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SIGNAL ESTIMATION FOR SECOND-ORDER VECTOR DIFFERENCE EQUATIONS[1]

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ABSTRACT

This paper considers a linear estimation problem for a stochastic process viewed as the output signal of a linear second-order vector difference equation (VDE) driven by a white-noise input. An innovations approach is applied directly to develop the one-stage prediction estimator and associated error covariances. It is shown that the estimator can be expressed as a second-order recursion that preserves the mathematical structure of the given signal model with innovations feedback loops. It is also shown that the innovations can be computed through a first-order recursion in terms of one-stage prediction estimates and the measurements.

1. INTRODUCTION

Discrete-time stochastic processes can be represented in various ways, for example using state-space realizations, transfer functions or vector difference equations. There is an extensive theory of recursive estimation based on state-space realizations [2,5]. Our objective here is to develop a theory of recursive estimation when the stochastic process is characterized in terms of linear second-order vector difference equations (VDE). Such representations have been used in [4] in an image processing context. Under certain sampling assumptions, models of elastic and mechanical systems can be described in terms of second-order vector difference equations [6]. Other applications where the signal dynamics are inherently ***second-order*** can, no doubt, be given.

A recursive equation for the one-stage prediction estimate is developed here, based on a linear second-order VDE signal model. The ***innovations approach*** [1,3] is applied directly to the second-order model thereby producing a recursion for the one-stage prediction estimate in second-order form. The resulting estimator is shown to to have the same mathematical structure as the given signal model with innovations feedback loops.

2. SIGNAL MODEL AND BASIC ASSUMPTIONS

The signal model considered here is a linear second-order VDE given by

$$x_{k+1} = A_k x_k + D_k x_{k-1} + \Gamma_k w_k \tag{1}$$

$$y_k = C_k x_k + E_k x_{k-1} + v_k \tag{2}$$

where k=1,2,3,... and x_0 and x_1 are the initial vectors, $\{x_k\}$ is an n–vector stochastic process. A_k, and D_k are real $n \times n$ index-varying matrices, $\{w_k\}$ is an r-vector zero mean gaussian white-noise process with covariance

$$E[w_k \cdot w_l^T] = Q_k \delta_{kl} \tag{3}$$

where, Q_k is an $r \times r$ index-varying matrix, and $\delta_{kl} = 1$ for $k=l$, $\delta_{kl} = 0$ for $k \neq l$, Γ_k is $n \times r$ index-varying matrix. $\{y_k\}$ is an m-vector output measurement process, C_k and E_k are real $m \times n$ measurement matrices; $\{v_k\}$ is an m-vector, zero mean gaussian white-noise process with covariance

$$E[v_k\, v_l^T] = R_k \delta_{kl} \tag{4}$$

Assume also the following

(1) The initial vectors x_0 and x_1 are jointly gaussian random vectors with means

$$E[x_0] = \bar{x}_0 \quad \text{, and} \quad E[x_1] = \bar{x}_1 \tag{5}$$

and covariances

$$E[(x_0-\bar{x}_0)(x_0-\bar{x}_0)^T] = \Sigma_{0|0} \tag{6}$$

$$E[(x_1-\bar{x}_1)(x_1-\bar{x}_1)^T] = \Sigma_{1|0} \tag{7}$$

$$E[(x_0-\bar{x}_0)(x_1-\bar{x}_1)^T] = \Pi_{1|0} \tag{8}$$

(2) The vectors x_0, x_1, $\{v_k\}$ and $\{w_k\}$ are mutually independent

(3) R_k is an $m \times m$ positive-definite matrix for each k.

The one-stage prediction estimator could be developed by converting the given second-order VDE to a state-variable form and applying Kalman filtering techniques. Our approach is to develop the estimator directly using an innovations approach [1,3].

3. EVOLUTION OF THE CONDITIONAL MEAN

It is desired to find a recursive equation for the conditional mean

$$\hat{x}_{k+1|k} = E[x_{k+1} | Y_k] \tag{9}$$

where $Y_k = \{y_1, y_2, \ldots, y_k\}$. The function $\hat{x}_{k|k-1}$ is referred to as the one-stage prediction estimate of x_k given Y_{k-1}. Define the set $\tilde{Y}_k = \{\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k\}$, where $\{\tilde{y}_k\}$ is called the innovations sequence of $\{y_k\}$ defined by

$$\tilde{y}_k = y_k - E[y_k | Y_{k-1}] \quad , \tilde{y}_1 = y_1 - C_1 \bar{x}_1 - E_1 \bar{x}_0 \tag{10}$$

where $E[y_k | Y_{k-1}]$ is the conditional mean of y_k given $Y_{k-1} = \{y_1, y_2, \ldots, y_{k-1}\}$. The innovations sequence defined by (10) has three properties that are exploited in our subsequent development. First, it has zero mean, second the set $\tilde{Y}_k$ constitutes an independent set; and third the sets Y_k and $\tilde{Y}_k$ span the same space. The proof of these properties is found in [2]. For simplicity in the subsequent formulas define $\tilde{y}_0 = 0$.

Without loss of generality, assume $\bar{x}_0 = \bar{x}_1 = 0$ in the subsequent development. Our approach is to make use of the properties of the innovations to write

$$\hat{x}_{k-1\,k} = E[x_{k+1} | \{\tilde{y}_k, \tilde{y}_{k-1}, Y_{k-2}\}]$$

$$= E[x_{k+1} | \tilde{y}_k] + E[x_{k+1} | \tilde{y}_{k-1}] + E[x_{k+1} | Y_{k-2}] \tag{11}$$

Next, each term of (11) is evaluated separately. The first term of (11) is evaluated as

$$E[x_{k+1} | \tilde{y}_k] = G_k^2 \tilde{y}_k \tag{12}$$

where the $n \times m$ gain matrix G_k^2 is defined by

$$G_k^2 \triangleq cov(x_{k+1}, \tilde{y}_k)[cov(\tilde{y}_k, \tilde{y}_k)]^{-1} \tag{13}$$

It can be shown that w_k and $\tilde{y}_{k-1}$ are independent so that the second term of (11) is

$$E[x_{k+1}|\tilde{y}_{k-1}] = E[A_k x_k + D_k x_{k-1} + \Gamma_k w_k | \tilde{y}_{k-1}]$$

$$= A_k E[x_k | \tilde{y}_{k-1}] + D_k G_{k-1}^1 \tilde{y}_{k-1} \tag{14}$$

where the $n \times m$ gain matrix G_k^1 is defined by

$$G_k^1 \triangleq cov(x_k, \tilde{y}_k)[cov(\tilde{y}_k, \tilde{y}_k)]^{-1} \tag{15}$$

Since w_k and Y_{k-2} are independent, the third term of (11) is evaluated as

$$\begin{aligned} E[x_{k+1}| Y_{k-2}] &= E[A_k x_k + D_k x_{k-1} + \Gamma_k w_k | Y_{k-2}] \\ &= A_k E[x_k | Y_{k-2}] + D_k E[x_{k-1}| Y_{k-2}] \\ &= A_k E[x_k | Y_{k-2}] + D_k E[x_{k-1}| Y_{k-2}] \\ &+ A_k E[x_k |\tilde{y}_{k-1}] - A_k E[x_k |\tilde{y}_{k-1}] \\ &= A_k \hat{x}_{k|k-1} + D_k \hat{x}_{k-1|k-2} - A_k E[x_k |\tilde{y}_{k-1}] \end{aligned}$$

Hence, substituting from (12), (14), and (16) into (11), the one stage prediction estimator can be written as

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k-1} + D_k \hat{x}_{k-1|k-2} + G_k^2 \tilde{y}_k + D_k G_{k-1}^1 \tilde{y}_{k-1} \tag{17}$$

where G_k^2, G_k^1 are as defined by (13) and (14) and $\{\tilde{y}_k\}$ is given by (10). In order to express $\tilde{y}_k$ in terms of one-stage prediction estimates, a first-order recurrence relation is developed for $\tilde{y}_k$ in a later section. Equations (13), (15), and (17) yield a recursive one-stage prediction estimator. It is interesting to note that (17) is a second-order recursion that preserves the form of the second-order VDE signal model with innovations feedback loops. A similar observation has been made in [4].

Given the one-stage prediction estimate $\hat{x}_{k|k-1}$, the filtered estimate defined by

$$\hat{x}_{k|k} \triangleq E[x_k | Y_k] \tag{18}$$

can be determined in terms of the one-stage prediction estimate as follows

$$\hat{x}_{k|k} = E[x_k \,|\, Y_{k-1}] + E[x_k \,|\, \tilde{y}_k]$$

$$= \hat{x}_{k|k-1} + G_k^{\,1} \tilde{y}_k \tag{19}$$

where $G_k^{\,1}$ is given by (15).

4. GAIN COMPUTATIONS

The estimator that has been developed in the preceding section involves two gain matrices $G_k^{\,1}$ and $G_k^{\,2}$ expressed in terms of the indicated covariances. These gain matrices are computed in this section.

The prediction and filtered error vectors are defined respectively by

$$\tilde{x}_{k|k-1} \triangleq x_k - \hat{x}_{k|k-1} \tag{20}$$

$$\tilde{x}_{k|k} \triangleq x_k - \hat{x}_{k|k} \tag{21}$$

Next observe that the innovations sequence $\{\tilde{y}_k\}$ can be written as

$$\tilde{y}_k = C_k \tilde{x}_{k|k-1} + E_k \tilde{x}_{k-1|k-1} + v_k \tag{22}$$

Define the covariance matrices by

$$\Sigma_{k|k-1} \triangleq E[\tilde{x}_{k|k-1} \tilde{x}_{k|k-1}^T] \tag{23}$$

$$\Sigma_{k|k} \triangleq E[\tilde{x}_{k|k} \tilde{x}_{k|k}^T] \tag{24}$$

$$\Pi_{k|k-1} \triangleq E[\tilde{x}_{k-1|k-1} \tilde{x}_{k|k-1}^T] \tag{25}$$

Now $cov(\tilde{y}_k, \tilde{y}_k)$ is evaluated as

$$cov(\tilde{y}_k, \tilde{y}_k) = C_k \Sigma_{k|k-1} C_k^T + C_k \Pi_{k|k-1}^T E_k^T + E_k \Pi_{k|k-1} C_k^T + E_k \Sigma_{k-1|k-1} E_k^T + R_k$$

$$\triangleq K_k \tag{26}$$

Clearly K_k is an $m \times m$ positive definite matrix.

Next G_k^1 can be evaluated as

$$G_k^1 = [\Sigma_{k|k-1} C_k^T + \Pi_{k|k-1}^T E_k^T] K_k^{-1} \tag{27}$$

G_k^2 is computed in a similar way as

$$G_k^2 = A_k G_k^1 + D_k [\Pi_{k|k-1} C_k^T + \Sigma_{k-1|k-1} E_k^T] K_k^{-1} \tag{28}$$

The gain matrices G_k^1 and G_k^2 thus depend on the covariance matrices $\Sigma_{k|k-1}$, $\Pi_{k|k-1}$, and $\Sigma_{k|k}$. Recursive formulas for these covariance matrices are developed in the next section.

5. EVOLUTION OF COVARIANCE MATRICES

In this section, recursive formulas for the previously mentioned covariance matrices are developed. First note, if (17) is subtracted from (1)

$$\tilde{x}_{k+1|k} = A_k \tilde{x}_{k|k-1} + D_k \tilde{x}_{k-1|k-2} + \Gamma_k w_k - G_k^2 \tilde{y}_k - D_k G_{k-1}^1 \tilde{y}_{k-1} \tag{29}$$

Also from (19) and (22)

$$\tilde{x}_{k+1|k+1} = \tilde{x}_{k+1|k} - G_{k+1}^1 \tilde{y}_{k+1} \tag{30}$$

and from (29) and (30)

$$\tilde{x}_{k+1|k} = A_k \tilde{x}_{k|k-1} + D_k \tilde{x}_{k-1|k-1} + \Gamma_k w_k - G_k^2 \tilde{y}_k \tag{31}$$

Then from (24) and (30), the evolution of $\Sigma_{k|k}$ can be shown to be given by

$$\Sigma_{k|k} = \Sigma_{k|k-1} - G_k^1 C_k \Sigma_{k|k-1} - G_k^1 E_k \Pi_{k|k-1} \tag{32}$$

Next from (23), (31) and some algebraic manipulations

$$\begin{aligned}\Sigma_{k+1|k} &= (D_k - G_k^2 E_k)(\Sigma_{k-1|k-1} D_k^T + \Pi_{k|k-1} A_k^T) \\ &+ (A_k - G_k^2 C_k)(\Pi_{k|k-1}^T D_k^T + \Sigma_{k|k-1} A_k^T) + \Gamma_k Q_k \Gamma_k^T\end{aligned} \tag{33}$$

Finally, a recursive formula for $\Pi_{k|k-1}$ is obtained in a similar way as

$$\Pi_{k+1|k} = \Sigma_{k|k} A_k^T + (\Pi_{k|k-1}^T - G_k^1 C_k \Pi_{k|k-1}^T - G_k^1 E_k \Sigma_{k-1|k-1}) D_k^T \tag{34}$$

The recursive equations (32)-(34) are initialized by the initial matrices (7)-(9).

6. INNOVATIONS REPRESENTATIONS

A useful characterization for the innovations in terms of the one-stage prediction estimates and the measurements is developed. The innovations sequence $\{\tilde{y}_k\}$, defined by (10), can be written as

$$\tilde{y}_k = y_k - C_k \hat{x}_{k|k-1} - E_k \hat{x}_{k-1|k-1} \ , \ \tilde{y}_1 = y_1 - C_1 \bar{x}_1 - E_1 \bar{x}_0 \tag{35}$$

Hence, $\tilde{y}_k$ is a linear combination of both the one-stage prediction and the filtered estimates. Since our estimator (17) is expressed entirely in terms of one-stage prediction estimates, it is desirable to characterize $\tilde{y}_k$ through one-stage prediction estimates only. This can be done by substituting from (19) into (35). This yields

$$\tilde{y}_k + E_k G_{k-1}^1 \tilde{y}_{k-1} = y_k - C_k \hat{x}_{k|k-1} - E_k \hat{x}_{k-1|k-2} \tag{36}$$

with

$$\tilde{y}_1 = y_1 - C_1 \bar{x}_1 - E_1 \bar{x}_0 \ , \ \tilde{y}_0 = 0 \tag{37}$$

which shows that the innovations satisfy a first-order recursion driven by the one-stage prediction estimates and the measurements.

7. MAIN RESULTS

The results can be summarized in the following theorem.

Theorem 1

The one-stage prediction estimator for the system (1) and (2), with the stated assumptions, is of the form

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k-1} + D_k \hat{x}_{k-1|k-2} + G_k^2 \tilde{y}_k + D_k G_{k-1}^1 \tilde{y}_{k-1} \tag{38}$$

for $k = 1,2,\ldots$, with initial vectors

$$\hat{x}_{0|-1} = \bar{x}_0 \quad \text{and} \quad \hat{x}_{1|0} = \bar{x}_1 \tag{39}$$

The innovations satisfy

$$\tilde{y}_k + E_k G_{k-1}^1 \tilde{y}_{k-1} 1 = y_k - C_k \hat{x}_{k|k-1} - E_k \hat{x}_{k-1|k-2} \ , \quad k = 2,3,\ldots \tag{40}$$

$$\tilde{y}_1 = y_1 - C_1 \bar{x}_1 - E_1 \bar{x}_0 \ , \ y_0 = 0 \tag{41}$$

The gains G_k^1 and G_k^2 are given respectively by (15) and (13) and the associated covariances are given by (32)-(34) and initialized by (7)-(9). In addition, the filtered estimate is given by equation

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + G_k^1 y_k \tag{42}$$

8. CONCLUSIONS

It has been demonstrated that the innovations approach can be applied directly to estimate signals described by linear second-order vector difference equations. This yields a recursive one-stage prediction estimator in second-order form that preserves the structure of the signal model with innovations feedback. It has also been shown that the innovations can be computed through a recurrence relation based on the knowledge of one-stage prediction estimates and the measurements.

This approach can be easily extended to obtain similar recursions for the filtered estimates and for the innovations expressed in terms of the filtered estimates and the measurements. In addition, the approach can be extended to the estimation of signals that are described in terms of higher-order vector difference equations.

9. REFERENCES

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