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AN INVESTIGATION OF TRANSPORT PROPERTIES OF PLASMAS USING THE LINEARIZED BOLTZMANN EQUATION

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I. INTRODUCTION

Let us assume that the plasma under consideration is representable by a set of Boltzmann equations, corresponding to each kind of components which constitute our system

$$\frac{\partial f_r}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f_r}{\partial \underset{\sim}{x}} + \underset{\sim}{a} \cdot \frac{\partial f_r}{\partial \underset{\sim}{v}} = \left(\frac{\delta f_r}{\delta t}\right)_{\text{coll.}} \quad , \quad r = 1, 2, \cdots, N$$

$$\underset{\sim}{a} = \underset{\sim}{a}^{\text{external}} + \underset{\sim}{a}^{\text{internal}}$$

$$\underset{\sim}{a}^{\text{int.}} = \frac{e_r}{m_r}\left[\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{v} \times \underset{\sim}{H}\right] , \tag{I-1}$$

coupled with the Maxwell's electromagnetic field equations

$$\underset{\sim}{\nabla} \times \underset{\sim}{H} - \frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{E} = \frac{4\pi}{c}\underset{\sim}{J} = \frac{4\pi}{c}\sum_{r=1}^{N} e_r \int_{\underset{\sim}{v}} \underset{\sim}{v} f_r \, d^3 v$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} + \frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{H} = 0$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{H} = 0$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 4\pi Q = 4\pi \sum_{r=1}^{N} e_r \int_{\underset{\sim}{v}} f_r \, d^3 v , \tag{I-2}$$

where the symbols used have their usual meanings.

We shall treat the non-linear interaction terms due to the internally induced fields as perturbations. Thus, setting

$$f_r = f_r^{(0)} + f_r^{(1)}$$

to the zeroth and the first order, we have

$$D f_r^{(0)} \equiv \left[\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} + \underset{\sim}{a}^{\text{ext.}} \cdot \frac{\partial}{\partial \underset{\sim}{v}}\right] f_r^{(0)} = \left(\frac{\delta f_r^{(0)}}{\delta t}\right)_{\text{coll.}} \tag{I-3}$$

and

$$D f_r^{(1)} + \phi(\underset{\sim}{x}, \underset{\sim}{v}, t) = \left(\frac{\delta f_r^{(1)}}{\delta t}\right)_{\text{coll.}} \tag{I-4}$$

respectively, where

$$\phi(\underline{x}, \underline{v}, t) = \underline{a}^{int.} \cdot \frac{\partial f_r^{(0)}}{\partial \underline{v}}.$$

For most plasma problems perhaps it is a good approximation to ignore the right hand sides of the Equations (3) and (4) completely. However, it does not seem appropriate to study the transport properties in the absence of such a dissipative mechanism as collisions. It is for this reason that we shall adopt a relaxation-type collision model, to which (perhaps because of its simplicity) a considerable amount of attention is given in the plasma literature,* in order to simulate the overall effect of these cumbersome terms to a reasonable extent. Thus we write

$$\left(\frac{\delta f_r}{\delta t}\right)_{coll.} \approx \frac{f_r^{(0)} - f_r}{\tau} \qquad \Rightarrow$$

$$\left(\frac{\delta f_r^{(0)}}{\delta t}\right)_{coll.} \approx 0 \;, \quad \left(\frac{\delta f_r^{(1)}}{\delta t}\right)_{coll.} \approx -\frac{f_r^{(1)}}{\tau}. \tag{I-5}$$

Here τ is a properly chosen "mean collision time", which, for the sake of brevity, will be assumed to be constant. It should be pointed out that extra care must be exercised in handling this exercised in handling this model, since the conservation laws are no longer satisfied in general.

One can show that the Equation (4), coupled with (5) can be put in an integral form:

$$f_r^{(1)}(\underline{x}, \underline{v}, t) - e^{-\frac{t}{\tau}} f_r^{(1)}(\xi(\underline{x}, \underline{v}, t), \gamma(\underline{x}, \underline{v}, t), 0)$$
$$= -\int_0^t e^{-\frac{s'}{\tau}} \phi(\xi(\underline{x}, \underline{v}, s'), \gamma(\underline{x}, \underline{v}, s'), t - s')\, ds', \tag{I-6}$$

* For a detailed discussion of this model see, for instance, Reference 1.

where the functions $\underline{\xi}(\underline{x},\underline{v},t)$ and $\underline{\gamma}(\underline{x},\underline{v},t)$ are to be determined by finding the integration constants of the solution of the Lagrange's subsidiary system corresponding to the unperturbed operator D, (cf., Reference 2), which is given by

$$\frac{dt}{ds} = 1 \quad , \quad \frac{d}{ds}\underline{x} = \underline{v} \quad , \quad \frac{d}{ds}\underline{v} = a^{ext.} ,$$

and

$$\frac{d}{ds} f_r^{(1)} = -\frac{1}{\tau} f_r^{(1)} - \phi . \tag{I-7}$$

Here we assumed that the solution of this system is of the form

$$\underline{x} = \underline{x}(\underline{\xi}, \underline{\gamma}, s) \quad , \quad \underline{v} = \underline{v}(\underline{\xi}, \underline{\gamma}, s) \tag{I-8a}$$

and the inverse

$$\underline{\xi} = \underline{\xi}(\underline{x}, \underline{v}, s) \quad , \quad \underline{\gamma} = \underline{\gamma}(\underline{x}, \underline{v}, s) \tag{I-8b}$$

exists. We shall refer to the set (8) as "the unperturbed orbit equations with respect to the parameter s."

The Equation (6) is particularly suitable for the initial value problems. A special case of this form has been applied to the related instability problems by Rosenbluth and Rostoker.[3] An asymptotic form of (6) can also be deduced by assuming the system is perturbed at $t = -\infty$. When the initial perturbation of $f_r^{(1)}$ satisfies an order property,* it can be shown that

$$f_r^{(1)}(\underline{x}, \underline{v}, t) = -\int_0^\infty e^{-\frac{s'}{\tau}} \phi(\underline{\xi}', \underline{\gamma}', t-s')\, ds' \tag{I-9}$$

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* For details see Appendix C.

A special case of this form has already been deduced in connection with the plasma microconductivity studies.[4]

In what follows we shall restrict ourselves to the case in which

$$\underline{a}^{ext.} = \frac{e_r}{m_r}\left[\underline{E}_0 + \frac{1}{c}\,\underline{v}\times\underline{H}_0\right], \tag{I-10}$$

where $\underline{E}_0$ and $\underline{H}_0$ are externally applied, constant fields. One can show that the Equation (7) may be solved in this case, giving:

$$\begin{aligned}\underline{x} &= \underline{\xi} + \underline{\gamma}\cdot\underline{\underline{G}}^r + \underline{\mathcal{E}}^r\cdot\underline{\underline{M}}^r\\ \underline{v} &= \underline{\gamma}\cdot\underline{\underline{R}}^r + \underline{\mathcal{E}}^r\cdot\underline{\underline{G}}^r\end{aligned} \tag{I-11}$$

and

$$\begin{aligned}\underline{\xi} &= \underline{x} - \underline{\underline{G}}^r\cdot\underline{v} + \underline{\underline{M}}^r\cdot\underline{\mathcal{E}}^r\\ \underline{\gamma} &= \underline{\underline{R}}^r\cdot\underline{v} - \underline{\underline{G}}^r\cdot\underline{\mathcal{E}}^r\end{aligned} \tag{I-12}$$

where

$$\underline{H}_0 = H_0\,\hat{\underline{z}}, \qquad \underline{\mathcal{E}}^r = \frac{e_r}{m_r}\underline{E}_0, \qquad \Omega^r = \frac{e_r H_0}{m_r c}$$

$$\underline{\underline{R}}^r(t) = (R^r_{ij}) = \begin{pmatrix} \cos\Omega^r t & -\sin\Omega^r t & 0\\ \sin\Omega^r t & \cos\Omega^r t & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{aligned}\underline{\underline{G}}^r(t) &= \int_0^t \underline{\underline{R}}^r(s')\,ds'\\ \underline{\underline{M}}^r(t) &= \int_0^t \underline{\underline{G}}^r(s')\,ds'\end{aligned} \tag{I-13}$$

The Equation (3), with the approximation (5) implies that the unperturbed distribution $f_r^{(o)}$ must remain constant on the unperturbed orbit. In other words, $f_r^{(o)}$ is invariant under the transformation (12). Thus, it can be any arbitrary function of $\underline{\xi}$ and $\underline{\gamma}$, i.e.,

$$f^{(0)}(\underline{x},\underline{v},t) = f^{(0)}\left(\underline{x}-\underline{\underline{G}}(t)\cdot\underline{v}+\underline{\underline{M}}(t)\cdot\underline{\mathcal{E}}\ ,\ \underline{\underline{R}}(t)\cdot\underline{v}-\underline{\underline{G}}(t)\cdot\underline{\mathcal{E}}\right) \tag{I-14}$$

(Hereafter we shall drop the index r whenever no ambiguity arises. We shall also suppress all terms involving the initial perturbations explicitly, and then take them into account subsequently as a whole whenever needed.)

Using the notation introduced above, the first order equation, in tensor notation, can be written as

$$\begin{aligned} f^{(1)}(\underline{x},\underline{v},t) &= -\frac{1}{m}\int_0^t ds'\, e^{-\frac{s'}{\tau}}\, \frac{\partial f^{(0)}}{\partial \gamma'_k}\, F_k^{(1)'} \\ &= -\frac{1}{m}\int_0^t ds'\, e^{-\frac{s'}{\tau}}\, F_j^{(1)'}\left[\frac{\partial f^{(0)}}{\partial x_k}G_{jk}(s') + \frac{\partial f^{(0)}}{\partial v_k}R_{jk}(s')\right], \end{aligned} \tag{I-15}$$

where the prime on the Lorentz force denotes

$$\begin{aligned} F_k^{(1)'} &\equiv F_k^{(1)}\left(\underline{x}-\underline{\underline{G}}(s')\cdot\underline{v}+\underline{\underline{M}}(s')\cdot\underline{\mathcal{E}}\ ,\ \underline{\underline{R}}(s')\cdot\underline{v}-\underline{\underline{G}}(s')\cdot\underline{\mathcal{E}}\ ,\ t-s'\right) \\ &\equiv e\left[E_k^{(1)}(\underline{\xi}(s'),t-s') + \frac{1}{c}\,\epsilon_{kij}\,\gamma_i(s')\,H_j^{(1)}(\underline{\xi}(s'),t-s')\right] \end{aligned} \tag{I-16}$$

II. THE FIRST ORDER TRANSPORT PROPERTIES

Let $\psi(\underline{v})$ be any microscopic property of physical interest. We define the first order averages as

$$n\langle\psi\rangle^{(1)} \equiv \int_{\underline{v}} d^3v\, \psi\, f^{(1)} \equiv \int_{\underline{v}} d^3v \int_0^t ds'\, e^{-\frac{s'}{\tau}}\, \hat{\psi}\ , \tag{II-1}$$

where

$$\hat{\psi} = -\frac{1}{m}\left[\frac{\partial f^{(0)}}{\partial x_k} G'_{jk} + \frac{\partial f^{(0)}}{\partial v_k} R'_{jk}\right] F_j^{(1)'}\psi\ .$$

The latter may be put in the following form:

$$\begin{aligned}\hat{\psi} = -\frac{1}{m}\Big[\psi\, G'_{jk}\frac{\partial}{\partial x_k}\left(f^{(0)}F_j^{(1)'}\right) - \\ -\frac{\partial\psi}{\partial v_k} R'_{jk}\cdot\left(f^{(0)}F_j^{(1)'}\right)\Big]\ ,\end{aligned} \tag{II-2}$$

using the definition of the derivative of a distribution defined on the whole velocity space (in the sense of L. Schwartz),

$$\frac{\partial f^{(0)}}{\partial v_j}\psi = -f^{(0)}\frac{\partial\psi}{\partial v_j}$$

(In the conventional analysis this step may be accomplished by remembering that the averaged quantity $\langle\psi\rangle^{(1)}$ will remain unaffected after such a substitution, provided

$$\lim_{|\underline{v}|\to\infty} f^{(0)}(\underline{v}) = 0$$

rapidly enough to insure that

$$\lim_{|\underline{v}|\to\infty} \psi(\underline{v})\, f^{(0)}(\underline{v}) = 0\ .)$$

In particular one obtains the following:

i) Mass density $(\psi = m, \quad n\langle m\rangle^{(1)} \equiv \rho^{(1)})$

$$\hat{\rho} = -G'_{pk}\frac{\partial}{\partial x_k}(f^{(0)}F_p^{(1)'}) . \tag{II-3}$$

ii) Mass current $(\psi = m v_k, \quad n\langle m v_k\rangle^{(1)} \equiv j_k^{(1)})$

$$\hat{j}_k = v_k\hat{\rho} + R'_{pk}\, f^{(0)}F_p^{(1)'} . \tag{II-4}$$

Eliminating the field term, one gets

$$\hat{\rho} = -G'_{pk}\frac{\partial}{\partial x_p}\left[\hat{j}_k - v_k\hat{\rho}\right]$$

$$\hat{\psi} = \frac{1}{m}\left(\psi - v_k\frac{\partial\psi}{\partial v_k}\right)\hat{\rho} + \frac{1}{m}\frac{\partial\psi}{\partial v_k}\hat{j}_k , \tag{II-5}$$

which shows that all the first order averages can be expressed in terms of the "integrands" of $\rho^{(1)}$ and $j_k^{(1)}$.

iii) Residual stress tensor $(\psi = m v_i v_k, \quad n\langle m v_i v_k\rangle^{(1)} \equiv \Psi^{(1)\,res.})$

$$\hat{\Psi}_{ik}^{res.} = -v_i v_k\hat{\rho} + \left[v_i\hat{j}_k + v_k\hat{j}_i\right] , \tag{II-6}$$

$$\hat{p}^{res.} \equiv \frac{1}{3}\hat{\Psi}_{kk}^{res}$$

$$= -\frac{v^2}{3}\hat{\rho} + \frac{2}{3}v_k\hat{j}_k \tag{II-7}$$

iv) Residual heat flow $\left(\psi = \frac{1}{2} m v^2 v_k \ , \quad n\langle \frac{1}{2} m v^2 v_k \rangle^{(1)} \equiv q_k^{(1)\,res.}\right)$

$$\hat{q}_k^{res.} = \frac{3}{2} v_k \hat{p}^{res.} + \frac{1}{2} v^2 (\hat{j}_k - v_k \hat{\rho}) .$$

(II-8)

The residual values given above are related to the first order averages by the following relations:

$$\Psi_{ik}^{(1)} = \Psi_{ik}^{(1)\,res.} + \langle v_i \rangle^{(0)} \langle v_k \rangle^{(0)} \rho^{(1)} - \langle v_i \rangle^{(0)} j_k^{(1)} - \langle v_k \rangle^{(0)} j_i^{(1)}$$

$$p^{(1)} = p^{(1)\,res.} + \frac{1}{3} \langle v_k \rangle^{(0)} \langle v_k \rangle^{(0)} \rho^{(1)} - \frac{2}{3} \langle v_k \rangle^{(0)} j_k^{(1)}$$

$$q_k^{(1)} = q_k^{(1)\,res.} - \frac{3}{2} \frac{\Theta^{(0)}}{m} j_k^{(1)} - \frac{1}{\rho^{(0)}} \Psi_{kp}^{(0)} j_p^{(1)} - \frac{3}{2} \langle v_k \rangle^{(0)} p^{(1)} - \frac{1}{2} \langle v_p \rangle^{(0)} \langle v_p \rangle^{(0)} j_k^{(1)}$$
$$- \langle v_k \rangle^{(0)} \langle v_p \rangle^{(0)} j_p^{(1)} - \langle v_p \rangle^{(0)} \Psi_{pk}^{(1)} + \langle v_p \rangle^{(0)} \langle v_p \rangle^{(0)} \langle v_k \rangle^{(0)} \rho^{(1)} .$$

(II-9)

where

$$\Psi_{ik}^{tot.} = \Psi_{ik}^{(0)} + \Psi_{ik}^{(1)}$$
$$\equiv m \int_{\underline{v}} (v_i - \langle v_i \rangle)(v_k - \langle v_k \rangle) f \, d^3 v$$

$$q_k^{tot.} = q_k^{(0)} + q_k^{(1)}$$
$$\equiv \frac{1}{2} m \int_{\underline{v}} (\underline{v} - \langle \underline{v} \rangle)^2 (v_k - \langle v_k \rangle) f \, d^3 v$$

$$p^{(0)} \equiv n^{(0)} \Theta^{(0)} \quad , \quad \rho^{(0)} \equiv n^{(0)} m$$

(II-10)

In particular, if $f^{(o)}$ is a symmetric function of $\underline{v}$, one simply gets

$$\Psi_{ik}^{(1)} = \Psi_{ik}^{(1)res}, \quad p^{(1)} = p^{(1)res}, \qquad q_k^{(1)} = q_k^{(1)res} - \frac{3}{2}\frac{\Theta^{(o)}}{m} j_k^{(1)} - \frac{1}{\rho^{(o)}} \Psi_{kj}^{(o)} j_j^{(1)} .$$

(Hereafter the superscript (o) corresponding to the zeroth order averages will be dropped whenever no ambiguity arises.)

III. A SPECIAL CASE

In this section we shall consider a special case, which is given considerable amount of attention in the plasma literature, and which assumes the following conditions for the unperturbed state:

i) Steady state,

ii) uniform in space,

iii) $\underline{E}_o = 0$, $Q_o = 0$.

Then the Equation (I-14) reduces to

$$f^{(0)}(\underline{v}) = f^{(0)}(\underline{\underline{R}}(t)\cdot\underline{v}) \Rightarrow f^{(0)} = f^{(0)}(u, w) \tag{III-1}$$

where w, u, and α are the cylindrical coordinates in the velocity space

$$v_1 = w\cos\alpha \ , \quad v_2 = w\sin\alpha \ , \quad v_3 = u \quad ; \quad \underline{H}_o = H_o\,\hat{\underline{e}}_3 \ .$$

Clearly, Maxwell-Boltzmann (MB) distribution is a special case of (1), which reads

$$f^{(0)}_{MB} = n\left(\frac{m}{2\pi\Theta}\right)^{\frac{3}{2}} \exp\left[-\frac{m}{2\Theta}(u^2+w^2)\right] . \tag{III-2}$$

In this latter case one can easily derive the following relations:

$$\begin{aligned}
\bar{\rho} &= -en\, G'_{pk}\frac{\partial \bar{E}'_p}{\partial x_k} = -G'_{kp}\frac{\partial \bar{J}_p}{\partial x_k} - 2\frac{\Theta}{m}\frac{\partial^2\bar{\rho}}{\partial x_p \partial x_k}M'_{pk} \\
\bar{J}_k &= en\, R'_{pk}\bar{E}'_p - \frac{\Theta}{m}G'_{pk}\frac{\partial\bar{\rho}}{\partial x_p} \\
\bar{\Psi}_{ik} &= \frac{\Theta}{m}\bar{\rho}\,\delta_{ik} - \frac{\Theta}{m}\left[G'_{pi}\frac{\partial \bar{J}_k}{\partial x_p} + G'_{pk}\frac{\partial \bar{J}_i}{\partial x_p}\right] - \left(\frac{\Theta}{m}\right)^2 G'_{pi}G'_{qk}\frac{\partial^2\bar{\rho}}{\partial x_p\partial x_q} \\
\bar{p} &= \frac{5}{3}\frac{\Theta}{m}\bar{\rho} + \frac{2}{3}\left(\frac{\Theta}{m}\right)^2 M'_{pq}\frac{\partial^2\bar{\rho}}{\partial x_p \partial x_q} = \frac{4}{3}\frac{\Theta}{m}\bar{\rho} - \frac{1}{3}\frac{\Theta}{m}G'_{pq}\frac{\partial \bar{J}_q}{\partial x_p} \\
\bar{q}_k &= -\left(\frac{\Theta}{m}\right)^2 G'_{pk}\frac{\partial\bar{\rho}}{\partial x_p} + \left(\frac{\Theta}{m}\right)^2 M'_{pq}\frac{\partial^2 \bar{J}_k}{\partial x_p\partial x_q}
\end{aligned} \tag{III-3}$$

where

$$\bar{\chi} \equiv \int_{\underline{v}} d^3v\, \hat{\chi} \qquad , \qquad \chi^{(1)} \equiv \int_0^t ds'\, e^{-\frac{s'}{\tau}}\, \bar{\chi}(\underline{x},t,s')$$

$$\bar{E}'_k = \int_{\underline{v}} d^3v\, E_k^{(1)}(\underline{x} - \underline{\underline{G}}'\cdot\underline{v}\, , t-s')\, \hat{f}^{(0)}(v) \quad ,$$

$$\hat{f}^{(0)} = \frac{1}{n} f^{(0)} . \qquad \text{(III-3a)}$$

We have already pointed out the difficulty encountered in dealing with the moment equations of the Boltzmann equation when the collision model (5-I) is adopted to represent the short range interactions. Nevertheless, whenever $\tau \to \infty$ (i.e., the collisionless case) this issue does not exist. In this latter case the moment equations of the perturbed state, when $f^{(0)} \in MB$, read

$$\frac{\partial \rho^{(1)}}{\partial t} + \frac{\partial j_k^{(1)}}{\partial x_k} = 0$$

$$\frac{\partial j_k^{(1)}}{\partial t} + \frac{\partial}{\partial x_p}\Psi_{kp}^{(1)} + \Omega\, \varepsilon_{3pk}\, j_p^{(1)} - en\, E_k^{(1)} = 0$$

$$\frac{\partial p^{(1)}}{\partial t} + \frac{2}{3}\frac{\partial q_k^{(1)}}{\partial x_k} + \frac{5}{3}\frac{\Theta}{m}\frac{\partial j_k^{(1)}}{\partial x_k} = 0 . \qquad \text{(III-4)}$$

It can be shown that (4) is satisfied by the set (3) when $\frac{1}{\tau} = 0$. Although in the case of finite τ, by a suitable interpretation of the amount of the properties transferred by collisions, the above mentioned difficulty may be overcome, it will not be necessary for us to introduce it here since we shall be using the Equation (3) explicitly without referring to (4). Some considerations will be given later to a possible employment of (4) simultaneously with (3).

Let us consider the more general case, by allowing $f^{(0)}(u,w)$ to be an arbitrary function of its arguments. Then the first order averages may be studied by making use of a Laplace-Fourier (FL) analysis in time and space, respectively. We define

$$\chi^{FL}(\underline{k},p) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty dt\, e^{-pt} \int_{\underline{x}} d^3x\, e^{-i\underline{k}\cdot\underline{x}}\, \chi(\underline{x},t) \tag{III-5}$$

Hence, one obtains

$$f^{FL}(\underline{k},\underline{v},p) = -\frac{e}{m}\frac{\partial f^{(0)}}{\partial v_j} E_k^{FL} \int_0^\infty N_{kj}(t')\exp\left[y(t')\right]dt', \tag{III-6}$$

where*

$$N_{kj} = R_{kj} + i\,\frac{1}{p}\,\delta^{sq}_{k\ell}\, k_\ell\, v_d\, R_{sj} R_{qd},$$

$$y(t) = -i\,\underline{k}\cdot\underline{\underline{G}}\cdot\underline{v} - \left(p+\frac{1}{\tau}\right)t. \tag{III-7}$$

Thus

$$\psi(\underline{v})\, f^{FL} = \frac{e}{m} E_k^{FL} f^{(0)} \int_0^\infty dt'\, N'_{ks}\, e^{y'}\left[\frac{\partial\psi}{\partial v_s} - i\,k_\ell G'_{\ell s}\psi\right]. \tag{III-8}$$

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* The N dimensional Kronecker delta may be defined as

$$\delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \mathrm{Det}\left(\delta_{r_i s_j}\right) = \begin{vmatrix} \delta_{r_1 s_1} & \delta_{r_1 s_2} & \cdots & \delta_{r_1 s_n} \\ \vdots & & & \vdots \\ \delta_{r_n s_1} & \delta_{r_n s_2} & \cdots & \delta_{r_n s_n} \end{vmatrix},$$

(in our problem clearly N = 3), where $r_1, r_2, \ldots, r_n$ and $s_1, s_2, \ldots, s_n$ are positive integers $\leq$ N.

It can easily be shown that, upon performing the azimuthal angle integration, one obtains

$$\int_{\underline{v}} d^3v \; e^{-i\underline{k}\cdot\underline{\underline{G}}\cdot\underline{v}} f^{(0)}(u,\omega) = (2\pi)^{\frac{3}{2}} f^{FH_0}(\nu,\kappa) ,$$

where

$$\chi^{FH_0}(\nu,\kappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du \, e^{-\nu u} \int_0^{\infty} \omega \, d\omega \, J_0(\kappa\omega)\, \chi(u,\omega),$$

$$\nu = k_{\parallel} t \quad , \qquad \kappa = \frac{2}{\Omega} k_{\perp} \sin\frac{\Omega t}{2}$$

$$k_{\perp} = \sqrt{k_1^2 + k_2^2} \quad , \qquad k_{\parallel} = k_3 . \tag{III-9}$$

Hence,

$$(2\pi)^{-\frac{3}{2}} \int_{\underline{v}} v_{j_1} v_{j_2} \cdots v_{j_n} \, e^{-i\underline{k}\cdot\underline{\underline{G}}\cdot\underline{v}} f^{(0)} d^3v = (i)^n \, G^{-1}_{j_1 p_1} G^{-1}_{j_2 p_2} \cdots G^{-1}_{j_n p_n} \frac{\partial^n f^{FH_0}_{(\nu,\kappa)}}{\partial k_{p_1} \partial k_{p_2} \cdots \partial k_{p_n}} ,$$

which indicates that the first order average of any property that can be exhibited in terms of the powers of v_k, is expressible by the Fourier-Hankel transform of $f^{(0)}$ and its derivatives. In particular one has

$$\rho^{FL} = -ien(2\pi)^{\frac{3}{2}} E_k^{FL} k_s \int_0^{\infty} dt \, e^{-(p+\frac{1}{\tau})t} \times$$
$$\times \left[G_{ks} - \frac{1}{p} \delta^{\ell r}_{kj} k_j G_{\ell s} G^{-1}_{nr} \frac{\partial}{\partial k_n} \right] \hat{f}^{FH_0}_{(\nu,\kappa)}$$

$$j_k^{FL} = en(2\pi)^{\frac{3}{2}} \; E_\ell^{FL} \int_0^{\infty} dt \; e^{-(p+\frac{1}{\tau})t} \times$$
$$\times \left[R_{\ell k} + k_n \left(G_{\ell n} G^{-1}_{km} - \frac{1}{p} \delta^{sg}_{\ell n} R_{sk} G^{-1}_{mg} \right) \frac{\partial}{\partial k_m} - \right.$$
$$\left. - k_n k_m \frac{1}{p} \delta^{sg}_{\ell n} G_{sm} G^{-1}_{pg} G^{-1}_{kj} \frac{\partial^2}{\partial k_p \partial k_j} \right] \hat{f}^{FH_0}_{(\nu,\kappa)} . \tag{III-10}$$

Let us finally note that the FH_o transforms for some special $\hat{f}^{(0)}(u,w)$ read as follows:

$$\hat{f}^{(0)} = \delta(u)\,\frac{\delta(w)}{2\pi(w)} \quad \Longrightarrow \quad \hat{f}^{FH_o} = (2\pi)^{-\frac{3}{2}}$$

$$\hat{f}^{(0)} \in MB \quad \Longrightarrow \quad \hat{f}^{FH_o} = (2\pi)^{-\frac{3}{2}}\ \exp\left[-\frac{\Theta}{m}\,\underline{k}\cdot\underline{\underline{M}}\cdot\underline{k}\right] \tag{III-11}$$

IV. ELECTROMAGNETIC PROPERTIES OF THE PLASMA

A. Electric Conductivity

In this section we shall assume that the conditions of the special case discussed in the last section are satisfied. Accordingly, one can introduce the FL analysis as before. The electric current density now can be computed as

$$J_k^{FL} = \sum_r \frac{e}{m} j_k^{FL}$$

If one defines the conductivity as being the tensorial relation between the FL components of the induced current and the electric field, one gets the following, by using the Equation (III-10):

$$J_k^{FL} = \sigma_{kj} E_j^{FL} , \tag{IV-1}$$

where

$$\sigma_{kj} = \sum_r \frac{\omega_p^2}{4\pi} (2\pi)^{\frac{3}{2}} \int_0^\infty dt \, e^{-(p+\frac{1}{\tau})t} \, S_{kj}(\hat{f}^{FH_0})$$

$$S_{kj}(\hat{f}^{FH_0}) = \Big[R_{jk} + k_p \big(G_{jp} G^{-1}_{kn} - \frac{1}{p} \delta^{lm}_{jp} R_{lk} G^{-1}_{nm}\big) \frac{\partial}{\partial k_n} - \frac{1}{p} k_p k_q \delta^{ls}_{jp} G_{lq} G^{-1}_{ms} G^{-1}_{kn} \frac{\partial^2}{\partial k_m \partial k_n} \Big] \hat{f}^{FH_0}(\nu, \kappa)$$

$$\omega_p^2 = \frac{4\pi n e^2}{m} \tag{IV-2}$$

In particular, if one assumes that $\underline{E}^{(1)}$ is constant, then it can easily be shown that (2) reduces asymptotically to the well-known result:

$$\sigma_{kj} = \sum_r \sigma_0 \int_0^\infty \frac{dt}{\tau} e^{-\frac{t}{\tau}} R_{jk}$$

$$= \sum_r \sigma_0 \begin{pmatrix} \frac{1}{1+\Omega^2\tau^2} & + \frac{\Omega\tau}{1+\Omega^2\tau^2} & 0 \\ \frac{-\Omega\tau}{1+\Omega^2\tau^2} & \frac{1}{1+\Omega^2\tau^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{IV-3}$$

where

$$\sigma_0 = \frac{1}{4\pi} \omega_p^2 \tau .$$

Here $\underline{\underline{\sigma}}$ is independent of k and p, so that (1) can be inverted easily. (e.g., Reference 5; or Reference 4) It can easily be verified that asymptotically the same result is obtained for the initially "cold" plasma case, i.e., $\hat{f}^{FH_0} = (2\pi)^{-\frac{3}{2}}$.

If $f^{(0)} \in MB$, then one gets

$$\sigma_{kj} = \sum \sigma_0 \int_0^\infty \exp[\Phi] \left\{ R_{jk} - \frac{\Theta}{m} k_p G_{pk} G_{jq} k_q \right\} \frac{dt}{\tau}, \tag{IV-4}$$

where

$$\Phi = -(p + \frac{1}{\tau}) t - \frac{\Theta}{m} \underline{k} \cdot \underline{\underline{M}} \cdot \underline{k} .$$

Apart from notation this is the result of L. Mower.[6]

B. Field Equation and Dispersion Relation

In this subsection we shall follow closely, but in somewhat generalized manner, the method of Bernstein in deducing a formal solution to the special case of Section III, (cf., Reference 7).

First, let us compute the electric charge density, using the Equation (III-10)

$$Q^{FL} = \sum_r \frac{e}{m} \rho^{FL}$$

$$= \sum_r \frac{\omega_p^2}{4\pi i} (2\pi)^{\frac{3}{2}} E_k^{FL} k_j \int_0^\infty dt \, e^{-(p+\frac{1}{\tau})t} \, Q_{jk}(\hat{f}^{FH_0}),$$

$$Q_{jk}(\hat{f}^{FH_0}) \equiv \left[G_{kj} - \frac{1}{p} k_m \delta_{km}^{s\ell} G_{sj} G_{n\ell}^{-1} \frac{\partial}{\partial k_n}\right] \hat{f}^{FH_0}(\nu, \kappa) . \quad \text{(IV-5)}$$

If we substitute the Equations (1) and (5) in the FL transform of the Maxwell's equations, which after eliminating the magnetic field terms as usual give

$$(k^2 c^2 + p^2) E_\ell^{FL} + i k_\ell c^2 4\pi Q^{FL} + 4\pi p J_\ell^{FL}$$

$$= \zeta_\ell = \text{initial conditions} , \quad \text{(IV-6)}$$

we obtain

$$T_{\ell k} E_k^{FL} = \zeta_\ell$$

$$E_j^{FL} = T_{j\ell}^{-1} \zeta_\ell \quad \text{(IV-7)}$$

$$= 3 \, \frac{\delta_{\ell q s}^{j p r} \, T_{pq} \, T_{rs} \, \zeta_\ell}{\delta_{pqr}^{ik\ell} \, T_{ip} \, T_{kq} \, T_{\ell r}} ,$$

where the generic term of $\underline{\underline{T}}$ is given by

$$T^r_{jk} = (k^2c^2 + p^2)\,\delta_{jk} + c^2\omega_p^2\, k_j k_\ell\, (2\pi)^{\frac{3}{2}} \int_0^\infty dt\, e^{-(p+\frac{1}{\tau})t}\, Q_{\ell k}(\hat{f}^{FH_0})$$
$$+ \omega_p^2\, p\, (2\pi)^{\frac{3}{2}} \int_0^\infty dt\; e^{-(p+\frac{1}{\tau})t}\; S_{jk}(\hat{f}^{FH_0}) \,,$$

$$\underline{\underline{T}} = \sum_r T^r_{jk} \,. \qquad \text{(IV-8)}$$

It is to be understood that the right hand side of (6), namely $\underline{\zeta}$, is to contain all the terms involving the initial perturbations of the distribution functions, if any, that have been consistently suppressed in our formulations, as well as the initial perturbations of the electromagnetic fields that may be introduced into our system.

The Equation (7) provides a formal solution to our initial value problem, since it enables one, at least in principle, to compute the transport properties of the plasma under consideration in terms of the initial perturbations only, when it is substituted in the Equation (III-3), although the complexity of the mathematical structure of the operations involved considerably reduces its practical applicability. Nevertheless, the questions involving the plasma oscillations (of course in the linearized sense) may be treated with considerable success for many cases of interest.[7] When the numerator of the Equation (7) is an analytic function of p, the dispersion relation is given by

$$\mathrm{Det}(\underline{\underline{T}}) \equiv \frac{1}{3!}\, \delta^{ijk}_{pqr}\, T_{ip} T_{jq} T_{kr} = 0 \qquad \text{(IV-9)}$$

A detailed study of this equation is not our concern here. However, we shall consider some famous particular cases, which are given some special attention in the literature (e.g., Reference 7 and Reference 5), in order to check the consistency of our formulations.

First, let us consider an initially cold plasma, for which case the Equation (8) reduces to

$$T^r_{jk} = (k^2c^2 + p^2)\,\delta_{jk} + c^2\omega_p^2 k_j k_l \int_0^\infty dt\, e^{-(p+\frac{1}{\tau})t}\, G_{kl}$$
$$+ \omega_p^2 p \int_0^\infty dt\; e^{-(p+\frac{1}{\tau})t}\; R_{kj} \tag{IV-10}$$

Moreover, assuming $H_0 = 0$, $\frac{1}{\tau} = 0$; one gets

$$T_{jk} = \left(k^2c^2 + p^2 + \sum_r \omega_p^2\right)\delta_{jk} + c^2 k_j k_k \frac{1}{p^2}\sum_r \omega_p^2 \tag{IV-11}$$

and the dispersion relation is obtained as follows:

$$\mathrm{Det}(\underline{T}) = \left(k^2c^2 + p^2 + \sum_r \omega_p^2\right)^2 \frac{1}{p^2}\left(p^2 + \sum_r \omega_p^2\right)\left(k^2c^2 + p^2\right) = 0 \tag{IV-12}$$

the first factor of which corresponds to the well-known phase velocity of Langmuir

$$V_{ph.}^2 = \frac{\omega^2}{k^2} = \frac{c^2}{1 - \dfrac{\sum \omega_p^2}{\omega^2}}\,, \qquad (p = i\omega) \tag{IV-13}$$

As a second application let us consider $\underline{E}^{(1)}$ as being pure transversal such that $\underline{k}\cdot\underline{E}^{FL}=0$ and $\underline{k}\times\underline{H}_0=0$. Again assuming an initially cold, collisionless plasma we get

$$T_{jk} = (k^2c^2+p^2)\,\delta_{jk} + p\sum_r \omega_p^2 \int_0^\infty dt\, e^{-pt} R_{kj}(t). \tag{IV-14}$$

Thus the dispersion relation in this case reads

$$\left(k^2c^2 + p^2 + p^2 \sum_r \frac{\omega_p^2}{p^2+\Omega^2}\right)^2 + \left(p\sum_r \frac{\omega_p^2\,\Omega}{p^2+\Omega^2}\right)^2 = 0 \tag{IV-15}$$

associated with the phase velocity

$$V_{ph}^2 = \frac{\omega^2}{k^2} = \frac{c^2}{1-\sum_r \frac{\omega_p^2}{\omega^2}\frac{1}{1\pm\frac{\Omega}{\omega}}}\,, \qquad (p=i\omega) \tag{IV-16}$$

The structure of $\underline{\underline{T}}$ implies that $\underline{E}^{(1)}$ clearly is circularly polarized. This is Spitzer's[5] result of ordinary and extraordinary waves corresponding to the signs + and -, respectively. If we assume a binary plasma with singly ionized ions, and the electron to ion mass ratio, μ, to be much less than unity, we get

$$V_{ph}^2 \cong \frac{c^2}{1-\dfrac{1}{\dfrac{\omega(\omega\mp\Omega_{(-)})}{\omega_{p(-)}^2}-\dfrac{a^2}{c^2}}}\,, \tag{IV-17}$$

where

$$a^2 = \frac{H_0^2}{4\pi \rho_{(+)}} = \frac{\Omega^2_{(+)}}{\omega^2_{p(+)}} c^2 \qquad \text{(Alfven's velocity)}. \qquad \text{(IV-18)}$$

It is seen that if the following conditions are satisfied, we have:

i) $$\frac{\omega}{\Omega_{(-)}} \ll \mu \ll \frac{\omega^2_{p(-)}}{\Omega^2_{(-)}} \qquad \Longrightarrow \qquad V_{ph} \sim a$$

or

ii) $$\omega = \Omega_{(-)}\,,\ \frac{a^2}{c^2} \ll 1 \qquad \text{(for the extraordinary waves)} \Longrightarrow V_{ph} \sim a.$$

This case corresponds to the so-called magnetohydrodynamic (or Alfven) waves.

Finally, let us consider the longitudinal oscillations. Here, applying the assumption that $\underline{H}^{(1)}$ is ignorable, so that a pure longitudinal oscillation can be isolated, we get (e.g., Reference 8)

$$\underline{k} \times \underline{E}^{FL} = 0 \qquad \Longrightarrow \qquad \underline{E}^{FL} = -i\,\underline{k}\,\phi^{FL}$$

In this case the Equation (5) is more appropriate to work with, since it gives the result more readily.

$$\phi^{FL} = \frac{\text{Initial Conditions}}{k^2 + (2\pi)^{\frac{3}{2}}\, k_j k_\ell \sum_r \omega_p^2 \int_0^\infty dt\, e^{-(p+\frac{1}{\tau})t}\, G_{j\ell}\, \hat{f}^{FH_0}(\nu, \kappa)}\,. \qquad \text{(IV-19)}$$

If the numerator is an analytic function of p, then the dispersion relation is given as

$$\frac{k_j k_\ell}{k^2} \sum_r \omega_p^2 \int_0^\infty dt\, e^{-(p+\frac{1}{\tau})t}\, G_{j\ell}\, \hat{f}^{FH_0}(\nu, \kappa) = -(2\pi)^{-\frac{3}{2}} \qquad \text{(IV-20)}$$

This result corresponds to the Harris' dispersion relation,[8] but it is given in the closed form. In particular, if one sets $H_0 = 0$, $\frac{1}{\tau} = 0$, it can easily be shown that the relation (20) reduces to the well-known Vlassov equation

$$1 + \sum_r \omega_p^2 \int d^3v \, \frac{\hat{f}^{(0)}(u,w)}{(p + i\,\underline{k}\cdot\underline{v})^2} = 0 \, , \tag{IV-21}$$

which is studied by many authors. (e.g., Reference 9.)

The Equation (19) seems to have computational advantages. Harris[8] in his letter considered the instabilities induced by the anisotropic unperturbed distributions, while Bernstein[7] has shown the absence of the exponentially increasing solutions whenever $f^{(0)} \in MB$. Thus it seems of some interest to investigate the class of functions which gives rise to such instabilities. Harris has considered two particular functions only. This, perhaps, is due to the long computational procedure, involving infinite series ranging from $-\infty$ to $+\infty$. It seems that, instead of specifying $f^{(0)}(u,w)$, as Harris did, one may select the function $f^{FH_0}(\nu,\kappa)$ which allows instabilities, using (20), then compute $f^{(0)}(u,w)$ by inverting the transform, if necessary.

V. PLASMA PARAMETERS

A. "Mobility" and "Diffusion" Tensors ($f^{(0)} \in MB$)

Let us assume that the unperturbed state of our plasma satisfies the conditions of the special case considered in Section III, and $f^{(0)} \in MB$. Then it can be shown that the first order transport properties in the FL space may be expressed as

$$
\begin{aligned}
\bar{\rho}^{FL} &= -en\, i k_j E_k^{FL} G_{kj} = -i k_j G_{jk} \bar{j}_k^{FL} + 2 \frac{\Theta}{m} k_j k_k M_{jk} \bar{\rho}^{FL} \\
\bar{j}_k^{FL} &= en R_{jk} E_j^{FL} - i \frac{\Theta}{m} k_j G_{jk} \bar{\rho}^{FL} \\
\bar{\Psi}_{kj}^{FL} &= \frac{\Theta}{m} \bar{\rho}^{FL} \delta_{kj} - \frac{\Theta}{m} i k_\ell \left[G_{\ell k} \bar{j}_j^{FL} + G_{\ell j} \bar{j}_k^{FL} \right] + \left(\frac{\Theta}{m}\right)^2 k_\ell k_n G_{\ell k} G_{nj} \bar{\rho}^{FL} \\
\bar{p}^{FL} &= \frac{5}{3} \frac{\Theta}{m} \bar{\rho}^{FL} - \frac{2}{3} \left(\frac{\Theta}{m}\right)^2 k_\ell k_j M_{\ell j} \bar{\rho}^{FL} = \frac{4}{3} \frac{\Theta}{m} \bar{\rho}^{FL} - \frac{1}{3} \frac{\Theta}{m} i k_j G_{j\ell} \bar{j}_\ell^{FL} \\
\bar{q}_k^{FL} &= -\left(\frac{\Theta}{m}\right)^2 i k_j G_{jk} \bar{\rho}^{FL} - \left(\frac{\Theta}{m}\right)^2 k_\ell k_j M_{\ell j} \bar{j}_k^{FL} ,
\end{aligned}
\tag{V-1}
$$

where

$$
\begin{aligned}
\chi^{FL} &= \int_0^\infty \exp\left[\Phi(t')\right] \bar{\chi}^{FL}(\underline{k}, p, t')\, dt' \\
\Phi(t) &= -\left(p + \frac{1}{\tau}\right) t - \frac{\Theta}{m} \underline{k} \cdot \underline{\underline{M}}(t) \cdot \underline{k} .
\end{aligned}
\tag{V-2}
$$

Clearly, knowing the solution of the field equation [cf. the Equation (IV-7)] corresponding to the initial value problem, one can formally compute the transport properties of interest by making use of (1). But due to the complexity of the mathematical operations involved, the above described method seems to involve almost formidable labor in order to deduce physically interpretable analytical expressions for these properties. Nevertheless, in this particular case under consideration, one can define certain tensors (in a sense, operators), resembling the general character of the classical concepts, such as mobility, diffusion, viscosity, thermal conductivity ..., by observing the mathematical structure of the Equation (1), which may be approximated, if necessary, by an averaging process suitable to the physical conditions of a given problem. In what follows we shall restrict our attention to the mobility and the diffusion only, although the extension of the general procedure employed to the other parameters does not involve much complexity. Therefore, only the first two equations of (1) need to be considered.

Let us introduce the tensors K^{FL}_{jk} and D^{FL}_{jk} as follows:

$$K^{FL}_{jk} \equiv \frac{e}{m} \int_0^\infty \exp[\Phi'] \, R'_{kj} \, dt'$$

$$D^{FL}_{jk} \equiv \frac{\Theta}{m} \frac{\int_0^\infty \exp[\Phi'] \, G'_{kj} \, \bar{\rho}^{FL} \, dt'}{\int_0^\infty \exp[\Phi'] \, \bar{\rho}^{FL} \, dt' \quad [\equiv \rho^{FL}]}, \tag{V-3}$$

which we shall refer to as "the 'mobility' and the 'diffusion' tensors in the FL space," respectively. Thus, by integrating the second equation of (1), one obtains

$$j^{FL}_k = nm \, K^{FL}_{k\ell} E^{FL}_\ell - i \, k_\ell D^{FL}_{k\ell} \rho^{FL}, \tag{V-4}$$

the mathematical form of which justifies our terminology to some extent.

We observe that the mobility tensor defined above is given explicitly is terms of the known functions only. However, this is not the case for the diffusion tensor, in general, which can be written in terms of the induced field as

$$D_{jk}^{FL} = \frac{\Theta}{m} \frac{k_p E_q^{FL} \int_0^\infty \exp[\Phi'] G'_{kj} G'_{qp} \, dt'}{k_p E_q^{FL} \int_0^\infty \exp[\Phi''] G''_{qp} \, dt''} , \tag{V-4}$$

or in terms of the initial conditions as

$$D_{jk}^{FL} = \frac{\Theta}{m} \frac{k_m \delta_{s\ell n}^{pqr} T_{ps} T_{q\ell} \zeta_n \int_0^\infty \exp[\Phi] G_{kj} G_{rm} \, dt}{k_m \delta_{s\ell n}^{pqr} T_{ps} T_{q\ell} \zeta_n \int_0^\infty \exp[\Phi] G_{rm} \, dt} . \tag{V-5}$$

Nevertheless, there are some special cases of interest for which the field dependence in (4) [therefore the initial conditions in (5)] cancels out. Among these, the following cases may be mentioned.

Case I. $\underline{H}_0 = 0$ Hence $G_{kj} = t\,\delta_{kj}$, thus

$$D_{jk}^{FL} = \frac{\Theta}{m} \frac{\int_0^\infty \exp[\Phi] \, t^2 \, dt}{\int_0^\infty \exp[\Phi] \, t \, dt}$$

where

$$\Phi(t) = -(p + \tfrac{1}{\tau})t - \frac{\Theta}{m}\frac{k^2 t^2}{2} .$$

Upon performing the indicated integration one obtains

$$D_{jk}^{FL} = \frac{2\Theta}{m}\,\frac{\tau}{1+p\tau}\left[\frac{b\,(1+2b^2)\frac{\sqrt{\pi}}{2}\,\mathrm{erfc}\,(b) - b^2 e^{-b^2}}{e^{-b^2} - 2b\frac{\sqrt{\pi}}{2}\,\mathrm{erfc}\,(b)}\right]\delta_{jk} \tag{V-6}$$

where $b = \sqrt{\frac{m}{2\Theta}}\,\frac{1+p\tau}{\tau k}$. For $b \gg 1$ on has

$$D_{jk}^{FL} \cong \frac{2\Theta}{m}\,\frac{\tau}{1+p\tau}\,\frac{\;\cdot\; + \frac{3}{4b^2}}{1 - \frac{3}{2b^2}} \rightarrow \frac{2\Theta}{m}\tau \quad \text{as} \quad p \to 0,\ k \to 0\,. \tag{V-7}$$

<u>Case II</u>. Since one has

$$\underline{E}^{FL}\cdot\underline{\underline{G}}\cdot\underline{k} = \frac{\sin\Omega t}{\Omega}(\underline{k}\cdot\underline{E}^{FL}) + \frac{1-\cos\Omega t}{\Omega}(\underline{k}\times\underline{E}^{FL})_3 + \frac{\Omega t - \sin\Omega t}{\Omega}\,k_3 E_3^{FL}$$

then in the following subcases the diffusion tensor is unaffected by the induced fields:

i) Pure transverse $\underline{E}^{(1)}$, perpendicular to $\underline{H}_0$, i.e., $E_3^{(1)} = 0$,

and

(a) $Q^{(1)} = 0\,,\quad \frac{\partial}{\partial t}H_3^{(1)} \neq 0$ or

(b) $Q^{(1)} \neq 0\,,\quad \frac{\partial}{\partial t}H_3^{(1)} = 0\,.$

ii) $E_3^{(1)} \neq 0$, but $Q^{(1)} = 0$ and $\frac{\partial}{\partial t}H_3^{(1)} = 0$ both.

There are also some cases (to be considered later) involving the linearly polarized $\underline{E}^{(1)}$ and a fixed direction of propagation, in which the same property may be observed.

It is of some relevance to note that a diffusion tensor may be defined in this particular case without referring to the FL transformation. To do this we consider the second equation of (III-3) in the integrated form, which is given as

$$j_k^{(1)} = \int_0^t ds' \, e^{-\frac{s'}{\tau}} \left[en \, R'_{jk} \bar{E}'_j - \frac{\Theta}{m} G'_{jk} \frac{\partial \bar{\rho}}{\partial x_j} \right] . \tag{V-8}$$

Identifying the second term of the right hand side as the diffusion current, we can define

$$D_{jk} = \frac{\Theta}{m} \, \frac{\int_0^t ds' \, e^{-\frac{s'}{\tau}} \, G'_{kj} \, \bar{\rho}}{\int_0^t ds' \, e^{-\frac{s'}{\tau}} \, \bar{\rho} \quad [\equiv \rho^{(1)}]} . \tag{V-9}$$

There are some special cases of interest, which will be considered later, for which the latter form seems to be more suitable for a given approximation procedure, than the relation (5) which defines the diffusion tensor in the FL space in terms of the initial conditions. However, because of the fact that in general $\bar{E}'_k$ [cf., the Equation (III-3a)] has different values for every particular species under consideration, a concept of the mobility tensor based on the coefficient of $\bar{E}'_k$ in the first term of the right hand side of (8) seems no more relevant, unless some additional assumptions are introduced. This will be done in Section VI.

Some general considerations will be given later to the case in which the conditions of the special case of Section III are not satisfied.

B. Ambipolar Diffusion

Due to the different diffusion rates of the charged particles in an initially neutral plasma, a space charge is developed soon after the introduction of the perturbations. This space charge provides a restoring mechanism that ultimately may form a different type of diffusion in which the charged particles diffuse with equal rates, essentially preserving the charge neutrality. (cf., Reference 10 and Reference 11.) This process, which will be called the "ambipolar diffusion" will be our next subject.

Let us consider a binary plasma which consists of electrons and singly charged ions. Thus the Equation (4) can be written for each type of particles by setting

$$\rho^{FL(\mp)} = m^{(\mp)} \; n^{FL(\mp)}$$

$$j_k^{FL(\mp)} = n\, m^{(\mp)} \; \omega_k^{FL(\mp)} , \tag{V-10}$$

where n is the unperturbed density which is the same for both particles, and the signs (+) and (-) denote the ions and the electrons, respectively. Thus, we have

$$n\, \omega_j^{FL(+)} = n\, \kappa_{jk}^{FL(+)} E_k^{FL} - i\, k_k D_{jk}^{FL(+)} \; n^{FL(+)}$$

$$n\, \omega_j^{FL(-)} = n\, \kappa_{jk}^{FL(-)} E_k^{FL} - i\, k_k D_{jk}^{FL(-)} \; n^{FL(-)} . \tag{V-11}$$

By making use of the commutativity property of $\underline{\underline{K}}^{FL(+)}$ and $\underline{\underline{K}}^{FL(-)}$, which can be shown without much difficulty, one can eliminate the ohmic current terms in (11), to get

$$n\left[K_{jk}^{FL(-)}\omega_k^{FL(+)} - K_{jk}^{FL(+)}\omega_k^{FL(-)}\right] = -i\,k_k\left[K_{jq}^{FL(-)}D_{qk}^{FL(+)}n^{FL(+)} - K_{jq}^{FL(+)}D_{qk}^{FL(-)}n^{FL(-)}\right]. \tag{V-12}$$

Furthermore, as is customary in the elementary theory of the ambipolar diffusion, one can assume the following conditions:

$$\begin{aligned} n^{FL(+)} &= n^{FL(-)} \equiv n^{FL} \\ \underline{\omega}^{FL(+)} &= \underline{\omega}^{FL(-)} \equiv \underline{\omega}^{FL}\ , \end{aligned} \tag{V-13}$$

which gives

$$n\,\omega_j^{FL} = -i\,k_\ell\,\left[D_{j\ell}^{AMB.}\right]^{FL}\,n^{FL}\ , \tag{V-14}$$

where

$$\left[\underline{\underline{D}}^{AMB.}\right]^{FL} = \frac{1}{\left[\underline{\underline{K}}^{FL(-)} - \underline{\underline{K}}^{FL(+)}\right]}\cdot\left[\underline{\underline{K}}^{FL(-)}\cdot\underline{\underline{D}}^{FL(+)} - \underline{\underline{K}}^{FL(+)}\cdot\underline{\underline{D}}^{FL(-)}\right], \tag{V-15}$$

or

$$\left[D_{jk}^{AMB.}\right]^{FL} = \frac{1}{2}\,\frac{\delta_{pqr}^{jh\ell}\left[K_{hq}^{FL(-)} - K_{hq}^{FL(+)}\right]\left[K_{\ell r}^{FL(-)} - K_{\ell r}^{FL(+)}\right]}{\mathrm{Det}\left[\underline{\underline{K}}^{FL(-)} - \underline{\underline{K}}^{FL(+)}\right]}\left[K_{ps}^{FL(-)}D_{sk}^{FL(+)} - K_{ps}^{FL(+)}D_{sk}^{FL(-)}\right]. \tag{V-16}$$

Inverting the FL transform in the Equation (12), one obtains

$$n\omega_k^{(1)} = -\int_{\underline{x}'}\int_0^t d^3x'\, dt'\; D_{kj}^{AMB}(\underline{x}',t')\, \frac{\partial n^{(1)}}{\partial x_j}(\underline{x}-\underline{x}',t-t')\,. \tag{V-17}$$

Thus, one may write

$$n\omega_k^{(1)} = -\frac{\partial}{\partial x_j}\left[\langle D_{kj}^{AMB}\rangle\, n^{(1)}\right] \quad , \tag{V-18}$$

where

$$\langle D_{kj}^{AMB}\rangle = \frac{\int_{\underline{x}'}\int_0^t d^3x'\, dt'\; D_{kj}^{AMB}(\underline{x}',t')\, n^{(1)}(\underline{x}-\underline{x}',t-t')}{\int_{\underline{x}'}\int_0^t d^3x'\, dt'\; n^{(1)}(\underline{x}',t')}\,. \tag{V-19}$$

Making use of the continuity equation one can deduce a "diffusion" equation as

$$\frac{\partial n^{(1)}}{\partial t} = \frac{\partial^2}{\partial x_k\,\partial x_j}\left[\langle D_{kj}^{AMB}\rangle\, n^{(1)}\right]\,. \tag{V-20}$$

Finally, it can be pointed out that for sufficiently smoothly varying functions $n^{(1)}$, one has approximately

$$\langle \underline{\underline{D}}^{AMB}\rangle \approx \lim_{\substack{k\to 0\\ p\to 0}} \left[\underline{\underline{D}}^{AMB}\right]^{FL}\,.$$

This limiting process can be seen to correspond to the case in which the higher derivatives of $n^{(1)}$ may be ignorable. Some further discussion on this matter will be given in Section VI.

C. "Diffusion Tensor" in the Case of the Anisotropic $f^{(o)}$

The concept of the diffusion tensor introduced earlier for the case of the Maxwellian $f^{(o)}$ becomes somewhat ambiguous in the anisotropic case, since the current Equation (II-4) does not seem to have an explicit dependence on $\underline{\nabla}\rho^{(1)}$ (or rather $\underline{\nabla}\hat{\rho}$). It is of some interest to note that whenever $f^{(0)}$ satisfies the differential equation

$$\frac{\partial}{\partial \gamma_k} f^{(0)}(\underline{\xi},\underline{\gamma}) = -A_{kp}(\underline{\xi})\gamma_p f^{(0)}(\underline{\xi},\underline{\gamma}) - B_k(\underline{\xi}) f^{(0)}(\underline{\xi},\underline{\gamma}) , \quad (k=1,2,3.) \tag{V-21}$$

such a dependence can be exhibited. Clearly, (21) has a solution if $A_{kj} = A_{jk}$. Substituting (21) into the current equation, by eliminating the term $\upsilon_k \hat{\rho}$, one gets after some manipulations

$$\hat{j}_k = -\tilde{A}_{kp} G_{qp} \frac{\partial \hat{\rho}}{\partial x_q} + \tilde{B}_k \hat{\rho} + \tilde{V}_k \tag{V-22}$$

where

$$\begin{aligned}
\tilde{A}_{kj} &= R_{pk} A^{-1}_{pq} R_{qj} \\
\tilde{B}_k &= \mathcal{E}_p G_{pk} - R_{pk} A^{-1}_{pq} B_q \\
\tilde{V}_k &= R_{pk}\left[f^{(0)} F_p^{(1)'}\right] - \frac{e}{c}\varepsilon_{pqr} G_{pl} R_{qs} \frac{\partial}{\partial x_l}\left[\tilde{A}_{ks} f^{(0)} H_r^{(1)'}\right] ,
\end{aligned} \tag{V-23}$$

provided $\underline{\underline{A}}^{-1}$ exists. The Equation (22) allows one to define a diffusion tensor as

$$D_{jk} = \frac{\int_0^t ds' \, e^{-\frac{s'}{\tau}} G'_{kp} \int_{\upsilon} d^3\upsilon \, \tilde{A}_{jp}(\underline{\xi}') \hat{\rho}}{\int_0^t ds' \, e^{-\frac{s'}{\tau}} \int_{\upsilon} d^3\upsilon \, \hat{\rho} \quad [\equiv \rho^{(1)}]} , \tag{V-24}$$

where $$\underline{\xi}' = \underline{x} - \underline{\underline{G}}(s') \cdot \underline{v} + \underline{\underline{M}}(s') \cdot \underline{\mathcal{E}} \, .$$

This definition is to be compared with the Equation (V-9), rather than the Equation (V-3).

The general solution of (21) is readily found to be

$$f^{(0)} = C(\underline{\xi}) \exp\left[-\frac{1}{2} \gamma_j A_{jk} \gamma_k - \gamma_k B_k \right] \tag{V-25}$$

A consistency condition, which expresses the invariance property along the unperturbed orbits, has to be satisfied to insure that $f^{(0)}$ is also a solution of the zeroth order equation. This condition may be written as

$$\begin{aligned} &\frac{\partial}{\partial s}\left[\ln C(\underline{x} - \underline{\underline{G}} \cdot \underline{v} + \underline{\underline{M}} \cdot \underline{\mathcal{E}}) \right] \\ &\quad = \frac{\partial}{\partial s}\left[\frac{1}{2} A_{jk} (R_{je} v_e - G_{je} \mathcal{E}_e)(R_{kn} v_n - G_{kn} \mathcal{E}_n) - (R_{je} v_e - G_{je} \mathcal{E}_e) B_j \right] . \end{aligned} \tag{V-26}$$

where s is the unperturbed orbit parameter. If we assume that the zeroth state is i) uniform, ii) steady, iii) $\underline{\mathcal{E}} = 0$, as considered in Section III, we get

$$A_{jk} R_{jm} \mathring{R}_{kn} v_m v_n - \mathring{R}_{jk} B_j v_k = 0 \, ,$$

(the dot denotes the derivative with respect to s), which implies

$$\mathring{R}_{jk} B_j = 0$$

$$A_{pq} (R_{pj} \mathring{R}_{qk} + R_{pk} \mathring{R}_{qj}) = A_{pq} \frac{d}{ds}(R_{pj} R_{qk}) = 0$$

Therefore

$$\underline{B} = (0, 0, B),$$

$$\underline{\underline{A}} = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{33} \end{pmatrix}. \tag{V-27}$$

Clearly, the Maxwellian distribution is a special case of this class, corresponding to

$$B = 0, \quad A_{11} = A_{33} = \frac{m}{\Theta}.$$

If we assume no mass flow along $\underline{H}_0$, i.e., B = 0, (25) now can be written as*

$$f^{(0)} = n\left(\frac{m}{2\pi}\right)^{\frac{3}{2}} \frac{1}{\Theta_\perp\sqrt{\Theta_{\|}}} \exp\left[-\frac{m}{2\Theta_\perp}w^2 - \frac{m}{2\Theta_{\|}}u^2\right], \tag{V-28}$$

where $\frac{3}{2}\Theta_{\|}$ and $\frac{3}{2}\Theta_\perp$ correspond to the average kinetic energies, parallel and perpendicular to $\underline{H}_0$, respectively, such that the total average kinetic energy is

$$\frac{3}{2}\Theta = \frac{1}{2}\left(\Theta_{\|} + 2\Theta_\perp\right).$$

In the case when $\Theta_{\|}$ is "large" enough, so that for $\left(\frac{mu^2}{2\Theta_{\|}} \ll 1\right)$ we can write

$$\exp\left[-\frac{m}{2\Theta_{\|}}u^2\right] \approx \frac{1}{1 + \frac{m}{2\Theta_{\|}}u^2}.$$

---

* The first order transport properties corresponding to a zeroth order distribution as given in (28) have been computed, but because of their length, they will not be given here.

Upon re-normalization, (28) reduces to

$$f^{(0)} = \frac{n}{\pi^2} \frac{m}{2\Theta_\perp} \sqrt{\frac{m}{2\Theta_\parallel}} \; \frac{\exp\left[-\frac{m}{2\Theta_\perp} w^2\right]}{1 + \frac{m}{2\Theta_\parallel} u^2} . \tag{V-29}$$

Apart from notation, this is one of the distributions which is considered by Harris in his Letter,[8] who has shown the existence of the instabilities induced by such a $f^{(o)}$.

In the absence of an equation such as (21), in an effort to give a meaning to the diffusion tensor, one may proceed as follows. Combining the Equations (II-4 and II-5) properly, one has

$$\hat{j}_k - G'_{pq} v_k \frac{\partial \hat{j}_q}{\partial x_p} = G'_{pq} v_q v_k \frac{\partial \hat{\rho}}{\partial x_p} + R'_{jk} f^{(0)} F^{(1)\prime}_j .$$

Thus, whenever the second term of the left hand side is ignorable, again, in a sense, a concept of the diffusion tensor may be introduced as

$$D_{jk} = - \frac{\int_0^t ds' \, e^{-\frac{s'}{\tau}} G'_{kp} \int_{\underline{v}} d^3v \, v_j v_p \hat{\rho}}{\rho^{(1)}} .$$

VI. ATTEMPTED EVALUATION OF THE EARLIER WORKS ON PLASMA DIFFUSION

A. General Remarks

The diffusion of the charged particles in a weakly or fully ionized plasma has been of considerable interest in plasma research for some time. The results of the earlier experiments on arc plasmas[12] have indicated the diffusion rates of much larger amplitudes than that of the predicted values which are derived in the context of the classical collision-diffusion theory. In an attempt to explain this discrepancy, Bohm and co-workers have postulated (Reference 12, page 201), a new mechanism of diffusion which is produced by the "random electrostatic fields arising from turbulent type plasma oscillations."[13] Although the details of the theory of this so-called "drain-diffusion" mechanism have not been made available, an approximate formula for the coefficient of the diffusion across an externally applied magnetic field was given as

$$D_{\perp} \cong \frac{10^8 \, \Theta \quad (\text{ev})}{16 \, H_0 \ (\text{oersted})} , \tag{VI-1}$$

which predicts $D_{\perp}$ to the same order of amplitude as that of the experimentally observed values.

However, later studies at the ORNL (for a review, see Reference 14), have indicated that $D_{\perp}$ seems to be proportional to H_0^{-2}, rather than H_0^{-1} as the Equation (1) implies. This result, being in agreement with the prediction of the collision theory in the large magnetic field range (more precisely $\Omega\tau \gg 1$), led Simon[15] to the conclusion that the diffusion process, in this particular case, is not ambipolar in character, and it is the electron "short-circuit" through the end plates of the arc chamber that mainly maintains the charge neutrality, which is due to the much larger mobilities in the direction of the axis of the arc chamber, along which the magnetic field is applied. On the basis of this consideration, Simon derived the following formula:

$$D_{\perp} \cong D_{\perp}^{(+)} \approx \frac{D_o^{(+)}}{[\Omega_{(+)}\tau_{(+)}]^2} \propto D_{\text{experimental}}, \qquad \text{(VI-2)}$$

where the index o denotes the case in which the magnetic field is zero.

In an attempt to eliminate the short-circuit effect, a series of experiments has been performed in Sweden using a long-thin chamber geometry,(16) the result of which seems to indicate that the diffusion of the charged particles increases for the magnetic fields greater than a critical value, to the intensities comparable to the diffusion rate in the absence of the magnetic field, which is no longer interpretable by the ordinary collision theory. Lehnert(16) concluded that this perhaps can be considered as an indication of the existence of an a la Bohm "drain-diffusion" mechanism, for the arguments based on the "short-circuit" effect as introduced by Simon do not seem to have relevance in this case. The similar long-thin geometry experiments at ORNL and Los Alamos have been reportedly proposed, nevertheless no experimental result has been made available yet.

In the following part of this section we shall make attempts to evaluate the ambipolar diffusion under the influence of strong magnetic field intensities, in an effort to explain the above described discrepancy between the theory and the experiment in the context of the formalism developed earlier. It will be shown that this discrepancy may be attributed to the "transverse" diffusion which is characterized by the off-diagonal elements of the tensorial plasma parameters, and which is seemingly ignored in the earlier treatments. To the same order of approximation the "direct" diffusion, i.e., the one characterized by the diagonal elements of the plasma parameters only, will be shown to give identical result to that of the collision theory.

Finally, by making use of our formulations, attempts will be made to re-establish the common result of two different approaches in estimating the diffusion of fully ionized gases across a magnetic field,[17-18] both indicating that $D_\perp$ is proportional to H_o^{-4}.

B. A Study of the Diffusion Processes in Arc Plasmas

Let us consider a weakly ionized gas mixture, in which the transport properties of interest remain substantially unhindered due to the encounters between the charged particles, such that the mean collision time, τ, can be characterized merely by the short range interactions between the charged particles under consideration and the neutral molecules which more abundantly exist in the system, and which will be assumed to remain unperturbed in an equilibrium state for all time. This presumably being the case for most arc plasmas, it seems necessary to distinguish, in general, the mean collision times corresponding to the ion-neutral and the electron-neutral encounters. Nevertheless, as the experimental results indicate, these two parameters are of the same order of magnitude, so that for the present purpose of estimation this distinction will be assumed to be ignorable. Clearly, this assumption is by no means a restriction to the general procedure employed, and can easily be relaxed.

In the case when the external field $\underline{H}_0$ is zero, the Equation (V-9) can be written as

$$D_{jk} = D_0 \delta_{jk} , \tag{VI-3}$$

where

$$D_0 = \frac{\Theta}{m} \bar{\tau}_0$$

$$\bar{\tau}_0 \equiv \frac{\int_0^t ds' e^{-\frac{s'}{\tau}} s' \bar{\rho}}{\int_0^t ds' e^{-\frac{s'}{\tau}} \bar{\rho}} \cdot = \alpha_0 \tau \tag{VI-4}$$

It can easily be seen that $\bar{\tau}_0$, which we shall refer to as "the modified mean collision time," asymptotically approaches to τ, whenever $\bar{\rho}$ can be treated as a constant. Hence, for most diffusion problems of interest α_0 may be considered of order unity.

The significance of this representation is two fold; first, the form of (4) closely resembles the result of the elementary collision theory, since, by setting

$$\bar{\lambda} = \alpha_0 \lambda = \bar{\tau} v_{ave.} = \bar{\tau} \sqrt{\langle v^2 \rangle^{(0)}} = \alpha_0 \tau \sqrt{\frac{3\Theta}{m}} ,$$

we get

$$D_0 = \bar{\lambda} \frac{v_{ave.}}{3} = \alpha_0 \frac{\lambda v_{ave.}}{3} \tag{VI-5}$$

(cf., Reference 11, Chapter 9) Secondly, as can be seen from the definition of $\bar{\tau}_o$, the overall effect of the internally induced fields is contained in an appropriately chosen collision parameter. (At least this is true for the diffusion process when $H_o = 0$.) This perhaps may be interpreted as an indication of a possible connection between the two different approaches which have been used in attacking the plasma problems; one of them, which is essentially due to Vlassov, is employed in this work. The other is the one in which the range of the internally induced fields is assumed to be screened by a cut-off distance, such as the Debye radius, so that their interactions with the charged particles may be treated in the context of the binary collision formalism by means of properly chosen reaction cross sections (e.g., Reference 5, Chapter 5; and Reference 19), such that the conventional methods of solving the Boltzmann equation[19,20] become applicable, (cf., Reference 21 and its references).

In the presence of an externally applied field $\underline{H}_o$, the above given interpretation seems no longer relevant, unless the modified mean collision time is agreed to be a tensorial quantity. It is for this reason that this case perhaps can be best studied by being divided into sections, as the diffusion along and across the magnetic field. The former presents no difficulty, since it does not differ appreciably from the case in which the field is absent. Hence, accordingly we may define

$$D_3 = \alpha_3 \frac{\Theta}{m} \tau = \bar{\alpha}_3 D_o \qquad \text{(VI-6)}$$

where

$$\bar{\alpha}_3 = \frac{\alpha_3}{\alpha_0} ,$$
$$= \frac{1}{\bar{\tau}_0} \frac{1}{\rho^{(1)}} \int_0^t ds' e^{-\frac{s'}{\tau}} s' \bar{\rho} .$$

The diffusion across the magnetic field, which is represented by the projection of the diffusion tensor in the (x,y) plane, namely $\underline{\underline{D}}^{\perp}$, may be subdivided as: (a) direct diffusion, (b) transverse diffusion, as indicated before, which will be the subject of the next subsection.

C. Diffusion Across the Magnetic Field ($\Omega\tau \gg 1$)

Let us first note that the Equation (V-9) asymptotically reduces to

$$D_{\text{direct}} \cong D_0 \frac{1}{1+\Omega^2\tau^2} \sim D_0 \frac{1}{\Omega^2\tau^2}$$
$$D_{\text{transverse}} \cong D_0 \frac{\Omega\tau}{1+\Omega^2\tau^2} \sim D_0 \frac{1}{\Omega\tau} \qquad \text{(VI-7)}$$

whenever $\bar{\rho}$ can be treated as a constant. This result is identical to that of the one which is deduced in the framework of the elementary collision-diffusion theory (cf., Reference 20, Chapter 18; Reference 10, page 396). It can be shown that a similar result is obtained in the case of the strong magnetic field and the slowly varying functions $\bar{\rho}$ along the unperturbed orbits such that the change over a Larmor period may be ignored, i.e.,

$$|\bar{\rho}(\underline{x}, t, s) - \bar{\rho}(\underline{x}, t, s + \frac{2\pi}{\Omega})| < \varepsilon . \qquad \text{(VI-8)}$$

Now, dividing the path of the integrations involved in the Equation (V-9) as

$$D^{\perp}_{jk} \cong \frac{\Theta}{m} \frac{\sum_{n=0}^{N} \int_{2n\pi/\Omega}^{2(n+1)\pi/\Omega} ds\, e^{-\frac{s}{\tau}}\, G^{\perp}_{kj}(s)\;\; \bar{\rho}(\underline{x},t,s)}{\sum_{n=0}^{N} \int_{2n\pi/\Omega}^{2(n+1)\pi/\Omega} ds\;\; e^{-\frac{s}{\tau}}\;\; \bar{\rho}(\underline{x},t,s)}\,, \qquad \text{(VI-9)}$$

and applying the mean value theorem to each subintegral with the condition (8) we get

$$D^{\perp}_{jk} \approx \frac{\Theta}{m}\; \frac{\int_0^{2\pi/\Omega} G^{\perp}_{kj}\; e^{-\frac{s}{\tau}} ds}{\tau\,(1-e^{-\frac{2\pi}{\Omega\tau}})} \;=\; D_0 \begin{pmatrix} \frac{1}{1+\Omega^2\tau^2} & +\frac{\Omega\tau}{1+\Omega^2\tau^2} \\ -\frac{\Omega\tau}{1+\Omega^2\tau^2} & \frac{1}{1+\Omega^2\tau^2} \end{pmatrix}, \qquad \text{(VI-10)}$$

where N is the closest integer to $[\Omega t - 1]$, $(\Omega t \gg 1)$.

The Equation (10) indicates that in the context of the above delineated approximation formalism the direct and the transverse diffusions are proportional to H_0^{-2} and H_0^{-1}, respectively, as expected in the elementary theory. It is of some interest to note that the transverse diffusion coefficient gives, in this case,

$$D_{\text{transverse}} \approx \left|\frac{D_0}{\Omega\tau}\right| = \frac{c}{|e|}\,\frac{\Theta}{H_0} \qquad \text{(VI-11)}$$

$$= \frac{1}{Z}\,10^{8}\, \frac{\Theta\ \text{(ev)}}{H_0\ \text{(oersted)}}\,,$$

where Z denotes the ionization degree of the particular species under consideration. Apart from a numerical factor (~ 10), this is identical to the Equation (1), which gives the approximate form of the Bohm's drain-diffusion coefficient.

In order to estimate the ambipolar diffusion, it seems necessary to introduce a mobility tensor in the range in which the previously indicated assumptions hold. The difficulty encountered in the definition of such a tensor, without referring to an FL (or similar) analysis as considered earlier, has already been pointed out. However, as the gyration radii of the unperturbed orbits corresponding to different species decrease with increasing H_0, the quantity

$$\bar{E}_k^r = \frac{1}{n_r} \int_{\underline{v}} E_k^{(1)}(\underline{x} - \underline{\underline{G}}^r \cdot \underline{v}) \, f_r^{(0)}(v) \, d^3v \qquad \text{(VI-12)}$$

may be expected to be quite insensitive in r, i.e., it does not differ appreciably from one kind of particles to the other, provided $\underline{E}^{(1)}$ is a sufficiently smooth function of $\underline{x}$, and the zeroth state is sufficiently cold, e.g., $\frac{\Theta k^2}{m \Omega^2} \ll 1$, where $\frac{1}{k}$ is a characteristic length of the system such as $n / | \underline{\nabla} n |$.

If this is true, then by replacing $\bar{E}_k^r$ by E_k in the Equation (V-8), the mobility tensor may be estimated asymptotically as

$$K_{jk} \cong \frac{e}{m} \int_0^\infty e^{-\frac{s'}{\tau}} R_{kj}(s') \, ds', \qquad \text{(VI-13)}$$

which, after the indicated integrals are performed, gives (cf., Reference 10, page 394)

$$\kappa_{jk} = \kappa_0 \begin{pmatrix} \frac{1}{1+\Omega^2\tau^2} & +\frac{\Omega\tau}{1+\Omega^2\tau^2} & 0 \\ \frac{-\Omega\tau}{1+\Omega^2\tau^2} & \frac{1}{1+\Omega^2\tau^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{VI-14}$$

where

$$\kappa_0 = \frac{e}{m}\tau .$$

The diagonal elements again are seen to resemble closely the result of the elementary theory (cf., Reference 11, Chapter 9). Furthermore, we also observe that κ_0 and D_0 satisfy the Einstein relation

$$\kappa_0 = \frac{e}{\Theta} D_0 \tag{VI-15}$$

The above result alternatively can be obtained from the Equation (III-3) by setting p = 0 and $\underline{k}$ = 0. We, therefore, write the ohmic current in the form

$$n \kappa_{jk} E_k ,$$

so that for a binary mixture of charged particles the total current equations can be given as

$$n w_k^{(\mp)} = n \kappa_{kj}^{(\mp)} E_j - D_{kj}^{(\mp)} \frac{\partial n^{(\mp)}}{\partial x_j} . \tag{VI-16}$$

The ambipolar diffusion tensor now can be obtained by following the procedure of Section (V-B) closely. Thus, accordingly, we eliminate the ohmic currents in (16) by using the commutativity of the tensors $\underline{\underline{\kappa}}^{(+)}$ and $\underline{\underline{\kappa}}^{(-)}$, and, introducing the conditions of the Equation (V-13), we obtain

$$\underline{\underline{D}}^{AMB.} = \left[\underline{\underline{\kappa}}^{(-)} - \underline{\underline{\kappa}}^{(+)}\right]^{-1} \cdot \left[\underline{\underline{\kappa}}^{(-)} \cdot \underline{\underline{D}}^{(+)} - \underline{\underline{\kappa}}^{(+)} \cdot \underline{\underline{D}}^{(-)}\right] . \tag{VI-17}$$

By inserting the estimated values of $\underline{\underline{D}}^{(\mp)}$ and $\underline{\underline{\kappa}}^{(\mp)}$ from (6), (10), and (14), and making the further assumptions that the ions being singly charged, and the mass ratio

$$\mu = \frac{m_{(-)}}{m_{(+)}} \ll 1 \ ,$$

we get, after some straight forward but somewhat tedious manipulations,

$$\underline{\underline{D}}_{\perp}^{AMB} \cong 2 D_0^{(+)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{VI-18}$$

Moreover, in the absence of $\underline{H}_0$, it can easily be shown that the ambipolar diffusion coefficient is given as

$$D_0 = \frac{\kappa_0^{(-)} D_0^{(+)} - \kappa_0^{(+)} D_0^{(-)}}{\kappa_0^{(-)} - \kappa_0^{(+)}} \cong 2 D_0^{(+)} , \tag{VI-19}$$

(cf., Reference 10, loc. cit.) so that, being of the same order of magnitude, the Equations (18) and (19) seem to explain the result of the long tube experiments, without referring to an additional diffusion mechanism, such as drain-diffusion.

Finally we note that, if the off-diagonal elements of the tensors involved in (17) are ignored, the result is found to be identical to the value expected in the elementary theory: (cf., Reference 10, loc.cit.)

$$D^{AMB} \cong 2 D_0^{(-)} / \Omega_{(-)}^2 \tau^2 . \tag{VI-20}$$

D. Diffusion in Fully Ionized Plasmas

In general, the study of the diffusion processes in a completely ionized plasma is perhaps somewhat simpler than the case in which the ionization is but partially accomplished, since in the latter case additional distribution functions representing the neutral molecules, and the corresponding Boltzmann equations, have to be further introduced into our system of coupled equations. However, this fact does not seem to provide any particular advantage if a relaxation-type model is to be adopted (as is done here) to replace the complicated collision integrals, in which the only coupling of the distribution functions of the neutrals appears. Therefore, one has to make a proper choice for the mean collision time, perhaps depending on the particular problem under consideration. As the earlier studies on the plasma diffusion indicate that alike particle encounters do not contribute anything (to the first order) to the diffusion current, (because of the fact that the motion of the "guiding centers" is substantially unhindered due to these types of interactions), perhaps it would be reasonable to choose τ as being the mean time between the unlike particle collisions, provided such a concept is meaningful. However, since the purpose of the present section merely is the re-derivation of the result of the formerly mentioned works, the study of the general case will not be our concern here; rather we shall adopt the particular special case considered there.

In an effort to resolve the well-known paradox that appears in the magnetohydrodynamical treatment of the diffusion process, which states that there can be no net current (or mass velocity) in a simple gas of charged particles along the direction of the density gradient, Simon(17) has considered the effect of the off-diagonal elements in the first order stress tensor which is developed in the kinetic theory in the context of the Enskog-Chapman formalism.[20] His result indicates that under certain circumstances like particle collisions may make a considerable contribution to the diffusion rate.* However, in this case the Fick's law apparently does not hold, since

$$\varpi_1 \cong \frac{3}{32} \frac{r_0^4}{\tau} \frac{d}{dx}\left[\frac{1}{n}\frac{d^2 n}{dx^2}\right] , \qquad \text{(VI-21)}$$

where r_o is the Larmor radius given as

$$r_0 = \langle v \rangle_\perp / \Omega \quad = \frac{1}{\Omega}\sqrt{\frac{2\Theta}{m}}$$

Moreover,[21] exhibits a dependence such that the diffusion rate varies as H_o^{-4}. Longmire and Rosenbluth[18] have shown that apart from a factor of 4/3, the same result is obtained by making use of a Fokker-Planck approach.

The special case considered in Simon's paper, which also will be adopted here, assumes that (i) only one kind of particles exists in the system, i.e., simple gas, (ii) the functions involved vary with x only, (iii) the induced electric field is linearly polarized along the x axis, (iv) the z component of the current is constant, i.e., $\frac{d}{dx} j_3^{(1)} = 0$, (v) the system is in the steady state.

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* Also see Reference 22.

The moment equations given in the Equation (III-4) reduce in this case to

$$\frac{d}{dx}\, j_1^{(1)} = 0$$

$$\frac{d}{dx}\, \Psi_{11}^{(1)} - \Omega j_2^{(1)} = en\, E^{(1)}$$

$$\frac{d}{dx}\, \Psi_{12}^{(1)} + \Omega j_1^{(1)} = 0$$

$$\frac{d}{dx}\, \Psi_{13}^{(1)} = 0 \, . \qquad \text{(VI-22)}$$

The difficulty encountered in the employment of these equations has already been pointed out when the collisions are represented by the relaxation model. Thus, in order to justify the employment of (22), let us suppose that we have ignored the collision interactions in the Boltzmann equations completely. Clearly, the solution of this latter problem can be easily obtained by simply setting $\frac{1}{\tau} = 0$ in the former results. Now we can use the relation derived for the stress tensor in the Equation (III-3), which gives in this case

$$\overline{\Psi}_{11} = \frac{\Theta}{m}\,\bar{\rho} - \left(\frac{\Theta}{m}\right)^2 G_{11}^2\, \frac{d^2}{dx^2}\,\bar{\rho}$$

$$\overline{\Psi}_{12} = -\frac{\Theta}{m}\, G_{11}\, \frac{d}{dx}\,\bar{j}_2 - \left(\frac{\Theta}{m}\right)^2 G_{11} G_{12}\, \frac{d^2}{dx^2}\,\bar{\rho}$$

$$\overline{\Psi}_{13} = \cdots = 0 \, .$$

(VI-23)

In order to take the collisions into account, we introduce an averaging process which resembles closely the one which is customary in the elementary diffusion theory:

$$\langle \chi \rangle_\tau = \int_0^\infty e^{-\frac{s}{\tau}}\, \chi(s)\, \frac{ds}{\tau} \qquad \text{(VI-24)}$$

Averaging the coefficients in (23) accordingly, and integrating along the unperturbed orbits we obtain

$$\Psi_{11}^{(1)} = \frac{\Theta}{m}\rho^{(1)} - \left(\frac{\Theta}{m}\right)^2 \langle G_{11}^2 \rangle_\tau \frac{d^2}{dx^2}\rho^{(1)}$$

$$\Psi_{12}^{(1)} = -\frac{\Theta}{m}\langle G_{11}\rangle_\tau \frac{d}{dx} j_2^{(1)} - \left(\frac{\Theta}{m}\right)^2 \langle G_{11} G_{12}\rangle_\tau \frac{d^2\rho^{(1)}}{dx^2}, \qquad \text{(VI-25)}$$

and after some straightforward elimination procedures with the use of Poisson's equation

$$\frac{d}{dx}E^{(1)} = 4\pi \frac{e}{m}\rho^{(1)},$$

we get $(\Omega\tau \gg 1)$:

$$\omega_1^{(1)} = \frac{1}{16}\frac{r_0^4}{\tau}\frac{d^3}{dx^3}\left(\frac{n^{(1)}}{n}\right) - \frac{1}{8}\frac{r_0^6}{\tau}\frac{d^5}{dx^5}\left(\frac{n^{(1)}}{n}\right) - \frac{1}{2}\frac{r_0^2}{\tau}\frac{\omega_p^2}{\Omega^2}\frac{d}{dx}\left(\frac{n^{(1)}}{n}\right)$$

$$\omega_2^{(1)} = \frac{1}{2}r_0^2\Omega\frac{d}{dx}\left(\frac{n^{(1)}}{n}\right) - \frac{1}{8}r_0^4\Omega\frac{d^3}{dx^3}\left(\frac{n^{(1)}}{n}\right) - c\frac{E^{(1)}}{H_0}. \qquad \text{(VI-26)}$$

Remembering that Simon's result does not contain the terms of higher order in $\gamma = r_0 k$, where $\frac{1}{k}$ is a characteristic length of the system, and the term corresponding to the electric field is ignored, then (21) is to be compared with the first term of the right hand side of $\omega_1^{(1)}$ only. Therefore, apart from a numerical factor of 3/2, the linearized form of Simon's result is equivalent to the one which is obtained here to the same order of approximation. Since our equations were linearized to start with, this result should not be considered as surprising. The origin of the factor 3/2 is not known. Perhaps it is inherent in the approximations used in the two methods.

The order property indicated in Simon's paper, i.e.,

$$\frac{\omega_2^{(1)}}{\omega_1^{(1)}} = \vartheta\left(\frac{\Omega\tau}{k^2}\right)$$

can be easily seen to be also satisfied by (26). Clearly, the last term in the right hand side of $\omega_2^{(1)}$ represents the linearized Hall current.

On the other hand, the relation given for the perturbed pressure in the Equation (III-3), after the indicated approximation procedure is employed, reads

$$p^{(1)} = \frac{5}{3}\frac{\Theta}{m}\rho^{(1)} + \frac{2}{3}\left(\frac{\Theta}{m}\right)^2 \langle M_{11}\rangle_\tau \frac{d^2\rho^{(1)}}{dx^2}$$

$$\langle M_{11}\rangle_\tau \approx \frac{1}{\Omega^2} \quad . \tag{VI-27}$$

Introducing the concept of the perturbed temperature as follows:

$$p_{tot.} \equiv n_{tot.}\Theta_{tot.} = (n+n^{(1)})(\Theta+\Theta^{(1)}) = p + p^{(1)}$$

$$\therefore \quad \frac{p^{(1)}}{p} = \frac{n^{(1)}}{n} + \frac{\Theta^{(1)}}{\Theta}, \quad \text{where} \qquad p = n\Theta, \text{ we get}$$

$$r_0^2\frac{d^2}{dx^2}\left(\frac{n^{(1)}}{n}\right) = 3\frac{p^{(1)}}{p} - 5\frac{n^{(1)}}{n} = 5\frac{\Theta^{(1)}}{\Theta} - 2\frac{p^{(1)}}{p} = 3\frac{\Theta^{(1)}}{\Theta} - 2\frac{n^{(1)}}{n}, \tag{VI-28}$$

which shows that to the first order in γ the adiabatic law is obeyed. Inserting (28) in (26), to the fifth order in γ we obtain

$$\omega_1^{(1)} \cong -\frac{r_0^2}{2\tau}\left[\frac{1}{4} + \frac{\omega_p^2}{\Omega^2}\right]\frac{d}{dx}\left(\frac{n^{(1)}}{n}\right) + \frac{3}{16}\frac{r_0^2}{\tau}\frac{d}{dx}\left(\frac{\Theta^{(1)}}{\Theta}\right)$$

$$\omega_2^{(1)} \cong \frac{3}{4}r_0^2\Omega\frac{d}{dx}\left(\frac{n^{(1)}}{n}\right) - \frac{3}{8}r_0^2\Omega\frac{d}{dx}\left(\frac{\Theta^{(1)}}{\Theta}\right) - \frac{cE^{(1)}}{H_0}. \tag{VI-29}$$

Thus, we observe that to this order of approximation the Fick's law still is satisfied, provided the effect of the thermal diffusion is properly identified. We have

$$D_1 = \frac{r_0^2}{2\tau}\left[\frac{1}{4} + \frac{\omega_p^2}{\Omega^2}\right]$$

$$k_{T,1} = \frac{3}{2 + 8(\omega_p^2/\Omega^2)}$$

$$D_2 = \frac{3}{4} r_0^2 \Omega$$

$$k_{T,2} = \frac{1}{2}$$

where k_T is the thermal-diffusion ratio.

VII. DISCUSSION AND CONCLUSIONS

The theory developed here in an attempt to investigate the transport properties of plasmas essentially is a combination of the various techniques which have been employed in attacking some special plasma problems. Among them Bernstein's formal solution of the first order equation, which was deduced in his studies concerning the plasma oscillations, and the investigations performed by Drummond and Mower on the microconductivity properties of the plasma may be mentioned.

Making use of the integral representation of the first order distribution, we were able to show that the perturbed values of the transport properties may be exhibited explicitly in terms of the "integrands" of the first order mass density and mass current, instead of the perturbed Lorentz force as has been used in the earlier studies. In the framework of this new representation, the relations established among the integrands of these properties enabled us to descry some tensorial functionals operating on these above mentioned integrands, such that the resemblance between the mathematical structures of these relations and the results of the ordinary kinetic theory provided a means for an identification of the plasma parameters.

In order to exhibit the effect of the anisotropy in the velocity distribution of the unperturbed state, a Fourier-Hankel analysis has been used, with which a seemingly more efficient study of some of the problems of interest may be performed, as has been pointed out earlier. The transport properties of the plasma are shown to be expressed explicitly in terms of these transforms and their derivatives.

We have given a considerable amount of attention to a special case in which some of the unperturbed properties of the plasma are suitably specified, and for this particular case we have deduced explicit expressions representing the diffusion properties of the plasma in terms of the initial conditions only, in the context of a Fourier-Laplace analysis performed on the position and time variables, respectively. However, because of the fact that for most cases of interest these representations turn out to be so complicated as to be almost absurd, in order to deduce any physically meaningful, analytically tractable result, it becomes inevitable to employ some kind of approximation formalism which provides accessible interpretations for the general character of the process under investigation without having to refer to an integral transform technique, such as FL analysis.

Two seemingly distinct methods of approximation, which nevertheless give essentially identical results, have been discussed in our previous work. Both provided sufficiently convincing agreements with the respective results of the treatments in which various theories have been employed that are available to plasma investigators.

It is believed that adequate evidence has been presented in this work leading to the conclusion that it is possible to investigate the behavior of the transport parameters of plasmas in some given physical conditions by making use of the mathematical model employed here, although there has been some doubt expressed in the recent literature about this matter. Moreover, this belief is also found to be supported (to some extent) by the fact that our attempted estimation of the ambipolar diffusion rates across an externally applied strong magnetic field appears to correspond to that of the experimentally observed results, and seems to provide a possible explanation for the discrepancy that has been found in the previous theoretical approaches, so that as a result of this postulation of a new mechanism of diffusion was considered necessary.

APPENDIX A

A REMARK ON THE n-th ORDER PERTURBATION FORMALISM

Let us assume that our system is near to a state, which will be denoted by the superscript (o), and the departure from this state can be represented by a dimensionless parameter γ, such that the solution of the Equations (I-1) and (I-2) may be exhibited in the form of a power series as

$$f_r = f_r^{(0)} + \gamma f_r^{(1)} + \gamma^2 f_r^{(2)} + \cdots . \tag{A-1}$$

The collision model (I-5) now may be written as

$$\left(\frac{\delta f_r}{\delta t}\right)_{Coll.} \cong \frac{f_r^{(0)} - f_r}{\tau} = -\frac{1}{\tau}\left[\gamma f_r^{(1)} + \gamma^2 f_r^{(2)} + \cdots\right] . \tag{A-2}$$

We introduce the quantities for (k = 0,1,2,...)

$$Q^{(k)} = \sum_r e_r \int_{\underline{v}} f_r^{(k)} \, d^3v \, , \qquad \underline{J}^{(k)} = \sum_r e_r \int_{\underline{v}} \underline{v} \, f_r^{(k)} \, d^3v \tag{A-3}$$

so that

$$Q = Q^{(0)} + \gamma Q^{(1)} + \cdots \quad , \qquad \underline{J} = \underline{J}^{(0)} + \gamma \underline{J}^{(1)} + \cdots . \tag{A-4}$$

Moreover, denoting the solution of the Maxwell's equation corresponding to the charge and current densities, $Q^{(k)}$ and $\underline{J}^{(k)}$, respectively, by $\underline{E}^{(k)}$ and $\underline{H}^{(k)}$, i.e.,

$$\underline{\nabla} \times \underline{H}^{(k)} - \frac{1}{c}\frac{\partial}{\partial t}\underline{E}^{(k)} = \frac{4\pi}{c}\underline{J}^{(k)} \quad , \qquad \underline{\nabla} \cdot \underline{H}^{(k)} = 0$$

$$\underline{\nabla} \times \underline{E}^{(k)} + \frac{1}{c}\frac{\partial}{\partial t}\underline{H}^{(k)} = 0 \; , \qquad \underline{\nabla} \cdot \underline{E} = 4\pi \, Q^{(k)} , \tag{A-5}$$

clearly we have

$$\underline{E} = \underline{E}^{(0)} + \gamma\, \underline{E}^{(1)} + \cdots , \qquad \underline{H} = \underline{H}^{(0)} + \gamma\, \underline{H}^{(1)} + \cdots . \tag{A-6}$$

Here we shall assume that in the zero-order state the effect of the internally induced fields is ignorable. Accordingly we set $\underline{E}^{(0)} = \underline{H}^{(0)} = 0$, hence $Q^{(0)} = 0$, $\underline{J}^{(0)} = 0$. Substituting in the Boltzmann equation, and setting the coefficient of γ^k equal to zero, we obtain

$$Df_r^{(0)} = 0 \quad , \qquad Df_r^{(1)} + \underline{a}^{(1)} \cdot \frac{\partial f_r^{(0)}}{\partial \underline{v}} + \frac{1}{\tau} f_r^{(1)} = 0 \tag{A-7}$$

$$Df_r^{(k)} + \underline{a}^{(k)} \cdot \frac{\partial f_r^{(0)}}{\partial \underline{v}} + \frac{1}{\tau} f_r^{(k)} = - \sum_{j=1}^{k-1} \underline{a}^{(j)} \cdot \frac{\partial f_r^{(k-j)}}{\partial \underline{v}} \tag{A-8}$$

where

$$\underline{a}^{(0)} = 0 \; , \qquad \underline{a}^{(j)} = \frac{e_r}{m_r} \left[\underline{E}^{(j)} + \frac{1}{c}\, \underline{v} \times \underline{H}^{(j)} \right] , \quad j = 1, 2, \cdots .$$

Now, if we assume that the Equation (8) coupled with (5) has been solved for $k \leq n-1$ i.e., the right hand side of (8) already is at our disposal, then it is seen that the problem reduces to the one which is already considered in the form of the first order equation [cf., the Equation (I-4)], provided a modified definition of function ϕ is introduced as follows:

$$\phi^{(n)} = \underline{a}^{(n)} \cdot \frac{\partial f_r^{(0)}}{\partial \underline{v}} + \sum_{j=1}^{n-1} \underline{a}^{(j)} \cdot \frac{\partial f_r^{(n-j)}}{\partial \underline{v}} , \tag{A-9}$$

which enables us to write the Equation (8) in the form of the integral equation

$$f_r^{(n)}(\underline{x}, \underline{v}, t) - e^{-\frac{t}{\tau}} f_r^{(n)}(\underline{\xi}(\underline{x},\underline{v},t), \gamma(\underline{x},\underline{v},t), 0) = - \int_0^t e^{-\frac{s'}{\tau}} \phi^{(n)}(\underline{\xi}', \gamma', t-s')\, ds' . \tag{A-10}$$

Now, the latter may be treated in exactly the same manner as in the case of n=1; therefore, by induction we conclude that the solution of the non-linear system (I-1) and (I-2) with the model (I-5) can be successively approximated to any order without introducing much complexity to the first order problem, provided the series (1) converges fast enough to insure the applicability of this method.

The above delineated method of deducing a formal solution of the non-linear Boltzmann equation coupled with the Maxwell's equations perhaps can be criticized as being too elaborate on the basis of the fact that another source of non-linearity in the Boltzmann equation, i.e., the contribution of the short range interactions, has been treated by means of a linear model, which, in general, does not yield the conservation equations. However, since no such difficulty arises in the case when the collisions are absent (or ignorable), as is pointed out in the text on several occasions, the effort to obtain the non-linear effects of the internally induced fields seems to be justified to a reasonable extent for plasmas in which such a condition occurs. Although when the questions involving the transport properties of the plasma are under consideration this condition does not seem to be relevant, it is perhaps possible to shed some more light on the understanding of these phenomena, in a somewhat sophisticated manner, by the employment of an approximation procedure, such as indicated in the Equation (VI-24), --which presumably (and hopefully) provides a sufficiently appropriate, resonably accessible treatment of the collision interactions--to the results obtained by using the formal solution of the collisionless case as described above.

APPENDIX B

SOME PROPERTIES OF THE TENSORS $\underline{\underline{R}}$, $\underline{\underline{G}}$, AND $\underline{\underline{M}}$.

The tensors $\underline{\underline{R}}$, $\underline{\underline{G}}$, and $\underline{\underline{M}}$ as defined in the Equation (I-13) have some operational properties which make the tensorial representation introduced earlier a powerful technique in handling the problem under consideration. Some of these properties will be given below without proof since most of them are almost trivial in character. We have

$$R_{kj}(s) = R_{jk}(-s) = R^{-1}_{kj}(-s) = R^{-1}_{jk}(s) , \tag{B-1}$$

$$R_{k\ell}(s)\, R_{\ell j}(t) = R_{k\ell}(t)\, R_{\ell j}(s) = R_{kj}(s+t) , \tag{B-2}$$

$$G_{kj}(-s) = -G_{jk}(s) , \qquad M_{kj}(s) = + M_{jk}(-s) , \tag{B-3}$$

$$\begin{aligned} G_{k\ell}(s)\, R_{\ell j}(t) &= R_{k\ell}(t)\, G_{\ell j}(s) = \\ &= G_{kj}(s+t) - G_{kj}(t) , \end{aligned} \tag{B-4}$$

$$\begin{aligned} G_{kj}(s) - G_{kj}(t) &= R_{k\ell}(t) G_{mj}(s) - R_{k\ell}(s)\, G_{\ell j}(t) \\ &= G_{k\ell}(s)\, R_{\ell j}(t) - G_{k\ell}(t)\, R_{\ell j}(s) , \end{aligned} \tag{B-5}$$

$$G_{k\ell}(s)\, R_{j\ell}(s) = G_{\ell j}(s)\, R_{\ell k}(s) = G_{j\ell}(s) \tag{B-6}$$

$$\Omega\, \varepsilon_{3jk}\, G_{ik} = \Omega\, \varepsilon_{3ki}\, G_{kj} = R_{ij} - \delta_{ij} \tag{B-7}$$

$$\Omega\, \varepsilon_{3jk}\, R_{ik} = \Omega\, \varepsilon_{3ki}\, R_{kj} = \dot{R}_{ij} \equiv \frac{d}{ds} R_{ij} \tag{B-8}$$

$$\Omega\, \varepsilon_{3ki} M_{kj}(s) = \Omega\, \varepsilon_{3jk} M_{ik}(s) = G_{ij}(s) - s\,\delta_{ij}\,, \qquad \text{(B-9)}$$

$$\Omega\, \varepsilon_{3kj} [G_{ik} + G_{ki}] = \delta^{ij}_{mn}\, R_{nm}\,, \qquad \text{(B-10)}$$

$$\Omega\, \varepsilon_{3k\ell}\, G_{ik} G_{j\ell} = \delta^{ij}_{mn}\, G_{nm} = \Omega\, \varepsilon_{3kj} [M_{ik} + M_{ki}]\,, \qquad \text{(B-11)}$$

$$G_{km} G_{jm} = G_{mk} G_{mj} = M_{jk} + M_{kj}\,, \qquad \text{(B-12)}$$

$$\begin{aligned} M_{jk}(t-s) &= M_{jk}(t) + M_{kj}(s) - G_{mk}(t)\, G_{mj}(s) \\ &= M_{jk}(t) + M_{kj}(s) - G_{jm}(s)\, G_{km}(t)\,. \end{aligned} \qquad \text{(B-13)}$$

Alternatively one can write

$$R_{ij}(s) = \begin{pmatrix} \cos\Omega s & -\sin\Omega s & 0 \\ \sin\Omega s & \cos\Omega s & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(B-14)}$$

$$= \cos\Omega s\, \delta_{ij} + \sin\Omega s\, \varepsilon_{3ji} + (1-\cos\Omega s)\, \delta_{3i}\delta_{3j}\,.$$

$$G_{ij}(s) = \begin{pmatrix} \dfrac{\sin\Omega s}{\Omega} & -\dfrac{1-\cos\Omega s}{\Omega} & 0 \\ \dfrac{1-\cos\Omega s}{\Omega} & \dfrac{\sin\Omega s}{\Omega} & 0 \\ 0 & 0 & s \end{pmatrix} \qquad \text{(B-15)}$$

$$= \frac{1}{\Omega}\sin\Omega s\, \delta_{ij} + \frac{1}{\Omega}(1-\cos\Omega s)\, \varepsilon_{3ji} + \frac{1}{\Omega}(\Omega s - \sin\Omega s)\, \delta_{3i}\delta_{3j}\,.$$

$$M_{ij}(s) = \begin{pmatrix} \dfrac{1-\cos\Omega s}{\Omega^2} & -\dfrac{\Omega s - \sin\Omega s}{\Omega^2} & 0 \\ \dfrac{\Omega s - \sin\Omega s}{\Omega^2} & \dfrac{1-\cos\Omega s}{\Omega^2} & 0 \\ 0 & 0 & \dfrac{s^2}{2} \end{pmatrix} \tag{B-16}$$

$$= \frac{1}{\Omega^2}(1-\cos\Omega s)\,\delta_{ij} + \frac{1}{\Omega^2}(\Omega s - \sin\Omega s)\,\varepsilon_{3ji} + \frac{1}{\Omega^2}\left[\frac{\Omega^2 s^2}{2} - (1-\cos\Omega s)\right]\delta_{3i}\delta_{3j}$$

$$G^{-1}_{ij}(s) = \begin{pmatrix} \frac{\Omega}{2}\cot\frac{\Omega s}{2} & \frac{\Omega}{2} & 0 \\ -\frac{\Omega}{2} & \frac{\Omega}{2}\cot\frac{\Omega s}{2} & 0 \\ 0 & 0 & 1/s \end{pmatrix} \tag{B-17}$$

$$= \frac{\Omega}{2}\cot\frac{\Omega s}{2}\,\delta_{ij} + \frac{\Omega}{2}\,\varepsilon_{3ij} + \left(\frac{1}{s} - \frac{\Omega}{2}\cot\frac{\Omega s}{2}\right)\delta_{3i}\delta_{3j}$$

Also

$$G^{-1}_{j\ell}(s)\,R_{\ell k}(s) = R_{j\ell}(s)\,G^{-1}_{\ell k}(s) = G^{-1}_{kj}(s)\,. \tag{B-18}$$

APPENDIX C

DERIVATION OF THE INTEGRAL FORM OF THE BOLTZMANN EQUATION

Here we shall assume that the condition (I-10) is satisfied, although the derivation of the integral form of the Boltzmann equation can be accomplished in a much more general case. First, let us consider the unperturbed orbit Equation (I-11). To show that it is the solution of the Equation (I-7) let us differentiate it with respect to t (or s) to get

$$\frac{d}{dt} x_k = \frac{d}{ds} x_k = R_{jk} \gamma_j + G_{jk} \mathcal{E}_j = v_k \tag{C-1}$$

and differentiating again

$$\begin{aligned}
\frac{d}{dt} v_k &= -\Omega\, \varepsilon_{3\ell k} R_{j\ell} \gamma_j + R_{jk} \mathcal{E}_j \\
&= -\Omega\, \varepsilon_{3\ell k} \left[v_\ell - G_{j\ell} \mathcal{E}_j \right] + R_{jk} \mathcal{E}_j \\
&= -\Omega\, \varepsilon_{3\ell k} v_\ell - \left[R_{jk} - \delta_{jk} \right] \mathcal{E}_j + R_{jk} \mathcal{E}_j
\end{aligned}$$

$\therefore$

$$\frac{d}{dt} \underline{v} = \frac{e}{m} \underline{E}_0 + \frac{e}{mc} \underline{v} \times \underline{H}_0 = \underline{a}^{ext}. \quad (\text{Q.E.D.}) . \tag{C-2}$$

Moreover, we observe that $\underline{x} = \underline{\xi}$, $\underline{v} = \underline{\gamma}$ at $s = 0$.

Now let us consider a function $f(\underline{x},\underline{v},t)$, and define

$$\tilde{f}(\underline{\xi},\underline{\gamma},\vartheta,s) \equiv f\left(\underline{x}(\underline{\xi},\underline{\gamma},s),\, \underline{v}(\underline{\xi},\underline{\gamma},s),\, \vartheta + s\right) \tag{C-3}$$

on the basis of the transformation indicated in the Equation (I-11).

Differentiating (3) with respect to s, clearly we get

$$\frac{\partial}{\partial s}\tilde{f} = \frac{\partial f}{\partial \underline{x}} \cdot \frac{d}{ds}\underline{x} + \frac{\partial f}{\partial \underline{v}} \cdot \frac{d}{ds}\underline{v} + \frac{\partial f}{\partial t} \equiv Df \tag{C-4}$$

which essentially proves the last equation of (I-7), since we have

$$Df = -\frac{1}{\tau} f - \phi \, . \tag{C-5}$$

Thus in the transformed space we can write

$$\frac{\partial \tilde{f}}{\partial s} + \frac{1}{\tau}\tilde{f} = -\tilde{\phi} \quad \Rightarrow \quad \frac{\partial}{\partial s}\left[e^{\frac{s}{\tau}}\tilde{f}\right] = -e^{\frac{s}{\tau}}\tilde{\phi} \tag{C-6}$$

and integrating from s = 0 to s = t, we get

$$f(\underline{x}(\underline{\xi},\underline{\gamma},t), \underline{v}(\underline{\xi},\underline{\gamma},t), \vartheta+t) - e^{-\frac{t}{\tau}} f(\underline{x}(\underline{\xi},\underline{\gamma},0), \underline{v}(\underline{\xi},\underline{\gamma},0), \vartheta)$$

$$= -\int_0^t e^{-\frac{t-s''}{\tau}} \phi(\underline{x}(\underline{\xi},\underline{\gamma},s''), \underline{v}(\underline{\xi},\underline{\gamma},s''), \vartheta+s'')\, ds'' \, . \tag{C-7}$$

Now, re-labeling

$$\begin{aligned} \underline{x} &= \underline{\xi} + \underline{\gamma}\cdot\underline{\underline{G}}(t) + \underline{\mathcal{E}}\cdot\underline{\underline{M}}(t) \\ \underline{v} &= \underline{\gamma}\cdot\underline{\underline{R}}(t) + \underline{\mathcal{E}}\cdot\underline{\underline{G}}(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \underline{\xi} &= \underline{x} - \underline{\underline{G}}(t)\cdot\underline{v} + \underline{\underline{M}}(t)\cdot\underline{\mathcal{E}} \\ \underline{\gamma} &= \underline{\underline{R}}(t)\cdot\underline{v} - \underline{\underline{G}}(t)\cdot\underline{\mathcal{E}} \end{aligned}$$

with $\vartheta = 0$, and changing the integration variable $s' = t - s''$ we have

$$f(\underline{x},\underline{v},t) - e^{-\frac{t}{\tau}} f(\underline{x} - \underline{\underline{G}}(t)\cdot\underline{v} + \underline{\underline{M}}(t)\cdot\underline{\mathcal{E}}, \underline{\underline{R}}(t)\cdot\underline{v} - \underline{\underline{G}}(t)\cdot\underline{\mathcal{E}}, 0)$$

$$= -\int_0^t ds'\, e^{-\frac{s'}{\tau}} \phi(\underline{\xi} + \underline{\gamma}\cdot\underline{\underline{G}}(t-s') + \underline{\mathcal{E}}\cdot\underline{\underline{M}}(t-s'), \underline{\gamma}\cdot\underline{\underline{R}}(t-s') + \underline{\mathcal{E}}\cdot\underline{\underline{G}}(t-s'), t-s')$$

$$= -\int_0^t ds'\, e^{-\frac{s'}{\tau}} \phi(\underline{x} - \underline{\underline{G}}(s')\cdot\underline{v} + \underline{\underline{M}}(s')\cdot\underline{\mathcal{E}}, \underline{\underline{R}}(s')\cdot\underline{v} - \underline{\underline{G}}(s')\cdot\underline{\mathcal{E}}, t-s') \tag{C-8}$$

Here, in the last step, we have made use of the Equations (2, 4, and 13 - Appendix B). On the other hand, if we re-label

$$\underline{x}(\underline{\xi},\underline{\gamma},0) \equiv \underline{\xi} \rightarrow \underline{x} \quad , \quad \underline{v}(\underline{\xi},\underline{\gamma},0) \equiv \underline{\gamma} \rightarrow \underline{v}$$

in (7), and take the limit at $t \rightarrow -\infty$, we obtain

$$\lim_{t\rightarrow-\infty} \left[e^{\frac{t}{\tau}} f(\underline{x}+\underline{v}\cdot\underline{\underline{G}}(t)+\underline{\mathcal{E}}\cdot\underline{\underline{M}}(t),\ \underline{v}\cdot\underline{\underline{R}}(t)+\underline{\mathcal{E}}\cdot\underline{\underline{G}}(t),\ \vartheta+t) \right]$$

$$-f(\underline{x},\underline{v},\vartheta)$$

$$=-\int_0^{-\infty} e^{\frac{s}{\tau}} \phi(\underline{x}+\underline{v}\cdot\underline{\underline{G}}(s)+\underline{\mathcal{E}}\cdot\underline{\underline{M}}(s),\ \underline{v}\cdot\underline{\underline{R}}(s)+\underline{\mathcal{E}}\cdot\underline{\underline{G}}(s),\ \vartheta+s)\,ds.$$

Now, changing the integration variable as s' = -s, and identifying $\vartheta \rightarrow t$ we get the asymptotic representation

$$f(\underline{x},\underline{v},t) = -\int_0^{\infty} e^{-\frac{s'}{\tau}} \phi(\underline{x}-\underline{\underline{G}}'\cdot\underline{v}+\underline{\underline{M}}'\cdot\underline{\mathcal{E}},\ \underline{\underline{R}}'\cdot\underline{v}-\underline{\underline{G}}'\cdot\underline{\mathcal{E}},\ t-s')\,ds', \tag{C-9}$$

provided

$$\lim_{s\rightarrow\infty} \left[e^{-\frac{s}{\tau}} f(\underline{x}-\underline{\underline{G}}(s)\cdot\underline{v}+\underline{\underline{M}}(s)\cdot\underline{\mathcal{E}},\ \underline{\underline{R}}(s)\cdot\underline{v}-\underline{\underline{G}}(s)\cdot\underline{\mathcal{E}},\ t-s) \right] = 0 . \tag{C-10}$$

APPENDIX D

FIELD EQUATION WITHOUT INTRODUCING THE INTEGRAL TRANSFORM

We shall restrict our attention to the special case of section III in order not to complicate the problem unduly, although the general procedure employed requires no such restriction, provided the unperturbed orbit equations are available so that the integral representation already developed can be used. We shall also consistently ignore the dissipative effects, by neglecting the collisions, for the reason indicated above.

Let us compute $4\pi \underline{\nabla} Q + \frac{4\pi}{c^2}\frac{\partial}{\partial t}\underline{J}$. First, we consider the first term.

$$4\pi \frac{\partial Q}{\partial x_j} = -\sum_r \frac{\partial}{\partial x_j}\left[\omega_p^2 \int_0^t ds\, G_{k\ell}(t-s)\, \frac{\partial \bar{E}_k(\underline{x},t,s)}{\partial x_\ell}\right]$$

$$= -\sum_r \omega_p^2\, G_{k\ell} * \frac{\partial^2 \bar{E}_k}{\partial x_\ell \partial x_j}, \qquad \text{(D-1)}$$

where

$$\bar{E}_k = \frac{1}{n}\int_{\underline{v}} d^3v\, E_k(\underline{x}-\underline{\underline{G}}(t-s)\cdot\underline{v}, t)\, f^{(0)}(v)\;, (f^{(0)}\in MB), \qquad \text{(D-2)}$$

and where the generalized convolution notation is defined as

$$h(s,t) * g(s,t) = \int_0^t h(s,t-s)\, g(s,t-s)\, ds. \qquad \text{(D-3)}$$

The second term gives

$$4\pi \frac{\partial}{\partial t} J_j = \frac{\partial}{\partial t}\sum_r \omega_p^2\Big[\int_0^t ds\, R_{kj}(t-s)\, \bar{E}_k(\underline{x},t,s)$$

$$+ \frac{\Theta}{m}\int_0^t ds\, G_{k\ell}(t-s) G_{mj}(t-s) \frac{\partial^2}{\partial x_\ell \partial x_m}\bar{E}_k(\underline{x},t,s)\Big],$$

which after some manipulations can be written as

$$4\pi \frac{\partial J_j}{\partial t} = \sum_r \Big\{ \omega_p^2 E_j(\underline{x},t) - \omega_p^2 \Omega\, \varepsilon_{3\ell j} R_{k\ell} * \bar{E}_k$$
$$+ \omega_p^2 \frac{\Theta}{m} \left[R_{kj} G_{pq} + R_{kp} G_{qj} + R_{qj} G_{kp} \right] * \frac{\partial^2}{\partial x_q \partial x_p} \bar{E}_k$$
$$+ \omega_p^2 \left(\frac{\Theta}{m}\right)^2 \left[G_{kp} G_{qj} G_{\ell n} \right] * \frac{\partial^4}{\partial x_p \partial x_q \partial x_\ell \partial x_n} \bar{E}_k \Big\} . \tag{D-4}$$

Now, inserting (1) and (4) into Maxwell's equation we get

$$\square^2 E_j \equiv \nabla^2 E_j - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E_j = \zeta_j + \frac{1}{c^2} E_j \sum_r \omega_p^2$$
$$- \sum_r \omega_p^2 G_{k\ell} * \frac{\partial^2 \bar{E}_k}{\partial x_\ell \partial x_j} - \frac{1}{c^2} \varepsilon_{3\ell j} \sum_r \omega_p^2 \Omega R_{k\ell} * \bar{E}_k$$
$$+ \frac{1}{c^2} \sum_r \frac{\Theta}{m} \omega_p^2 \left[R_{kj} G_{pq} + R_{kp} G_{qj} + R_{qj} G_{kp} \right] * \frac{\partial^2 \bar{E}_k}{\partial x_q \partial x_p}$$
$$+ \frac{1}{c^2} \sum_r \left(\frac{\Theta}{m}\right)^2 \omega_p^2 \left[G_{kp} G_{qj} G_{\ell n} \right] * \frac{\partial^4 \bar{E}_k}{\partial x_p \partial x_q \partial x_\ell \partial x_n} , \tag{D-5}$$

where ζ_j is an explicit function of $f^{(1)}(\underline{x}, \underline{v}, 0)$ only.

It can be shown that the FL transform of (5), making use of the second moment equation of (III-4), gives the field equation deduced in section IV. Although this deduction will not be attempted here, to exhibit the consistency of (5), we shall re-examine some of the particular cases studied in section IV. First, for initially cold plasma (5) reads

$$\Box^2 E_j = \frac{1}{c^2} E_j \sum_r \omega_p^2 - \frac{1}{c^2} \varepsilon_{3\ell j} \sum_r \omega_p^2 \Omega R_{k\ell} * E_k$$

$$- \sum_r \omega_p^2 G_{k\ell} * \frac{\partial^2 E_k}{\partial x_\ell \partial x_k} + \zeta_j . \tag{D-6}$$

Moreover, if $H_o = 0$ we get

$$\Box^2 E_j - \frac{1}{c^2} E_j \sum_r \omega_p^2 + \int_0^t (t-s) \frac{\partial^2 E_k(\underline{x}, s)}{\partial x_k \partial x_j} ds \sum_r \omega_p^2 = \zeta_j \tag{D-7}$$

Differentiating twice with respect to t

$$c^2 \nabla^2 \frac{\partial^2 E_j}{\partial t^2} - \frac{\partial^4 E_j}{\partial t^4} - \frac{\partial^2 E_j}{\partial t^2} \sum_r \omega_p^2 + c^2 \frac{\partial^2 E_k}{\partial x_k \partial x_j} \sum_r \omega_p^2 = c^2 \frac{\partial^2 \zeta_j}{\partial t^2}$$

and setting

$$\underline{E} = \underline{E}^o \exp[i\underline{k}\cdot\underline{x} - i\omega t] \quad , \quad \underline{\zeta} = \underline{\zeta}^o \exp[i\underline{k}\cdot\underline{x} - i\omega t]$$

we obtain

$$\left[\omega^2(k^2c^2 - \omega^2 - \sum_r \omega_p^2)\delta_{ij} - c^2 k_i k_j \sum_r \omega_p^2\right] E_i^0 = - c^2\omega^2 \zeta_j^o \tag{D-8}$$

which is identical to the Equation (IV-11).

As a second application, we consider again an initially cold plasma, and the pure transverse wave case propagating along $\underline{H}_0$, i.e., $\underline{\nabla}\cdot\underline{E}=0$, $\underline{E}\cdot\underline{H}_0=0$, $\frac{\partial E_k}{\partial x}=\frac{\partial E_k}{\partial y}=0$. Then (6) can be put in the form

$$\frac{\partial^2 E_j}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 E_j}{\partial t^2}-\frac{1}{c^2}\sum_r \omega_p^2 \frac{\partial^2}{\partial t^2}\int_0^t G_{kj}(s)\, E_k(z,t-s)\,ds=\zeta_j . \tag{D-9}$$

The Equation (9) exhibits the fact that the plasma medium is dielectric in character* (at least in this case) if we generalize the concept of the dielectricity as being tensorial and hereditary. In order to study this effect we expand $\underline{E}$ into Taylor series

$$\frac{\partial^2 E_j}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 E_j}{\partial t^2}-\frac{1}{c^2}\sum_r \omega_p^2\frac{\partial^2}{\partial t^2}\left[E_k M_{kj}^{(0)}+\frac{\partial E_k}{\partial t}M_{kj}^{(1)}+\frac{\partial^2 E_k}{\partial t^2}M_{kj}^{(2)}+\cdots\right]=\zeta_j ,$$

where

$$\underline{\underline{M}}^{(n)}=\int_0^t \frac{s^n}{n!}(-1)^n\,\underline{\underline{G}}(s)\,ds , \qquad \underline{\underline{M}}^{(0)}\equiv\underline{\underline{M}} .$$

Ordering with respect to the time derivatives, we get

$$\begin{aligned}\frac{\partial^2 E_j}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 E_j}{\partial t^2}-\frac{1}{c^2}\Big[&E_k\sum_r\omega_p^2 R_{kj}+\frac{\partial E_k}{\partial t}\sum_r\omega_p^2(G_{kj}-tR_{kj})\\&+\frac{\partial^2 E_k}{\partial t^2}\sum_r\omega_p^2\Big(M_{kj}-tG_{kj}+\frac{t^2}{2}R_{kj}\Big)\\&+\sum_{n=3}^{\infty}\frac{\partial^n E_k}{\partial t^n}\sum_r\omega_p^2\Big(M_{kj}^{(n-2)}+(-1)^{n-1}\frac{t^{n-1}}{(n-1)!}G_{kj}+(-1)^n\frac{t^n}{n!}R_{kj}\Big)\Big]\\&=\zeta_j .\end{aligned} \tag{D-10}$$

* See, for instance, Reference 23.

Although, taking the FL transform, the dispersion relation (IV-16) can be obtained without much difficulty, it seems of some interest to employ the averaging process which will be discussed in what follows. This process will be performed in two steps; first, the time dependent coefficients in (10) will be approximated by using the Equation (VI-24), secondly, we shall take the limit as $\tau \rightarrow \infty$. After performing these above indicated manipulations on obtains

$$\frac{\partial^2 E_j}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_j}{\partial t^2} = \zeta_j + \frac{\omega^2_{p(+)}}{c^2}\left[\frac{1}{\Omega^2_{(+)}}\frac{\partial^2 E_j}{\partial t^2} - \frac{1}{\Omega^4_{(+)}}\frac{\partial^4 E_j}{\partial t^4} + \cdots\right]$$
$$+ \frac{\omega^2_{p(+)}}{c^2}\left[\frac{1}{\Omega^3_{(+)}}\frac{\partial^3 E_k}{\partial t^3} - \frac{1}{\Omega^5_{(+)}}\frac{\partial^5 E_k}{\partial t^5} + \cdots\right]\varepsilon_{3kj} \tag{D-11}$$

Here we considered a binary plasma with singly charged ions, and neglected the terms of the higher order in $\mu = \frac{m_{(-)}}{m_{(+)}}$. Now, if we substitute $\underline{E} \rightarrow E_1 + i\, E_2 = E^\circ \exp [\, ikz \mp i\omega t\,]$, in order to examine the circularly polarized waves, after summing the resulting geometric series we can easily obtain Spitzer's dispersion relation discussed earlier [cf., the Equation (VI-16)].

It is of some interest to note that in the case when the terms having the higher order derivatives of $\underline{E}$ are ignorable (for instance when $\lambda = \frac{\omega}{\Omega_{(+)}} \ll 1$), then (11) identically reduces to the magnetohydrodynamic-wave equation

$$\frac{\partial^2 E_j}{\partial z^2} - \frac{1}{c^2}\left(1 + \frac{4\pi\rho c^2}{H_0^2}\right)\frac{\partial^2 E_j}{\partial t^2} = \zeta_j \tag{D-12}$$

(see, for instance, Reference 5, Chapter 4), corresponding to the dielectric constant $\varepsilon = 1 + 4\pi\rho c^2 H_0^{-2}$. This result, as most of our earlier results, exhibits a complete agreement with the predictions obtained by the simple first order orbit theoretical considerations (cf., Reference 23) in their common applicability region, namely $\lambda = \frac{\omega}{\Omega_{(+)}} \ll 1$, so that the physical properties of the plasma do not vary appreciably during an ion Larmor period.

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