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OPTIMAL LOW THRUST ESCAPE VIEWED AS A RESONANCE PHENOMENON*

by

Robert A. Jacobson

and

William F. Powers

The University of Michigan
Ann Arbor, Michigan

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Abstract

In this study a complete second order perturbation solution to a modified optimal low thrust escape problem is presented. The optimal thrust direction is shown to be tangential to first order, and oscillatory to second order with period equal to that of the initial circular reference orbit. The improvement of the optimal trajectory over a tangential thrust escape trajectory is shown to be a second order resonance type effect.

I. Introduction

The problem of low thrust escape from an initial circular orbit has been studied by many researchers using a wide variety of methods. Escape using a specified thrust program such as tangential, circumferential, or radial has been studied from both a numerical and an approximate analytical viewpoint (References 1-12) Escape using an optimal control, determined by the calculus of variations, has also been solved numerically,[13-15] but little analytical work has been done in this area.[16] On the other hand, numerous studies of the optimal close-orbit transfer problem, both analytical and numerical, have been reported (References 12,17-23).

In this study a modification of the problem of minimum time escape from an initial near circular orbit under low constant thrust acceleration will be considered. As a result of previous numerical studies, it is known that tangential thrust is near optimal, and that the optimal control angle exhibits an oscillatory behavior with a period near that of the osculating orbital period, and with a mean value near tangential. The main purpose of this analysis is to explain: (1) the relationship between the optimal and the tangential controls, and (2) the physical significance of the oscillatory behavior of the optimal steering angle.

II. Formal Problem Definition

The specific problem to be studied is as follows: given a space vehicle in an initial near-circular orbit with energy E_0, find the control angle program which will take the vehicle to a specified energy level E_f in minimum time. The vehicle is assumed to be subject to a low constant thrust acceleration engine, the gravity field is inverse square, and all motion is confined to the initial orbital plane. The equations of motion are:

$$\dot{r} = V \sin\gamma$$

$$\dot{\theta} = \frac{V}{r} \cos\gamma$$

$$\dot{V} = -\frac{\mu}{r^2} \sin\gamma + a \cos\phi$$

$$\dot{\gamma} = \left(\frac{V}{r} - \frac{\mu}{r^2 V}\right) \cos\gamma + \frac{a}{V} \sin\phi$$

where

r = radial distance

θ = polar angle

V = total velocity magnitude

γ = flight path angle measured from the local horizontal

ϕ = control angle measured from the velocity direction

a = thrust acceleration

as shown in Figure 1.

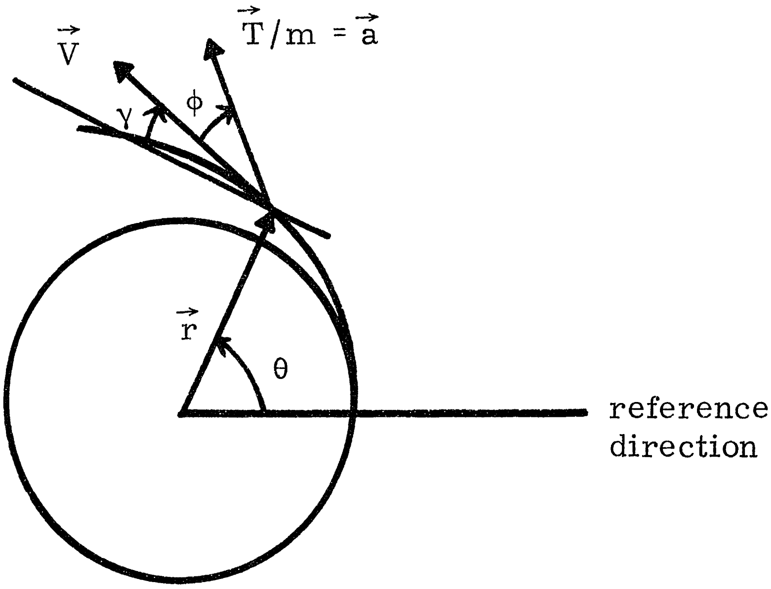

Figure 1

The optimal control program is obtained by application of the calculus of variations, where the performance index and the Hamiltonian are

$$J = t_f - t_0$$

$$H = \lambda_1 V \sin\gamma + \lambda_2 \frac{V}{r} \cos\gamma - \lambda_3 \frac{\mu}{r^2} \sin\gamma + \lambda_4\left(\frac{V}{r} - \frac{\mu}{r^2 V}\right) \cos\gamma$$
$$+ a(\lambda_3 \cos\phi + \frac{\lambda_4}{V} \sin\phi) ,$$

and the optimal control is defined by

$$\sin\phi = \frac{\lambda_4}{\sqrt{\lambda_4^2 + \lambda_3^2 V^2}}$$

$$\cos\phi = \frac{\lambda_3 V}{\sqrt{\lambda_4^2 + \lambda_3^2 V^2}}$$

The multiplier equations are

$$\dot{\lambda}_1 = \lambda_2 \frac{V}{r^2}\cos\gamma - 2\lambda_3\frac{\mu}{r^3}\sin\gamma + \lambda_4\left(\frac{V}{r^2} - \frac{2\mu}{r^3 V}\right)$$

$$\dot{\lambda}_2 = 0$$

$$\dot{\lambda}_3 = -\lambda_1\sin\gamma - \lambda_2\frac{1}{r}\cos\gamma - \lambda_4\left(\frac{1}{r} + \frac{\mu}{r^2V^2}\right)\cos\gamma + a\,\frac{\lambda_4}{V^2}\sin\phi$$

$$\dot{\lambda}_4 = -\lambda_1 V\cos\gamma + \lambda_2\frac{V}{r}\sin\gamma + \lambda_3\frac{\mu}{r^2}\cos\gamma + \lambda_4\left(\frac{V}{r} - \frac{\mu}{r^2V}\right)\sin\gamma$$

The boundary conditions are

$$t_0 = 0 \qquad , \quad \tfrac{1}{2}V^2(t_f) - \frac{\mu}{r(t_f)} = E_f$$

$$r(0) = R_0 \qquad , \quad \lambda_2(t_f) = 0$$

$$\theta(0) = 0 \qquad , \quad \lambda_4(t_f) = 0$$

$$V(0) = \sqrt{\frac{\mu}{R_0}} = R_0\omega_0 \qquad , \quad H(t_f) = 1$$

$$\gamma(0) = \frac{2a}{R_0\omega_0^2} \qquad , \quad \lambda_1(t_f) - \frac{\mu\lambda_3(t_f)}{r^2(t_f)V(t_f)} = 0$$

where the last four terminal conditions are given by the transversality conditions, and the particular choice of the initial flight path angle will be explained later.

III. Approximate Analytical Solution

Since θ does not influence the problem and $\lambda_2(t) \equiv 0$, only the sixth order system defined by $(r, v, \gamma, \lambda_1, \lambda_3, \lambda_4)$ will be considered. A solution in the form of an expansion in powers of a, the thrust acceleration, will be assumed as follows:

$$r = r_0 + ar_1 + a^2 r_2 + \cdots$$

$$V = V_0 + aV_1 + a^2V_2 + \cdots$$

$$\gamma = \gamma_0 + a\gamma_1 + a^2\gamma_2 + \cdots$$

$$\lambda_1 = a\lambda_{11} + a^2\lambda_{12} + \cdots$$

$$\lambda_3 = a\lambda_{31} + a^2\lambda_{32} + \cdots$$

$$\lambda_4 = a\lambda_{41} + a^2\lambda_{42} + \cdots$$

$$\sin\phi = \beta_0 + a\beta_1 + a^2\beta_2 + \cdots$$

$$\cos\phi = \alpha_0 + a\alpha_1 + a^2\alpha_2 + \cdots$$

IIIa. Zero order state variable solutions

After substituting the above expansions into the system of equations and grouping terms in powers of a, the zero order state equations are

$$\dot{r}_0 = V_0 \sin\gamma_0$$

$$\dot{V}_0 = -\frac{\mu}{r_0^2}\sin\gamma_0$$

$$\dot{\gamma}_0 = \frac{1}{V_0}\left(\frac{V_0^2}{r_0} - \frac{\mu}{r_0^2}\right)\cos\gamma_0$$

The solution of these equations subject to the given initial conditions is

$$r_0(t) = R_0$$

$$V_0(t) = R_0\omega_0$$

$$\gamma_0(t) = 0$$

That is, the zero order solution is a circular orbit of radius R_0.

IIIb. First order state and multiplier solutions

The first order system of equations obtained from the expansions is

$$\dot{r}_1 = V_1 \sin\gamma_0 + V_0\gamma_1\cos\gamma_0$$

$$\dot{V}_1 = \frac{\mu}{r_0^2}\left(2\frac{\mu r_1}{r_0}\sin\gamma_0 - \gamma_1\cos\gamma_0\right) + \alpha_0$$

$$\dot{\gamma}_1 = \left(2\frac{V_1}{r_0} - \frac{V_0 r_1}{r_0^2} + 2\frac{\mu r_1}{r_0^3 V_0}\right)\cos\gamma_0 - \frac{1}{V_0}\left(\frac{V_0^2}{r_0} - \frac{\mu}{r_0^2}\right)\left(\gamma_1\sin\gamma_0 + \frac{V_1}{V_0}\cos\gamma_0\right)$$
$$+ \frac{1}{V_0}\beta_0$$

$$\dot{\lambda}_{11} = \frac{1}{V_0}\left(\frac{V_0^2}{r_0^2} - 2\frac{\mu}{r_0^3}\right)\lambda_{41}$$

$$\dot{\lambda}_{31} = -\left(\frac{1}{r_0} + \frac{\mu}{r_0^2 V_0^2}\right)\lambda_{41}$$

$$\dot{\lambda}_{41} = -V_0\lambda_{11} + \frac{\mu}{r_0^2}\lambda_{31}$$

$$\alpha_0 = \left[1 + \left(\frac{\lambda_{41}}{\lambda_{31}V_0}\right)^2\right]^{-\frac{1}{2}}, \quad \beta_0 = \frac{\lambda_{41}}{\lambda_{31}V_0}\left[1 + \left(\frac{\lambda_{41}}{\lambda_{31}V_0}\right)^2\right]^{-\frac{1}{2}}$$

Since the multiplier equations are independent of the first order state equations, they may be solved easily when the zero order state solution is known. The general solution is

$$\lambda_{11}(t) = c_2 + \frac{c_1}{R_0}\cos(\omega_0 t + \beta)$$

$$\lambda_{31}(t) = \frac{c_2}{\omega_0} + \frac{2}{R_0\omega_0} c_1 \cos(\omega_0 t + \beta)$$

$$\lambda_{41}(t) = c_1 \sin(\omega_0 t + \beta)$$

From the expansion of the terminal conditions in powers of a we find to first order at t_f that

$$\lambda_{11}(t_f) - \omega_0 \lambda_{31}(t_f) = 0$$

$$\lambda_{41}(t_f) = 0$$

$$\lambda_{31}(t_f) = \frac{1}{a^2}$$

which implies

$$\lambda_{11}(t) = \frac{\omega_0}{a^2}$$

$$\lambda_{31}(t) = \frac{1}{a^2}$$

$$\lambda_{41}(t) = 0$$

It follows immediately that

$$\begin{array}{ll} \alpha_0 = 1 & \\ & \Longrightarrow \\ \beta_0 = 0 & \end{array} \quad \begin{array}{l} \cos\phi = 1 \\ \sin\phi = 0 \end{array}$$

Therefore, the optimal "escape" control program to first order is tangential thrust.

The first order state equations may now be solved using the tangential control program. The general first order state solution is

$$r_1(t) = c_3 + \frac{2}{\omega_0} t + R_0 c_1 \sin\omega_0 t - R_0 c_2 \cos\omega_0 t$$

$$V_1(t) = -\tfrac{1}{2}\omega_0 c_3 - t + R_0\omega_0 c_2 \cos\omega_0 t - R_0\omega_0 c_1 \sin\omega_0 t$$

$$\gamma_1(t) = \frac{2}{R_0\omega_0^2} + c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$

The initial conditions for this system are

$$r_1(0) = 0$$

$$V_1(0) = 0$$

$$\gamma_1(0) = \frac{2}{R_0\omega_0^2}$$

which implies

$$r_1(t) = \frac{2}{\omega_0} t$$

$$V_1(t) = -t$$

$$\gamma_1(t) = \frac{2}{R_0 \omega_0^2}$$

It should be noted that the above set of initial conditions, which correspond to an orbit of low eccentricity, was first suggested by Lawden[16] in an effort to simplify the higher order solutions by elimination of the oscillatory first order motion of the escape spiral.

IIIc. Second order state and multiplier solutions

After substitution of the zero order solution into the second order equations, we have

$$\dot{r}_2 = R_0 \omega_0 \gamma_2 + \gamma_1 V_1$$

$$\dot{V}_2 = -R_0 \omega_0^2 \gamma_2 + 2\omega_0^2 r_1 \gamma_1$$

$$\dot{\gamma}_2 = \frac{\omega_0}{R_0} r_2 + \frac{2}{R_0} V_2 - \frac{1}{R_0^2 \omega_0} V_1^2 - \frac{3}{R_0^2} r_1 V_1 - \frac{2\omega_0}{R_0^2} r_1^2 + \frac{1}{R_0 \omega_0} \beta_1$$

$$\dot{\lambda}_{12} = -\frac{\omega_0}{R_0} \lambda_{42} - 2\omega_0^2 \gamma_1 \lambda_{31}$$

$$\dot{\lambda}_{32} = \frac{2}{R_0} \lambda_{42} - \lambda_{11} \gamma_1$$

$$\dot{\lambda}_{42} = -R_0 \omega_0 \lambda_{12} + R_0 \omega_0^2 \lambda_{32} - V_1 \lambda_{11} - 2\omega_0^2 \lambda_{31} r_1$$

$$\beta_1 = \frac{\lambda_{42}}{\lambda_{31} V_0}$$

The multiplier equations are independent of the state equations as in the first order case, and may be solved easily when the first order solutions are known. The general solution is

$$\lambda_{12}(t) = \omega_0 A_2 - \frac{3}{R_0 a^2} t + \frac{A_1}{R_0} \cos(\omega_0 t + \beta)$$

$$\lambda_{32}(t) = A_2 + \frac{2A_1}{R_0 \omega_0} \cos(\omega_0 t + \beta)$$

$$\lambda_{42}(t) = -\frac{1}{\omega_0 a^2} + A_1 \sin(\omega_0 t + \beta)$$

The second order terminal conditions obtained from the expansion of the boundary conditions in powers of a are

$$\lambda_{12}(t_f) = \omega_0 \left(\lambda_{32} - \lambda_{31} \frac{r_1}{r_0} - \lambda_{31} \frac{V_1}{V_0} \right)_{t_f} = 0$$

$$\lambda_{32}(t_f) = 0$$

$$\lambda_{42}(t_f) = 0$$

which implies

$$A_1 \sin(\omega_0 t_f + \beta) = \frac{1}{\omega_0 a^2} > 0$$

$$A_2 + \frac{2A_1}{R_0\omega_0} \cos(\omega_0 t_f + \beta) = 0$$

$$A_1 \cos(\omega_0 t_f + \beta) = 0$$

The second order multiplier solution is then

$$\lambda_{12}(t) = -\frac{3}{R_0 a^2} t - \frac{1}{R_0\omega_0 a^2} \sin\omega_0 (t - t_f)$$

$$\lambda_{32}(t) = -\frac{2}{R_0\omega_0^2 a^2} \sin\omega_0 (t - t_f)$$

$$\lambda_{42}(t) = -\frac{1}{\omega_0 a^2} [1 - \cos\omega_0 (t - t_f)]$$

From the control angle expansion given in the Appendix, it follows that

$$\tan\phi = \sin\phi = -\frac{a}{R_0\omega_0^2} [1 - \cos\omega_0 (t - t_f)]$$

The optimal control angle is, therefore, oscillatory with frequency ω_0 and amplitude of order a.

The second order state equations may now be solved using the oscillatory control program. The complete solution with zero initial conditions on the second order state is

$$r_2(t) = \frac{3}{R_0\omega_0^2} t^2 - \frac{18 - \cos\omega_0 t_f}{2R_0\omega_0^4} (1 - \cos\omega_0 t)$$

$$+ \frac{1}{2R_0\omega_0^4} [\cos\omega_0 (t - t_f) + \omega_0 t \sin\omega_0 (t - t_f) - \cos\omega_0 t_f]$$

$$V_2(t) = \frac{18 - \cos\omega_0 t_f}{2R_0\omega_0^3} (1 - \cos\omega_0 t)$$

$$- \frac{1}{2R_0\omega_0^3} [\cos\omega_0 (t - t_f) + \omega_0 t \sin\omega_0 (t - t_f) - \cos\omega_0 t_f]$$

$$\gamma_2(t) = \frac{8}{R_0^2\omega_0^3}t + \frac{1}{2R_0^2\omega_0^3}t\cos\omega_0(t-t_f) - \left(\frac{18-\cos\omega_0 t_f}{2R_0^2\omega_0^4}\right)\sin\omega_0 t$$

In the above expressions it can be seen that "resonance" type terms of the form $t \sin t$ and $t \cos t$ have been introduced by the control input which oscillates at the natural frequency of the second order solution. It will be shown that it is precisely these terms which make the optimal "better" than the tangential escape trajectory.

We now have a complete second order expansion for the state variables and multipliers of the optimal control problem as defined in section II. The only remaining unknown is the final time which may be found by application of the terminal energy condition. Clearly, the solution as given will not hold to escape, i.e., $E_f = 0$, since the various terms in the expansion will become large and invalidate the assumed solution form. But if E_f is near E_0, then the solution should be accurate. Furthermore, the control angle for this energy increase problem should behave like that in the initial portion of the escape trajectory.

IV. Energy Increase Comparison

The rate of change of energy is

$$\dot{E} = aV\cos\phi$$

The small parameter expansion form of this equation is

$$\dot{E} = a[V_0\alpha_0 + a(V_0\alpha_1 + V_1\alpha_0) + a^2(V_2\alpha_0 + \alpha_2 V_0 + \alpha_1 V_1)]$$

since

$$V = V_0 + aV_1 + a^2V_2$$

$$\cos\phi = \alpha_0 + a\alpha_1 + a^2\alpha_2$$

From the solution in Section III and the angle expansion given in the Appendix, it follows that

$$\alpha_0 = 1$$

$$\alpha_1 = 0 \qquad \text{(optimal)}$$

$$\alpha_2 = -\frac{1}{2}\frac{1}{R_0^2\omega_0^4}[1-\cos\omega_0(t-t_f)]^2$$

For the tangential thrust program $\cos\phi \equiv 1$, which implies

$$\alpha_0 = 1$$
$$\alpha_1 = 0 \qquad \text{(tangential)}$$
$$\alpha_2 = 0$$

The rates of change of energy on the optimal and the tangential trajectories are then

$$\dot{E}|_{opt} = a[V_0 + aV_1 + a^2(V_2 + V_0\alpha_2)]_{opt}$$

$$\dot{E}|_{tan} = a[V_0 + aV_1 + a^2 V_2]_{tan}$$

Since there is no difference between the optimal and the tangential trajectories in the zero and first order solutions,

$$V_0|_{opt} = V_0|_{tan}$$

$$V_1|_{opt} = V_1|_{tan}$$

The second order expression for the velocity along the tangential trajectory can be determined easily. Since $\sin\phi \equiv 0$ along the tangential trajectory, then $\beta_1 \equiv 0$. After substituting $\beta_1 \equiv 0$ into the second order equations of motion above, the second order tangential state solution can be obtained. The second order tangential velocity is

$$V_2|_{tan} = \frac{8}{R_0\omega_0^3}(1 - \cos\omega_0 t)$$

Upon integration of the $\dot{E}$-equations, the energy changes along the two trajectories at any time are

$$\Delta E|_{opt} = a\Big\{R_0\omega_0 t - \tfrac{1}{2}at^2 + \frac{a^2}{4R_0\omega_0^4}[(33 + 2\cos\omega_0(t - t_f))\omega_0 t - 36\sin\omega_0 t$$
$$+ 2\cos\omega_0 t_f \sin\omega_0 t - \tfrac{1}{2}\sin 2\omega_0(t - t_f) - \tfrac{1}{2}\sin 2\omega_0 t_f]\Big\}$$

$$\Delta E|_{tan} = a\Big\{R_0\omega_0 t - \tfrac{1}{2}a^2t^2 + \frac{a^2}{4R_0\omega_0^4}[32\omega_0 t - 32\sin\omega_0 t]\Big\}$$

The energy difference between the two trajectories at any time is

$$\Delta(\Delta E) = \frac{a^3}{4R_0\omega_0^4}\left\{[1 + 2\cos\omega_0(t - t_f)]\omega_0 t - 4\sin\omega_0 t + 2\cos\omega_0 t_f \sin\omega_0 t - \tfrac{1}{2}\sin 2\omega_0(t - t_f) - \tfrac{1}{2}\sin 2\omega_0 t_f\right\}$$

At the final time

$$\Delta(\Delta E)_f = \frac{a^3}{4R_0\omega_0^4}\left[3\omega_0 t_f + \tfrac{1}{2}\sin 2\omega_0 t_f - 4\sin\omega_0 t_f\right]$$

It can be shown that $\Delta(\Delta E)_f$ is positive for all $\omega_0 t_f$. In fact for small $\omega_0 t_f$, the series expansion of the trigonometric expressions can be used to show that

$$\Delta(\Delta E)_f \cong \frac{a^3}{40R_0\omega_0^4}(\omega_0 t_f)^5$$

Therefore, at the time the optimal trajectory reaches the specified terminal energy, the energy level is higher than the energy level at the corresponding time on the tangential trajectory. This implies that the optimal trajectory will reach a specified terminal energy level faster than the tangential trajectory. It must be noted, however, that at intermediate times the energy on the tangential may be greater than on the optimal. In other words, the tangential trajectory may reach intermediate energy levels sooner than the optimal. This phenomenon should not be entirely unexpected since the minimum time trajectory to a given energy level is not the minimum time trajectory to all lower energy levels. It is only the minimum time trajectory from the initial state to all of the states occurring along the optimal.

It is well known that the tangential thrust program at each instant maximizes the rate of energy increase along a trajectory[15]. Therefore, it is reasonable to ask how the optimal manages to improve on the tangential. Since it has been shown that the optimal is tangential to first order, higher-order terms must produce the difference. In looking at the second order solution, we find that the difference between the tangential and optimal velocities may or may not be positive at any given time. However, the difference between their mean values taken over a revolution-to-go, from $t = t_f - (2N+2)\pi/\omega_0$ to $t = t_f - 2N\pi/\omega_0$, is greater than zero, i.e.,

$$\overline{V_2}\Big|_{opt} - \overline{V_2}\Big|_{tan} = \overline{\Delta V} = \frac{3a^2}{2R_0\omega_0^3} > 0.$$

We conclude then that on the average the velocity is higher along the optimal trajectory. If the tangential velocity is compared to the component of the optimal velocity in the optimal thrust direction , the mean value of the optimal velocity component is also found to exceed the mean value of the tangential velocity, i.e.,

$$\overline{V \cos \phi}\Big|_{opt} - \overline{V}\Big|_{tan} = \frac{3a^2}{4R_0\omega_0^3} > 0$$

Therefore, not only does the optimal velocity exceed the tangential velocity on the average, but also its component in the direction of thrust exceeds the tangential velocity. Recalling that the rate of energy increase depends only upon the thrust acceleration and the velocity component along the thrust vector, we see that the optimal improves on the tangential by maintaining a higher velocity component in the direction of thrust. The key to the higher velocity on the optimal is the existence of the "resonance" type terms in the optimal velocity expression which have been introduced by the control angle oscillations.

Considering the motion from a physical viewpoint, on the escape spiral the low thrust engine does work on the spacecraft causing its energy to increase. The vehicle spirals outward increasing its potential energy and decreasing its kinetic energy. The rate of energy increase depends highly upon the vehicle's velocity, i.e., its kinetic energy, and therefore decreases as the vehicle moves out. The tangential thrust program maximizes the rate of energy increase at each point along the trajectory but makes no direct effort to control the vehicle's velocity. The optimal thrust program, on the other hand, causes the energy to increase in such a way that the rate of increase of potential energy and rate of decrease of kinetic energy are reduced. The higher kinetic energy on the optimal then gives the vehicle more capability for increasing its energy as it moves out. The vehicle uses this additional capability in the latter portion of the trajectory to add more energy than it could on a tangential thrust trajectory, and in this way achieves escape in less time. The oscillation in the optimal control angle is a result of the trade-off between keeping the rate of energy increase high and the rate of kinetic energy decrease low.

V. Numerical Results

In order to test the accuracy of the approximate analytic solution, a comparison was made with an exact optimum energy increase trajectory. The exact solution was generated by numerically solving the two point boundary value problem using a secant iteration method. The initial values of the analytic multipliers were used as first guesses in the iteration scheme and seemed to work quite well. The initial values for the state variables were

$$r(0) = 6.67817 \times 10^6 \text{ meters}$$
$$V(0) = 7.72580 \times 10^6 \text{ meters/sec}$$
$$\gamma(0) = 2.19518 \times 10^{-2} \text{ radians}$$

These correspond to an initial orbit with the following eccentricity and energy

$$e_0 = 2.0 \times 10^{-3}$$
$$E_0 = -2.98440 \times 10^7 \text{ newton-meters/kg}$$

The specified terminal energy was

$$E_f = -2.86218 \times 10^7 \text{ newton-meters/kg}$$

which corresponds to $\omega_0 t_f = 6\pi$ in the analytic solution. Since $\omega_0 = 1.15687 \times 10^{-3}$, the analytic $t_f = 1.629353 \times 10^4$ seconds. The terminal time found in the numerical solution was $t_f = 1.629351 \times 10^4$ seconds. In Figures 2 and 3 a comparison between the analytical and numerical solutions is made. The state variables are in close agreement, as are the costate variables, with the primary differences appearing as a slight mean value offset in the velocity costate, and as a small period discrepancy in both the velocity and flight path angle costates. The optimal control angles are also close with only slight differences in period and amplitude.

A solution was also carried out using a terminal energy of

$$E_f = -2.58695 \times 10^7 \text{ newton-meters/kg}$$

In the analytic solution this energy level occurs at $\omega_0 t_f = 20\pi$, or $t_f = 5.43117 \times 10^4$ seconds. Numerical results give a final time, $t_f = 5.43116 \times 10^4$ seconds; and again the analytical and numerical state and costate variables differed only slightly. The determination of the full limitations of this approximation are currently under study.

VI. Conclusions

As a result of this analysis we reaffirm the well known fact that for low thrust spiral escape trajectories, tangential thrust is nearly time optimal, and in fact, is optimal to first order in the thrust-acceleration expansion solution. In addition, we now conclude that the observed oscillation in the optimal control angle is a second order resonance type phenomenon which reduces the velocity loss and therefore increases the rate of energy gain along the trajectory.

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Appendix

In the solution of the systems of equations it was found easier to consider separate expansions of the control angle functions rather than to use the multiplier expressions directly. The optimal control program is defined by

$$\tan\phi = \frac{\lambda_4}{\lambda_3 V}$$

By assuming expansions of the form

$$V = V_0 + aV_1 + a^2 V_2 + \cdots$$

$$\lambda_3 = a\lambda_{31} + a^2\lambda_{32} + a^3\lambda_{33} + \cdots$$

$$\lambda_4 = a\lambda_{41} + a^2\lambda_{42} + a^3\lambda_{43} + \cdots$$

we obtain

$$\tan\phi = \eta_0 + a\eta \ + a^2\eta_2 + \cdots$$

where

$$\eta_0 = \frac{\lambda_{41}}{\lambda_{31} V_0}$$

$$\eta_1 = \frac{1}{\lambda_{31} V_0}\left[\lambda_{42} - \lambda_{41}\left(\frac{V_1}{V_0} + \frac{\lambda_{32}}{\lambda_{31}}\right)\right]$$

$$\eta_2 = \frac{1}{\lambda_{31} V_0}\left[\lambda_{43} - \lambda_{42}\left(\frac{V_1}{V_0} + \frac{\lambda_{32}}{\lambda_{31}}\right) + \lambda_{41}\left(\frac{\lambda_{32} V_1}{\lambda_{31} V_0} + \frac{\lambda_{32}}{\lambda_{31}^2} + \frac{V_1^2}{V_0^2} - \frac{\lambda_{33}}{\lambda_{31}} - \frac{V_2}{V_0}\right)\right]$$

Next consider

$$\sin\phi = \frac{\lambda_4}{\sqrt{\lambda_4^2 + \lambda_3^2 V^2}} \quad , \quad \cos\phi = \frac{\lambda_3 V}{\sqrt{\lambda_4^2 + a_3^2 V^2}} \quad ,$$

which can be written as

$$\cos\phi = [1 + \tan^2\phi]^{-\frac{1}{2}}$$

$$\sin\phi = \tan\phi \cos\phi$$

Substituting in the expression for $\tan\phi$ and expanding in a Taylor's series about $a = 0$:

$$\cos\phi = \alpha_0 + a\alpha_1 + a^2\alpha_2 + \cdots$$

where

$$\alpha_0 = (1 + \eta_0^2)^{-\frac{1}{2}}$$

$$\alpha_1 = -\eta_0\eta_1 (1 + \eta_0^2)^{-\frac{3}{2}}$$

$$\alpha_2 = -\tfrac{1}{2}[3\eta_0^2\eta_1^2 (1 + \eta_0^2)^{-\frac{5}{2}} - (\eta_1^2 + 2\eta_0\eta_2)(1 + \eta_0^2)^{-\frac{3}{2}}] .$$

Using the expansions for tan ϕ and cos ϕ we obtain

$$\sin\phi = \beta_0 + a\beta_1 + a^2\beta_2 + \cdots$$

where

$$\beta_0 = \eta_0(1 + \eta_0^2)^{-\frac{1}{2}}$$

$$\beta_1 = \eta_1 (1 + \eta_0^2)^{-\frac{1}{2}} - \eta_1\eta_0(1 + \eta_0^2)^{-\frac{3}{2}}$$

$$\beta_2 = \eta_2 (1 + \eta_0^2)^{-\frac{1}{2}} - \tfrac{1}{2}(-3\eta_0\eta_1^2 + 2\eta_0^2\eta_2)(1 + \eta_0^2)^{-\frac{3}{2}} + \tfrac{3}{2}\eta_0^3\eta_1^2 (1 + \eta_0^2)^{-\frac{5}{2}}$$

Figure 2. Radial Distance, Total Velocity, and Their Corresponding Costates

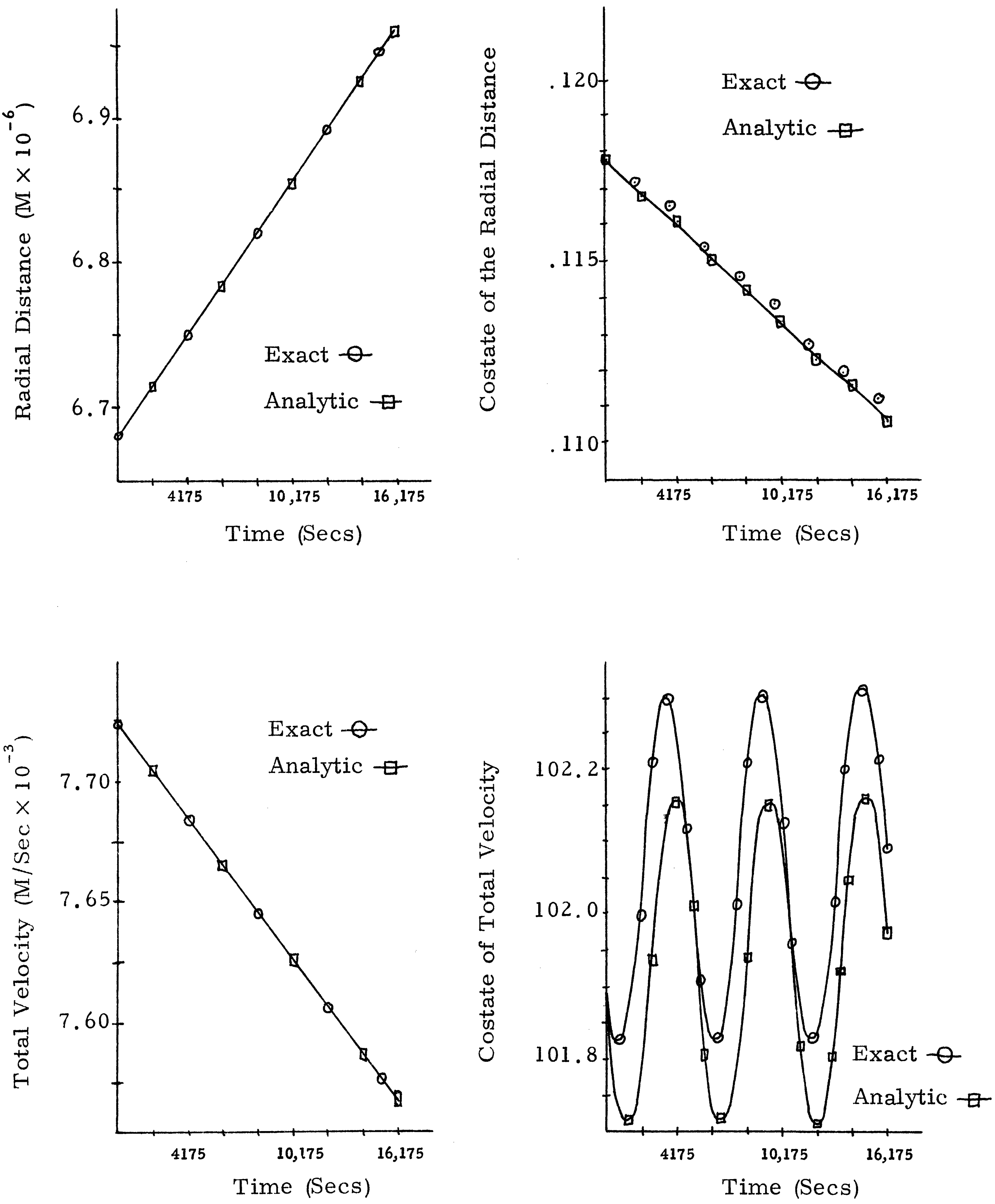

Figure 3. Flight Path Angle, its Costate, and the Control Angle