# Optimal Production Policies for Multistage Systems with Setup Costs and Uncertain Capacities

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**Abstract**

The increased complexity of modern manufacturing has lead to uncertainties in production processes. Factors, such as unplanned machine maintenance, tool unavailability and complex process adjustments, make it difficult to maintain a predictable level of output. To be effective, an appropriate production model must incorporate these uncertainties into the representation of the production process. This paper considers a one-time production of an application-specific product which must follow a fixed routing through the manufacturing system. The flow of items can be modeled as a multi-stage serial production line. The productive capacity is uncertain at each stage and the decision to produce at any stage incurs a significant setup cost. Semifinished products have little value and inability to satisfy the demand incurs a penalty for each unit of unmet demand. We show that the optimal production policy for this system can be characterized by two critical numbers, which can be computed apriori based on the cost parameters and distributional information for all downstream stages. Sensitivity of the critical numbers is also explored.

## 1 Introduction

The increased sophistication of modern manufacturing processes in many high-tech industries has lead to an increase in internal uncertainties in these manufacturing systems. The use of state-of-the-art technology, intricate equipment, complex tooling requirements, involved process control, reliance on specialized operator skills and greater adherence to performance specifications, are all factors that make it difficult to maintain a predictable level of output. Often, the newer, more profitable products face even greater uncertainty because the organization has not yet accumulated enough learning experience with the equipment and processes to reduce the variance. Nevertheless, production needs to be carried out in an economic manner, despite the inherent variabilities in the system.

We present a model for planning production in a multi-stage serial production system where the aggregate productive capacity at each stage is uncertain. That is, the output quantity at any stage is the minimum of input quantity and the realized productive capacity, which is random. Production is carried out to satisfy an uncertain one-time demand for the final product. Each unit of unsatisfied demand incurs a penalty, so does the the disposal of any unused material. A decision to produce at any stage incurs a setup cost plus a processing cost for each unit produced. Due to limited time until shipment, there is only one opportunity to produce at any stage, i.e., a production shortfall can not be compensated by another production run. Production control must decide, for each stage, how much to produce *after* the output from the immediately preceding stage becomes available. One must take into account capacity uncertainties at all downstream stages, together with the demand uncertainties and costs, to arrive at an economic production decision. Our analysis shows that the optimal production policy for each stage can be characterized simply by two critical numbers. If the available input exceeds the *lower* critical number, one tries to produce as much as possible, but no more than the *upper* critical number. If, on the other hand, available input is less than the lower critical number, one chooses not to produce. The interrelationship among critical numbers and their sensitivity to various cost parameters as well as capacities and demand distributions are also explored.

There are two fundamentally different models in the literature to represent internal uncertainties. *Yield* models focus on output loss due to process imperfections while *capacity* models deal with production loss due to resource unavailability. In random yield models, one identifies the defective units *after* processing the entire input quantity and incurring the production cost. In uncertain capacity models, one may not be able to process the entire input material due to resource constraints; unused input incurs no production cost. To further illustrate the differences between the two models, note that increasing the input batch size always increases the expected output in an uncertain yield model. This is not necessarily true for the uncertain capacity model where, no matter how large the input batch size, the expected output can not exceed the average productive capacity. Henig and Gerchak (1990) propose a general representation of production output which can be used to model yield as well as capacity uncertainty.

Yield models have been studied rather extensively in recent years. Interested readers are referred to Yano and Lee (1991) for an excellent survey. We discuss here only those works which are closely related to the present paper. Lee and Yano (1988) analyze a single-period, single-product, serial production system, similar to the one considered here, but without set-up cost. The demand is known and the yield at each stage is a random multiple of input batch size (referred as stochastically proportional yield by Henig and Gerchak, 1990). Lee and Yano (1988) show that the optimal production policy for each stage can be characterized by a single critical number representing the target input quantity. The policy stipulates that one should input the target quantity, if enough is available; otherwise one should input whatever is available. An identical result for the random capacity case can be obtained from our model by setting the setup costs at all stages to zero.

A number of generalizations have been attempted for the Lee and Yano (1988) model. Yano (1986a) shows that structural results similar to those in Lee and Yano (1988) are valid even when the demand is random. Wein (1992) allows for the rework of defective items at a cost. She shows that the optimal production/rework policy can be characterized by two critical numbers. Barad and Braha (1991) allow for the procurement of semifinished items in a multi-stage binomial yield model. They establish the optimality of a two-critical-number policy where the second critical number specifies a procurement target for semifinished items. Yano (1986b) presents an extension of the Lee and Yano (1988) model by incorporating a setup cost at each stage of production. For a single-stage problem, she demonstrates that a two-critical-number policy, similar to that proposed in this paper, is optimal. She also demonstrates the optimality of a similar policy for a two-stage system, but only under certain restrictive conditions. The form of the optimal policy for general serial production systems with random yield and setup costs remains an open research problem worthy of investigation. This paper addresses the same problem for the case where production stages are subjected to, not yield, but capacity uncertainties.

The analysis of Lee and Yano (1988) model for the multi-period scenario is quite involved and is unlikely to yield a structurally simple policy. In fact, the dynamic problem is quite complex even for a single-stage problem, as shown by Gerchak, Vickson and Parlar (1988). As a result, recent research efforts have focussed on the study of such systems under sub-optimal but tractable operating policies. Tang (1990) develops operating characteristics for Lee and Yano (1988) model under partial restoration rule and provides interesting insights about the impact of uncertainty. Denardo and Lee (1991) extend Tang's approach to allow rework and unreliable machines. Gong and Matsuo (1990) formulate a linear control problem that tries to stabilize work-in-process and to smooth production. The development of models for systems with setup costs is still lacking in the literature.

The aggregate capacity model used in this paper captures parsimoniously the cumulative impact of varying availabilities of numerous productive resources. Hopp, Spearman and Duenyas (1993) have used this representation to model the total amount of regular-time capacity available in any period. They demonstrate that the distribution of aggregate capacity plays a central role in setting production quotas for a pull manufacturing system. Ciarallo, Akella and Morton (1994) consider finite and infinite-horizon models for a single-stage production system with uncertain capacity and uncertain demand. They show that a single critical number, which represents the order-up-to point, is sufficient to characterize the optimal policy in a multi-period setting. This is in contrast to stochastically proportional yield models where, under similar circumstances, the optimal policy is known to have no structurally simple form (see Gerchak et al. 1988).

The mathematical instrument used to model the uncertainty in aggregate capacity here is identical to those in Ciarallo, Akella and Morton (1994). In contrast to their single-stage, multi-period model, we analyze a multi-stage, single-period model with setup cost at each stage of production. In Ciarallo et al., order-up-to policies are derived from the analysis of a cost function which is quasi-convex. In spite of this non-convexity, the cost-to-go is shown to be convex. The presence of setup cost in our model destroys the quasi-convex unimodal structure of the cost function so effectively utilized by Ciarallo et al. in their analysis. We derive the two-critical-number policy from the analysis of a cost function which is neither unimodal nor quasi-convex. In fact, the nature of this cost function changes from concave-increasing, to convex-decreasing, convex-increasing, simply increasing, and finally to concave-increasing.

The contribution of this paper is threefold. First, the optimality of a simple two-critical-number policy is established for an important class of manufacturing problems. Considering the unusual behavior of the cost function involved, our proof of optimality is somewhat novel. Second, the sensitivity of the critical numbers to cost parameters is explored, which reveal many interesting structural properties of the system. Finally, the impact of uncertainties on the production line is studied by changing the distributions of demand and capacities. The interaction between internal uncertainties and demand uncertainty is also revealed by the recursive equation used for the computation of critical numbers.

This paper is organized as follows. Section 2 presents a mathematical description of the problem and its formulation as a dynamic optimization model. Section 3 analyzes a single-stage problem and shows that a two-critical-number policy is optimal. This result is extended in Section 4 to a multi-stage problem. The sensitivity of the critical numbers to cost parameters, as well as to capacity and demand uncertainties, is explored in Sections 5 and 6, respectively. A numerical example is presented in Section 7. The paper concludes with some final remarks in Section 8.

## 2 Problem Description and Formulation

Consider an $N$-stage serial production system shown in Figure 1. Let the stages be numbered such that the final stage of production is denoted as stage 1 while the stage of production to be performed first is denoted as stage $N$. The production is aimed towards satisfying a single uncertain requirement or demand for a product at the final stage. Let $Z$ be the demand random variable with c.d.f. $Q(z)$ and p.d.f. $q(z)$.

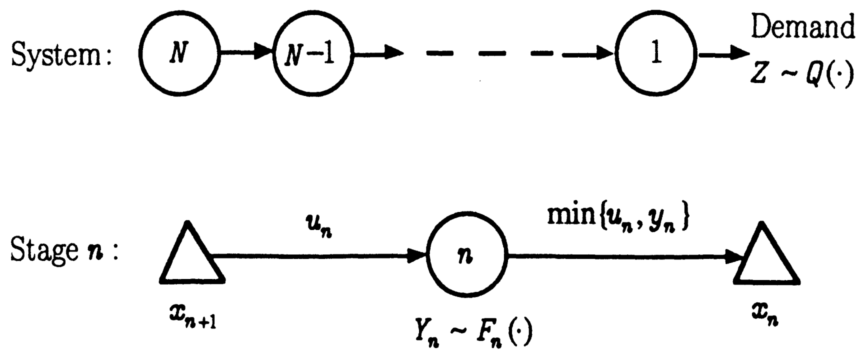


Figure 1: A Serial Production System

At any stage, the productive capacity may be uncertain due to a number of factors such as machine failures, tool unavailability, time spent in process, parameter readjustments, etc. Let the random variable $Y_n$ represent the available productive capacity at stage $n$ with c.d.f. $F_n(y_n)$ and p.d.f. $f_n(y_n)$. We assume that all the random variables are mutually independent and their c.d.f.'s are continuous and twice differentiable.

The problem is to determine a *planned* production quantity, $u_n$, at each stage of production such that the expected total cost is minimized. The *actual* production output from a stage may be less than the planned production quantity. This will occur whenever the capacity realization, $y_n$, falls below the production target, $u_n$. Otherwise, the complete production target is accomplished successfully. In general, the production output of stage $n$, $x_n$, is given by $min\{u_n, y_n\}$.

We assume that the initial inventory of all semi-finished items is zero; the analysis can be extended to accommodate positive initial inventories. The planned production quantity at stage $n$ is obviously constrained by the actual production output from stage $n + 1$, which, in turn, depends upon the planned production quantity at stage $n + 1$, etc. That is, $u_n \leq x_{n+1} = min\{u_{n+1}, y_{n+1}\}$. We solve this problem dynamically by delaying the specification of the planned input quantity at a stage until the production output from the stage before becomes known. In other words, $u_N, u_{N-1}, \ldots, u_1$ must be determined sequentially after receiving the output from the stage before.

The following costs are considered in this model. The decision to produce at stage $n$ incurs an "out-of-pocket" setup cost $K_n$ independent of the production quantity. In addition, each unit of item actually produced at stage $n$ incurs a cost $w_n$. Each unit of unsatisfied demand incurs a penalty $\pi$. There is a cost $h_n$ for disposing a unit of item processed at stage $n$

but not used by stage $n-1$. The cost of disposing leftover raw material is represented by $h_{N+1}$. Since this model is especially applicable to one-time production of unique products, it is assumed that semi-finished items have little salvage value. Disposal of raw material and finished product, on the other hand, may bring a net cash inflow; this is modeled by allowing $h_1$ and $h_{N+1}$ to be negative. All other parameters and costs are assumed to be non-negative.

The following two cost conditions are necessary to ensure that it is profitable to produce and that production is motivated only by the desire to satisfy the demand.

CONDITION 1 $h_2 + \pi > w_1$.

If Condition 1 did not hold, one would simply dispose the input material at cost $h_2$ and incur a penalty $\pi$ for not meeting the demand, rather than process an item at a higher cost $w_1$. Clearly, this makes it unprofitable to produce anything at the final stage.

CONDITION 2 $w_n + h_n > h_{n+1}$ *for* $n = 1, \ldots, N$.

This states that it is less expensive to dispose an item at one stage than to process it and then dispose it at the next stage.

Let $C_n(x_{n+1})$ be the expected cost of operating an $n$-stage system with available input $x_{n+1}$, assuming that the best input decision is used at stage $n$ through stage 1. Then, $C_N(x_{N+1})$ represents the minimum expected cost to operate the whole system, where the available raw material is $x_{N+1}$ at stage $N$. A dynamic programming formulation for the problem can now be given as

$$C_0(x_1) \equiv E_Z\Big\{h_1 \max\{0, x_1 - Z\} + \pi \max\{0, Z - x_1\}\Big\}, \tag{1}$$

and

$$\begin{aligned} C_n(x_{n+1}) = \min_{0 \le u_n \le x_{n+1}} & E_{Y_n}\Big\{h_{n+1}(x_{n+1} - \min\{u_n, Y_n\}) + K_n \delta(u_n) \\ & + w_n \min\{u_n, Y_n\} + C_{n-1}(\min\{u_n, Y_n\})\Big\}, \quad n = 1, \ldots, N \end{aligned} \tag{2}$$

where

$$\delta(u_n) = \begin{cases} 1 & \text{if } u_n > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $u_n^*$ be the optimal value of $u_n$. In our analysis we will often utilize the following representation for $C_n(x_{n+1})$

$$C_n(x_{n+1}) = \min_{0 \le u_n \le x_{n+1}} \Big\{h_{n+1}x_{n+1} + K_n \delta(u_n) + \gamma_n(u_n)\Big\} \quad n = 1, \ldots, N, \tag{3}$$

where $\gamma_n(u_n)$ is a function only of $u_n$ and is given by

$$\begin{aligned} \gamma_n(u_n) = & \int_0^{u_n} [(w_n - h_{n+1})y_n + C_{n-1}(y_n)]\, dF_n(y_n) \\ & + \bar{F}_n(u_n)[(w_n - h_{n+1})u_n + C_{n-1}(u_n)], \end{aligned} \tag{4}$$

where $\bar{F}_n(u_n) = 1 - F_n(u_n)$.

The function $\gamma_n(u_n)$ plays a central role in defining the structure of the optimal policy for this problem. The following observation follows directly from (3).

**Lemma 1** *If $\gamma_n(0) < \gamma_n(u_n) + K_n$ for all $u_n$, then it is not worthwhile to produce at stage $n$. Thus, $u_n^* = 0$.*

The following lemma points to another intuitive result. If no input material is available at stage $n$, then the minimum expected cost for stage $n$ through stage 1 is simply the expected penalty for not meeting the demand.

**Lemma 2** $C_n(0) \;=\; \gamma_n(0) \;=\; \pi E[Z]$ *for all $n$.*

*Proof:* From (3), $C_n(0) = \gamma_n(0)$ and from (4), $\gamma_n(0) = C_{n-1}(0)$. The result follows by induction, since $C_0(0) = \pi E[Z]$ from (1). ■

These observations will be utilized during the analysis in coming sections.

## 3 The Single-Stage Problem

In this section we analyze the single-stage problem for the model introduced in Section 2. This analysis will provide important insights in understanding the multi-stage problem. We begin by rewriting (3) as

$$C_1(x_2) \;=\; \min_{0 \le u_1 \le x_2} \left\{ h_2 x_2 + K_1 \delta(u_1) + \gamma_1(u_1) \right\}, \tag{5}$$

where $\gamma_1(u_1)$ is obtained by substituting (1) into (4),

$$\begin{aligned} \gamma_1(u_1) \;=\; & \int_0^{u_1} \left[ (w_1 - h_2) y_1 + h_1 \int_0^{y_1} (y_1 - z)\, dQ(z) + \pi \int_{y_1}^{\infty} (z - y_1)\, dQ(z) \right] dF_1(y_1) \\ & + \bar{F}_1(u_1) \left[ (w_1 - h_2) u_1 + h_1 \int_0^{u_1} (u_1 - z)\, dQ(z) + \pi \int_{u_1}^{\infty} (z - u_1)\, dQ(z) \right]. \end{aligned}$$

We first investigate the nature of $\gamma_1(\cdot)$ since it plays a central role in the minimization in (5). The first two derivatives of $\gamma_1(u_1)$ are given by

$$\gamma_1'(u_1) \;=\; \frac{d\gamma_1}{du_1} \;=\; \bar{F}_1(u_1) g_1(u_1), \tag{6}$$

$$\gamma_1''(u_1) \;=\; \frac{d^2\gamma_1}{du_1^2} \;=\; \bar{F}_1(u_1)(h_1 + \pi) q(u_1) - f_1(u_1) g_1(u_1), \tag{7}$$

$$\text{where} \quad g_1(u_1) \;=\; (h_1 + \pi) Q(u_1) + w_1 - h_2 - \pi. \tag{8}$$

$g_1(u_1)$ is non-decreasing since Conditions 1 and 2 together imply that $(h_1 + \pi) > 0$. Define $S_1$ such that $g_1(S_1) = 0$, i.e.,

$$S_1 \;=\; Q^{-1}\left( \frac{h_2 + \pi - w_1}{h_1 + \pi} \right). \tag{9}$$

$S_1$ is non-negative and finite because $(h_2 + \pi - w_1) > 0$ and $(h_2 + \pi - w_1) < (h_1 + \pi)$ from Conditions 1 and 2. Then, $g_1(u_1)$ is negative in $(0, S_1)$ and positive in $(S_1, \infty)$. The nature of $\gamma_1(\cdot)$ can now be characterized using (6) and (7):

1. For $u_1 \in (0, S_1)$, $\gamma_1'(u_1) \leq 0$ and $\gamma_1''(u_1) \geq 0$, hence $\gamma_1(u_1)$ is decreasing and convex in this region.

2. For $u_1 \in (S_1, \infty)$, $\gamma_1'(u_1) \geq 0$ and hence $\gamma_1(u_1)$ is increasing in this region. Nothing further can be concluded about the nature of $\gamma_1(u_1)$ since $\gamma_1''(u_1)$ may alternate its sign in different sub-intervals.

From these observations, it is clear that $\gamma_1(u_1)$ attains its global minimum at $u_1 = S_1$. The behavior of $\gamma_1(u_1)$ is shown graphically in Figure 2.

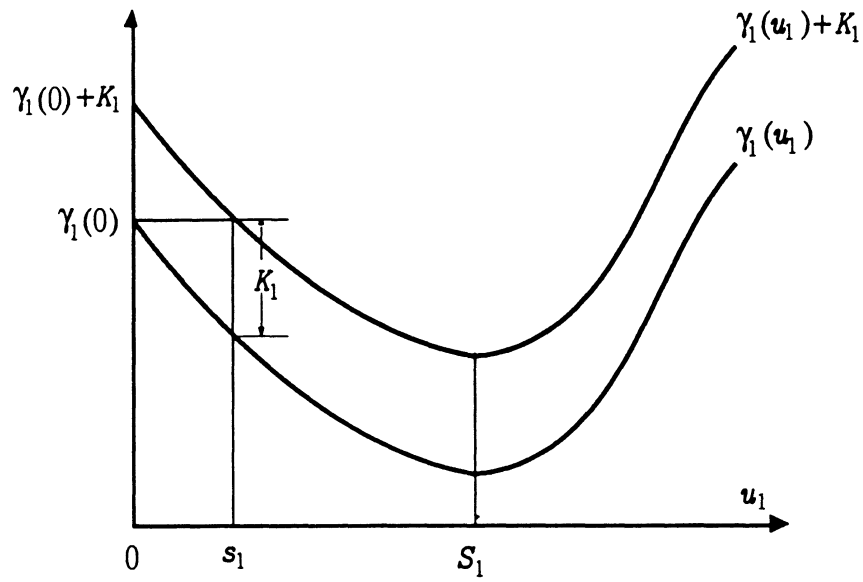


Figure 2: The form of $\gamma_1(u_1)$

Whenever $S_1 = 0$, the problem is trivial since it is optimal not to produce. Note that from (9), $S_1 > 0$ iff

$$\begin{aligned} h_2 + \pi - w_1 &> (h_1 + \pi) Q(0), \\ \text{or} \quad w_1 + h_1 - h_2 &< (h_1 + \pi) Pr\{Z > 0\}. \end{aligned}$$

That is, if the effective cost of processing a unit, $(w_1 + h_1 - h_2)$, exceeds the marginal saving derived from making a unit available, one is better off not processing anything.

Now consider the minimization in (5), in particular the term $(K_1 \delta(u_1) + \gamma_1(u_1))$. The nature of $(K_1 + \gamma_1(u_1))$ is identical to that of $\gamma_1(u_1)$ and it attains its global minimum at $S_1$ with value $K_1 + \gamma_1(S_1)$. If $\gamma_1(0) < K_1 + \gamma_1(S_1)$, it is not worthwhile to produce, as per Lemma 1. An excessively high setup cost makes the problem trivial, since one always chooses not to produce. If, on the other hand, $\gamma_1(0) > K_1 + \gamma_1(S_1)$, since $\gamma_1(u_1)$ is decreasing in region $(0, S_1)$, there exists a unique $s_1 \in (0, S_1)$ such that

$$\gamma_1(0) = K_1 + \gamma_1(s_1). \tag{10}$$

It follows (see Figure 2) from the definition of $s_1$ and the decreasing nature of $\gamma_1(u_1)$ over $(0, S_1)$ that

$$\gamma_1(0) \leq K_1 + \gamma_1(u_1) \quad \text{for } u_1 \leq s_1, \tag{11}$$

$$\text{and} \quad \gamma_1(0) > K_1 + \gamma_1(u_1) \quad \text{for } s_1 < u_1 \leq S_1. \tag{12}$$

Based upon the available input material $x_2$, the optimal policy can now be characterized in terms of the two critical numbers $s_1$ and $S_1$. For $x_2 \leq s_1$, it is not worthwhile to produce because the setup cost, $K_1$, will offset the expected savings, $(\gamma_1(0) - \gamma_1(u_1))$, derived from production. This follows from (11), since $u_1 \leq x_2 \leq s_1$. For $s_1 < x_2 \leq S_1$, the advantage gained by producing, $(\gamma_1(0) - \gamma_1(u_1))$, can offset the setup cost, $K_1$, *provided* one plans to produce more than $s_1$. This follows from (12). In fact, since $\gamma_1(u_1)$ is decreasing in range $(s_1, S_1)$, it pays to set the production target as high as possible; the optimal policy simply inputs all the available material. Finally, for $x_2 > S_1$, one sets the production target at $S_1$, the point where $(K_1 + \gamma_1(u_1))$ attains its global minimum.

The optimal policy for stage 1 and the nature of $\gamma_1(u_1)$ are now summarized in the following theorem.

THEOREM 1 *For the model stated in Section 2, if it is worthwhile to produce, then*

1. *the optimal policy for stage 1 is*

$$u_1^*(x_2) = \begin{cases} 0 & \text{if } x_2 \in (0, s_1) \\ x_2 & \text{if } x_2 \in (s_1, S_1) \\ S_1 & \text{if } x_2 \in (S_1, \infty), \end{cases} \tag{13}$$

*where critical numbers $S_1$ and $s_1$ are solutions to equations (9) and (10) respectively and satisfy the relationship, $0 \leq s_1 \leq S_1 < \infty$;*

2. $\gamma_1(u_1)$ *is* $\begin{cases} \text{decreasing and convex} & \text{in range } (0, S_1) \\ \text{increasing} & \text{in range } (S_1, \infty). \end{cases}$

The upper critical number, $S_1$ is the production target, i.e., the largest quantity one would ever process at stage 1. The lower critical number, $s_1$, represents the smallest input quantity one is willing to process at stage 1. Note that $S_1$, given by (9), is the same newsboy solution one would expect if the problem were formulated without any capacity constraint.

This observation is consistent with that in Ciarallo et al. regarding the order-up-to point in a single-stage single-period model with capacity uncertainty. The lower critical number, $s_1$, however, *does* depend on the capacity distribution. In effect, given an input quantity, the decision as to whether one should produce is intrinsically linked to capacity uncertainty. However, the production target is independent of capacity. Once a decision is made to produce, one simply hopes that enough capacity will be available

The cost-to-go, $C_1(x_2)$, is obtained by substituting $u_1^*$ from (13) into (5)

$$C_1(x_2) = \begin{cases} h_2x_2 + \gamma_1(0) & \text{if } x_2 \in (0, s_1) \\ h_2x_2 + K_1 + \gamma_1(x_2) & \text{if } x_2 \in (s_1, S_1) \\ h_2x_2 + K_1 + \gamma_1(S_1) & \text{if } x_2 \in (S_1, \infty). \end{cases}$$

One can easily show that, unlike $C_0(\cdot)$, $C_1(\cdot)$ is not convex. As a result, the mathematical structure of the multi-stage problem presented in the next section is quite different than that for a single stage problem.

# 4 The Multi-Stage Problem

The analysis of the single-stage problem, presented in the last section, was considerably simplified because the terminal cost-to-go, $C_0(x_1)$, is convex. This is no longer true for the multi-stage problem that we analyze in this section. The key results of this paper are contained in

**Theorem 2** *For the model stated in Section 2, if it is worthwhile to produce, then*

1. *the optimal policy for stage n is*

$$u_n^*(x_{n+1}) = \begin{cases} 0 & \text{if } x_{n+1} \in (0, s_n) \\ x_{n+1} & \text{if } x_{n+1} \in (s_n, S_n) \\ S_n & \text{if } x_{n+1} \in (S_n, \infty), \end{cases}$$

   *where critical numbers $s_n$ and $S_n$ are solutions to equations*

$$\gamma_n(s_n) + K_n = \gamma_n(0), \tag{14}$$
$$\text{and} \quad \gamma_n'(S_n) = 0,$$

   *respectively and satisfy the relationship*

$$0 \le s_{n-1} \le s_n \le S_n \le S_{n-1} \le \infty;$$

2. $$\gamma_n(\cdot) \text{ is } \begin{cases} \text{increasing and concave} & \text{in range } (0, s_{n-1}) \\ \text{decreasing and convex} & \text{in range } (s_{n-1}, S_n) \\ \text{increasing} & \text{in range } (S_n, S_{n-1}) \\ \text{increasing and concave} & \text{in range } (S_{n-1}, \infty). \end{cases}$$

*Proof:* The proof will be by induction on $n$. Define $s_0 \equiv 0$ and $S_0 \equiv \infty$. Then, for stage 1 all the properties are true from Theorem 1. To prove that these properties hold for the general case, suppose that Theorem 2 is true for stage $n-1$. Then, the following properties must hold:

$$u^*_{n-1}(x_n) = \begin{cases} 0 & \text{if } x_n \in (0, s_{n-1}) \\ x_n & \text{if } x_n \in (s_{n-1}, S_{n-1}) \\ S_{n-1} & \text{if } x_n \in (S_{n-1}, \infty), \end{cases} \tag{15}$$

where critical numbers $s_{n-1}$ and $S_{n-1}$ satisfy

$$\gamma_{n-1}(s_{n-1}) + K_{n-1} = \gamma_{n-1}(0), \tag{16}$$

$$\gamma'_{n-1}(S_{n-1}) = 0, \tag{17}$$

$$s_{n-2} \le s_{n-1} \le S_{n-1} \le S_{n-2}, \tag{18}$$

and $$\gamma_{n-1}(\cdot) \text{ is } \begin{cases} \text{increasing and concave} & \text{in range } (0, s_{n-2}) \\ \text{decreasing and convex} & \text{in range } (s_{n-2}, S_{n-1}) \\ \text{increasing} & \text{in range } (S_{n-1}, S_{n-2}) \\ \text{increasing and concave} & \text{in range } (S_{n-2}, \infty). \end{cases} \tag{19}$$

Figure 3(a) illustrates the nature of $\gamma_{n-1}(u_{n-1})$ and the relationship among critical numbers as indicated in the induction hypotheses (16)–(19).

Since the nature of $\gamma_n(\cdot)$ plays a fundamental role in determining the form of the optimal policy, we first explore its behavior. To this end, we first differentiate $\gamma_n(u_n)$ in (4) to get

$$\gamma'_n(u_n) = \bar{F}_n(u_n)[w_n - h_{n+1} + C'_{n-1}(u_n)]. \tag{20}$$

The term $C'_{n-1}(\cdot)$ can be obtained by first substituting $u^*_{n-1}$ from (15) into (3) to get

$$C_{n-1}(x_n) = \begin{cases} h_n x_n + \gamma_{n-1}(0) & \text{if } x_n \in (0, s_{n-1}) \\ h_n x_n + K_{n-1} + \gamma_{n-1}(x_n) & \text{if } x_n \in (s_{n-1}, S_{n-1}) \\ h_n x_n + K_{n-1} + \gamma_{n-1}(S_{n-1}) & \text{if } x_n \in (S_{n-1}, \infty), \end{cases} \tag{21}$$

which can then be differentiated to yield

$$C'_{n-1}(x_n) = \begin{cases} \gamma'_{n-1}(x_n) + h_n & \text{if } x_n \in (s_{n-1}, S_{n-1}) \\ h_n & \text{otherwise.} \end{cases}$$

$\gamma'_n(u_n)$ can now be written as

$$\gamma'_n(u_n) = \begin{cases} \bar{F}_n(u_n) g_n(u_n) & \text{if } u_n \in (s_{n-1}, S_{n-1}) \\ \bar{F}_n(u_n)[w_n + h_n - h_{n+1}] & \text{otherwise,} \end{cases} \tag{22}$$

where $$g_n(u_n) = w_n + h_n - h_{n+1} + \gamma'_{n-1}(u_n). \tag{23}$$

Differentiating $\gamma'_n(u_n)$, one readily obtains

$$\gamma''_n(u_n) = \begin{cases} \bar{F}_n(u_n)\gamma''_{n-1}(u_n) - f_n(u_n) g_n(u_n) & \text{if } u_n \in (s_{n-1}, S_{n-1}) \\ -f_n(u_n)[w_n + h_n - h_{n+1}] & \text{otherwise.} \end{cases} \tag{24}$$

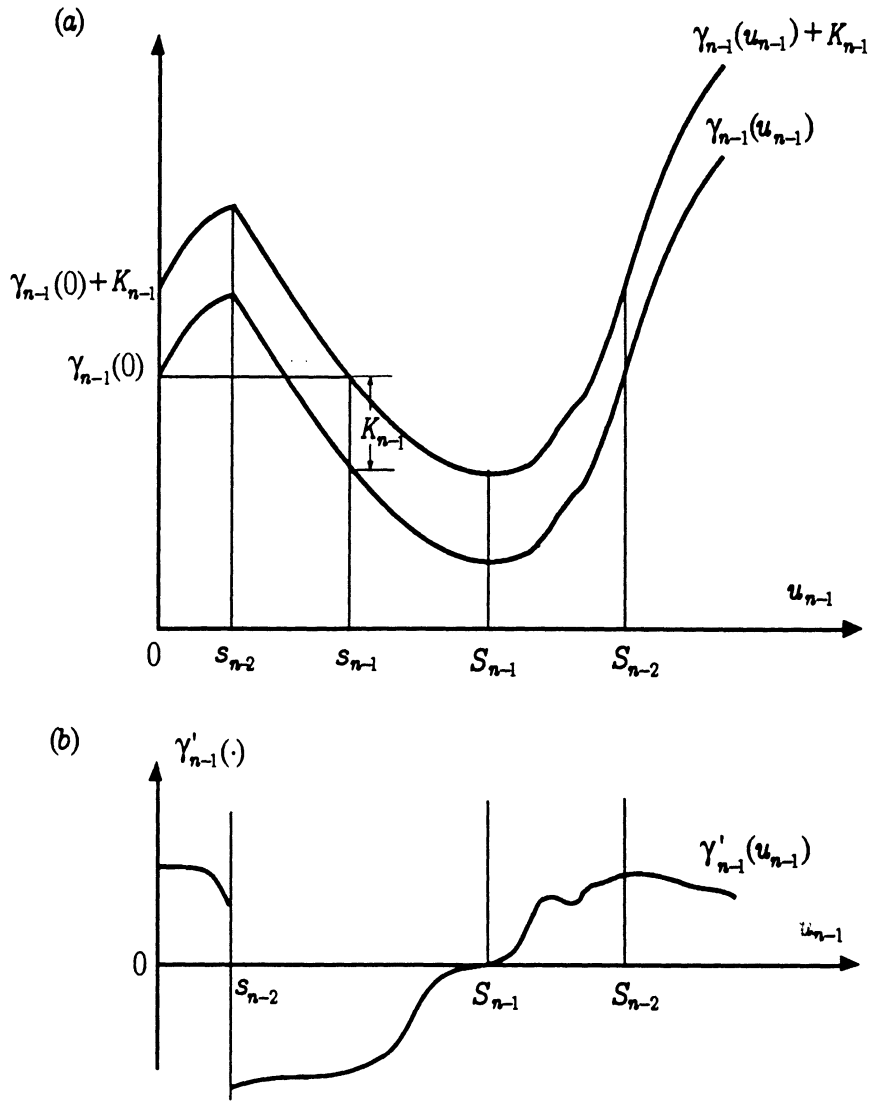


Figure 3: (a) The form of $\gamma_{n-1}(u_{n-1})$ and (b) its derivative

We first investigate the behavior of $\gamma_n(u_n)$ for $u_n \notin (s_{n-1}, S_{n-1})$. From Condition 2, the term $(w_n + h_n - h_{n+1})$ is always positive. Equations (22) and (24) then imply that $\gamma_n'(u_n) > 0$ and $\gamma_n''(u_n) < 0$, and hence $\gamma_n(u_n)$ is increasing and concave outside the interval $(s_{n-1}, S_{n-1})$.

To explore the behavior of $\gamma_n(u_n)$ in the interval $(s_{n-1}, S_{n-1})$, we need to first understand the behavior of $\gamma_{n-1}'(u_n)$ since it controls the behavior of $g_n(u_n)$. From the properties of $\gamma_{n-1}(\cdot)$ indicated in (19), it can be inferred that (i) for $u_n \in (s_{n-2}, S_{n-1})$, $\gamma_{n-1}'(u_n) < 0$, $\gamma_{n-1}''(u_n) > 0$, and (ii) for $u_n > S_{n-1}$, $\gamma_{n-1}'(u_n) > 0$. These imply that $\gamma_{n-1}'(u_n)$ is negative and increasing in range $(s_{n-2}, S_{n-1})$, crosses zero from below at $S_{n-1}$ and remains positive for $u_n > S_{n-1}$, as illustrated in Figure 3(b).

Now consider the behavior of $g_n(u_n)$ over the interval of interest, $(s_{n-1}, S_{n-1})$. From (23), $g_n(u_n) =$ positive constant$+\gamma_{n-1}'(u_n)$. But $\gamma_{n-1}'(u_n)$ is increasing in range $(s_{n-2}, S_{n-1})$.

From (18), $(s_{n-1}, S_{n-1}) \subseteq (s_{n-2}, S_{n-1})$, i.e., the interval $(s_{n-1}, S_{n-1})$ is a sub-interval of $(s_{n-2}, S_{n-1})$. Hence, $g_n(u_n)$ is increasing over $(s_{n-1}, S_{n-1})$. The zero-crossing property of $g_n(u_n)$ is of fundamental interest. But before we explore that, we propose the following lemma which rules out the circumstances under which the problem is trivial. The proof and further economic interpretation can be found in the Appendix.

**Lemma 3** *If $(w_n + h_n - h_{n+1}) > -\gamma'_{n-1}(s^+_{n-1})$, then it is not worthwhile to produce at stage $n$. That is, $u^*_n = 0$.*

Consider $g_n(u_n)$ now. It is increasing over interval $(s_{n-1}, S_{n-1})$. From Lemma 3 and (23), $g_n(s^+_{n-1}) < 0$. Also, $g_n(S_{n-1}) > 0$ by substituting (17) in (23) and using Condition 2. Hence, there exists an $S_n$, $s_{n-1} \leq S_n \leq S_{n-1}$, such that

$$g_n(S_n) = 0. \tag{25}$$

Clearly, $g_n(u_n) < 0$ for $u_n \in (s_{n-1}, S_n)$ and $g_n(u_n) > 0$ for $u_n \in (S_n, S_{n-1})$. Also, from (19), $\gamma''_{n-1}(u_n) > 0$ for $u_n \in (s_{n-2}, S_{n-1}) \supseteq (s_{n-1}, S_{n-1})$. It follows from (22) and (24) that (i) $\gamma'_n(S_n) = 0$, (ii) for $u_n \in (s_{n-1}, S_n)$, $\gamma'_n(u_n) < 0$ and $\gamma''_n(u_n) > 0$, and (iii) for $u_n \in (S_n, S_{n-1})$, $\gamma'_n(u_n) > 0$, but $\gamma''_n(u_n)$ may be positive or negative. Together, these properties imply that $\gamma_n(\cdot)$ (i) has a local minimum at $S_n$, (ii) is decreasing and concave over interval $(s_{n-1}, S_n)$, and (iii) is increasing over interval $(S_n, S_{n-1})$. Recall that $\gamma_n(\cdot)$ is increasing and concave for $u_n \leq s_{n-1}$ and $u_n \geq S_{n-1}$. This completes the characterization of $\gamma_n(\cdot)$.

We now characterize the form of the optimal policy. Note that $\gamma_n(u_n)$ for $u_n \in (0, s_{n-1})$ achieves its minimum at $\gamma_n(0)$ while $\gamma_n(u_n)$, $u_n \in (s_{n-1}, \infty)$, achieves its minimum at $\gamma_n(S_n)$. If $\gamma_n(0) \leq \gamma_n(S_n)$, it is optimal not to produce at stage $n$. If, on the other hand, $\gamma_n(S_n) \leq \gamma_n(0)$, i.e., $S_n$ is the global minimum of $\gamma_n(\cdot)$, then it *may* be worthwhile to produce. The answer depends on the setup cost, $K_n$. If $K_n \geq \gamma_n(0) - \gamma_n(S_n)$, i.e., the setup is more expensive than the maximum benefit achievable from production, then one chooses simply not to produce, irrespective of the available input material $x_{n+1}$. If, on the other hand, $K_n < \gamma_n(0) - \gamma_n(S_n)$, there exists a unique $s_n \in (s_{n-1}, S_n)$ such that

$$\gamma_n(0) \quad = \quad K_n + \gamma_n(s_n). \tag{26}$$

Note that a point $s_n$ satisfying (26) can not lie in interval $(0, s_{n-1})$ since $\gamma_n(\cdot)$ is increasing over this interval and $\gamma_n(s_n) < \gamma_n(0)$. It follows from the definition of $s_n$ and the decreasing nature of $\gamma_n(u_n)$ over $(s_{n-1}, S_n)$ that

$$\gamma_n(0) \;\leq\; K_n + \gamma_n(u_n) \quad \text{for all } u_n \leq s_n, \tag{27}$$

$$\text{and} \qquad \gamma_n(0) \;>\; K_n + \gamma_n(u_n) \quad \text{for all } s_n < u_n \leq S_n. \tag{28}$$

Based upon the available input material $x_{n+1}$, the optimal policy can now be characterized in terms of the two critical numbers $s_n$ and $S_n$. For $x_{n+1} \leq s_n$, it is not worthwhile to produce because the setup cost, $K_n$, will offset the expected savings, $(\gamma_n(0) - \gamma_n(u_n))$, derived from production. This follows from (27), since $u_n \leq x_{n+1} \leq s_n$. For $s_n < x_{n+1} \leq S_n$, the advantage gained by producing, $(\gamma_n(0) - \gamma_n(u_n))$, can offset $K_n$ *provided* one plans to produce more than $s_n$. This follows from (28). In fact, since $\gamma_n(u_n)$ is decreasing in range $(s_n, S_n)$, it pays to set the production target as high as possible; the optimal policy simply inputs all the available material. Finally, for $x_{n+1} > S_n$, one sets the production target at $S_n$, the point where $(K_n + \gamma_n(u_n))$ attains its global minimum. Q.E.D. ■

The lower critical number, $s_n$, represents the smallest input quantity one is willing to process at stage $n$. In this sense, $s_n$ is a measure of the *effective* setup cost at stage $n$. The upper critical number, $S_n$, is the maximum desired output from stage $n$. The interrelationship among critical numbers is of significance for their efficient computation and for understanding how system operates. An immediate corollary of Theorem 2 is

**Corollary 1** *For the model stated in Section 2, the optimal operating policy is characterized by a sequence of critical numbers, $\{s_n, S_n\}$, such that*

$$0 \leq s_1 \leq s_2 \leq \ldots \leq s_N \leq S_N \leq \ldots \leq S_2 \leq S_1 < \infty.$$

This relationship has many intuitive implications for the operation of the serial system under study:

1. The lower critical number is zero at a stage *only if* it is zero at *all* downstream stages.

2. The lower critical number increases as one moves upstream.

3. The upper critical number decreases as one moves upstream. As a result, for $n = N-1, \ldots, 2, 1$, one has a binary choice: (i) input everything if $x_{n+1} > s_n$, or (ii) input nothing if $x_{n+1} < s_n$. In effect, the system has a single-critical-number policy for all stages except stage $N$, which must follow a two-critical-number policy.

4. The upper critical number at *any* stage is greater than the lower critical number for *all* stages. That is, the largest lower critical number is smaller than the smallest upper critical number.

5. The sequence of intervals $\{(s_n, S_n)\}$ is imbedded, i.e.,

$$(s_N, S_N) \subseteq (s_{N-1}, S_{N-1}) \subseteq \cdots \subseteq (s_2, S_2) \subseteq (s_1, S_1) \subseteq (0, \infty). \tag{29}$$

## 5 Sensitivity Analysis

This section explores how the critical numbers, $s_n$ and $S_n$, change as the cost parameters are varied. Whenever possible, we have tried to characterize the sensitivity quantitatively, but in many instances, we could only give a qualitative characterization.

### 5.1 Sensitivity Analysis for the Upper Critical Number, $S_n$

The upper critical number, $S_n$, was defined in Section 4 as the zero of the recursive function $\gamma_n'(\cdot)$ in interval $(s_{n-1}, S_{n-1})$. For the purpose of sensitivity analysis, it will be helpful to express $\gamma_n'(\cdot)$ explicitly in terms of the other model parameters. Consider $\gamma_n'(u_n)$ for $u_n \in (s_{n-1}, S_{n-1})$. Substituting (23) into (22),

$$\gamma_n'(u_n) = \bar{F}_n(u_n)(w_n + h_n - h_{n+1}) + \bar{F}_n(u_n)\gamma_{n-1}'(u_n), \quad \text{for } u_n \in (s_{n-1}, S_{n-1}). \qquad (30)$$

Similarly, for $u_n \in (s_{n-2}, S_{n-2})$, $\gamma_{n-1}'(u_n)$ is given by

$$\gamma_{n-1}'(u_n) = \bar{F}_{n-1}(u_n)(w_{n-1} + h_{n-1} - h_n) + \bar{F}_{n-1}(u_n)\gamma_{n-2}'(u_n),$$

which can be substituted in (30) to yield

$$\begin{aligned}\gamma_n'(u_n) &= \bar{F}_n(u_n)(w_n + h_n - h_{n+1}) + \bar{F}_n(u_n)\bar{F}_{n-1}(u_n)(w_{n-1} + h_{n-1} - h_n) \\ &\quad + \bar{F}_n(u_n)\bar{F}_{n-1}(u_n)\gamma_{n-2}'(u_n), \qquad \text{for } u_n \in (s_{n-1}, S_{n-1})\end{aligned}$$

Note that this substitution is valid because intervals $(s_{n-1}, S_{n-1}) \subseteq (s_{n-2}, S_{n-2})$ as indicated in (29). By recursive substitution, we can obtain $\gamma_n'(u_n)$ in terms of the model parameters alone. The result is

$$\gamma_n'(u_n) = \sum_{k=1}^{n}\prod_{j=k}^{n}\bar{F}_j(u_n)(w_k + h_k - h_{k+1}) - \prod_{j=1}^{n}\bar{F}_j(u_n)\bar{Q}(u_n)(\pi + h_1) \quad \text{for } u_n \in (s_{n-1}, S_{n-1}), \qquad (31)$$

where $\bar{Q}(u_n) = 1 - Q(u_n)$ and we have used the definition of $g_1(u_1)$ given in (8). Recall that the upper critical number, $S_n$, is the solution to equation $\gamma_n'(u_n) = 0$, $u_n \in (s_{n-1}, S_{n-1})$. That is, $S_n$ satisfies

$$\gamma_n'(S_n) = \sum_{k=1}^{n}\prod_{j=k}^{n}\bar{F}_j(S_n)(w_k + h_k - h_{k+1}) - \prod_{j=1}^{n}\bar{F}_j(S_n)\bar{Q}(S_n)(\pi + h_1) = 0. \qquad (32)$$

Equation (32) defines $S_n$ in terms of model parameters; other critical numbers are not present in this equation. The upper critical number for any stage can be computed using (32) in a non-recursive fashion. Note that $S_n$ is independent of the capacity distribution for stage $n$, but it does depend on the capacity distributions for all the downstream stages.

Also, the absence of the setup cost parameters, $K_i$'s, in (32) implies that the upper critical numbers do not depend on setup cost.

Using the implicit-function theorem, the sensitivity of $S_n$ with respect to any parameter $p$ can be expressed by

$$\frac{\partial S_n}{\partial p} = -\frac{\partial \gamma_n'(S_n)/\partial p}{\partial \gamma_n'(S_n)/\partial S_n}.$$

From (24), for $u_n \in (s_{n-1}, S_{n-1})$,

$$\frac{\partial \gamma_n'(S_n)}{\partial S_n} = \gamma_n''(S_n) = \bar{F}_n(S_n)\gamma_{n-1}''(S_n) - f_n(S_n)g_n(S_n).$$

But $g_n(S_n) = 0$ from definition of $S_n$, which makes the last term of the above equation vanish. Hence

$$\frac{\partial S_n}{\partial p} = -\frac{\partial \gamma_n'(S_n)/\partial p}{\bar{F}_n(S_n)\gamma_{n-1}''(S_n)}. \tag{33}$$

The sensitivity of $S_n$ to changes in cost parameters can now be obtained by taking partial derivative of (32) with respect to $h_i$, $w_i$, $\pi$ and $K_i$ respectively and then substituting the results into (33). We get

**Theorem 3** *The sensitivity of the upper critical number, $S_n$, with respect to*

1. *disposal cost, $h_i$, is*

$$\frac{\partial S_n}{\partial h_i} = \begin{cases} -Q(S_n)\prod_{j=1}^{n-1}\bar{F}_j(S_n)/\gamma_{n-1}''(S_n) & \text{if } i = 1 \\ -F_{i-1}(S_n)\prod_{j=i}^{n-1}\bar{F}_j(S_n)/\gamma_{n-1}''(S_n) & \text{if } 1 < i < n \\ -F_{n-1}(S_n)/\gamma_{n-1}''(S_n) & \text{if } i = n \\ 1/\gamma_{n-1}''(S_n) & \text{if } i = n+1 \\ 0 & \text{otherwise;} \end{cases}$$

2. *unit production cost, $w_i$, is*

$$\frac{\partial S_n}{\partial w_i} = \begin{cases} -\prod_{j=i}^{n-1}\bar{F}_j(S_n)/\gamma_{n-1}''(S_n) & \text{if } i < n \\ -1/\gamma_{n-1}''(S_n) & \text{if } i = n \\ 0 & \text{otherwise;} \end{cases}$$

3. *penalty for not satisfying the demand, $\pi$, is*

$$\frac{\partial S_n}{\partial \pi} = \bar{Q}(S_n)\prod_{j=1}^{n-1}\bar{F}_j(S_n)/\gamma_{n-1}''(S_n);$$

4. *setup cost, $K_i$, is*

$$\frac{\partial S_n}{\partial K_i} = 0.$$

Note that $\gamma''_{n-1}(S_n) > 0$ since $\gamma_{n-1}(u_n)$ is convex in $(s_{n-1}, S_{n-1})$ and $S_n \in (s_{n-1}, S_{n-1})$ by Theorem 2. We make the following observations based on Theorem 3:

1. The upper critical number at a stage (i) *decreases* with an increase in disposal cost at a downstream inventory location, (ii) *increases* with an increase in disposal cost at the inventory location immediately upstream and (iii) *remains unaffected* by changes in disposal cost at all other upstream inventory locations.

2. The sensitivity of the upper critical number at stage $n$, with respect to changes in disposal cost at a downstream inventory location $i$, $\partial S_n/\partial h_i$, is directly proportional to the probability that the entire input $S_n$ gets through *all* the production stages following stage $n$, up to and including stage $i$, and is stopped at location $i$ (due to insufficient capacity at stage $i-1$, or insufficient demand, if $i = 1$). That is,
$$\frac{\partial S_n}{\partial h_i} \propto \Pr\{Y_{n-1} > S_n;\ Y_{n-2} > S_n;\ \dots ;Y_i > S_n;\ Y_{i-1} < S_n\} \text{ for } 1 < i \le n,$$
and $$\frac{\partial S_n}{\partial h_1} \propto \Pr\{Y_{n-1} > S_n;\ Y_{n-2} > S_n;\ \dots ;Y_1 > S_n;\ Z < S_n\}.$$
Clearly, the farther the inventory location $i$ from stage $n$, the smaller the likelihood that the entire input $S_n$ reaches location $i$, and hence the weaker the dependence of $S_n$ on $h_i$.

3. The upper critical number at a stage *decreases* with an increase in production cost at that stage or at any other downstream stages, but it remains unaffected by the changes in production costs at all upstream stages. Moreover, $\partial S_n/\partial w_i$ is proportional to the probability that input $S_n$ is successfully processed at all stages following stage $n$, up to and including stage $i$. Based on observation 2,
$$\frac{\partial S_n}{\partial h_i} = \begin{cases} \Pr\{Y_{i-1} < S_n\}\, \partial S_n/\partial w_i & \text{if } 1 < i \le n \\ \Pr\{Z < S_n\}\, \partial S_n/\partial w_1 & \text{if } i = 1, \end{cases}$$
also, for $i < j \le n$,
$$\frac{\partial S_n}{\partial w_i} = \Pr\{Y_i > S_n;\ Y_{i+1} > S_n;\ \dots ;Y_{j-1} > S_n\}\, \frac{\partial S_n}{\partial w_j} < \frac{\partial S_n}{\partial w_j},$$
i.e., the upper critical number at a stage is far more sensitive to the changes in production cost at a nearby downstream stage than those at a stage farther downstream.

4. The upper critical number at a stage *increases* as the penalty for not satisfying the demand increases. Moreover, the sensitivity $\partial S_n/\partial \pi$ is proportional to the probability that input $S_n$ is successfully processed at all downstream stages, and still falls short of satisfying the demand. Based on observation 2, we have
$$\frac{\partial S_n}{\partial \pi} = -\Pr\{Z > S_n\}\, \frac{\partial S_n}{\partial w_1},$$

i.e., the upper critical number is less sensitive to changes in production cost than to changes in stockout penalty; the negative sign signifies that their effects are in opposite direction.

5. The upper critical number is equally sensitive to changes in disposal cost at the immediately preceding inventory location and to the production cost at the stage under consideration. In fact the changes in either of these parameters affects the upper critical number the most, as can be seen from,

$$\frac{\partial S_n}{\partial w_n} = -\frac{\partial S_n}{\partial h_{n+1}} = -\frac{1}{\gamma''_{n-1}(S_n)} = \textit{Constant of proportionality}.$$

## 5.2 Sensitivity Analysis for Lower Critical Number, $s_n$

The proofs of the following sensitivity results for the lower critical number are somewhat involved and are relegated to the Appendix.

**Theorem 4** *The lower critical number at a stage is zero if and only if the production at that stage incurs no setup cost and the same is true for all downstream stages. That is,* $s_n = 0$ *if and only if* $K_i = 0$*, for all* $i \leq n$.

Whenever the lower critical number, $s_n$, is zero, the optimal policy at stage $n$ is of *produce-up-to* form given by a single critical number, $S_n$,

$$u_n^*(x_{n+1}) = \begin{cases} x_{n+1} & \text{if } x_{n+1} \leq S_n \\ S_n & \text{if } x_{n+1} > S_n. \end{cases}$$

According to Theorem 4, a nonzero setup cost at a stage makes the lower critical number positive not only for that stage, but for all upstream stages. If the optimal policy for a stage is of produce-up-to form, the optimal policy for all downstream stages must also be of produce-up-to form.

The following theorem shows that any increase in the setup cost at a stage will also increase the effective setup costs at all the earlier stages.

**Theorem 5** *An increase in setup cost at a stage leads to an increase in lower critical number at that stage as well as at all preceding stages.*

The lower critical number at a stage (i) increases with an increase in setup cost at that stage or at any other succeeding stages, and (ii) remains unaffected by setup cost changes at preceding stages.

**Theorem 6** *An increase in stockout penalty leads to a decrease in lower critical numbers for all stages.*

As the shortage penalty increases, one is more likely to carry out production (due to a decrease in lower critical number) and one tends to produce in a larger quantity (due to an increase in upper critical number) at all stages, to avoid the higher shortage penalty. A reduction in unit production cost has an analogous affect on all upstream stages as indicated in the theorem below.

THEOREM 7 *A decrease in the unit production cost at a stage leads to a decrease in the lower critical number at that stage as well at all preceding stages.*

The lower critical number at a stage (i) decreases with a decrease in unit production cost at that stage or at any other succeeding stages, and (ii) remains unaffected by the changes in unit production costs at preceding stages.

THEOREM 8 *A decrease in disposal cost at an inventory location leads to an increase in lower critical number for the stage next to the inventory location.*

Any decrease in disposal cost at the inventory location before a stage increases the lower critical number and decreases the upper critical number at the stage. However, the decrease in the disposal cost will increase the critical numbers at all the upstream stages.

## 6 The Impact of Uncertainties

We now turn our attention to examine the effect of demand and capacity uncertainties on the optimal policy. We explore this by examining changes in critical numbers as the demand or capacity distribution is changed stochastically. Our results are summarized in the following two theorems; their proofs can be found in the Appendix.

THEOREM 9 *As the demand increases stochastically, the upper critical number increases but the lower critical number decreases at every stage of the system.*

An increase in demand tends to increase the expected shortage penalty. To avoid this extra penalty, one is less reluctant to start production and is willing to produce more.

THEOREM 10 *A stochastic increase in capacity at any stage leads to an increase in the upper critical numbers for all upstream stages, but the upper critical numbers for all downstream stages, including that for the current stage, remain unchanged.*

As capacity decreases stochastically at a stage, there is larger probability that capacity will take on smaller values. Since it is more likely that only a smaller input quantity can be processed at this stage, the maximum desired output from upstream stages are curtailed.

The capacity changes at a stage, however, does not affect the maximum desired output from that stage. To explain this, suppose that the maximum desired output for a stage was determined assuming an infinite capacity at that stage. In case of capacity uncertainty, one simply hopes to produce this desired amount. Reducing the maximum desired output in anticipation of a low capacity realization guarantees lower output for all capacity realizations. This can be no better than letting the realized capacity limit the output.

# 7 A Numerical Example

The purpose of this example is (i) to demonstrate that the two critical numbers can be computed efficiently, (ii) to validate that the sequence of critical numbers, $\{s_n, S_n\}$, is imbedded, and (iii) to illustrate the nature of $\gamma_n(\cdot)$ numerically.

Consider a three-stage serial production system with the following costs,

$$\begin{array}{lllll} & w_3 = 30 & w_2 = 10 & w_1 = 15 & \\ h_4 = 10 & h_3 = 20 & h_2 = 25 & h_1 = 50 & \pi = 200 \\ & K_3 = 25000 & K_2 = 0 & K_1 = 45000. & \end{array}$$

Demand as well as capacities are assumed to follow a lognormal density function

$$\phi(a|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma a}\exp[\frac{-(\ln a - \mu)^2}{2\sigma^2}] \quad \text{for all } a > 0,$$

where $\mu$ and $\sigma$ are parameters of the lognormal distribution. That is, $q(z) = \phi(z|\mu_d, \sigma_d)$ and $f_n(y_n) = \phi(y_n|\mu_n, \sigma_n)$, $n = 1, 2, 3$. The parameters of these distributions are

$$\begin{array}{llll} \mu_3 = 8.5 & \mu_2 = 8.3 & \mu_1 = 8.5 & \mu_d = 7.3 \\ \sigma_3 = .2 & \sigma_2 = .5 & \sigma_1 = .3 & \sigma_d = .5. \end{array}$$

The upper critical numbers, $S_n$'s, can be computed using (32). Note that this computation need not be carried out in a recursive fashion, i.e., the upper critical numbers can be obtained independent of each other. In fact, since the implementation of the optimal policy requires only $S_3$ (see Observation 3 for Corollary 1), there is no need to compute any other upper critical numbers. For the purpose of comparison, we report below all the $S_n$'s

$$S_3 = 1708, \quad S_2 = 2177, \quad S_1 = 2434.$$

The lower critical numbers are calculated recursively using (14) to obtain

$$s_3 = 453, \quad s_2 = 231, \quad s_1 = 214.$$

Note that $s_2$ is greater than zero despite the fact that $K_2$ is zero. This is because the effective setup cost at stage 2 is greater than zero due to a positive setup cost at stage 1. Observe also that the critical numbers satisfy the monotonicity property indicated in Corollary 1. To illustrate the nature of $\gamma_n$, a plot of $\gamma_3(u_3)$ is shown in Figure 4.

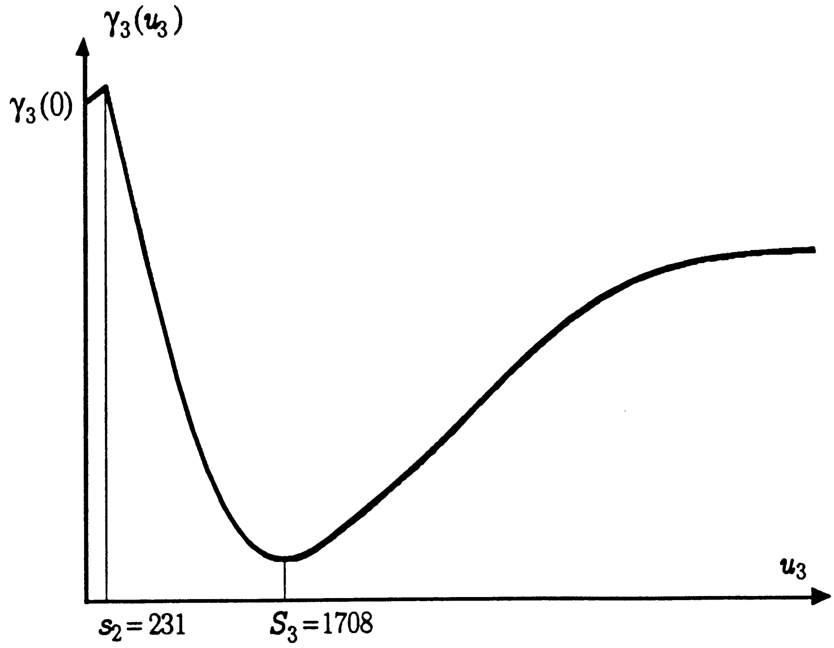


Figure 4: A plot of $\gamma_3(u_3)$ for the numerical example

# 8 Concluding Remarks

A major goal of this investigation was to explore how capacity uncertainties affect production decisions in a multi-stage system with setup costs. Despite the unwieldy nature of the cost functions involved, we were able to establish the optimality of a simple two-critical-number policy. The critical numbers for successive stages were shown to be monotonic. This fact was exploited for efficient computation of critical numbers. The monotonicity of critical numbers also lead us to the interesting conclusion that production for all but the very first stage, can effectively be controlled using only the lower critical numbers. In our effort to understand how production decisions are affected by cost parameters, we explored the sensitivity of critical numbers to these parameters. The impact of demand and capacity uncertainties were also studied by letting their distributions change stochastically. These results provided us with further insights on the interrelationship among various stages of the system.

# Appendix

**Proof of Lemma 3:** Suppose $(w_n + h_n - h_{n+1}) > -\gamma'_{n-1}(s^+_{n-1})$ then, from (23), $g_n(s^+_{n-1}) > 0$. Since $g_n(u_n)$ is increasing over interval $(s_{n-1}, S_{n-1})$, it remains positive over this entire interval. As a result, from (22), $\gamma'_n(u_n)$ is always positive. That is, $\gamma_n(0) < \gamma_n(u_n)$ for all $u_n$, and it is not worthwhile to produce at stage $n$. Q.E.D. ∎

*Further Explanation for Lemma 3:* Recall that for the single-stage problem, an equivalent condition is $(w_1 + h_1 - h_2) > (\pi + h_1)Pr\{Z > 0\}$ or $g_1(0) > 0$, which implies that $S_1 = 0$. Condition 2 and Lemma 3 together specify lower and upper bounds on quantity $(w_n + h_n - h_{n+1})$ which can be explained as follows. If $(w_n + h_n - h_{n+1}) < 0$, one has an unnecessary incentive to process the input at stage $n$, just for the sake of disposing the output at the next stage. On the other hand, as $(w_n + h_n - h_{n+1})$ increases, it provides an increasing disincentive for production, and beyond a point, it may well become totally uneconomical to carry out production at stage $n$. In general, if the effective cost of processing a unit, $(w_n + h_n - h_{n+1})$, exceeds the maximum marginal benefit derived from making the unit available at the next stage, $-\gamma'_{n-1}(s^+_{n-1})$, one simply chooses not to produce.

**Proof of Theorem 4:** From (10), we know that $s_1 = 0$ if and only if $K_1 = 0$ since $s_1 \in (0, S_1)$ and $\gamma_1(\cdot)$ is decreasing in $(0, S_1)$ by Theorem 1. Assume $K_1 = 0$. From Theorem 2, $\gamma_2(\cdot)$ is decreasing in $(s_1 = 0, S_2)$. Then, from (26), $s_2 = 0$ if and only if $K_2 = 0$ since $s_2 \in (0, S_2)$ by definition. If $K_1 \neq 0$, then $s_1 > 0$. By Theorem 2, $s_2 > s_1 > 0$. Therefore $K_1$ must be zero to have $s_2 = 0$. To proceed by induction, assume $s_{n-1} = \ldots = s_2 = s_1 = 0$ if and only if $K_{n-1} = \ldots = K_2 = K_1 = 0$. By Theorem 2, $\gamma_n(\cdot)$ is decreasing in $(s_{n-1} = 0, S_n)$. By the definition of the lower critical number, $s_n = 0$ if and only if $K_n = s_{n-1} = 0$ since $s_{n-1} \in (0, S_n)$ by definition. Suppose $K_i \neq 0$ for some $i < n$. Then, $s_{n-1} > 0$ by assumption. By Theorem 2, $s_n > s_{n-1} > 0$. Therefore, $K_i$ for all $i < n$ must be zero to have $s_n = 0$. QED. ∎

**Proof of Theorem 5:** Suppose the setup cost at stage $i$ increases from $K_i$ to $\hat{K}_i$. Let the corresponding functions $\gamma_n(\cdot)$, $C_n(\cdot)$, etc., be represented by $\hat{\gamma}_n(\cdot)$, $\hat{C}_n(\cdot)$, etc. Similarly, let the new critical numbers be represented by $\hat{s}_n$ and $\hat{S}_n$, for all $n$. Recall that the upper critical numbers are not affected by changes in setup costs (Property 4 of Theorem 3), i.e., $S_n = \hat{S}_n$ for all $n$. Since the change at stage $i$ does not affect decision for any of the downstream stage, hence $s_j = \hat{s}_j$ for all $j < i$. From (26), for stage $i$

$$\gamma_i(s_i) \;=\; \gamma_i(0) - K_i,$$

$$\hat{\gamma}_i(\hat{s}_i) = \hat{\gamma}_i(0) - \hat{K}_i.$$

Since $\gamma_i(0) = \hat{\gamma}_i(0)$ by Lemma 2 and $K_i < \hat{K}_i$, hence $\gamma_i(s_i) > \hat{\gamma}_i(\hat{s}_i)$. Observe from (3) and (4) that $\gamma_i(\cdot)$ is not a function of $K_i$, i.e., $\gamma_i(u_i) = \hat{\gamma}_i(u_i)$ for all $u_i$. Then, $\gamma_i(\hat{s}_i) = \hat{\gamma}_i(\hat{s}_i) < \gamma_i(s_i)$. By Theorem 2, $\gamma_i(\cdot)$ is decreasing in range $(s_{i-1} = \hat{s}_{i-1}, S_i)$, and both $s_i$ and $\hat{s}_i$ belong to interval $(s_{i-1}, S_i)$. Hence, $\hat{s}_i > s_i$ since $\gamma_i(\hat{s}_i) < \gamma_i(s_i)$.

For stage $i+1$, from (4),

$$\begin{aligned}\hat{\gamma}_{i+1}(u_{i+1}) &= \int_0^{u_{i+1}} [(w_{i+1} - h_{i+2})y_{i+1} + \hat{C}_i(y_{i+1})]\, dF_{i+1}(y_{i+1}) \\ &\quad + \bar{F}_{i+1}(u_{i+1})[(w_{i+1} - h_{i+2})u_{i+1} + \hat{C}_i(u_{i+1})].\end{aligned}$$

Then, we can derive

$$\begin{aligned}\hat{\gamma}_{i+1}(u_{i+1}) - \gamma_{i+1}(u_{i+1}) &= \int_0^{u_{i+1}} [\hat{C}_i(y_{i+1}) - C_i(y_{i+1})]\, dF_{i+1}(y_{i+1}) \\ &\quad + \bar{F}_{i+1}(u_{i+1})[\hat{C}_i(u_{i+1}) - C_i(u_{i+1})], \qquad \forall u_{i+1}. \qquad \text{(A.1)}\end{aligned}$$

Consider (A.1) for $u_{i+1} \in (\hat{s}_i, S_i)$. Since $\hat{s}_i \geq s_i$ and $\hat{S}_i = S_i$, the first term can be divided into three sub-intervals $(0, s_i)$, $(s_i, \hat{s}_i)$ and $(\hat{s}_i, u_{i+1})$, and then by substituting (21) into (A.1), we have

$$\begin{aligned}&\hat{\gamma}_{i+1}(u_{i+1}) - \gamma_{i+1}(u_{i+1}) \\ &= \int_0^{s_i} [\hat{\gamma}_i(0) - \gamma_i(0)]\, dF_{i+1}(y_{i+1}) + \int_{s_i}^{\hat{s}_i} [\hat{\gamma}_i(0) - \gamma_i(y_{i+1}) - K_i]\, dF_{i+1}(y_{i+1}) \\ &\quad + \int_{\hat{s}_i}^{u_{i+1}} [\hat{K}_i - K_i]\, dF_{i+1}(y_{i+1}) + \bar{F}_{i+1}(u_{i+1})[\hat{K}_i - K_i]. \qquad \text{(A.2)}\end{aligned}$$

The last two terms of (A.2) are positive since $\hat{K}_i \geq K_i$. Observing that $\gamma_i(0) = \hat{\gamma}_i(0)$ from Lemma 2, the first term becomes zero and the integrand of the second term can be rewritten as $\gamma_i(s_i) - \gamma_i(y_{i+1})$ by the definition of $s_i$. By Theorem 2, $\gamma_i(\cdot)$ is decreasing in $(s_i, \hat{s}_i) \subseteq (s_{i-1}, S_i)$. Therefore, $\gamma_i(s_i) > \gamma_i(y_{i+1})$ for $y_{i+1} \in (s_i, \hat{s}_i)$. Hence, the second term of (A.2) is also positive. As a result, $\hat{\gamma}_{i+1}(u_{i+1}) > \gamma_{i+1}(u_{i+1})$ in $(\hat{s}_i, S_{i+1})$. By Lemma 2 and from the definition of the lower critical number, we have

$$\gamma_{i+1}(s_{i+1}) = \gamma_{i+1}(0) - K_{i+1} = \hat{\gamma}_{i+1}(0) - K_{i+1} = \hat{\gamma}_{i+1}(\hat{s}_{i+1}).$$

Hence, $\gamma_{i+1}(s_{i+1}) = \hat{\gamma}_{i+1}(\hat{s}_{i+1}) > \gamma_{i+1}(\hat{s}_{i+1})$ since $\hat{s}_{i+1} \in (\hat{s}_i, S_{i+1})$ by Theorem 2. Notice that $\gamma_{i+1}(\cdot)$ is decreasing in $(\hat{s}_i, S_{i+1}) \subseteq (s_i, S_{i+1})$ by Theorem 2. Therefore, $s_{i+1} < \hat{s}_{i+1}$. In order to proceed by induction, we need $\hat{\gamma}_{i+1}(u_{i+1}) > \gamma_{i+1}(u_{i+1})$ in $(\hat{s}_{i+1}, S_{i+2}) \subseteq (\hat{s}_i, S_{i+1})$ from (29). The rest of the proof for stage $n > i+1$ is analogous to the proof at stage $i+1$. However, instead of the term $(\hat{K}_i - K_i)$ in (A.2), it will become $\hat{\gamma}_{n-1}(u_n) - \gamma_{n-1}(u_n)$ for stage $n > i+1$.

By induction, assume $s_{n-1} \le \hat{s}_{n-1}$ and $\gamma_{n-1}(u_{n-1}) \le \hat{\gamma}_{n-1}(u_{n-1})$ in range $(\hat{s}_{n-1}, S_n)$ for $n > i$. For stage $n$, from (4) we can write

$$\begin{aligned}\hat{\gamma}_n(u_n) - \gamma_n(u_n) &= \int_0^{u_n} [\hat{C}_{n-1}(y_n) - C_{n-1}(y_n)]\, dF_n(y_n) \\ &\quad + \bar{F}_n(u_n)[\hat{C}_{n-1}(u_n) - C_{n-1}(u_n)]. \qquad \text{(A.3)}\end{aligned}$$

Consider (A.3) for $u_n \in (\hat{s}_{n-1}, S_{n-1})$. Since $\hat{s}_{n-1} \ge s_{n-1}$ and $\hat{S}_{n-1} = S_{n-1}$, the first term can be divided into three sub-intervals $(0, s_{n-1})$, $(s_{n-1}, \hat{s}_{n-1})$ and $(\hat{s}_{n-1}, u_n)$, and then by substituting (21) into (A.3), we have

$$\begin{aligned}&\hat{\gamma}_n(u_n) - \gamma_n(u_n) \\ &= \int_0^{s_{n-1}} [\hat{\gamma}_{n-1}(0) - \gamma_{n-1}(0)]\, dF_n(y_n) + \int_{s_{n-1}}^{\hat{s}_{n-1}} [\hat{\gamma}_{n-1}(0) - \gamma_{n-1}(y_n) - K_{n-1}]\, dF_n(y_n) \\ &\quad + \int_{\hat{s}_{n-1}}^{u_n} [\hat{\gamma}_{n-1}(y_n) - \gamma_{n-1}(y_n)]\, dF_n(y_n) + \bar{F}_n(u_n)[\hat{\gamma}_{n-1}(u_n) - \gamma_{n-1}(u_n)]. \qquad \text{(A.4)}\end{aligned}$$

The last two terms of (A.4) are positive since $\hat{\gamma}_{n-1}(u_n) \ge \gamma_{n-1}(u_n)$. Observing that $\gamma_{n-1}(0) = \hat{\gamma}_{n-1}(0)$ by Lemma 2, the first term becomes zero and the integrand of the second term can be rewritten as $\gamma_{n-1}(s_{n-1}) - \gamma_{n-1}(y_n)$ by the definition of $s_{n-1}$. By Theorem 2, $\gamma_{n-1}(\cdot)$ is decreasing in $(s_{n-1}, \hat{s}_{n-1}) \subseteq (s_{n-2}, S_{n-1})$. Therefore, $\gamma_{n-1}(s_{n-1}) > \gamma_{n-1}(y_n)$ for $y_n \in (s_{n-1}, \hat{s}_{n-1})$. Hence, the second term of (A.4) is also positive. As a result, $\hat{\gamma}_n(u_n) > \gamma_n(u_n)$ in $(\hat{s}_{n-1}, S_n)$. By Lemma 2 and from the definition of the lower critical number, we have

$$\gamma_n(s_n) = \gamma_n(0) - K_n = \hat{\gamma}_n(0) - K_n = \hat{\gamma}_n(\hat{s}_n).$$

Hence, $\gamma_n(s_n) = \hat{\gamma}_n(\hat{s}_n) > \gamma_n(\hat{s}_n)$ since $\hat{s}_n \in (\hat{s}_{n-1}, S_n)$ by Theorem 2. Notice that $\gamma_n(\cdot)$ is decreasing in $(\hat{s}_{n-1}, S_n) \subseteq (s_{n-1}, S_n)$ by Theorem 2. Therefore, $s_n < \hat{s}_n$. In order to complete the proof by induction, $\hat{\gamma}_n(u_n) > \gamma_n(u_n)$ in $(\hat{s}_n, S_{n+1}) \subseteq (\hat{s}_{n-1}, S_n)$ from (29). QED. ■

**Proof of Theorem 6:** Suppose the penalty cost $\pi$ increases to $\hat{\pi}$, and the associated $\gamma_n(\cdot)$, $s_n$ and $S_n$ become $\hat{\gamma}_n(\cdot)$, $\hat{s}_n$ and $\hat{S}_n$, respectively. The following lemma must hold for all stages

LEMMA **A.1** *For all stage $n$,*

$$\int_0^{s_{n-1}} \bar{F}_n(u_n)[w_n + h_n - h_{n+1}]\, du_n + \int_{s_{n-1}}^{s_n} \gamma_n'(u_n)\, du_n = -K_n. \qquad \text{(A.5)}$$

**Proof:** By the definition of $s_n$, consider

$$-K_n = \gamma_n(s_n) - \gamma_n(0) = \int_0^{s_n} \gamma_n'(u_n)\, du_n,$$

by definition of integration. From (22), $\gamma_n'(u_n)$ can be decomposed into two sub-intervals $(0, s_{n-1})$ and $(s_{n-1}, s_n)$ instead of $(0, s_n)$. Hence, equation (26) can be rewritten as (A.5). QED. □

By Condition 2, the integrand in the first term of (A.5) is positive, and since $\gamma_n(\cdot)$ is decreasing in $(s_{n-1}, S_n)$ by Theorem 2, the integrand in the second term is negative. Hence, equation (A.5) can be interpreted as follows: the increment of $\gamma_n(\cdot)$ from 0 to $s_{n-1}$ minus the decrement from $s_{n-1}$ to $s_n$, is always equal to $-K_n$.

Consider (A.5) for the case when the penalty is $\hat{\pi}$

$$\int_0^{\hat{s}_{n-1}} \bar{F}_n(u_n)[w_n + h_n - h_{n+1}]\, du_n + \int_{\hat{s}_{n-1}}^{\hat{s}_n} \hat{\gamma}_n'(u_n)\, du_n \;=\; -K_n. \tag{A.6}$$

Since $-K_n$ is a constant, the difference between the left-hand-sides of (A.5) and of (A.6) is zero. We will prove the theorem by induction. For each stage $n$, we will show if $s_n \geq \hat{s}_n$ is not true, then the difference between (A.5) and (A.6) will turn out to be negative. By contradiction, $s_n \geq \hat{s}_n$.

For stage 1, the first terms of both (A.5) and (A.6) vanish since $s_0 = \hat{s}_0 \equiv 0$. Then, by subtracting (A.6) from (A.5), we have

$$\int_0^{s_1} \gamma_1'(u_1)\, du_1 - \int_0^{\hat{s}_1} \hat{\gamma}_1'(u_1)\, du_1 \;=\; 0. \tag{A.7}$$

In order to prove by contradiction, suppose $s_1 < \hat{s}_1$. Then, equation (A.7) can be rewritten as

$$\int_0^{s_1} [\hat{\gamma}_1'(u_1) - \gamma_1'(u_1)]\, du_1 + \int_{s_1}^{\hat{s}_1} \hat{\gamma}_1'(u_1)\, du_1 \;=\; 0. \tag{A.8}$$

Since $\hat{\pi} \geq \pi$, thereby $\gamma_1'(u_1) \geq \hat{\gamma}_1'(u_1)$ in range $(0, s_1) \subset (0, \infty)$ from (31). Then, the first term of (A.8) is negative. The second term is also negative since the integrand, $\hat{\gamma}_1'(u_1)$, is negative in range $(s_1, \hat{s}_1) \subseteq (0, \hat{s}_1)$ by Theorem 1. Hence, the left-hand-side of (A.8) becomes negative. This breaks the equality in (A.8). Therefore, $s_1 \geq \hat{s}_1$.

By induction, assume $s_{n-1} \geq \hat{s}_{n-1}$. For stage $n$, in order to prove by contradiction, suppose $s_n < \hat{s}_n$. Hence, $\hat{s}_{n-1} < s_{n-1} < s_n < \hat{s}_n$ since $s_{n-1} \leq s_n$ by Theorem 2. Subtract (A.6) from (A.5)

$$\begin{aligned} -\int_{\hat{s}_{n-1}}^{s_{n-1}} \bar{F}_n(u_n)[w_n + h_n - h_{n+1}]\, du_n + \int_{\hat{s}_{n-1}}^{s_{n-1}} \hat{\gamma}_n'(u_n)\, du_n \\ + \int_{s_{n-1}}^{s_n} [\hat{\gamma}_n'(u_n) - \gamma_n'(u_n)]\, du_n + \int_{s_n}^{\hat{s}_n} \hat{\gamma}_n'(u_n)\, du_n \;=\; 0. \end{aligned} \tag{A.9}$$

The first term of (A.9) is negative since $w_n + h_n - h_{n+1} > 0$ by Condition 2. By Theorem 2, $\hat{\gamma}_n'(\cdot)$ is negative in range $(\hat{s}_{n-1}, \hat{S}_n)$, which contains intervals $(\hat{s}_{n-1}, s_{n-1})$ and $(s_n, \hat{s}_n)$.

Therefore, the second and fourth terms of (A.9) are both negative. From (31), consider $\gamma_n'(u_n)$ in $(s_{n-1}, S_{n-1})$ and $\hat{\gamma}_n'(u_n)$ in $(\hat{s}_{n-1}, \hat{S}_{n-1})$. Since $\hat{\pi} \geq \pi$, hence $\hat{\gamma}_n'(u_n) \leq \gamma_n'(u_n)$ in range $(s_{n-1}, S_{n-1}) = (s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})$ since $S_{n-1} < \hat{S}_{n-1}$ by Property 3 of Theorem 3. By Theorem 2, $(s_{n-1}, S_{n-1})$ contains interval $(s_{n-1}, s_n)$. Therefore, the third term of (A.9) is also negative. Hence, the left-hand-side of (A.9) is negative. This breaks the equality in (A.9). As a result, $\hat{s}_n \leq s_n$. QED. ■

**Proof of Theorem 7:** Suppose the unit production cost $w_i$ at stage $i$ decreases to $\hat{w}_i$, and the associated $\gamma_n(\cdot)$, $s_n$ and $S_n$ become $\hat{\gamma}_n(\cdot)$, $\hat{s}_n$ and $\hat{S}_n$, respectively. By Lemma A.1, rewrite (A.5) for stage $i$,

$$\int_0^{s_{i-1}} \bar{F}_i(u_i)[w_i + h_i - h_{i+1}]\, du_i + \int_{s_{i-1}}^{s_i} \gamma_i'(u_i)\, du_i = -K_i. \tag{A.10}$$

For the change from $w_i$ to $\hat{w}_i$, equation (A.10) becomes

$$\int_0^{\hat{s}_{i-1}} \bar{F}_i(u_i)[\hat{w}_i + h_i - h_{i+1}]\, du_i + \int_{\hat{s}_{i-1}}^{\hat{s}_i} \hat{\gamma}_i'(u_i)\, du_i = -K_i. \tag{A.11}$$

Since $-K_i$ is a constant, the difference between the left-hand-sides of (A.10) and of (A.11) is zero. We will prove the theorem by induction. For each stage $n$, we will show that if $s_n \geq \hat{s}_n$ is not true, then the difference between (A.10) and (A.11) will turn out to be negative. By contradiction, $s_n \geq \hat{s}_n$.

By subtracting (A.11) from (A.10) and observing $s_{i-1} = \hat{s}_{i-1}$(since $s_{i-1}$ is not a function of $w_i$), we have

$$\int_0^{s_{i-1}} \bar{F}_i(u_i)[\hat{w}_i - w_i]\, du_i + \int_{s_{i-1}}^{s_i} [\hat{\gamma}_i'(u_i) - \gamma_i'(u_i)]\, du_i + \int_{s_i}^{\hat{s}_i} \hat{\gamma}_i'(u_i)\, du_i = 0. \tag{A.12}$$

The first term is negative since $\hat{w}_i < w_i$, and the third term is also negative since $\hat{\gamma}_i'(\cdot)$ is negative in range $(s_i, \hat{s}_i) \subseteq (s_{i-1}, \hat{S}_i)$ by Theorem 2. From (31), consider $\gamma_i'(u_i)$ in $(s_{i-1}, S_{i-1})$ and $\hat{\gamma}_i'(u_i)$ in $(\hat{s}_{i-1}, \hat{S}_{i-1})$. Since $\hat{w}_i \leq w_i$, hence $\hat{\gamma}_i'(u_i) \leq \gamma_i'(u_i)$ in range $(s_{i-1}, S_{i-1}) = (s_{i-1}, S_{i-1}) \cap (\hat{s}_{i-1}, \hat{S}_{i-1})$ since $S_{i-1} < \hat{S}_{i-1}$ by Property 2 of Theorem 3. By Theorem 2, $(s_{i-1}, S_{i-1}) \supseteq (s_{i-1}, s_i)$. Therefore, the second term of (A.12) is also negative. Hence, the left-hand-side of (A.12) is negative. This breaks the equality in (A.12). As a result, $\hat{s}_i \leq s_i$.

By induction, assume $s_{n-1} \geq \hat{s}_{n-1}$. For stage $n > i$, consider (A.5) and (A.6), which is the version of (A.5) after $w_i$ shifts to $\hat{w}_i$. Suppose $s_n < \hat{s}_n$. Hence, $\hat{s}_{n-1} < s_{n-1} < s_n < \hat{s}_n$ since $s_{n-1} \leq s_n$ by Theorem 2. Notice that both integrands in the first term of (A.5) and (A.6) are the same. Subtract (A.6) from (A.5). Then, we get (A.9). The first term of (A.9) is negative since $w_n + h_n - h_{n+1}$ by Condition 2. By Theorem 2, $\hat{\gamma}_n'(\cdot)$ is negative in range $(\hat{s}_{n-1}, \hat{S}_n)$, which contains intervals $(\hat{s}_{n-1}, s_{n-1})$ and $(s_n, \hat{s}_n)$. Therefore, the second and fourth terms of (A.9) are both negative. From (31), consider $\gamma_n'(u_n)$ in $(s_{n-1}, S_{n-1})$ and $\hat{\gamma}_n'(u_n)$ in $(\hat{s}_{n-1}, \hat{S}_{n-1})$. Since $\hat{w}_i \leq w_i$, hence $\hat{\gamma}_n'(u_n) \leq \gamma_n'(u_n)$ in range $(s_{n-1}, S_{n-1}) = (s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})$ since $S_{n-1} < \hat{S}_{n-1}$ by Property 2 of Theorem 3. By Theorem 2, $(s_{n-1}, S_{n-1})$ contains interval $(s_{n-1}, s_n)$. Therefore, the third term of (A.9) is also negative. Hence, the left-hand-side of (A.9) is negative. This breaks the equality in (A.9). As a result, $\hat{s}_n \leq s_n$. QED. ■

**Proof of Theorem 8:** Suppose the disposal cost $h_{n+1}$ at inventory location $n+1$ increases to $\hat{h}_{n+1}$, and the associated $\gamma_n(\cdot)$, $s_n$ and $S_n$ become $\hat{\gamma}_n(\cdot)$, $\hat{s}_n$ and $\hat{S}_n$, respectively. By Lemma A.1, consider (A.5) for stage $n$ when the disposal cost at inventory location $n+1$ is $\hat{h}_{n+1}$,

$$\int_0^{\hat{s}_{n-1}} \bar{F}_n(u_n)[w_n + h_n - \hat{h}_{n+1}]\, du_n + \int_{\hat{s}_{n-1}}^{\hat{s}_n} \hat{\gamma}_n'(u_n)\, du_n \;=\; -K_n. \tag{A.13}$$

Since $-K_n$ is a constant, the difference between the left-hand-sides of (A.5) and of (A.13) is zero. In order to prove $s_n \geq \hat{s}_n$ by contradiction, we will suppose that $s_n < \hat{s}_n$ for stage $n$, and then show that the difference becomes negative. Suppose $s_n < \hat{s}_n$. Notice that $s_{n-1}$ is not a function of $h_{n+1}$, i.e., $s_{n-1} = \hat{s}_{n-1}$. By subtracting (A.13) from (A.5), we have

$$\int_0^{s_{n-1}} \bar{F}_n(u_n)[h_{n+1} - \hat{h}_{n+1}]\, du_n + \int_{s_{n-1}}^{s_n} [\hat{\gamma}_n'(u_n) - \gamma_n'(u_n)]\, du_n + \int_{s_n}^{\hat{s}_n} \hat{\gamma}_n'(u_n)\, du_n = 0. \tag{A.14}$$

The first term is negative since $\hat{h}_{n+1} > h_{n+1}$, and the third term is also negative since $\gamma_n'(\cdot)$ is negative in range $(s_n, \hat{s}_n) \subseteq (s_{n-1}, \hat{S}_n)$ by Theorem 2. From (31), consider $\gamma_n'(u_n)$ in $(s_{n-1}, S_{n-1})$ and $\hat{\gamma}_n'(u_n)$ in $(\hat{s}_{n-1}, \hat{S}_{n-1})$. Since $\hat{h}_{n+1} > h_{n+1}$, hence $\hat{\gamma}_n'(u_n) \leq \gamma_n'(u_n)$ in range $(s_{n-1}, S_{n-1}) = (s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})$ since $S_{n-1} < \hat{S}_{n-1}$ by Property 1 of Theorem 3. By Theorem 2, $(s_{n-1}, S_{n-1})$ contains interval $(s_{n-1}, s_n)$. Therefore, the second term of (A.14) is also negative. Hence, the left-hand-side of (A.14) is negative. This breaks the equality in (A.14). As a result, $\hat{s}_n \leq s_n$. ■

**Proof of Theorem 9:** Suppose that the demand distribution changes from $Q(z)$ to $\hat{Q}(z)$ such that $\bar{Q}(z) \leq \bar{\hat{Q}}(z)$ for all $z$, and the associated $g_n(\cdot)$, $\gamma_n(\cdot)$, $s_n$ and $S_n$ become $\hat{g}_n(\cdot)$, $\hat{\gamma}_n(\cdot)$, $\hat{s}_n$ and $\hat{S}_n$, respectively. We first show inductively that $S_n \leq \hat{S}_n$ for all stages. By substituting (22) into (31) for $u_n \in (s_{n-1}, S_{n-1})$, one obtains

$$\begin{aligned} g_n(u_n) &= (w_n + h_n - h_{n+1}) + \sum_{k=1}^{n-1} \prod_{j=k}^{n-1} \bar{F}_j(u_n)(w_k + h_k - h_{k+1}) \\ &\quad - \prod_{j=1}^{n-1} \bar{F}_j(u_n)\bar{Q}(u_n)(\pi + h_1). \end{aligned} \tag{A.15}$$

Since $\hat{s}_0 = s_0 \equiv 0$ and $\hat{S}_0 = S_0 \equiv \infty$, both $g_1(u_1)$ and $\hat{g}_1(u_1)$ are defined in $(0, \infty)$ in (A.15). Since $\bar{Q}(u_n) \leq \bar{\hat{Q}}(u_n)$ hence $g_1(u_1) \geq \hat{g}_1(u_1)$ in range $(0, \infty)$. Then, $\hat{g}_1(S_1) \leq g_1(S_1) = 0$ by definition of $S_1$. Hence, $S_1 \leq \hat{S}_1$ since function $\hat{g}_1(u_1)$ is increasing and $\hat{g}_1(\hat{S}_1) = 0$. By induction, assume $S_{n-1} \leq \hat{S}_{n-1}$. For stage $n$,

1. in case that $S_n < \hat{s}_{n-1}$: By Theorem 2, $S_n \leq \hat{S}_n$ since $\hat{s}_{n-1} \leq \hat{S}_n$ .

2. in case that $S_n > \hat{s}_{n-1}$: From (A.15), consider $g_n(u_n)$ in $(s_{n-1}, S_{n-1})$ and $\hat{g}_n(u_n)$ in range $(\hat{s}_{n-1}, \hat{S}_{n-1})$. Since $\bar{Q}(u_n) \leq \bar{\hat{Q}}(u_n)$, hence $g_n(u_n) \geq \hat{g}_n(u_n)$ in range $(max\{s_{n-1}, \hat{s}_{n-1}\}, S_{n-1}) = (s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})$. By Theorem 2, both $S_n$ and $\hat{S}_n$ belong to the interval $(max\{s_{n-1}, \hat{s}_{n-1}\}, S_{n-1})$. Hence, $\hat{g}_n(S_n) \leq g_n(S_n) = 0$ from (23). Therefore, $\hat{S}_n \geq S_n$ since $\hat{g}_n(\cdot)$ is increasing in $(S_n, \hat{S}_{n-1}) \subseteq (\hat{s}_{n-1}, \hat{S}_{n-1})$.

Now, we need to prove that $s_n \geq \hat{s}_n$ for all stages. From (31), while the demand density shifts from $Q(z)$ to $\hat{Q}(z)$, we have $\hat{\gamma}'_n(u_n) \leq \gamma'_n(u_n)$ in range $\{(s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})\}$ since $\hat{g}_n(u_n) \leq g_n(u_n)$ in the same range. This result is analogous to the one in the proof of Theorem 6, where the penalty cost increases from $\pi$ to $\hat{\pi}$. Following the proof of Theorem 6 by using (A.5)–(A.9), one can show that $s_n \geq \hat{s}_n$. QED. ■

**Proof of Theorem 10:** Assume the capacity density at stage $i$ change from $F_i(y_i)$ to $F_i(y_i)$ such that $\bar{F}_i(y_i) \leq \bar{\hat{F}}_i(y_i)$ for all $y_i$, and the associated $g_n(\cdot)$, $s_n$ and $S_n$ become $\hat{g}_n(\cdot)$, $\hat{s}_n$ and $\hat{S}_n$, respectively. From (A.15), $g_i(u_i)$ is not a function of $F_i(\cdot)$. Hence, from (25) $S_i = \hat{S}_i$. From (22),

$$\begin{aligned} \gamma'_i(u_i) &= \bar{F}_i(u_i) g_i(u_i) \quad \text{if } u_i \in (s_{i-1}, S_{i-1}), \\ \hat{\gamma}'_i(u_i) &= \bar{\hat{F}}_i(u_i) g_i(u_i) \quad \text{if } u_i \in (\hat{s}_{i-1}, \hat{S}_{i-1}). \end{aligned}$$

Then, $\gamma'_i(u_i) \geq \hat{\gamma}'_i(u_i)$ in range $\{(s_{i-1}, S_{i-1}) \cap (\hat{s}_{i-1}, \hat{S}_{i-1})\}$. From (23),

$$\begin{aligned} g_{i+1}(u_{i+1}) &= w_{i+1} + h_{i+1} - h_{i+2} + \gamma'_i(u_{i+1}), \\ \hat{g}_{i+1}(u_{i+1}) &= w_{i+1} + h_{i+1} - h_{i+2} + \hat{\gamma}'_i(u_{i+1}). \end{aligned}$$

Hence, $g_{i+1}(u_{i+1}) \geq \hat{g}_{i+1}(u_{i+1})$ in range $\{(s_{i-1}, S_{i-1}) \cap (\hat{s}_{i-1}, \hat{S}_{i-1})\}$. By recursive substitution and (29), we obtain $\gamma_n(u_n) \geq \hat{\gamma}_n(u_n)$ and $g_n(u_n) \geq \hat{g}_n(u_n)$ in range $\{(s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})\}$ for $n > i$.

By induction, assume $S_{n-1} \leq \hat{S}_{n-1}$ for $n - 1 > i$. Then, $g_n(u_n) \geq \hat{g}_n(u_n)$ in range $\{(s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, S_{n-1})\}$. For stage $n$, if $S_n < \hat{s}_{n-1}$, then $S_n \leq \hat{S}_n$ since $\hat{s}_{n-1} \leq \hat{S}_n$ by Theorem 2. If, on the other hand, $S_n > \hat{s}_{n-1}$, then $S_n \in \{(s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, S_{n-1})\}$ by Theorem 2. Hence, $\hat{g}_n(S_n) \leq g_n(S_n) = 0$ from (23). Therefore, $\hat{S}_n \geq S_n$ since $\hat{g}_n(\cdot)$ is increasing in $(S_n, \hat{S}_{n-1}) \subseteq (\hat{s}_{n-1}, \hat{S}_{n-1})$. ■

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