U-Shaped Lines with Switchover Times and Costs

Seyed M.R. Iravani
Department of Industrial & Operations Engineering
University of Michigan Ann Arbor, MI 48109

Morton J.M. Posner
Department of Mechanical & Industrial Engineering
University of Toronto, Canada

John A. Buzacott
Schulich School of Business
York University, North York, Canada

Technical Report 97-13 October 1997

U-Shaped Lines with Switchover Times and Costs

SEYED M.R. IRAVANI
Department of Industrial and Operations Engineering
University of Michigan, Ann Arbor, Michigan 48109-2117

MORTON J. M. POSNER
Department of Mechanical and Industrial Engineering
University of Toronto, Toronto, Canada

JOHN A. BUZACOTT
Schulich School of Business
York University, North York, Canada

Abstract

The advantages of U-shaped production lines over traditional production lines have been encouraging more manufacturing companies to employ these lines in order to increase their flexibility in adapting to changes in demand. In this paper, a general definition for U-shaped lines is presented and different versions of these lines are introduced. Then, considering the fact that a U-shaped line is actually a tandem queue attended by moving servers, the effect of switching costs and walking times are examined by decomposing a U-shaped line into a number of tandem queues each attended by a moving server.

Keywords: U-shaped lines, tandem queues, switchover times, optimal cost policies

1 Introduction

Traditional production or assembly lines are based on establishing a processing sequence for parts being produced on the line. Parts move in a smooth, simple, logical and direct path through a sequence of work stations each comprised of special purpose equipment and single-functional workers. Although characterized by relatively high production rates, these lines have their own limitations. For example; machine stoppages halt the line, the slowest station paces the line and the line requires general supervision. However, the most important limitation of traditional production or assembly lines is the inherent inflexibility in changing the production rate. One way of increasing the flexibility of these lines in adapting to demand changes is through attaining flexibility in deciding on the number of workers. This is called *Shojinka* in Japanese (Monden [11]). Shojinka in the Toyota production system means to increase or decrease the number of workers at a shop as the production demand changes. Shojinka actually increases productivity by adjusting and rescheduling human resources. This concept has created new flexible lines in which there are fewer workers than stations in the line, and the workers walk to adjacent stations to continue work on an item.

Two main factors in Shojinka are:

- multi-functional workers, and
- U-shaped layout.

Multi-functional workers have the proper skills to work in different work stations in the line, and under a U-shaped layout, the walking times of workers between stations are reduced; and furthermore, it is easier to broaden or narrow the range of jobs for which each worker is responsible.

It should be noted that in some U-shaped lines the number of workers is the same as the number of work stations. These lines, which we prefer to call *traditional U-shaped lines*, are designed in a U shape for better material flow (entry and exit are located in the same side, close to each other), layout design considerations, or for better supervision and control. In this chapter, by U-shaped lines we only mean lines in which there are fewer workers than stations, whether the line is U-shaped or not.

Recently, U-shaped lines have become very popular. Some reasons for their popularity over traditional lines are as follows (Monden [11], Miltenburg [10], Bartholdi and Eisenstein [5] and Japanese Management Association [9]):

- Flexibility to increase or decrease the necessary number of workers when adapting to changes in production quantities (changes in demand).
- Having multi-functional workers rotating through stations in the line allows more workers to participate in efforts to improve the process.
- Workers stay more alert by rotating through a variety of tasks as compared to repeating a single short cycle task.
- The number of stations is always less or equal to that required on a traditional line, because there are more possibilities for grouping tasks into stations on a U-shaped line.
- The worker moves the material automatically as a part of the task, so that usually no special material-handling equipment is necessary.

Also, the advantages of U-shaped lines over traditional batch production in shops with functional layout can be summarized as: lower inventories, simpler material handling, easier production planning and control, opportunities of team work and problem solving, better control of quality, and so on.

These advantages have increased the number of U-shaped lines in manufacturing companies. However, based on the nature of products and activities, different versions of U-shaped lines with different features have been designed. In this paper a general definition for U-shaped lines is presented and different versions of U-shaped lines are introduced. Then, the literature on U-shaped lines is studied; and finally, considering the fact that a U-shaped line is actually a tandem queue attended by moving servers, the effect of switching costs and walking times are examined by decomposing a U-shaped line into a number of tandem queues each attended by a moving server.

2 U-Shaped Lines

Based on the production resources and the level of technology which were required in manufacturing units, different types of U-shaped lines have been designed to satisfy the requirements of those units. Before introducing these lines, we present our general definition for a U-shaped line which is used and referred to in this chapter as follows:

"A U-shaped line is a production or assembly system in which there are machines or tools available to perform N different operations on an item, and all items that enter the system require the sequence of operations $1, 2, \ldots, N$. There are M *workers (operators)*, $M < N$, who use the machines and tools to perform operations on items. Each different operation is usually performed at a different place which is called a *work station*. Therefore, U-shaped lines actually consist of N work stations sequenced from 1 to N in a line which is usually configured in a U shape. Workers move among work stations to process items according to *operational rules*. The operational rules determine each worker's next action when an operation is completed. According to the operational rules, worker m ($m \in \mathcal{M}_U$ where $\mathcal{M}_U = \{1, 2, \ldots, M\}$) may perform operations at a specific zone or the set of work stations N_m, where $\bigcup_{m=1}^{M} N_m = \mathcal{N}_U$ where $\mathcal{N}_U = \{1, 2, \ldots, N\}$. In other words, worker m is restricted to work in a specific zone in the line which contains work stations in set N_m. This zone is called the *working zone* of worker m. It should be noted that for two workers i and j, we may have $N_i \cap N_j = \emptyset$. This means that the working zones of workers i and j may overlap." Figure 1 shows a typical U-shaped line with $N = 12$ work stations and $M = 3$ workers.

Our definition characterizes different aspects of U-shaped lines for better understanding of the elements involved in these lines. In the next sections we present a new classification of U-shaped lines with the objective of organizing the previous research and providing a framework for further studies on these lines.

2.1 Worker-Oriented U-Shaped Lines

In a *worker-oriented U-shaped line* the operations on items require the presence of a worker during the whole operation. In other words, one item at a time can be processed in each work

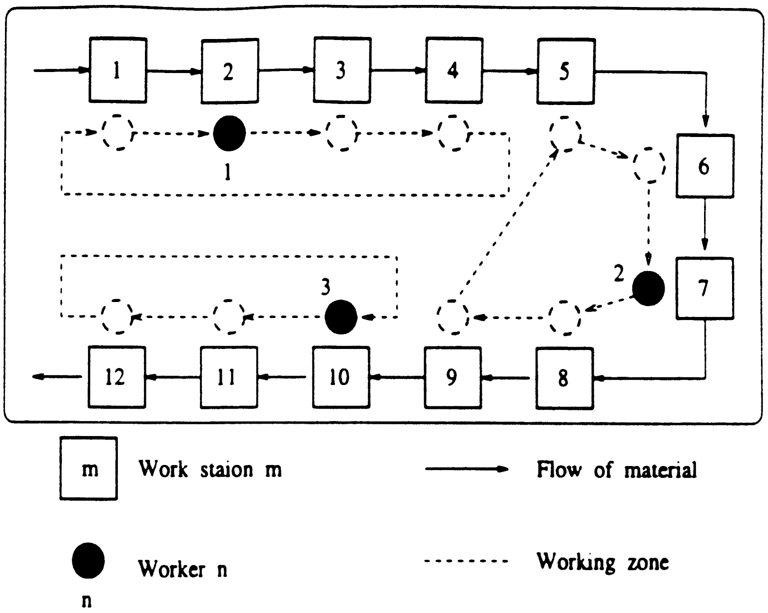

Figure 1. A U-shaped line with $N = 12$ work stations and $M = 3$ workers.

station and exactly one worker is required during the operation in that station. Typically, in these types of U-shaped lines, the machines or tools which are used are not highly automated or advanced, and the line is actually established based on the workers' skills and team work rather than the capabilities of the equipment in the line.

Bucket brigade production systems are one type of worker-oriented U-shaped lines. In a bucket brigade production system, exactly one worker is required during the operation on an item of a batch. Each worker processes his batch from station to station until he reaches a busy station where his successor is working on his batch. In this case, the worker must wait until the next station becomes available, and then he continues to work on his batch. When the last worker completes his batch on the last work station, the line resets. The reset process is actually a takeover process as follows: the last worker returns and takes over the batch of his predecessor, who in turn returns and takes over the batch of his predecessor, and so on, until the first worker starts a new batch at the first station. This idea was first commercialized in the apparel industry by "Aisin Seiki Co. Ltd." a subsidiary of Toyota, and named *Toyota Sewn Products Management System* or TSS (Bartholdi and Eisenstein [4]). TSS lines were first developed in Japan in 1970's and are widely used in apparel and textile industries. However, the first implementation of TSS in the US was in 1986. [1]

[1] In Riverside Fashions of Norris, South Carolina (Bartholdi and Eisenstien [4]).

TSS is actually a bucket brigade production system in which the bucket size is 1. The operational rule in TSS lines consists of two separate rules (forward and backward rules), and if the workers are numbered from 1 to M in the direction of product flow, then each worker must independently follows these rules (adapted from Bartholdi and Eisenstien [4]):

Forward rule: Process your item at successive work stations taking into account that at any station the worker with the higher index has priority. If your successor takes over your item, or if you are the last worker and you complete processing your item in the last station, follow the backward rule.

Backward rule: Walk back and begin to work on the item of your predecessor, or if you are the first worker, pick up raw material and start a new item in the first station. Follow the forward rule.

It is assumed that there is always enough raw material in front of the first work station. In TSS lines (bucket brigades) there is always a possibility for workers to be blocked in forward movements. In this case, they are not allowed to pass their successor or return to take over the work of their predecessor.

2.2 Machine-Oriented U-Shaped Lines

In each work station n of a *machine-oriented U-shaped line*, there is a machine which performs operation n $(n \in \mathcal{N}_U)$ on the items. These machines are advanced and are able to perform the main operation automatically on an item after being set up by an operator. Here, the workers are machine operators, and the number of operators, M, is less than the number of machines, N. Therefore, each operator is in charge of at least one machine. The difference between worker-oriented and machine-oriented U-shaped lines is that in worker-oriented U-shaped lines one worker is required during the whole processing time of an item in a station; however, in machine-oriented U-shaped lines, when an item is attached to a machine and the machine is turned on by an operator, the item can be processed automatically without the operator. Therefore, the operator can switch to another machine to continue his job and the item will be detached by the same or another operator later. In a worker-oriented U-shaped line, the number of busy work stations is at most equal to the number of workers; but, in a machine-oriented U-shaped line this number can be more than the number of operators. One example of the machine-oriented U-shaped line is the *single unit production and conveyance* ("Ikko-Nagashi" in Japanese), which is applied to a production line without conveyors to manufacture different kinds of relatively small parts (Monden [11]). In this line, one operator is in charge of N machines. When the operator visits a machine, his job is to wait for the processing of the preceding item if it is not complete, detach the processed item from the machine, attach the item which he has brought, turn the machine on and transfer the detached item to the next machine. According to this operational rule and considering that the line has only one operator, the new item enters the system only after one completed product exits. The work in process in the system is constant, and to increase the production rate more operators may be allocated to the system and zones assigned to each operator.

2.3 Static Working Zones Vs Dynamic Working Zones

Suppose in a machine-oriented U-shaped line that N_m is the set of machines which are assigned to operator m ($m \in \mathcal{M}_U$), and $N_i \cap N_j = \emptyset$ for $\forall i \neq j \in \mathcal{M}_U$. Then, each machine is actually assigned to one of the operators. In other words, the sets of machines assigned to specific operators are mutually exclusive, and therefore, the operators' working zones have no common area. These working zones which remain fixed and unchanged during the operation of the line are called *static working zones.* In worker-oriented U-shaped lines, the static working zones appear when only one specific worker is allowed to work in each working station. This means that operation n is always performed by worker m.

Sometimes static working zones are defined based on the fraction of work completed on an item instead of the number of operations performed on an item. In other words, the boundaries of a static working zone may not be the end point of an operation. If a boundary covers a fraction of an operation in a work station, then the remaining fraction belongs to another working zone. This means that there exists a work station which is used by two workers: but these workers perform separate fractions of an operation on an item in that station.

In U-shaped lines with *dynamic working zones*, set N_m is not a fixed set and may change in time. This means that each operator (worker) may be in charge of different machines (may work in different work stations) in each cycle. Therefore, for each worker different working zones are created in each cycle. Bucket brigades are a typical example of U-shaped lines with dynamic working zones. In bucket brigades each operation is not always performed by the same worker. U-shaped lines with dynamic working zones such as bucket brigades actually eliminate the possibility of starvation in a work station and create a self-organized line. On the other hand, workers in U-shaped lines with dynamic zones must be more skillful, because they must be able to work in more stations compared with the case of static zones designed for the same number of workers in a given U-shaped line.

2.4 Sequenced Working Zones Vs Mixed Working Zones

Consider a U-shaped line in which each worker is in charge of consecutive work stations. This usually means that the servers are not allowed to pass each other in the line. Thus, the U-shaped line will consist of M working zones which are located in a prescribed sequence. We call these zones as *sequenced working zones* (Figure 2.a). In U-shaped lines with sequenced working zones, each zone is a multi-stage tandem queue attended by a moving server. The sequenced zones may either change in different cycles (dynamic working zones) or remain the same all the time (static working zones). However, in both cases, the working zones can be always numbered from 1 to M in the direction of production flow. A TSS line is an example of a U-shaped line with sequenced dynamic working zones.

In U-shaped lines with *mixed working zones*, the work stations assigned to at least one worker are not necessarily consecutive work stations. In other words, at least one worker is allowed to skip some work stations and go from work station i to j ($j > n > i$, $i, j, n \in \mathcal{N}_U$), where another worker is in charge of work station n (Figure 3.b).

2.5 U-Shaped Lines with Multi-Functional Machines

Consider the single multi-functional machine scheduling problem in which a single machine is able to perform N different operations, one at a time. If this machine is used to process items which require the sequence of operations $1, 2, \ldots, N$, then this system is indeed a U-shaped line with N work stations and 1 worker. The setup time required after operation

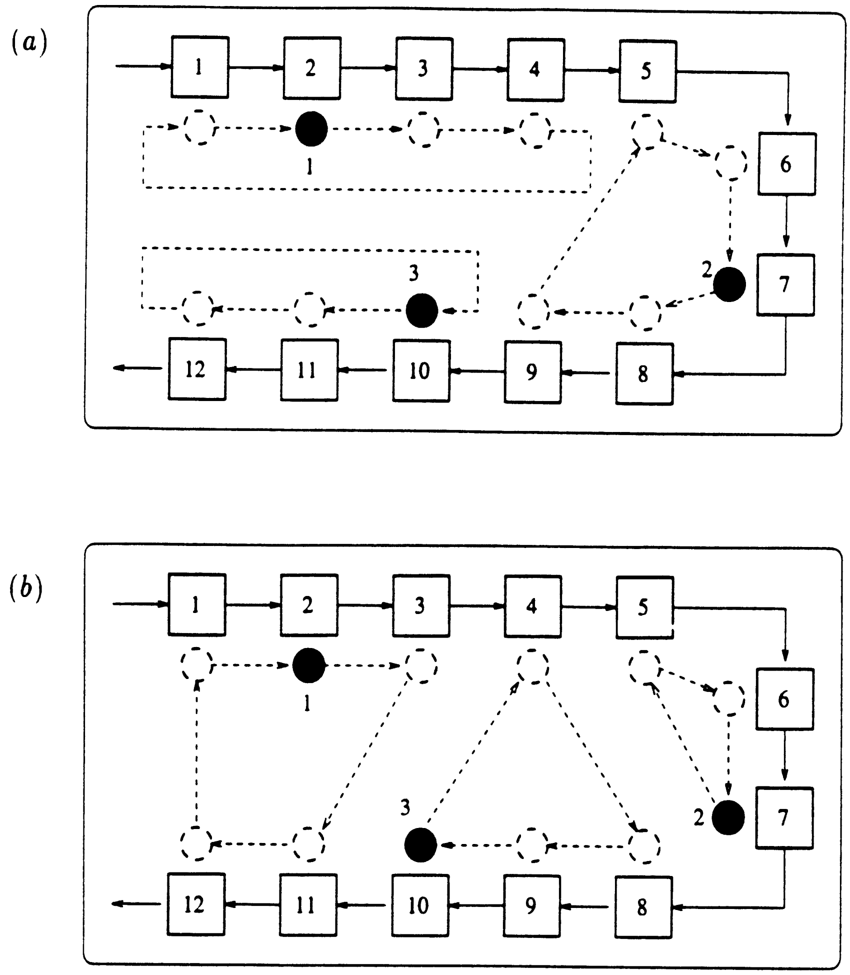

Figure 2. A U-shaped line with (*a*) Sequenced working zones. (*b*) Mixed working zones.

i to start operation j can be considered as the walking time between work stations i and j. Now, if M ($M < N$) similar machines are used and the set of operations N_m is assigned to machine m, where $\bigcup_{m=1}^{M} N_m = \mathcal{N}_U$, then for one operator per machine, this system may be considered a U-shaped line with N work stations and M workers.

U-shaped lines with multi-functional machines are not usually configured as a U-shaped line or at least the number of work stations is not quite clear to an observer. The walking times in these lines, which are actually the setup times, are typically greater than walking times in worker-oriented and machine-oriented U-shaped lines. In worker-oriented U-shaped lines, there are usually no setup times, and walking times are relatively small, while in machine-oriented U-shaped lines, the setup times are the times required to set the machine to perform the same operation on the next item. This time is usually less than the time required to set the machine to perform another operation on the next item (the setup time in a multi-functional machine).

3 Literature Review

In the category of worker-oriented U-shaped lines, Schoer, Wang and Ziemke [14] published the first paper on TSS lines in 1991. They analyzed a particular TSS line through simulation to achieve some statistics they needed, but did not reach any general conclusions about TSS lines. However, the first comprehensive paper on TSS lines (bucket brigades) was presented by Bartholdi and Eisenstein [4] in the context of apparel industry, in which they introduced a sufficient condition to achieve the maximum production rate. They assumed that all items are identical, requiring the same total processing time, and work station j performs a fraction p_j of the total processing time of an item. They define the state of the system as vector $X = (x_1, x_2, \ldots, x_M)$, where x_m, the position of worker m in the line, is determined by the fraction of the cumulated work completed so far on an item being processed by worker m to the total work required for that item in the line. Therefore, $0 \leq x_1 \leq x_2 \leq \cdots \leq x_M \leq 1$, because workers are not allowed to pass one another. They modeled worker m by work velocity v_m which can be interpreted as the number of complete items that worker m can produce per unit time, while working alone in the TSS line. For deterministic processing times and almost zero walking times, they showed that if workers are sequenced from slowest to fastest ($v_1 < v_2 < \cdots < v_M$), then the dynamic working zones in the line converge to static working zones, and if each worker is never blocked, then the working zone of worker m is bounded in the interval of work content $[l_m^{(w)}, u_m^{(w)}]$, where

$$l_m^{(w)} = \frac{\sum_{i=1}^{m-1} v_i}{\sum_{i=1}^{M} v_j} \quad (1)$$

$$u_m^{(w)} = \frac{\sum_{i=1}^{m} v_i}{\sum_{i=1}^{M} v_j}, \quad (2)$$

and the production rate of the line reaches to its maximum rate $\sum_{i=1}^{M} v_j$. In other words, by sequencing workers from slowest to fastest, the TSS line balances itself. On the other hand, they considered that if workers are not sequenced from slowest to fastest, then: (i) the TSS line can fail to balance itself, (ii) adding a worker to the line can decrease the production rate, and (iii) increasing the velocity of a worker may decrease the production rate. However, in TSS lines where workers are sequenced from slowest to fastest (a balanced TSS line), adding or speeding up a worker never decreases the production rate.

In their next paper with Bunimovich [2], Bartholdi and Eisenstein analyzed the behaviour of bucket brigade production lines with 2 and 3 workers. They simplified the model by assuming that the work content is spread continuously through the line rather than clumped in discrete amounts at work stations. Thus, when a faster worker follows a slower one, he will not be blocked because the next work station is occupied by the slower one. He will be blocked when his item reaches the same state of completion as that of the slower worker, whereupon he remains blocked continuously with his work velocity decreasing to the velocity of the slower worker. For bucket brigades with two workers, they concluded that if $v_1 < v_2$, then the system reaches to its maximum production rate $v_1 + v_2$. However, in bucket brigades where $v_1 > v_2$, the faster worker is continually blocked and the production rate is $2v_2$. In bucket brigades with three workers, they labeled workers so that $v_{min} < v_{mid} < v_{max}$, and they concluded the following:

- If the last worker in the line is the fastest one, then the line will achieve the maximum production rate $v_{min} + v_{mid} + v_{max}$ (irrespective of sequence of v_{min} and v_{mid}).

- If the last worker in the line is the slowest one, then the line will achieve the smallest production rate $3v_{min}$ (irrespective of sequence of v_{max} and v_{mid}).

- If the worker are sequenced as (**slowest, fastest, mid**), then the line will achieve either maximum production rate $v_{min} + v_{mid} + v_{max}$ or production rate $2(v_{min} + v_{mid})$.

- If the workers are sequenced as (**fastest, slowest, mid**), then the system displays a complex behaviour and will achieve a suboptimal production rate.

Bartholdi et al [2] also established the necessary condition $v_1 < v_M$ for a bucket brigade to balance itself.

The application of bucket brigades in order-picking systems in warehouses is described in Bartholdi, Bunimovich and Eisenstein [3]. In the order-picking version of bucket brigades, the workers are pickers, the items are orders and the working stations are different bays of the flowracks. The papers which describe orders are picked up by the first picker, who opens a box, and then slides it along the line as he moves, picking up different items to put into the box. This is the same job that other pickers do, except that each picker receives his box from his predecessor according to the bucket brigade operational rule. The main issue in the order-picking version of bucket brigades is that the order picking times are not deterministic, because the amount of work varies from order to order. However, Bartholdi, Bunimovich and Eisenstein claimed that although they have not been able to establish the proof for the stochastic case, nevertheless an array of evidence, including plausible explanation, simulation and field experiments have confirmed that if workers are sequenced from slowest to fastest, the bucket brigade will continuously and spontaneously (re)balance the work with the result that the average pick-up (production) rate is maximized. All results presented in Bartholdi et al [4, 2, 3] are based on the assumption that in bucket brigades the walking times between work stations are significantly less than processing times in work stations.

In their last paper, Bartholdi and Eisenstein [5] examined 150 TSS lines and presented some useful comments to help managers design bucket brigades. Some of these results are summarized as follows:

- The number of workers in a bucket brigade line must be less than $1/p_{max}$, where p_{max} is the largest fraction of total work to be done in a work station.
- Bucket brigades with a small number of workers perform better than bucket brigades with large numbers of workers. In the apparel industry, experience has shown that team effectiveness is reduced if the team has more than ten members; while three to six members is most common.
- Processing in large buckets reduces the variance of the work at each station and the chance of blocking.
- The bucket brigade production lines are mostly recommended when
    1. there are significant changes in demand,
    2. work stations are inexpensive relative to labor costs,
    3. the work in different work stations mostly need a single skill,
    4. the workers can move easily among work stations and the work takeover process can be done without difficulties.

Zavadlov, McClain and Thomas [15] analyzed different worker-oriented U-shaped lines through an exporatory approach using both Markovian and simulation models. The first model that they analyzed was a U-shaped line with three work stations, two workers and mixed static working zones. Assuming exponential processing times, the system is modeled as a Morkov chain and through a numerical example, the effect of variations in processing times and WIP are determined. They found that lower CVs and higher WIP levels caused lower idle time and increased throughput. The second model is a U-shaped line with four working stations and two workers. They compared the performances of sequenced and mixed working zones and they found that, in their example, the mixed working zone balances the mean workload, whereas the sequenced working zone incurs an idle time. They also concluded that doubling the buffer capacity improves the efficiency of systems with mixed working zones, more than the efficiency of systems with sequenced working zones. In their third model, they analyzed a U-shaped line with three work stations and two workers in which the working zones are sequenced and dynamic. Considering different operational rules, the efficiency of the line was examined through a Markov chain formulation of the system. For a longer line, they examined the effects of the relative size of the shared tasks in a system of nine work stations and five workers. Finally, they analyzed the performance of the two U-shaped lines with 12 and 9 work stations, and 6 and 5 workers, respectively, and they concluded that the system with no specific assignment is the most flexible line. In this system the workers can work on any machine if needed.

Miltenburg and Wijngaard [10] considered the line balancing problem of the worker-oriented U-shaped line with zero walking times and deterministic processing times. They introduced an optimization problem to find the optimal balance which defines the optimal static working zones for the U-shaped line with a minimum number of workers for a given set of tasks and cycle time. A dynamic programming procedure for computing the optimal balance was presented along with two heuristic procedures which represent extensions of well-known heuristics for the traditional line balancing problem. Finally, these heuristics were evaluated through their performances on well-known line balancing problems in the literature.

In the category of machine-oriented U-shaped lines, Ohno and Nakade [13] considered the single unit production and conveyance system with N machines, single operator and deterministic and constant processing, operation and walking times. Operation times are actually considered as the time required for detaching the processed item, putting it on a chute, and attaching a new item. They derived the operator's waiting time at each machine in each cycle and the cycle times. They also studied the same problem with M operators and obtained the overall cycle time of the line for given static working zones. To find the optimal static working zones for the operators which minimize the overall cycle time of the line, they formulated the problem as a combinatorial optimization problem, and then analyzed the optimal static working zones for the problem with two workers. Finally, they derived the average throughput of the line for the stochastic version of this problem with a single operator.

In another paper by Nakade and Ohno [12], two systems of machine-oriented U-shaped lines with a single operator were considered, in which the stochastic processing, operation and walking times are comparable in the sense of an increasing convex order. It is shown that as the operator is more skillful in the operation, in the sense of this order, the cycle time is shorter. They derived the expected cycle time for the line in which the processing times are Erlang random variables, and also obtained an upper and lower bound for the expected cycle time for the line with generally distributed processing times.

Most of the literature on U-shaped lines has not considered a switching cost when the worker moves (switches) from one work station to another, or if walking times are significant compared to processing times. Therefore, the only cost involved is the holding cost in the work station, and this can be minimized if a worker processes a single item instead of a batch, and completes that item, moving from station to station. In the next sections we examine the effect of switching costs and significant walking times on the optimal batch size in some classes of U-shape lines.

4 U-Shaped Lines with Multi-Functional Machines and Switching Costs

Consider a worker-oriented U-shaped line with N work stations and M workers who work in static and sequenced working zones. Consider i_m and j_m as the first and the last work stations in set N_m, sequenced according to the increasing order of the work station number. Also, suppose a processed item is called a *type j item* when it is processed in work station $j-1$ and then requires processing at work station j. Hence, type 1 items are actually the raw materials and type $N+1$ items are final products.

In a U-shaped line with static and sequenced working zones, there is always a chance for work station i_m to be starved. One way to eliminate this possibility is to have enough type i_m items in the buffer of work station i_m $(m \in \mathcal{M}_U)$. Applying this policy, the U-shaped line can be decomposed into M mutually independent multi-stage tandem queues, each attended by a moving server. This usually occurs in U-shaped lines with multi-functional machines, in which machine m performs all operations in set N_m and one operator is in charge of each machine. Switching costs and walking times from work station i to j are actually the setup costs and setup times, respectively, when operation j must be done after operation i. Since setup costs and setup times are involved, it may be optimal for operator m to complete a batch of items before switching to the next operation. To find the optimal batch size we need to introduce the following definitions and symbols:

- A *greedy and exhaustive policy in stages i to j ($j \geq i$)*, is a policy in which the server applies a greedy and exhaustive policy in stages $i, i+1, \ldots, j$, and switches from stage k to $k+1$ ($i \leq k \leq j-1$) after each exhaustion epoch.

- $i \uparrow j$ indicates a $(j-i+1)$-stage tandem queue ($j \geq i$) consisting of stages $i, i+1, \ldots, j$, and buffer of stage $j+1$.

- $(i \overset{k}{\triangleright} j)$ indicates a policy which serves a batch of size k in queue $i \uparrow j$.

- $\overset{r}{\odot} (i \overset{k}{\triangleright} j)$ indicates that policy $(i \overset{k}{\triangleright} j)$ is repeated r times.

To find the optimal batch size for operator m which minimizes the total average holding and switching costs, consider the multi-stage tandem queue $i_m \uparrow j_m$ with random service times S_j in work station j, random walking times D_{ij} from work station i to j, holding cost h_j in work station j and switching cost K_{ij} whenever the operator switches from operation i to j $(i \neq j;\ i, j \in N_m)$. If operator m completes a batch of size k in a cycle by applying a greedy and exhaustive policy in stages i_m to j_m, then the total average holding and switching costs in a cycle, $\mathcal{C}[i_m \overset{k}{\triangleright} j_m]$, is

$$\begin{aligned} \mathcal{C}[i_m \overset{k}{\triangleright} j_m] &= \frac{k(k+1)}{2} \sum_{r=i_m}^{j_m} h_r \overline{S}_r + \frac{k(k-1)}{2} \sum_{r=i_m}^{j_m} h_{r+1} \overline{S}_r \\ &\quad + \sum_{r=i_m}^{j_m-1} K_{r,r+1} + K_{j_m,i_m} + k \sum_{r=i_m}^{j_m} h_{r+1} \overline{D}_{r,r+1} \ . \end{aligned} \tag{3}$$

Since it is assumed that there are always enough type i_m items available in the buffer of work station i_m, therefore, the holding costs of these item are not considered in (3). Suppose $TC_m(1|k)$ is the total average holding and switching cost per produced item in queue $i_m \uparrow j_m$ when batch of size k is completed in each cycle; then,

$$\begin{aligned} TC_m(1|k) &= \frac{\mathcal{C}[i_m \overset{k}{\triangleright} j_m]}{k} \\ &= k\mathcal{H}_{i_m,j_m} + \frac{1}{k}\ddot{\mathcal{K}}_{i_m,j_m} + \mathcal{HD}_{i_m,j_m} \ , \end{aligned} \tag{4}$$

where

$$\mathcal{H}_{i_m,j_m} = \frac{1}{2}\sum_{r=i_m}^{j_m}(h_r + h_{r+1})\overline{S}_r \qquad (5)$$

$$\ddot{\mathcal{K}}_{i_m,j_m} = \sum_{r=i_m}^{j_m-1} K_{r,r+1} + K_{j_m,i_m} \qquad (6)$$

$$\mathcal{HD}_{i_m,j_m} = \frac{1}{2}\sum_{r=i_m}^{j_m}(h_r - h_{r+1})\overline{S}_r + h_{r+1}\overline{D}_{r,r+1} \ . \qquad (7)$$

Therefore, k^* will be the optimal batch size in queue $i_m \uparrow j_m$, if

$$TC_m(1|k^*) - TC_m(1|k^* + 1) < 0$$

$$TC_m(1|k^*) - TC_m(1|k^* - 1) < 0 \ ,$$

or

$$\mathcal{H}_{i_m,j_m} - \frac{\ddot{\mathcal{K}}_{i_m,j_m}}{k^*(K^* + 1)} > 0 \qquad (8)$$

$$\mathcal{H}_{i_m,j_m} - \frac{\ddot{\mathcal{K}}_{i_m,j_m}}{k^*(K^* - 1)} < 0 \ , \qquad (9)$$

upon using (4). Combining (8) and (9), we conclude that the optimal batch size which minimizes the total average holding and switching costs per produced item in queue $i_m \mid j_m$ is the integer k^* satisfying

$$k^*(k^* - 1) < \frac{\ddot{\mathcal{K}}_{i_m,j_m}}{\mathcal{H}_{i_m,j_m}} < k^*(k^* + 1) \ . \qquad (10)$$

It is clear that the optimality condition (10) is independent of the walking times. This is so because the total average holding cost during the walking times is constant for any single item of a batch, and is equal to $\sum_{r=i_m}^{j_m} h_{r+1}\overline{D}_{r,r+1}$.

Example 1

Consider a U-shaped line with three multifunctional machines where $N_1 = \{1, 2\}$, $N_2 = \{3, 4, 5\}$, $N_3 = \{6, 7, 8, 9, 10\}$ and

$$h = [2 \ \ 3 \ \ 5 \ \ 7 \ \ 10 \ \ 10 \ \ 11 \ \ 14 \ \ 15 \ \ 18]$$

$$\overline{S} = [9 \ \ 5 \ \ 1 \ \ 2 \ \ 3 \ \ 8 \ \ 2 \ \ 1 \ \ 3 \ \ 1]$$

$$K = [250 \ \ 200 \ \ 45 \ \ 80 \ \ 75 \ \ 90 \ \ 60 \ \ 70 \ \ 20 \ \ 100] \ ,$$

in which K_i is the switching cost to operation (stage) i ($i = 1, 2, \ldots, 10$). Assuming that there are always enough items available in the buffer of each machine, the optimal batch size k_1^* for machine 1 (queue $1 \uparrow 2$) can be obtained using (10) as follows:

$$\begin{aligned}
\ddot{\mathcal{K}}_{1,2} &= K_1 + K_2 \\
&= 450 \\
\mathcal{H}_{1,2} &= \frac{1}{2}\sum_{r=1}^{2}(h_r + h_{r+1})\overline{S}_r \\
&= 42.5\ .
\end{aligned}$$

Since k_1^* must satisfy

$$K_1^*(K_1^* - 1) < \frac{450}{42.5} < K_1^*(K_1^* + 1)\ ,$$

therefore, $k_1^* = 3$.

Using the same approach for machine 2 (queue $3 \uparrow 5$) and machine 3 (queue $6 \uparrow 10$), we get

$$K_2^*(K_2^* - 1) < \frac{200}{53} < K_2^*(K_2^* + 1)$$

$$K_3^*(K_3^* - 1) < \frac{340}{182} < K_3^*(K_3^* + 1)$$

which leads to $k_2^* = 2$ and $k_3^* = 1$ □

5 Worker-Oriented U-Shaped Lines with Sequenced Static Working Zones

Consider a worker-oriented U-shaped line with N work stations and M workers in which:

- Working zones are static and sequenced, so that worker m ($m \in \mathcal{M}_U$) is in charge of tandem work stations i_m to j_m (tandem queue $i_m \uparrow j_m$).
- Walking times are insignificant compared with the processing times in work stations.
- Items arrive at the buffer of work station 1 according to a random process, and the buffer of work station i_m is supplied by work station j_{m-1}, $m = 2, 3, \ldots, M$.
- the holding cost rate is h_i per unit time in work station i, and the switching cost K_{ij} is charged whenever a worker switches from work station i to j.

The differences between this model and the model in Section 4 are: (i) in this model walking times are actually considered to be zero, (ii) the items in the buffer of work station i_m are received from work station j_{m-1}; thus, there exists a chance for work station i_m to be starved during the operation. Since static and sequenced working zones are considered, the U-shaped line can be decomposed into M multi-stage tandem queues, each attended by a moving server. Suppose that worker m applies a limited policy in work station i_m and a greedy and exhaustive policy in work stations $i_m + 1$ to j_m. Then the optimal limit M_L^{*m} which minimizes the total average holding and switching cost in work stations i_m to j_m (queue $i_m \uparrow j_m$) can be approximated as follows:

1. For $m = 1$, since there are always enough items (raw materials) available in work station 1, the multi-stage tandem queue $i_1 \uparrow j_1$ with a limited policy behaves like the multi-stage tandem queue in Section 4, and so the optimal limit M_L^{*1} is actually the optimal batch size k^* which can be obtained using (10).

2. Multi-stage tandem queues $i_m \uparrow j_m$, $2 \leq m \leq M$ are $G/G_1 - G_2 - \cdots - G_{j_m - i_m + 1}/1$ queues with zero switchover times, a limited policy in stage i_m and a greedy and exhaustive policy in stages $i_m + 1$ to j_m. These models are the same as those introduced in Iravani, Posner and Buzacott [7], except that here the arrival process is not Poisson. The arrival process to queue $i_m \uparrow j_m$ is actually the departure process from queue $i_{m-1} \uparrow j_{m-1}$. Nevertheless, the optimal limits of the limited policy in high traffic intensity can be used as an approximator for the optimal limits in lower traffic intensities, even for systems with non-Poissonian arrival process for the following reasons:

   - In Iravani, Posner and Buzacott [7, 8], it is shown that the sufficient and necessary conditions for optimality of limits 1 and 2 in a limited policy are independent of the arrival process.

   - In $G/G_1 - G_2 - \cdots - G_N/1$ under low traffic intensity, limited policies with limit $M_L \geq 2$ behave almost the same, and therefore, the total average cost $TC(M_L)$ is very flat at the bottom; consequently, the difference in total average cost between the optimal limit and the limit which is optimal in high traffic intensity is insignificant (see Iravani, posner and Buzacott [8]). On the other hand, as ρ increases, the optimal limit in systems with lower traffic intensities approach the optimal limit in the same system with high traffic intensity.

Therefore, when $2 \le m \le M$, the optimal limit M_L^{*m} can be approximated for queue $i_m \uparrow j_m$ using Proposition 1 which yields the optimal limits of the limited policy in high traffic intensity.

Proposition 1

Consider a multi-stage tandem queue $i_m \uparrow j_m$, $(i_m < j_m)$ with zero switchover times in which a limited policy with limit M_L^m in stage i_m and a greedy and exhaustive policy in stages $i_m + 1$ to j_m are applied. Then assuming a high traffic intensity, the optimal limit M_L^{*m} satisfies

$$\frac{M_L^{*m}(M_L^{*m}-1)}{2} \le \frac{\ddot{\mathcal{K}}_{i_m,j_m}}{\sum_{r=i_m+1}^{j_m}(h_r - h_{i_m})\overline{S}_r + \sum_{r=i_m+1}^{j_m} h_r \overline{S}_{r-1}} \le \frac{M_L^{*m}(M_L^{*m}+1)}{2} . \quad (11)$$

Proof

Since in high traffic the limited policy with limit M_L^m actually serves M_L^m customers in each cycle, therefore to conclude (11) we now compare two limited policies with limits k and $k+1$ over the time during which $k(k+1)$ customers are served. Let $h_{j_m+1} = 0$; then for $M_L^m = k$ during $k+1$ cycles, we have

$$C[\overset{k+1}{\odot}(i_m \overset{k}{\triangleright} j_m)] = (k+1)C[i_m \overset{k}{\triangleright} j_m] + kK_{j_m,i_m} + \frac{k^2 k(k+1)}{2} h_{i_m} \sum_{r=i_m}^{j_m} \overline{S}_r , \quad (12)$$

and similarly, for $M_L^m = k+1$ during k cycles, we have

$$C[\overset{k}{\odot}(i_m \overset{k+1}{\triangleright} j_m)] = kC[i_m \overset{k+1}{\triangleright} j_m] + (k-1)K_{j_m,i_m} + \frac{(k+1)^2 k(k-1)}{2} h_{i_m} \sum_{r=i_m}^{j_m} \overline{S}_r, \quad (13)$$

so that

$$\begin{aligned} C[\overset{k}{\odot}(i_m \overset{k+1}{\triangleright} j_m)] - C[\overset{k+1}{\odot}(i_m \overset{k}{\triangleright} j_m)] &= kC[i_m \overset{k+1}{\triangleright} j_m] - (k+1)C[i_m \overset{k}{\triangleright} j_m] \\ &\quad - K_{j_m,i_m} - \frac{k(k+1)}{2} h_{i_m} \sum_{r=i_m}^{j_m} \overline{S}_r . \end{aligned} \quad (14)$$

On the other hand,

$$C[i_m \overset{k+1}{\triangleright} j_m] = \frac{(k+1)(k+2)}{2} \sum_{r=i_m}^{j_m} h_r \overline{S}_r + \frac{k(k+1)}{2} \sum_{r=i_m+1}^{j_m} h_r \overline{S}_{r-1} + \ddot{\mathcal{K}}_{i_m,j_m} \quad (15)$$

and

$$\mathcal{C}[i_m \overset{k}{\triangleright} j_m] = \frac{k(k+1)}{2} \sum_{r=i_m}^{j_m} h_r \overline{S}_r + \frac{k(k-1)}{2} \sum_{r=i_m+1}^{j_m} h_r \overline{S}_{r-1} + \ddot{\mathcal{K}}_{i_m,j_m} . \tag{16}$$

Therefore,

$$k\mathcal{C}[i_m \overset{k+1}{\triangleright} j_m] - (k+1)\mathcal{C}[i_m \overset{k}{\triangleright} j_m] = \frac{k(k+1)}{2}(\sum_{r=i_m}^{j_m} h_r \overline{S}_r + \sum_{r=i_m+1}^{j_m} h_r \overline{S}_{r-1}) - \ddot{\mathcal{K}}_{i_m,j_m} . \tag{17}$$

and by substituting (17) into (14), we obtain

$$\begin{aligned} \mathcal{C}[\overset{k}{\odot} (i_m \overset{k+1}{\triangleright} j_m)] - \mathcal{C}[\overset{k+1}{\odot} (i_m \overset{k}{\triangleright} j_m)] &= \frac{k(k+1)}{2}[\sum_{r=i_m}^{j_m} (h_r - h_{i_m})\overline{S}_r + \sum_{r=i_m+1}^{j_m} h_r \overline{S}_{r-1}] \\ &\quad -\ddot{\mathcal{K}}_{i_m,j_m} . \end{aligned} \tag{18}$$

Since $\lfloor \overset{k}{\odot} (i_m \overset{k+1}{\triangleright} j_m) \rfloor = \lfloor \overset{k+1}{\odot} (i_m \overset{k}{\triangleright} j_m) \rfloor$, thus, in high traffic, systems with limit $M_L^m = k$ have lower average cost than systems with limit $M_L^m = k+1$, provided

$$\mathcal{C}[\overset{k}{\odot} (i_m \overset{k+1}{\triangleright} j_m)] - \mathcal{C}[\overset{k+1}{\odot} (i_m \overset{k}{\triangleright} j_m)] \geq 0,$$

or, equivalently,

$$\frac{k(k+1)}{2} \geq \frac{\ddot{\mathcal{K}}_{i_m,j_m}}{\sum_{r=i_m+1}^{j_m} (h_r - h_{i_m})\overline{S}_r + \sum_{r=i_m+1}^{j_m} h_r \overline{S}_{r-1}} . \tag{19}$$

Using the same approach and comparing $\mathcal{C}[\overset{k-1}{\odot} (i_m \overset{k}{\triangleright} j_m)]$ and $\mathcal{C}[\overset{k}{\odot} (i_m \overset{k-1}{\triangleright} j_m)]$, it can be concluded that if

$$\frac{k(k-1)}{2} \leq \frac{\ddot{\mathcal{K}}_{i_m,j_m}}{\sum_{r=i_m+1}^{j_m} (h_r - h_{i_m})\overline{S}_r + \sum_{r=i_m+1}^{j_m} h_r \overline{S}_{r-1}} , \tag{20}$$

then the system with limit $M_L^m = k$ has lower average cost than a system with limit $M_L^m = k-1$. Combining (19) and (20), the proof is complete. □

It should be noted that applying an optimal limited policy by worker m in tandem queue $i_m \uparrow j_m$ $(m \in \mathcal{M}_U)$ does not mean that the U-shape line is optimized. Applying an optimal limited policy only guarantees the local optimization in static working zones given that limited policies must be implemented in working zones.

Example 2

Consider a worker-oriented U-shaped line with $N = 10$ work stations and $M = 3$ workers who work in sequenced and static working zones $N_1 = \{1,2\}$, $N_2 = \{3,4,5\}$ and $N_3 = \{6,7,8,9,10\}$. Also, suppose that the switching costs, holding costs and average service times in this line are the same as for Example 1. If each worker applies a limited policy in his working zone and items in the buffer of work stations 3 and 6 are supplied by work stations 2 and 5, respectively, then using (11), the optimal limit M_L^{*m} for working zone m ($m = 1,2,3$) must satisfy the following:

$$\frac{M_L^{*1}(M_L^{*1}-1)}{2} \leq \frac{450}{32} \leq \frac{M_L^{*1}(M_L^{*1}+1)}{2}$$
$$\frac{M_L^{*2}(M_L^{*2}-1)}{2} \leq \frac{200}{46} \leq \frac{M_L^{*2}(M_L^{*2}+1)}{2}$$
$$\frac{M_L^{*3}(M_L^{*3}-1)}{2} \leq \frac{340}{214} \leq \frac{M_L^{*3}(M_L^{*3}+1)}{2},$$

which lead to $M_L^{*1} = 5$, $M_L^{*2} = 3$ and $M_L^{*3} = 2$. □

6 Bucket Brigades with Switching Costs

In the model introduced in Bartholdi and Eisenstein [4] for bucket brigades with M workers and N work stations, the position of worker m is expressed as the cumulative fraction x_m of work completed on her item (batch). Thus, the position of worker m is actually a real number between zero and one. On the other hand, the total processing time of an item (batch) in the line is normalized to one time unit so that the processing time requirement in work station n is p_n, which is a fixed percentage of the total standard work content of the product. Therefore, if the standard processing time in work station n is $S_i^{(st)}$ ($n \in \mathcal{N}_U$), then the interval of work content $[l_n^{(s)}, u_n^{(s)}]$,

$$l_n^{(s)} = \frac{\sum_{r=1}^{n-1} S_j^{(st)}}{\sum_{r=1}^{N} S_j^{(st)}} \tag{21}$$

$$u_n^{(s)} = \frac{\sum_{r=1}^{n} S_j^{(st)}}{\sum_{r=1}^{N} S_j^{(st)}}, \tag{22}$$

actually refers to work station n.

Suppose the work velocity of a standard worker who completes processing of an item in its standard time is set at 1, and the velocity of the slower and faster workers are scaled according to this standard worker. Then a worker with work velocity $v = 0.5$ is half as fast as the standard worker. The standard worker needs exactly time $S_n^{(st)}$ to complete processing of an item in work station n; however, a worker with work velocity $v \neq 1$ does the same operation in $S_n^{(st)}/v$.

According to Bartholdi and Eisenstein [4], if the workers of a bucket brigade with deterministic processing times are sequenced from slowest to fastest, then dynamic working zones converge to static working zones, and if workers are never blocked, then the static working zone of worker m is bounded in the interval of work content $[l_m^{(w)}, u_m^{(w)}]$. A bucket brigade under these assumption is called a ***balanced bucket brigade***. Corollary 1 describes how a balanced bucket brigade can be analyzed in terms of tandem queues attended by a moving server.

Corollary 1

A balanced bucket brigade with N work stations and M workers behaves like M identical parallel N-stage tandem queues, each attended by a moving server who applies a greedy and exhaustive policy in stages 1 to N; and as each server moves from stage 1 to N, his work velocity increases in consecutive intervals of work content. □

Corollary 1 actually decomposes a balanced bucket brigade regarding to the number of workers (number of batches being processed in the line), M, rather than the working zones of workers. Each batch is processed in stages 1 to N independent of the other $M - 1$ batches which are processed simultaneously in the line. In other words, it can be considered that M batches are processed in M parallel N-stage tandem queues. These M parallel tandem queues are identical and all servers have work velocity v_m in the interval of work content $[l_m^{(w)}, u_m^{(w)}]$. As described in Section 2.3, the boundaries of working zones, $l_m^{(w)}$ and $u_m^{(w)}$ $(m \in \mathcal{M}_U)$, do not necessarily refer to the boundaries of some work stations, $l_i^{(s)}$ and $u_j^{(s)}$ $(i, j \in \mathcal{N}_U)$. In other words, worker $m + 1$ may take over the item of worker m in the middle of its processing in one of the work stations. This means that an item may be processed in two (or more) work velocities in one working station. Figure 3 presents a typical example showing how interval $[l_m^{(w)}, u_m^{(w)}]$ describes the static working zone of worker

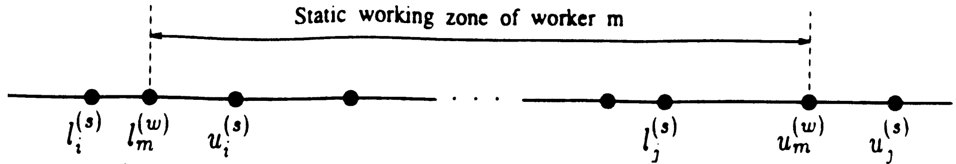

Figure 3. The static working zone of worker m in a bucket brigade.

m.

Suppose Figure 3 refers to a TSS line; then worker m is actually in charge of work stations $i+1, i+2, \ldots, j-1$. He also takes over the work of his predecessor (worker $m-1$) in work station i when fraction $l_m^{(w)}$ of work (compared to the total work required for an item in the line) on his item is complete. On the other hand, his item is taken over by his successor (worker $m+1$) in work station j when fraction $u_m^{(w)}$ of work on the item is complete. Since all items in work stations $i+1, i+2, \ldots, j-1$ are processed by worker m at work velocity v_m, the actual processing time on an item in work stations n, $\overline{S}_n$, is

$$\overline{S}_n = \frac{S_n^{(st)}}{v_n} \qquad ; \quad n = i+1, i+2, \ldots, j-1. \tag{23}$$

However, the actual processing times $\overline{S}_i$ and $\overline{S}_j$ of an item in work stations i and j, respectively, are obtained from

$$\overline{S}_i = \left(\frac{l_m^{(w)} - l_i^{(s)}}{u_i^{(s)} - l_i^{(s)}}\right)\frac{S_i^{(st)}}{v_{m-1}} + \left(1 - \frac{l_m^{(w)} - l_i^{(s)}}{u_i^{(s)} - l_i^{(s)}}\right)\frac{S_i^{(st)}}{v_m} \tag{24}$$

$$\overline{S}_j = \left(\frac{u_m^{(w)} - l_j^{(s)}}{u_j^{(s)} - l_j^{(s)}}\right)\frac{S_j^{(st)}}{v_m} + \left(1 - \frac{u_m^{(w)} - l_j^{(s)}}{u_j^{(s)} - l_j^{(s)}}\right)\frac{S_j^{(st)}}{v_{m+1}} . \tag{25}$$

Considering different static working zones, the same approach can be used to obtain the actual processing time $\overline{S}_n$ in work station n $(n \in \mathcal{N}_U)$. Therefore, based on Corollary 1, the TSS line with deterministic processing times in each work station can be decomposed into M parallel $G/D_1 - D_2 - \cdots - D_N/1$ queues in which the service time in stage i is $\overline{S}_i$ $(i = 1, 2, \ldots, N)$.

Now suppose that Figure 3 refers to a bucket brigade. Since worker m is in charge of work stations $i+1, i+2, \ldots, j-1$, all items in a batch will be processed at the same work velocity v_m in these stations, and equation (23) is now also true for bucket brigades. However, equations (24) and (25) will only be true if $\overline{S}_i$ and $\overline{S}_j$ are considered the actual processing times of a batch in work stations i and j, respectively. To find the actual processing times of items in a batch in work station j, let q_j be the fraction of the job in stage j (Figure 3) which is done at work velocity v_m; then,

$$q_j = \frac{u_m^{(w)} - l_j^{(s)}}{u_j^{(s)} - l_j^{(s)}} \ . \tag{26}$$

If a batch consisting of k items must be processed in work station j, then $kq_j S_j^{(st)}$ is that portion of the total standard work $kS_j^{(st)}$ in stage j which is completed at work velocity v_m, and the remaining part, $k(1-q_j)S_j^{(st)}$, is completed at work velocity v_{m+1}. This means that fraction kq_j of items in a batch are processed in stage j in actual time $kq_j S_j^{(st)}/v_m$ and the remaining fraction $k(1-q_j)$ of items are processed in actual time $k(1-q_j)S_j^{(st)}/v_{m+1}$.

Let J_i be the work station in which the the work velocity changes from v_i to v_{i+1} during the operation in that stage, and let $\mathcal{J}$ ($\mathcal{J} = \{J_1, J_2, \ldots, N+1\}$) be the set of these work stations. Here, $N+1$ is considered to be a hypothetical stage where the server changes work velocity from v_M to v_1.

Corollary 2 describes the changes in the work velocity in the work stations of a balanced bucket brigade.

Corollary 2

Let N_m be the set of work stations in the working zone of worker m. Then in a balanced bucket brigade the work velocity in work stations $n \in N_m \cap \mathcal{J}^c$ ($\mathcal{J}^c$ is the complement of set $\mathcal{J}$) is the constant v_m. Now let $\Im[u]$ be the largest integer less than u. Then, in a balanced bucket brigade with bucket size k, the work velocity in work station $J_m \in \mathcal{J}$, $m \in \mathcal{M}_U$ is:

- v_m, for the first $\Im[kq_{J_m}]$ items in the batch,
- v_m, for the fraction $kq_{J_m} - \Im[kq_{J_m}]$ of work on the $(\Im[kq_{J_m}]+1)$th item,
- v_{m+1}, for the remaining fraction $1-(kq_{J_m} - \Im[kq_{J_m}])$ of work on the $(\Im[kq_{J_m}]+1)$th item, and
- v_{m+1}, for the last $k - \Im[kq_{J_m}] - 1$ items. □

Remark 1

Corollary 2 only considers bucket brigades in which at most one change in work velocity may occur in each work station. In other words, each worker is in charge of at least one operation (work station) in the line, which is a realistic assumption. □

According to Corollary 1, the balanced bucket brigades with deterministic processing times and bucket size k can be considered as M parallel $G/D_1 - D_2 - \cdots - D_N/1$ queues in which servers apply greedy and exhaustive policies in stages 1 to N to process buckets of size k and change their work velocities according to Corollary 2. Thus, finding the optimal batch size for the balanced bucket brigade when switching costs are involved is equivalent to finding the optimal batch size in one of these identical queues. Since there are always enough items available in stage 1, obtaining the optimal batch size is almost similar to the model in Section 4. However, the difference is that here in each queue the server changes his work velocity in some stages while processing an item of a batch. Therefore, for the batch of size k in each parallel N-stage tandem queue, we have

$$\mathcal{C}[1 \overset{k}{\triangleright} N] = \mathcal{C}_h[1 \overset{k}{\triangleright} N] + \sum_{r=1}^{N-1} K_{r,r+1} + K_{N1} \ , \tag{27}$$

where $\mathcal{C}_h[\gamma]$ is the average holding cost of applying policy γ. Therefore,

$$\begin{aligned} \mathcal{C}_h[1 \overset{k}{\triangleright} N] &= \mathcal{C}_h[1 \overset{k}{\triangleright} J_1 - 1] + \sum_{J_i \in \mathcal{J}} \mathcal{C}_h[J_i \overset{k}{\triangleright} J_i + 1] \\ &\quad + \sum_{J_i \in \mathcal{J}} \mathcal{C}_h[J_i + 2 \overset{k}{\triangleright} J_{i+1} - 1] \ , \end{aligned} \tag{28}$$

and

$$\mathcal{C}_h[1 \overset{k}{\triangleright} J_1 - 1] = \frac{k(k+1)}{2} \sum_{r=1}^{J_1-1} h_r \overline{S}_r + \frac{k(k-1)}{2} \sum_{r=1}^{J_1-1} h_{r+1} \overline{S}_r \tag{29}$$

$$\mathcal{C}_h[J_i + 2 \overset{k}{\triangleright} J_{i+1} - 1] = \frac{k(k+1)}{2} \sum_{r=J_i+2}^{J_{i+1}-1} h_r \overline{S}_r + \frac{k(k-1)}{2} \sum_{r=J_i+2}^{J_{i+1}-1} h_{r+1} \overline{S}_r \ . \tag{30}$$

However, $\mathcal{C}_h[J_i \overset{k}{\triangleright} J_i + 1]$ is different because the server changes his work velocity during the operation in stage J_i, $(J_i \in \mathcal{J})$. Therefore, considering Corollary 2 and defining $a_{J_i} = \Im[kq_{J_i}]$, we will have

$$\begin{aligned} \mathcal{C}_h[J_i \overset{k}{\triangleright} J_i + 1] &= h_{J_i}[\ (\sum_{r=k-a_{J_i}+1}^{k} r)\frac{S_{J_i}^{(st)}}{v_i} + (\sum_{r=1}^{k-a_{J_i}-1} r)\frac{S_{J_i}^{(st)}}{v_{i+1}}\ ] \\ &\quad + h_{J_i}(k - a_{J_i})[\ (kq_{J_i} - a_{J_i})\frac{S_{J_i}^{(st)}}{v_i} + (1 - kq_{J_i} + a_{J_i})\frac{S_{J_i}^{(st)}}{v_{i+1}}\ ] \\ &\quad + h_{J_i+1}[\ (\sum_{r=1}^{a_{J_i}-1} r)\frac{S_{J_i}^{(st)}}{v_i} + (\sum_{r=a_{J_i}+1}^{k-1} r)\frac{S_{J_i}^{(st)}}{v_{i+1}}\ ] \end{aligned}$$

$$+h_{J_i+1}a_{J_i}[\,(kq_{J_i}-a_{J_i})\frac{S_{J_i}^{(st)}}{v_i}+(1-kq_{J_i}+a_{J_i})\frac{S_{J_i}^{(st)}}{v_{i+1}}\,]$$

$$+h_{J_i+1}[\frac{k(k+1)}{2}]\frac{S_{J_i+1}^{(st)}}{v_{i+1}}+h_{J_i+2}[\frac{k(k-1)}{2}]\frac{S_{J_i+1}^{(st)}}{v_{i+1}}\ ,$$

which, after some algebra, yields

$$\begin{aligned} C_h[J_i \overset{k}{\triangleright} J_i+1] &= h_{J_i}\{\ \frac{S_{J_i}^{(st)}}{v_i}[\frac{a_{J_i}(a_{J_i}+1)}{2}+kq_{J_i}(k-a_{J_i})]\} \\ &+h_{J_i}\{\frac{S_{J_i}^{(st)}}{v_{i+1}}[\frac{k(k+1)}{2}-\frac{a_{J_i}(a_{J_i}+1)}{2}-kq_{J_i}(k-a_{J_i})]\} \\ &+h_{J_i+1}\{\ \frac{S_{J_i}^{(st)}}{v_i}[\frac{a_{J_i}(a_{J_i}-1)}{2}+a_{J_i}(kq_{J_i}-a_{J_i})]\} \\ &+h_{J_i+1}\{\frac{S_{J_i}^{(st)}}{v_{i+1}}[\frac{k(k-1)}{2}-\frac{a_{J_i}(a_{J_i}-1)}{2}-a_{J_i}(kq_{J_i}-a_{J_i})]\} \\ &+h_{J_i+1}\{\frac{S_{J_i+1}^{(st)}}{v_{i+1}}[\frac{k(k+1)}{2}]\}+h_{J_i+2}\{\frac{S_{J_i+1}^{(st)}}{v_{i+1}}[\frac{k(k-1)}{2}]\}\ . \end{aligned} \qquad (31)$$

Substituting (29) - (31) into (28), and (28) into (27), the total costs for batch of size k in each of the parallel queues are obtained. Therefore, the total cost per produced item when the batch size is k, $TC(1|k)$, is

$$TC(1|k)=\frac{C_h[1 \overset{k}{\triangleright} N]}{k}\ ,$$

and the optimal batch size in each of the M parallel queues, or the optimal bucket size in the balanced bucket brigade will be the integer k^* satisfying

$$\begin{cases} TC(1|k^*)-TC(1|k^*+1)<0 \\ TC(1|k^*)-TC(1|k^*-1)<0\ . \end{cases}$$

Remark 2

A simpler approach to approximate the optimal bucket size k^* can be used by assuming a constant actual processing time $\overline{S}_{J_i}$ for all items of a batch in work station $J_i \in \mathcal{J}$ as following

$$\overline{S}_{J_i}=q_{J_i}\frac{S_{J_i}^{(st)}}{v_i}+(1-q_{J_i})\frac{S_{J_i}^{(st)}}{v_{i+1}}\ .$$

Thus, optimality condition (10) can be used to find the optimal bucket size k^*. □

Remark 3

Bartholdi and Eisenstein [3] claimed that even though they don't have the proof, they believe that if workers are sequenced from slowest to fastest, a bucket brigade with stochastic processing times balances itself and the production rate reaches to maximum. If this is true, then the optimal bucket size which minimizes the total average holding and switching costs can also be obtained for the stochastic version of bucket brigades using the same approach as Section 6 by considering $\overline{S}_n$ as the average actual processing time of an item in work station n $(n \in \mathcal{N}_U)$. $\square$

Example 3

Consider a bucket brigade with $N = 10$ work stations and $M = 3$ workers with work velocities $v_1 = 0.9$, $v_2 = 1$ and $v_3 = 1.2$. Also, consider that the holding costs, switching costs and standard processing times of an item in each work station are the same as h, K and $\overline{S}$ in Example 1. Suppose that the workers are sequenced from slowest to fastest and the bucket brigade balances itself with no blocking; then the working zone of worker m, namely the interval of work content $[l_m^{(w)}, u_m^{(w)}]$, will be, for $m = 1, 2, 3$,

$$[l_1^{(w)}, u_1^{(w)}] = [0.00,\ \frac{0.9}{3.1}] = [0.00,\ 0.29]$$

$$[l_2^{(w)}, u_2^{(w)}] = [0.29,\ \frac{1.9}{3.1}] = [0.29,\ 0.61]$$

$$[l_3^{(w)}, u_3^{(w)}] = [0.61,\ \frac{3.1}{3.1}] = [0.61,\ 1.00]\ .$$

However, work station n can be represented by interval of work content $[l_n^{(s)}, u_n^{(s)}]$ as follows:

$$\begin{aligned}[l_n^{(s)}, u_n^{(s)}] &= [\frac{\sum_{r=1}^{n-1} S_r^{(st)}}{\sum_{r=1}^{10} S_r^{(st)}}, \frac{\sum_{r=1}^{n} S_r^{(st)}}{\sum_{r=1}^{10} S_r^{(st)}}] \\ &= [\frac{\sum_{r=1}^{n-1} S_r^{(st)}}{35}, \frac{\sum_{r=1}^{n} S_r^{(st)}}{35}]\ .\end{aligned}$$

Therefore,

$$[l_1^{(s)}, u_1^{(s)}] = [0.00,\ 0.26] \quad [l_2^{(s)}, u_2^{(s)}] = [0.26,\ 0.40] \quad [l_3^{(s)}, u_3^{(s)}] = [0.40,\ 0.43]$$

$$[l_4^{(s)}, u_4^{(s)}] = [0.43,\ 0.46] \quad [l_5^{(s)}, u_5^{(s)}] = [0.46,\ 0.57] \quad [l_6^{(s)}, u_6^{(s)}] = [0.57,\ 0.80]$$

$$[l_7^{(s)}, u_7^{(s)}] = [0.80,\ 0.86] \quad [l_8^{(s)}, u_8^{(s)}] = [0.86,\ 0.89] \quad [l_9^{(s)}, u_9^{(s)}] = [0.89,\ 0.97]$$

$$[l_{10}^{(s)}, u_{10}^{(s)}] = [0.97,\ 1.00]$$

Figure 4 shows the working zones of the workers and work stations in terms of the intervals of work content. According to Figure 4, $\mathcal{J} = \{2, 6, 11\}$, which means that the

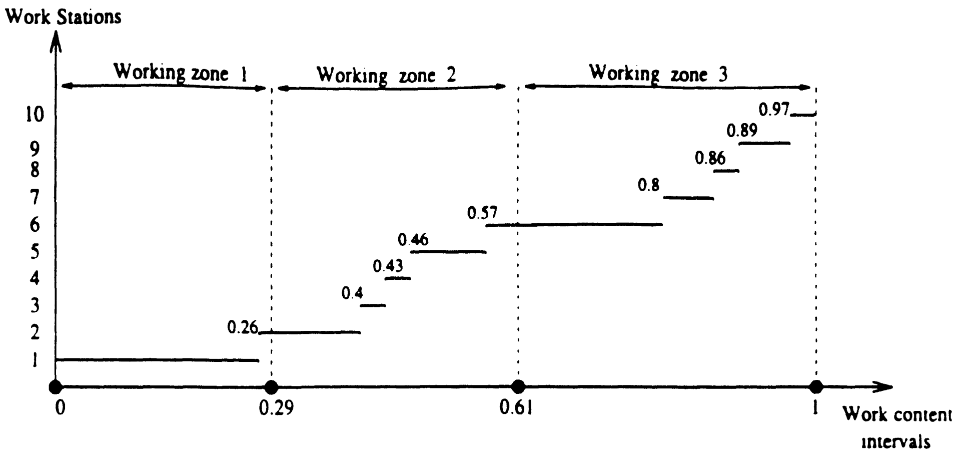

Figure 4. Working zones of balanced bucket brigade in example 8.3.

work velocities are changed in work stations 2, 6 and 11 from 0.9 to 1, 1 to 1.2 and 1.2 to 0.9, respectively. Hence

$$q_2 = \frac{u_1^{(w)} - l_2^{(s)}}{u_2^{(s)} - l_2^{(s)}} = \frac{0.29 - 0.26}{0.40 - 0.26} = 0.21$$

$$q_6 = \frac{u_2^{(w)} - l_6^{(s)}}{u_6^{(s)} - l_6^{(s)}} = \frac{0.61 - 0.57}{0.80 - 0.57} = 0.17$$

Using Corollary 1, the actual processing times in work stations $n \in \mathcal{J}^C$, $\overline{S}_n$, are

$$\overline{S}_1 = \frac{9}{0.9} = 10 \qquad \overline{S}_3 = \frac{1}{1} = 1 \qquad \overline{S}_4 = \frac{2}{1} = 2 \qquad \overline{S}_5 = \frac{3}{1} = 3$$

$$\overline{S}_7 = \frac{2}{1.2} = 1.67 \qquad \overline{S}_8 = \frac{1}{1.2} = 0.83 \qquad \overline{S}_9 = \frac{3}{1.2} = 2.5 \qquad \overline{S}_{10} = \frac{1}{1.2} = 0.83$$

Considering (27), we will have

$$\begin{aligned} \mathcal{C}[1 \overset{k}{\triangleright} 10] &= \mathcal{C}_h[1 \overset{k}{\triangleright} 10] + \sum_{r=1}^{9} K_{r,r+1} + K_{10,1} \\ &= \mathcal{C}_h[1 \overset{k}{\triangleright} 10] + 990 \,, \end{aligned} \tag{32}$$

where

$$\begin{aligned} \mathcal{C}_h[1 \overset{k}{\triangleright} 10] &= \mathcal{C}_h[1 \overset{k}{\triangleright} 1] + \mathcal{C}_h[2 \overset{k}{\triangleright} 3] + \mathcal{C}_h[4 \overset{k}{\triangleright} 5] \\ &\quad + \mathcal{C}_h[6 \overset{k}{\triangleright} 7] + \mathcal{C}_h[8 \overset{k}{\triangleright} 10] \,. \end{aligned} \tag{33}$$

On the other hand,

$$\mathcal{C}_h[1 \overset{k}{\triangleright} 1] = \frac{k(k+1)}{2} h_1 \overline{S}_1 + \frac{k(k-1)}{2} h_2 \overline{S}_1$$

$$= \quad 25k^2 - 5k \ , \qquad (34)$$

$$\mathcal{C}_h[4 \overset{k}{\triangleright} 5] = \frac{k(k+1)}{2}(h_4\overline{S}_4 + h_5\overline{S}_5) + \frac{k(k-1)}{2}(h_5\overline{S}_4 + h_6\overline{S}_5)$$

$$= \quad 47k^2 - 3k \ , \qquad (35)$$

$$\mathcal{C}_h[8 \overset{k}{\triangleright} 10] = \frac{k(k+1)}{2}(h_8\overline{S}_8 + h_9\overline{S}_9 + h_{10}\overline{S}_{10}) + \frac{k(k-1)}{2}(h_9\overline{S}_8 + h_{10}\overline{S}_9)$$

$$= \quad 60.76k^2 + 3.31k \ . \qquad (36)$$

However, if $a_2 = \Im[kq_2]$ and $a_6 = \Im[kq_6]$, then according to (32) we have

$$\begin{aligned} \mathcal{C}_h[2 \overset{k}{\triangleright} 3] &= h_2\{\frac{5}{0.9}[\frac{a_2(a_2+1)}{2} + 0.21k(k-a_2)]\} \\ &+h_2\{\frac{5}{1}[\frac{k(k+1)}{2} - \frac{a_2(a_2+1)}{2} - 0.21k(k-a_2)]\} \\ &+h_3\{\frac{5}{0.9}[\frac{a_2(a_2-1)}{2} + a_2(0.21k - a_2)]\} \\ &+h_3\{\frac{5}{1}[\frac{k(k-1)}{2} - \frac{a_2(a_2-1)}{2} + a_2(0.21k - a_2)]\} \\ &+h_3\{\frac{1}{1}[\frac{k(k+1)}{2}]\} + h_4\{\frac{1}{1}[\frac{k(k-1)}{2}]\} \\ &= 26.35k^2 - 6k + 0.23a_2k - 0.56a_2(a_2-1) \ , \end{aligned} \qquad (37)$$

and

$$\begin{aligned} \mathcal{C}_h[6 \overset{k}{\triangleright} 7] &= h_6\{\frac{8}{1}[\frac{a_6(a_6+1)}{2} + 0.17k(k-a_6)]\} \\ &+h_6\{\frac{8}{1.2}[\frac{k(k+1)}{2} - \frac{a_6(a_6+1)}{2} - 0.17k(k-a_6)]\} \\ &+h_7\{\frac{8}{1}[\frac{a_6(a_6-1)}{2} + a_6(0.17k - a_6)]\} \\ &+h_7\{\frac{8}{1.2}[\frac{k(k-1)}{2} - \frac{a_6(a_6-1)}{2} + a_6(0.17k - a_6)]\} \\ &+h_7\{\frac{2}{1.2}[\frac{k(k+1)}{2}]\} + h_8\{\frac{2}{1.2}[\frac{k(k-1)}{2}]\} \\ &= 93.1k^2 - 5.83k + 0.23a_6k - 0.67a_2(a_2-1) \ . \end{aligned} \qquad (38)$$

Substituting (34) - (38) into (33), after some algebra we get

$$\begin{aligned} \mathcal{C}_h[1 \overset{k}{\triangleright} 10] &= 252.21k^2 - 16.52k + 0.23k(a_2 + a_6) \\ &\quad -0.56a_2(a_2+1) - 0.67a_6(a_6+1) + 990 \ . \end{aligned}$$

Therefore,

$$\begin{aligned} TC[1|k] &= 252.21k - \frac{1}{k}(0.56a_2(a_2+1) + 0.67a_6(a_6+1) - 990) \\ &\quad +0.23(a_2 + a_6) - 16.52 \ , \end{aligned}$$

which implies that the optimal bucket size is $k^* = 2$ with $TC[1|k^* = 2] = 982.9$. □

7 Conclusion

The amount of literature on U-shaped production lines started to increase in 1991 in an attempt to analyze the behaviour of the TSS lines. The advantages of U-shaped production lines over traditional production lines encourage more manufacturing companies to employ these lines every day. Therefore, further investigations on the performance of these lines will be required in the near future. In order to establish a framework for the analysis of U-shaped lines, we presented a general definition and classification for these lines. We also showed that these lines can be analyzed in several ways by decomposing them into tandem queues each attended by a moving server. The decomposition may create completely disjointed tandem queues (as in Section 4 and 5), or exactly similar tandem queues (as in section 6). We decomposed three different types of U-shaped lines to examine the effect of switching cost and walking time on the batch size through the line.

Further studies on U-shaped lines can be carried out on the different issues of their design. These would represent an attempt to evaluate the performance of a specific line, or to find optimal characteristics of the line such as the optimal number of work stations, optimal number of workers, optimal working zones and optimal operational rules.

References

[1] Anon, *Today's FMS*, Vought Corp., (1985).

[2] Bartholdi III, J. J., Bunimovich L. A. and D. D. Eisenstein. Dynamic of 2-and-3-Worker Bucket Brigade Production Lines. (Submitted for Publication), (1996).

[3] Bartholdi III, J. J., Bunimovich L. A. and D. D. Eisenstein. Bucket Brigades: A Self-Balancing Order Picking System for a Warehouse. (Submitted for Publication), (1996).

[4] Bartholdi III, J. J. and D. D. Eisenstein. A Production Line That Balances Itself. *Oper. Res.*, **44** (1996) 21-34.

[5] Bartholdi III, J. J., Eisenstein D. D., Jacobs-Blecha C. and H. D. Ratliff. Design of Bucket Brigade Production Lines. (Submitted for Publication), (1996).

[6] Hall R. W., *Zero Inventories*, Dow Jones-Irwin Inc., Homewood, Illinois (1983).

[7] Iravani S. M. R., Posner M. J. M. and J. A. Buzacott. A Two-Stage Tandem Queue Attended by a Moving Server with Holding and Switching Costs. To appear in *Queueing Systems*.

[8] Iravani S. M. R., Posner M. J. M. and J. A. Buzacott. On Gated-Limited Policy in Tandem Queues Attended by a moving server. Working Paper.

[9] Japanese Management Association, *CANON Production System*, Productivity Press, Stamford, CT (1987).

[10] Miltenburge C. J. and J. Wijngaard. The U-Line Line Balancing Problem. *Management Sci.* **40** (1994) 1378-1388.

[11] Monden M., *Toyota Production system; An integrated Approach to Just-In-Time*, Industrial Engineering and Management Press, Georgia (1993).

[12] Nakade K. and K. Ohno. Analysis of Cycle Times in a U-Shaped Production Line. Department of Systems Engineering, Nagoya Institute of Technology (1995).

[13] Ohno K. and K. Nakade. Analysis and Optimization of a U-Shaped Production Line. Department of Systems Engineering, Nagoya Institute of Technology (1995).

[14] Schroer B. G., Wang J. and M. C. Ziemke. A Look at TSS Through Simulation. *Bobbin Mag.* March (1991) 138-144.

[15] Zavadlov E., McClain J. O. and L. J. Thomas. Self-Buffering, Self-Balancing, Self-flushing Production Lines. *Management Science* **42** (1996) 1151-1164.