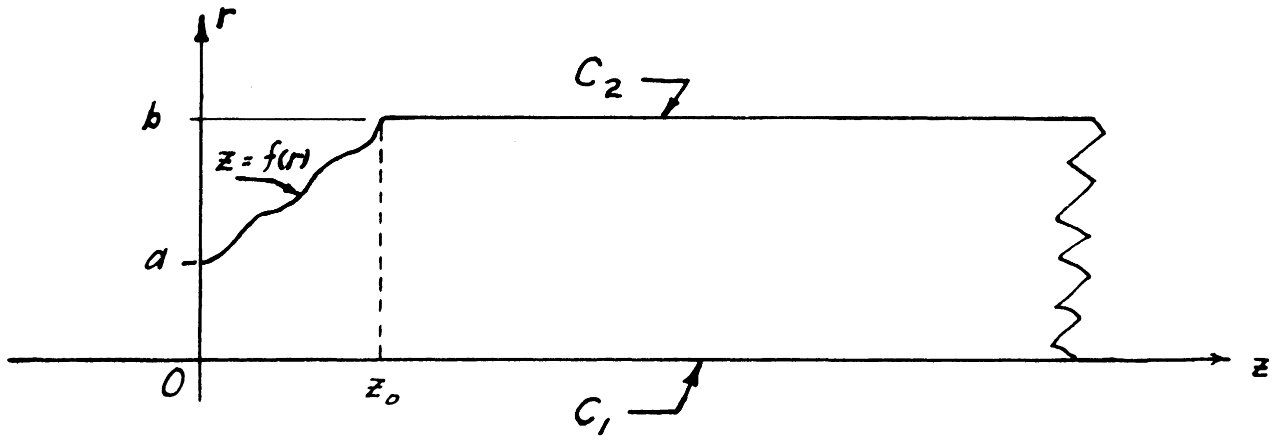

Figure 17. A Semi-Infinite Shaft with a General Meridian Section

where the constant k_2 is determined as usual by the twisting couple M through the relation

(17.4) $M = 2\pi\mu k_2$.

The separation of variables yields

(17.5) $$\begin{bmatrix} e^{\alpha z} \\ e^{-\alpha z} \end{bmatrix} \times \begin{bmatrix} r^2 J_2(\alpha r) \\ r^2 Y_2(\alpha r) \end{bmatrix} = F$$

where α is a separation constant, and $J_2(\alpha r)$ and $Y_2(\alpha r)$ are Bessel functions of order two, and of the first and second kind respectively. They have the definitions [30]

$$J_2(\alpha r) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(2+n)!} \left(\frac{\alpha r}{2}\right)^{2+2n},$$

$$Y(\alpha r) = 2\left[\gamma + \ln \frac{\alpha r}{2}\right] J_2(\alpha r) - \frac{4}{(\alpha r)^2} - 1 - \sum_{n=1}^{\infty} \frac{(-1)^n (H_{n+2} + H_n)}{n!\,(n+2)!} \left(\frac{\alpha r}{2}\right)^{2+2n}$$

where

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

and γ is Euler's constant with the definition

$$\gamma = \lim_{n \to \infty} (H_n - \ln n).$$

We are interested only in bounded solutions, so we have the well known functions [3]

$$F(z, r) = e^{-\alpha z} r^2 J_2(\alpha r).$$

We note that this solution satisfies the boundary conditions on C_1. The boundary condition on the straight part $(r = b)$ of C_2 requires that

(17.6) $J_2(\alpha b) = 0$.

Let $\alpha_i (i = 1, 2, \ldots)$ denote the positive roots of equation (17.6) arranged in ascending order. Then the countably infinite family of functions

$$(17.7) \qquad F_i(z, r) = e^{-\alpha_i z} r^2 J_2(\alpha_i r), \qquad (i = 1, 2, \ldots)$$

vanishes on the axis C_1 of the shaft as required and also vanishes on the straight portion of the outer boundary C_2 of the meridian section. Of course linear combinations of $F_i(z, r)$ also have these properties.

On that part of C_2 which is not straight, we have $0 \leqq z = f(r) \leqq z_o$. We define functions $F_i(r)$ on this part of C_2 by the relations

$$(17.8) \qquad \begin{aligned} F_i(r) &\equiv F_i(f(r), r) \\ &\equiv e^{-\alpha_i f(r)} r^2 J_2(\alpha_i r), \qquad (i = 1, 2, \ldots). \end{aligned}$$

These functions are defined for $a \leqq r \leqq b$ and will be referred to extensively.

From the functions $F_i(r)$, we shall construct an orthonormal system of functions and continue them throughout the meridian section of the shaft. From these, we shall then construct an infinite series solution of the fundamental differential equation. By a proper choice of certain Fourier coefficients, it will then be possible to satisfy all of the necessary boundary conditions.

18. Linear Independence of the Functions $F_i(r)$. In order to construct an orthonormal system of functions from the functions $F_i(r)$, it is necessary to establish their linear independence.

A finite set of functions $g_1(x), g_2(x), \ldots, g_n(x)$ is said to be linearly dependent over a fundamental interval $x_o \leqq x \leqq x_1$ if there exists constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$\sum_{i=1}^{n} c_i \, g_i(x) \equiv 0, \qquad (x_o \leqq x \leqq x_1).$$

The functions are said to be linearly independent otherwise. An infinite set of functions $g_1(x)$, $g_2(x)$, ... is said to be linearly independent over a fundamental interval if every finite subset of the functions is linearly independent over the fundamental interval.

The linear independence of the functions $F_i(r)$ is a delicate matter for arbitrary arguments $f(r)$ in the exponential factors, or even when $f(r)$ is monotone increasing. The linear independence is obvious when $f(r)$ is constant, since it then follows from the linear independence of the Bessel functions. We shall demonstrate that the functions $F_i(r)$ are linearly independent in the special case when $f(r)$ is a linear monotone increasing function and postulate that the monotone increasing property of $f(r)$ is sufficient for the linear independence of the $F_i(r)$.

Let us use contradiction to show that the functions

$$(18.1) \qquad F_i(r) = e^{-\alpha_i(r-a)} r^2 J_2(\alpha_i r), \qquad (i = 1, 2, \ldots)$$

are linearly independent on the interval $a \leq r \leq b$. Let n be any fixed positive integer and suppose that $F_1(r)$, $F_2(r)$, ..., $F_n(r)$ are linearly dependent. This means that

$$(18.2) \qquad \sum_{i=1}^{n} c_i F_i(r) \equiv 0, \qquad a \leq r \leq b,$$

for some choice of the constants c_i all of which are not zero. This is equivalent to the requirement that

$$(18.3) \qquad \sum_{i=1}^{n} b_i e^{-\alpha_i r} J_2(\alpha_i r) = 0$$

where $b_i = c_i e^{\alpha_i a}$.

Now $e^{-\alpha_i r}$ and $J_2(\alpha_i r)$ $(i = 1, 2, \ldots)$ are analytic functions of r on $a \leq r \leq b$ so that each product $e^{-\alpha_i r} J_2(\alpha_i r)$ is an analytic function of r. Hence, each term of equation (18.3) has a unique series expansion and, the coefficient of each power of r must vanish. The desired power series expansions are obtained from the products of the power series representations for $e^{-\alpha_i r}$ and $J_2(\alpha_i r)$, $i = 1, 2, \ldots n$. This yields, for $i = 1, 2, \ldots$,

$$8(\alpha_i r)^{-2} e^{-\alpha_i r} J_2(\alpha_i r)$$

$$= 1 - \alpha_i r + (\frac{1}{2!} - \frac{1}{2.6})(\alpha_i r)^2 - (\frac{1}{3!} - \frac{1}{2.6})(\alpha_i r)^3$$

$$+ (\frac{1}{4!} - \frac{1}{2!2.6} + \frac{1}{2.4.6.8})(\alpha_i r)^4 + \ldots .$$

When these expressions are substituted into equation (18.3), the coefficient for each power of r equated to zero, and common factors removed, we have

$$(18.4) \qquad \sum_{i=1}^{n} b_i(\alpha_i)^{j+2} = 0, \qquad j = 0, 1, 2, \ldots .$$

Every equation in the infinite system (18.4) must be satisfied by the n constants b_i. Hence, we may consider, equivalently, the infinite system

$$(18.5) \qquad \sum_{i=1}^{n} b_i(\alpha_i)^{2j+2} = 0, \qquad j = 0, 1, 2, \ldots .$$

Now α_i are the positive ascending zeros of $J_2(\alpha_i b)$. The system (18.5) is precisely that which follows from the equation

$$(18.6) \qquad \sum_{i=1}^{n} b_i \, J_2(\alpha_i r) = 0, \qquad a \leqq r \leqq b.$$

However, the Bessel functions are linearly independent on $a \leqq r \leqq b$ so that equation (18.6) has no nontrivial solution for the constants b_i. Accordingly, (18.5) has no nontrivial solutions and our assumption is false. Thus, the functions $F_1(r)$, $F_2(r)$, ..., $F_n(r)$ are linearly independent on $a \leqq r \leqq b$. Since this is true for any integer n, the infinite system $F_1(r)$, $F_2(r)$, ... is linearly independent on $a \leqq r \leqq b$ as was to be shown.

We have demonstrated above that the set of nonconstant $f(r)$, for which the system $F_1(r)$, $F_2(r)$, ... is linearly independent, is not empty. We assume, in that which follows, that the system $F_1(r)$, $F_2(r)$, ... is linearly independent when $f(r)$ is defined by equations (17.2).

19. <u>The Orthonormal System.</u> To construct the orthonormal system, we adapt the following standard notations [32]. The inner product (G, H) of two functions $G(x)$ and $H(x)$ on an interval $x_0 \leqq x \leqq x_1$ is the integral

$$(19.1) \qquad (G, H) = \int_{x_0}^{x_1} G(x)\, H(x)\, dx.$$

The norm $\| G \|$ of a function $G(x)$ is

$$(19.2) \qquad \| G \| = (G, G).$$

A set of functions $\{\psi_n\}$ is said to be an orthogonal system on the fundamental interval if

$$\begin{aligned} (\psi_m, \psi_n) &= 0 \qquad (m \neq n) \\ &= \|\psi_n\| \quad (m = n) \ . \end{aligned}$$

An orthogonal system is said to be orthonormal if $\|\psi_n\| = 1$, $(n = 1, 2, \ldots)$.

We now construct an orthonormal system $\{\varphi_n\}$ from the functions F_1, F_2, ... of equation (17.8). We follow the procedure set forth in the Bateman Manuscript Project [34]. Toward this end, let us write

$$(19.3) \quad \begin{cases} \psi_1(r) = F_1(r), \\ \psi_n(r) = \begin{vmatrix} (F_1, F_1) & (F_1, F_2) & \cdots & (F_1, F_n) \\ \cdot & \cdot & \cdot & \cdot \\ (F_{n-1}, F_1) & (F_{n-1}, F_2) & \cdots & (F_{n-1}, F_n) \\ F_1(r) & F_2(r) & \cdots & F_n(r) \end{vmatrix}, \quad (n = 2, 3, \ldots), \end{cases}$$

where the inner products are computed for the interval $a \leqq r \leqq b$. Then we have

$$(\psi_m, \psi_n) = 0 \quad (m \neq n)$$
$$= \|\psi_n\| \quad (m = n),$$

so that the system $\{\psi_n(r)\}$ is orthogonal, for the interval $a \leq r \leq b$.

Let us now introduce the Gram determinant G_n which is defined by the equations

$$(19.4) \quad \begin{cases} G_0 = 1, \\ G_n = \begin{vmatrix} (F_1, F_1) & \cdots & (F_n, F_1) \\ \cdot & \cdots & \cdot \\ (F_1, F_n) & \cdots & (F_n, F_n) \end{vmatrix}, (n = 1, 2, \ldots), \end{cases}$$

where the inner products are again computed for the fundamental interval $a \leqq r \leqq b$. Then, we have an orthonormal system $\{\varphi_n(r)\}$ defined by the equations

$$(19.5) \qquad \varphi_n(r) = (G_{n-1}\, G_n)^{-1/2}\, \psi_n(r), \quad (n = 1, 2, \ldots),$$

on the fundamental interval $a \leqq r \leqq b$.

20. <u>The Series Expansion of an Arbitrary Function in Terms of the Orthonormal System</u>. Let us consider now the question of expanding an arbitrary function $h(r)$ into an infinite series of the orthonormal functions $\{\varphi_n\}$, namely,

$$(20.1) \quad h(r) = \sum_{n=1}^{\infty} c_n\, \varphi_n(r)\ ,$$

where c_n are constants. Since φ_n are orthonormal, it is readily seen that

$$(20.2) \qquad c_n = (h, \varphi_n), \qquad (n = 1, 2, \ldots) .$$

If the series (20.1) is actually to represent the function $h(r)$ uniquely, the series must converge. It is common practice to consider mean square convergence for orthonormal systems. This convergence problem is extremely complex for our particular orthonormal system.

When the discussion for mean square convergence is restricted to Riemann square integrable functions, the uniform convergence of the series (20.1) and the uniqueness of the expansion coefficients (20.2) must be considered. The uniform convergence question could possibly be evaded by an enlargment of the function space under consideration to include Lebesgue square integrable functions. However, this would introduce other difficulties.

The validity of the series representation (20.1) might also be examined by the construction of a boundary value problem for the system $\{\varphi_n\}$. In this case the differential equation satisfied by $\varphi_n(r)$ is

$$(20.3) \qquad F'' + (2f' - 3/r)\,F' + \alpha[\alpha + f'' + (f')^2 - 3/r]F = 0,$$

where $f(r)$ is the monotone increasing function defined by equation (17.2), and $\alpha \equiv \alpha_n$, $(n = 1, 2, \ldots)$. One boundary condition is $F(b) = 0$ and another condition is to be specified. The coefficient of F causes a serious difficulty. The change of dependent variable $G = e^{\alpha f(r)} F$ transforms the differential equation to a Bessel's equation which has well known solutions. This gives little of the desired information about the functions $F_n(r)$ or the orthonormal system $\{\varphi_n\}$. However, it does lead to an interesting problem of a more general nature, namely, given a set of functions $h_i(x)$, $(i = 1, 2, \ldots)$, which are solutions to a Sturm-Liouville system, what restrictions are necessary, on an arbitrary set of functions $g_i(x)$, $(i, 1, 2, \ldots)$, to insure that the product set

$g_i(x)\ h_i(x)$, $(i = 1, 2, \ldots)$ retains the desirable properties of the given functions.

The above mentioned possible avenues of approach have all been investigated extensively. The problems arising in each case have been extremely complex and any one of them might constitute a suitable thesis topic. Hence, we shall restrict attention to monotone increasing functions $f(r)$, and shall assume that for such functions the necessary convergence properties are present in the associated series expansions.

21. A Series Solution for the Torsion Problem of a Semi-Infinite Shaft with a General Meridian Section. We continue our orthonormal functions $\varphi_n(r)$ of equations (19.5) throughout the meridian section of the shaft. Accordingly we let

$$\text{(21.1)} \quad \begin{cases} \psi_1(z, r) = F_1(z, r), \\ \psi_n(z, r) = \begin{vmatrix} (F_1, F_1) & (F_1, F_2) & \ldots & (F_1, F_n) \\ \vdots & \vdots & & \vdots \\ (F_{n-1}, F_1) & (F_{n-1}, F_2) & \ldots & (F_{n-1}, F_n) \\ F_1(z, r) & F_2(z, r) & \ldots & (F_n(z, r) \end{vmatrix}, \quad (n = 2, 3, \ldots), \end{cases}$$

for $0 \leqq z$ and $0 \leqq r \leqq b$, where $F_i(z, r)$, $(i = 1, 2, \ldots)$, are defined by equations (17.7), and the inner products (F_i, F_j) are defined by the equations

$$(F_i, F_j) = \int_a^b F_i(r)\ F_j(r)\ dr, \quad (i, j = 1, 2, \ldots) .$$

Then, we let

$$\text{(21.2)} \qquad \varphi_n(z, r) = (G_{n-1}G_n)^{-1/2}\ \psi_n(z, r), \quad (n = 1, 2, \ldots),$$

where G_n, $(n = 1, 2, \ldots)$ is the Gram determinant of equations (19.4). When $z = f(r)$ on C_2, we have

$$(21.3) \qquad \varphi_n(f(r), r) \equiv \varphi_n(r), \quad (n = 1, 2, \ldots),$$

for $a \leqq r \leqq b$, where $\{\varphi_n(r)\}$ is the orthonormal system of equations (19.5).

Let us now return to the boundary value problem for the torsion problem under consideration. We try for a solution in the form

$$(21.4) \qquad F(z, r) = \delta[r^4 - \sum_{n=1}^{\infty} c_n \varphi_n(z, r)],$$

where δ and c_n, $(n = 1, 2, \ldots)$ are arbitrary constants. That $F(z, r)$ is a solution of the fundamental differential equation follows because the differential equation is linear, r^4 is a solution, and $\varphi_n(z, r)$ are linear combinations of the functions $F_i(z, r)$ from (17.8) which are solutions.

It will be recalled that $F_i(z, r) = 0$ on the curve C_1, Figure 17, so the boundary condition (17.3) on this curve is satisfied. Also, $F_i(z, r)$ vanish on the straight portion of C_2, Figure 17, so the boundary condition on this line is satisfied if

$$(21.5) \qquad b^4 = k_2,$$

where k_2 is a constant related to the twisitng couple. The boundary condition on the remaining portion of C_2 is satisfied, formally, if

$$\delta(r^4 - \sum_{n=1}^{\infty} c_n \varphi_n(r)) = k_2 = \delta b^4,$$

or if

$$(21.6) \qquad \sum_{n=1}^{\infty} c_n \varphi_n(r) = r^4 - b^4, \qquad (a \leqq r \leqq b).$$

It is thus necessary that

$$c_n = (\varphi_n(r), r^4 - b^4), \quad (n = 1, 2, \ldots) .$$

As noted in Section 19, we shall assume that the infinite series of (21.6) converges sufficiently to represent the indicated function.

The constant k_2 is related to the twisitng couple M through equation (17.4). From this we obtain

$$\delta = \frac{M}{2\kappa\mu b^4},$$

and so the required formal solution to the torsion problem under consideration is, from (21.4)

$$(21.7) \quad F(z, r) = \frac{M}{2\kappa\mu b^4}[r^4 - \sum_{n=1}^{\infty} c_n \varphi_n(z, r)].$$

We note that as z becomes infinite, the solution (21.7) approaches the form

$$F = \frac{Mr^4}{2\kappa\mu b^4}$$

which is the familiar solution for the torsion of a uniform circular shaft of radius b.

In terms of rectangular cartesian coordinates, the equations (4.4) for the contravariant component of displacement are

$$u^o{}_{,z} = r^{-3} F_{,r}, \quad u^o{}_{,r} = -r^{-3} F_{,z}.$$

If we assume furthermore that termwise differentiation and integration are permissable for the series in the solution (21.7), these last equations yield formally

$$(21.8) \quad u^o = \frac{M}{2\kappa\mu b^4}[4z + r^{-1} \sum_{n=1}^{\infty} c_n(G_{n-1}G_n)^{-1/2} \sum_{m=1}^{n} C_{mn} e^{-\alpha_m t} J_1(\alpha_m r) + c]$$

where C_{mn}, $(m, n = 1, 2, \ldots)$, is the cofactor of $F_m(z, r)$ in the definitions (21.1), and the arbitrary constant c corresponds to a rigid body rotation of the shaft about its axis. From (21.8), the physical component of displacement $u = r u^{o}$ in the direction of the parametric line of x^{o} follows directly.

In terms of polar cylindrical coordinates (θ, z, r), the general equations (4.7) for the stress components become

$$(21.9) \qquad \tau_{\theta z} = u r^{-2} F_{,r} , \quad \tau_{\theta r} = - r^{-2} F_{,z} ,$$

where $\tau_{\theta z}$ and $\tau_{\theta r}$ are physical components of stress. A direct calculation with $F(z, r)$ from the solution (21.7) yields formally

$$(21.10) \quad \begin{aligned} \tau_{\theta z} &= \frac{M}{2 \varkappa b^4} [4r - \sum_{n=1}^{\infty} c_n (G_{n-1} G_n)^{-1/2} \sum_{m=1}^{n} C_{mn} \alpha_m e^{-\alpha_m z} J_1(\alpha_m r)], \\ \tau_{\theta r} &= \frac{M}{2 \varkappa b^4} \sum_{n=1}^{\infty} c_n (G_{n-1} G_n)^{-1/2} \sum_{m=1}^{n} C_{mn} \alpha_m e^{-\alpha_m z} J_2(\alpha_m r), \end{aligned}$$

where C_{mn} is defined as for equation (21.8).

It is of interest to look at a general term of the series appearing above in equations (21.5), (21.7), (21.8) and (21.10). For this, we let β_{jn} $(j = 1, 2, 3, 4, 5)$ denote the resultant of the appropriate operation on $F_n(z, r)$, $(n = 1, 2, \ldots)$, as follows:

$$(21.11) \quad \begin{cases} \beta_{1n} = e^{-\alpha_n f(r)} r^2 J_2(\alpha_n r) & \text{for equation (21.5),} \\ \beta_{2n} = e^{-\alpha_n z} r^2 J_2(\alpha_n r) & \text{for the solution (21.7),} \\ \beta_{3n} = \frac{1}{r} e^{-\alpha_n z} J_1(\alpha_n r) & \text{for the displacement (21.8),} \\ \beta_{4n} = \alpha_n e^{-\alpha_n z} J_1(\alpha_n r) & \text{for } \tau_{\theta z}, \text{(21.10),} \\ \beta_{5n} = \alpha_n e^{-\alpha_n z} J_2(\alpha_n r) & \text{for } \tau_{\theta r}, \text{(21.10).} \end{cases}$$

Then, we recall the constants c_n from equations (21.6) in the form

$$c_n = (G_{n-1}G_n)^{-\frac{1}{2}} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ a_1 & \cdots & a_n \end{vmatrix}, \quad (n = 1, 2, \ldots),$$

where

$$a_{ij} = (F_i, F_j), \quad (i = 1, 2, \ldots, n - 1; \quad j = 1, 2, \ldots n),$$

$$a_i = (F_i, r^4 - b^4), \quad (i = 1, 2, \ldots, n).$$

Thus, a general nth term for the series mentioned above has the form

$$(21.12) \quad \frac{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ a_1 & \cdots & a_n \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1,n-1} \\ \cdot & \cdots & \cdot \\ a_{n-1,1} & \cdots & a_{n-1\ n-1} \end{vmatrix}} \; \frac{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ \beta_{j1} & \cdots & \beta_{jn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}}, \quad (j = 1,2,3,4,5; n=1,2,\ldots).$$

It thus appears that analytical use of the above series for computational purposes is likely to be involved.

BIBLIOGRAPHY

1. I.S. Sokolnikoff, Mathematical Theory of Elasticity, second Edition, McGraw-Hill, New York, 1956, pp. 89 - 90.

2. I.S. Sokolnikoff, Mathematical Theory of Elasticity, second Edition, McGraw-Hill, New York, 1956, p. 86.

3. I.S. Sokolnikoff, Mathematical Theory of Elasticity, second Edition, McGraw-Hill, New York, 1956, pp. 190 - 193.

4. J.L. Synge, The problem of Saint Venant for a cylinder with free sides, Quarterly of Applied Mathematics, Vol. 2, 1945, pp. 307 - 317.

5. T.J. Higgins, Stress analysis of shafting exemplified by Saint Venant's torsion problem, Experimental Stress Analysis, Vol. 3, No. 1, 1945, pp. 94 - 101.

6. J.H. Michell, The uniform torsion and flexure of incomplete tores, with applications to helical springs, Proceedings of the London Mathematical Society, Vol. 31, 1900, pp. 130 - 136.

7. A. Föppl, Über die Torsion von runden Stäben nich veränderlichem Durchmesser, Sitzungsherichte der mathematische - physihalischen Klasse; Akademie der Wissenschaften, Muchen, Vol. 35, 1905, pp. 249 - 262.

8. A. Willers, Die Torsion eines Rotationskorpers um sein Achse, Zeitschrift für Mathematik und Physike, Vol. 55, 1907, pp. 225 - 263.

9. J. Larmor, "The influence of flaws and air cavities on strength of materials, The London, Edinburg and Dublin Philosophical Magazine and Journal of Science, Vol. 33 - 5th series, 1892, pp. 70 - 78.

10. A. Timpe, Die Torsion von Umdrehungskorpen, Mathematische Annalen, Vol. 71, 1912, pp. 480 - 509.

11. A. Timpe, Torsionfreie Achsensymetrische Deformation von Umdrehungenskörpen und ihre Inversion, Zeitschrift für Angewandte Mathematiks und Mechaniks, Vol. 28, 1948, pp. 161-166.

12. A. Timpe, Spannungsfunkion für die von Kugel-und Kegelflächen begrenzten Körper und Kuppelproblem, Zeitschrift für Angewandte Mathematiks und Mechaniks, Vol. 30, 1950, pp. 50-56.

13. A. Timpe, Brüchenlösungen beim Problem der achiensymmetrischen Torsion, Zeitschrift für Angewandte Mathematiks und Mechaniks, Vol. 32, 1952, pp. 226 - 227.

14. G.E. Hay, The Torsion of shafts of circular cross sections, unpublished.

15. E. Mellan, Ein Beitraq zur Torsion von Rotationskorpern, Technische Blätter (Prague), Vol. 52, 1920, pp. 417 - 419, 427 - 429.

16. T. Pöschl, Bisheriqe Lösungen des Torsionsproblems für Drehkörper, Zeitschrift für Angewandte Mathematics und Mechaniks, Vol. 2, 1922, pp. 137 - 147.

17. H. Neuber, Theory of Notch Stresses, J.W. Edwards, Ann Arbor, Michigan, 1946, pp. 124 - 127.

18. H. Neuber, Theory of Notch Stresses, J.W. Edwards, Ann Arbor, Michigan, 1946, pp. 99 - 101.

19. R. Sonntog, Formula to compute the maximum shear stress due to pure torsion at the transition points of stepped shafts, Zeitschrift für Angewandte Mathematics und Mechaniks, Vol. 34, 1954, pp. 19 - 36.

20. B.A. Sokolow, Problems of elastic torsion of bars, Appl. Math. Mech., (Akad. Nauk. SSSR Prikl. Mat. Mech.), Vol. 8, 1944, pp. 468 - 474.

21. M. Abbassi, Torsion of circular shafts of variable diameter, Journal of Applied Mechanics, Vol. 22, 1955, 530 - 532.

22. M. Abbassi, Simple solutions of Saint Venant Torsion Problem using Tchebycheff Poloynomials, Quarterly of Applied Mathematics, Vol. 14, 1956, pp. 75 - 81.

23. P.P. Chattarji, A note on the torsion of circular shafts of variable diameter. Journal of Applied Mechanics, Vol. 79, 1957, pp. 477 - 478.

24. H.J. Reissner and G.J. Wennagil, Torsion of noncylindrical shafts of circular cross section, Journal of Applied Mechanics, Vol. 17, 1950, pp. 275 - 282.

25. P. Brousse, Etude d'equations aux derivees partielles rencontrées dans la theory des phenomènes de torsion, Publ. Sci. Tech. Ministere de l'Air, no 257, Paris, 1952.

26. Chih Bing Ling, Torsion of a circular cylinder having a spherical cavity, Quarterly of Applied Mathematics, Vol. 10, 1952, pp. 149 - 156.

27. S.C. Das, On the effect of a small spherical cavity in a semi infinite solid under stresses produced by a couple on the plane boundary, Bulletin Calcutta Mathematical Society, Vol. 45, 1953, pp. 89 - 93.

28. H. Poritsky, Stress fields of axially symmetric shafts in torsion and related fields, Proc. Symposia Appl. Math., Vol. 3, 1950, pp. 163 - 186, McGraw Hill Book Co., New York.

29. G. Weiss and L.E. Payne, Torsion of a shaft with a toroidal Cavity, Journal of Applied Physics, Vol. 25, 1954, pp 1321 - 1328.

30. G.N. Watson, A treatise on the theory of Bessel Functions, second edition, The University Press, Cambridge, 1944, pp. 15 and 62.

31. G.N. Watson, A treatise on the theory of Bessel Functions, second edition, The University Press, Cambridge, 1944, p. 214.

32. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 1, Interscience Publishers Inc., New York, 1953, p. 49.

33. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 1, Interscience Publishers Inc., New York, 1953, p. 505.

34. Bateman Manuscript Project, Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York, 1953, p. 154.

35. J.M.H. Olmsted, Completeness and Parseval's Equation, The American Mathematical Monthly, Vol. 65, 1958, pp. 343 - 345.

36. W. Rudin, Principles of Mathematical Analysis, McGraw-Hill Book Co., New York, 1953, p. 214.