

An Analog Eigenvalue Technique

by

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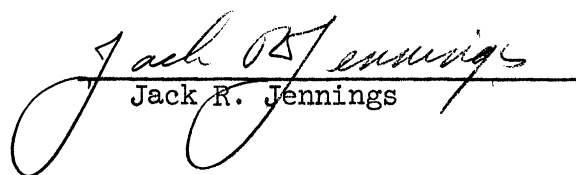

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INTRODUCTION

The problem dealt with in this thesis is that of finding the characteristic roots or eigenvalues and corresponding eigenvectors of real matrices using the electronic differential analyzer.

A reasonably complete list of uses of a knowledge of the eigenvalues of an algebraic matrix would be very large. Such eigenvalues may be interpreted as the natural frequencies of mechanical or electrical systems, as buckling load for elastic structures, as approximate natural frequencies of continuous systems represented by difference equations, etc. Furthermore, for each eigenvalue so interpreted, there corresponds an eigenvector representing a particular shape or configuration of the physical system involved.

A technique was developed which may be used for the reduction of any real matrix with real roots to the diagonal form of its latent roots. The technique differs from existing methods of solving this type of problem using analog computers in the following respects.

1. It is systematic. One eigenvector and the corresponding eigenvalue are obtained at a time and it is known beforehand which eigenvalue (in order of size) is obtained.
2. It uses comparatively little equipment. Previous sorts of analog matrix reducers were all essentially modifications of simultaneous equation solvers with expensive, hand-manipulated equipment added to make them applicable to the matrix eigenvalue problem. The added equipment usually included at least one piece of non-linear apparatus for each eigenvector component plus a set of ganged linear elements to be set to the eigenvalues. The technique to be described in this thesis works without ganged eigenvalue potentiometers.
3. It is completely automatic. No action by the operator is required except for setting the coefficient potentiometers to the values of the matrix elements. Scaling of these elements is done without making trial runs on the computers.
4. It has a greater range of application than previous analog matrix reducer

apparatus. Heretofore, such analog computers have been restricted to dealing with small real symmetric matrices. The range of application of the new technique is substantially broader than this. The extent of this range, the reasons for the limitation of the range, and the type of failure produced when the attempt is made to exceed this range are all derived analytically in the text.

5. It may be adapted (with considerable equipment) to obtaining the complete diagonal matrix and the complete modal matrix in a single computer run.

6. Its iterative numerical analogue might profitably be employed as a digital computer process.

In spite of the above list of virtues, it seems necessary to justify the effect that has been directed to solving on the electronic differential analyzer a problem which seems properly a part of the digital computer's domain. A careful reading of the literature on the subject will indicate, however, that the algebraic eigenvalue problem is hardly a completely owned part of any computer's domain. Programs for large digital computers in fact usually produce results whose accuracy and utility decrease rapidly with increasing matrix size.⁽¹⁾ While the present investigation makes no claims to having cured these difficulties, it does, at least, have the comparative virtues of easy setup and fast determination of results. Its chief application may, in fact, lie in finding good starting values for iterative digital computer improvement. It is also expected that it will be useful in obtaining partial reduction of moderately large matrices for selected eigenvectors and eigenvalues.

The development to be followed in the thesis is mainly theoretical. There are several reasons for this. First of all, the range of application of the technique and its inherent limitations could be well defined only by a theoretical study. Secondly, a relatively few computer runs can be used to illustrate that the computer will solve the problem where the theory says that it should and will show the type of failure produced in situations where the process should theoretically fail. Third, the theoretical development gives a good insight into the analagous

(1) see for example, Wilkinson, J.H., "Calculation of Eigenvectors by Method of Lanczos," Computer Journal, I, 3, Oct. 1958, pp. 148-152.

numerical techniques. Lastly the theoretical development produced a few mathematical results which if not entirely new were at least unknown to me at the start of this project.

The notation employed will be essentially that used in Guillemin.⁽¹⁾ A square matrix array of n^2 elements will be represented as

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3n} \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix} \quad (1)$$

Where individual elements are indicated, the first subscript will indicate the row in which the element appears, and the second subscript will indicate the column in which the element is located. A column or row matrix will be indicated as

$$\begin{bmatrix} x \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad (2)$$

or

$$\underline{x} = \underline{x_1 \ x_2 \ \cdot \ \cdot \ \cdot \ x_n} \quad (3)$$

respectively. Such column or row matrices may be considered as n-dimensional

(1) Guillemin, E.A., The Mathematics of Circuit Analysis, John Wiley and Sons, Inc. New York, 1949

vectors having components along mutually orthogonal axes equal to the element values, with the subscript indicating the number of the axis along which the component is measured.

With such a vector interpretation of column or row matrices, the square matrix is interpreted as a linear vector operator which may be used to transform one vector into another as

$$[A] x = y \quad (4)$$

by the usual law for matrix multiplication; namely,

$$y_i = \sum_{j=1}^n a_{ij} x_j \quad (5)$$

If the $[A]$ matrix be supposed composed of a set of n row vectors, then each component of the y vector in equation (4) is equal to the scalar product of the original x vector and one of the n vectors α_i of the set

$$[A] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} \quad (6)$$

Solving equation (4) for x ,

$$x = [A]^{-1} y \quad (7)$$

where $[A]$ inverse,

$$[A]^{-1} = \begin{bmatrix} A_{11} & A_{21} & A_{31} & \cdot & \cdot & \cdot & A_{n1} \\ A_{12} & A_{22} & A_{32} & \cdot & \cdot & \cdot & A_{n2} \\ A_{13} & A_{23} & A_{33} & \cdot & \cdot & \cdot & A_{n3} \\ A_{1n} & A_{2n} & A_{3n} & \cdot & \cdot & \cdot & A_{nn} \end{bmatrix} \frac{1}{A} \quad (8)$$

where A_{ij} is the cofactor of the element a_{ij} of $[A]$; i.e., A_{ij} is $(-1)^{i+j}$ times the determinant of the array left after crossing out the i th row and j th columns of $[A]$, and where A is the determinant of the complete array $[A]$. The matrix

product

$$[A] [A]^{-1} = [A]^{-1} [A] = [I] \quad (9)$$

where the elements δ_{ij} of the identity matrix $[I]$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad (10)$$

and where the elements c_{ij} of a product of matrices with elements a_{ij} and b_{ij} are given as

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} \quad (11)$$

if

$$[C] = [A] [B] \quad (12)$$

and where

$$c_{ij} = \sum_{k=1}^n b_{ik} a_{kj} \quad (13)$$

if

$$[C] = [B] [A] \quad (14)$$

since matrix multiplication is in general not commutative. Note that $[A]^{-1}$ is defined only if A , the determinant of $[A]$ is not zero. Where the inverse does not exist, the matrix is said to be singular.

The matrix obtained by interchanging corresponding rows and columns of $[A]$ is designated "A transpose" and written

$$[A]_t$$

with elements

$$a_{ij} = a_{ji} \quad (15)$$

If

$$[A] x = y \quad (16)$$

and

$$[x \ x] = [y \ y], \quad (17)$$

then,

$$\underline{x} \cdot x = \underline{x} [A]_t [A] x. \quad (18)$$

It follows that, in this case,

$$[A]_t [A] = [I] = [A]^{-1} [A] \quad (19)$$

and that therefore

$$[A]_t = [A]^{-1} \quad (20)$$

Equation (17) may be read, "The lengths of the original vector x and the transformed vector y are the same." The transformation thus amounts to only a rotation (with perhaps reflection) of the vector x . Such a transformation is called an orthogonal transformation and a matrix producing such a transformation must have the property (20). This property can be written in terms of the elements of the orthogonal matrix $[A]$ as

$$a_{ij} = \frac{A_{ij}}{A} \quad (21)$$

Since the rule for obtaining products of determinants is identical with that for determining products of matrices, (20) requires that the determinant of the transformation matrix,

$$A = \frac{1}{A}$$

and, therefore,

$$A = \pm 1. \quad (22)$$

The matrix $[A]$ may be transformed by multiplication with other square matrices of the same order (number of rows). A transformation of particular interest is that which corresponds to identical transformations of coordinates for a vector and for its transform under the matrix $[A]$. If

$$[A] x = y \quad (23)$$

where

$$x = [c] \xi \quad (24)$$

and

$$y = [c] \eta, \quad (25)$$

then,

$$[A] [c] \xi = [c] \eta \quad (26)$$

and

$$[C]^{-1} [A] [C] \xi = [B] \xi = \eta. \quad (27)$$

The transformation

$$[B] = [C]^{-1} [A] [C] \quad (28)$$

is called a similarity transformation of the matrix $[A]$, and the matrices $[A]$ and $[B]$ are said to be similar.

One of the central problems in what follows will be to find means of obtaining a modal matrix $[L]$ associated with a matrix $[A]$ such that the similarity transformation

$$[L]^{-1} [A] [L] = [\Lambda] \quad (29)$$

where $[\Lambda]$ is a diagonal matrix with elements

$$\lambda_{ij} = 0, \quad i \neq j. \quad (30)$$

The elements λ_{ii} are called the latent roots or eigenvalues of the matrix $[A]$. The columns of the modal matrix are called the principal directions or eigenvectors of the matrix $[A]$ and the corresponding coordinate transformation

$$x = [L] \xi \quad (31)$$

is called a principal axis transformation. The eigenvectors have the property

$$[A] x_i = \lambda_{ii} x_i \quad (32)$$

where x_i is the eigenvector corresponding to eigenvalue λ_{ii} . Equation (32) indicates that an eigenvector and its transform have the same direction but, in general, different lengths. The ratio of the lengths of the eigenvector and its transform is the eigenvalue associated with that particular eigenvector. The process of searching for the eigenvectors of $[A]$ may be pictured geometrically as moving an x vector about while observing the y vector which is the transform of x under the process characterized by the matrix $[A]$. There may be positions of the x vector in which it is aligned with the y vector. These are the real eigenvector positions, and these are the items sought by the new process to be described. Note that since the equation set 33 is homogeneous, only the "directions" of the eigenvectors may be found. Any one component of any eigenvector may be arbitrarily assigned. It is precisely this aspect of the eigenvector

determination which has caused the most trouble with existing techniques and apparatus. The new technique offers some relief from this difficulty.

Equation (32) may be rewritten

$$\left(\begin{bmatrix} [A] & -\lambda[I] \end{bmatrix} x \right)_i = 0, \quad (33)$$

which can have non-zero solutions x_i only if the determinant

$$\left| \begin{bmatrix} [A] & -\lambda[I] \end{bmatrix} \right| = 0. \quad (34)$$

Equation (34) is called the characteristic equation for the matrix $[A]$. Its left-hand member, which is an n th degree polynomial in λ is called the characteristic function for the matrix $[A]$. The roots of this polynomial equation are the eigenvalues for $[A]$.

This completes the list of essential operations and the notations to be used for them in what follows. Other results from elementary matrix theory will be mentioned when needed.

While it is true that the geometric interpretation of the matrix eigenvalue problem adopted in this thesis is not the only interpretation possible, it is also true that the point of view adopted has no peculiar shortcomings and is as simple as any possible. Furthermore, while geometric language is employed freely for spaces of dimension 4 or more for which there is no "real" geometry, the specialization of the results to ordinary 3-space usually offers a clear understanding of the mathematical operations involved in recompense for the sins committed in n -space.

EXISTING COMPUTER TECHNIQUES FOR MATRIX
DIAGONALIZATION

Mechanically assisted means for the reduction of an arbitrary matrix to the diagonal equivalent of its latent roots may be conveniently classified according to the type of computer, whether digital or analog, intended to be used to effect the reduction. This classification is essentially complete and mutually exclusive. The two different machine methods have very little in common.

Digital Techniques

Several numerical techniques for the similarity transformation of a matrix to its diagonal equivalent have been in use for a comparatively long time. The advent of the high-speed electronic digital computer has spurred improvements in these processes and the development of new processes.

Since no direct method exists for the reduction of a matrix of order higher than four to the diagonal equivalent of its latent roots, all of the numerical techniques and digital programs for effecting such a reduction are iterative and relaxation routines. Such a routine is initiated by "guessing" more or less arbitrarily, at a solution of the problem. The calculation routine then develops a second guess which, hopefully, is closer to the solution than the first. The second guess is used to compute a third, and so forth. The process is terminated either after a fixed number of repetitions of this process or else when a calculated guess differs by not more than a fixed amount from the previous one. If the process leads to a unique solution, it is said to converge. Convergence is thus a central problem of numerical matrix reduction routines. Improving the rate of convergence or rate of error reduction, is the principal aim of the newer routines.

Willers¹ gives sufficient conditions for convergence of elementary indirect numerical routines. Householder² has presented extensive analyses of the convergence problem in certain matrix routines. The analogue computer problem analogous to the convergence problem in digital routines is the problem of stability. Determination of stability conditions for an analogue computer setup seems ordinarily more easy than the determination of conditions for convergence of a numerical routine.

The initial rate of convergence of the numerical routines - that is the convergence reflected by the first few steps - is a function of the elements of the matrix. Since the techniques are all equivalent to similarity transformations of the matrix intended to reduce the off-diagonal elements to zero and the diagonal elements to eigenvalues, initial convergence rate is materially increased if the off-diagonal elements are small compared to the diagonal elements. A preliminary process for reducing certain large off-diagonal elements to zero is therefore often very helpful. The Jacobi transformation³ is such a process.

Jacobi Transformation

The Jacobi transformation is a similarity transformation which may be used to reduce any pair of symmetrically located off-diagonal elements of a matrix to zero provided the elements of the matrix satisfy certain requirements. If the selected off-diagonal elements to be reduced to zero be designated a_{ij} and a_{ji} , then the requirements are that a_{ii} , a_{ij} , a_{ji} , a_{jj} be all real and that

$$(a_{ii} - a_{jj})^2 + 4a_{ij}a_{ji}$$

be non-negative. These requirements are obviously always met if the matrix is real

¹ Willers, F.A., Practical Analysis, Dover Publications, Inc. N.Y. 1948

² Householder, A.S., "Approximate Solution of Matrix Problem", Assn Computing Machj-J, Vol 5, N3, July 1958, pp. 205-243

³ Willers, F.A., op. cit.

and symmetric. These requirements are in fact identical with the requirement that the eigenvalues of the matrix

$$\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

be real.

The Jacobi transformation is formally effected by defining quantities ρ , α , and Δ as

$$\begin{aligned} \rho \cos 2\alpha &= a_{ii} - a_{jj} \\ \rho \sin 2\alpha &= a_{ij} + a_{ji} \\ \rho \sin 2\Delta &= a_{ji} - a_{ij} \end{aligned} \quad (35)$$

and new coordinates ξ_i and ξ_j as

$$\begin{aligned} (\cos 2\Delta) x_i &= \cos(\alpha + \Delta) \xi_i + \sin(\alpha - \Delta) \xi_j \\ (\cos 2\Delta) x_j &= \sin(\alpha + \Delta) \xi_i - \cos(\alpha - \Delta) \xi_j \end{aligned} \quad (36)$$

All elements in row i and j and columns i and j are replaced by elements α_{mn} according to the formulae

$$\begin{aligned} \alpha_{ik} &= a_{ik} \cos(\alpha - \Delta) + a_{jk} \sin(\alpha - \Delta), \quad k = 1, 2, \dots, n \\ \alpha_{jk} &= a_{ik} \sin(\alpha - \Delta) - a_{jk} \cos(\alpha - \Delta), \quad k = 1, 2, \dots, n \\ \alpha_{ki} &= \cos 2\Delta = a_{ki} \cos(\alpha + \Delta) + a_{kj} \sin(\alpha + \Delta), \quad k = 1, 2, \dots, n \\ \alpha_{kj} &= \cos 2\Delta = a_{ki} \sin(\alpha - \Delta) - a_{kj} \cos(\alpha - \Delta), \quad k = 1, 2, \dots, n \end{aligned} \quad (37)$$

This makes $\alpha_{ij} = \alpha_{ji} = 0$.

The previously noted restrictions on the applicability of the process amount to guarantees that $\sin^2 \Delta$ be not greater than one.

The formal process as presented above becomes considerably more comprehensible if analyzed as a vector operation. The entire procedure amounts to transforming the principal minor matrix

$$\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

to the diagonal form of its latent roots by a similarity transformation of the entire original matrix. This is done by substituting for original coordinates x_i and x_j , the principal axis coordinates ξ_i and ξ_j in the non-principal subspace spanned by x_i and x_j . In this subspace, 2Δ is the angle between the ξ vectors, and α is the angle between their bisector and the x_i axis.

The Jacobi transformation is equivalent to the similarity transformation

$$[A] = [C]^{-1} [A] [C] \quad (38)$$

where

$$[C]$$

is the identity matrix except for the elements

$$\begin{aligned} c_{ii} &= \cos(\alpha + \Delta) \\ c_{ij} &= \sin(\alpha - \Delta) \\ c_{ji} &= \sin(\alpha + \Delta) \\ c_{jj} &= -\cos(\alpha - \Delta) \end{aligned} \quad (39)$$

$[C]^{-1}$ is thus also an identity matrix except for elements

$$\begin{aligned} c'_{ii} &= \frac{\cos(\alpha - \Delta)}{\cos 2\Delta} \\ c'_{ij} &= \frac{\sin(\alpha - \Delta)}{\cos 2\Delta} \\ c'_{ji} &= \frac{\sin(\alpha + \Delta)}{\cos 2\Delta} \\ c'_{jj} &= \frac{\cos(\alpha + \Delta)}{\cos 2\Delta} \end{aligned} \quad (40)$$

$$\frac{1}{\cos 2\Delta} [C]$$

is the modal matrix for

$$\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

The Jacobi transformation is not only a useful first step in preparing matrices for other numerical diagonalization routines, but also, it may be used repeatedly as an iterative technique to complete the diagonalization. When so used, the a_{ij} , a_{ji}

elements selected for elimination at each step are usually the biggest magnitude off-diagonal element and the similarly located transpose element. Since every element in the i^{th} and j^{th} rows is affected by the step which reduces a_{ij} and a_{ji} to zero, the diagonalization is in general not complete after a finite number of steps.

When the Jacobi transformation is employed as an iterative diagonalization routine, the modal matrix L for the original matrix

$$[A]$$

is generated as the product of the transformation matrices

$$[C]_i$$

i.e.,

$$[L] = [C]_1 [C]_2 \cdots [C]_n \cdots \quad (41)$$

The iterative Jacobi procedure may be pictured geometrically by considering the axis position change in the two-dimensional i - j subspace for the case of the symmetric matrix

$$[A].$$

For such a matrix, the eigenvectors are the principal axes of the surface

$$F = \underline{x} [A] x \quad (42)$$

with

$$\underline{x} x = 1.0 \quad (1) \quad (43)$$

This is seen to be the case by writing the equations for the extremes of F

$$\frac{\partial}{\partial x_i} (\underline{x} [A] x - \lambda (\underline{x} x)) = 0 \quad (2) \quad (44)$$

which are seen to be the characteristic equations for the matrix $[A]$. The surface F is a hyper-ellipsoid which intersects the i - j subspace in an ellipse. The ξ_i and ξ_j axes found by the Jacobi transformation are the principal axes of this ellipse. These may be reasonably expected to lie closer to principal subspaces of

¹ Guillemin, E.A., The Mathematics of Circuit Analysis, John Wiley and Sons, Inc., New York 1949

² Fox, Calculus of Variations, Oxford University Press

the matrix $[A]$ - that is, closer to principal planes of the hyper-ellipsoid,

$$F = \begin{bmatrix} \underline{x} & [A] & x \\ \underline{x} & x \end{bmatrix} = 1.0 - \quad (45)$$

than do the original x_i and x_j axes, although it may be appreciated that the conditions under which this is certain to be true are not trivial.

Method of Lanczos: ⁽¹⁾

A direct technique developed by Lanczos may be used to obtain a matrix similar to any real matrix

$$[A] = [a_{ij}] \quad (46)$$

with all elements of the transformed matrix zero whose row and column designations differ by more than one. Such a matrix has non-zero elements only along the major diagonal and adjacent to the major diagonal. The matrix is called "codiagonal" by Lanczos.

The matrix $[C]$ which effects the transformation of

$$[A]$$

to the codiagonal form is found by arbitrarily selecting a vector

$$c]_1$$

which will serve as the first column of

$$[C]$$

Successive column vectors are then found as

$$c]_2 = [A] c]_1 - \frac{c]_1 [A] c]_1}{c]_1 c]_1} c]_1 \quad c]_1 = [A] c]_1 - \alpha_1 c]_1$$

$$c]_3 = [A] c]_2 - \frac{c]_2 [A] c]_2}{c]_2 c]_2} c]_2 - \frac{c]_1 [A] c]_2}{c]_1 c]_1} c]_1$$

¹ Lanczos, C., "An Iteration Method for the Solution of the Eigenvalue Problem of Linear Differential and Integral Operators", Jorn. of Research of the National Bureau of Standards, V 45 pp. 255

$$\begin{aligned}
 c]_3 &= [A] c]_2 - \alpha_2 c]_2 - \beta_2 c]_1 \\
 c]_{i+1} &= [A] c]_i - \frac{c]_i [A] c]_i}{c]_i c]_i} c]_i - \frac{c]_{i-1} [A] c]_i}{c]_{i-1} c]_{i-1}} c]_{i-1} \\
 &= [A] c]_i - \alpha_1 c]_i - \beta_{i-1} c]_{i-1}
 \end{aligned} \tag{47}$$

The preceding equations may be rewritten

$$[A] c]_i = c]_{i+1} + \alpha_i c]_i + \beta_i c]_{i-1} \tag{48}$$

so that

$$[A] \begin{bmatrix} c]_1 & c]_2 & \dots & c]_n \end{bmatrix} = \begin{bmatrix} c]_1 & c]_2 & \dots & c]_n \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_2 & 0 & \dots & 0 & 0 \\ 1 & \alpha_2 & \beta_3 & \dots & 0 & 0 \\ 0 & 1 & \alpha_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{n-1} & \beta_n \\ 0 & 0 & 0 & \dots & 1 & \alpha_n \end{bmatrix} \tag{49}$$

which may be written

$$[A] [C] = [C] [\alpha, \beta] \tag{50}$$

or

$$[C]^{-1} [A] [C] = [\alpha, \beta] \tag{51}$$

which shows that the codiagonal matrix

$$[\alpha, \beta]$$

is similar to

$$[A]$$

Relaxation Methods^{1,2} The method of Lanczos and the diagonalization technique using repeated Jacobi transformation are ordinarily designated as iterative techniques. The technique developed by Givens³ also belongs in this group. Iterative techniques are characterized by a finite number of required operations at each step of the process albeit there may be required indefinitely many steps. The name "relaxation

¹ Allen D. N. dt G., Relaxation Methods, McGraw Hill Brook Co., Inc. N.Y. 1954

² Southwell, R. V., Relaxation Methods in Engineering Science, Oxford Univ. Press, N. Y. 1946

³ Givens, W., "Numerical Computation of the Characteristic Roots and Vectors of a Real Symmetric Matrix", Oak Ridge National Laboratory Report No. 1574

method" is ordinarily reserved for those procedures which require not only an indefinite number of steps but which also require an indefinite number of operations per step. Two criteria for terminating computation are thus necessary, one to decide at what point a step is sufficiently well performed, a second to decide when no more steps are necessary and the entire process is complete.

The name "relaxation method" is an abbreviated form of the older designation "the method of systematic relaxation of constraints". The method was first applied to the engineering problem of calculating the deflection of a structure under load. If the structure were, for instance, a bridge truss, the truss members would be assumed constrained solidly between pairs of horizontal and vertical rails. A pair of rails at some selected joint would then be assumed removed and the deflection of that joint calculated. That joint would then be clamped again at its calculated deflected position and the process repeated on some other joint. Since unclamping any joint produced stress changes in the members connected to neighboring joints, the deflection at any particular joint would in general have to be calculated over and over. Eventually however, every joint clamp could be removed with negligible resulting movement. A decision as to what constituted negligible movement was necessary to decide when to terminate the process.

Processes similar to that described above have been developed for solving many sorts of problems in algebraic and differential equations. An example of use of the technique to solve the algebraic matrix eigenvalue problem is the Rayleigh¹ method.

The Rayleigh method, as presented by Allen, is applicable only to symmetric, positive-definite matrices. The most general problem soluble by the method is the diagonalization of one symmetric positive definite matrix with respect to another.

¹ Allen, D. N. de G.; Op. Cit.

That is, the method may be used to find positive values, λ and non-zero vectors

$x]$

for which

$$\left([A] - \lambda [B] \right) x] = 0 \quad (52)$$

Use is made of the fact that the "Rayleigh quotient",

$$q = \frac{\underline{x} \begin{bmatrix} A & x \\ B & x \end{bmatrix}}{\underline{x} \begin{bmatrix} B & x \end{bmatrix}} \quad (53)$$

is stationary at a solution

$$q = \lambda. \quad (54)$$

This is seen to be so by setting the partial derivative of the Rayleigh quotient with respect to each component of

$x]$

equal to zero to obtain the equations

$$\left([A] - q [B] \right) x] = 0 \quad (55)$$

which are seen to be identical with the eigenvalue equations. This identification requires that

$$\begin{aligned} [A] &= [A]_t \\ [B] &= [B]_t \end{aligned} \quad (56)$$

and

$$[B] x] \neq 0 \quad (57)$$

for

$$x] \neq 0 \quad (58)$$

The mechanics of the Rayleigh method consists of four steps, the last two of which are repeated over and over until no appreciable change occurs at some step. The four steps are the following:

- (1) Guess a value for $x]$. If no better approach exists, this may be done by making

$$x]_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad (59)$$

(2) Calculate a value q_1 of the Rayleigh quotient

$$q = \frac{\underline{x}]_1 [A] x]_1}{\underline{x}]_1 [B] x]_1} \quad (60)$$

For the choice of

$$x]_1$$

in step (1) this would give

$$q_1 = \frac{a_{11}}{b_{11}} \quad (61)$$

(3) Define "residuals"

$$r.] = \left([A] - q_1 [B] \right) \left(x]_1 + \Delta x \right) \quad (62)$$

and attempt to make

$$r.] = 0 \quad (63)$$

by proper choice of

$$\Delta x].$$

This is done by steps usually by making the biggest component of

$$r.]$$

equal to zero by a change in a component of

$$\Delta x].$$

Again, the component of

$$\Delta x].$$

selected is ordinarily the one having the biggest coefficient in the equation for the selected component of

$$r.]$$

This process is essentially only a means of approaching a solution

$$x]$$

of the equation set

$$\left([A] - q_1 [B] \right) x] = 0 \quad (64)$$

However, in general, the only solution of this equation set is

$$x] = 0. \quad (65)$$

Therefore, when the solution is seen tending toward the null solution - and this requires some judgement - the process is halted, the vector

$$x]_1 + \Delta x]$$

is "normalized", usually by making its biggest magnitude component a certain size, and the next step in the process is performed.

(4) Use the normalized vector

$$k \left(x]_1 + \Delta x] \right) = x]_2 \quad (66)$$

to calculate a new Rayleigh quotient

$$q_2 = \frac{\frac{x]_2}{\|x]_2\|} [A] \frac{x]_2}{\|x]_2\|}}{\frac{x]_2}{\|x]_2\|} [B] \frac{x]_2}{\|x]_2\|}} \quad (67)$$

Steps (3) and (4) are then repeated over and over until a solution is obtained.

$$q_k = \lambda_i \quad (68)$$

$$x]_k = x]_i \quad (69)$$

In addition to the restrictions on the Rayleigh method noted above, it is not known whether the calculated value of

$$\lambda_i$$

is the biggest, smallest or some intermediate member of the set

$$\left(\lambda_1, \lambda_2, \dots, \lambda_n \right).$$

An alternative technique which eliminates this last difficulty is available in the "intensification method"¹. The intensification method also has the comparative virtue that each step of the process may be carried to mathematical completion - the vectors constituting intermediate step solutions exist and in general are not zero. Inasmuch as this method is closely related to the analog technique to be exposed in the sequel, this method will be thoroughly described.

The intensification procedure may be used to find λ and

$$x]$$

in the equation set

$$\left([A] - \lambda [B] \right) x] = 0, \quad (70)$$

where

$$\begin{aligned} [A] &= [A]_t \\ [B] &= [B]_t \end{aligned} \quad (71)$$

¹ Allen, D. N. de G., op. cit

and

$$\lambda > 0 \quad (72)$$

The first step of the procedure is, as in the Rayleigh method, to guess a vector

$$x] = x] \quad (73)$$

A vector closer to the vector corresponding to the smallest eigenvalue λ is found by solving for

$$k x]''$$

from the equation set

$$[A] k x]'' = [B] x] \quad (74)$$

The vector

$$k x]''$$

is then "normalized" in some more or less arbitrary fashion by selecting the constant, "k".. A new vector

$$x]''''$$

is found from

$$[A] k^1 x]'''' = [B] x]'' \quad (75)$$

etc. The process is continued until a vector

$$x]^{(n+1)} = K x]^{(n)} \quad (76)$$

is obtained. Then

$$x]^{(n)} = x]_1 \quad (77)$$

where

$$\left([A] - \lambda [B] \right) x]_1 = 0 \quad (78)$$

and where

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_n .$$

This is to say, the intensification procedure finds the smallest positive eigenvalue, λ_1 , and the eigenvector

$$x]_1$$

corresponding to that eigenvalue. This may be shown as follows:

If all the λ 's of the set $(\lambda_1, \lambda_2, \dots, \lambda_n)$ are positive real, then

any arbitrary vector, $x]'$, may be expressed as a linear sum of the eigenvectors

$$x]' = a_1 x]_1 + a_2 x]_2 + \dots + a_n x]_n \quad (79)$$

where

$$\begin{aligned} [A] x]_1 &= \lambda_1 [B] x]_1 \\ [A] x]_2 &= \lambda_2 [B] x]_2 \\ &\vdots \\ [A] x]_n &= \lambda_n [B] x]_n \end{aligned} \quad (80)$$

If, then we compute

$$\begin{aligned} [A] x]'' &= [B] x]' \\ &= a_1 [B] x]_1 + a_2 [B] x]_2 + \dots + a_n [B] x]_n \end{aligned} \quad (81)$$

then

$$x]'' = \frac{a_1}{\lambda_1} x]_1 + \frac{a_2}{\lambda_2} x]_2 + \dots + \frac{a_n}{\lambda_n} x]_n \quad (82)$$

The ratio of the component

$$x]_1$$

in

$$x]''$$

to the other components

$$x]_i$$

is bigger than the corresponding ratio of components in

$$x]'$$

by the factor $\frac{\lambda_i}{\lambda_1}$.

By a sufficiently great number of repetitions of the intensification procedure,

this ratio may be made as large as desired and thus

$$\lim_{n \rightarrow \infty} x]^{(n)} = K x]_1 \quad (83)$$

The preceding statement requires that all the values

$$\lambda_1, \lambda_2, \dots, \lambda_n$$

be positive real although it does not really require that

$$[A] = [A]_t \quad (84)$$

nor

$$[B] = [B]_t. \quad (85)$$

The method is thus applicable to finding eigenvectors and eigenvalues of a real matrix

$$[A]$$

with respect to a real matrix

$$[B]$$

providing only that all such eigenvalues are positive real.

The intensification procedure finds the eigenvector corresponding to the smallest eigenvalues, λ_1 . The next smallest eigenvalue, λ_2 , and its corresponding eigenvector may be found by applying the method to an initial vector.

$$x]^{1,2} = x]' - \frac{x]_1 x]'}{x]_1 x]_1} x]_1 = a_2 x]_2 + a_3 x]_3 + \dots + a_n x]_n. \quad (86)$$

The vector

$$x]^{1,2}$$

has no

$$x]_1$$

eigenvector component. The convergence is thus to the member of the set

$$x]_2, x]_3, \dots, x]_n$$

corresponding to the smallest eigenvalue of the set

$$\lambda_2, \lambda_3, \dots, \lambda_n.$$

i.e., to

$$x]_2.$$

In the successive computational steps, errors usually produce

$$x]_1$$

components which have to be removed in the same manner used to remove this component from

$$x]'$$

to produce the starting vector

$$x]^{1,2}$$

Compared to the Rayleigh method, the intensification method is seen to be relatively orderly in that it produces an eigenvalue and corresponding eigenvector whose position in the complete set is known beforehand, and relatively straightforward in that each computational step has a unique well defined solution. This last fact allows the residuals in a relaxation solution to be "completely relaxed" (made as near zero as possible) instead of "partly relaxed" as in the Rayleigh method. The Rayleigh method has the comparative advantage that no modification of the basic procedure is necessary to find solutions other than the "gravest mode" solution. Furthermore, in justice to Rayleigh, it must be admitted that he did not present his technique in the same form that Allen does. Rayleigh¹ suggested the determinate technique for solving $n-1$ of the n homogeneous equations for $n-1$ vector components in terms of any one component. This procedure makes the convergence of the process a function of the equation selected to be "left out". An alternative may be found from considering the vector description of the process.

If matrices

$$[A]$$

and

$$[B]$$

be considered as sets of row vectors

$$[A] = \begin{bmatrix} \underline{\alpha}_1 \\ \underline{\alpha}_2 \\ \underline{\alpha}_n \end{bmatrix} \quad (87)$$

$$[B] = \begin{bmatrix} \underline{\beta}_1 \\ \underline{\beta}_2 \\ \underline{\beta}_n \end{bmatrix} \quad (88)$$

then the goal of the Rayleigh method is to find eigenvalues such that the vectors

$$\underline{\alpha}_i - \lambda \underline{\beta}_i, \quad i = 1, 2, \dots, n$$

are coplanar. In n -dimensional space, an $(n-1)$ dimensional plane may be passed through any n arbitrary points. This plane will be characterized by an equation

$$r_1 x_1 + r_2 x_2 + \dots + r_n x_n = \Gamma \quad (89)$$

¹ Lord Rayleigh, The Theory of Sound, Dover Publication, New York 1945, Vol I, p. 110.

linear in the n coordinate variables x_i . The coefficients γ_i , may in fact be found from substituting the coordinates of the n arbitrary points in the equation for the plane and solving the resulting n simultaneous equations. Any one of the $(n+1)$ unknowns

$$\gamma_1, \gamma_2, \dots, \gamma_n, \Gamma$$

may be assigned arbitrarily. This amounts only to multiplying the equation for the plane by a constant. Γ may be considered the scalar product of an arbitrary vector.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad (90)$$

extending from the coordinate system origin to a point in the plane, with the vector

$$\underline{\gamma} = \gamma_1, \gamma_2, \dots, \gamma_n, \quad (91)$$

since the equation of the plane may be written

$$\underline{\gamma} \cdot \mathbf{x} = \Gamma \quad (92)$$

is thus obviously normal to the plane. If Γ be chosen progressively smaller, the components of

$$\underline{\gamma}$$

must decrease, and thus Γ serves as a measure of the "size" of the vector

$$\underline{\gamma}$$

Provided the position of the plane is fixed, then as Γ is reduced toward zero, the vector

$$\underline{\gamma}$$

must also approach zero except in the case where the plane includes the origin of coordinates. In Allen's version of Rayleigh's process, the vector

$$\underline{\gamma}$$

plays the part of approximation to an eigenvector. It is to be found in spite of the facts that

$$\Gamma = 0 \quad (93)$$

and that the plane through the tips of the vectors

$$\underline{\alpha}_i - g_i \underline{\beta}_i$$

does not include the coordinate origin. There would seem no loss of generality and a considerable gain in computational simplicity, to result from making

$$\Gamma = \sqrt{N}, \quad (94)$$

say, so that, to find the j^{th} approximation to an eigenvector, we set

$$(\underline{\alpha}_{i_1} - q_j \underline{\beta}_{i_1}) x_j = 1.0, \quad i = 1, 2, \dots, N \quad (95)$$

Figure I shows the three planes

$$[\underline{r} \ x] = 0 \quad (96)$$

together with the vectors

$$\underline{\alpha}$$

and

$$\lambda \underline{\beta}$$

where

$$\begin{bmatrix} \underline{\alpha}_{i_1} \\ \underline{\alpha}_{i_2} \\ \underline{\alpha}_{i_3} \end{bmatrix} = [A] = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 6 \end{bmatrix} \quad (97)$$

$$\begin{bmatrix} \underline{\beta}_{i_1} \\ \underline{\beta}_{i_2} \\ \underline{\beta}_{i_3} \end{bmatrix} = [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (98)$$

giving, approximately

$$\begin{aligned} \lambda_1 &= 1.95 \\ \lambda_2 &= 3.73 \\ \lambda_3 &= 9.35. \end{aligned} \quad (99)$$

If any value

$$q_j \neq \lambda \quad (100)$$

were computed from the Rayleigh process just proposed, then the plane through the tips of the vectors

$$\underline{\alpha} - q_j \underline{\beta}$$

would not include the origin. The vector

$$x_j$$

normal to this plane would thus not be an eigenvector. If this vector be computed from the equations

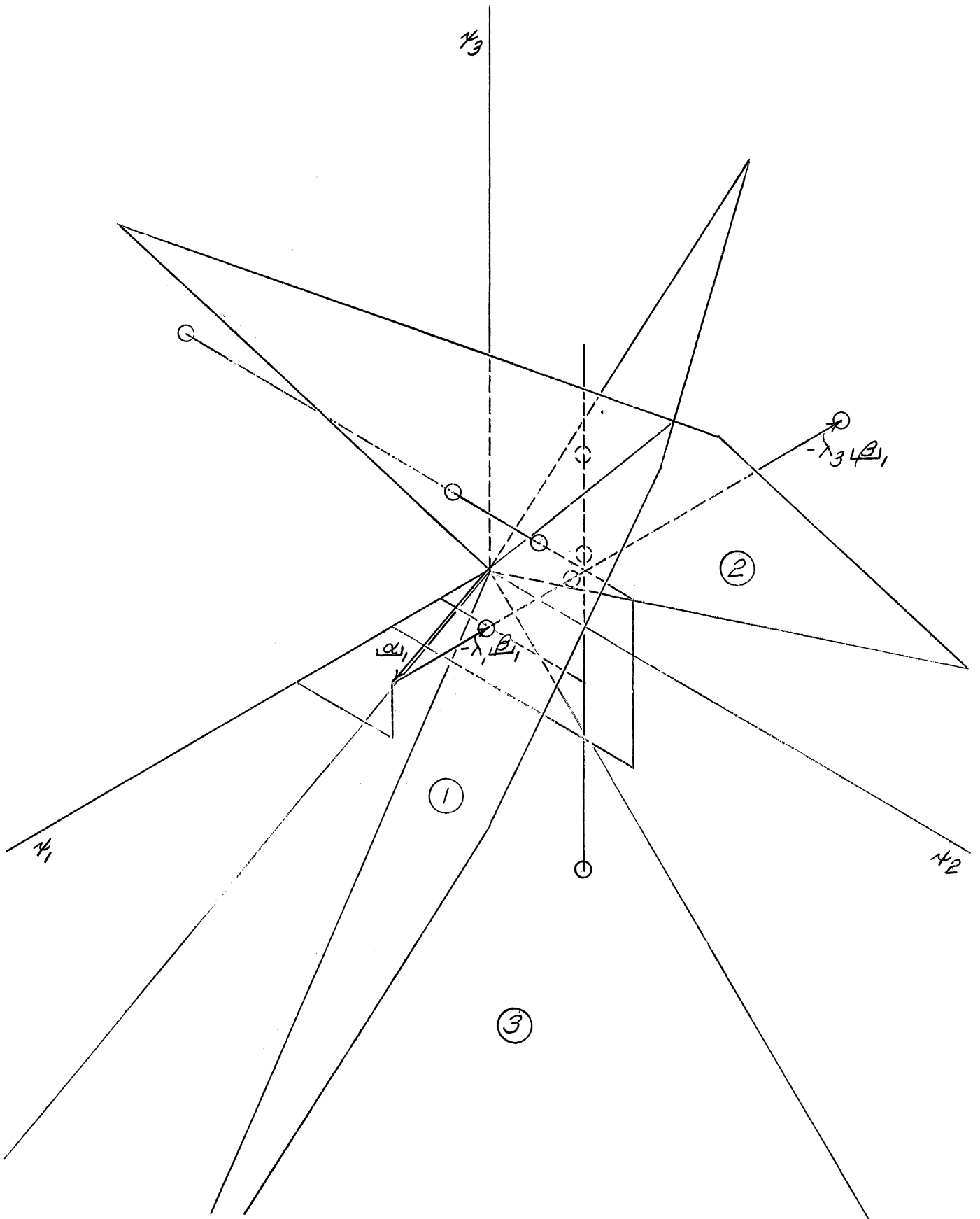


Figure I Principal Planes and Vector Sets

$$\left([A] - q_j [B] \right) x_j = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad (101)$$

then the "length" x , of the vector

$$x_j$$

must be the reciprocal of the distance of the plane from the coordinate origin.

The length of the vector

$$X = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \quad (102)$$

thus serves as a direct measure of the excellence of the approximation at each step of the process outlined.

Difficulty occurs in the process if two or more of the vectors

$$\underline{\alpha}_i - \lambda \underline{\beta}_i$$

are nearly coincident, in which case a small error in the computed value for λ can produce a plane through the tips of the vectors

$$\underline{\alpha}_i - g \underline{\beta}_i, \quad g \neq \lambda$$

markedly different from the plane of the vectors

$$\underline{\alpha}_i - \lambda \underline{\beta}_i.$$

This difficulty is common to all versions of the Rayleigh method.

The geometric picture of the intensification procedure is somewhat simpler than that for the Rayleigh method. The i^{th} step of the intensification procedure produces a vector

$$x^{(i)}$$

such that

$$[A] x^{(i)} = [B] x^{(i-1)} ;$$

that is, the vector found, transforms under

$$[A]$$

into the same vector as the transform under

$$[B]$$

of the preceding vector. Any starting vector

$$x^{(0)}$$

is thus led through a definite sequence of positions toward some position such that

$$[A] x^{(i)} = \lambda_j [B] x^{(i-1)} = \lambda_j [B] x_j = [A] x_j \quad (103)$$

which is a real eigenvector position. In the elementary problem where

$$[B] = [I] = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \quad (104)$$

the successive positions of the vector are given by successive "powers" of the operator

$$[A]^{-1}$$

on the vector; i.e.

$$\begin{aligned} x^{(1)} &= [A]^{-1} x^{(0)} \\ x^{(2)} &= [A]^{-1} x^{(1)} = [A]^{-2} x^{(0)} \\ \vdots \\ x^{(n)} &= [A]^{-n} x^{(0)} \end{aligned} \quad (105)$$

etc. In the case where

$$[B]$$

is not the identity matrix, the successive vector positions are given by powers of

$$[A]^{-1} [B] \quad (106)$$

operating on the original vector. A sequence of positions for

$$[A] = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 6 \end{bmatrix} \quad (107)$$

$$[B] = [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (108)$$

$$x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (109)$$

is shown in Figure II. Note the approximately geometric reduction of angular error and the monotone convergence (no circulation about the limiting position). Both characteristics are associated with the eigenvalues of the matrix, the first with the fact that the eigenvalues are distinct, the second with the fact that they are all real. The basis for these statements is developed in the discussion of stability of the similar analog process to be developed later.

The problem of convergence is a central one in numerical iterative processes. A thorough treatment of the problem in iterative solution of simultaneous linear algebraic equation sets, (including those "approximate" to differential equations and boundary values) is given by Householder¹. Householder considers two questions: First, to what field of problems may the matrix process be applied with guaranteed convergence? Second, how may the process be defined to yield as rapid convergence as possible? To answer the first question, that of simple convergence, Householder defines an error vector equal to the difference between the solution sought and the approximate solution obtained at any step of the process. "Norms" are then defined for this vector and also for the operation matrix. The first is called N_e , the second N_m . The process converges, then, if

$$\lim_{n \rightarrow \infty} N_m^n N_e = 0, \quad (110)$$

A "steepest descent" solution is obtained if for residuals

$$r_i = \sum_j a_{ij} x_j - b_i \quad (111)$$

and

$$S = 1/2 \sum_i r_i^2, \quad (112)$$

a change

$$\Delta x_j = -k \frac{\partial S}{\partial x_j} = -k \sum_i r_i \frac{\partial r_i}{\partial x_j} = -k \left[\underline{r} \alpha \right]_j \quad (113)$$

is made in the j^{th} component of the vector

$$\underline{x}],$$

¹ Householder, A.S., "Approximate Solution of Matrix Problems", Assn. Computing Machy - J, v5n3 July 1958, pp. 205-243. This reference contains an extensive bibliography

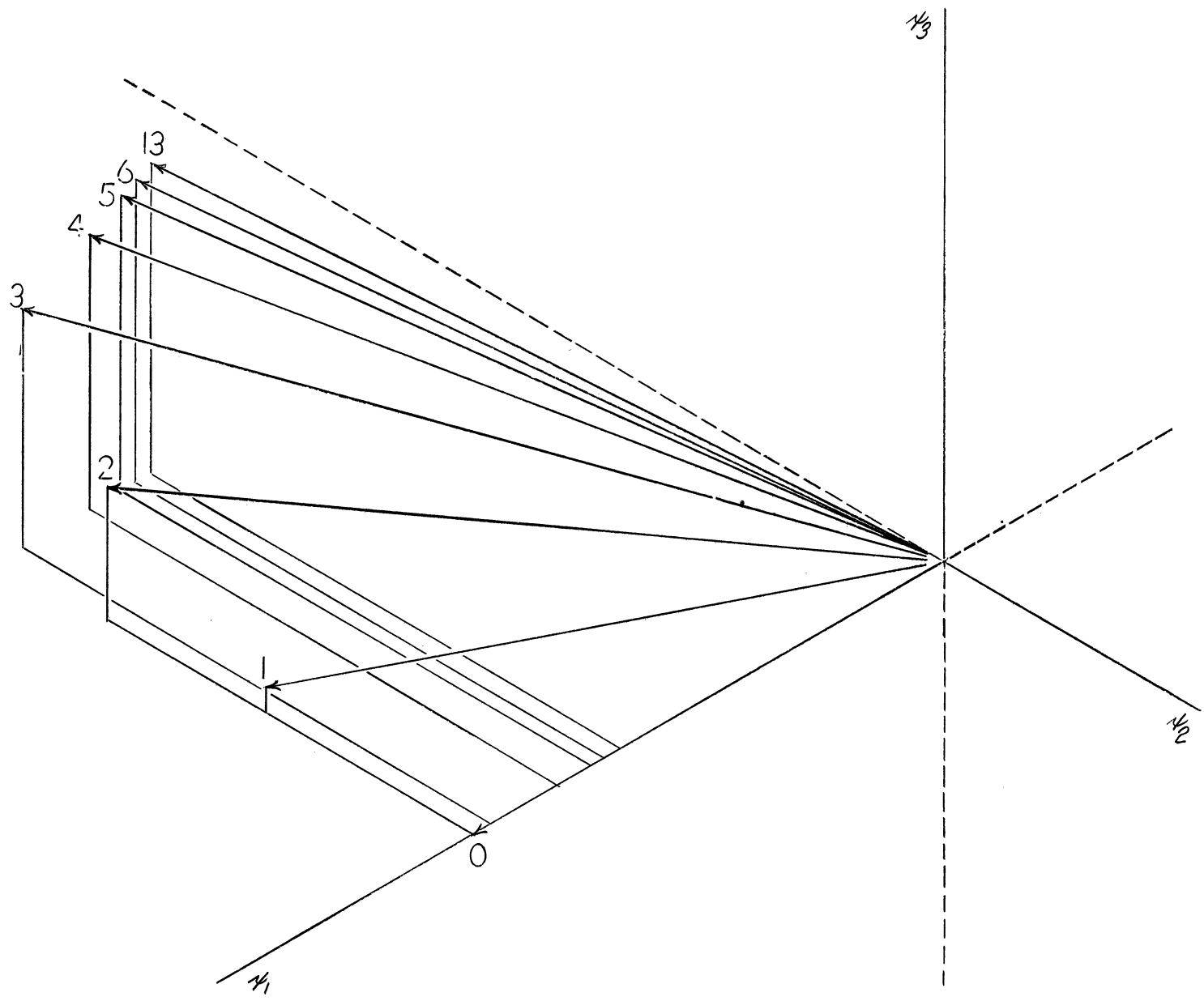


Figure II Intensification Procedure Iterants

where

$$\underline{r} = \underline{r_1, r_2, \dots, r_n}, \quad (114)$$

$$\alpha]_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}, \quad (115)$$

and

$$\underline{x}] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (116)$$

The function \underline{S} defines the crater down which the process path leads most steeply. While some such error function may be defined to make almost any convergent process seem optimum, the above definition is reasonable and uncomplicated. Certain other aspects of the discussion are, if not unreasonable, at least certainly not uncomplicated. The corresponding problem of stability in the continuous analog process is considerably simpler.

Analog Techniques

A number of analog computers have been designed to find eigenvalues and eigenvectors of real symmetric positive definite matrices. Certain of these will be described in the following paragraphs. Almost without exception, such machines have been modified versions of machines whose basic intent was the solution of sets of simultaneous determinate algebraic equations. The stability of such equipments is of primary importance. Stability can, in such cases, be inferred from that of the simultaneous equation apparatus. The next few paragraphs will present a review of several papers on this particular topic, after which the eigenvalue machines will be described in detailed.

An analysis of the analog equation solver stability problem is given in Cederbaum, I; and Fuchs, A. "On the Stability of Linear Algebraic Equation Solvers", Actes des Journees Internationale de Calcul Analogique, Presses Academiques Europeens, Strasbourg, 1959, pp. 174-178.

A model equation solver employing coefficient pots, standard voltage supplies, and summing operational amplifiers is proposed and analysed. The model is as shown below.

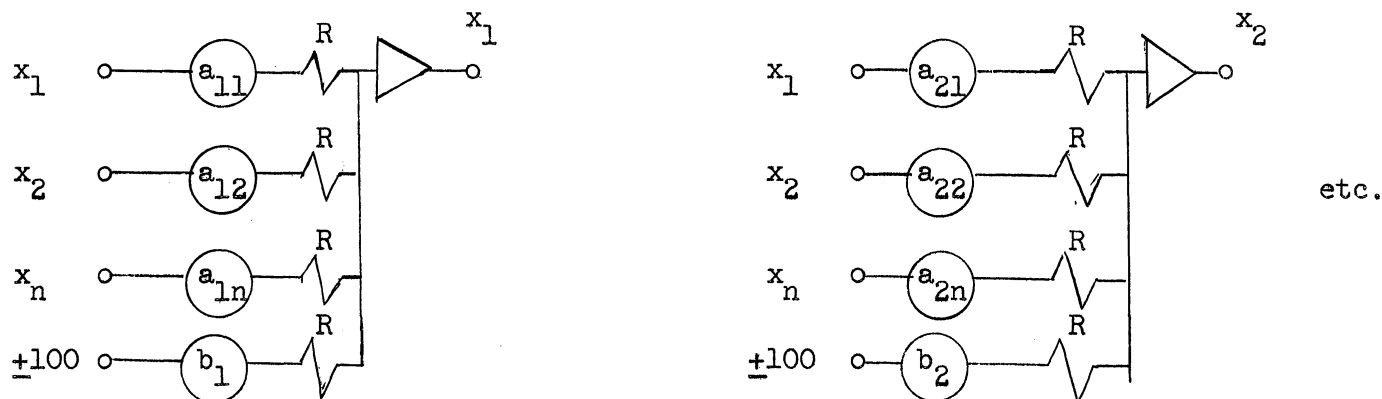


Figure III
Model Equation Solver

to solve the equation set

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\ a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n &= b_n \end{aligned} \quad (117)$$

or

$$[A] x = b \quad (118)$$

The model is obviously not very general since it immediately requires

$$a_{ij} > 0, \text{ for all } i \text{ and } j,$$

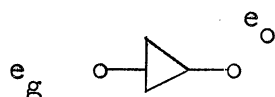
a condition not often encountered in simultaneous equation sets. Apparently the model was chosen not for its practicability but primarily because it leads to a simple stability analysis in terms of the input-output relationship or "transfer function" of the high-gain amplifiers employed. It may be noted in passing that this is not the only analog computer stability analysis that suffers from an impractical computer setup. Such a shortcoming seems, in fact, rather the rule than the exception. In any case, with the setup as shown, Cederbaum and Fuchs find a set of n characteristic equations

$$\lambda_i = - \frac{n+1}{G(p)}, \quad i = 1, 2, \dots, n \quad (119)$$

where λ_i is a characteristic root of the coefficient matrix

$$[A]$$

and where $G(p)$ is the operator form of the differential equation relating high-gain amplifier input and output, i.e.,



$$e_o = G(p) e_g \quad (120)$$

A Nyquist plot of $G(j\omega)$ for all of ω must then not encircle any point

$$-\frac{n+1}{\lambda_i}, i = 1, 2, \dots, N, \quad (121)$$

if the system is to be stable. Using a theorem a Parodi's, Fuchs then goes on to state sufficient criteria for stability in terms of the elements of the coefficient matrix and gains of the high-gain amplifier at two particular frequencies.

If Cederbaum and Fuchs's equation solver were made more practical by allowing coefficients of either sign, then the simple analysis presented would not apply. This is because additional inverting amplifiers would have to be used, each of whose outputs would constitute an additional independent voltage source, raising the degree of the characteristic equation for the system and obscuring the relationship between the system stability and the characteristic roots of the system. For the practical system each amplifier would be associated with an equation

$$e_i \frac{1}{G(p)} \sum_{j=1}^{n+1} \frac{1}{R_{ij}} + \sum_{j=1}^{n+1} \frac{\alpha_{ij} e_j}{R_{ij}} = 0 \quad (122)$$

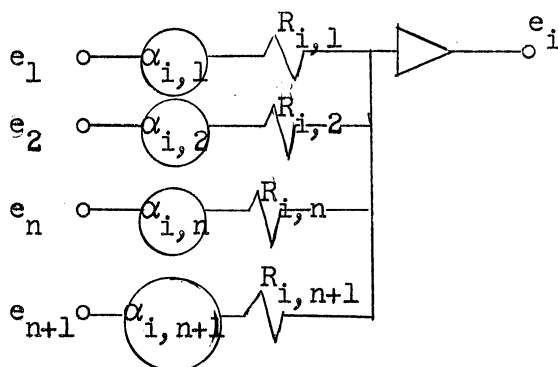


Figure IV
Illustration of General Summer Equation

where

$$\alpha_{i,n+1} e_{n+1} = b_i \quad (123)$$

and where not all the R's are necessarily identical. This formulation of the stability problem obviously applies to setups for solution of differential equations by merely replacing the R_{ij} 's with complex $Z_{ij}(p)$'s. Here again the technique adopted in the literature has been to simplify the model setup enough to make the analysis possible even if the resulting setup isn't practical. A few generalizations about stability even if somewhat imprecise, will be found helpful. First of all, for the sorts of amplifiers usually used in electronic differential analyzers, the effect on the characteristic roots of a given differential equation setup for solution on the analyzer produced by amplifier dynamic characteristics is the following: A slight shift of the equation characteristic roots is produced. A group of high frequency, high damping ratio roots is added by the amplifier dynamics.⁽¹⁾ Both these effects can ordinarily be neglected in the solutions generated on high-quality equipment.

The above generalization can be used to produce a good first approximation to a stability analysis for the practical equation solver. If the amplifiers producing solution variables x_i be transformed from summers to summing integrators by adding to each of these a unit feedback capacitor,

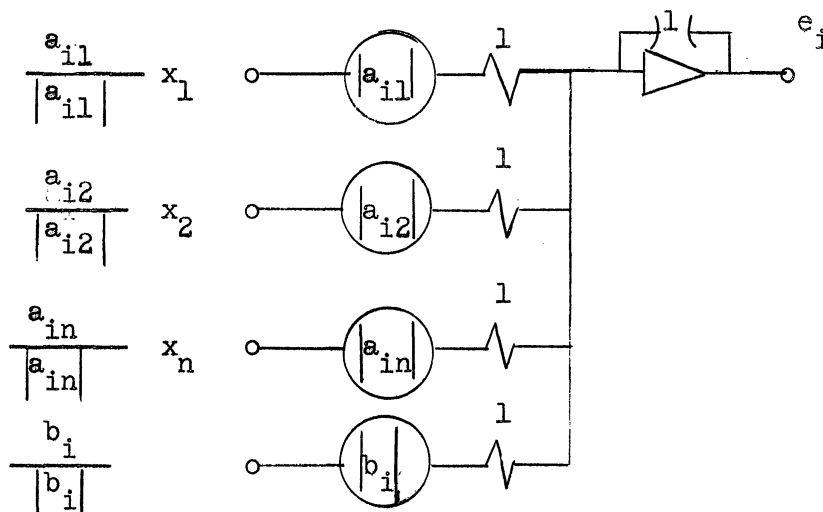


Figure V. Illustration of General Integrator Equation

¹ MacNee, A.B. "Some Limitations on the Accuracy of Electronic Differential Analyzers", Proc. of the IRE, V. 40 V 3, March 1952, pp. 303-308.

then, independent of how many inverters are employed to produce the equivalent of negative coefficients, the differential equation associated with the resulting setup has characteristic roots almost exactly the negatives of the coefficient matrix eigenvalues. In brief, the setup is stable if the matrix eigenvalues all have positive real parts. Furthermore, the steady-state outputs of the integrating amplifiers are, provided they exist, exactly the solution values for the algebraic equation set. The important proposition introduced here which will be used in the sequel is that if an analyzer setup correspond to a differential equation, then the setup will be stable if the differential equation have only roots with negative real parts. Inclusion of amplifier dynamics is maybe important to dynamic error analysis but is rarely of concern in static stability studies.

Mallock's Computer: Hartree¹ describes an a.c. computer developed by Mallock which uses various numbers of turns of wire on transformer cores to represent the coefficients of a set of simultaneous algebraic equations. The coils corresponding to coefficients of any one equation are connected in series with a coil representing the size of the constant term in the equation. The constant term coils are all on a core excited by an a.c. source. The arrangement is shown in Figure VI. The flux in each core will then be proportional to one of the unknowns, x_i , in the equation set

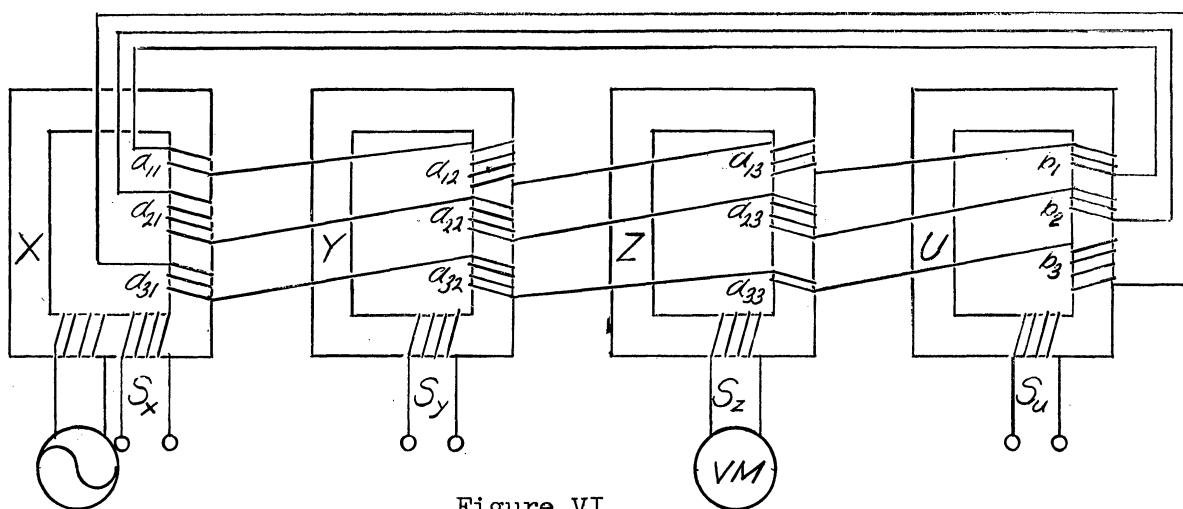


Figure VI.
Mallock's Computer

¹ Hartree, Douglas R., Calculatory Instruments and Machines, The University of Illinois Press, 1949.

$$[A] x = b \quad (124)$$

or, what is the same thing, the voltage across a standard size coil on each transformer can be read as the unknown. The voltages across the individual coefficient coils are proportional to individual terms, $a_{ij} x_j$, in the equation set. The currents, i_{ij} in the several circuits are proportional to the solution elements of the equation set

$$[A]_t i = kx \quad (125)$$

where k is a function of the magnetic permeability of the transformer coils.

Hartree states that the instrument can be used to solve the eigenvalue problem

$$[A] x = \lambda x \quad (126)$$

although this apparently requires considerable modification of the basic apparatus. The actual modification procedure is not described by Hartree.

Hughes and Wilson's Computer: An a.c. analog computer more explicitly designed to solve the elementary eigenvalue problem was developed by Hughes and Wilson. Since the passive elements of the computer are bilateral, the computer is useful only for symmetric matrices. Since only positive elements are used to represent eigenvalues, the device is further restricted to real eigenvalues. The arrangement of the computer is shown in Figure VII.

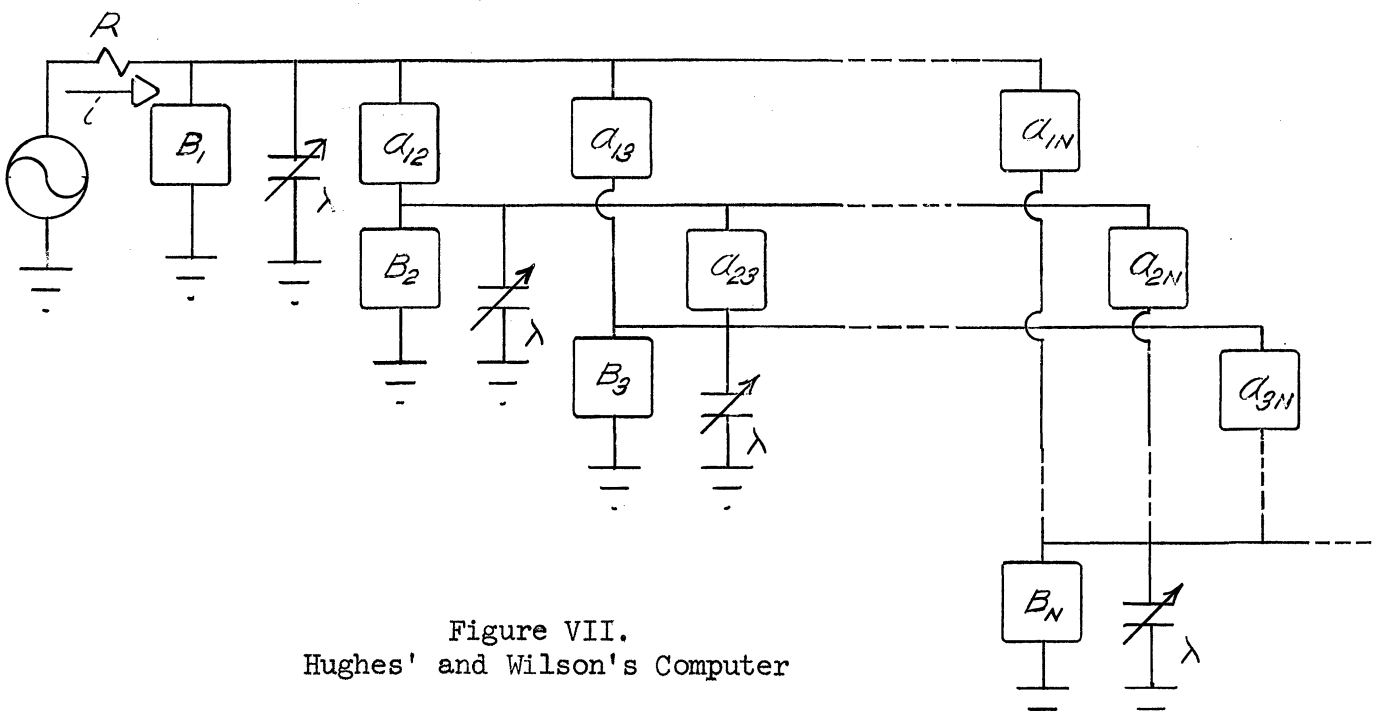


Figure VII.
Hughes' and Wilson's Computer

The x 's are represented by voltages at the several nodes. Positive coefficients are represented by capacitors, negative coefficients by inductors. Writing the equations for current flow to each node,

$$\begin{aligned}
 & -B_1 x_1 - \lambda^2 x_1 + a_{12}(x_2 - x_1) + a_{13}(x_3 - x_1) + \dots + a_{1n}(x_n - x_1) = i \\
 & (-B_1 - \sum_{j=2}^n a_{1j} - \lambda^2) x_1 + a_{12} x_2 + \dots + a_{1n} x_n = 0 \\
 & a_{21} x_1 + (-B_2 - \sum_{\substack{j=1 \\ j \neq 2}}^n a_{2j} - \lambda^2) x_2 + a_{23} x_3 + \dots + a_{2n} x_n = 0 \\
 & \vdots \\
 & a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3 + \dots + (-B_n - \sum_{j=1}^{n-1} a_{nj} - \lambda^2) x_n = 0
 \end{aligned} \tag{127}$$

So that if

$$B_i = - \sum_{j=1}^n a_{ij}, \tag{128}$$

then except for the finite right hand member of the first equation, the equation set is exactly

$$[A] x = \lambda x$$

The x 's will be small except when the capacitors are adjusted for resonance. In this condition the voltages become finite and the branch currents become large, making the current input i relatively negligible and the voltage equations approximately the eigenvalue equations. Dissipation in the passive circuit elements saves the system from destruction.

Adcock's Computer: A machine more like the majority of modern analog eigenvalue circuits is described by Adcock.¹ This machine employs a ganged potentiometer to be set to the several real positive eigenvalues, N^2 coefficient pots set to the matrix element values, and N servomultipliers, each with $N+1$ identical input elements. $N-1$ of the servomultipliers are used to find solutions of the $N-1$ determinate equations

$$\sum_j A_{ij} x_j - q x_i = 0, \quad i = 2, 3, \dots, N \tag{129}$$

where q represents the setting of the ganged eigenvalue potentiometers and where

¹ Adcock, W. A., "An Automatic Simultaneous Equation Solver and Its Use in Solving Secular Equations", Rev. Sci. Instr., 19:181-187, 1948.

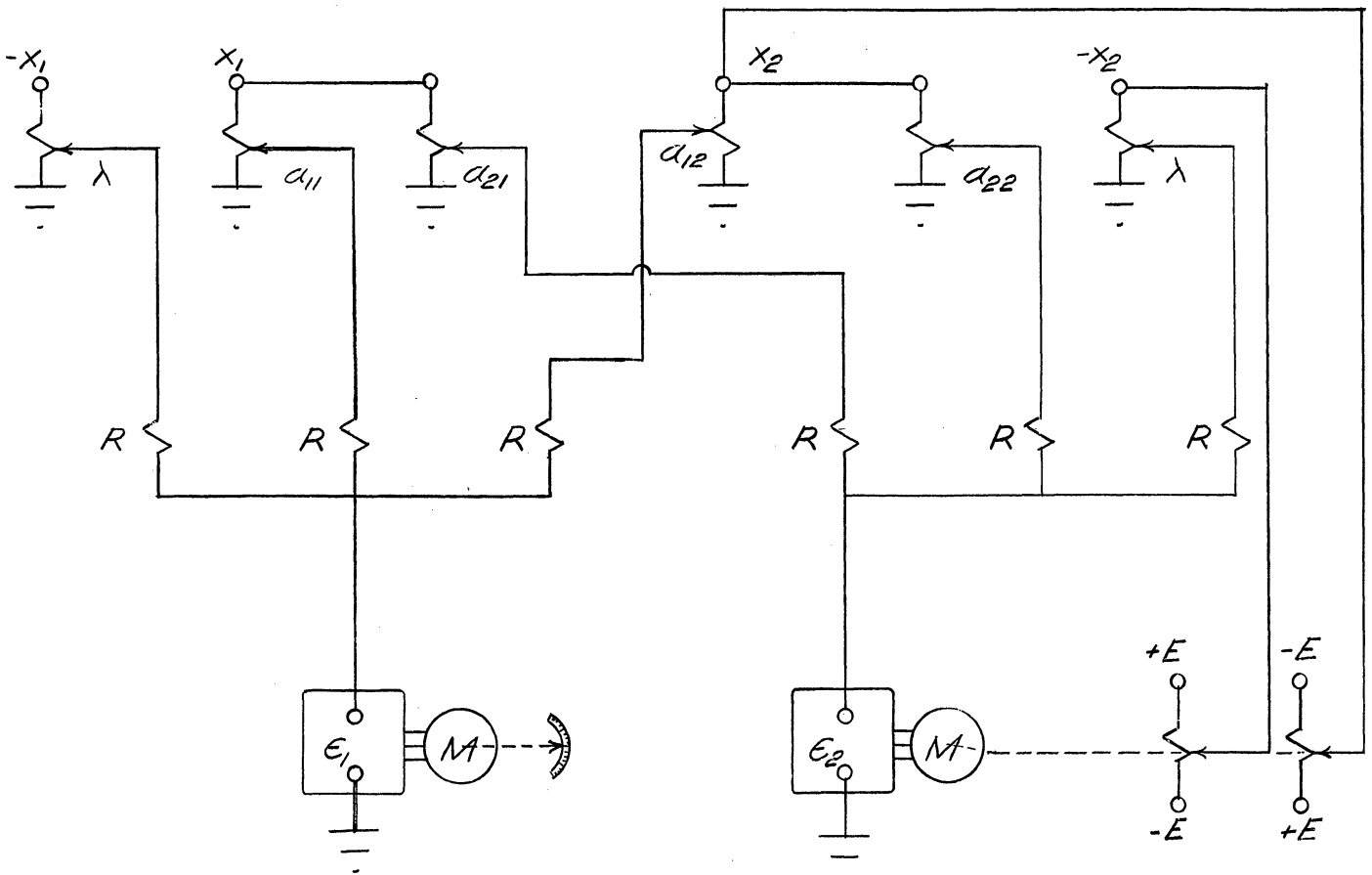


Figure VIII Adcock's Computer

some one element x_k is set arbitrarily. The remaining servomultiplier is used as an error indicator with output representing

$$\sum_j A_{ij} x_j - qx_1$$

When this quantity is made zero, then

$$q = \lambda \tag{130}$$

and

$$[A] x = \lambda x \tag{131}$$

An electronic differential analyzer circuit described by Harbert¹ differs from Adcock's machine only in the substitution of summing operational amplifiers for the N-1 servos used to compute x_2, x_3, \dots, x_n and in the substitution of a meter for the error indicating device in Adcock's machine. The same limitations that apply to Adcock's machine apply also to Harbert's differential analyzer version of it. It is apparent that any of the devices described could be set up in equivalent mathematical form on an electronic differential analyzer. The circuit to be described next differs in certain essentials from any of the preceding.

A New Technique for the Diagonalization of Symmetric Matrices

The technique to be described will be presented as a set of simultaneous non-linear differential equations suitable for solution on conventional electronic differential analyzer equipment. The set up of the problem is relatively straightforward and principal attention will therefore be devoted to the equations themselves. There are three pertinent questions regarding the equations which must be answered. First, do the equations have singular points corresponding to solutions of the algebraic eigenvalue problem? Second, are these singular points stable; i.e., will the analyzer behavior lead to a solution? Lastly, how may the problem parameters be scaled so as to guarantee a solution and so as to obtain the most nearly accurate solution

¹ Harbert, F. C. "Analog Computer Techniques", Electronic Engineering, V 34 N 384, 385, February 1960, pp. 74-77, March 1960, pp. 166-169.

which the machine can produce? These three aspects of the problem will be taken up serially, concentrating only on the equations for the first step of the diagonalization procedure. Finally the equations for the remaining steps in the procedure will be examined.

Equations for the First Step , $\begin{bmatrix} 0 \\ x \end{bmatrix} = y \begin{bmatrix} -x \\ y \end{bmatrix} x$

The form of the differential equations used to find an eigenvector

$$x \begin{bmatrix} \end{bmatrix}_n ,$$

and an eigenvalue λ_n , satisfying the elementary eigenvalue equations

$$[A] x \begin{bmatrix} \end{bmatrix}_n = \lambda_n x \begin{bmatrix} \end{bmatrix}_n = y \begin{bmatrix} \end{bmatrix}_n \quad (132)$$

seems most reasonable if the process that led to the development of the equations is presented first.

It was known initially, that the intensification procedure, computing serially

$$x \begin{bmatrix} \end{bmatrix} (2) = [A]^{-1} x \begin{bmatrix} \end{bmatrix} (1)$$

$$x \begin{bmatrix} \end{bmatrix} (3) = [A]^{-1} x \begin{bmatrix} \end{bmatrix} (2) = [A]^{-2} x \begin{bmatrix} \end{bmatrix} (1)$$

etc., produced in the limit, at least in certain circumstances, an eigenvector

$$\lim_{n \rightarrow \infty} x \begin{bmatrix} \end{bmatrix}^{(n)} = x \begin{bmatrix} \end{bmatrix}_1 \quad (134)$$

where

$$[A] x \begin{bmatrix} \end{bmatrix}_1 = \lambda_1 x \begin{bmatrix} \end{bmatrix}_1 \quad (135)$$

It was known for instance that the procedure converged to the smallest value, λ_1 of the set of values $(\lambda_1, \lambda_2, \dots, \lambda_n)$ whenever the matrix was such that all the values of the set were positive and real. Positive realness is also a sufficient condition for the solubility of the equation set by an array of summing operational amplifiers. A set of simultaneous algebraic equations cannot in general be solved by such an array unless either the eigenvalues of $[A]$ all have positive real parts or unless either the matrix or the setup be modified in some way. Hand calculation of examples showed that positive realness of the matrix eigenvalues was not essential to convergence of the intensification procedure. An alternative procedure was therefore sought which would retain the great virtue of the intensification procedure of producing a unique

eigenvalue solution and which would also have a direct summing amplifier setup which was not unstable in circumstances where the numerical procedure converged.

A rather obvious solution of this problem is the iterative procedure

$$\begin{aligned} x^{(2)} &= [A] x^{(1)} \\ x^{(3)} &= [A] x^{(2)} = [A]^2 x^{(1)} \end{aligned} \quad (136)$$

etc., which must have the same properties with respect to the matrix

$$[A]^{-1}$$

that the intensification procedure has with respect to the matrix

$$[A] ;$$

i.e., the procedure must converge to a vector

$$x_n = \lim_{n \rightarrow \infty} x^{(n)} \quad (137)$$

where

$$[A] x_n = \lambda_n x_n \quad (138)$$

where λ_n is the largest positive eigenvalue of the set of eigenvalues of the matrix

$$[A].$$

Also the operational amplifier setup corresponding to the modified procedure is simply an "open-ended" array of summers -- one summer for each equation with no interconnection between summers -- about which there is no question of stability.

Equations (136),

$$x^{(i+1)} = [A] x^{(i)}, \quad (136)$$

might now be modified for automatic solution on the analyzer by giving to each component, x_i , of the vector

$$x]$$

a velocity, \dot{x}_i , proportional to the difference between x_i and i th component of the transform

$$[A] x]$$

of

$$x].$$

That is, we might solve the set

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} A \\ x \end{bmatrix} - x, \quad (139)$$

This will not work because the modulus,

$$\sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

of

$$\begin{bmatrix} x \end{bmatrix}$$

changes along with its "direction" (except in the singular case $\lambda_n = 1.0$).

In order to obtain a singular point not at the origin

$$\begin{bmatrix} x \end{bmatrix} = 0 \quad (140)$$

the linear equation set (139) must be made non-linear. A suitable procedure might be to make the velocity vector

$$\begin{bmatrix} \dot{x} \end{bmatrix}$$

lie in the plane of

$$\begin{bmatrix} x \end{bmatrix}$$

and

$$\begin{bmatrix} A \\ x \end{bmatrix}$$

and be normal to

$$\begin{bmatrix} x \end{bmatrix};$$

i.e., such that

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix} = 0, \quad (141)$$

This modification changes (139) to

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} \begin{bmatrix} A \\ x \end{bmatrix} - \begin{bmatrix} x \\ x \end{bmatrix} \begin{bmatrix} A \\ x \end{bmatrix} \cdot \begin{bmatrix} x \\ x \end{bmatrix}, \quad (142)$$

lies in the plane of

$$\begin{bmatrix} x \end{bmatrix}$$

and

$$\begin{bmatrix} A \\ x \end{bmatrix}$$

since it is a linear sum of these vectors. (142) also satisfies (141), as may be shown by direct substitution. Equations (142) may be described geometrically as a process wherein an arbitrary initial vector,

$$x \Big],$$

pursues its own transform,

$$[A] \ x]$$

while keeping its length,

$$\sqrt{x_1^2 + x_2^2 + \dots + x_N^2} = (\underline{x} \ x])^{1/2}, \quad (143)$$

constant. A remaining difficulty with (142) is the fact that this equation set has an infinite number of singular points. It has, in fact, a "singular line" which is the eigenvector direction. It may be transformed into a set with at most two stable finite singular points by making the indicated modulus

$$\underline{x} \ x] = 1.0 ; \quad (144)$$

i.e., by making

$$\begin{aligned} \overset{\circ}{x} &= [A] \ x] - \underline{x} [A] \ x] \ x] \\ \overset{\circ}{x} &= y] - \underline{x} y] \ x] \end{aligned} \quad (145)$$

where

$$y] = [A] \ x] \quad (146)$$

Stability: Equations (145) will now be shown to have the required stability properties, after which the problem of scaling for machine solution will be considered.

The unit sphere,

$$\underline{x} \ x] = 1.0, \quad (144)$$

may be considered a principal surface of the set

$$\overset{\circ}{x} = y] - \underline{x} y] \ x] \quad (145)$$

for

$$\begin{aligned} \underline{x} \ \overset{\circ}{x} &= \underline{x} y] - \underline{x} y] \ \underline{x} \ x] \\ &= 0, \quad \underline{x} \ x] = 1.0 \end{aligned} \quad (146)$$

This surface contains singular points of (145) where it intersects principal direction of

$$[A],$$

for

$$\dot{x} = 0 = y - \underline{x} y \quad x \quad (147)$$

implies

$$[A] x = \underline{x} [A] x \quad x \quad (148)$$

and thus, either

$$x = 0 \quad (149)$$

or else

$$\left| [A] - \underline{x} [A] x \right| [I] = 0 \quad (150)$$

$$\underline{x} [A] x = \lambda \quad (151)$$

and thus, from (148) and (151)

$$[A] x = \lambda x \quad (152)$$

If there are N distinct real values of λ , then (152) and (146) together define $2N$ singular points of the system (145). One other singular point is given by (149). There are thus $2N + 1$ finite singular points of the system. The steady-state position of the vector

$$x,$$

if such exist, must be at a stable point of the set of singular points.

To determine the stability of the singular points requires a linearization of the equations of motion in the neighborhood of each singular point to determine characteristic roots of the linear system in that neighborhood.

Near the origin

$$\begin{aligned} \dot{x} &= [A] x - \underline{x} [A] x \quad x \\ &= [A] x, \end{aligned} \quad (153)$$

approximately, retaining only the terms in the right-hand numbers linear in the components x_i . The characteristic roots for the singular point at the origin of coordinates are thus simply the eigenvalues of

$$[A].$$

Thus, the singular point at the origin will be unstable provided

$$\underline{[A]}$$

has any eigenvalue with positive real part.

Now in the neighborhood

$$\sqrt{\xi_1^2 + \xi_2^2 + \dots + \xi_N^2} = (\underline{\xi} \quad \underline{\xi})^{1/2} < \delta \quad (154)$$

of a singular point

$$\underline{x}]_i$$

where

$$\begin{aligned} [A] \underline{x}]_i &= \lambda_i \underline{x}]_i & (155) \\ \underline{x}]_i \underline{x}]_i &= 1.0 \end{aligned}$$

and

$$\underline{x}] = \underline{x}]_i + \underline{\xi}] \quad (156)$$

equation (145) becomes

$$\begin{aligned} \dot{\underline{x}}] &= \dot{\underline{\xi}}] = [A] (\underline{\xi}] + \underline{x}]_i) - (\underline{\xi}] + \underline{x}]_i) [A] (\underline{\xi}] + \underline{x}]_i) (\underline{\xi}] + \underline{x}]_i) \\ \dot{\underline{\xi}}] &= [A] \underline{\xi}] + \cancel{\lambda_i \underline{x}]_i} - \underline{\xi}] [A] \underline{\xi}] \underline{\xi}] - \underline{\xi}] [A] \underline{\xi}] \underline{x}]_i \end{aligned} \quad (157)$$

$$\begin{aligned} &- \underline{x}]_i [A] \underline{\xi}] \underline{\xi}] - \underline{x}]_i [A] \underline{\xi}] \underline{x}]_i - \underline{\xi}] [A] \underline{x}]_i \underline{\xi}] \\ &- \underline{\xi}] [A] \underline{x}]_i \underline{x}]_i - \underline{x}]_i [A] \underline{x}]_i \underline{\xi}] - \cancel{\underline{x}]_i [A] \underline{x}]_i \underline{x}]_i} \end{aligned} \quad (158)$$

or approximately

$$\begin{aligned} \dot{\underline{\xi}}] &= [A] \underline{\xi}] - \underline{x}]_i [A] \underline{\xi}] \underline{x}]_i - \underline{\xi}] [A] \underline{x}]_i \underline{x}]_i \\ &- \underline{x}]_i [A] \underline{x}]_i \underline{\xi}]. \end{aligned} \quad (159)$$

For

$$\underline{[A]}$$

symmetrical, (159) becomes

$$\dot{\xi}] = [A] \xi] - 2\lambda_i \underline{x}_i \xi] x]_i - \lambda_i \xi]. \quad (160)$$

The linear system (159) has a "radial" principal direction

$$\xi] = a x]_i \quad (161)$$

with

$$\begin{aligned} \dot{\xi}] &= a\lambda_i x]_i - 2a\lambda_i x]_i - a\lambda_i x]_i = -2a\lambda_i x]_i \\ &= -2\lambda_i \xi] \end{aligned} \quad (162)$$

It also has principal direction

$$\xi] = a x]_i + b x]_j \quad (163)$$

where

$$[A] x]_j = \lambda_j x]_j \quad (164)$$

and

$$a + b \underline{x}_i x]_j = 0 = \underline{\xi} x]_i \quad (165)$$

Substituting (163), (164), and (165) into (159)

$$\begin{aligned} \dot{\xi}] &= a\lambda_i x]_i + b\lambda_j x]_j - a\lambda_i x]_i - b\lambda_j \underline{x}_i x]_j x]_i \\ &\quad - a\lambda_i x]_i - b\lambda_i \underline{x}_j x]_i x]_i - a\lambda_i x]_i - b\lambda_i x]_j \\ &= -2a\lambda_i x]_i + b(\lambda_j - \lambda_i) x]_j + a\lambda_j x]_i + a\lambda_i x]_i \\ &= a(\lambda_j - \lambda_i) x]_i + b(\lambda_j - \lambda_i) x]_j \end{aligned} \quad (166)$$

$$\dot{\xi}] = (\lambda_j - \lambda_i) \xi] \quad (167)$$

Thus, for all λ 's real (as must be the case for

$$[A]$$

symmetric) and distinct, there are $N+1$ finite singular points, one at

$$x] = 0] \quad (168)$$

with characteristic values

$$\lambda_1, \lambda_2, \dots, \lambda_N$$

and with principal directions

$$x] = x]_i \quad (169)$$

where

$$[A] x]_i = \lambda_i x]_i. \quad (170)$$

The other N singular points are at

$$x] = x]_i \quad (171)$$

where

$$[A] x]_i = \lambda_i x]_i \quad (172)$$

and

$$\underline{x}_i x]_i = 1.0 \quad (173)$$

with characteristic values

$$\lambda_1 - \lambda_i, \lambda_2 - \lambda_i, \dots, \lambda_{i-1} - \lambda_i, -2\lambda_i, \lambda_{iH} - \lambda_i, \dots, \lambda_N - \lambda_i$$

and with characteristic directions

$$x] = a_1 x]_1 + b_1 x]_i, a_2 x]_2 + b_2 x]_i, \dots, a_{i-1} x]_{i-1} + b_{i-1} x]_i, \\ x]_i, a_{i+1} x]_{i+1} + b_{i+1} x]_i, \dots, a_N x]_N + b_N x]_i \quad (174)$$

where

$$a_j + b_j \underline{x}_i x]_j = 0 \quad (175)$$

and

$$[A] x]_j = \lambda_j x]_j. \quad (176)$$

Again, for

$$[A]$$

symmetric, the system has in general exactly two stable singular points; namely, those points where the principal direction

$$x] = k \ x]_N \quad (177)$$

corresponding to the largest positive value λ_N ,

$$\lambda_N > \lambda_{N-1} > \dots > \lambda_2 > \lambda_1, \quad (178)$$

$$\lambda_N > 0, \quad (179)$$

pierces the unit hypersphere

$$\underline{x} \ x] = 1.0. \quad (180)$$

The principal directions at this point lie along

$$x]_N$$

with characteristic root $-2\lambda_N$ and normal to

$$x]_N$$

in two-dimensional spaces defined by

$$x]_N$$

and by other principal directions

$$x]_i, \quad i = 1, 2, \dots, N-1,$$

where

$$[A \ x]_i = \lambda_i \ x]_i \quad (181)$$

with characteristic roots

$$\lambda_i - \lambda_N, \quad i = 1, 2, \dots, N-1.$$

The unit hypersphere

$$\underline{x} \ x] = 1.0 \quad (182)$$

is a separatrix, since, in the surface, the velocity,

$$\begin{bmatrix} 0 \\ \dot{x} \end{bmatrix},$$

is everywhere either tangent to the surface or else is zero.

Also, the lines

$$x] = k \ x]_i \quad (183)$$

$$[A \ x]_i = \lambda_i \ x]_i \quad (184)$$

$$\underline{x}_i \ x]_i = 1.0 \quad (185)$$

are principal directions (principal one-dimensional subspaces) since, then,

$$\begin{aligned} \begin{bmatrix} 0 \\ \dot{x} \end{bmatrix} &= [A \ x] - \underline{x} [A \ x] \ x] \\ &= k \lambda_i \ x]_i - k^3 \lambda_i \ x]_i = k \lambda_i (1-k^2) \ x]_i \end{aligned} \quad (186)$$

$$\begin{bmatrix} 0 \\ \dot{x} \end{bmatrix} = \lambda_i (1-k^2) \ x]. \quad (187)$$

And in fact, the two-dimensional subspaces

$$x] = a \ x]_i + b \ x]_j \quad (188)$$

where

$$[A \ x]_i = \lambda_i \ x]_i \quad (189)$$

$$[A \ x]_j = \lambda_j \ x]_j \quad (190)$$

$$\underline{x}_i \ x]_i = \underline{x}_j \ x]_j = 1.0 \quad (191)$$

are principal subspaces, since then,

$$\begin{aligned} \begin{bmatrix} 0 \\ \dot{x} \end{bmatrix} &= a \lambda_i \ x]_i + b \lambda_j \ x]_j - (a^3 \lambda_i + a^2 b \lambda_j \ \underline{x}_i \ x]_j \\ &\quad + a^2 b \lambda_i \ \underline{x}_j \ x]_i + a b^2 \lambda_j \ x]_i + (a^2 b \lambda_i + a b^2 \lambda_j \ \underline{x}_i \ x]_j \\ &\quad + a b^2 \lambda_i \ \underline{x}_j \ x]_i + b^3 \lambda_j \ x]_j \\ &= \alpha \ x]_i + \beta \ x]_j \end{aligned} \quad (192)$$

Thus, neither the unit hypersphere nor the principal two-dimensional subspaces spanned by pairs of eigenvectors can be crossed by a solution path. Each eigenvector thus forms an edge of $2N-2$ principal N -dimensional subspaces of the transformation

$$(\eta) \ x] = \begin{bmatrix} 0 \\ \dot{x} \end{bmatrix} = [A \ x] - \underline{x} [A \ x] \ x], \quad (193)$$

on each side of the coordinate origin and both inside and outside the unit hypersphere. An initial condition for the system must lie within one of these principal subspace "wedges" and the system must then move toward an edge (eigenvector) of the wedge. The appearance of the system boundaries for $N=3$ is sketched in Figure IX. A singular point, P, is shown in these figure, together with the principal directions of the linearized system in the neighborhood of P. Actually, this figure is a little more general than the case considered so far, since for

$$[A] = [A]_t$$

$$[A] x]_i = \lambda_i x]_i \quad (194)$$

$$[A] x]_j = \lambda_j x]_j ,$$

$$\underline{x}_i [A]_t = \lambda_i \underline{x}_i = \underline{x}_i [A] \quad (195)$$

$$\underline{x}_i [A] x]_j = \lambda_i \underline{x}_i x]_j ,$$

and

$$\underline{x}_i [A] x]_j = \lambda_j \underline{x}_i x]_j . \quad (196)$$

thus

$$(\lambda_i - \lambda_j) \underline{x}_i x]_j = 0 \quad (197)$$

and therefore

$$\underline{x}_i x]_j = 0, \lambda_i \neq \lambda_j, \text{ and } [A] = [A]_t . \quad (198)$$

Thus for a symmetric matrix, the principal planes of Figure IX would be drawn mutually orthogonal.

Scaling: Before equations (145),

$$\underline{\dot{x}}] = [A] x] - \underline{\dot{x}}] [A] x] x] \quad (145)$$

can be solved on an analog computer, the matter of the relative magnitudes of the quantities x_i and the computer voltages must be considered. Several matters can be resolved immediately. First, since the equations are non-linear, multipliers will be required. A sort of scaling of the dependent variable x_i

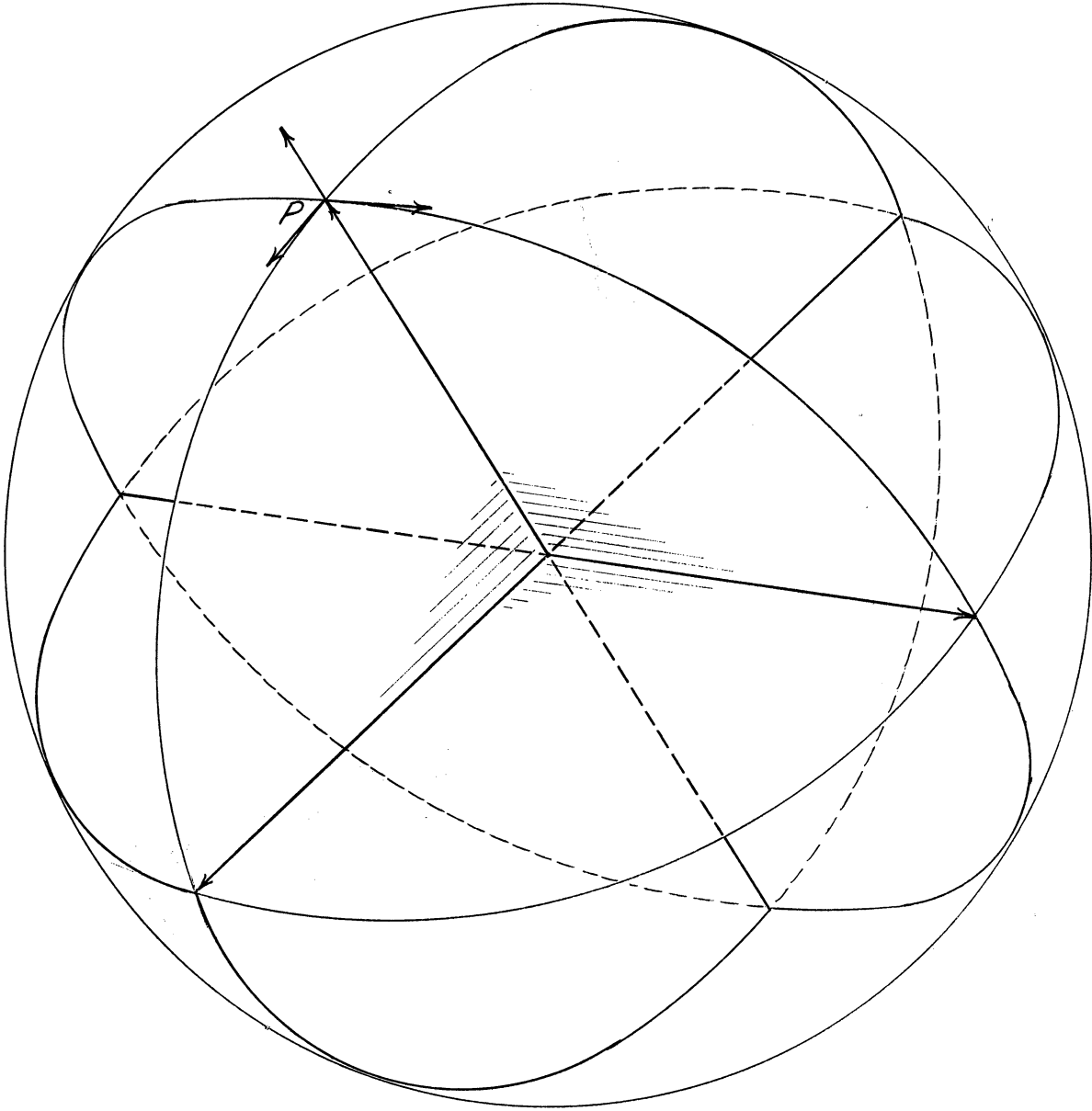


Figure IX Singular Points, Lines, and Surfaces for $N = 3$

called "machine-scaling"⁽¹⁾ is therefore highly convenient. This scale makes the unit of x equal to the unit used for multiplication, 100 volts on most electronic differential analyzers. Second, since the matrix elements are coefficients for the equation set (145), it will be convenient to represent these by coefficient potentiometers whose maximum setting is 1.0. This may be done consistently by multiplying all elements of the matrix by the same number. Choice of this number is tantamount to choice of a scale for the eigenvalues. This choice is aided by first drawing a computer model for equations (145) to show how this choice affects the set-up. Accordingly, Figure X shows the first computer setup employing equations (145). The particular form of the equations used was

$$\begin{aligned} \dot{x} &= y - \underline{x} y \quad x \\ y &= [A] x \end{aligned} \quad (145)$$

$$[A] = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \\ 2 & 3 & 4 \end{bmatrix} \quad (199)$$

The following points are clear from Figure X. Voltages corresponding to the individual components of

$$x$$

and

$$y = [A] x$$

are developed, as is the voltage corresponding to

$$\underline{x} y.$$

All of these voltages must be less than the maximum possible output voltages of the amplifiers. Scaling to make the voltages all less than the unit multiplier voltage (follow-up pot voltage) is in general allowable. Since

$$\sqrt{x_1^2 + x_2^2 + x_3^2} = (\underline{x} x)^{1/2} = 1.0 \quad (200)$$

(1) Korn, G. A., and Korn, T.M. Electronic Analog Computers, McGraw Hill Book Co., Inc., New York, 1952

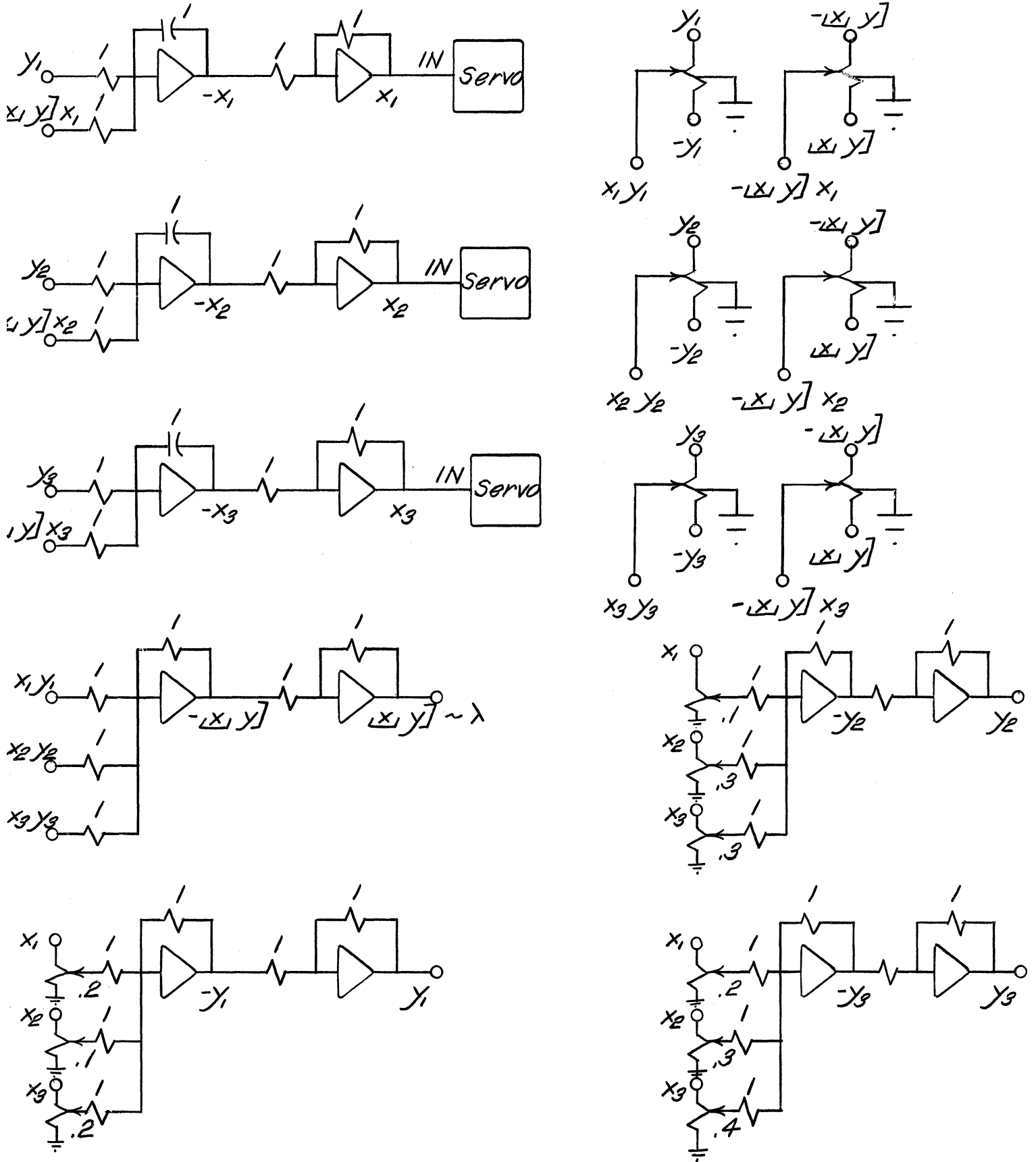


Figure X Computer Example Setup for Equations 145

in the steady state, the

$$x]$$

component voltages will be safely scaled. Again, in the steady-state,

$$y] = [A] x] = \lambda_{\max.} x] . \quad (201)$$

The

$$y]$$

component voltages can certainly be developed if

$$|\lambda_i| \leq 1.0, \quad i = 1, 2, \dots, N . \quad (202)$$

This condition may effectively be obtained by multiplying the matrix

$$[A]$$

by a scale factor k , such that

$$k |\lambda|_{\max.} \leq 1.0 \quad (203)$$

Then if

$$[A] x]_m = \lambda_{\max} x]_m , \quad (204)$$

$$k [A] x]_m = k \lambda_{\max.} x]_m = \lambda'_{\max} x]_m . \quad (205)$$

The factor k is thus a scale factor for the eigenvalues and does not affect the eigenvectors. A suitable value for k may be found from a theorem of Wilczynski, "The coefficients of a unique canonical form are invariants,"⁽¹⁾ and from a statement in Birkhoff and MacLane⁽²⁾

"One such invariant (under the group $[A] [P]^{-1} [A][P]$) is

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij} a_{ji} . "$$

(1) Wilczynski, Proc. Nat. Acad. Sci., 4, 1918, 300-5.

(2) Birkhoff, G. and MacLane, S., A Survey of Modern Algebra, The Macmillon Co., New York, 1958, p. 313

This invariant is a function of two of the coefficients of the characteristic function for

$$\begin{aligned} & [A] \quad . \quad \text{If} \\ & \left| [A] - \lambda [I] \right| = (-\lambda)^N + c_{N-1} (-\lambda)^{N-1} + \dots - c_1 \lambda + c_0 \end{aligned} \quad (206)$$

then

$$\begin{aligned} c_0 &= |A| \\ c_1 &= \sum_{i=1}^N |A_{ii}| \quad , \end{aligned} \quad (207)$$

where

$$|A_{ii}|$$

is the principal minor formed by crossing out row and column i from $|A|$, (208)

$$c_N = \sum_{i=1}^N a_{ii} \quad (209)$$

and

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij} a_{ji} = c_{N-1}^2 + (-1)^{N-1} 2c_{N-2} \quad (210)$$

Now, invariant (210) is obviously equal to the sum of the squares of the eigenvalues of

$$[A] ;$$

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij} a_{ji} = \sum_{i=1}^N |\lambda_i|^2 \quad (211)$$

The square root of this invariant must therefore be larger than any of the eigenvalue moduli;

$$\sqrt{\sum_{i=1}^N \sum_{j=1}^N a_{ij} a_{ji}} \geq |\lambda|_{\max} \quad , \quad (212)$$

The reciprocal of the square root (212) is therefore a satisfactory maximum value for the matrix multiplier, k . Letting

$$[A]' = \frac{1}{\sqrt{\sum_{i=1}^N \sum_{j=1}^N a_{ij} a_{ji}}} [A], \quad (213)$$

the eigenvalues, λ_i , of

$$[A]',$$

will obey the inequality

$$|\lambda_i| \leq 1.0 \quad (202)$$

and the eigenvectors of

$$[A]',$$

will be identical with those of

$$[A]$$

Furthermore, since

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij}^2 \geq \sum_{i=1}^N \sum_{j=1}^N a_{ij} a_{ji}, \quad (203)$$

a satisfactory scale for

$$[A]',$$

is

$$[A]' = \frac{1}{\sqrt{\sum_{i=1}^N \sum_{j=1}^N a_{ij}^2}} [A]. \quad (204)$$

With either of the two scales for

$$[A]',$$

$$[y_i \ y]_i = [x_i \ x]_i [A]_t [A] x_i = \lambda_i^2 \leq 1.0$$

in the steady state. Thus, the y_i voltages may be satisfactorily developed. Furthermore, with the above scales, and again in the steady state

$$\underline{x}_i \ y]_i = \lambda_i \underline{x}_i \ x]_i \leq 1.0 \quad (205)$$

and the $\underline{x}_i \ y]$ voltages may be formed by amplifier outputs. Finally, then, with

$$[A] = [A]_t, \quad (206)$$

the equations

$$\underline{x}] = y] - \underline{x} \ y] \ x] \quad (207)$$

$$y] = [A]^t \ x] \quad (208)$$

may be solved with an analog computer setup of the form shown in Figure X provided

$$[A]^t = k [A] \quad (209)$$

$$k \leq \frac{1}{\sqrt{\sum_{i=1}^N \sum_{j=1}^N a_{ij}^2}}. \quad (210)$$

For the problem shown in Figure X

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij}^2 = 57 \quad (211)$$

$$k = 0.1 \leq \frac{1}{\sqrt{57}} \sim 0.1325. \quad (212)$$

with this scale,

$$\begin{aligned} \lambda_1 &= .01944 \\ \lambda_2 &= .1387 \\ \lambda_3 &= .7418 \end{aligned} \quad (213)$$

approximately. The setup shown displays λ_3 and

$$\underline{x}]_3$$

as shown in Appendix II, Figure II.

Equations for Successive Steps

When the first eigenvalue and eigenvector have been successfully obtained using the intensification procedure, succeeding eigenvectors may be obtained by applying the original technique to a modified vector, the modification involving a computation using the previously obtained eigenvectors. Provided the matrix is symmetrical, all that is necessary is to subtract from the trial vector all the components of that vector along the previously obtained eigenvectors. Geometrically speaking, we know that the remaining eigenvectors must be orthogonal to those already found, consequently we look for these only in the (principal) subspace orthogonal to the eigenvectors already obtained. Trials of several analog variants of this procedure led to the conclusion that a straightforward formulation of the exact same technique is as stable and convenient as any of the easily formulated alternatives.

Accordingly, the eigenvector corresponding to the next largest member of the set of eigenvalues of a symmetric matrix is obtained from the formulae

$$\begin{bmatrix} 0 \\ \bar{x} \end{bmatrix} = y]' - \underline{x} y]' \quad x] \quad (214)$$

where

$$\begin{aligned} y]' &= y] - \underline{x}_N y] \quad x]_N \\ y] &= [A] x] \end{aligned} \quad (215)$$

and

$$\begin{aligned} \underline{x}_N x]_N &= 1.0 \\ [A] x]_N &= \lambda_N x]_N \end{aligned} \quad (216)$$

Now, since as shown in the development leading to equation (198), the eigenvectors of a symmetric matrix form a mutually orthogonal set, all the eigenvectors except

$$x]_N$$

lie in the principal subspace defined by

$$y]'$$

as given in equation (215). There are several means of implementing, on the computer, equations (214) and (215), one of the simplest of which follows from a result deduced in the following stability analysis of these equations.

The stability of equations (214) and (215) may be conveniently inferred from considering the equivalent set

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} y \\ -\underline{x} \end{bmatrix} \quad (214)$$

$$y' = [A]_1 x = [A] x - \underline{x}_N [A] x \quad (217)$$

This makes

$$[A]_1 = \left([I] - x \right)_N \underline{x}_N [A] \quad (218)$$

where

$$\underline{x}_N x \Big|_N = 1.0 \quad (219)$$

as may be verified by writing out equations (217). The formulation (217) makes the second step process identical with that for the first set except for the substitution of matrix

$$[A]_1$$

for

$$[A]$$

The stability properties of a setup corresponding to the second step equations follows immediately from the analysis of first step stability. The only remaining problem is then to obtain the eigenvalues of

$$[A]$$

If

$$[A] x \Big|_i = \lambda_i x \Big|_i$$

then

$$\begin{aligned} [A]_1 x \Big|_i &= [A] x \Big|_i - x \Big|_N \underline{x}_N [A] x \Big|_i \\ &= \lambda_i x \Big|_i - x \Big|_N \underline{x}_N \lambda_i x \Big|_i \end{aligned}$$

$$[A]_1 x]_i = \begin{cases} \lambda_i x]_i, & \lambda_i \neq \lambda_N \\ 0, & \lambda_i = \lambda_N \end{cases} \quad (220)$$

since

$$\underline{x}_N x]_i = 0, \quad \lambda_i \neq \lambda_N \quad (221)$$

and

$$\underline{x}_N x]_i = 1, 0, \lambda_i = \lambda_N. \quad (222)$$

The matrix

$$[A]_1$$

thus has the same eigenvectors as

$$[A]$$

It has also the same eigenvalues except that corresponding to

$$x]_N$$

it has the eigenvalue

$$\lambda'_N = 0. \quad (223)$$

The set (214) (215) (216) has then exactly two stable singular points

$$x] = x]_{N-1} \quad (224)$$

where

$$\begin{aligned} \underline{x}_{N-1} x]_{N-1} &= 1, 0 \\ [A] x]_{N-1} &= \lambda_{N-1} x]_{N-1} \\ \lambda_{N-1} &> \lambda_{N-2} > \dots > \lambda_2 > \lambda_1 \end{aligned} \quad (225)$$

provided only that

$$\lambda_{N-1} > 0. \quad (226)$$

The principal directions of the linearized system in the neighborhood of these singular points lie along the eigenvectors

$$x]_i, \quad i = 1, 2, \dots, N$$

where

$$[A] x]_i = \lambda_i x]_i \quad (227)$$

with characteristic roots of the linearized system

$$\lambda_1 - \lambda_{N-1}, \lambda_2 - \lambda_{N-1}, \dots, \lambda_{N-2} - \lambda_{N-1}, -2\lambda_{N-1}, -\lambda_{N-1}$$

respectively. The principal subspace boundaries are the same as for the first step. The stability characteristics are altered by substituting zero for eigenvalue λ_N . It may be noted that the singular points

$$\begin{aligned} & x]_N, \\ & [A] x]_N = \lambda_N x]_N \\ & \underline{x}_N x]_N = 1.0 \end{aligned} \quad (228)$$

which were stable for the first step are now unstable with linearized system characteristic roots

$$\lambda_1, \lambda_2, \dots, \lambda_{N-1}, 0.$$

The zero root appears troublesome, since it applies in any neighborhood of the line

$$x]_N$$

independent of location along that line. This appearance proves practically deceptive since the remaining characteristic roots in such a neighborhood normally includes at least one strongly unstable root.

Equations (214)-(216) may be mechanized in any of several ways. A straightforward technique would be to use a setup like Figure X with summers added to form components of

$$y]',$$

as shown in Figure XI

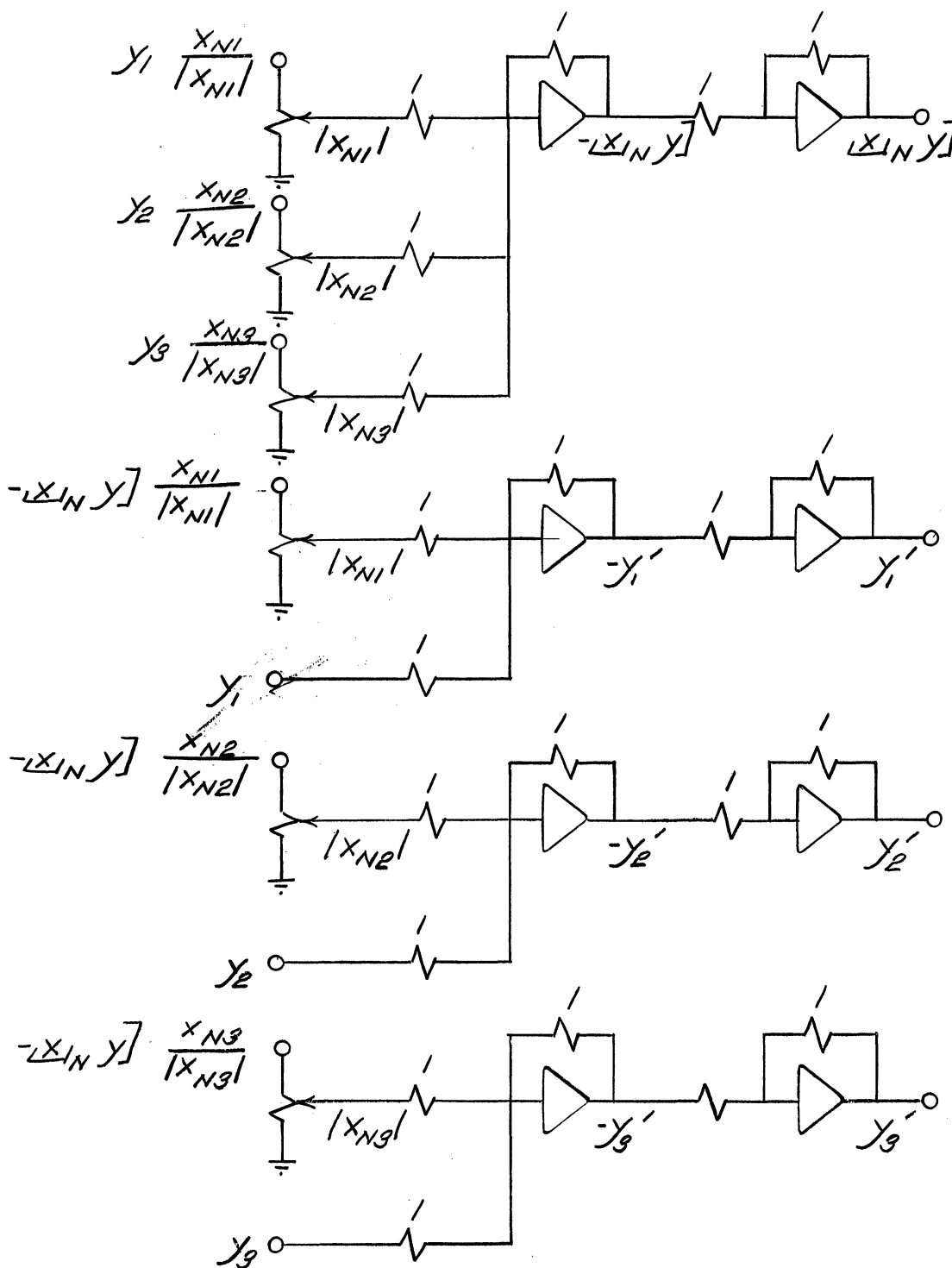


Figure XI Computer Example Setup for Equations for Second Step

This adds to the original system $2N$ potentiometers and $(2N+2)$ summers. If sufficient equipment is available, the entire circuit of Figure X could be duplicated and the additional pots needed to compute

$$y]' ,$$

obtained as multiplier pots from the part of the setup used to obtain λ_n and

$$x]_N .$$

If only a limited amount of equipment is available, the setup used to obtain λ_n could be modified to obtain λ_{n-1} and

$$x]_{N-1}$$

by resetting the coefficient pots from the elements of

$$[A]$$

to elements of

$$[A]' = ([I] - x]_N \underline{x}_N) [A] . \quad (229)$$

Succeeding eigenvalues λ_i and corresponding eigenvectors

$$x]_i$$

may be found as steady-state solutions of equations

$$\underline{0} = y]' - \underline{x} y]' x] \quad (230)$$

where

$$y]' = y] - \underline{x}_N y] x]_N - \underline{x}_{N-1} y] x]_{N-1} - \dots - \underline{x}_{i+1} y] x]_{i+1} \quad (231)$$

$$y] = [A] x] \quad (232)$$

which is equivalent to

$$y]' = [A]_i x] \quad (233)$$

where

$$[A]_i = ([I] - x]_N \underline{x}_N - x]_{N-1} \underline{x}_{N-1} - \dots - x]_{i+1} \underline{x}_{i+1}) [A] \quad (234)$$

These equations have exactly two stable singular points at

$$\begin{aligned}
 \underline{x}]_i &= x]_i \\
 [A] x]_i &= \lambda_i x]_i \\
 \underline{x}]_i x]_i &= 1.0
 \end{aligned}
 \tag{235}$$

provided

$$\lambda_i > 0 \tag{236}$$

where

$$\lambda_i > \lambda_{i-1} > \dots > \lambda_2 > \lambda_1 \tag{237}$$

For

$$[A]$$

symmetric, all of the eigenvalues may be obtained from the described procedure provided

$$\lambda_N > \lambda_{N-1} > \dots > \lambda_2 > \lambda_1 > 0. \tag{238}$$

If, on the other hand,

$$\lambda_{i-1} > 0 > \lambda_i, \tag{239}$$

the procedure will fail at the i th step. The remaining i eigenvalues may be obtained by applying the same process to the matrix

$$- [A]$$

which has the same eigenvectors as

$$[A]$$

and whose eigenvalues are the negatives of those of

$$[A].$$

Succinctly, the process works without modification for positive definite matrices and works with minor modification for symmetric indefinite matrices.

If only the moduli of the eigenvalues of an indefinite symmetric matrix be wanted, the reduction process may be applied to the matrix

$$[A]^2 = [A][A] \tag{240}$$

which has the same eigenvectors as

$$[A]$$

but whose eigenvalues are the squares of those for the corresponding eigenvectors in

$$[A].$$

This is seen to be the case since if

$$[A] x]_i = \lambda_i x]_i, \quad (241)$$

then

$$[A]^2 x]_i = \lambda_i [A] x]_i = \lambda_i^2 x]_i \quad (242)$$

and, in fact,

$$[A]^n x]_i = \lambda_i^n x]_i \quad (243)$$

For symmetric matrices λ_i is always real and hence, $\lambda_i^2 \geq 0$ (244)

The above statement offers incidentally a clear picture of the Cayley-Hamilton theorem from an operational point of view, for, if

$$[A] x]_i = \lambda_i x]_i, \quad (245)$$

where

$$\left| [A] - \lambda_i [I] \right| = F(\lambda_i) = 0, \quad i = 1, 2, \dots, N, \quad (246)$$

then for an arbitrary

$$x] = a_1 x]_1 + a_2 x]_2 + \dots + a_N x]_N, \quad (247)$$

$$\begin{aligned} F([A]) x] &= a_1 F([A]) x]_1 + a_2 F([A]) x]_2 + \dots \\ &+ a_N F([A]) x]_N \\ &= a_1 F(\lambda_1) x]_1 + a_2 F(\lambda_2) x]_2 + \dots + a_N F(\lambda_N) x]_N \\ &= 0. \end{aligned} \quad (248)$$

Thus since

$$x]$$

was arbitrary, it follows that

$$F([A]) = 0; \quad (249)$$

i.e., the matrix satisfies its own characteristic equation.

The suggested modification of the

$$[A]$$

matrix has no bearing on the amount of equipment required for the diagonalization of the matrix. The most economical technique with respect to equipment requirements is that of developing the eigenvectors

$$x]_i$$

and eigenvalues λ_i serially, resetting the matrix element coefficient pots each time to elements of the modified matrix

$$[A]_i = ([I] - x]_1 \quad \underline{x}_1 \quad - \quad x]_2 \quad \underline{x}_2 \quad \dots \quad x]_{i-1} \quad \underline{x}_{i-1}) [A] \quad (250)$$

The equipment required is

N integrating operational amplifiers
 3N + 2 summing " "
 N servomultipliers each with 2 multiplying pots
 N² coefficient pots.

$$x]_i$$

eigenvector components are displayed as amplifier output voltages and also as servo-multiplier dial readings. Eigenvalues, λ_i , are displayed as summing operational amplifier output voltages

$$\underline{x}_i \quad y]_i = \lambda_i \quad \underline{x}_i \quad x]_i = \lambda_i, \quad (251)$$

for

$$[A] \quad x]_i = \lambda_i \quad x]_i \quad (252)$$

and

$$\underline{x}_i \quad x]_i = 1.0 \quad (253)$$

Saving the computation of successive

$$[A]_i$$

matrices necessitates adding the circuit of Figure XI which increases the above list to

N integrating operation amplifiers
 N³ + 2N + 2 summing operational amplifiers
 N servomultipliers each with 2 multiplying pots
 N²(N+1)/2 coefficient potentiometers.

The number of summing amplifiers required is excessive even for only moderately large N . For this reason, the minimum equipment procedure was used for all experimental studies. The even more elaborate technique of solving for all eigenvalues and eigenvectors simultaneously requires

N^2 integrating operational amplifiers
 $6N^2$ - N summing operational amplifiers
 N^2 servomultipliers, N each with $s(N-i)$ multiplying pots
 N^3 coefficient potentiometers.

This procedure would involve setting up all N^2 equations of the set (230) (231) (232) simultaneously. This requires modifying the subscript notation of the equations to identify the different groups of N equations. Accordingly, (230) (231) (232) become

$$\overset{0}{x}]_{N-i} = y]_{N-i}' - \frac{x_{N-i}}{y}]_{N-i} y]_{N-i}' x]_{N-i}, \quad i = 0, 1, 2, \dots, N-1 \quad (254)$$

$$y]_{N-i}' = y]_{N-i} - \sum_{j=0}^{i-1} \frac{x_{N-j}}{y}]_{N-i} y]_{N-i}' x]_{N-j} \quad (255)$$

$$y]_{N-i} = [A] x]_{N-i} \quad (256)$$

where

$$x]_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{Ni} \end{bmatrix} \quad (257)$$

The notation differs from that used previously in that the subscripts identify variables rather than solution constants. A double subscript notation is used with the first subscript denoting the number of the elements in a vector, the second denoting the number of the vector. The steady state values of the elements, x_{ij} , so numbered are the elements of the modal matrix.

$$[L] = [x_{ij}] \quad (258)$$

where

$$[L]^{-1} [A][L] = \begin{bmatrix} \lambda_1 & 0 & 0 \dots 0 \\ 0 & \lambda_2 & 0 \dots 0 \\ 0 & 0 & \lambda_3 \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \dots \lambda_N \end{bmatrix} \quad (259)$$

The equipment requirements for solving equations (254), (255), (256) are as follows:

- (254) requires N^2 summing integrators
 $(N^2 + 2N)$ summers
 N^2 servomultipliers -- N each with
 $(2N-2i)$ multiplying pots, $i=0,1,2,\dots,N-1$.
- (255) requires $(3N^2 - N)$ summers
- (256) requires $(2N^2 - 2N)$ summers
 N^3 coefficient pots.

A circuit for the example of Figure X is shown as Figures XII through XVI. The complete equations for the circuit are written out below. Note from the form of the circuit that there is no new question of stability raised by the circuit since the preceding stability analysis applies completely.

In Figure XII, the solution for λ_3 and

$$x]_3$$

proceeds as before with no new inputs to this part of the circuit. Once this part of the circuit is near steady-state, the multiplying pots act in the same fashion as fixed coefficient pots to make the behavior of the part of the circuit solving for λ_2 and

$$x]_2$$

the same as in the serial procedure, etc. Note also that the circuit is very large for so modest a problem and that the added computation necessary in the simple serial procedure seems an attractive exchange for the added circuit complexity.

The equations for the circuit are

$$\begin{aligned} \ddot{x}_{13} &= y'_{13} - \underline{x}_3 y]_3' x_{13} && 3 \text{ summing integrators} \\ \ddot{x}_{23} &= y'_{23} - \underline{x}_3 y]_3' x_{23} && 3 \text{ summers} \\ \ddot{x}_{33} &= y'_{33} - \underline{x}_3 y]_3' x_{33} && 3 \text{ multiplier} \end{aligned}$$

$$\underline{x}_3 y]_3' = x_{13} y'_{13} + x_{23} y'_{23} + x_{33} y'_{33} \quad 2 \text{ summers}$$

$$y'_{13} = 0.2x_{13} + 0.1x_{23} + 0.2x_{33}$$

$$y'_{23} = 0.1x_{13} + 0.3x_{23} + 0.3x_{33} \quad 6 \text{ summers}$$

$$y'_{33} = 0.2x_{13} + 0.3x_{23} + 0.4x_{33} \quad 9 \text{ coefficient pots}$$

$$\overset{\circ}{x}_{12} = y'_{12} \underline{-x_2} y]'_2 x_{12} \quad 3 \text{ summing integrators}$$

$$\overset{\circ}{x}_{22} = y'_{22} \underline{-x_2} y]'_2 x_{22} \quad 3 \text{ summers}$$

$$\overset{\circ}{x}_{32} = y'_{32} \underline{-x_2} y]'_2 x_{32} \quad 3 \text{ multipliers}$$

$$\underline{-x_2} y]'_2 = x_{12} y'_{12} + x_{22} y'_{22} + x_{32} y'_{32} \quad 2 \text{ summers}$$

$$y'_{12} = y_{12} \underline{-x_3} y]'_2 x_{13}$$

$$y'_{22} = y_{22} \underline{-x_3} y]'_2 x_{23} \quad 6 \text{ summers}$$

$$y'_{32} = y_{32} \underline{-x_3} y]'_2 x_{33}$$

$$\underline{-x_3} y]'_2 = x_{13} y_{12} + x_{23} y_{22} + x_{33} y_{32} \quad 2 \text{ summers}$$

$$y_{12} = 0.2x_{12} + 0.1x_{22} + 0.2x_{32} \quad 6 \text{ summers}$$

$$y_{22} = 0.1x_{12} + 0.3x_{22} + 0.3x_{32}$$

$$y_{32} = 0.2x_{12} + 0.3x_{22} + 0.4x_{32} \quad 9 \text{ coefficient pots}$$

$$\overset{\circ}{x}_{11} = y'_{11} \underline{-x_1} y]'_1 x_{11} \quad 3 \text{ summing integrators}$$

$$\overset{\circ}{x}_{21} = y'_{21} \underline{-x_1} y]'_1 x_{21} \quad 3 \text{ summers}$$

$$\overset{\circ}{x}_{31} = y'_{31} \underline{-x_1} y]'_1 x_{31} \quad 3 \text{ multipliers}$$

$$\underline{-x_1} y]'_1 = x_{11} y'_{11} - x_{21} y'_{21} - x_{31} y'_{31} \quad 2 \text{ summers}$$

$$y'_{11} = y_{11} \underline{-x_3} y]'_1 x_{13} \underline{-x_2} y]'_1 x_{12}$$

$$y'_{21} = y_{21} \underline{-x_3} y]'_1 x_{23} \underline{-x_2} y]'_1 x_{22} \quad 6 \text{ summers}$$

$$y'_{31} = y_{31} \underline{-x_3} y]'_1 x_{33} \underline{-x_2} y]'_1 x_{32}$$

$$\underline{x}_3 y]_1 = x_{13} y_{11} + x_{23} y_{21} + x_{33} y_{31} \quad 2 \text{ summers}$$

$$\underline{x}_2 y]_1 = x_{12} y_{11} + x_{22} y_{21} + x_{32} y_{31} \quad 2 \text{ summers}$$

$$y_{11} = 0.2x_{11} + 0.1x_{21} + 0.2x_{31} \quad 6 \text{ summers}$$

$$y_{21} = 0.1x_{11} + 0.3x_{21} + 0.3x_{31}$$

$$y_{31} = 0.2x_{11} + 0.3x_{21} + 0.4x_{31} \quad 9 \text{ coefficient pots}$$

9 summing integrators

51 summers

9 servos

27 coefficient pots

$\sum =$

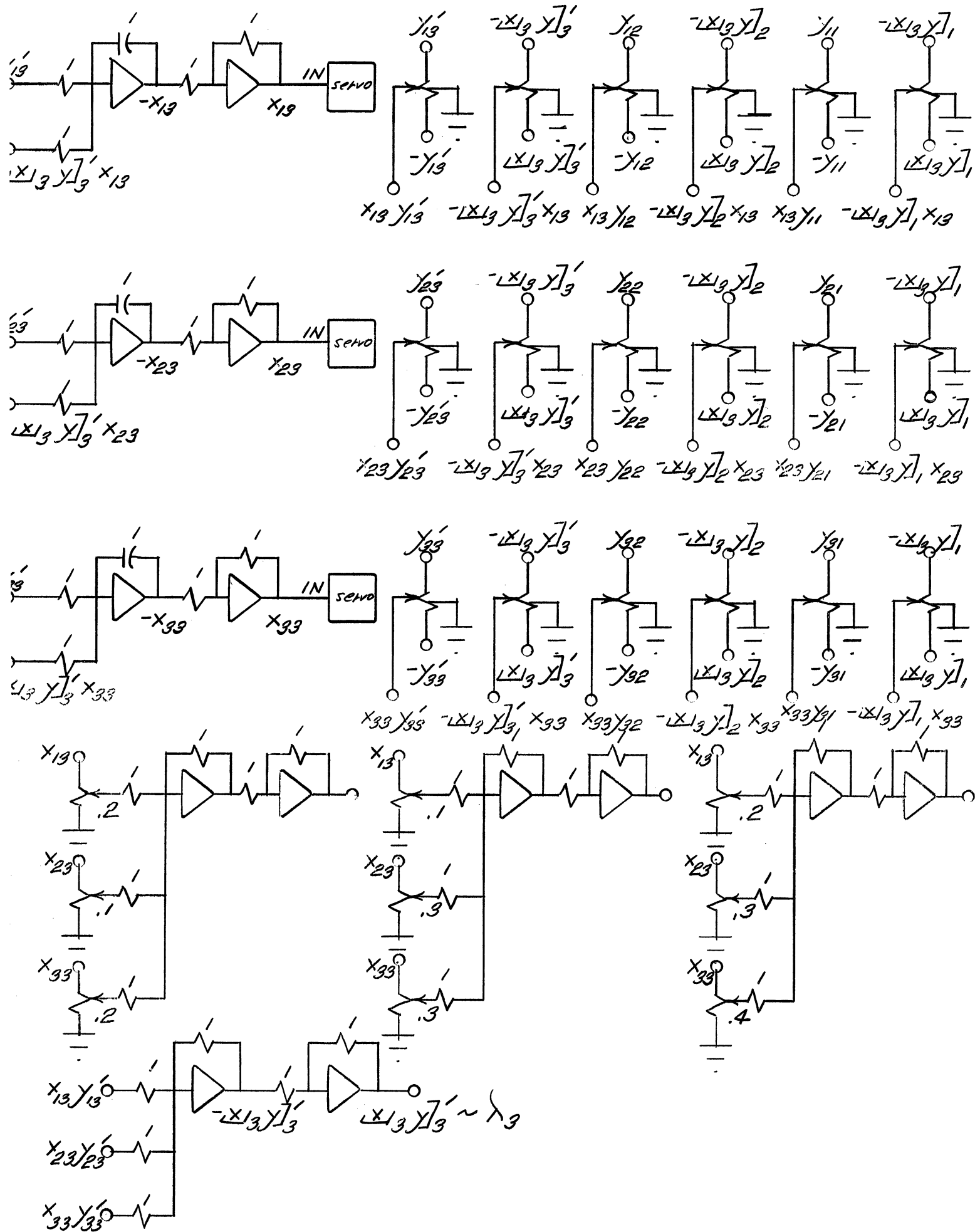


Figure XII One-Shot Circuit - Sheet 1

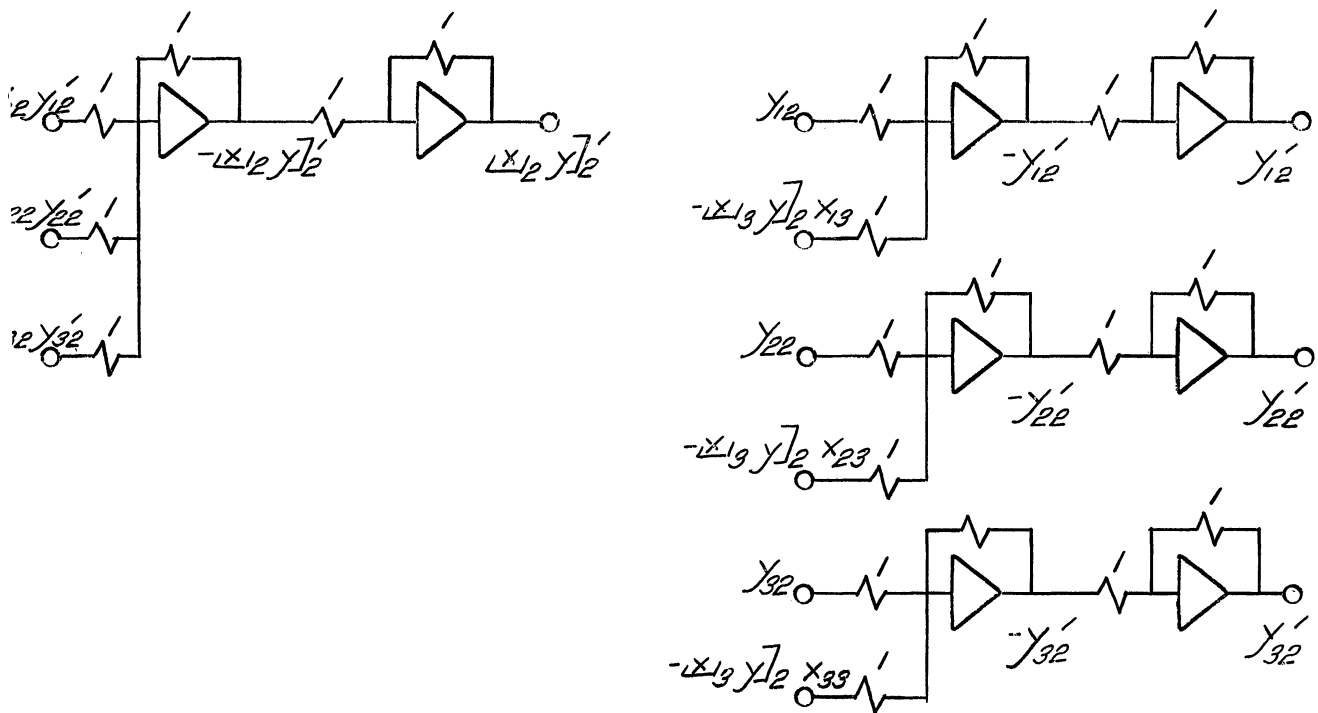
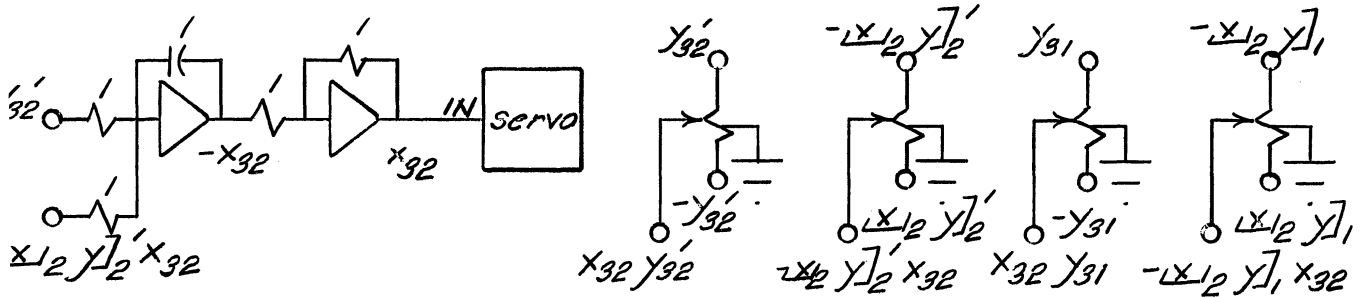
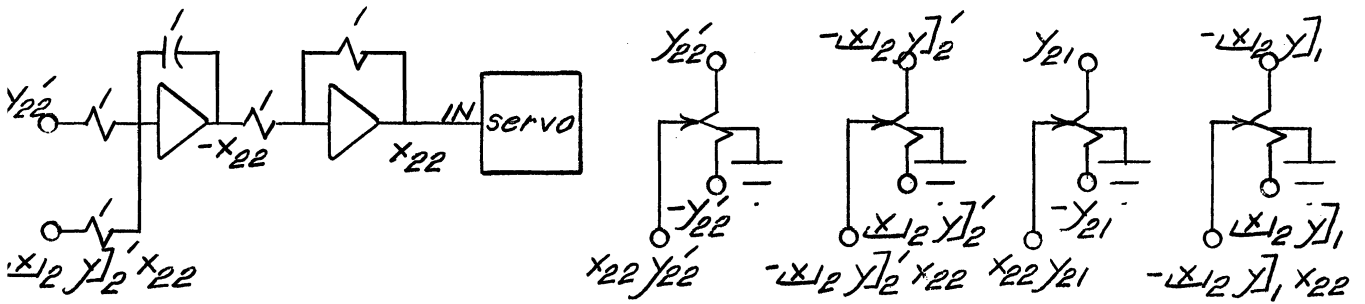
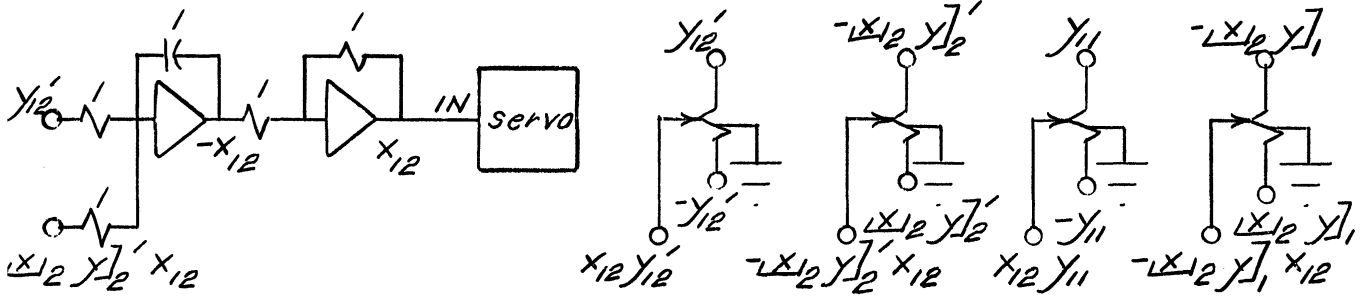


Figure XIII One-Shot Circuit - Sheet 2

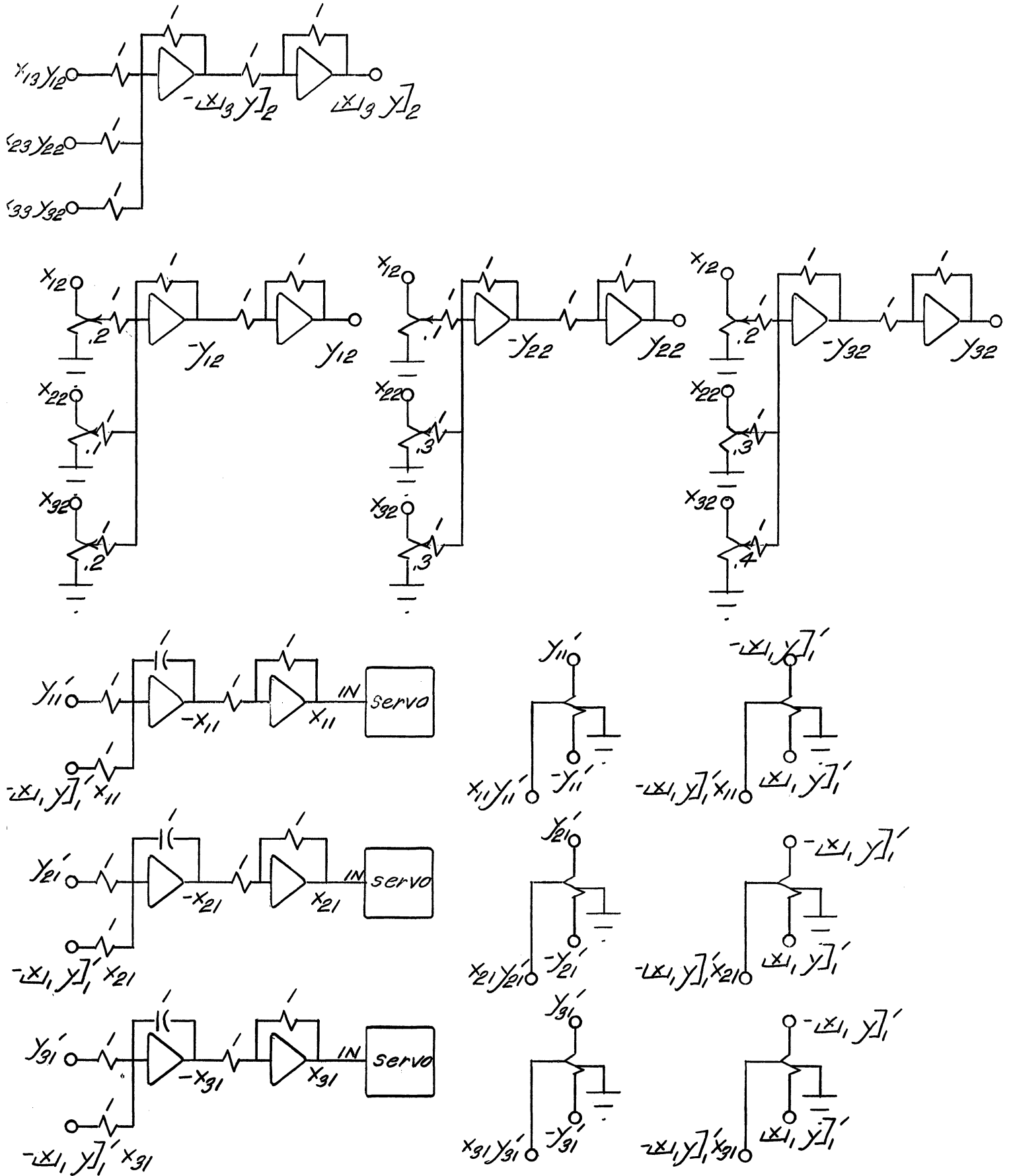


Figure XIV One-Shot Circuit - Sheet 3

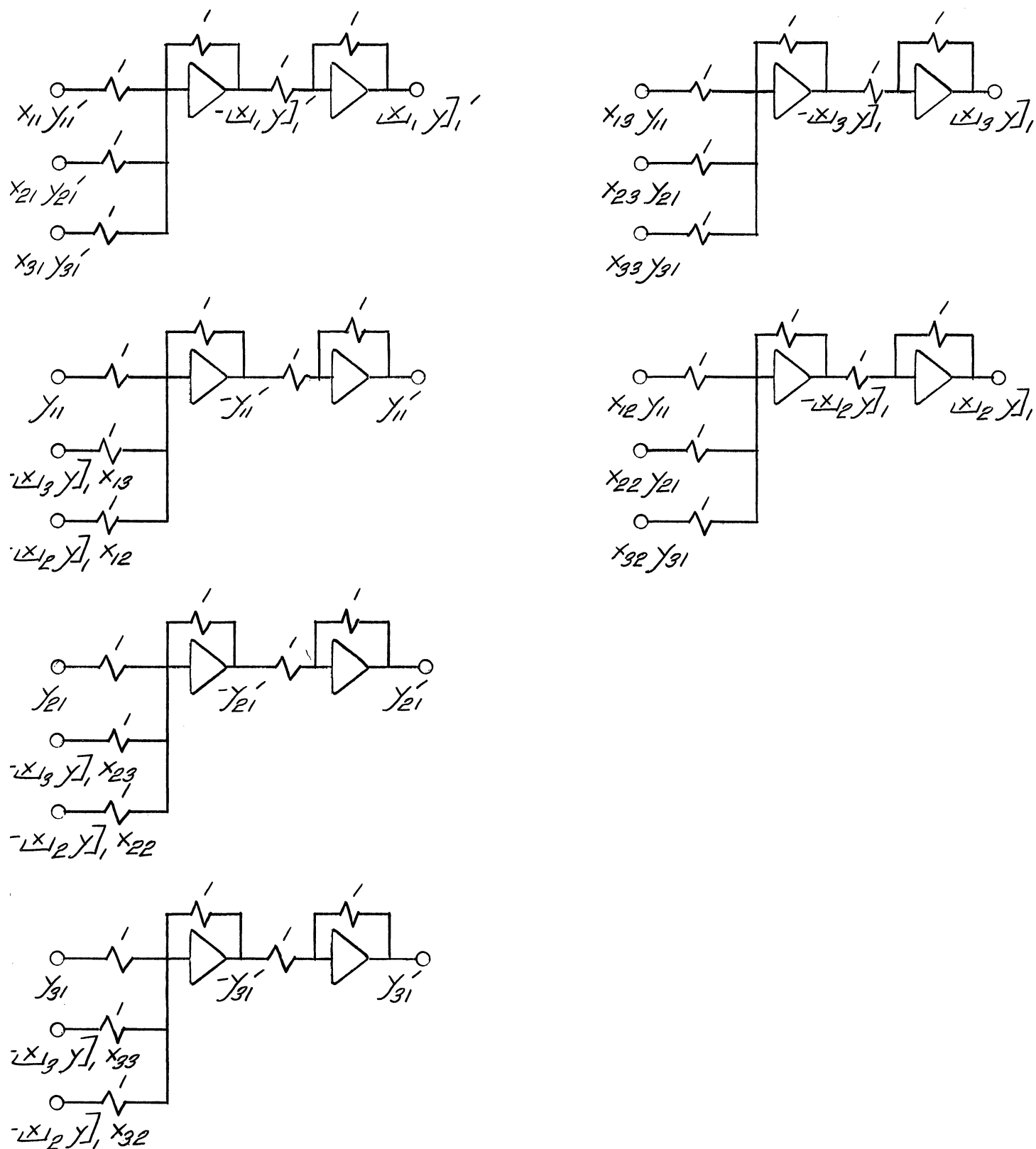


Figure XV One-Shot Circuit - Sheet 4

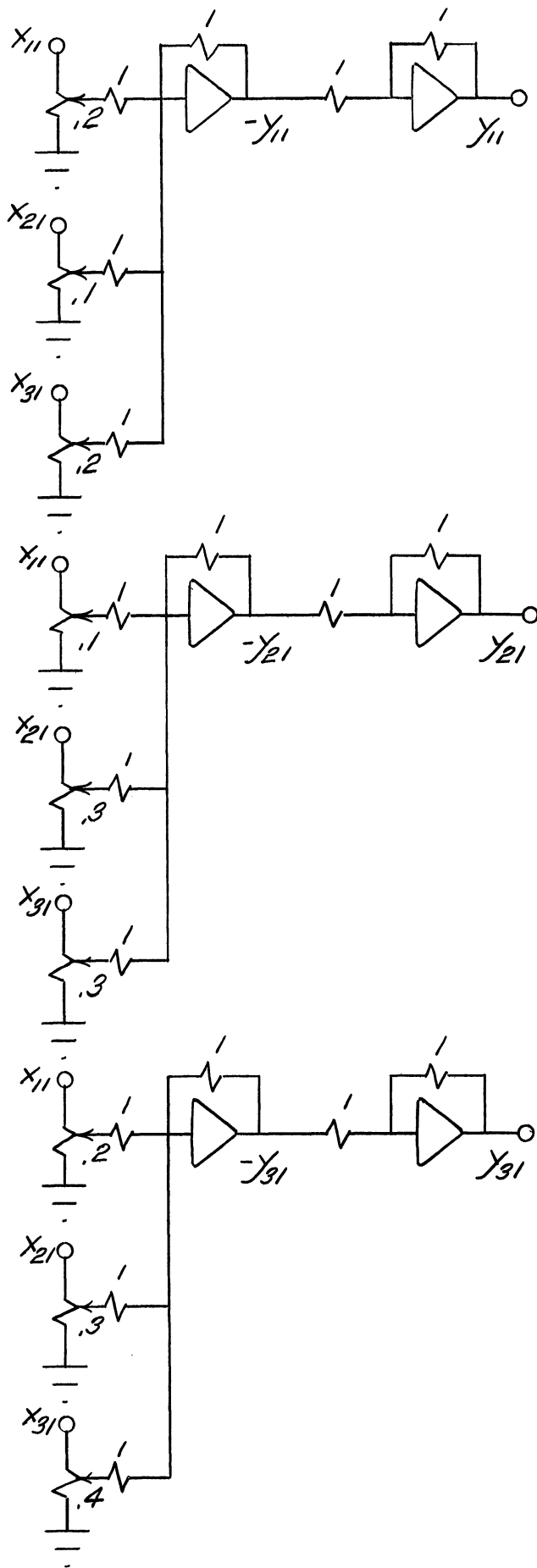


Figure XVI One-Shot Circuit - Sheet 5

Modification of the Technique for Asymmetric Matrices

Real symmetric matrices have real eigenvalues and real orthogonal eigenvectors. Asymmetric matrices have, in general, complex eigenvalues and complex eigenvectors. Since the diagonalization technique already described deals only with real quantities and relies for completion of the diagonalization process on the orthogonal property of the eigenvectors, considerable modification is necessary to adapt the process to the asymmetric case.

The case of an asymmetric matrix with real eigenvalues only requires the least modification. For this case, the equations for the first step

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ y \end{bmatrix} &= \begin{bmatrix} y \\ -x \end{bmatrix} + \begin{bmatrix} y \\ x \end{bmatrix} \\ y &= [A] x \end{aligned} \quad (260)$$

apply without modification. Furthermore, the stability analysis given previously for the first step still applies, since the condition of symmetry was nowhere necessary in that analysis. The system possesses $2N+1$ finite singular points at

$$x = 0 \quad (261)$$

and at

$$x = x_i \quad (262)$$

where

$$[A] x_i = \lambda_i x_i \quad (263)$$

and

$$\begin{bmatrix} x_i \\ x_i \end{bmatrix} = 1.0 \quad (264)$$

The linearized system in the neighborhood of the singular point at the origin has characteristic roots

$$r_i = \lambda_i, \quad i = 1, 2, \dots, N$$

The singular points corresponding to eigenvalue

$$\lambda = \lambda_i$$

have linearized characteristic roots

$$r = \lambda_1 - \lambda_i, \lambda_2 - \lambda_i, \dots, \lambda_{i-1} - \lambda_i, -2\lambda_i, \lambda_{i+1} - \lambda_i, \dots, \lambda_N - \lambda_i. \quad (267)$$

Two of these singular points at

$$x] = x]_N \quad (268)$$

where

$$[A] x]_N = \lambda_N x]_N \quad (269)$$

and

$$\lambda_N > \lambda_{N-1} > \dots > \lambda_1, \quad (270)$$

are completely stable provided

$$\lambda_N > 0. \quad (271)$$

So far the situation is exactly the same as for the symmetric case. However, an attempt to apply the equations for the second step

$$\begin{aligned} \overset{0}{x}] &= y] - \underline{x} y] x] \\ y]' &= y] - \underline{x}_N y] x]_N \\ y] &= [A] x] \end{aligned} \quad (272)$$

fails to produce the eigenvector

$$x]_{N-1}$$

since, in general

$$\underline{x}_N x]_{N-1} \neq 0.$$

A modified process may be developed using the results of the following section.

Principal Subspaces of $[A]$ and $[A]_t$

What is required for the succeeding steps of the process for asymmetric matrices is a vector,

$$x]_N' \neq x]_N$$

such that

$$\underline{x}_i \cdot x]_N' = 0, \quad i \neq N \quad (274)$$

Note first that

$$x]_N'$$

will in general be confined to one dimension, since the space spanned by

$$x]_i, \quad i \neq N$$

will be, in general, (N-1) -dimensional. This is to say

$$x]_N'$$

is a unique vector. Thus

$$\begin{aligned} 0 &= \lambda_i \underline{x}_i \cdot x]_N' = ([A] \cdot x]_i)_t \cdot x]_N' \\ &= \underline{x}_i \cdot [A]_t \cdot x]_N' \end{aligned} \quad (275)$$

whence

$$[A]_t \cdot x]_N' = k \cdot x]_N' = \lambda_N \cdot x]_N' \quad (276)$$

which is to say

$$x]_N'$$

is the eigenvector of

$$[A]_t$$

corresponding to eigenvalue

$$\lambda = \lambda_N \quad (277)$$

Briefly then, given a real eigenvalue

$$\lambda = \lambda_N \quad (278)$$

of the transpose matrix

$$[A]_t$$

the eigenvector

$$x]_{Nt}$$

corresponding to this eigenvalue is orthogonal to the principal subspace of

$$[A]$$

not containing the eigenvector of

$$[A]$$

corresponding to the root $\lambda = \lambda_N$.

The second step of the asymmetric matrix reduction procedure follows immediately as

$$\begin{bmatrix} 0 \\ \underline{x} \end{bmatrix} = \begin{bmatrix} y \\ -\underline{x} \end{bmatrix}' - \begin{bmatrix} y \\ \underline{x} \end{bmatrix} \quad (279)$$

where

$$\begin{aligned} \begin{bmatrix} y \\ \underline{x} \end{bmatrix}' &= \begin{bmatrix} y \\ -\underline{x} \end{bmatrix}_{Nt} \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{Nt} \\ \begin{bmatrix} A \\ \underline{x} \end{bmatrix} &= \begin{bmatrix} y \\ \underline{x} \end{bmatrix} \\ \begin{bmatrix} A \\ \underline{x} \end{bmatrix}_{Nt} &= \lambda_N \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{Nt} \end{aligned} \quad (280)$$

and

$$\underline{x}_{Nt} \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{Nt} = 1.0 \quad (281)$$

The equations for the i^{th} step are then

$$\begin{bmatrix} 0 \\ \underline{x} \end{bmatrix} = \begin{bmatrix} y \\ -\underline{x} \end{bmatrix}' - \begin{bmatrix} y \\ \underline{x} \end{bmatrix} \quad (282)$$

$$\begin{aligned} \begin{bmatrix} y \\ \underline{x} \end{bmatrix}' &= \begin{bmatrix} y \\ -\underline{x} \end{bmatrix}_{N,t} \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{N,t} - \begin{bmatrix} y \\ -\underline{x} \end{bmatrix}_{N-1,t} \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{N-1,t} - \dots \\ &\quad - \begin{bmatrix} y \\ -\underline{x} \end{bmatrix}_{N-i+1,t} \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{N-i+1,t} \end{aligned} \quad (283)$$

$$\begin{bmatrix} y \\ \underline{x} \end{bmatrix} = \begin{bmatrix} A \\ \underline{x} \end{bmatrix} \quad (284)$$

$$\begin{bmatrix} A \\ \underline{x} \end{bmatrix}_j = \lambda_j \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_j, \quad j = 1, 2, \dots, i-1, \quad (285)$$

$$\underline{x}_j \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_j = 1.0. \quad (286)$$

Equation (283) may be rewritten

$$\begin{bmatrix} y \\ \underline{x} \end{bmatrix}' = \begin{bmatrix} A \\ \underline{x} \end{bmatrix}_i \begin{bmatrix} y \\ \underline{x} \end{bmatrix} \quad (287)$$

$$\begin{aligned} \begin{bmatrix} A \\ \underline{x} \end{bmatrix}_i &= \left(\begin{bmatrix} I \\ -\underline{x} \end{bmatrix}_{Nt} \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{Nt} - \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{N-1,t} \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{N-1,t} - \dots \right. \\ &\quad \left. - \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{N-i+1,t} \begin{bmatrix} y \\ \underline{x} \end{bmatrix}_{N-i+1,t} \right) \begin{bmatrix} A \\ \underline{x} \end{bmatrix} \end{aligned} \quad (288)$$

The eigenvectors of

$$\begin{bmatrix} A \\ \underline{x} \end{bmatrix}_i$$

are identical with those for

$$\begin{bmatrix} A \\ \underline{x} \end{bmatrix}$$

but the eigenvalues

$$\lambda_j, j > 1$$

are zero for

$$[A]_i$$

rather than the original values.

The discussion of principal directions and characteristic roots in the neighborhood of singular points not at the origin was kept sufficiently general in the section on symmetric matrices to cover the present situation. Symmetry was not a requirement for the results then derived although realness of the matrix eigenvalues was required. If any of the matrix eigenvalues are complex, as they may be for real asymmetric matrices, then the stability of the singular points for equations (282)-(286) must be reinvestigated.

Complex Roots

First of all, equations (267), for the characteristic roots of the linearized system near a singular point,

$$r = \lambda_1 - \lambda_i, \lambda_2 - \lambda_i, \dots, \lambda_{i-1} - \lambda_i, -2\lambda_i, \lambda_{i+1} - \lambda_i, \dots, \lambda_N - \lambda_i, \quad (267)$$

still apply for λ_i real whether or not any of the other characteristic roots are complex. Thus the equations for the first step will have two completely stable singular points if

$$\begin{aligned} \lambda_N > \operatorname{Re} \lambda_j, \quad j = 1, 2, \dots, N-1 \\ \lambda_N > 0. \end{aligned} \quad (289)$$

If

$$\prod_m \lambda_k \neq 0$$

then there is no eigenvector corresponding to λ_k and its conjugate. There is, however, a principal plane corresponding to λ_k with the property that any vector in this plane transforms into another vector in the plane and that no vector in the plane transforms into itself under the transformation

$$y] = [A] x] . \quad (291)$$

To investigate the behavior of the system of equations (282)-(286), it is necessary to consider the properties of principal 2-dimensional subspaces of the matrix

$$[A] .$$

If, again, λ_k be a complex eigenvalue of

$$[A],$$

then a direct application of classic techniques for obtaining eigenvectors yields a corresponding complex eigenvector

$$x]_k = x]_{kr} + j x]_{ki} \quad (292)$$

where

$$j = \sqrt{-1}. \quad (293)$$

It is easily shown that the vectors

$$x]_{kr}$$

and

$$x]_{ki}$$

define a principal plane of the transformation

$$y] = [A] x] \quad (294)$$

since if

$$[A] x]_k = \lambda_k x]_k \quad (295)$$

then

$$[A] x]_{kr} = \lambda_{kr} x]_{kr} - \lambda_{ki} x]_{ki} = y]_{kr} \quad (296)$$

and

$$[A] x]_{ki} = \lambda_{kr} x]_{ki} + \lambda_{ki} x]_{kr} = y]_{ki} \quad (297)$$

which is to say, the transforms of the real and imaginary parts of a complex eigenvector both lie in the plane defined by the real and imaginary parts of that vector. Further more, there is no unique correspondence between eigenvalue

λ_k and any particular

$$x]_k$$

since if

$$x]_k$$

is a complex eigenvector, then so is

$$\lambda_k^j x]_k, \quad j = 1, 2, 3, \dots$$

Unless λ_k is a root of unity, the above set will be infinite.

Of course, the fact that a complex eigenvalue λ_k defines a principal plane is not a peculiarity of complex eigenvalues. There are $(N-1)$ principal planes through each real eigenvector of a set of N real eigenvectors. Each pair of complex conjugate eigenvalues likewise defines a principal plane although $(2N-2)$ of the $N(N-1)/2$ principal planes that would exist if all the eigenvalues were real, do not exist if the eigenvalues are instead complex. Thus, if one pictures a transition from real to complex eigenvectors, there does not occur an exchange of a principal plane for two principal vectors but rather merely a loss of $(2N-2)$ principal planes along with the two principal vectors. In the set of non-linear differential equations designed to produce eigenvectors as steady-state solution, one vector of the complex set of N vectors might be called a dominant plane since any velocity vector in the space would have a component directed toward this vector. The dominant vector in the space would have a component directed toward this vector. The dominant vector is thus the one having the largest real value. Similarly, the plane of the two eigenvalues could be called a dominant plane since every velocity vector in the space would have a component directed toward the plane, at least in the neighborhood of the plane. Furthermore, there will exist a dominant plane even if this plane is associated with complex conjugate eigenvalues rather than with a pair of real eigenvalues. This will occur if

$$\begin{aligned} \operatorname{Re} \lambda_k &> \operatorname{Re} \lambda_i, \quad i \neq k \\ \operatorname{Re} \lambda_k &> 0. \end{aligned} \tag{298}$$

This may be shown as follows:

If all the eigenvalues

$$\lambda_i, \quad i = 1, 2, \dots, N,$$

are real with the exception of the pair

$$\lambda_k$$

and its conjugate

$$\overline{\lambda_k}$$

then the singular points at

$$x] = x]_i, \quad i \neq k$$

$$\underline{x}_i \quad x]_i = 1.0 \quad (299)$$

where

$$[A] x]_i = \lambda_i \quad x]_i \quad (300)$$

will be unstable if

$$\text{Re } \lambda_k > \lambda_i \quad (301)$$

since they will have neighborhood principal roots

$$\lambda_k - \lambda_i$$

with

$$\text{Re } (\lambda_k - \lambda_i) > 0. \quad (302)$$

Thus all the system singular points will, under the assumption

$$\text{Re } \lambda_k > \lambda_i, \quad (303)$$

be unstable. The only possible stable region will then be the principal plane associated with the complex pair λ_k and $\bar{\lambda}_k$. This plane can next be shown to contain a stable limit cycle under the condition

$$\text{Re } \lambda_k > 0. \quad (304)$$

The plane corresponding to the conjugate pair $\lambda_k, \bar{\lambda}_k$ is a principal plane of the transformation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} \quad (305)$$

The transformation can be written in this plane in terms of new orthogonal axes,

ξ_1 and ξ_2 , as

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = [T] \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad (306)$$

where

$$[T] = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \quad (307)$$

and

$$(t_{11} + t_{22})^2 < 4D = 4 \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix} \quad (\text{condition for complex eigenvalues}) \quad (308)$$

Now form

$$\begin{aligned} \underline{\xi} \underline{\xi}^0 &= (1 - \underline{\xi} \underline{\xi}) \underline{\xi} [T] \underline{\xi} \\ \frac{\underline{\xi} \underline{\xi}}{1 - \underline{\xi} \underline{\xi}} &= \underline{\xi} [T] \underline{\xi} = \frac{r^0 r}{1 - r^2} \end{aligned} \quad (309)$$

where

$$\underline{\xi} \underline{\xi} = r^2$$

But

$$\begin{aligned} \frac{r^0}{r(1-r^2)} &= \frac{d}{dt} \log \frac{r}{\sqrt{|1-r^2|}} \\ &= \left[t_{11} \dot{\xi}_1^2 + t_{22} \dot{\xi}_2^2 + (t_{12} + t_{21}) \dot{\xi}_1 \dot{\xi}_2 \right] \frac{1}{r^2} \\ &= t_{11} \cos^2 \theta + t_{22} \sin^2 \theta + (t_{12} + t_{21}) \sin \theta \cos \theta \end{aligned}$$

where

$$\begin{aligned} \dot{\xi}_1 &= r \cos \theta \\ \dot{\xi}_2 &= r \sin \theta \end{aligned}$$

Hence, the net change in

$$\log \frac{r}{\sqrt{|1-r^2|}}$$

per trip around the coordinate origin along a solution path will be

$$\begin{aligned} \int_0^{2\pi} \left[t_{11} \cos^2 \theta + t_{22} \sin^2 \theta + (t_{12} + t_{21}) \sin \theta \cos \theta \right] d\theta \\ = \pi(t_{11} + t_{22}) = 2\pi \operatorname{Re} \lambda_k \end{aligned} \quad (313)$$

this is positive for

$$\operatorname{Re} \lambda_k > 0.$$

It is, of course, assumed that an increase in θ corresponds to an increase in time. Since this assumption is equivalent to an assumed sense for θ , the above result is completely general.

Now the function

$$\log \frac{r}{\sqrt{|1 - r^2|}}$$

has the appearance shown in Figure XVII. It is clear that an increase in this quantity amounts to an approach to the value

$$r = 1 \tag{315}$$

Hence, the limit cycle

$$r = 1 \tag{315}$$

is stable for the plane corresponding to λ_k for

$$\text{Re } \lambda_k > 0$$

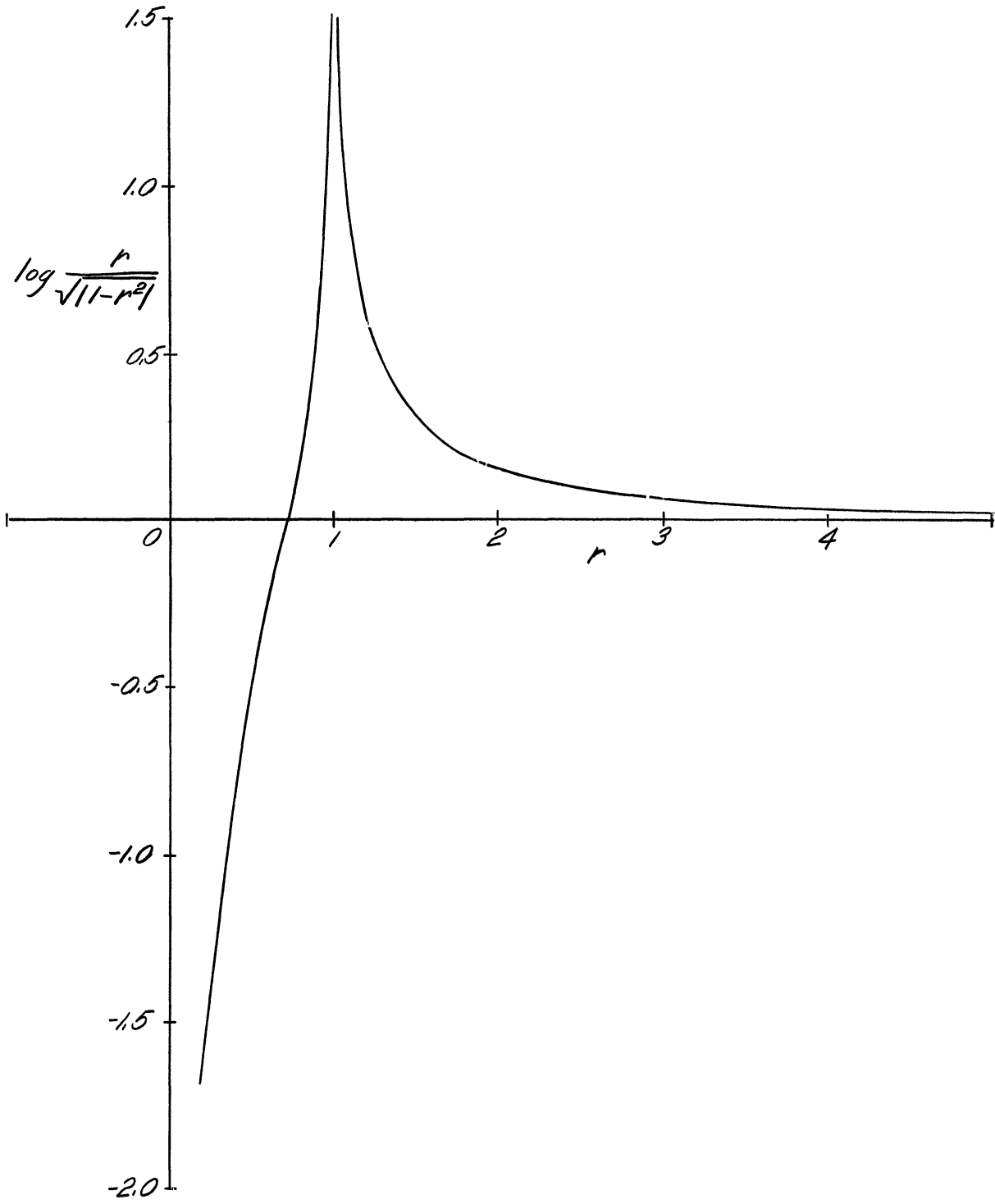


Figure XVII Radial Convergence Function

A limit cycle for a 3-dimensional system of the sort just considered is shown as Figure IV, Appendix II. Evidently, once a stage of the reduction procedure has been reached which produces such a limit cycle behavior, the process cannot be continued by direct application of any of the equations so far developed. To see what might be done, it is helpful to consider further some of the transformation properties of the matrix in a principal plane (which is also a principal plane of the non-linear position to velocity transformation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} \quad (317)$$

A real principal plane of the transformation

$$\begin{bmatrix} y \\ x \end{bmatrix} = [A] \begin{bmatrix} x \\ y \end{bmatrix} \quad (318)$$

contains both

$$\begin{bmatrix} y \\ x \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

for any

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

Assuming that orthogonal axes have been chosen in this plane, the transformation can then be represented by a 2nd order matrix. The relative positions of the vectors

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$\begin{bmatrix} y \\ x \end{bmatrix}$$

are of primary interest. Obviously if the angle between these two vectors is zero, then that position is an eigenvector. There may be an infinite number of such positions in the plane, or there may be two or one or none. These cases correspond respectively to the case of equal roots with a symmetric matrix, real and distinct roots, equal roots with an asymmetric matrix and complex roots with an asymmetric matrix. Also of interest are the positions of

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

for extreme differences between the positions of

x]

and

y] .

Taking

$$\theta = \arctan \frac{y_2}{y_1} - \arctan \frac{x_2}{x_1} ,$$

θ has extremes at

$$\begin{aligned} \frac{\partial \theta}{\partial x_1} = 0 &= \frac{1}{1 + \frac{y_2^2}{y_1^2}} \left(\frac{1}{y_1} \frac{\partial y_2}{\partial x_1} - \frac{y_2}{\partial y_1} \frac{\partial y_1}{\partial x_1} \right) - \frac{1}{1 + \frac{x_2^2}{x_1^2}} \left(- \frac{x_2}{x_1^2} \right) \\ &= \frac{\frac{\partial y_2}{\partial x_1} - y_2 \frac{\partial y_1}{\partial x_1}}{y_1^2 + y_2^2} + \frac{x_2}{x_1^2 + x_2^2} \\ &= \frac{a_{21} y_1 - a_{11} y_2}{y_1^2 + y_2^2} + \frac{x_2}{x_1^2 + x_2^2} \end{aligned} \quad (319)$$

where

$$y_1 = a_{11} x_1 + a_{12} x_2$$

$$y_2 = a_{21} x_1 + a_{22} x_2 \quad (320)$$

Now,

$$-a_{21} y_1 + a_{11} y_2 = |A| x_2 . \quad (321)$$

Thus for θ an extreme,

$$\frac{x_2}{x_1^2 + x_2^2} = \frac{|A|}{y_1^2 + y_2^2} x_2$$

$$\frac{y_1^2 + y_2^2}{x_1^2 + x_2^2} = |A| \quad (322)$$

Now, calling the two eigenvalues characterizing the principal plane λ_1 and λ_2 ,

$$\lambda_1 \lambda_2 = |A| \quad (323)$$

and thus for θ an extreme

$$\sqrt{\frac{y_1^2 + y_2^2}{x_1^2 + x_2^2}} = \frac{|y|}{|x|} = \sqrt{\lambda_1 \lambda_2} ; \quad (324)$$

which is to say, the ratio of the lengths of a vector and its transform is the geometric mean of the eigenvalues at a position such that the angle between the vector and its transform is either a maximum or a minimum. There are, in each principal plane, two such positions, independent of the nature of the eigenvalues. The important point here is that if the eigenvalues are complex, then the ratio of the length of the transform to the length of the vector is the modulus of the eigenvalue at positions of extreme θ . Thus, finally if

$x]$

is a unit vector, as it will be on the steady-state limit cycle in a principal plane corresponding to complex eigenvalues, then at extremes of θ ,

$$|y| = |\lambda|. \quad (325)$$

The modulus of a complex root could thus be obtained by adding equipment to compute

$$\theta = \arccos \frac{|x \cdot y|}{(|x \cdot y|)^{1/2}} \quad (326)$$

and arranging either manually or automatically to hold the computation at an extreme of this value, at which point could be read

$$(|x \cdot y|)^{1/2} = |\lambda|. \quad (327)$$

The diagonalization process can be continued beyond finding a dominant complex eigenvalue λ_k by using as equations for succeeding steps

$$\begin{aligned} \begin{bmatrix} 0 \\ x \end{bmatrix} &= y \begin{bmatrix} -x \\ y \end{bmatrix} \begin{bmatrix} x \\ \end{bmatrix} \\ y \begin{bmatrix} \\ \end{bmatrix} &= y \begin{bmatrix} \\ \end{bmatrix} - \sum_{i=1}^{k-1} \frac{x_{it} y}{x_{it} y} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \frac{x_{kt}}{y_{kt} y} \\ &- \frac{1}{(y_{kt} y)_{kt}} \begin{bmatrix} y_{kt} y \\ y \end{bmatrix} \begin{bmatrix} y \\ y \end{bmatrix}_{kt} \end{aligned}$$

where

$$x \begin{bmatrix} \\ \end{bmatrix}_{kt}$$

and

$$y \begin{bmatrix} \\ \end{bmatrix}_{kt}$$

are any vectors in the principal plane of the transpose matrix

$$[A]_t$$

corresponding to complex eigenvalue λ_k . This can be shown to work by showing that the orthogonality property of eigenvectors of

$$[A]_t$$

to the (N-1) - dimensional principal subspaces of

$$[A]$$

extends to 2-dimensional principal subspaces of

$$[A]_t$$

being orthogonal to (N-2) - dimensional principal subspaces of

$$[A]$$

Let λ_c and λ_{ct} be complex eigenvalues of

$$[A]$$

and

$$[A]_t$$

with corresponding complex eigenvectors

$$\mathbf{x}]_c = \mathbf{x}]_{cr} + j \mathbf{x}]_{ci} \quad (329)$$

and

$$\mathbf{x}]_{ct} = \mathbf{x}]_{ctr} + j \mathbf{x}]_{cti} \quad (330)$$

then

$$[A] \mathbf{x}]_c = \lambda_c \mathbf{x}]_c \quad (331)$$

and

$$[A]_t \mathbf{x}]_{ct} = \lambda_{ct} \mathbf{x}]_{ct} \quad (332)$$

Transposing the two members of the second equation and post-multiplying by

$$\begin{aligned} & \mathbf{x}]_c, \\ \underline{\mathbf{x}}]_{ct} [A] \mathbf{x}]_c &= \lambda_{ct} \underline{\mathbf{x}}]_{ct} \mathbf{x}]_c \end{aligned} \quad (333)$$

Premultiplying the first equation by

$$\begin{aligned} & \underline{\mathbf{x}}]_{ct}, \\ \underline{\mathbf{x}}]_{ct} [A] \mathbf{x}]_c &= \lambda_c \underline{\mathbf{x}}]_{ct} \mathbf{x}]_c \end{aligned} \quad (334)$$

Subtracting (333) from (334) ,

$$(\lambda_c - \lambda_{ct}) \underline{\mathbf{x}}]_{ct} \mathbf{x}]_c = 0 \quad (335)$$

and thus, either

$$\lambda_c = \lambda_{ct}; \quad (336)$$

i.e.,

$$\lambda_{cr} = \lambda_{ctr} \quad (337)$$

and

$$\lambda_{ci} = \lambda_{cti} \quad (338)$$

or else

$$\underline{x}_{ct} \ x]_c = 0 \quad (339)$$

which implies

$$\underline{x}_{ctr} \ x]_{cr} - \underline{x}_{cti} \ x]_{ci} = 0 \quad (340)$$

and

$$\underline{x}_{ctr} \ x]_{ci} + \underline{x}_{cti} \ x]_{cr} = 0 \quad (341)$$

Since

$$\underline{x}]_c$$

is not unique, either of the preceding equations can hold only if each of the terms in the left hand members is individually zero. Thus, the 2-dimensional principal subspace corresponding to a complex eigenvalue λ_K of

$$[A]_t$$

is normal to the (N-2) - dimensional principal subspace of

$$[A]$$

not corresponding to λ_K .

Repeated Roots

The case of repeated eigenvalues, which is a singular case between that for real and that for complex eigenvalues, offers distinctly different problems depending on whether the matrix is symmetric or asymmetric. In the symmetric case, no modification in the reduction procedure is necessary at all since in this case a straightforward application of the procedure outlined will produce orthogonal eigenvectors in the principal plane corresponding to the repeated eigenvalue. Except for being orthogonal, the orientation of the

two vectors in this plane is undefined. The singular points on the unit circle in the plane are neutrally stable along the tangent to the circle so that the computed position of the first eigenvector will be purely a matter of chance.

Thus any vector in the principal 2-d subspace corresponding to a repeated eigenvalue of a symmetric matrix is an eigenvector. On the other hand, only one vector in the principal 2-d subspace corresponding to a repeated eigenvalue of an asymmetric matrix is an eigenvector. There still exists, of course, two positions for which

$$\frac{|y|}{|x|} = \lambda, \quad (342)$$

one of these being a position of minimum angular displacement between

$$y]$$

and

$$x]$$

namely, the eigenvector position; the other being a position of maximum angle between

$$y]$$

and

$$x],$$

In order to apply the modified equations for succeeding steps, (328), it is necessary to hold the computer in the neighborhood of the position of maximum angle between

$$y]$$

and

$$x] .$$

This proves a very difficult trick practically inasmuch as this is the region of maximum rate of change of the

$$x]$$

vector on the limit cycle. Situations wherein an asymmetric matrix has nearly equal roots offer real practical difficulties in the reduction procedure.

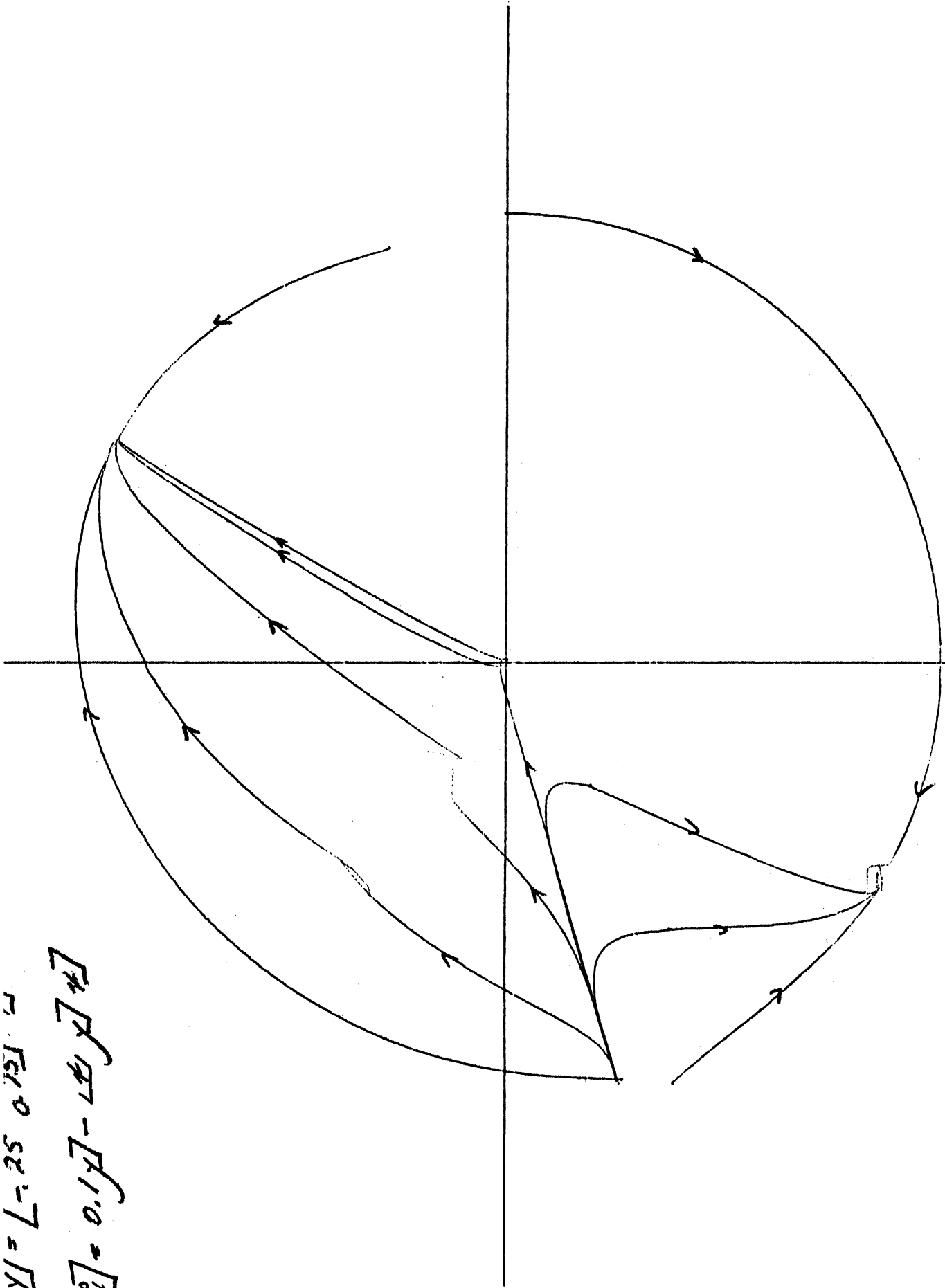
Figures XVIII through XXVII illustrate the vector field of equations (282) in a principal 2-d subspace for various characters of the eigenvalues characterizing that subspace. The figures were obtained by running trajectories of the transformation

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} y \\ -x \end{bmatrix} - \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a_{11} & 0.50 \\ -0.25 & 0.75 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned} \quad (343)$$

for various values of a_{11} . Both x- and y-plane trajectories are shown for each value of a_{11} . Figure XX shows the important case

$$\lambda_1 = 0 \quad (344)$$

which characterizes several principal 2-d subspaces in all but the first step of the general procedure. Note that the process indicates no strong tendency to "hang up" on the zero root eigenvector. Figure XXI shows the most significant property of the transformation; namely, that the transformed space is N-1 dimensional -- the 2-d space transforms into a line.



$\Delta = L = 25 \times 10^{-3} \times \dots$
 $\dots = 0.1 \times \dots - 1.25 \times \dots$

Figure XVIII X-Plane + Real Roots of Opposite Sign

$$y = [-.25 \ 0.75] \quad \wedge$$
$$z = 0.1y - 1.4y^2 \quad \wedge$$

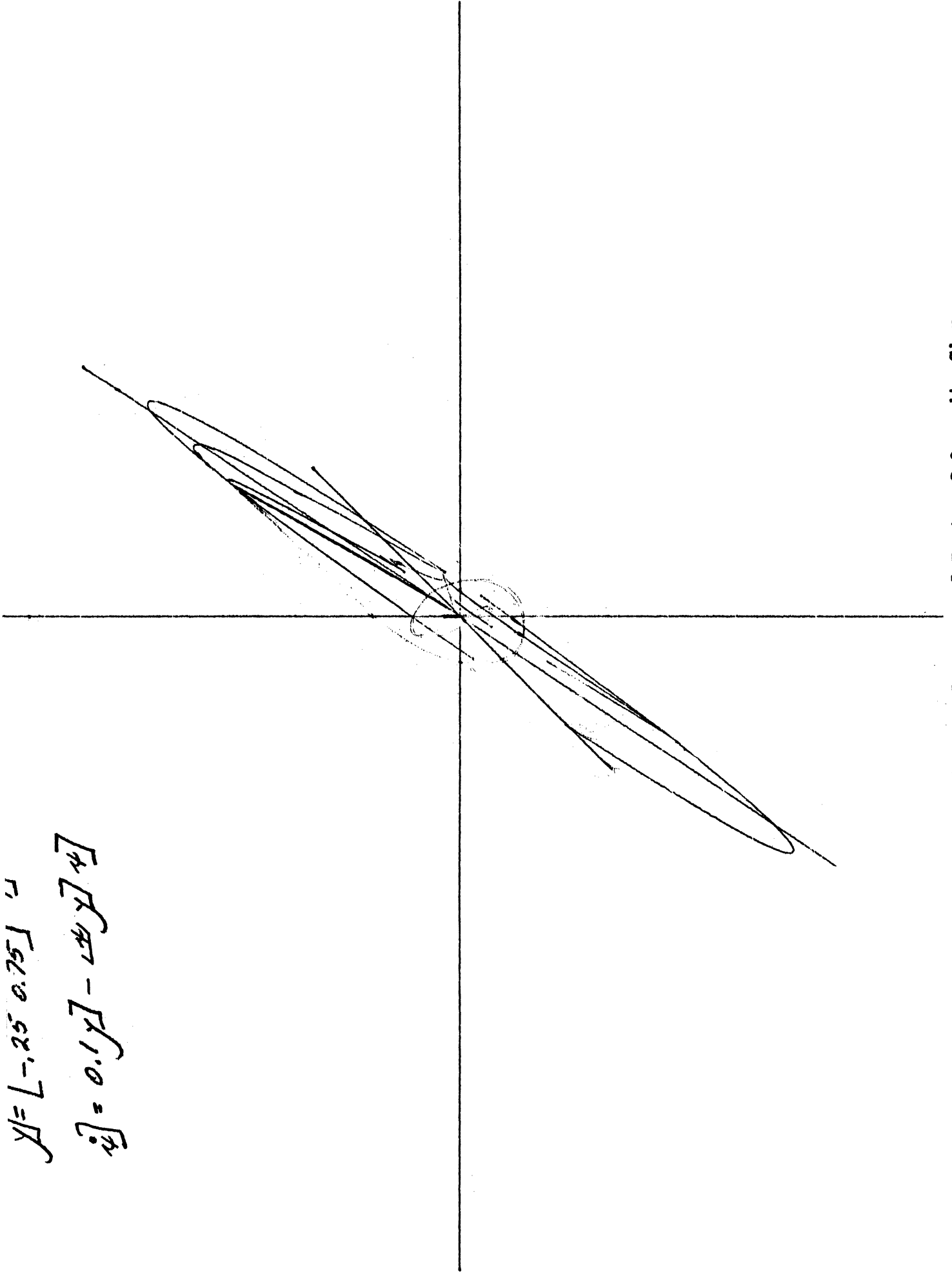


Figure XIX Y-Plane - Real Roots of Opposite Sign

$$\lambda = 0.591$$

$$p[\lambda] = \begin{bmatrix} -1.167 & .50 \\ -1.25 & .75 \end{bmatrix} \psi$$

$$q[\lambda] = 0.1\lambda - 1.4\lambda^2 \psi$$

$$\lambda = 0, \frac{2}{3}$$

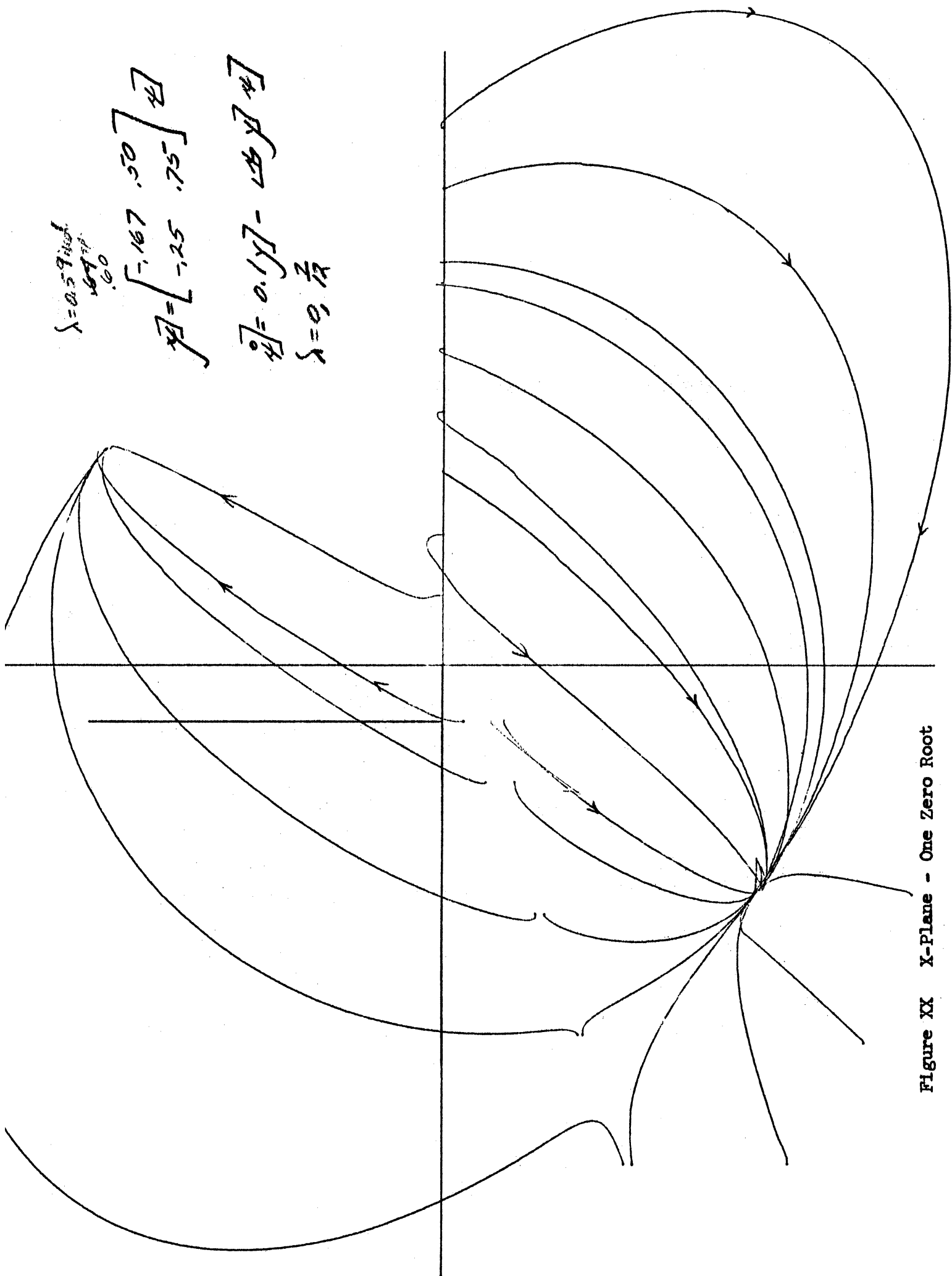


Figure XX X-Plane - One Zero Root

$$Y = \begin{bmatrix} -1.167 & 0.5 \\ -1.25 & 0.75 \end{bmatrix} u$$

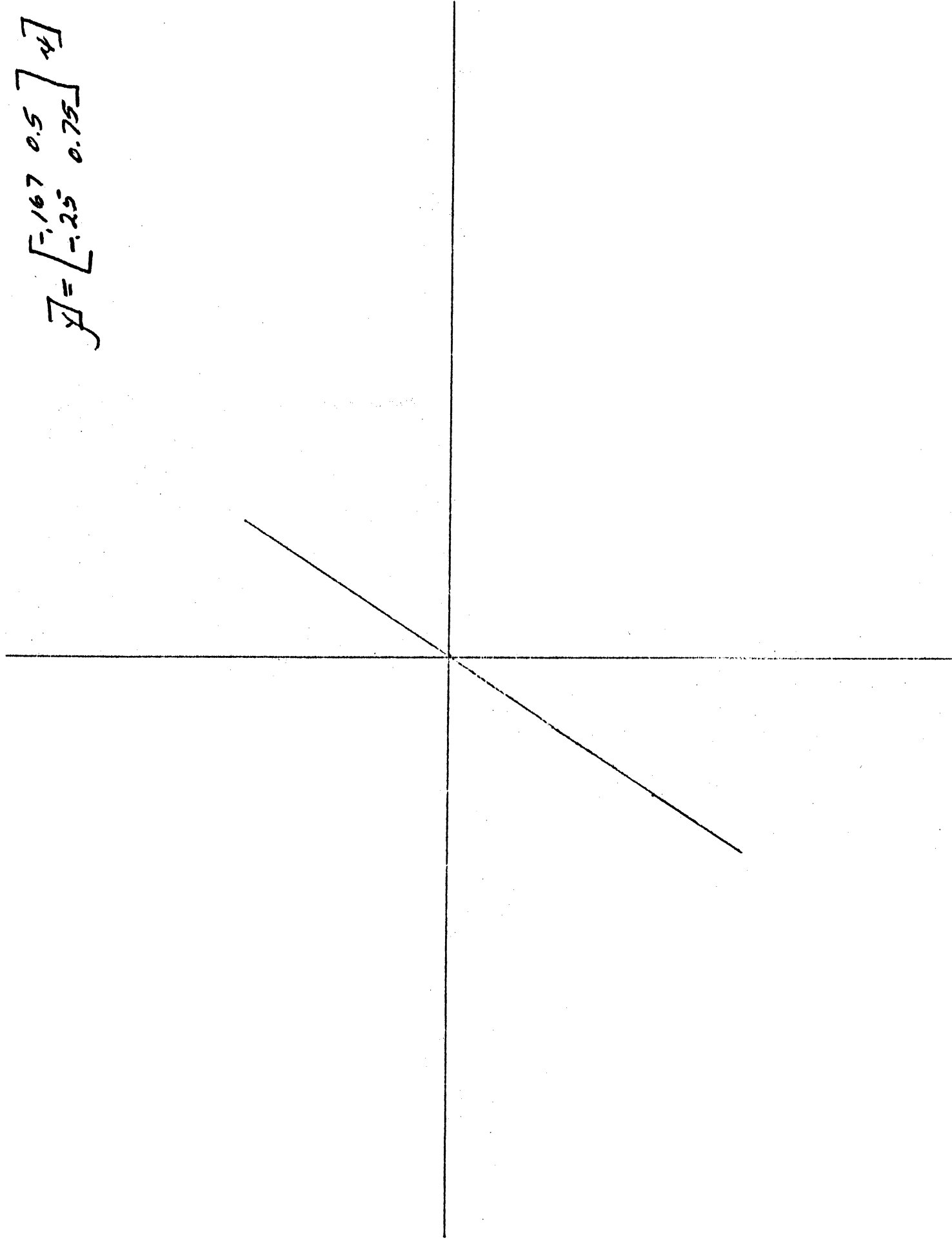


Figure XXI Y-Plane - One Zero Root

$s = [-1 \pm j \sqrt{1 - 0.25}]$

$\phi = 0.1y - 0.25y^2$

$\lambda_1 = 0.5$

$\lambda_2 = 0.25$

$\phi_1 = \frac{1}{2\sqrt{5}}$

$\phi_2 = \frac{1}{5\sqrt{2}}$

$2a_1 \phi_1 < 0$
for $\phi_2 (4 + 3\phi_2) < 0$

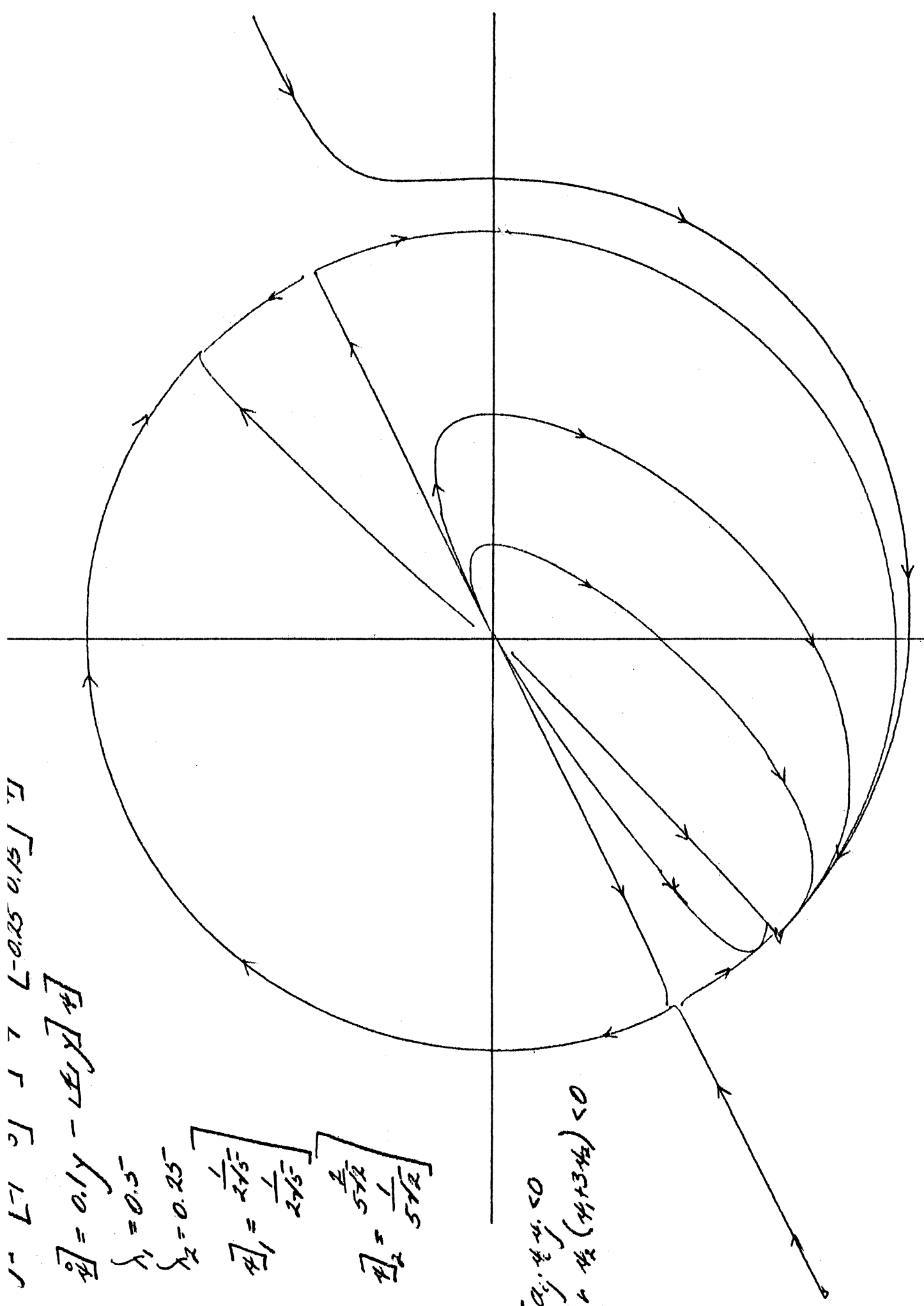


Figure XXII X-Plane - Real Positive Unequal Roots

$$A = \begin{bmatrix} -0.25 & 0.75 \\ 0.1 & -1.4 \end{bmatrix}$$

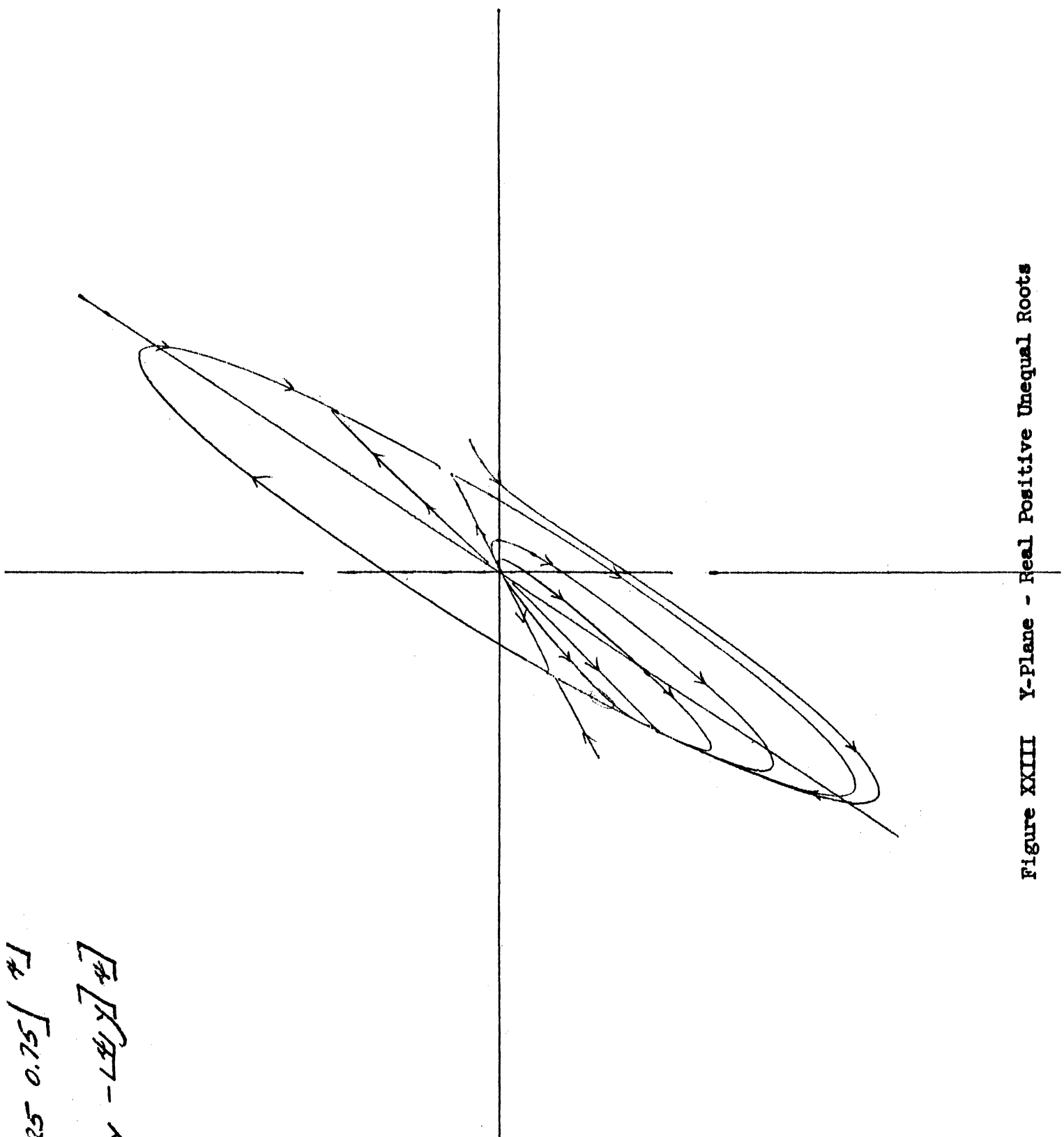


Figure XXIII Y-Plane - Real Positive Unequal Roots

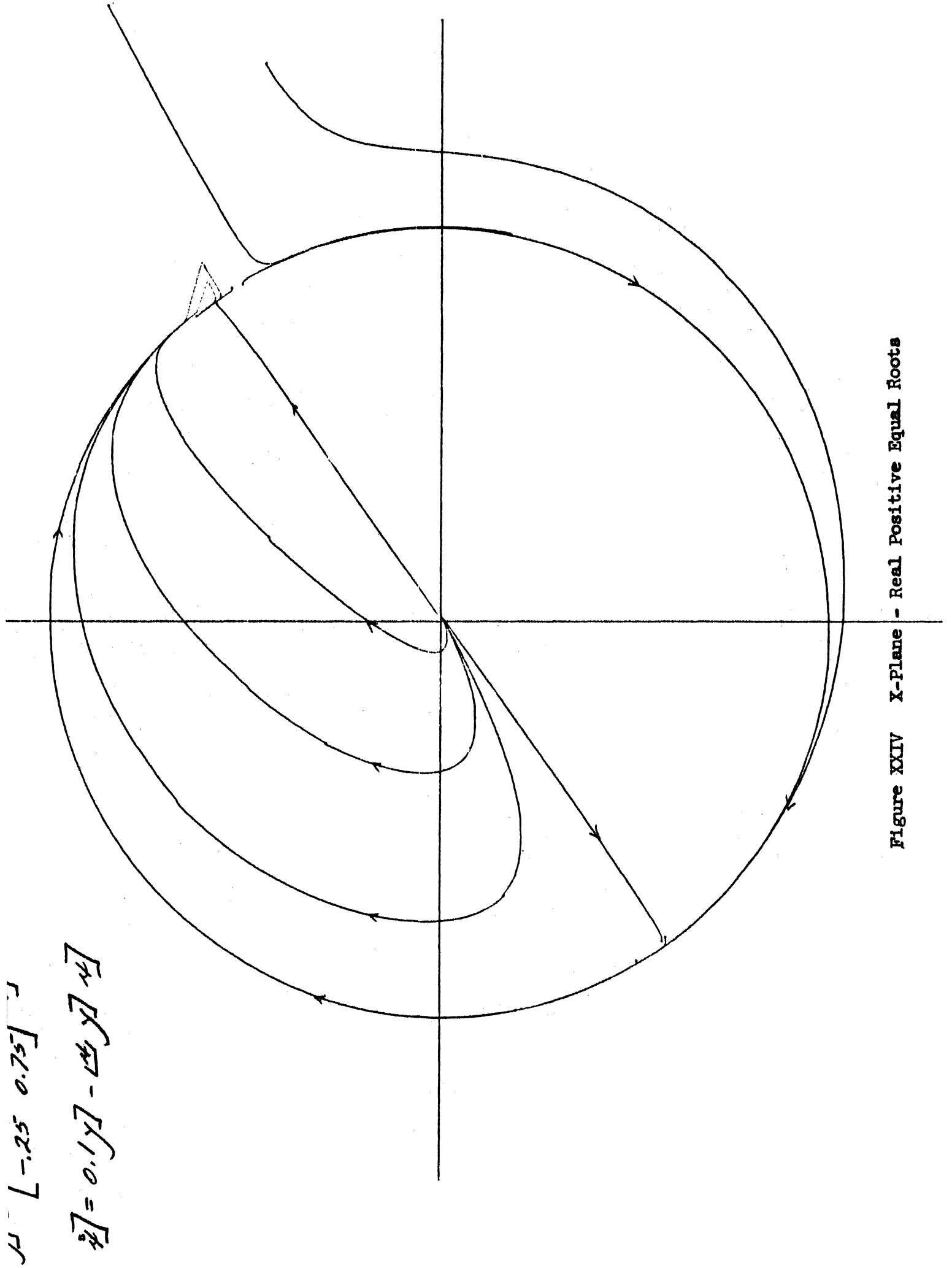


Figure XXIV X-Plane - Real Positive Equal Roots

$$\lambda = [-1.25 \quad 0.75]$$

$$\vec{v} = [0.1 \quad y] - [0.1 \quad y] \vec{v}$$

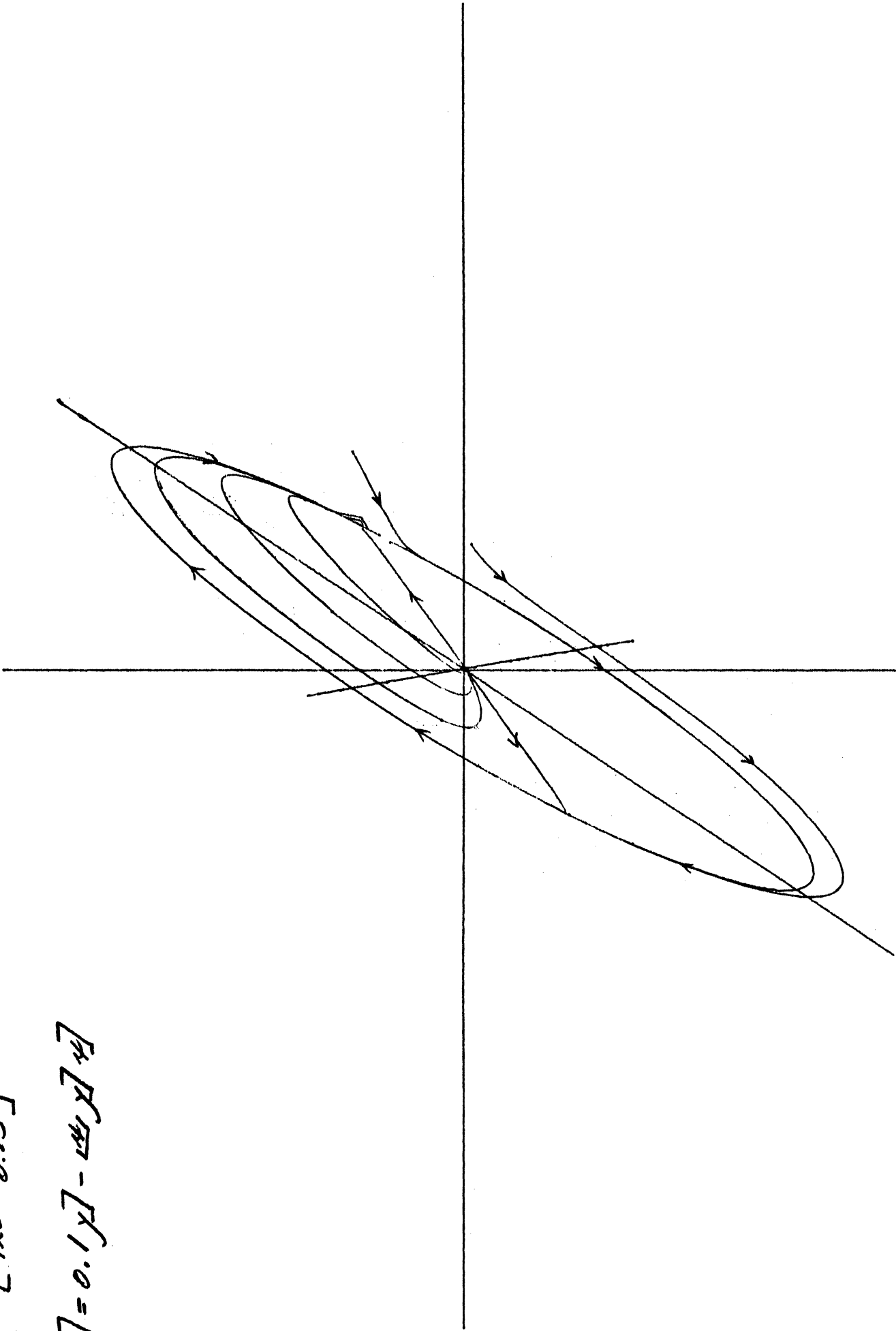


Figure XXV Y-Plane - Real Positive Equal Roots

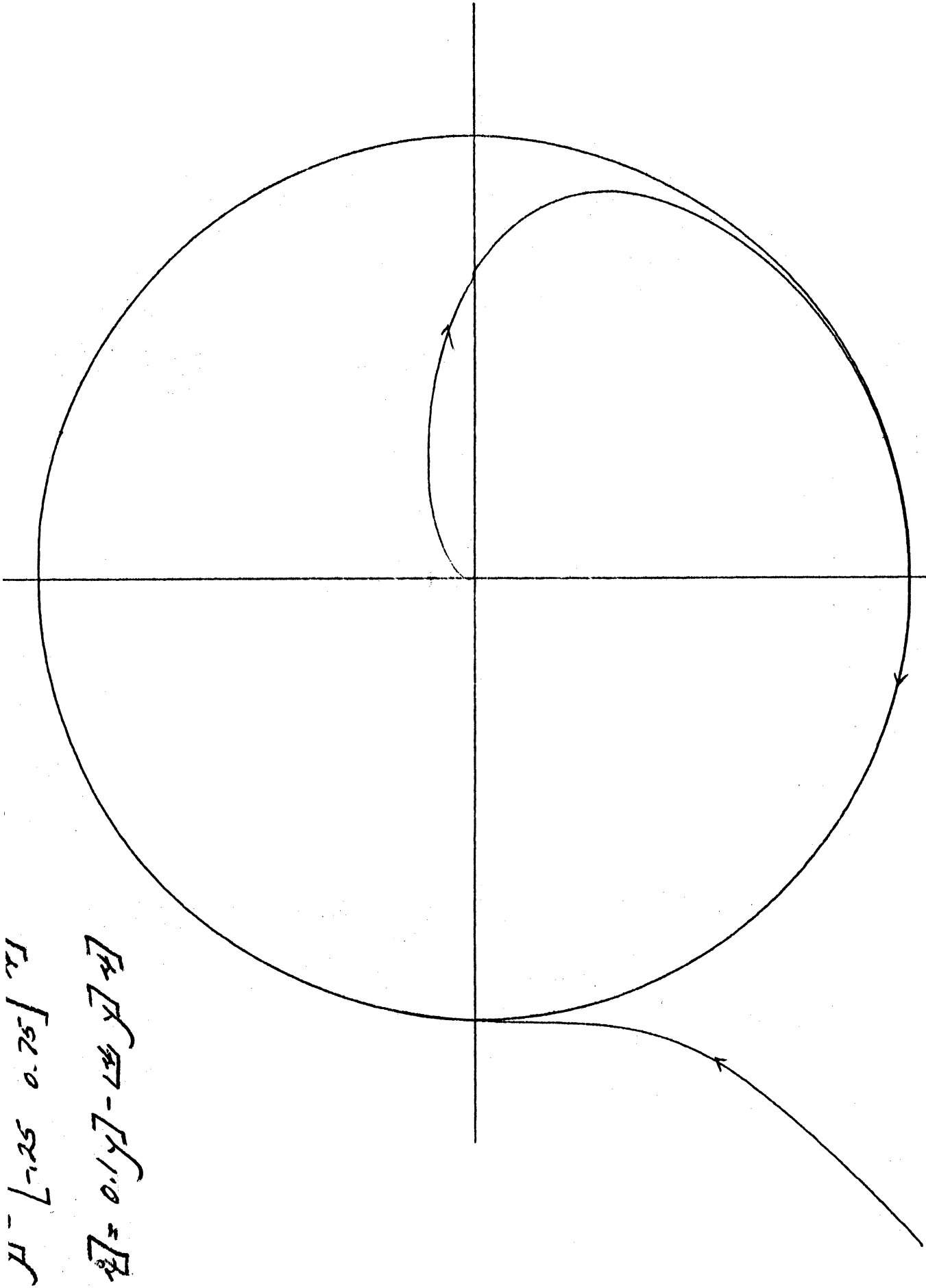


Figure XXVI X-Plane - Complex Roots

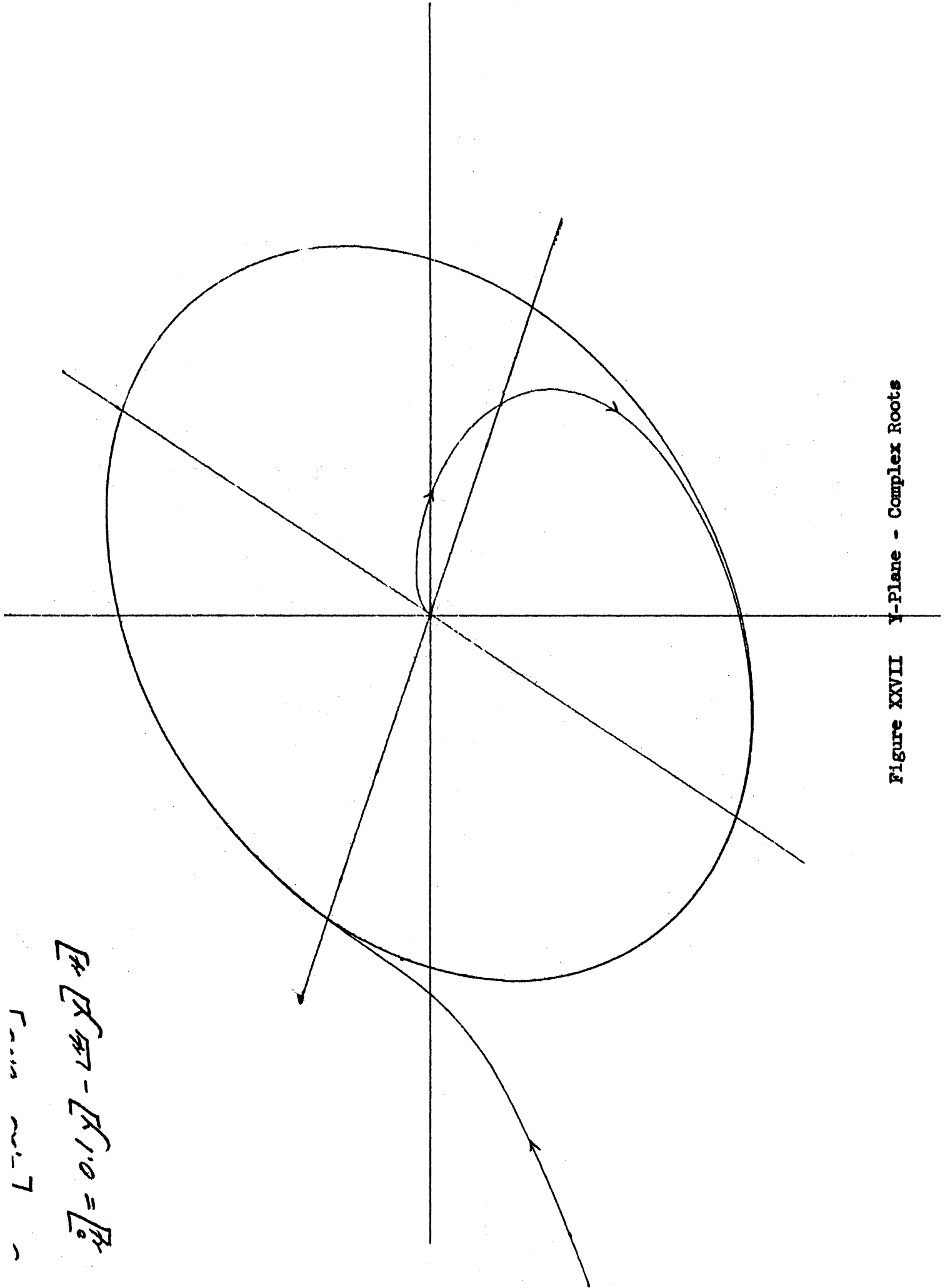


Figure XXVII Y-Plane - Complex Roots

Figure XXIII illustrates the important (if negative) property of the asymmetric transformation that the two eigenvectors are not orthogonal and are not conjugate diameters of the ellipse into which the circle

$$\underline{x} \cdot \underline{x} = 1.0 \quad (345)$$

transforms. This is basically the reason why the transpose field is needed for the complete reduction of asymmetric matrices. Figure XXIV shows the repeated eigenvalue field with the velocity vector directed in a single sense along the unit circle.

Figure XXVI displays the complex eigenvalue case with no singular point at all on the unit circle, which is (in this case) a stable limit cycle. This limit cycle is in general traversed at a non-uniform rate, for using

$$\dot{\underline{x}} = \underline{y} - \underline{x} \cdot \underline{y} \cdot \underline{x} \quad (346)$$

$$\underline{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \underline{x} \quad (347)$$

$$\theta = \arctan \frac{x_2}{x_1} \quad (348)$$

gives

$$\dot{x}_1 = [a_{11}(1-x_1^2) - a_{22}x_2^2 - (a_{12}+a_{21})x_1x_2]x_1 + a_{12}x_2 \quad (349)$$

$$\dot{x}_2 = [-a_{11}x_1^2 + a_{22}(1-x_2^2) - (a_{12}+a_{21})x_1x_2]x_2 + a_{21}x_1 \quad (350)$$

$$\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2 + x_2^2} \quad (351)$$

Thus

$$\frac{\partial \dot{\theta}}{\partial x_1} = \frac{(a_{22}-a_{11})x_2(x_1^2+x_2^2) + 2(a_{21}+a_{12})x_1x_2^2}{(x_1^2+x_2^2)^2} \quad (352)$$

$$= 0$$

if and only if

$$a_{22} = a_{11} = a \tag{353}$$

$$a_{12} = -a_{21} = b \tag{354}$$

$$\lambda = a + j b \tag{355}$$

This gives

$$\dot{x}_1 = a(1-x_1^2-x_2^2) x_1 + b x_2 \tag{356}$$

$$\dot{x}_2 = -b x_1 + a(1-x_1^2-x_2^2) x_2 \tag{357}$$

$$\dot{\theta} = -b \tag{358}$$

Since equation (358) holds independently of the size of the vector

x],

equations (356) and (357) may conveniently be used as design equations for a resolver with input θ as shown in Figure XXVIII.

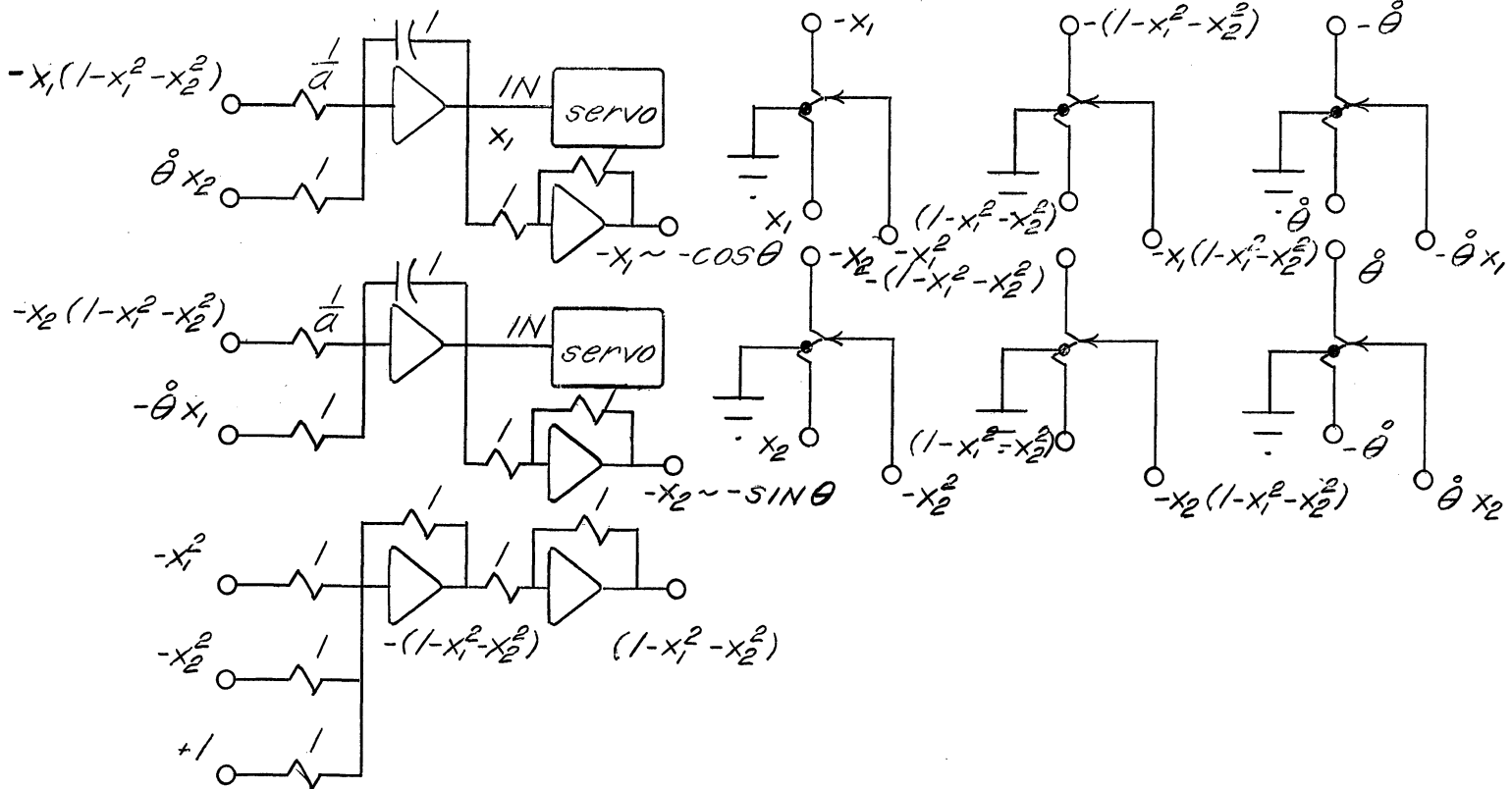


Figure XXVIII
Resolver with $\dot{\theta}$ Input

Modification of the Technique for the
Simultaneous Diagonalization of Two Matrices

Under certain circumstances, the techniques so far described may be modified to obtain solutions of the general eigenvalue problem

$$[A] u = \lambda [B] u \quad (359)$$

The following few paragraphs are devoted to describing the modifications required and to defining the circumstances in which such modification will yield solutions of equation (359).

Relationship to Elementary Problem

Equation (359) is seen to be equivalent to equation (32) provided

$$\begin{aligned} [A] u &= y \\ [B] u &= x \end{aligned} \quad (360)$$

Thus the eigenvalues of

$$[A]$$

with respect to

$$[B]$$

are identical with the eigenvalues of

$$[B]^{-1} [A]$$

provided, of course, that

$$[B]$$

is non-singular. Criteria for stability and scaling can thus be inferred from applying results already obtained to the matrix

$$[B]^{-1} [A]$$

except insofar as the new problem requires modified circuitry.

Computer Equations

Subject to the definitions, (360) and (361), the equations

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \quad (363)$$

$$\begin{bmatrix} A \\ B \end{bmatrix} u = y \quad (364)$$

$$\begin{bmatrix} B \\ A \end{bmatrix} u = x \quad (365)$$

will yield an eigenvector

$$u_N$$

as a steady-state solution, where

$$\begin{bmatrix} x_N \\ x_N \end{bmatrix} = 1.0$$

$$\begin{bmatrix} A \\ B \end{bmatrix} u_N = \lambda_N \begin{bmatrix} x_N \\ x_N \end{bmatrix} \quad (366)$$

$$\begin{bmatrix} B \\ A \end{bmatrix} u_N = \begin{bmatrix} x_N \\ x_N \end{bmatrix} .$$

Note that since

$$u_N$$

is not in general aligned with either

$$x_N$$

or

$$y_N$$

the equations of motion (363) force the

$$\begin{bmatrix} B \\ A \end{bmatrix}$$

transform of

$$u$$

to pursue the

$$\begin{bmatrix} A \\ B \end{bmatrix}$$

transform to a position where the two transforms are aligned although neither one is necessarily aligned with

$$u .$$

If both

$$\begin{bmatrix} A \\ B \end{bmatrix}$$

and

$$[B]$$

are symmetric, succeeding step equations are

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ A \\ B \end{bmatrix} &= \begin{bmatrix} y \\ u \\ u \end{bmatrix}' - \begin{bmatrix} x \\ y \\ x \end{bmatrix}' \\ &= \begin{bmatrix} y \\ u \\ u \end{bmatrix}' - \sum_{j=1}^{i-1} \begin{bmatrix} u_{N-j} \\ u_{N-j} \\ u_{N-j} \end{bmatrix} \begin{bmatrix} u \\ u \\ u \end{bmatrix}_{N-j} \end{aligned} \quad (367)$$

where

$$\begin{aligned} [A] u_{N-j} &= \lambda_{N-j} x_{N-j} \\ [B] u_{N-j} &= x_{N-j} \end{aligned} \quad (368)$$

Equations (367) may be rewritten as

$$[A]' u = y \quad (367')$$

where

$$[A]' = \left([I] - \sum_{j=1}^{i-1} \frac{u_{N-j} u_{N-j}}{u_{N-j} u_{N-j}} \right) [A] \quad (367'')$$

If either

$$[A]$$

or

$$[B]$$

is asymmetric, succeeding steps are

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ A \\ B \end{bmatrix} &= \begin{bmatrix} y \\ u \\ u \end{bmatrix}' - \begin{bmatrix} x \\ y \\ x \end{bmatrix}' \\ &= \begin{bmatrix} y \\ u \\ u \end{bmatrix}' - \sum_{j=1}^{i-1} \begin{bmatrix} u_{N-j,t} \\ u_{N-j,t} \\ u_{N-j,t} \end{bmatrix} \begin{bmatrix} u \\ u \\ u \end{bmatrix}_{N-j,t} \end{aligned} \quad (369)$$

where

$$\begin{aligned} [B]_t x_{N-j,t} &= \lambda_{N-j} u_{N-j,t} \\ [A]_t x_{N-j,t} &= u_{N-j,t} \end{aligned} \quad (370)$$

Stability:

A new stability question is introduced with the above equations since the vector

$$u]$$

is obtained by solving a set of simultaneous algebraic equations. This difficulty was avoided in the case

$$[B] = [I] \quad (371)$$

essentially by solving for the eigenvalues with largest real parts first instead of following the intensification procedure directly to solve first for eigenvalues with smallest real parts. In the general case where

$$[B] \neq [I] \quad (372)$$

the solution of a set of simultaneous equations is unavoidable.

A straightforward electronic differential analyzer setup to solve a set of simultaneous algebraic equations may or may not be stable depending on the nature of the coefficient matrix, and, to some extent, on the design of the operational amplifier. Positive realness of the eigenvalues of the coefficient matrix is in general sufficient, though not necessary, to guarantee stability. A favorite technique with designers of algebraic equation solvers is to modify the set

$$[B] u = x \quad (373)$$

to

$$x \overset{0}{u} = [B]_{\tau} (x) - [B] u \quad (374)$$

which has positive real characteristic roots since the matrix

$$[B]_{\tau} [B]$$

has necessarily positive real eigenvalues only. Solving equations (374) rather than (373) requires doubling the number of operational amplifiers and the number of coefficient pots, which seems a small price for guaranteed stability. The time constant τ in equations (374) may be made small (of the order of milliseconds) in order to avoid spoiling the dynamic behavior of the rest of the system.

Scaling:

A new scaling problem is also introduced in equations (363), (364), (365) since there are, in this set, three vectors to be generated; namely,

$$\underline{u} = [B]^{-1} \underline{x} \quad (375)$$

$$\underline{x} = [B] \underline{u} \quad (376)$$

and

$$\underline{y} = [A] \underline{u} \quad (377)$$

Equation (363) guarantees that in the steady-state

$$\underline{x} \underline{x} = 1.0 \quad (378)$$

In order to keep

$$\underline{u} \underline{u} \leq 1.0 \quad (379)$$

and

$$\underline{y} \underline{y} \leq 1.0 \quad (380)$$

we require (from (375) and (377)), all eigenvalues of

$$[A]$$

and

$$[B]^{-1}$$

to have modulus not greater than 1.0. Using a result of Hadamard's⁽¹⁾, this may be done by making

$$[A]' = k_A [A] \quad (381)$$

$$[B]' = k_B [B] \quad (382)$$

where

$$k_A \leq \frac{1}{\left(\sum_j |a_{ij}| \right)_{\max.}} \quad (383)$$

$$k_B \leq \frac{1}{\left(\sum_{\substack{j \\ i \neq j}} |b_{ij}| \right)_{\min.}} \quad (384)$$

(1) Parodi, Maurice M., "Sur Quelques Propriétés des Valeurs Caractéristiques des Matrices Carrées", Mém Sci. Math., Gauthier - Villars, Paris, 1952

This follows from Hadamard's result that an eigenvalue λ_k of a matrix

$$[A]$$

is bounded in modulus by

$$\left(\sum_{\substack{j \\ i \neq j}} |a_{ij}| \right)_{\min.} \equiv \lambda_k \equiv \left(\sum_{\substack{j \\ i \neq j}} |a_{ij}| \right)_{\max.} \quad (385)$$

APPENDIX I

A Related Problem - Function Factoring
on the Analog Computer

An article by Pike and Silverberg¹ furnishes a technique for design of generators of functions of two (or more) independent variables. A slight modification of the techniques previously described in this dissertation provides convenient computation of the less manageable parts of the computations necessary for such design.

The gist of the article is that two-variable function generators with outputs

$$z(x, y)$$

may be made from single variable function generators with outputs

$$g_k(x)$$

and

$$h_k(y)$$

according to the formula

$$z_{ij} \sim a + g_{i1} + h_{j1} + \sum_{k=2}^q g_{ik} h_{jk} = \zeta_{ij} \quad (1)$$

Data for the design consists of a set of mn values

$$z_{ij}, \quad i=1, 2, \dots, m; \quad j=1, 2, \dots, n.$$

The constant, a and the sets of values, g_k and h_k , are chosen so that

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (z_{ij} - \zeta_{ij})^2$$

is a minimum. This leads to the design formulae

¹ Pike, E. W. and Silverberg, T.R., "Designing Mechanical Computers", Machine Design, July 1952, pp. 131-137; August 1952, pp. 159-163

$$a = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n z_{ij} \quad (2)$$

$$g_{i1} = \frac{1}{n} \sum_{j=1}^n r_{ij1} \quad (3)$$

$$h_{j1} = \frac{1}{m} \sum_{i=1}^m r_{ij1} \quad (4)$$

$$g_{ik} = \frac{\sum_{j=1}^n r_{ijk} h_{jk}}{\sum_{j=1}^n h_{jk}^2}, k = 2, 3, 4, \dots \quad (5)$$

$$h_{jk} = \frac{\sum_{i=1}^m r_{ijk} g_{ik}}{\sum_{i=1}^m g_{ik}^2}, k = 2, 3, 4, \dots \quad (6)$$

where

$$r_{ij1} = z_{ij} - a \quad (7)$$

$$r_{ij2} = r_{ij1} - g_{i1} - h_{j1} \quad (8)$$

$$r_{i,j,k+1} = r_{ijk} - g_{ik} - h_{jk}, k = 2, 3, 4, \dots \quad (9)$$

The calculation procedure is straightforward except for formulas (5) and (6) which require an iterative routine for solution. The iterative routine involves starting with incorrect elements of one set (either g_k or h_k) and recomputing elements of the two sets until consistent sets are obtained. The computer technique to be described formulates this procedure by assigning to each element of the two inconsistent sets a time rate of change proportional to the difference between the two members of the defining equation. Since the modulus of the two sets is undefined, the requirement that, in the steady-state,

$$\sum_{i=1}^m g_{ik}^2 = 1.0 \quad (10)$$

is added. The computer equations are then

$$\frac{dg_{ik}}{dt} = \frac{\sum_{j=1}^n r_{ijk} h_{jk}}{\sum_{j=1}^n h_{jk}^2} - g_{ik} \quad (11)$$

$$\frac{dh_{jk}}{dt} = \sum_{i=1}^m r_{ijk} g_{ik} - h_{jk} \quad (12)$$

Solution of this set of equations requires

$n + 1$ multipliers

$3(m+n) + 1$ operational amplifiers

$2mn$ coefficient potentiometers

Scaling is chosen to make

$$|r_{ijk}| \leq 1.0$$

A computer circuit to solve this problem for a typical 4×4 array is shown in Figures AI1 through AI4.

The gain K is adjusted experimentally to make

$$P < K^2 \sum_{j=1}^n h_{jk}^2 \leq 1.0$$

with P as large as possible and with K a conveniently available computing circuit element. An example was run with $K = 1/2$ and

$$\begin{aligned} r_{i,j,k+1} &= r_{113} = -.3910 & r_{123} &= -.0508 & r_{133} &= -.1968 & r_{143} &= 0.6386 \\ & r_{213} &= -.0570 & r_{223} &= -.1142 & r_{233} &= -.7310 & r_{243} &= 0.9022 \\ & r_{313} &= 0.1890 & r_{323} &= 0.1030 & r_{333} &= 0.6118 & r_{343} &= -.9038 \\ & r_{413} &= 0.2590 & r_{423} &= 0.0620 & r_{433} &= 0.3160 & r_{443} &= -.6370 \end{aligned}$$

Steady state values read from the computer were

$$\begin{aligned} g_{12} &= -.3730 & h_{12} &= 0.1958 \times 2 = \text{(comp. reading)} \\ & & & & & & & \times 1/k \\ g_{22} &= -.6015 & h_{22} &= 0.0859 \times 2 \\ g_{32} &= 0.5814 & h_{32} &= 0.4966 \times 2 \\ g_{42} &= 0.3910 & h_{42} &= -.7777 \times 2 \end{aligned}$$

Equations.

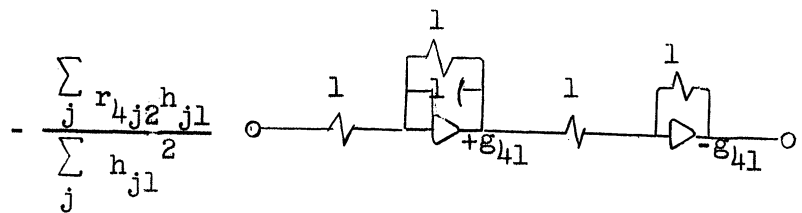
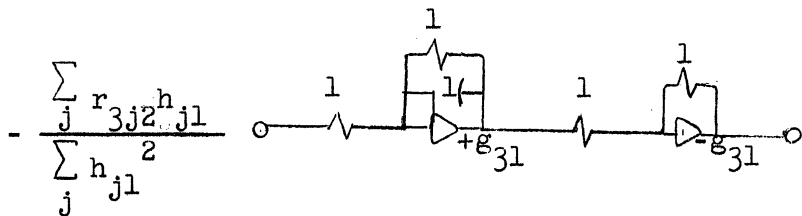
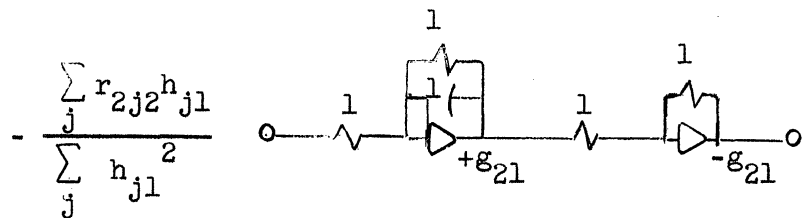
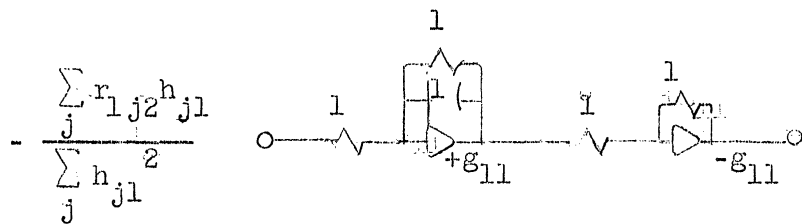
$$g_{i1}^0 = \frac{\sum_{j=1}^n r_{ij2} h_{j1}}{\sum_{j=1}^n h_{j1}^2} - g_{i1}$$

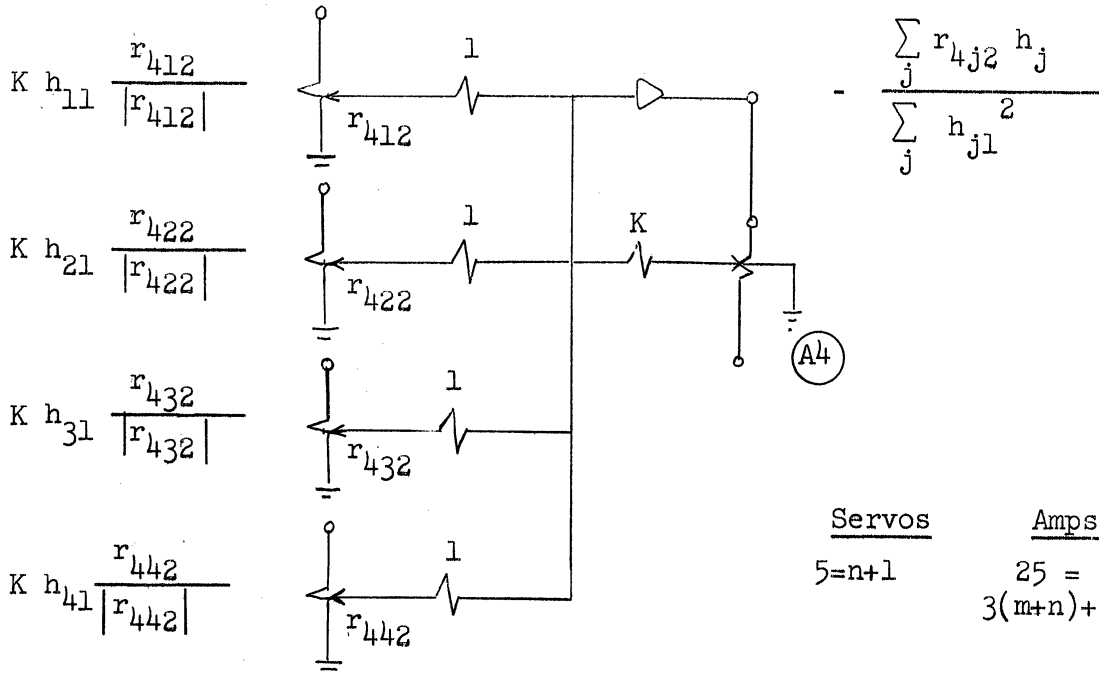
$$h_{j1}^0 = \sum_{i=1}^m r_{ij2} g_{i1} - h_{j1}$$

make

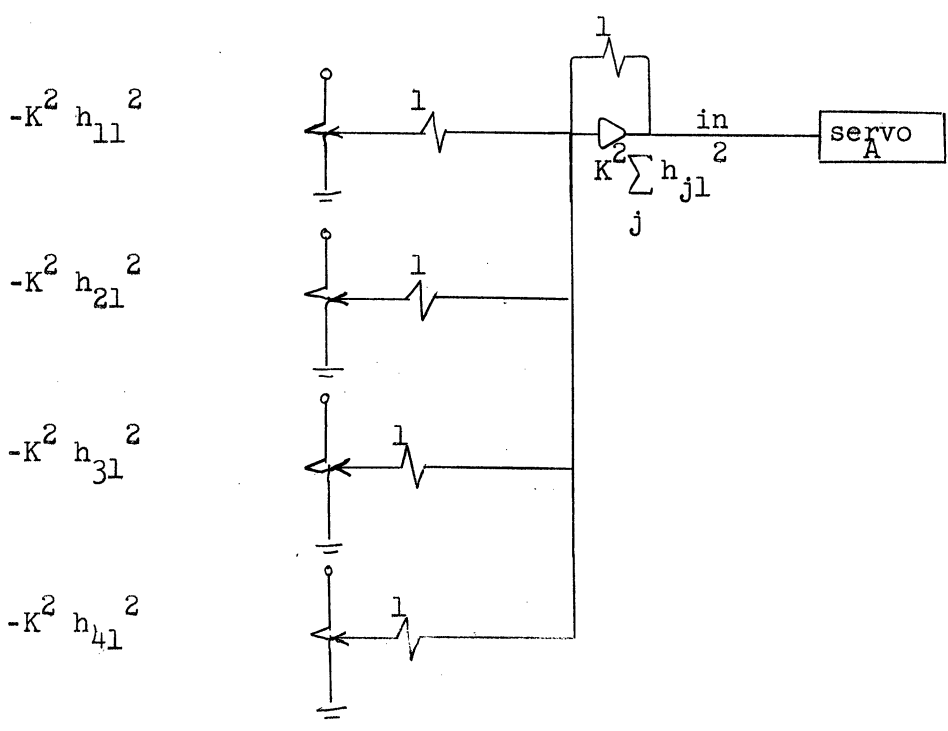
$$\sum_{i=1}^m g_{i1}^2 = 1.0 .$$

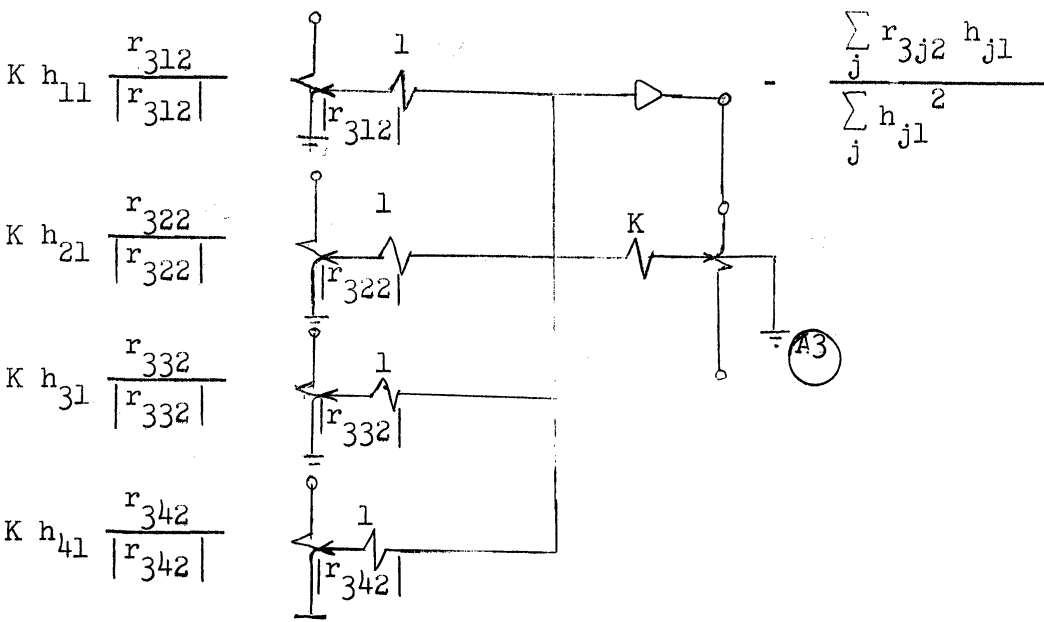
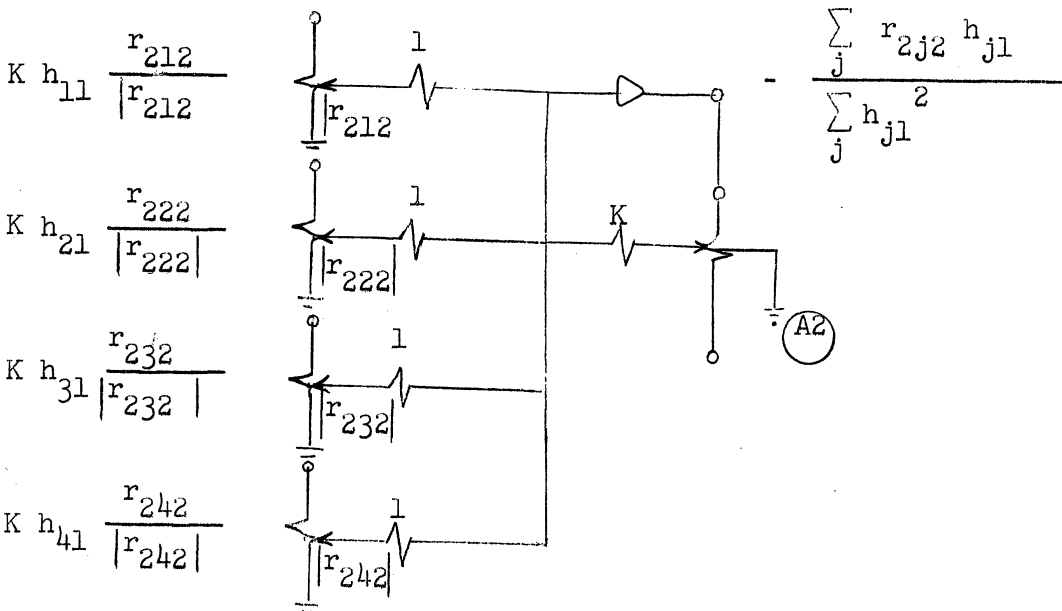
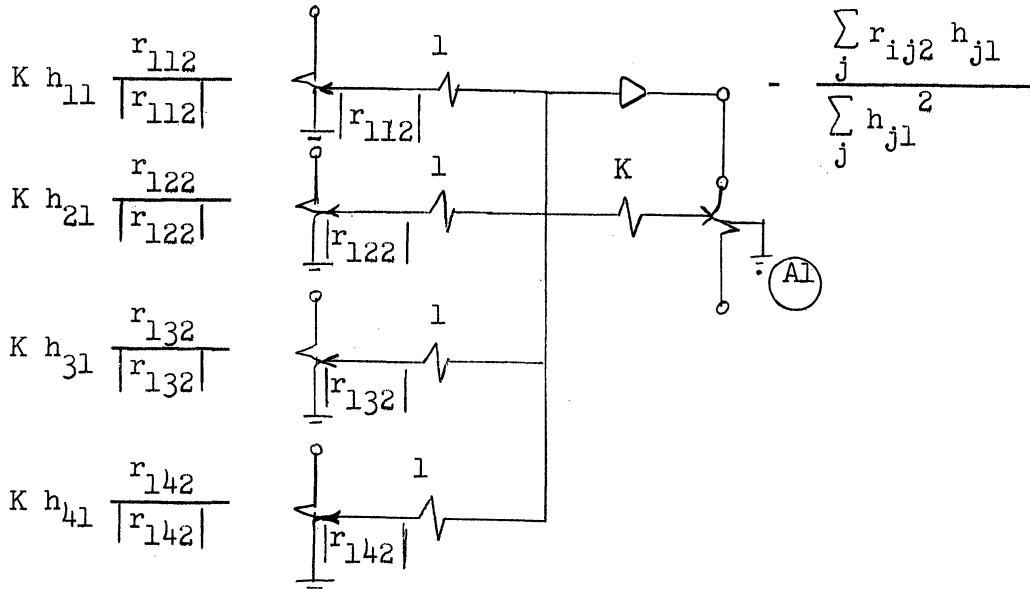
k is adjusted to make $k^2 \sum_{j=1}^n h_{j1}^2 \leq 1.0$. 4 x 4 example

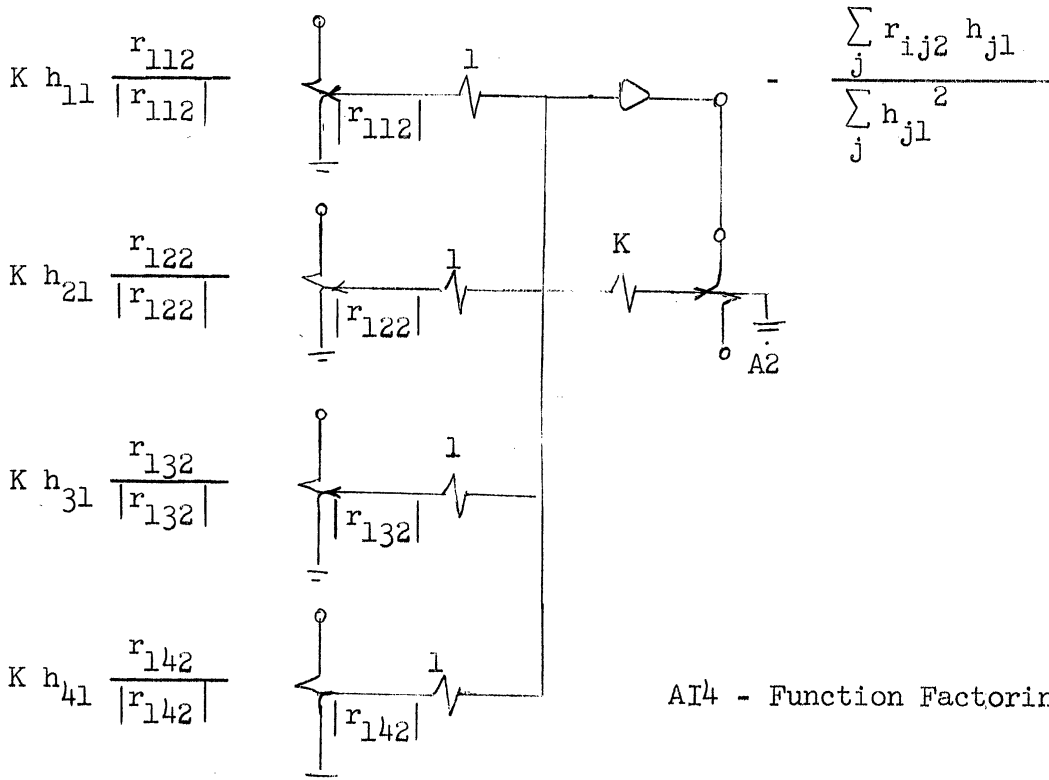
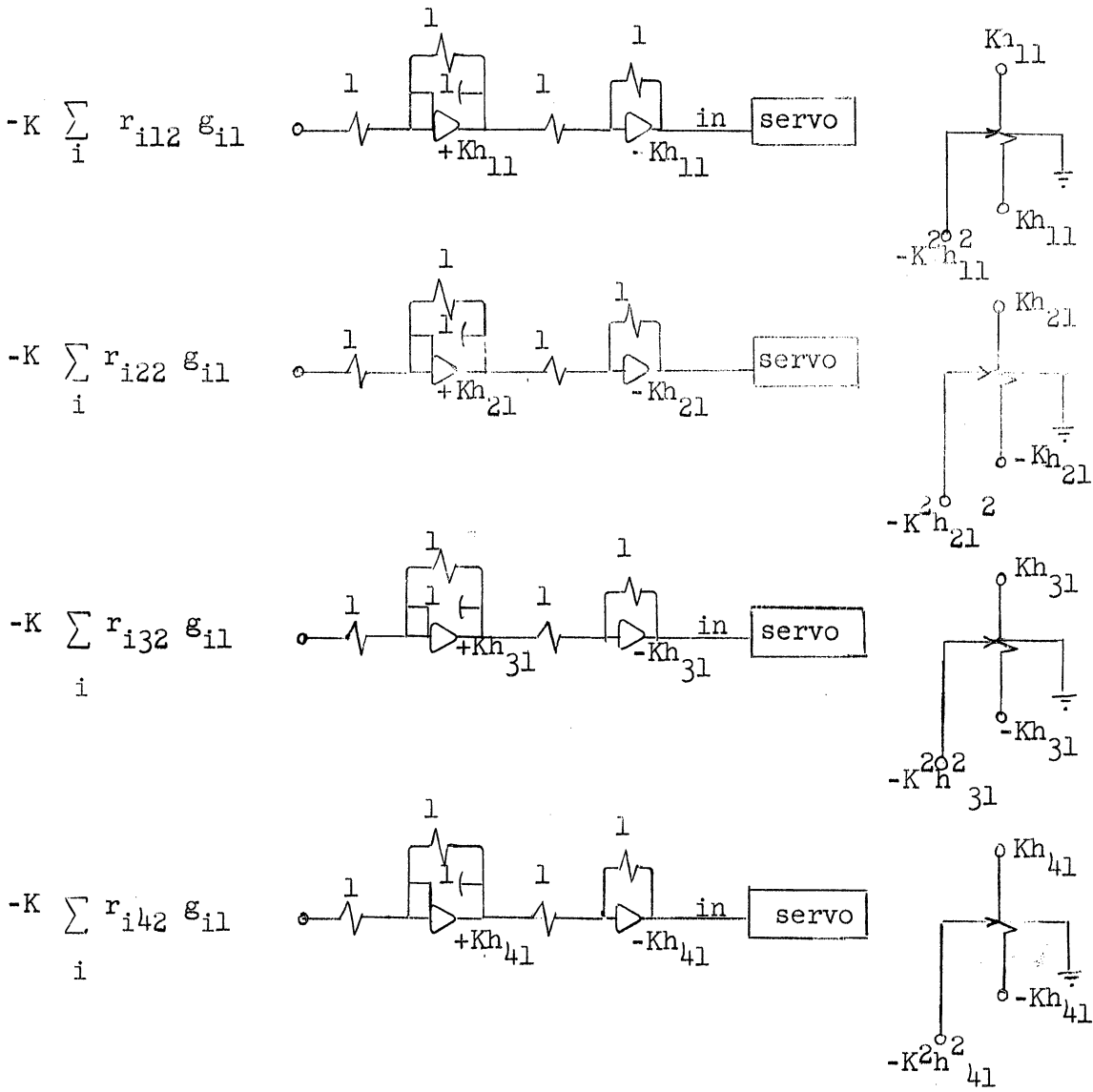


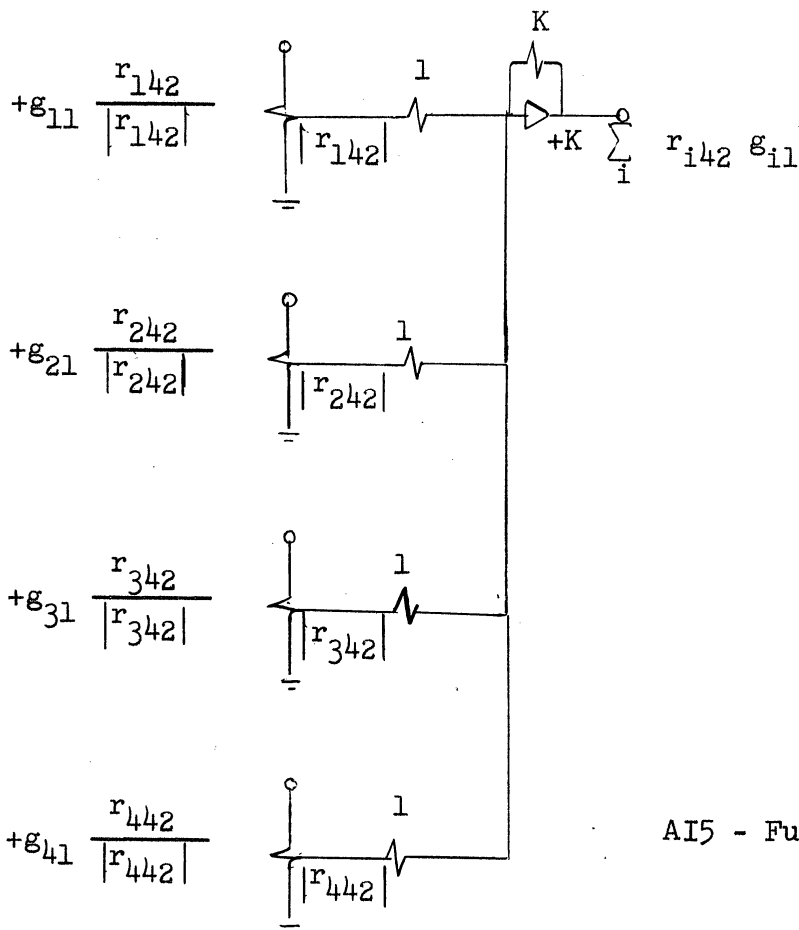
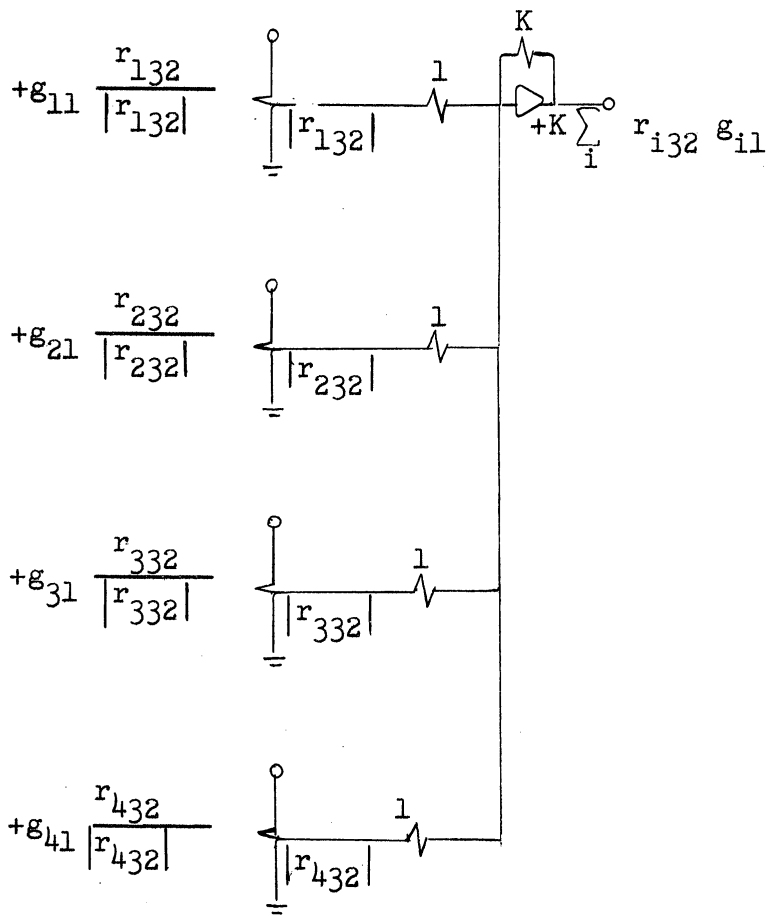


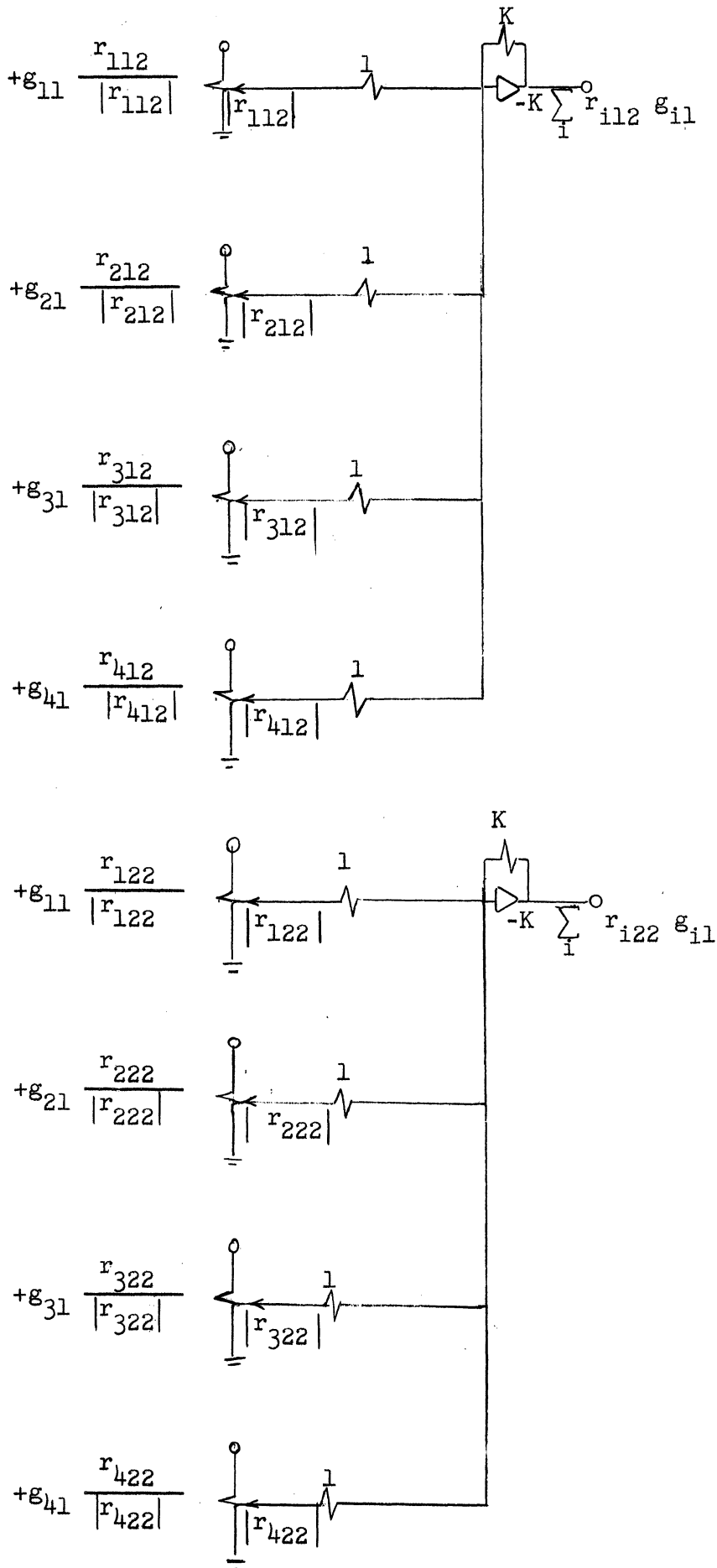
<u>Servos</u>	<u>Amps</u>	<u>Pots</u>
5=n+1	25 =	32=2mn
	3(m+n)+1	











Hand calculated values were

$$\begin{array}{ll} g_{12} = -.3749 & h_{12} = 0.1966 \times 2 \\ g_{22} = -.6027 & h_{22} = 0.0862 \times 2 \\ g_{32} = +.5844 & h_{32} = 0.4980 \times 2 \\ g_{42} = +.3933 & h_{42} = -.7809 \times 2 \end{array}$$

The biggest discrepancy is less than 0.6% of the correct value.

Computer records Figure AI5 through AI3 show the transient behavior of the computer solving this problem for four different sets of initial conditions. The system is obviously stable, relatively insensitive to initial condition changes, and reaches sufficiently close to steady-state conditions in about ten seconds. Changes in potentiometer settings to accommodate different $r_{i,j,k+1}$ sets takes about 15 minutes. Alternatively, the process of finding elements of the seks g_k and h_k may be pictured as a process for finding eigenvectors of a symmetric matrix, thus giving a possibility of using the same techniques employed in the text.

Acquisition of product factors g_{ik} and h_{jk} may be considered as a method for finding the m components g_{ik} ($i = 1, 2, \dots, m$) of a vector $\bar{g}]_k$ and n scalar multipliers h_{jk} ($j = 1, 2, \dots, n$) from a set of n vectors $\bar{r}]_{j,k}$ each with m components $r_{i,j,k}$ ($i = 1, 2, \dots, m$). These n vectors will occupy not more than $n-1$ dimensions since their sum is zero ($\sum_i r_{i,j,k} = 0$). They can occupy not more than m dimensions since this is the number of dimensions of the space in which they are embedded. Thus finally they will occupy no more dimensions than the smaller of m and $n-1$. The conditions defining $\bar{g}]_k$ and h_{jk} ($j = 1, 2, \dots, n$) are that $\sum_{i=1}^m \sum_{j=1}^n (r_{i,j,k} - g_{ik} h_{jk})^2$ be a minimum. This can be written $\sum_{i=1}^m \sum_{j=1}^n r_{i,j,k+1}^2$ must be a minimum where $r_{i,j,k+1} = r_{i,j,k} - g_{ik} h_{jk}$. Now $\sum_{i=1}^m \sum_{j=1}^n r_{i,j,k+1}^2 = \sum_{j=1}^n (\bar{r}]_{j,k} - \bar{g}_k h_{jk})^2 = \sum_{j=1}^n \bar{r}]_{j,k+1} \bar{r}]_{j,k+1}$, which is to say that the sums of the squared magnitudes of the remainder vectors formed by subtracting from the original remainder vectors $\bar{r}]_{j,k}$ the vectors $h_{jk} \bar{g}]_k$ must be a minimum. Now this sum of squares can be interpreted as the moment of inertia about the $\bar{g}]_k$ axis of unit masses at the tips of the $\bar{r}]_{j,k}$ vectors. This moment of inertia will be a minimum for $\bar{g}]_k$ the principal axis of minimum moment of inertia. Furthermore, the $\bar{g}]_{k+1}$ vector will be the axis of minimum moment of inertia for a set of unit masses at the tips of the $\bar{r}]_{j,k+1}$ vectors which is also the principal axis of next to minimum moment of inertia for the set of unit masses at the tips of the $\bar{r}]_{j,k}$

vectors. The successive \bar{g} vectors are thus the principal axes of moment of inertia of the set of unit masses at the tips of the $\bar{r}]_{j,k}$ vectors. These axes exist, are orthogonal, and there can be no more than m of them. The symmetric inertia tensor from which they can be obtained can be written

$$\begin{array}{c}
 \left[\begin{array}{c}
 \sum_{\substack{i=1 \\ i \neq 1}}^m \sum_{j=1}^m r_{ij}^2 - \sum_{j=1}^n r_{1j} r_{2j} \cdots - \sum_{j=1}^n r_{mj} r_{ij} \\
 \\
 - \sum_{j=1}^n r_{2j} r_{1j} \sum_{\substack{i=1 \\ i \neq 2}}^m \sum_{j=1}^n r_{ij}^2 \cdots - \sum_{j=1}^n r_{2j} r_{mj} \\
 \\
 \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\
 \\
 - \sum_{j=1}^n r_{mj} r_{1j} - \sum_{j=1}^n r_{mj} r_{2j} \cdots - \sum_{\substack{i=1 \\ i \neq m}}^m \sum_{j=1}^n r_{ij}^2
 \end{array} \right]
 \end{array}$$

The \bar{g} vectors are the eigenvectors of this matrix. If $n-1 < m$, then there will be $m-n+1$ equal maximum eigenvalues. The first \bar{g} vector ($\bar{g}]_2$) will correspond to the fundamental eigenvalue. The λ 's represent moments of inertia about the corresponding eigenvector axes.

The above analysis obviously applies without change if $\bar{g}]_k$, h_{jk} and $\bar{r}]_{j,k}$ are replaced by $h]_k$, g_{ik} and $\bar{r}]_{i,k}$ respectively. From this last fact we deduce that we require no more than $m-1$ or n product sets, whichever is fewer. Therefore, considering this and the corresponding statement made previously, no more than $m-1$ or $n-1$ sets (whichever is fewer) will ever be required for a perfect and complete reproduction of the original data.

The preliminary steps of the function-factoring process may likewise be examined from the point of view of their effect on a vector set of function values. The first such steps consists of subtracting from the original function values their average value. We define a set

$$r_{ijl} = f_{ij} - a$$

where

$$a = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f_{ij} .$$

If now we consider the f_{ij} as a set of vectors $f]_j$ each with m components, then the preceding step forms a new set $r]_{j1}$ by subtracting from each $f]_j$ the vector $a]_1$ where the vector $1]$ is the vector with unit positive components along each axis.

The next step is to form

$$r_{ij2} = r_{ij1} - g_{i1} - h_{j1}$$

where

$$g_{i1} = \frac{1}{n} \sum_{j=1}^n r_{ij1}$$

$$h_{j1} = \frac{1}{m} \sum_{i=1}^m r_{ij1}$$

or equivalently

$$r]_{j2} = r]_{j1} - g]_1 - h_{j1} \bar{1}]$$

where

$\bar{g}]$ is the vector to the center of gravity of the $r]_{j1}$ vectors and where $m h_{j1} = r]_{j1} \cdot \bar{1}]$ is the scalar product of the $r]_{j1}$ vector and the $\bar{1}]$ vector.

Since

$$g]_1 \cdot \bar{1}] = \frac{1}{n} \sum_{i=1}^m \sum_{j=1}^n r_{ij1} = 0,$$

$$r]_{j2} \cdot \bar{1}] = r]_{j1} \cdot \bar{1}] - g]_1 \cdot \bar{1}] - h_{j1} \bar{1}] \cdot \bar{1}]$$

$$r]_{j2} \cdot \bar{1}] = m h_{j1} - 0 - m h_{j1} = 0$$

which is to say, the $\bar{r}]_{j2}$ vectors all lie in the $m-1$ dimensional subspace normal to the $\bar{1}]$ vector. This may be otherwise stated

$$\sum_{j=1}^m r_{ij2} = 0.$$

Also, since

$$\sum_{j=1}^n r_{ij2} = 0,$$

then

$$\sum_{j=1}^n r]_{j2} = 0$$

which is to say, the vector sum of the $r]_{j2}$ vectors is zero. The $r]_{j2}$ vectors thus occupy not more than $m-1$ or $n-1$ dimensions whichever is fewer.

Example: At this stage a 3×3 set of data points would present a picture like Figure 1. In this figure we have taken

$$\begin{aligned} r_2(x_1, y_1) = r_{112} &= 1 & r_{122} &= -3 & r_{132} &= 2 \\ r_2(x_2, y_1) = r_{212} &= 2 & r_{222} &= -1 & r_{232} &= -1 \\ r_2(x_3, y_1) = r_{312} &= -3 & r_{322} &= 4 & r_{332} &= -1 \end{aligned}$$

so that $r]_{j2} = r]_{12}$ has components $1, 2, -3$, $r]_{22}$ has components $-3, -1, 4$, etc. In the figure, the observer is looking toward the origin from a point in the first octant equidistant from the axes. The plane of the $r]$ vectors, $x_1 + x_2 + x_3 = 0$, is thus normal to the line of sight. The moment of inertia matrix is formed as

$$I_{x_1 x_1} = \sum_{i=2}^3 \sum_{j=1}^3 r_{ij}^2 = 2^2 + 1^2 + 1^2 + 3^2 + 4^2 + 1^2 = 32$$

$$I_{x_1 x_2} = - \sum_{j=1}^3 r_{1j} r_{2j} = - \left[1 \times 2 + (-3)(-1) + 2(-1) \right] = -3 = I_{x_2 x_1}$$

etc.

so that, finally

$$I_g = \begin{bmatrix} 32 & -3 & 17 \\ -3 & 40 & 9 \\ 17 & 9 & 20 \end{bmatrix}$$

The eigenvalues of this matrix, from

$$\begin{vmatrix} 32 & -\lambda & -3 & 17 \\ -3 & 40 & -\lambda & 9 \\ 17 & 9 & 20 & -\lambda \end{vmatrix} = 0$$

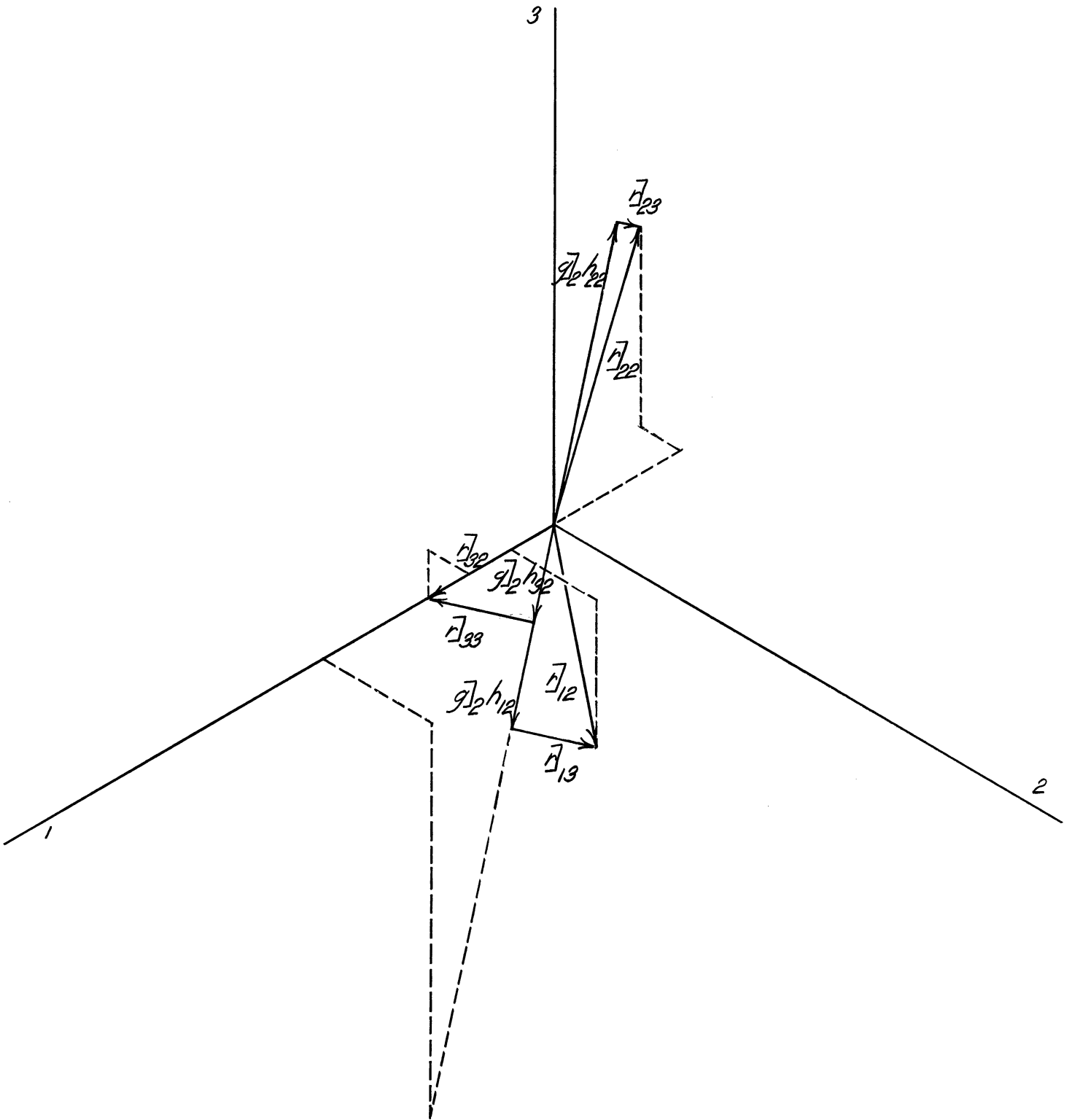


Figure I6

Function Factoring Vector Sets

are

$$\lambda = 5.56, 40.47, 45.97 .$$

The eigenvector corresponding to $\lambda_1 = 5.56$ is

$$g]_2 = \begin{bmatrix} 0.542 \\ 0.257 \\ -.799 \end{bmatrix} = \begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \end{bmatrix}$$

Thus

$$h_{12} = g]_2 \circ r]_{12} = 1 \cdot 0.542 + 2 \cdot 0.257 + (-3) (-.799) = 3.453$$

$$h_{22} = g]_2 \circ r]_{22} = -5.079$$

$$h_{32} = g]_2 \circ r]_{32} = 1.626.$$

Recalculating the elements of $g]_2$ using the above values of h_{j2} yields exactly the original set. The eigenvalue 5.56 is the moment of inertia about the $g]_2$ axis or, equivalently, it is the mean square error of fit after the terms $g_{i2} h_{j2}$ are subtracted from the elements r_{ij2} . The $g]_2 h_{j2}$ vectors are shown in Figure 1. Also shown are the remainders

$$r]_{j3} = r]_{j2} - g]_2 h_{j2} .$$

The $r]_{j3}$ are colinear so that subtracting one more product $g]_3 h_{j3}$ will annihilate them. $g]_3$ is the eigenvector

$$g]_3 = \begin{bmatrix} g_{13} \\ g_{23} \\ g_{33} \end{bmatrix} = \begin{bmatrix} -0.611 \\ +0.774 \\ -.165 \end{bmatrix}$$

corresponding to $\lambda_2 = 40.47$. The 40.47 would be the mean square error of fit if the vectors $h_{j3} g]_3$ were subtracted from the set $r]_{j2}$ without subtracting the set $h_{j2} g]_2$. The vector $g]_3$ corresponds to a zero eigenvalue of the moment of inertia matrix.

$$I_{g_2} = \begin{bmatrix} 3.49 & 2.63 & -.559 \\ 2.63 & 2.22 & 0.710 \\ -.559 & 0.710 & 5.41 \end{bmatrix}$$

of the remainder set

$$[r_{ij3}] = \begin{bmatrix} -.872 & -.247 & 1.118 \\ 1.112 & -.305 & -1.418 \\ -.241 & -.058 & 0.299 \end{bmatrix}$$

The eigenvector corresponding to the eigenvalue 45.97 of the matrix I_{g1} is the vector

$$\begin{bmatrix} 0.578 \\ 0.578 \\ 0.578 \end{bmatrix}$$

corresponding to the line of sight of the observer in Figure 1. Remainder components in this direction were eliminated by the first 3 steps of the function-factoring process.

Two important points are made clear by the preceding discussion. First, the best choice for a starting set of values for the iteration procedure for finding g_{ik} and h_{jk} is that row or column of $r_{i,j,k}$ which contains the biggest numbers. Not only will this set lie in the proper subspace containing g_{ik} or h_{jk} but also it must lie reasonably near the set being looked for since obviously the axis through the origin of minimum moment of inertia must lie reasonably close to the mass point furthest removed from the origin.

Second, the function factoring iterative procedure converges slowly whenever the remainder set is such that the corresponding moment of inertia ellipsoid is nearly a surface of revolution. In such a case the axis of minimum moment of inertia may be nearly arbitrary in two or more dimensions. Practically, the difficulty is not serious since, in such a case, it really wouldn't make much difference which way the axis pointed as long as it was in the subspace whose intersection with the momental ellipsoid was the hypersurface equivalent of an only slightly distorted sphere.

APPENDIX II

A Three-Dimensional Display TechniqueStereoscopic Projection on the Analog Computer

At a certain point of the development of the preceding work, it became important to have available a readily perceived display of a curve in space. In spite of the obvious utility of such a device, this seems to have remained one of the many problems which anyone could solve but which, somehow, no one ever got around to solving. There follows a set of equations which may be set up on an electronic differential analyzer for stereoscopic projections onto a plane of a point specified by components along three axes. Objects thus transformed may be plotted separately to large scale for photographing and viewing through an optical stereoscope or they may be plotted as small side-by-side displays to be viewed with the exercise of some ocular gymnastics.

Viewing a stereoscopic projection necessarily involves some eyestrain since the eyes must be focused and converged inconsistently. That is, the eyes must be aimed at the point in space to be viewed but they must be focused on the projection plane. This difficulty is minimized if the projection plane is passed through the approximate center of the object to be viewed. The specialization that the projection plane be passed through the three dimensional coordinate origin will therefore be assumed.

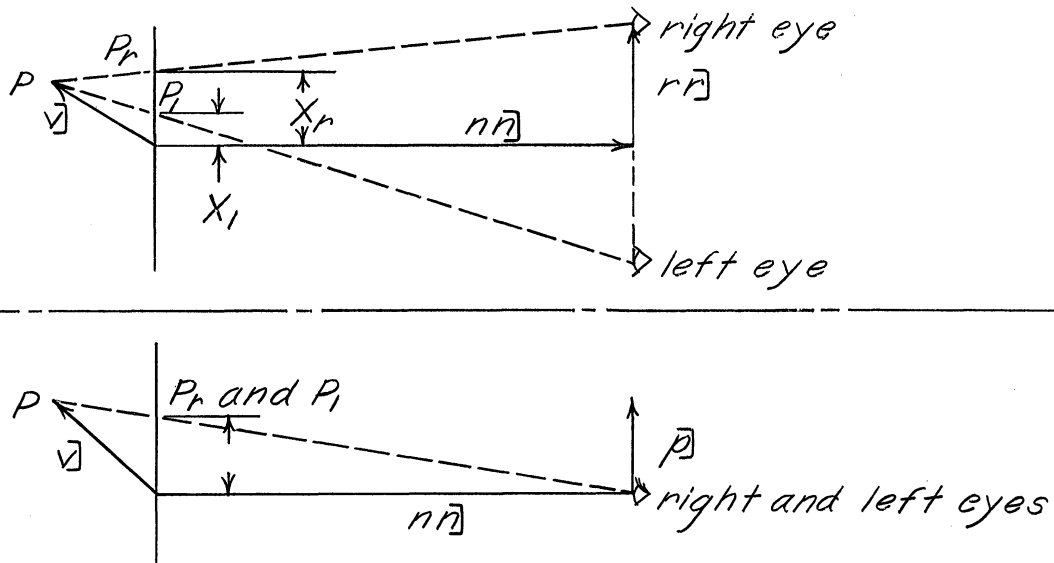


Figure I

AIII - Stereoscopic Display Diagram

The basic arrangement is shown in Figure I. A projection plane parallel to a line between the eyes is assumed. On this are drawn orthogonal axes, X and Y with the X axis parallel to the line between the eyes. Then, for any point P in space, the lines of sight joining P to the pupils pierces the projection plane at a unique distance Y above the X axis. If \underline{n} be a unit vector normal to the projection plane directed toward the observer and \underline{p} a unit vector along the Y axis in the projection plane, and if \underline{v} be the distance from pupil to projection plane, then calling \underline{v} the vector to the point P from the origin of the projection plane,

$$\frac{Y}{\underline{v} \cdot \underline{p}} = \frac{n}{n - \underline{n} \cdot \underline{v}}$$

or

$$Y = \frac{\underline{v} \cdot \underline{p}}{1 - \frac{\underline{v} \cdot \underline{n}}{n}} \quad (1)$$

Letting \underline{r} be a unit vector along the positive X axis of the projection plane, and calling one-half the interpupillary distance r , the X projection for the right eye becomes

$$\frac{r - X_r}{r - \underline{v} \cdot \underline{r}} = \frac{n}{n - \underline{n} \cdot \underline{v}}$$

$$X_r = \frac{\underline{v} \cdot \underline{r} - r}{1 - \frac{\underline{v} \cdot \underline{n}}{n}} + r \quad (2)$$

Similarly, the left eye projection X_l is

$$\frac{r + X_l}{\underline{v} \cdot \underline{r} - r} = \frac{n}{n - \underline{n} \cdot \underline{v}}$$

$$X_l = \frac{\underline{v} \cdot \underline{r} + r}{1 - \frac{\underline{v} \cdot \underline{n}}{n}} - r \quad (3)$$

If the projections are to be made side by side, then they must be separated some distance less than $2r$. This may be done by adding some quantity slightly less than r to X_r and subtracting a similar amount from X_l .

A convenient specialization of the above equations is that obtained from looking toward the origin from a point centered in the first octant. Taking

$$n = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$r = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$p = \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

gives

$$Y = \frac{\frac{1}{\sqrt{6}} (2x_3 - x_1 - x_2)}{1 - \frac{1}{n\sqrt{3}} (x_1 + x_2 + x_3)}$$

$$X_r = \frac{\frac{1}{\sqrt{2}} (x_2 - x_1) - r}{1 - \frac{1}{n\sqrt{3}} (x_1 + x_2 + x_3)} + r + \zeta$$

$$X_1 = \frac{\frac{1}{\sqrt{2}} (x_2 + x_1) + r}{1 - \frac{1}{n\sqrt{3}} (x_1 + x_2 + x_3)} - r - \zeta$$

With a multiplier scale of 100 volts per unit, and a plotter scale of 10 volts per inch.

$$r = 1.25 \text{ in.} \times 10 \frac{\text{v}}{\text{in.}} \times \frac{1 \text{ unit}}{100 \text{ v.}} = 0.125 \text{ units}$$

For a focal distance of 20 inches

$$n = 2.0 \text{ units}$$

Thus,

$$Y = \frac{\frac{1}{6} (2x_3 - x_1 - x_2)}{1 - \frac{1}{2\sqrt{3}} (x_1 + x_2 + x_3)}$$

$$X_r = \frac{\frac{1}{2} (x_2 - x_1) - 0.125}{1 - \frac{1}{2\sqrt{3}} (x_1 + x_2 + x_3)} + 0.23, \text{ (letting } \zeta = 0.105)$$

$$X_l = \frac{\frac{1}{2} (x_2 - x_1) + 0.125}{1 - \frac{1}{2\sqrt{3}} (x_1 + x_2 + x_3)} - 0.23$$

These were the transformation equations used in making Figures II, III, and IV. In addition, "throttle" control of the integrators in the setup of the differential equations whose solution paths are plotted was employed to make possible good enough simultaneous control of computer and plotter to allow showing visibility in the plots. This required preliminary plots to ascertain which line lay in front at each intersection of two lines in each of the two plots. The final plot was made by lifting the plotter pen each time the solution path passed behind another line.

$$\begin{bmatrix} .1 & .3 & .3 \\ .2 & .3 & .4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.156 \begin{bmatrix} 4 \\ -10 \\ 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

20 cc.

7-21-59
[Signature]

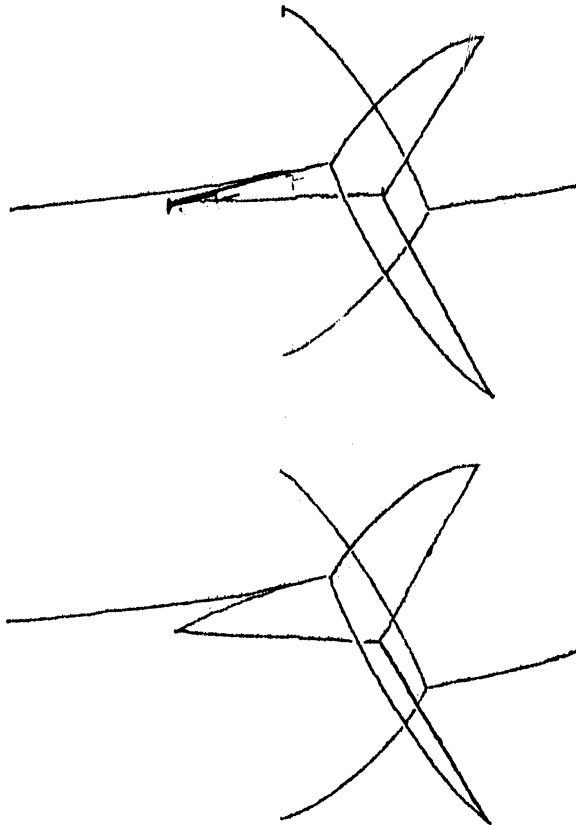


Figure II
 A112 - First Step Stereoscopic Display with Various Initial Conditions

$$\begin{bmatrix} .7 & .3 & .3 \\ .2 & .3 & .7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.156 \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Handwritten signature and date: *Handwritten signature* 7-21-59

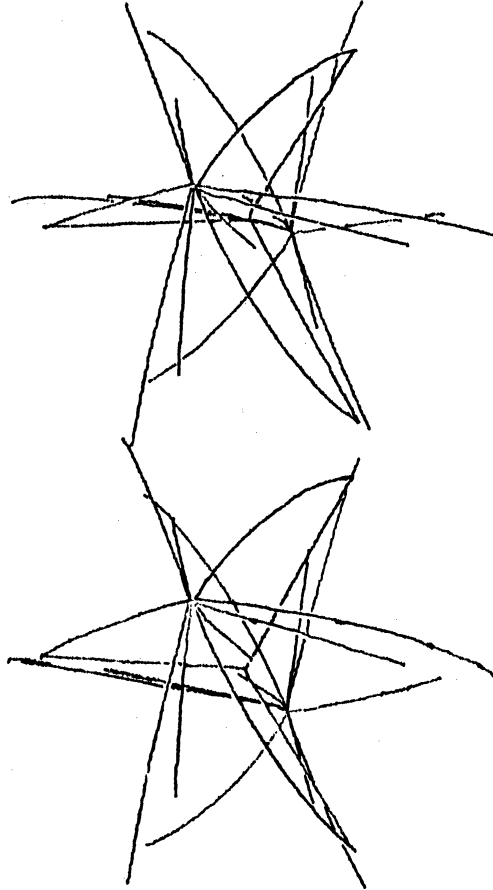


Figure III
 AII3 - First Step Stereoscopic Display with All
 Initial Conditions in Unit Sphere

$$\begin{bmatrix} 0.386 & -0.092 & 0.214 \\ 0.308 & 0.521 & 0.103 \\ 0.057 & -0.112 & 0.191 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.156 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 7.28 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

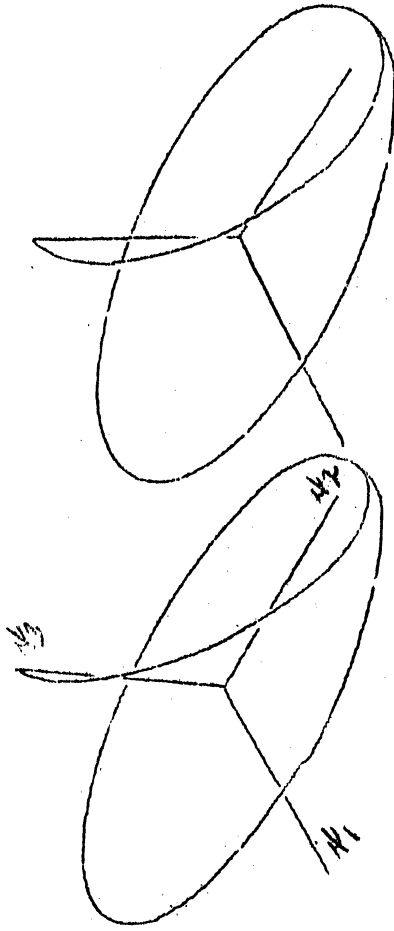


Figure IV
A114 - First Step Stereoscopic Display of
Complex Dominant Root Case

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