LQG Control with an H_∞ Performance Bound: A Riccati Equation Approach

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Abstract—**An LQG control-design problem involving a constraint on H_∞ disturbance attenuation is considered. The H_∞ performance constraint is embedded within the optimization process by replacing the covariance Lyapunov equation by a Riccati equation whose solution leads to an upper bound on L_2 performance. In contrast to the pair of separated Riccati equations of standard LQG theory, the H_∞-constrained gains are given by a coupled system of three modified Riccati equations. The coupling illustrates the breakdown of the separation principle for the H_∞-constrained problem. Both full- and reduced-order design problems are considered with an H_∞ attenuation constraint involving both state and control variables. An algorithm is developed for the full-order design problem and illustrative numerical results are given.**

I. INTRODUCTION

THE fundamental differences between Wiener–Hopf–Kalman (WHK) control design (for example, LQG theory [1]) and H_∞ control theory [2]–[4] can be traced to the modeling and treatment of uncertain exogenous disturbances. As explained by Zames in the classic paper [2], LQG design is based upon a stochastic noise disturbance model possessing a fixed covariance (power spectral density), while H_∞ theory is predicated on a deterministic disturbance model consisting of bounded power (square-integrable) signals. Since LQG design utilizes a quadratic cost criterion, it follows from Plancherel's theorem that WHK theory strives to minimize the L_2 norm of the closed-loop frequency response, while H_∞ theory seeks to minimize the worst-case attenuation. For systems with poorly modeled disturbances which may possess significant power within arbitrarily small bandwidths, H_∞ is clearly appropriate, while for systems with well-known disturbance power spectral densities, WHK design may be less conservative.

In addition to the fact that H_∞ design embodies many classical design objectives [5], it also presents a natural tool for modeling plant uncertainty in terms of normed H_∞ plant neighborhoods. In contrast, the H_2 topology has been shown in [6] to be too weak for a practical robustness theory, while the H_∞ norm is not only suitable for robust stabilization but is also conveniently submultiplicative. Within the WHK state-space theory, however, the appropriate robustness model appears not to be a nonparametric normed plant neighborhood as in H_∞ theory, but rather a parametric uncertainty model. The principal technique for bounding the effects of real parameters within state-space models is Lyapunov function theory (see, e.g., [7]–[16] and the references therein). Such structured uncertainties are difficult to capture nonconservatively within H_∞ theory except with specialized refinements [17].

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In spite of the fundamental differences between WHK design and H_∞ theory, a significant connection was discovered in [18]. Specifically, Petersen observed that a modified algebraic Riccati equation developed for parameter-robust full-state-feedback control can be reinterpreted to yield controllers satisfying H_∞ disturbance attenuation bounds. This relationship was further explored in [19] where it was shown that the H_∞-optimal static full-state-feedback controller is also optimal over the class of dynamic full-state-feedback controllers. The results of [18]–[20] thus solve the standard problem considered in [3] and [4] for the full-state-feedback case.

The extension of these results to the standard problem for dynamic output-feedback compensation, however, was not given in [18]–[20]. Within the realm of quadratic robust stabilization, the dynamic output-feedback problem was addressed in [7]. The results of [7] involve a pair of decoupled modified Riccati equations along with an auxiliary inequality. Using different techniques, a more complete solution was obtained in [13] and [14] involving a coupled system of three modified Riccati equations for full-order dynamic compensation and a coupled system of four modified Riccati and Lyapunov equations in the fixed-order (i.e., reduced-order) case as in [21]. The results of [13] and [14] thus raise the following question: What is the relevance of this system of coupled design equations to the problem of H_∞ disturbance attenuation via fixed-order compensation?

To begin to address this question, the goal of the present paper is to develop a design methodology for fixed-order, i.e., full- and reduced-order, L_2 optimal control which includes as a design constraint a bound on H_∞ disturbance attenuation. There are three principal motivations for developing such a theory. First, the results of [18]–[20] present full-state-feedback controllers whose form is directly analogous to the standard LQR solution. However, no L_2 interpretation was provided in [18]–[20] to explain this similarity. The present paper thus provides an L_2 interpretation within the context of an H_∞ design constraint. A novel feature of this mathematical formulation is the dual interpretation of the disturbances. That is, within the context of L_2 optimality the disturbances are interpreted as white noise signals while, simultaneously, for the purpose of H_∞ attenuation the very same disturbance signals have the alternative interpretation of deterministic L_2 functions. This dual interpretation is unique to the present paper since stochastic modeling plays no role in [18]–[20]. We also note recent results obtained in [22] which essentially show that the H_2 plant topology can be induced by imposing L_2 and L_∞ topologies on the disturbance and output spaces, respectively. For further investigation into the relationships between L_2 and H_∞ control, see [22a].

The second motivation for our approach is the simultaneous treatment of both L_2 and H_∞ performance criteria which quantitatively demonstrates design tradeoffs. Specifically, in order to enforce the H_∞ constraint we derive an upper bound for the L_2 criterion. Minimization of this upper bound shows that the enforcement of an H_∞ disturbance attenuation constraint leads directly to an increase in the L_2 performance criterion.

The third motivation for our approach is to provide a characterization of fixed-order dynamic output-feedback control-

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lers yielding specified disturbance attenuation. Existing optimal H_∞ design methods tend to yield high-order controllers. Intuitively, solving the fixed-order design equations for progressively smaller H_∞ disturbance attenuation constraints should, in the limit, yield an H_∞-optimal controller over the class of fixed-order stabilizing controllers. Although our main result gives sufficient conditions, we also state hypotheses under which these conditions are also necessary (Proposition 4.1). It should also be noted that the inherent coupling among the modified Riccati equations shows that the classical separation principle of LQG theory is not valid for the H_∞-constrained full- and reduced-order design problems.

In the full-order case involving equalized L_2 and H_∞ performance weights, we also show that the H_∞-constrained gains are given by two rather than three equations (Section V). These two equations are precisely those given in [26] for the pure H_∞ problem without an L_2 interpretation. Since the results of [26] are necessary as well as sufficient, these connections show that our sufficient conditions (at least in this special case) are also necessary. The authors are indebted to Prof. J. C. Doyle for pointing out these relationships and to D. Mustafa for providing a preprint of [45] which further clarifies these connections.

Besides establishing connections with robust stabilizability in state-space systems, an immediate benefit of the modified Riccati equation characterization of H_∞-constrained controllers is the opportunity to develop novel computational algorithms for controller synthesis. To this end a continuation algorithm has been developed for solving the coupled system of three modified Riccati equations. In a numerical study (see Section VIII) we have demonstrated convergence of the algorithm and reasonable computational efficiency. Homotopy methods were suggested for the coupled Riccati equations because of their demonstrated effectiveness in related problems which also involve coupled modified Riccati equations [23]–[25]. Since H_∞ control problems are solvable by established numerical methods [4], it should be stressed that the objective of these numerical studies is not to make direct comparisons with existing H_∞ synthesis algorithms, but rather to demonstrate solvability of the coupled modified Riccati equations.

The contents of the paper are as follows. After presenting notation at the end of this section, the statement of the H_∞-constrained LQG control problem is given in Section II. The principal result of Section II (Lemma 2.1) shows that if the algebraic Lyapunov equation for the closed-loop covariance is replaced by a modified Riccati equation possessing a nonnegative-definite solution, then the closed-loop system is asymptotically stable, the H_∞ disturbance attentuation constraint is satisfied, and the L_2 performance is bounded above by an auxiliary cost function. The problem of determining compensator gains which minimize this upper bound subject to the Riccati equation constraint is considered in Section III as the auxiliary minimization problem. Necessary conditions for the auxiliary minimization problem (Theorem 3.1) are given in the form of a coupled system of three modified Riccati equations. In Section IV the necessary conditions of Theorem 3.1 are combined with Lemma 2.1 to yield sufficient conditions for closed-loop stability, H_∞ disturbance attenuation, and bounded L_2 performance. In Section V we derive alternative forms of the design equations and specialize the results to the simpler case in which the LQG weights are equal to the H_∞ weights. To achieve further design flexibility, the reduced-order control-design problem is considered in Section VI. A simplified qualitative analysis of the full-order design equations is given in Section VII to highlight important features with regard to existence and multiplicity of solutions. Finally, a numerical algorithm is presented in Section VIII along with illustrative numerical results. A series of designs is obtained to illustrate tradeoffs between the L_2 and H_∞ aspects and the conservatism of the L_2 performance bound. Although in the present paper the numerical results are limited to the case of full-order dynamic compensation, reduced-order designs have been obtained in [27] using Theorem 6.1.

Notation

Note: All matrices have real entries.

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$	Real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value
I_r, $(\ )^T$, $0_{r\times s}$, 0_r	$r \times r$ identity matrix, transpose, $r \times s$ zero matrix, $0_{r\times r}$
tr, $\rho(\ )$	Trace, spectral radius
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \le Z_2$, $Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r$, $Z_2 - Z_1 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$
$n, m, l, n_c, p, q, q_\infty$; $\tilde{n}$	Positive integers; $n + n_c (n_c \le n)$
$x, u, y, x_c, \tilde{x}$	$n, m, l, n_c, \tilde{n}$-dimensional vectors
$\tilde{x}$	$\begin{bmatrix} x \\ x_c \end{bmatrix}$
A, B, C	$n \times n$, $n \times m$, $l \times n$ matrices
A_c, B_c, C_c	$n_c \times n_c$, $n_c \times l$, $m \times n_c$ matrices
$\tilde{A}$	$\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}$
$w(\cdot)$	p-dimensional standard white noise
D_1, D_2	$n \times p$, $l \times p$ matrices; $D_1 D_2^T = 0$
V_1, V_2	$D_1 D_1^T$, $D_2 D_2^T$; $V_2 \in \mathbb{P}^l$
$\tilde{D}$	$\begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}$
$\tilde{V}$	$\begin{bmatrix} V_1 & 0_{n\times n_c} \\ 0_{n_c\times n} & B_cV_2B_c^T \end{bmatrix}$
E_1, E_2	$q \times n$, $q \times m$ matrices; $E_1^TE_2 = 0$
$\tilde{E}$	$[E_1\ \ E_2C_c]$
R_1, R_2	$E_1^TE_1$, $E_2^TE_2$; $R_2 \in \mathbb{P}^m$
$\tilde{R}$	$\begin{bmatrix} R_1 & 0_{n\times n_c} \\ 0_{n_c\times n} & C_c^TR_2C_c \end{bmatrix} = \tilde{E}^T\tilde{E}$
$E_{1\infty}, E_{2\infty}$	$q_\infty \times n$, $q_\infty \times m$ matrices; $E_{1\infty}^TE_{2\infty} = 0$
$\tilde{E}_\infty$	$[E_{1\infty}\ \ E_{2\infty}C_c]$
$R_{1\infty}, R_{2\infty}$	$E_{1\infty}^TE_{1\infty}$, $E_{2\infty}^TE_{2\infty}$
$\tilde{R}_\infty$	$\begin{bmatrix} R_{1\infty} & 0_{n\times n_c} \\ 0_{n_c\times n} & C_c^TR_{2\infty}C_c \end{bmatrix} = \tilde{E}_\infty^T\tilde{E}_\infty$
Σ, $\bar{\Sigma}$	$BR_2^{-1}B^T$, $C^TV_2^{-1}C$
β, γ	Nonnegative constant, positive constant

II. Statement of the Problem

In this section we introduce the LQG dynamic output-feedback control problem with constrained H_∞ disturbance attenuation between the plant and sensor disturbances and the state and control variables. Without the L_2 performance criterion, the problem considered here essentially corresponds to the standard problem of [3] and [4]. For simplicity we restrict our attention to controllers of order $n_c = n$ only, i.e., controllers whose order is equal to the dimension of the plant. This constraint is removed in Section VI where controllers of reduced order are considered. Hence, throughout Sections II–V the controller dimension n_c and closed-loop plant dimension $\tilde{n} \triangleq n + n_c$ should be interpreted as n and $2n$, respectively. Controllers of order greater than n are generally of less interest in practice and thus are not considered in this paper.

H_∞-Constrained LQG Control Problem: Given the nth-order stabilizable and detectable plant

$$\dot{x}(t)=Ax(t)+Bu(t)+D_1w(t), \tag{2.1}$$

$$y(t)=Cx(t)+D_2w(t) \tag{2.2}$$

determine an nth-order dynamic compensator

$$\dot{x}_c(t)=A_cx_c(t)+B_cy(t), \tag{2.3}$$

$$u(t)=C_cx_c(t) \tag{2.4}$$

which satisfies the following design criteria:

i) the closed-loop system (2.1)–(2.4) is asymptotically stable, i.e., $\tilde{A}$ is asymptotically stable;

ii) the $q_\infty \times p$ closed-loop transfer function

$$H(s) \triangleq \tilde{E}_\infty(sI_{\tilde{n}}-\tilde{A})^{-1}\tilde{D} \tag{2.5}$$

from $w(t)$ to $E_{1\infty}x(t) + E_{2\infty}u(t)$ satisfies the constraint

$$\|H(s)\|_\infty \le \gamma \tag{2.6}$$

where $\gamma > 0$ is a given constant; and

iii) the performance functional

$$J(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \mathbb{E}[x^T(t)R_1x(t)+u^T(t)R_2u(t)] \tag{2.7}$$

is minimized.

Note that the closed-loop system (2.1)–(2.4) can be written as

$$\dot{\tilde{x}}(t)=\tilde{A}\tilde{x}(t)+\tilde{D}w(t) \tag{2.8}$$

and that (2.7) becomes

$$J(A_c, B_c, C_c)=\lim_{t\to\infty} \mathbb{E}[(\tilde{E}\tilde{x}(t))^T(\tilde{E}\tilde{x}(t))]$$
$$=\lim_{t\to\infty} \mathbb{E}[\tilde{x}^T(t)\tilde{R}\tilde{x}(t)]. \tag{2.9}$$

Remark 2.1: Since (A, B, C) is assumed to be stabilizable and detectable the set of nth-order stabilizing compensators is nonempty.

Remark 2.2: It is easy to show that the performance functional (2.7) is equivalent to the more familiar expression involving an averaged integral, i.e.,

$$J(A_c, B_c, C_c)=\lim_{t\to\infty} \frac{1}{t}\, \mathbb{E}\left\{\int_0^t [x^T(s)R_1x(s) + u^T(s)R_2u(s)]\, ds\right\}.$$

Remark 2.3: For convenience we assume $D_1D_2^T = 0$, which effectively implies that the plant disturbance and sensor noise are uncorrelated.

Remark 2.4: One may also consider a general L_2 optimization problem of the form min $\|T - UQV\|_2$, where Q is a parameterization of stabilizing controllers. In this case, without a constraint on the MacMillan degree of Q, it may be possible to satisfy (2.6) with smaller values of γ.

Note that the problem statement involves both L_2 and H_∞ performance weights. In particular, the matrices R_1 and R_2 are the L_2 weights for the state and control variables. By introducing L_2-weighted variables

$$z(t)=E_1x(t),\quad v(t)=E_2u(t)$$

the cost (2.7) can be written as

$$J(A_c, B_c, C_c)=\lim_{t\to\infty} \mathbb{E}[z^T(t)z(t)+v^T(t)v(t)].$$

For convenience we thus define $R_1 \triangleq E_1^TE_1$ and $R_2 \triangleq E_2^TE_2$ which appear in subsequent expressions. Although an L_2 cross-weighting term of the form $2x^T(t)R_{12}u(t)$ can also be included, we shall not do so here to facilitate the presentation.

For the H_∞ performance constraint, the transfer function (2.5) involves weighting matrices $E_{1\infty}$ and $E_{2\infty}$ for the state and control variables. The matrices $R_{1\infty} \triangleq E_{1\infty}^TE_{1\infty}$ and $R_{2\infty} \triangleq E_{2\infty}^TE_{2\infty}$ are thus the H_∞ counterparts of the L_2 weights R_1 and R_2. Although we do not require that $R_{1\infty}$ and $R_{2\infty}$ be equal to R_1 and R_2, we shall require in the next section that $R_{2\infty} = \beta^2R_2$, where the nonnegative scalar β is a design variable. Finally, the condition $E_{1\infty}^TE_{2\infty} = 0$ precludes an H_∞ cross-weighting term which again facilitates the presentation.

Before continuing, it is useful to note that if $\tilde{A}$ is asymptotically stable for a given compensator (A_c, B_c, C_c), then the performance (2.7) is given by

$$J(A_c, B_c, C_c)=\operatorname{tr} \tilde{Q}\tilde{R} \tag{2.10}$$

where the steady-state closed-loop state covariance defined by

$$\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^T(t)] \tag{2.11}$$

satisfies the $\tilde{n} \times \tilde{n}$ algebraic Lyapunov equation

$$0=\tilde{A}\tilde{Q}+\tilde{Q}\tilde{A}^T+\tilde{V}. \tag{2.12}$$

Remark 2.5: Using (2.10) and (2.12) it can be shown that the L_2 cost criterion (2.7) can be written in terms of the L_2 norm of the impulse response of the closed-loop system. Specifically, by writing $\tilde{Q}$ satisfying (2.12) as

$$\tilde{Q}=\int_0^\infty e^{\tilde{A}t}\tilde{V}e^{\tilde{A}^Tt}\, dt$$

(2.10) becomes

$$J(A_c, B_c, C_c)=\int_0^\infty \|\tilde{E}e^{\tilde{A}t}\tilde{D}\|_F^2\, dt$$

where $\|\cdot\|_F$ denotes the Frobenius matrix norm

The key step in enforcing the disturbance attenuation constraint (2.6) is to replace the algebraic Lyapunov equation (2.12) by an algebraic Riccati equation which overbounds the closed-loop steady-state covariance. Justification for this technique is provided by the following result.

Lemma 2.1: Let (A_c, B_c, C_c) be given and assume there exists $\mathcal{Q} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ satisfying

$$\mathcal{Q} \in \mathbb{N}^{\tilde{n}} \tag{2.13}$$

and

$$0=\tilde{A}\mathcal{Q}+\mathcal{Q}\tilde{A}^T+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q}+\tilde{V}. \tag{2.14}$$

Then

$$(\tilde{A}, \tilde{D}) \text{ is stabilizable} \tag{2.15}$$

if and only if

$$\tilde{A} \text{ is asymptotically stable.} \tag{2.16}$$

In this case,

$$\|H(s)\|_\infty \le \gamma \tag{2.17}$$

and

$$\tilde{Q} \le \mathcal{Q}. \tag{2.18}$$

Consequently,

$$J(A_c, B_c, C_c) \leq \mathcal{J}(A_c, B_c, C_c, \mathcal{Q}) \tag{2.19}$$

where

$$\mathcal{J}(A_c, B_c, C_c, \mathcal{Q}) \triangleq \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{2.20}$$

Proof: It follows from [28, Theorem 3.6] that (2.15) implies that $(\tilde{A}, [\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + \tilde{V}]^{1/2})$ is also stabilizable. Using the assumed existence of a nonnegative-definite solution to (2.14) and [28, Lemma 12.2], it now follows that $\tilde{A}$ is asymptotically stable. The converse is immediate. The proof of (2.17) follows from a standard manipulation of (2.14); for details see [29, Lemma 1]. To prove (2.18), subtract (2.12) from (2.14) to obtain

$$0 = \tilde{A}(\mathcal{Q} - \tilde{Q}) + (\mathcal{Q} - \tilde{Q})\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} \tag{2.21}$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\mathcal{Q} - \tilde{Q} = \int_0^\infty e^{\tilde{A}t}[\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q}]e^{\tilde{A}^Tt}\, dt \geq 0. \tag{2.22}$$

Finally, (2.19) follows immediately from (2.18). □

Remark 2.6: Note that (2.15) is actually a closed-loop disturbability condition which is not concerned with control as such. This condition guarantees that the system does not possess undisturbed unstable modes. Of course, if $\tilde{V}$ is positive definite or $(\tilde{A}, \tilde{D})$ is controllable, then (2.15) is satisfied.

Lemma 2.1 shows that the H_∞ disturbance attenuation constraint is automatically enforced when a nonnegative-definite solution to (2.14) is known to exist and $\tilde{A}$ is asymptotically stable. Furthermore, all such solutions provide upper bounds for the actual closed-loop state covariance $\tilde{Q}$ along with a bound on the L_2 performance criterion. Next, we present a partial converse of Lemma 2.1 which guarantees the existence of a unique minimal nonnegative-definite solution to (2.14) when (2.17) is satisfied. The minimal solution is desirable since it yields the least performance bound in (2.19). This was first pointed out in [45].

Lemma 2.2: Let (A_c, B_c, C_c) be given, suppose $\tilde{A}$ is asymptotically stable, and assume the disturbance attenuation constraint (2.17) is satisfied. Then there exists a unique nonnegative-definite solution $\mathcal{Q}$ satisfying (2.14) and such that $\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty$ is asymptotically stable. Furthermore, this solution is also minimal.

Proof: The result is an immediate consequence of [30, pp. 150 and 167], using Theorems 3 and 2, along with the dual version of [28, Lemma 12.2]. The proof of minimality is given in [29]. □

Remark 2.7: To further clarify the relationships between the L_2 and H_∞ aspects of the problem, we note that the closed-loop system can be represented by two possibly different transfer functions. Specifically, with respect to the L_2 cost criterion, the closed-loop transfer function between disturbances and controlled variables is given by the triple $(\tilde{A}, \tilde{D}, \tilde{E})$ while for the H_∞ constraint the closed-loop transfer function (2.5) corresponds to the triple $(\tilde{A}, \tilde{D}, \tilde{E}_\infty)$.

Finally, it can be shown that the closed-loop Riccati equation (2.14) also enforces a constraint on the norm of the Hankel operator corresponding to the closed-loop system $(\tilde{A}, \tilde{D}, \tilde{E}_\infty)$ when $\mathcal{Q}$ is positive definite. Thus, let $\tilde{P} \in \mathbb{N}^{\tilde{n}}$ denote the solution to

$$0 = \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + \tilde{R}_\infty \tag{2.23}$$

and note that $\tilde{P}$ and $\tilde{Q}$ [satisfying (2.12)] are the observability and controllability Gramians, respectively, of the system $(\tilde{A}, \tilde{D}, \tilde{E}_\infty)$. As shown in [31], the norm of the Hankel operator corresponding to $(\tilde{A}, \tilde{D}, \tilde{E}_\infty)$ is given by $\lambda_{\max}^{1/2}(\tilde{P}\tilde{Q})$.

Proposition 2.1: Suppose there exists $\mathcal{Q} \in \mathbb{P}^{\tilde{n}}$ satisfying (2.14) and such that (2.15) or, equivalently, (2.16) holds. Then

$$\lambda_{\max}^{1/2}(\tilde{P}\tilde{Q}) \leq \gamma. \tag{2.24}$$

Proof: Since $\mathcal{Q}$ is assumed to be invertible, (2.14) is equivalent to

$$0 = \gamma^2\tilde{A}^T\mathcal{Q}^{-1} + \gamma^2\mathcal{Q}^{-1}\tilde{A} + \gamma^2\mathcal{Q}^{-1}\tilde{V}\mathcal{Q}^{-1} + \tilde{R}_\infty. \tag{2.25}$$

Subtracting (2.23) from (2.25) shows that $\gamma^2\mathcal{Q}^{-1} - \tilde{P} \geq 0$, or, equivalently, $\gamma^2 I_{\tilde{n}} \geq \mathcal{Q}^{1/2}\tilde{P}\mathcal{Q}^{1/2}$. Thus,

$$\gamma^2 \geq \lambda_{\max}(\mathcal{Q}^{1/2}\tilde{P}\mathcal{Q}^{1/2}) = \lambda_{\max}(\tilde{P}^{1/2}\mathcal{Q}\tilde{P}^{1/2}) \geq \lambda_{\max}(\tilde{P}^{1/2}\tilde{Q}\tilde{P}^{1/2}) = \lambda_{\max}(\tilde{P}\tilde{Q})$$

which yields (2.24). □

III. The Auxiliary Minimization Problem and Necessary Conditions for Optimality

As discussed in the previous section, the replacement of (2.12) by (2.14) enforces the H_∞ disturbance attenuation constraint and yields an upper bound for the L_2 performance criterion. That is, given a compensator (A_c, B_c, C_c) for which there exists a nonnegative-definite solution to (2.14), the actual L_2 performance $J(A_c, B_c, C_c)$ of the compensator is guaranteed to be no worse than the bound given by $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$. Hence, $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$ can be interpreted as an auxiliary cost which leads to the following mathematical programming problem.

Auxiliary Minimization Problem: Determine $(A_c, B_c, C_c, \mathcal{Q})$ which minimizes $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$ subject to (2.13) and (2.14).

It follows from Lemma 2.1 that the satisfaction of (2.13) and (2.14) along with the generic condition (2.15) leads to: 1) closed-loop stability; 2) prespecified H_∞ performance attenuation; and 3) an upper bound for the L_2 performance criterion. Hence, it remains to determine (A_c, B_c, C_c) which minimizes $J(A_c, B_c, C_c, \mathcal{Q})$, and thus provides an optimized bound for the actual L_2 performance $J(A_c, B_c, C_c)$. Rigorous derivation of the necessary conditions for the auxiliary minimization problem requires additional technical assumptions. Specifically, we restrict $(A_c, B_c, C_c, \mathcal{Q})$ to the open set

$$\mathfrak{X} \triangleq \{(A_c, B_c, C_c, \mathcal{Q}) : \mathcal{Q} \in \mathbb{P}^{\tilde{n}},\ \tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty \text{ is asymptotically stable, and } (A_c, B_c, C_c) \text{ is controllable and observable}\}. \tag{3.1}$$

Remark 3.1: The set $\mathfrak{X}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the auxiliary minimization problem. Specifically, the requirement that $\mathcal{Q}$ be positive definite replaces (2.13) by an open set constraint, the stability of $\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty$ serves as a normality condition, and (A_c, B_c, C_c) minimal is a nondegeneracy condition. Note that the stabilizability condition (2.15) and stability condition (2.16) play no role in determining solutions of the auxiliary minimization problem.

The following result presents the necessary conditions for optimality in the auxiliary minimization problem. The proof of this result is given in the Appendix as a special case of the corresponding result for reduced-order dynamic compensation considered in Section VI. As mentioned previously, we assume that $R_{2\infty} = \beta^2 R_2$, where $\beta \geq 0$. Furthermore, for arbitrary $\hat{Q}$, $P \in \mathbb{N}^n$ define

$$S \triangleq (I_n + \beta^2\gamma^{-2}\hat{Q}P)^{-1}. \tag{3.2}$$

Since the eigenvalues of $\hat{Q}P$ coincide with the eigenvalues of the nonnegative-definite matrix $P^{1/2}\hat{Q}P^{1/2}$, it follows that $\hat{Q}P$ has nonnegative eigenvalues. Thus, the eigenvalues of $I_n + \beta^2\gamma^{-2}\hat{Q}P$ are all greater than one so that the above inverse exists.

Theorem 3.1: If $(A_c, B_c, C_c, \mathfrak{Q}) \in \mathfrak{X}$ solves the auxiliary minimization problem then there exist $Q, P, \hat{Q} \in \mathbb{N}^n$ such that

$$A_c = A - Q\bar{\Sigma} - \Sigma PS + \gamma^{-2} QR_{1\infty}, \tag{3.3}$$

$$B_c = QC^T V_2^{-1}, \tag{3.4}$$

$$C_c = -R_2^{-1} B^T PS, \tag{3.5}$$

$$\mathfrak{Q} = \begin{bmatrix} Q+\hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{bmatrix} \tag{3.6}$$

and such that $Q, P, \hat{Q}$ satisfy

$$0 = AQ + QA^T + V_1 + \gamma^{-2} QR_{1\infty} Q - Q\bar{\Sigma}Q, \tag{3.7}$$

$$0 = (A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty})^T P + P(A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty}) + R_1 - S^T P\Sigma PS, \tag{3.8}$$

$$0 = (A - \Sigma PS + \gamma^{-2} QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma PS + \gamma^{-2} QR_{1\infty})^T + \gamma^{-2}\hat{Q}(R_{1\infty} + \beta^2 S^T P\Sigma PS)\hat{Q} + Q\bar{\Sigma}Q. \tag{3.9}$$

Furthermore, the auxiliary cost is given by

$$\mathcal{J}(A_c, B_c, C_c, \mathfrak{Q}) = \text{tr}\,[(Q+\hat{Q})R_1 + \hat{Q}S^T P\Sigma PS]. \tag{3.10}$$

Conversely, if there exist $Q, P, \hat{Q} \in \mathbb{N}^n$ satisfying (3.7)–(3.9), then $(A_c, B_c, C_c, \mathfrak{Q})$ given by (3.3)–(3.6) satisfies (2.13) and (2.14) with auxiliary cost (2.20) given by (3.10).

Remark 3.2: If Q and $\hat{Q}$ are nonnegative definite, then the fact that the definiteness condition (2.13) is satisfied can easily be seen by writing $\mathfrak{Q}$ as

$$\mathfrak{Q} = \begin{bmatrix} Q & 0_n \\ 0_n & 0_n \end{bmatrix} + \begin{bmatrix} \hat{Q}^{1/2} \\ \hat{Q}^{1/2} \end{bmatrix} \begin{bmatrix} \hat{Q}^{1/2} \\ \hat{Q}^{1/2} \end{bmatrix}^T.$$

As mentioned in Section II, it is desirable to determine solutions Q and $\hat{Q}$ which yield the minimal solution to (2.14).

Remark 3.3: Setting $\beta = 0$, or equivalently, $E_{2\infty} = 0$, specializes Theorem 3.1 to the cheap H_∞ control case in which H_∞ attenuation between disturbances and controls is not constrained. In this case $S = I_n$, $\mathfrak{Q}$ is given by (3.6), and (3.3)–(3.5) become

$$A_c = A - Q\bar{\Sigma} - \Sigma P + \gamma^{-2} QR_{1\infty}, \tag{3.11}$$

$$B_c = QC^T V_2^{-1}, \tag{3.12}$$

$$C_c = -R_2^{-1} B^T P \tag{3.13}$$

where Q satisfies (3.7), and (3.8) and (3.9) become

$$0 = (A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty})^T P + P(A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty}) + R_1 - P\Sigma P, \tag{3.14}$$

$$0 = (A - \Sigma P + \gamma^{-2} QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma P + \gamma^{-2} QR_{1\infty})^T + \gamma^{-2}\hat{Q}R_{1\infty}\hat{Q} + Q\bar{\Sigma}Q. \tag{3.15}$$

Finally, the auxiliary cost reduces to

$$\mathcal{J}(A_c, B_c, C_c, \mathfrak{Q}) = \text{tr}\,[(Q+\hat{Q})R_1 + \hat{Q}P\Sigma P]. \tag{3.16}$$

Numerical solution of (3.7), (3.14), and (3.15) is discussed in Section VIII.

Remark 3.4: Note that if both $\beta = 0$ and $R_{1\infty} = 0$, then Theorem 3.1 specializes to the standard LQG result.

Theorem 3.1 presents necessary conditions for the auxiliary minimization problem which explicitly synthesize extremal controllers (A_c, B_c, C_c). These necessary conditions comprise a system of three modified algebraic Riccati equations in variables Q, P, and $\hat{Q}$. The Q and P equations are similar to the estimator and regulator Riccati equations of LQG theory, while the $\hat{Q}$ equation has no counterpart in the standard theory. Note that the Q equation is decoupled from the P and $\hat{Q}$ equations and thus can be solved independently. The P equation, however, depends on Q. Thus, regulator/estimator separation holds in only one direction which clearly shows that the certainty equivalence principle is no longer valid for the L_2/H_∞ design problem. Furthermore, since the P and $\hat{Q}$ equations are coupled, they must be solved simultaneously. Finally, note that if the H_∞ disturbance attenuation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then the P equation becomes decoupled from the $\hat{Q}$ equation and thus the $\hat{Q}$ equation becomes superfluous. Furthermore, the remaining Q and P equations separate and coincide with the standard LQG result.

IV. Sufficient Conditions for H_∞ Disturbance Attenuation

In this section we combine Lemma 2.1 with the converse of Theorem 3.1 to obtain our main result guaranteeing closed-loop stability, H_∞ disturbance attenuation, and an optimized L_2 performance bound.

Theorem 4.1: Suppose there exist $Q, P, \hat{Q} \in \mathbb{N}^n$ satisfying (3.7)–(3.9) and let $(A_c, B_c, C_c, \mathfrak{Q})$ be given by (3.3)–(3.6). Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ attenuation constraint

$$\|H(s)\|_\infty \le \gamma \tag{4.1}$$

and the L_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c) \le \text{tr}\,[(Q+\hat{Q})R_1 + \hat{Q}S^T P\Sigma PS]. \tag{4.2}$$

Proof: The converse portion of Theorem 3.1 implies that $\mathfrak{Q}$ given by (3.6) satisfies (2.13) and (2.14) with auxiliary cost given by (3.10). It now follows from Lemma 2.1 that the stabilizability condition (2.15) is equivalent to the asymptotic stability of $\tilde{A}$, the H_∞ disturbance attenuation constraint (2.17) holds, and the performance bound (2.19), which is equivalent to (4.2), holds. □

Remark 4.1: In applying Theorem 4.1 it is not actually necessary to check (2.15) which holds generically. Rather, it suffices to check the stability of $\tilde{A}$ directly which is guaranteed to be equivalent to (2.15).

In applying Theorem 4.1 the principal issue concerns conditions on the problem data under which the coupled Riccati equations (3.7)–(3.9) possess nonnegative-definite solutions. Clearly, for γ sufficiently large, (3.7)–(3.9) approximate the standard LQG result so that existence is assured. The important case of interest, however, involves small γ so that significant H_∞ disturbance attenuation is enforced. Thus, if (4.1) can be satisfied for a given $\gamma > 0$, it is of interest to know whether one such controller can be obtained by solving (3.7)–(3.9). Lemma 2.2 guarantees that (2.14) possesses a solution for any controller satisfying (2.17). Thus, our sufficient condition will also be necessary as long as the auxiliary minimization problem possesses at least one extremal over $\mathfrak{X}$. When this is the case we have the following immediate result.

Proposition 4.1: Let γ^* denote the infimum of $\|H(s)\|_\infty$ over all stabilizing nth-order dynamic compensators and suppose that the auxiliary minimization problem has a solution for all $\gamma > \gamma^*$. Then for all $\gamma > \gamma^*$ there exist $Q, P, \hat{Q} \in \mathbb{N}^n$ satisfying (3.7)–(3.9).

Unlike the standard LQG result involving a pair of separated Riccati equations, the new result enforcing H_∞ disturbance attenuation involves a nonstandard coupled system of three modified Riccati equations. The asymmetry of these equations suggests the possibility of a dual result in which the modifications to the standard P and Q Riccati equations are reversed. Such a dual result will generally be different from Theorem 4.1 and thus will yield an improved bound for particular problems. This point was demonstrated in [16] for the problem of robust performance analysis. Due to space limitations, however, we give only a brief outline of the dual H_∞ results. Note that $J(A_c, B_c, C_c)$ given by (2.10) is also given by

$$J(A_c, B_c, C_c) = \operatorname{tr} \tilde{P}\tilde{V} \tag{4.3}$$

where $\tilde{P} \in \mathbb{N}^{\tilde{n}}$ is the unique solution to (2.23) with $\tilde{R}_\infty$ replaced by $\tilde{R}$. Next, utilizing (4.3) in place of (2.10), the H_∞ disturbance attenuation constraint (2.6) can now be enforced by replacing (2.23) by the Riccati equation

$$0 = \tilde{A}^T\mathcal{P} + \mathcal{P}\tilde{A} + \gamma^{-2}\mathcal{P}\tilde{V}_\infty\mathcal{P} + \tilde{R} \tag{4.4}$$

where $\tilde{V}_\infty$ has the same form as $\tilde{V}$ but may involve weights $V_{1\infty}$ and $V_{2\infty}$. Note that (4.4) is merely the dual of (2.14). We also require the condition dual to (2.15) given by

$$(\tilde{E}, \tilde{A}) \text{ is detectable} \tag{4.5}$$

and that $\tilde{A} + \gamma^{-2}\tilde{V}_\infty\mathcal{P}$ be asymptotically stable. Once again, the sufficient conditions for H_∞ disturbance attenuation involve a coupled system of three modified Riccati equations dual to (3.7)–(3.9). Similar remarks apply to the reduced-order case given by Theorem 6.1 below. Finally, if $\tilde{R}_\infty = \tilde{R}$ and $\tilde{V}_\infty = \tilde{V}$, then it can be shown that $\operatorname{tr} \mathcal{Q}\tilde{R} = \operatorname{tr} \mathcal{P}\tilde{V}$ and thus the solutions to the primal and dual problems coincide.

V. Alternative Forms of the Riccati Equations

In this section we develop alternative forms of the Riccati equations (3.7)–(3.9). These alternative forms provide further insight into the structure of (3.7)–(3.9) and, in certain cases, are simpler and thus are easier to solve computationally. This section also provides connections between our approach and [26].

First we note that the gains (3.3), (3.5), and (3.10) do not depend upon P and $\hat{Q}$ individually, but rather only upon the term $Z \triangleq PS$. Thus, it is of interest to know whether (3.8) and (3.9) can be transformed to yield an equation which characterizes Z directly. The following result summarizes useful properties of Z.

Lemma 5.1: Let $P, \hat{Q} \in \mathbb{N}^n$ and define $Z \triangleq PS$. Then $Z = Z^T = S^TP$, where $S^T = (I_n + \beta^2\gamma^{-2}P\hat{Q})^{-1}$, and Z is nonnegative definite. If, in addition, P is positive definite, then Z is positive definite and

$$Z = (P^{-1} + \beta^2\gamma^{-2}\hat{Q})^{-1}. \tag{5.1}$$

Proof: The result (5.1) is immediate. The remaining results can be obtained by replacing P by $P + \epsilon I_n$, where $\epsilon > 0$, and taking the limit as $\epsilon \to 0$. □

Proposition 5.1: Let $Q \in \mathbb{N}^n$ and suppose there exist $P \in \mathbb{P}^n$ and $\hat{Q} \in \mathbb{N}^n$ satisfying (3.8) and (3.9). Then $Z \triangleq PS$ satisfies

$$\begin{aligned} 0 = {} & (A + \gamma^{-2}QR_{1\infty} + \gamma^{-2}\hat{Q}[R_{1\infty} - \beta^2R_1])^TZ \\ & + Z(A + \gamma^{-2}QR_{1\infty} + \gamma^{-2}\hat{Q}[R_{1\infty} - \beta^2R_1]) \\ & + R_1 - Z(\Sigma + \beta^2\gamma^{-4}\hat{Q}[R_{1\infty} - \beta^2R_1]\hat{Q})Z \\ & + \beta^2\gamma^{-2}ZQ\bar{\Sigma}QZ \end{aligned} \tag{5.2}$$

and (3.9) is equivalent to

$$\begin{aligned} 0 = {} & (A - \Sigma Z + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma Z + \gamma^{-2}QR_{1\infty})^T \\ & + \gamma^{-2}\hat{Q}(R_{1\infty} + \beta^2Z\Sigma Z)\hat{Q} + Q\bar{\Sigma}Q. \end{aligned} \tag{5.3}$$

Furthermore, (3.3), (3.5), and (3.10) become

$$A_c = A - Q\bar{\Sigma} - \Sigma Z + \gamma^{-2}QR_{1\infty}, \tag{5.4}$$

$$C_c = -R_2^{-1}B^TZ, \tag{5.5}$$

$$\mathcal{J}(A_c, B_c, C_c, \mathcal{Q}) = \operatorname{tr}[(Q + \hat{Q})R_1 + \hat{Q}Z\Sigma Z]. \tag{5.6}$$

Proof: Using the identities

$$P = (I_n - \beta^2\gamma^{-2}Z\hat{Q})^{-1}Z = Z(I_n - \beta^2\gamma^{-2}\hat{Q}Z)^{-1}$$

which follow from (5.1), equation (5.2) can be obtained by forming the new equation

$$(I_n - \beta^2\gamma^{-2}Z\hat{Q})(3.8)(I_n - \beta^2\gamma^{-2}\hat{Q}Z) + \beta^2\gamma^{-2}Z(3.9)Z. \tag{5.7}$$

Finally, (5.3)–(5.5) are restatements of (3.9), (3.3), and (3.5) with $Z = PS$. □

Having obtained a single equation (5.2) for $Z = PS$ by combining (3.8) and (3.9) for P and $\hat{Q}$, it is of interest to know whether (3.8) for P can be recovered from (5.2) and (5.3).

Proposition 5.2: Let $Q \in \mathbb{N}^n$, $\beta > 0$, suppose there exist $Z \in \mathbb{P}^n$ and $\hat{Q} \in \mathbb{N}^n$ satisfying (5.2) and (5.3), and assume that

$$\rho(Z\hat{Q}) < \beta^{-2}\gamma^2. \tag{5.8}$$

Then $P \triangleq (Z^{-1} - \beta^2\gamma^{-2}\hat{Q})^{-1}$ is positive definite and satisfies (3.8). Furthermore, P satisfies $Z = PS$.

Proof: If (5.8) holds, then it can be shown that P as defined above is positive definite. Reversing the proof of Proposition 5.1, (3.8) can be recovered by forming

$$(I_n - \beta^2\gamma^{-2}Z\hat{Q})^{-1}[(5.2) - \beta^2\gamma^{-2}Z(5.3)Z](I_n - \beta^2\gamma^{-2}\hat{Q}Z)^{-1}. \quad \square$$

Although Proposition 5.2 allows us to reconstruct (3.8) for P, it can only be utilized when (5.8) holds. This fact raises a question as to the sufficiency of (3.7), (5.2), and (5.3) in the absence of (3.8). It turns out that the matrices P and Z need not actually satisfy (3.8) and (5.2) to enforce the H_∞ performance constraint (2.17) since only the Q and $\hat{Q}$ equations are required. Rather, P can be viewed as a parameterization of Z which, in turn, is a parameterization of the gains A_c and C_c given by (5.4) and (5.5) which yield a controller satisfying the desired H_∞ performance. These observations are summarized by the following result which does not require that Z be obtained by solving (5.2).

Proposition 5.3: Let $Z \in \mathbb{N}^n$ and suppose there exist $Q, \hat{Q} \in \mathbb{N}^n$ satisfying (3.7) and (5.3). Then $(A_c, B_c, C_c, \mathcal{Q})$ given by (5.4), (3.4), (5.5), and (3.6) satisfy (2.13) and (2.14). Thus, (2.15) and (2.16) are equivalent, and, in this case, (2.17) and (2.19) hold.

Proof: The result follows by direct verification of (2.14). □

Proposition 5.3 shows that the H_∞ constraint (2.17) is enforced for arbitrary $Z \in \mathbb{N}^n$ as long as (3.7) and (5.3) can be solved for Q and $\hat{Q}$. The price we pay for using arbitrary Z is that we no longer are assured that Z is obtained from (5.2) or from $Z = PS$ where P satisfies (3.8). Since P arises from the Lagrange multiplier for the constraint (2.14) [see (A.3)], it follows that an arbitrary choice of P (or Z) may fail to minimize the L_2 auxiliary cost (2.20). Thus, regarding P and Z as free parameters effectively ignores the L_2 aspect of Theorem 4.1.

It is also of interest to introduce yet another transformation of (3.7)–(3.9) by defining

$$Y \triangleq (Z^{-1} + \beta^2\gamma^{-2}Q)^{-1} = (P^{-1} + \beta^2\gamma^{-2}[Q + \hat{Q}])^{-1} \tag{5.9}$$

when P is positive definite. As in Lemma 5.1, Y is also positive definite.

Proposition 5.4: Let $Q \in \mathbb{N}^n$ and suppose there exist $P \in \mathbb{P}^n$ and $\hat{Q} \in \mathbb{N}^n$ satisfying (3.8) and (3.9). Then Y defined by (5.9) satisfies

$$\begin{aligned} 0 = {} & (A + \gamma^{-2}[Q + \hat{Q}][R_{1\infty} - \beta^2R_1])^TY \\ & + Y(A + \gamma^{-2}[Q + \hat{Q}][R_{1\infty} - \beta^2R_1]) \\ & + R_1 + \beta^2\gamma^{-2}YV_1Y - Y\Sigma Y \\ & - \beta^2\gamma^{-4}Y(Q + \hat{Q})(R_{1\infty} - \beta^2R_1)(Q + \hat{Q})Y \end{aligned} \tag{5.10}$$

and (3.9) is equivalent to

$$0=(A-\Sigma[Y^{-1}-\beta^2\gamma^{-2}Q]^{-1}+\gamma^{-2}QR_{1\infty})\hat{Q}$$
$$+\hat{Q}(A-\Sigma[Y^{-1}-\beta^2\gamma^{-2}Q]^{-1}+\gamma^{-2}QR_{1\infty})^T$$
$$+\gamma^{-2}\hat{Q}(R_{1\infty}+\beta^2[Y^{-1}-\beta^2\gamma^{-2}Q]^{-1}$$
$$\cdot\Sigma[Y^{-1}-\beta^2\gamma^{-2}Q]^{-1})\hat{Q}+Q\bar{\Sigma}Q. \quad (5.11)$$

Furthermore, (3.3), (3.5), and (3.10) become

$$A_c=A-Q\bar{\Sigma}-\Sigma(Y^{-1}-\beta^2\gamma^{-2}Q)^{-1}+\gamma^{-2}QR_{1\infty}, \quad (5.12)$$

$$C_c=-R_2^{-1}B^T(Y^{-1}-\beta^2\gamma^{-2}Q)^{-1}, \quad (5.13)$$

$$\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})=\text{tr}\,[(Q+\hat{Q})R_1$$
$$+\hat{Q}(Y^{-1}-\beta^2\gamma^{-2}Q)^{-1}\Sigma(Y^{-1}-\beta^2\gamma^{-2}Q)^{-1}]. \quad (5.14)$$

Proof: To obtain (5.10), form

$$Y[Z^{-1}(5.2)Z^{-1}+\beta^2\gamma^{-2}(3.7)]Y. \qquad \square$$

The following result allows us to recover (3.8) for P from (5.10) and (5.11).

Proposition 5.5: Let $Q \in \mathbb{N}^n$, $\beta > 0$, suppose there exist $Y \in \mathbb{P}^n$ and $\hat{Q} \in \mathbb{N}^n$ satisfying (5.10), (5.11), and assume that

$$\rho(Y[Q+\hat{Q}])<\beta^{-2}\gamma^2. \quad (5.15)$$

Then $P \triangleq (Y^{-1} - \beta^2\gamma^{-2}[Q + \hat{Q}])^{-1}$ is positive definite and satisfies (3.8).

Proof: The result follows by reversing the proof of Proposition 5.4. $\square$

By specializing further, it is possible to achieve even greater simplification. Specifically, we consider the case in which the L_2 and H_∞ weights are equalized, i.e.,

$$R_{1\infty}=R_1,\ \beta=1. \quad (5.16)$$

In this case it is always possible to eliminate (5.3) and (5.11) by noting that they are satisfied by $\hat{Q} = \gamma^2Z^{-1}$ and $\hat{Q} = \gamma^2Y^{-1} - Q$, respectively. However, although this solution enforces the H_∞ constraint, it can be seen from the resulting form of $\mathcal{J}$ that this solution does not correspond to the minimal solution $\mathcal{Q}$ of (2.14). Hence, we impose additional assumptions which allow us to directly characterize the solution which yields the minimal performance bound. We are indebted to D. Mustafa for clarifying this point in [45] where it is also shown that the auxiliary cost (2.20) is equivalent to an entropy integral.

Proposition 5.6: Assume (5.16) is satisfied, suppose there exist $Q \in \mathbb{N}^n$ and $Z_\infty \in \mathbb{P}^n$ satisfying

$$0=AQ+QA^T+V_1+\gamma^{-2}QR_{1\infty}Q-Q\bar{\Sigma}Q, \quad (5.17)$$

$$0=(A+\gamma^{-2}QR_{1\infty})^TZ_\infty+Z_\infty(A+\gamma^{-2}QR_{1\infty})$$
$$+R_{1\infty}-Z_\infty\Sigma Z_\infty+\gamma^{-2}Z_\infty Q\bar{\Sigma}QZ_\infty \quad (5.18)$$

and such that

$$A+\gamma^{-2}QR_{1\infty}+(\gamma^{-2}Q\bar{\Sigma}Q-\Sigma)Z_\infty \text{ is asymptotically stable} \quad (5.19)$$

and

$$(A+\gamma^{-2}QR_{1\infty}+Z_\infty^{-1}R_{1\infty},\ \gamma^{-1}[R_{1\infty}+Z_\infty\Sigma Z_\infty]^{1/2}) \text{ is observable.} \quad (5.20)$$

Furthermore, let (A_c, B_c, C_c) be given by

$$A_c=A-Q\bar{\Sigma}-\Sigma Z_\infty+\gamma^{-2}QR_{1\infty}, \quad (5.21)$$

$$B_c=QC^TV_2^{-1}, \quad (5.22)$$

$$C_c=-R_{2\infty}^{-1}B^TZ_\infty. \quad (5.23)$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|H(s)\|_\infty\le\gamma \quad (5.24)$$

and the L_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c)\le\text{tr}[QR_{1\infty}+Q\bar{\Sigma}QZ_\infty]. \quad (5.25)$$

Proof: First note that it follows from (5.18) that

$$-(A+\gamma^{-2}QR_{1\infty}+Z_\infty^{-1}R_{1\infty})=Z_\infty[A+\gamma^{-2}QR_{1\infty}$$
$$+(\gamma^{-2}Q\bar{\Sigma}Q-\Sigma)Z_\infty]Z_\infty^{-1} \quad (5.26)$$

and thus (5.19) implies that $-(A + \gamma^{-2}QR_{1\infty} + Z_\infty^{-1}R_{1\infty})$ is asymptotically stable. It now follows from (5.20) that there exists $N \in \mathbb{P}^n$ satisfying

$$0=-(A+\gamma^{-2}QR_{1\infty}+Z_\infty^{-1}R_{1\infty})^TN-N(A+\gamma^{-2}QR_{1\infty}$$
$$+Z_\infty^{-1}R_{1\infty})+\gamma^{-2}(R_{1\infty}+Z_\infty\Sigma Z_\infty). \quad (5.27)$$

It can now be shown that $\hat{Q} = \gamma^2Z_\infty^{-1} - N^{-1}$ satisfies (5.3) with $\beta = 1$ and $Z = Z_\infty$. Furthermore, (5.8) is satisfied so that the hypotheses of Theorem 4.1 are verified. The expression (5.25) now follows by direct substitution. $\square$

Finally, we consider a simplified version of Proposition 5.4.

Proposition 5.7: Assume (5.16) is satisfied and suppose there exist $Q \in \mathbb{N}^n$ and $Y_\infty \in \mathbb{P}^n$ satisfying

$$0=AQ+QA^T+V_1+\gamma^{-2}QR_{1\infty}Q-Q\bar{\Sigma}Q, \quad (5.28)$$

$$0=A^TY_\infty+Y_\infty A+R_{1\infty}+\gamma^{-2}Y_\infty V_1Y_\infty-Y_\infty\Sigma Y_\infty, \quad (5.29)$$

$$\rho(QY_\infty)<\gamma^2 \quad (5.30)$$

and such that

$$A+(\gamma^{-2}V_1-\Sigma)Y_\infty \text{ is asymptotically stable} \quad (5.31)$$

and

$$(A+Y_\infty^{-1}R_{1\infty},\ \gamma^{-1}[R_{1\infty}+(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}$$
$$\cdot\Sigma(Y_\infty^{-1}-\gamma^{-2}Q)]^{1/2}) \text{ is observable.} \quad (5.32)$$

Furthermore, let (A_c, B_c, C_c) be given by

$$A_c=A-Q\bar{\Sigma}-\Sigma(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}+\gamma^{-2}QR_{1\infty}, \quad (5.33)$$

$$B_c=QC^TV_2^{-1}, \quad (5.34)$$

$$C_c=-R_{2\infty}^{-1}B^T(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}. \quad (5.35)$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|H(s)\|_\infty\le\gamma \quad (5.36)$$

and the L_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c)\le\text{tr}[QR_{1\infty}+Q\bar{\Sigma}Q(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}]. \quad (5.37)$$

Proof: The proof is similar to the proof of Proposition 5.6 with $\hat{Q} = \gamma^2Y_\infty - Q - \hat{N}^{-1}$, where $\hat{N}$ satisfies

$$0=-(A+Y_\infty^{-1}R_{1\infty})^T\hat{N}-\hat{N}(A+Y_\infty^{-1}R_{1\infty})$$
$$+\gamma^{-2}[R_{1\infty}+(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}\Sigma(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}]. \quad \square$$

Remark 5.1: The solutions Q and Y_∞ of (5.28) and (5.29) are analogous to the matrices Y_∞ and X_∞ of [26], while (5.30) corresponds to condition 5.2(iii) of [26]. Note that by letting $\gamma \to \infty$, (5.25) and (5.37) coincide with 5-77a of [1] and the LQG result is recovered.

Remark 5.2: It is interesting to note that (5.17) and (5.18) with controller gains (5.21)–(5.23) are already known since they are identical to the optimality conditions for the linear-exponential-of-quadratic-Gaussian problem treated in [33] (see also [34] and [35]). Specifically, see (3.1) and (4.1) on pp. 603 and 609, respectively. As shown in [33], minimizing the criterion

$$J=\lim_{t\to\infty} \mathbb{E}\mu e^{\mu/2(x^TR_1x+u^TR_2u)}$$

leads to the pair of modified Riccati equations (5.17) and (5.18) with γ^{-2} replaced by μ. This implies that the exponential-of-quadratic design problem effectively enforces a bound of $\mu^{-1/2}$ on the H_∞ norm of the closed-loop transfer function. There also exist fundamental connections with the problem of entropy maximization [43]–[45].

VI. Extensions to Reduced-Order Dynamic Compensation

In this section we extend Theorem 4.1 by expanding the formulation of Section III to allow the compensator to be of fixed dimension n_c which may be less than the plant order n. Hence, in this section define $\tilde{n} = n + n_c$, where $n_c \le n$. As in [21] this constraint leads to an oblique projection which introduces additional coupling in the design equations along with an additional equation. The following lemma is required.

Lemma 6.1: Let $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and suppose rank $\hat{Q}\hat{P} = n_c$. Then there exist $n_c \times n$ G, Γ, and $n_c \times n_c$ invertible M, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that

$$\hat{Q}\hat{P}=G^TM\Gamma, \tag{6.1}$$

$$\Gamma G^T=I_{n_c}. \tag{6.2}$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^T\Gamma, \tag{6.3}$$

$$\tau_\perp \triangleq I_n-\tau \tag{6.4}$$

are idempotent and have rank n_c and $n - n_c$, respectively.

Proof: Conditions (6.1)–(6.4) are a direct consequence of [36, Theorem 6.2.5]. □

Theorem 6.1: Let $n_c \le n$, suppose there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying

$$0=AQ+QA^T+V_1+\gamma^{-2}QR_{1\infty}Q-Q\bar{\Sigma}Q+\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T, \tag{6.5}$$

$$0=(A+\gamma^{-2}[Q+\hat{Q}]R_{1\infty})^TP+P(A+\gamma^{-2}[Q+\hat{Q}]R_{1\infty})+R_1 -S^TP\Sigma PS+\tau_\perp^TS^TP\Sigma PS\tau_\perp, \tag{6.6}$$

$$0=(A-\Sigma PS+\gamma^{-2}QR_{1\infty})\hat{Q}+\hat{Q}(A-\Sigma PS+\gamma^{-2}QR_{1\infty})^T +\gamma^{-2}\hat{Q}(R_{1\infty}+\beta^2S^TP\Sigma PS)\hat{Q}+Q\bar{\Sigma}Q-\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T, \tag{6.7}$$

$$0=(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})^T\hat{P}+\hat{P}(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty}) +S^TP\Sigma PS-\tau_\perp^TS^TP\Sigma PS\tau_\perp, \tag{6.8}$$

$$\text{rank } \hat{Q}=\text{rank } \hat{P}=\text{rank } \hat{Q}\hat{P}=n_c \tag{6.9}$$

and let $(A_c, B_c, C_c, \mathcal{Q})$ be given by

$$A_c=\Gamma(A-Q\bar{\Sigma}-\Sigma PS+\gamma^{-2}QR_{1\infty})G^T, \tag{6.10}$$

$$B_c=\Gamma QC^TV_2^{-1}, \tag{6.11}$$

$$C_c=-R_2^{-1}B^TPSG^T, \tag{6.12}$$

$$\mathcal{Q}=\begin{bmatrix} Q+\hat{Q} & \hat{Q}\Gamma^T \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^T \end{bmatrix}. \tag{6.13}$$

Then, $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|H(s)\|_\infty\le\gamma \tag{6.14}$$

and the L_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c)\le\text{tr}[(Q+\hat{Q})R_1+\hat{Q}S^TP\Sigma PS]. \tag{6.15}$$

Remark 6.1: It is easy to see that Theorem 6.1 is a direct generalization of Theorem 4.1. To recover Theorem 4.1, set $n_c = n$ so that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. In this case the last term in each of (6.5)–(6.8) can be deleted and (6.8) becomes superfluous. Furthermore, (6.5)–(6.7) now reduce to (3.7)–(3.9), as expected. If, furthermore, $\beta = 0$ then $S = I_n$ so that (6.5)–(6.7) now reduce to the cheap H_∞ control case given by (3.7), (3.14), and (3.15). Alternatively, setting $\gamma = \infty$ and retaining the reduced-order constraint $n_c < n$ yields the result of [21].

Remark 6.2: By introducing a new variable $Z = PS = (P^{-1} + \beta^2\gamma^{-2}\hat{Q})^{-1}$ as in Section V, (6.6) becomes

$$0=(A+\gamma^{-2}QR_{1\infty}+\gamma^{-2}\hat{Q}[R_{1\infty}-\beta^2R_1])^TZ +Z(A+\gamma^{-2}QR_{1\infty}+\gamma^{-2}\hat{Q}[R_{1\infty}-\beta^2R_1]) +R_1-Z(\Sigma+\beta^2\gamma^{-4}\hat{Q}[R_{1\infty}-\beta^2R_1]\hat{Q})Z +\tau_\perp^TZ\Sigma Z\tau_\perp+\beta^2\gamma^{-2}Z(Q\bar{\Sigma}Q-\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T)Z \tag{6.16}$$

which specializes to (5.2) when $n_c = n$, i.e., $\tau_\perp = 0$. When (5.16) holds, (6.16) becomes

$$0=(A+\gamma^{-2}QR_{1\infty})^TZ_\infty+Z_\infty(A+\gamma^{-2}QR_{1\infty}) +R_{1\infty}-Z_\infty\Sigma Z_\infty+\tau_\perp^TZ_\infty\Sigma Z_\infty\tau_\perp +\gamma^{-2}Z_\infty(Q\bar{\Sigma}Q-\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T)Z_\infty. \tag{6.17}$$

Analogous equations for Y defined by (5.9) can also be developed.

VII. Analysis of the Design Equations

Before developing numerical algorithms, it is instructive to analyze the design equations to determine existence and multiplicity of nonnegative-definite solutions. Although a detailed theoretical analysis remains an area for future research, in this section we present a simplified treatment which highlights important asymptotic properties of the equations. it turns out that several key properties are discernible by considering the scalar case $n = 1$. Although many of these properties can be developed for general n, the simplified scalar case suffices for obtaining a useful qualitative analysis. Here we consider only (3.7), (3.14), and (3.15).

Since the Q equation (3.7) is decoupled from (3.14) and (3.15), it can be analyzed separately. It is easy to see that there exists a unique nonnegative solution for $\gamma > (R_1/\bar{\Sigma})^{1/2}$ as in the case of a standard Riccati equation with stabilizability and detectability hypotheses. Furthermore, it can be seen that for

$$(R_1/[\bar{\Sigma}+(A^2/V_1)])^{1/2}<\gamma<(R_1/\bar{\Sigma})^{1/2}$$

there exist two nonnegative solutions when A is stable and zero nonnegative solutions when A is unstable. Below this lower bound for γ, nonnegative solutions Q do not exist. This result thus indicates (as in LQG theory [42]) a lower bound to the achievable H_∞ disturbance attenuation as determined by the sensor noise intensity V_2 appearing in $\bar{\Sigma}$.

Since the P and $\hat{Q}$ equations (3.14) and (3.15) are coupled, they must be analyzed jointly. Since (3.15) is a standard Riccati equation it follows under generic hypotheses that it possesses exactly one nonnegative-definite solution for all values of Q and $\hat{Q}$. The analysis of the $\hat{Q}$ equation is, however, more involved. It can be shown that the existence of real solutions is a complicated function of γ, Q, and P. When real solutions do exist, it follows that there exist either zero or two nonnegative-definite solutions. To obtain further qualitative insight into the solutions P and $\hat{Q}$, we fix γ and allow $R_2 \to 0$, that is, the cheap L_2 control case. It thus follows that $P \sim (R_1\Sigma)^{1/2}$ and that either $\hat{Q} \sim 2\gamma^2(\Sigma/R_1)^{1/2}$ or $\hat{Q} \sim 1/2\bar{\Sigma}Q^2(\Sigma R_1)^{-1/2}$, which correspond to the previously mentioned pair of solutions satisfying (3.15). This result thus indicates that an arbitrarily small H_∞ disturbance attenuation constraint γ can be achieved [subject to the solvability of (3.7)] by sufficiently increasing the L_2 controller authority. That is, since solutions exist in the cheap L_2 control case, the H_∞ disturbance attenuation constraint is achievable. The ability to achieve small γ is also attributable to the fact that since $\beta = 0$, H_∞ disturbance attenuation to the control variables is not limited in (3.7), (3.14), and (3.15) as in Theorems 3.1 and 6.1. Of course, as is well known, it is not possible to make $\gamma \to 0$ by letting $\Sigma \to \infty$ and $\bar{\Sigma} \to \infty$ when the system possesses nonminimum phase zeros. Also, note that both of the asymptotic solutions to (3.15) are guaranteed to yield the bound (4.1). The solution of interest, however, is $\hat{Q} = O(\Sigma^{-1/2})$ since it clearly yields a lower value of $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$ than $\hat{Q} = O(\Sigma^{1/2})$.

VIII. Numerical Algorithm and Illustrative Results

In this section we describe a numerical algorithm which has been developed and implemented for solving the coupled Riccati equations (3.7), (3.14), and (3.15). We also present numerical results for an illustrative example.

Coupled modified Riccati equations arise in a variety of problems and homotopic continuation methods have been shown to be particularly successful [23]–[25]. To solve (3.7), (3.14), and (3.15) we have implemented a simplified continuation method involving the constraint constant γ. The idea is to exploit the fact that for large γ the problem is approximated by LQG which provides a reliable starting solution. The continuation parameter γ is then successively decreased until either a desired value of γ is achieved or no further decrease is possible. This algorithm is now summarized. Let $\epsilon > 0$ denote a convergence criterion.

Algorithm 8.1: To solve (3.7), (3.14), and (3.15), perform the following steps:

Step 1: Initialize $\gamma > 0$.

Step 2: Solve (3.7) for Q.

Step 3: Let $k = 0$, $\hat{Q}_0 = 0$.

Step 4: Solve (3.14) for $P_{k+1} = P$ with $\hat{Q} = \hat{Q}_k$.

Step 5: Solve (3.15) for $\hat{Q}_{k+1} = \hat{Q}$ with $P = P_{k+1}$.

Step 6: If $k \geq 1$ check for $\|P_{k+1} - P_k\| < \epsilon$ and $\|\hat{Q}_{k+1} - \hat{Q}_k\| < \epsilon$.

Step 7: If convergence is not achieved in Step 6 (or $k = 0$) let $k \leftarrow k + 1$ and go to Step 4; otherwise decrease γ and go to Step 2.

Steps 2, 4, and 5 were carried out using a standard Riccati solver [37] which proved to be reliable even when the quadratic term was indefinite or nonnegative definite. For instance, for the example considered below, the term $\gamma^{-2}R_1 - \bar{\Sigma}$ was indefinite for all finite γ. The crucial step in the algorithm is the decreasing of γ in Step 7. Significant effort was devoted to providing a smooth transition to smaller values of γ without sacrificing computational efficiency. The development of more sophisticated continuation algorithms remains an area for future research.

The example considered was formulated in [38] and was considered extensively in [24], [25], and [39] to compare reduced-order design methods. The example is interesting since it possesses a complex pair of nonminimum phase zeros due to the fact that the physical system (coupled rotating disks) has noncolocated sensors and actuators. The plant is of eighth order and has two neutrally stable poles. The problem data are as follows:

$$n = n_c = 8, \quad m = l = 1, \quad q = p = 2,$$

$$A = \begin{bmatrix} -0.161 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6.004 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5822 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -9.9835 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.4073 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3.982 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0.0064 \\ 0.00235 \\ 0.0713 \\ 1.0002 \\ 0.1045 \\ 0.9955 \end{bmatrix} \qquad C = [1 \ \ 0_{1\times 7}]$$

$$E_1 = E_{1\infty} = 10^{-3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0.55 & 11 & 1.32 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_{2\infty} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \beta = 0,$$

$$D_1 = [B \ \ 0_{8\times 1}], \quad D_2 = [0 \ \ 1].$$

With the problem data as given, the LQG controller was found to yield a closed-loop H_∞ performance of 1.39 (i.e., 2.87 dB above unity gain). Using Algorithm 8.1 we obtained a solution for $\gamma = 0.52$ for a net H_∞ performance improvement of 8.7 dB (see Fig. 1). Note that this result is consistent with [3, Proposition 8.1] which implies that the maximum ratio of the H_∞ performance of the optimal L_2 controller to the H_∞ performance of the optimal H_∞ controller can be no more than twice the number of right-half-plane zeros, which for the present problem with two nonminimum phase zeros corresponds to a factor of 4 (i.e., 12 dB).

Our numerical experience revealed two interesting features. First, the loop between Steps 4 and 6 converged reliably. However, a critical value $\gamma_{\min}$ of γ was invariably found below which solutions could not be computed. This value $\gamma_{\min}$ appears to represent the best achievable H_∞ performance for the given L_2 weights. Second, for each value of $\gamma \geq \gamma_{\min}$ for which a solution was computed, the actual H_∞ performance was close to this value revealing that the H_∞ bound is tight. For example, the actual worst-case attenuation of the $\gamma = 0.52$ design shown in Fig. 1 is 0.511. Controller characteristics are given in Table I and are plotted in Fig. 2 for several values of γ. Note that in each case the L_2 performance bound is within 30 percent of the actual L_2 performance.

IX. Further Extensions

The results obtained herein can readily be extended in several directions. These include the treatment of parameter uncertainties [13]–[15], [46], extensions to controllers with static feedthrough [32], and the inclusion of cross-weighting terms ($x^T(t)R_{12}u(t)$) and noise correlation ($D_1D_2^T \neq 0$). Finally, as mentioned in Remark 5.2, connections with the exponential-of-quadratic cost criterion [33]–[35] and entropy maximization [43]–[45] are of interest.

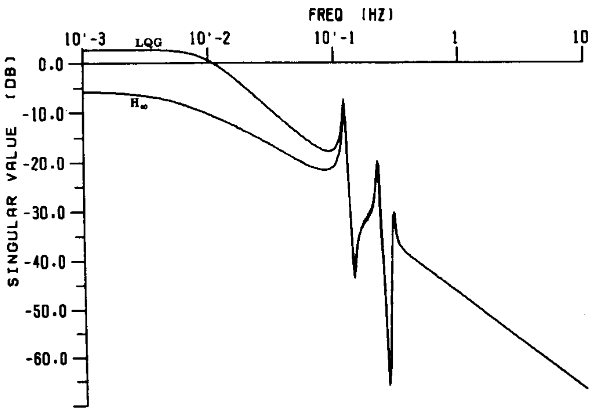

Fig. 1.

TABLE I

H_∞ Attenuation Constraint γ	Actual H_∞ Attenuation $\|H(s)\|_\infty$	L_2 Performance Bound $J(A_c, B_c, C_c, \mathcal{Q})$	Actual L_2 Performance $J(A_c, B_c, C_c)$
∞ (LQG)	1.39	—	.143
2	1.18	.159	.146
1.5	1.06	.171	.151
1.0	.855	.204	.168
.9	.797	.217	.176
.8	.732	.236	.187
.7	.661	.262	.203
.52	.511	.299	.262

Appendix
Proof of Theorem 6.1

To optimize (2.20) over the open set $\mathcal{X}$ subject to the constraint (2.14), form the Lagrangian

$$\mathcal{L}(A_c, B_c, C_c, \mathcal{Q}, \mathcal{P}, \lambda) \triangleq \operatorname{tr}\{\lambda \mathcal{Q}\tilde{R} + [\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty \mathcal{Q} + \tilde{V}]\mathcal{P}\} \quad \text{(A.1)}$$

where the Lagrange multipliers $\lambda \geq 0$ and $\mathcal{P} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are not both zero. We thus obtain

$$\frac{\partial \mathcal{L}}{\partial \mathcal{Q}} = (\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty)^T \mathcal{P} + \mathcal{P}(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty) + \lambda \tilde{R}. \quad \text{(A.2)}$$

Setting $\partial\mathcal{L}/\partial\mathcal{Q} = 0$ yields

$$0 = (\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty)^T \mathcal{P} + \mathcal{P}(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty) + \lambda \tilde{R}. \quad \text{(A.3)}$$

Since $\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty$ is assumed to be stable, $\lambda = 0$ implies $\mathcal{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, $\mathcal{P}$ is nonnegative definite.

Now partition $\tilde{n} \times \tilde{n}$ $\mathcal{Q}\mathcal{P}$ into $n \times n$, $n \times n_c$, and $n_c \times n_c$ subblocks as

$$\mathcal{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}.$$

Thus, with $\lambda = 1$ the stationarity conditions are given by

$$\frac{\partial \mathcal{L}}{\partial \mathcal{Q}} = (\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty)^T \mathcal{P} + \mathcal{P}(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty) + \tilde{R} = 0, \quad \text{(A.4)}$$

$$\frac{\partial \mathcal{L}}{\partial A_c} = P_{12}^T Q_{12} + P_2 Q_2 = 0, \quad \text{(A.5)}$$

$$\frac{\partial \mathcal{L}}{\partial B_c} = P_2 B_c V_2 + (P_{12}^T Q_1 + P_2 Q_{12}^T) C^T = 0, \quad \text{(A.6)}$$

$$\frac{\partial \mathcal{L}}{\partial C_c} = R_2 C_c Q_2 + \beta^2\gamma^{-2} R_2 C_c (P_1 Q_{12} + P_{12} Q_2)^T Q_{12} + B^T(P_1 Q_{12} + P_{12} Q_2) = 0. \quad \text{(A.7)}$$

Expanding (2.14) and (A.4) yields

$$0 = AQ_1 + Q_1 A^T + BC_c Q_{12}^T + Q_{12} C_c^T B^T + \gamma^{-2} Q_1 R_{1\infty} Q_1 + \beta^2\gamma^{-2} Q_{12} C_c^T R_2 C_c Q_{12}^T + V_1, \quad \text{(A.8)}$$

$$0 = AQ_{12} + Q_{12} A_c^T + BC_c Q_2 + Q_1 C^T B_c^T + \gamma^{-2} Q_1 R_{1\infty} Q_{12} + \beta^2\gamma^{-2} Q_{12} C_c^T R_2 C_c Q_2, \quad \text{(A.9)}$$

$$0 = A_c Q_2 + Q_2 A_c^T + B_c C Q_{12} + Q_{12}^T C^T B_c^T + \gamma^{-2} Q_{12}^T R_{1\infty} Q_{12} + \beta^2\gamma^{-2} Q_2 C_c^T R_2 C_c Q_2 + B_c V_2 B_c^T, \quad \text{(A.10)}$$

$$0 = A^T P_1 + P_1 A + C^T B_c^T P_{12}^T + P_{12} B_c C + \gamma^{-2} R_{1\infty}(P_1 Q_1 + P_{12} Q_{12}^T)^T + \gamma^{-2}(P_1 Q_1 + P_{12} Q_{12}^T) R_{1\infty} + R_1, \quad \text{(A.11)}$$

$$0 = A^T P_{12} + P_{12} A_c + C^T B_c^T P_2 + P_1 B C_c + \gamma^{-2} R_{1\infty}(P_{12}^T Q_1 + P_2 Q_{12}^T)^T + \beta^2\gamma^{-2}(P_1 Q_{12} + P_{12} Q_2) C_c^T R_2 C_c, \quad \text{(A.12)}$$

$$0 = A_c^T P_2 + P_2 A_c + P_{12}^T B C_c + C_c^T B^T P_{12} + C_c^T R_2 C_c. \quad \text{(A.13)}$$

Lemma A.1: Q_2 and P_2 are positive definite.

Proof: By a minor extension of results from [40], (A.10) can be rewritten as

$$0 = (A_c + B_c C Q_{12} Q_2^+) Q_2 + Q_2 (A_c + B_c C Q_{12} Q_2^+)^T + \Psi$$

where

$$\Psi \triangleq \gamma^{-2} Q_{12}^T R_{1\infty} Q_{12} + \beta^2\gamma^{-2} Q_2 C_c^T R_2 C_c Q_2 + B_c V_2 B_c^T$$

and Q_2^+ is the Moore-Penrose or Drazin generalized inverse of Q_2. Next note that since (A_c, B_c) is controllable it follows from [28, Lemma 2.1 and Theorem 3.6] that $(A_c + B_c C Q_{12} Q_2^+, \Psi^{1/2})$ is also controllable. Now, since Q_2 and Ψ are nonnegative definite, [28, Lemma 12.2] implies that Q_2 is positive definite. Using (A.13), similar arguments show that P_2 is positive definite. □

Since R_2, V_2, Q_2, P_2 are invertible, (A.5)–(A.7) can be written as

$$-P_2^{-1} P_{12}^T Q_{12} Q_2^{-1} = I_{n_c}, \quad \text{(A.14)}$$

$$B_c = -P_2^{-1}(P_{12}^T Q_1 + P_2 Q_{12}^T) C^T V_2^{-1}, \quad \text{(A.15)}$$

$$C_c[I_{n_c} + \beta^2\gamma^{-2}(Q_{12}^T P_1 + Q_2 P_{12}^T) Q_{12} Q_2^{-1}] = -R_2^{-1} B^T (P_1 Q_{12} + P_{12} Q_2) Q_2^{-1}. \quad \text{(A.16)}$$

Now define the $n \times n$ matrices

$$Q \triangleq Q_1 - Q_{12} Q_2^{-1} Q_{12}^T, \quad P \triangleq P_1 - P_{12} P_2^{-1} P_{12}^T,$$

$$\hat{Q} \triangleq Q_{12} Q_2^{-1} Q_{12}^T, \quad \hat{P} \triangleq P_{12} P_2^{-1} P_{12}^T,$$

$$\tau \triangleq -Q_{12} Q_2^{-1} P_2^{-1} P_{12}^T$$

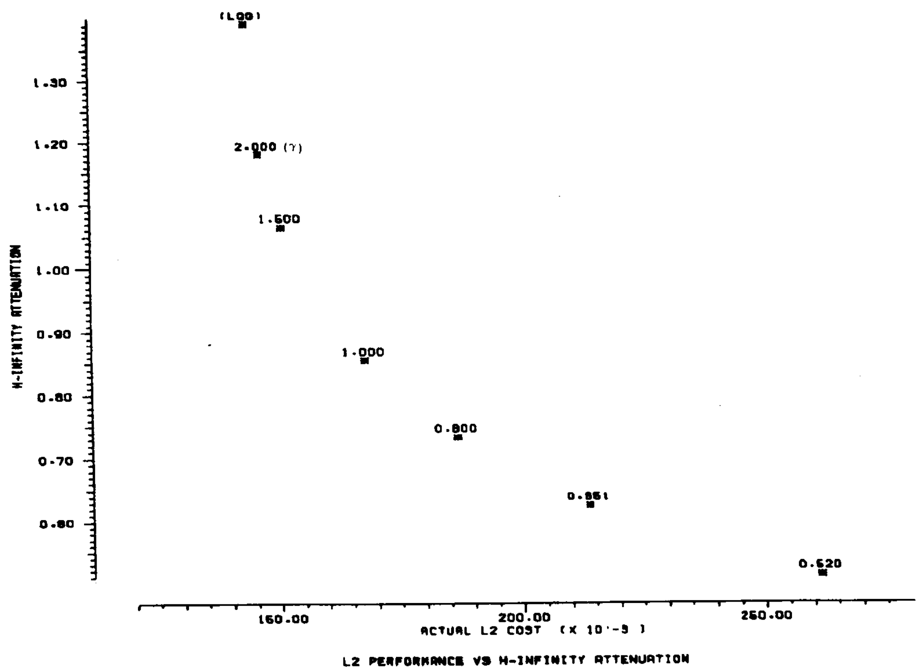

Fig. 2.

and the $n_c \times n$, $n_c \times n_c$, and $n_c \times n$ matrices

$$G \triangleq Q_2^{-1}Q_{12}^T,\quad M \triangleq Q_2P_2,\quad \Gamma \triangleq -P_2^{-1}P_{12}^T.$$

Note that $\tau = G^T\Gamma$.

Clearly, Q, P, $\hat{Q}$, and $\hat{P}$ are symmetric and $\hat{Q}$ and $\hat{P}$ are nonnegative definite. To show that Q and P are also nonnegative definite, note that Q is the upper left-hand block of the nonnegative definite matrix $\tilde{\mathcal{Q}}\mathcal{Q}\tilde{\mathcal{Q}}^T$, where

$$\tilde{\mathcal{Q}} \triangleq \begin{bmatrix} I_n & -Q_{12}Q_2^{-1} \\ 0_{n_c\times n} & I_{n_c} \end{bmatrix}.$$

Similarly, P is nonnegative definite.

Next note that with the above definitions (A.14) is equivalent to (6.2) and that (6.1) holds. Hence, $\tau = G^T\Gamma$ is idempotent, i.e., $\tau^2 = \tau$.

It is helpful to note the identities

$$\hat{Q} = Q_{12}G = G^TQ_{12}^T = G^TQ_2G,\quad \hat{P} = -P_{12}\Gamma = -\Gamma^TP_{12}^T = \Gamma^TP_2\Gamma, \tag{A.17}$$

$$\tau G^T = G^T,\quad \Gamma\tau = \Gamma, \tag{A.18}$$

$$\hat{Q} = \tau\hat{Q},\quad \hat{P} = \hat{P}\tau, \tag{A.19}$$

$$\hat{Q}\hat{P} = -Q_{12}P_{12}^T. \tag{A.20}$$

Using (A.14) and Sylvester's inequality, it follows that

$$\text{rank } G = \text{rank } \Gamma = \text{rank } Q_{12} = \text{rank } P_{12} = n_c.$$

Now using (A.17) and Sylvester's inequality yields

$$n_c = \text{rank } Q_{12} + \text{rank } G - n_c \le \text{rank } \hat{Q} \le \text{rank } Q_{12} = n_c$$

which implies that rank $\hat{Q} = n_c$. Similarly, rank $\hat{P} = n_c$, and rank $\hat{Q}\hat{P} = n_c$ follows from (A.20).

The components of $\mathcal{Q}$ and $\mathcal{P}$ can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G, and Γ as

$$Q_1 = Q + \hat{Q},\quad P_1 = P + \hat{P}, \tag{A.21}$$

$$Q_{12} = \hat{Q}\Gamma^T,\quad P_{12} = -\hat{P}G^T, \tag{A.22}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^T,\quad P_2 = G\hat{P}G^T. \tag{A.23}$$

Next note that by using (A.21)–(A.23) it can be shown that the right-hand coefficient of C_c in (A.16) is given by

$$\hat{S} \triangleq I_{n_c} + \beta^2\gamma^{-2}\Gamma\hat{Q}PG^T.$$

To prove that $\hat{S}$ is invertible use (A.19) and (6.3) and note that

$$I_{n_c} + \beta^2\gamma^{-2}\Gamma\hat{Q}PG^T = I_{n_c} + \beta^2\gamma^{-2}\Gamma\hat{Q}\tau^TPG^T$$
$$= I_{n_c} + \beta^2\gamma^{-2}(\Gamma\hat{Q}\Gamma^T)(GPG^T).$$

Since $\Gamma\hat{Q}\Gamma^T$ and GPG^T are nonnegative definite, their product has nonnegative eigenvalues (see Lemma 5.1). Thus, each eigenvalue of $I_{n_c} + \beta^2\gamma^{-2}\Gamma\hat{Q}PG^T$ is real and is greater than unity. Hence, $\hat{S}$ is invertible. Now note that by using (6.2) and (6.3) it can be shown that

$$G^T\hat{S}^{-1} = SG^T.$$

The expressions (6.11), (6.12), and (6.13) follow from (A.15), (A.16), and the definition of $\mathcal{Q}$. Next, computing either Γ(A.9)–(A.10) or G(A.12) + (A.13) yields (6.10). Substituting (A.21)–(A.23) into (A.8)–(A.13) and the expression for A_c into (A.9), (A.10), (A.12), and (A.13) it follows that (A.10) = Γ(A.9) and (A.13) = G(A.12). Thus, (A.10) and (A.13) are superfluous and can be omitted. Thus, (A.8)–(A.13) reduce to

$$0 = AQ + QA^T + V_1 + \gamma^{-2}(Q+\hat{Q})R_{1\infty}(Q+\hat{Q})$$
$$+ \beta^2\gamma^{-2}\hat{Q}S^TP\Sigma PS\hat{Q}$$
$$+ (A - \Sigma PS)\hat{Q} + \hat{Q}(A - \Sigma PS)^T, \tag{A.24}$$

$$0=[(A-\Sigma PS)\hat{Q}+\hat{Q}(A-\Sigma PS)^T+Q\bar{\Sigma}Q+\gamma^{-2}(Q+\hat{Q})R_{1\infty}(Q+\hat{Q})-\gamma^{-2}QR_{1\infty}Q+\beta^2\gamma^{-2}\hat{Q}S^TP\Sigma PS\hat{Q}]\Gamma^T, \quad (A.25)$$

$$0=(A+\gamma^{-2}[Q+\hat{Q}]R_{1\infty})^TP+P(A+\gamma^{-2}[Q+\hat{Q}]R_{1\infty})+R_1+(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})^T\hat{P}+\hat{P}(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty}), \quad (A.26)$$

$$0=[(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})^T\hat{P}+\hat{P}(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})+S^TP\Sigma PS]G^T. \quad (A.27)$$

Next, using (A.24) + $G^T\Gamma$(A.25)G − (A.25)G − [(A.25)G]^T and $G^T\Gamma$(A.25)G − (A.25)G − [(A.25)G]^T yields (6.5) and (6.7). Similarly, using (A.26) + Γ^TG(A.27)Γ − (A.27)Γ − [(A.27)Γ]^T and Γ^TG(A.27)Γ − (A.27)Γ − [(A.27)Γ]^T yields (6.6) and (6.8).

Finally, to prove the converse we use (6.5)–(6.13) to obtain (2.14) and (A.4)–(A.7). Let A_c, B_c, C_c, G, Γ, τ, Q, P, $\hat{Q}$, $\hat{P}$, $\mathcal{Q}$ be as in the statement of Theorem 6.1 and define Q_1, Q_{12}, Q_2, P_1, P_{12}, P_2 by (A.21)–(A.23). Using (6.2), (6.11), and (6.12) it is easy to verify (A.6) and (A.7). Finally, substitute the definitions of Q, P, $\hat{Q}$, $\hat{P}$, G, Γ, and τ into (6.5)–(6.8) using (6.2), (6.3), and (A.19) to obtain (2.14) and (A.4). Finally, note that

$$\mathcal{Q}=\begin{bmatrix} Q & 0_{n\times n_c} \\ 0_{n_c\times n} & 0_{n_c}\end{bmatrix}+\begin{bmatrix} I_n \\ \Gamma\end{bmatrix}\hat{Q}[I_n \ \Gamma^T]$$

which shows that $\mathcal{Q} \geq 0$. □

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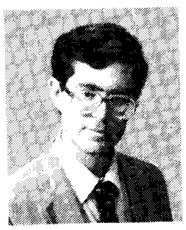

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