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A BENCHMARK PROBLEM FOR NONLINEAR CONTROL DESIGN

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1. INTRODUCTION

This paper describes a nonlinear control design problem involving the nonlinear interaction of a translational oscillator and an eccentric rotational proof mass. This problem provides a benchmark for examining nonlinear control design techniques within the framework of a nonlinear fourth-order dynamical system. The problem is posed in the spirit of the linear benchmark problem described in Reference 1.

This system was originally studied as a simplified model of a dual-spin spacecraft to investigate the resonance capture phenomenon.[2] More recently, it has been studied to investigate the utility of a rotational proof-mass actuator for stabilizing translational motion.[3–5] Viewed in this way, the rotational/translational proof-mass actuator (RTAC) has the feature that the nonlinearities associated with the actuator stroke limitation are implicit in the system dynamics. In contrast, the stroke limitation constraint must be considered separately in linear translational proof-mass actuators.[6] A similar system has been studied as a rotating unbalanced mass (RUM) actuator in References 7 and 8.

2. PROBLEM STATEMENT

The system shown in Figure 1 represents a translational oscillator with an eccentric rotational proof-mass actuator. The oscillator consists of a cart of mass M connected to a fixed wall by a linear spring of stiffness k. The cart is constrained to have one-dimensional travel. The proof-mass actuator attached to the cart has mass m and moment of inertia I about its centre of mass, which is located a distance e from the point about which the proof mass rotates. The motion occurs in a horizontal plane, so that no gravitational forces need to be considered. In Figure 1, N denotes the control torque applied to the proof mass, and F is the disturbance force on the cart.

Let q and $\dot{q}$ denote the translational position and velocity of the cart, and let θ and $\dot{\theta}$ denote the angular position and velocity of the rotational proof mass, where $\theta = 0$ is perpendicular to the

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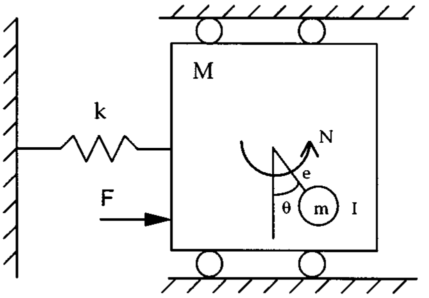

Figure 1. Rotational actuator to control a translational oscillator

motion of the cart, and $\theta = 90^\circ$ is aligned with the positive q direction. The equations of motion are given by

$$(M + m)\ddot{q} + kq = -me(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + F$$

$$(I + me^2)\ddot{\theta} = -me\ddot{q}\cos\theta + N$$

With the normalizations[3]

$$\xi \triangleq \sqrt{\frac{M+m}{I+me^2}}\,q, \qquad \tau \triangleq \sqrt{\frac{k}{M+m}}\,t$$

$$u \triangleq \frac{M+m}{k(I+me^2)}N, \qquad w \triangleq \frac{1}{k}\sqrt{\frac{M+m}{I+me^2}}\,F,$$

the equations of motion become

$$\ddot{\xi} + \xi = \varepsilon\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right) + w$$

$$\ddot{\theta} = -\varepsilon\ddot{\xi}\cos\theta + u$$

where ξ is the normalized cart position, and w and u represent the non-dimensionalized disturbance and control torque, respectively. In the normalized equations, the symbol $(\dot{\cdot})$ represents differentiation with respect to the normalized time τ. The coupling between the translational and rotational motions is represented by the parameter ε which is defined by

$$\varepsilon \triangleq \frac{me}{\sqrt{(I+me^2)(M+m)}}$$

Letting $x = [x_1, x_2, x_3, x_4]^T = [\xi, \dot{\xi}, \theta, \dot{\theta}]^T$, the non-dimensional equations of motion in first-order form are given by

$$\dot{x} = f(x) + g(x)\,u + d(x)\,w,$$

where

$$f(x) = \begin{bmatrix} x_2 \\ \dfrac{-x_1 + \varepsilon x_4^2 \sin x_3}{1 - \varepsilon^2 \cos^2 x_3} \\ x_4 \\ \dfrac{\varepsilon \cos x_3 (x_1 - \varepsilon x_4^2 \sin x_3)}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ \dfrac{-\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \\ 0 \\ \dfrac{1}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix}, \quad d(x) = \begin{bmatrix} 0 \\ \dfrac{1}{1 - \varepsilon^2 \cos^2 x_3} \\ 0 \\ \dfrac{-\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix}$$

The nonlinear benchmark problem can now be stated as follows.

Nonlinear benchmark problem

Design a controller that satisfies the following criteria:

1. The closed-loop system is stable (e.g. locally or globally).
2. The closed-loop system exhibits good settling behaviour for a class of initial conditions.
3. The closed-loop system exhibits good disturbance rejection compared to the uncontrolled oscillator for a class of disturbance signals.
4. The control effort should be reasonable (e.g. maximum torque).

It may be interesting to consider the following, optional objective:

5. The controller should not distinguish between the values θ and $\theta \bmod 2\pi$, since these values represent the same rotational configuration.

The requirements 1–4 for stabilization, free response, disturbance rejection and control effort are not precisely stated. Instead, each designer is given some freedom to interpret these issues individually. Requirement 5 avoids 'unwinding', i.e. the use of control effort to move the system from $[0, 0, 2\pi n, 0]^T$ to $[0, 0, 0, 0]^T$. Additionally, each designer may impose additional constraints on the problem as desired, where such features serve to highlight the capabilities of particular nonlinear control design methods.

3. LABORATORY TESTBED

A laboratory-scale version of the nonlinear benchmark problem has been constructed.[9] The parameters for a nominal configuration are given in Table I.

Table I

Description	Parameter	Value	Units
Cart mass	M	1·3608	kg
Arm mass	m	0·096	kg
Arm eccentricity	e	0·0592	m
Arm inertia	I	0·0002175	kg m^2
Spring stiffness	k	186·3	N/m
Coupling parameter	ε	0·200	—

The physical configuration of the system necessitates the constraint

$$|q| \leqslant 0{\cdot}025 \text{ m}$$

In addition, the control torque is limited by $N \leqslant 0.100$ N m continuous, although somewhat higher torques can be tolerated for short periods.

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