

Lyapunov-based backward-horizon adaptive stabilization

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SUMMARY

In this paper we develop a discrete-time adaptive stabilization algorithm based on a one-step backward-horizon cost criterion. By optimizing the cost with respect to the update step size, we obtain a gain update law that guarantees convergence of the plant states. The convergence proof is based on a modified Lyapunov technique. We extend the algorithm to include integral control for rejecting constant disturbances and we present an experimental application to DC motor positioning system. Copyright © 2003 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Although there is no precise definition of adaptive control, one can say intuitively that an adaptive controller operates by adjusting parameters in response to the behaviour of the plant. For continuous-time systems, direct adaptive control algorithms have been developed based on Lyapunov methods [1–5]. The proofs of stability and convergence often depend on the existence of a reference stabilizing controller (called a ‘dummy gain matrix’ in Reference [3, p. 67]), although knowledge of a stabilizing controller is not needed. For output feedback, these results are generally limited to minimum phase systems with known relative degree.

Direct adaptive control algorithms have also been developed for discrete-time systems [1, 4–13]. However, unlike the continuous-time case, these discrete-time results are based on RLS or LMS algorithms rather than Lyapunov methods. In particular, the approach developed in Reference [6] is based on a convergence result called the Key Technical Lemma (Lemma 6.2.1, pp. 181–182, [12]) which can be applied to RLS or projection-based adaptive control methods. This approach is extended to certain classes of non-minimum phase plants in References [14, 15] and to plants with disturbances in Reference [16]. Extensions of this approach to smooth stabilization with unknown high frequency gain are given in References [17, 18].

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Lyapunov synthesis for adaptive control is more straightforward in continuous time because the Lyapunov candidate can usually be chosen such that the derivative is linear in the error states [19]. Nevertheless, Lyapunov techniques have been used for discrete-time direct adaptive control algorithms in References [20–23]. The work in References [20, 21] is based on an RLS approach for model-reference adaptive control in which a cost function based on past input–output data is minimized with respect to the current controller parameters. Such an approach is retrospective in the sense that it optimizes controller performance based on past data. A conceptually similar approach is used in Reference [22], where the controller update gradient is based on a window of past data. In Reference [23], a one-step-ahead cost function is used to determine the optimal control signal; however, implementation of this algorithm depends on the choice of two positive-definite matrices that need to satisfy an *a priori* unverifiable stability condition. In addition, the Lyapunov function for stability analysis of the update law in Reference [23] is based on a parameter identification problem and thus does not explicitly involve the states of the plant and controller.

In this paper, we develop an alternative approach for full-state feedback based on a modified Lyapunov technique and an adaptive step size. We begin by considering an update law for the feedback gain matrix based on minimizing a time-dependent cost function that involves the state at the current time step. The gradient of the cost function with respect to the gain matrix at the previous time step provides an update direction, while the step-size in the gradient direction is chosen such that the distance from the updated gain matrix to the optimal gain is minimized. This optimization is a one-step backward-horizon procedure because the current gain matrix, which affects the state at the next time step, is updated based on the prior cost function involving the current state. An analogous step size is used in References [24–26] within the context of Reference [6], and also in Reference [22] as a key element in an adaptive disturbance rejection algorithm.

We present the main results in Section 2. Implementation issues are discussed in Section 3. In Section 4, we augment the adaptive stabilization controller with an integrator to reject step disturbances. Numerical examples with single input and multiple input plants are presented in Section 5, experimental results are given in Section 6, and conclusions are in Section 7.

2. ADAPTIVE STABILIZATION ALGORITHM

Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k \quad (1)$$

where $x_k \in \mathfrak{R}^{n_x}$, $u_k \in \mathfrak{R}^{n_u}$ and $k = 0, 1, \dots$ denotes the time step. We assume that the pair (A, B) is stabilizable and $\text{rank}(B) = n_u$. Furthermore, we assume there exists $K_s \in \mathfrak{R}^{n_u \times n_x}$ such that $A_s \triangleq A + BK_s$ is asymptotically stable and known. However, we do not assume that we have sufficient knowledge of A and B to actually determine K_s . Therefore, our objective is to determine a full-state-feedback control law of the form:

$$u_k = K_k x_k \quad (2)$$

such that the origin of the closed-loop system (1), (2) is attractive with respect to x_k . The adaptive gain matrix K_k is updated at each time step k to yield the next gain matrix K_{k+1} .

In certain cases these assumptions can be satisfied with minimal knowledge of the system parameters. For instance, for a single input system in companion form, we do not require knowledge of the last row of A . Additional details as well as a multiple input example are given in Section 5.

To derive an adaptively stabilizing control law, we consider the one-step cost function

$$J_k(K) \triangleq \frac{1}{2} \varepsilon_k^T(K) P \varepsilon_k(K) \quad (3)$$

where $\varepsilon_k(K) \triangleq x_{k+1}(K) - x_{k+1}(K_s)$, $x_{k+1}(K) = (A + BK)x_k$, the state at time $k + 1$ when the gain matrix K is used at time k , and $P \in \mathfrak{R}^{n \times n}$ is a positive-definite matrix. Note that $x_{k+1}(K_s) = A_s x_k$. We also define

$$\hat{J}_k(K) \triangleq \|K - K_s\|_F^2 \quad (4)$$

Let $\sigma_{\max}(A)$ denote the maximum singular value of A , let I_n denote the $n \times n$ identity matrix, and let Z^+ denote the set of nonnegative integers.

Lemma 1

Consider the gain update law

$$K_{k+1}(\eta_k) = K_k - \eta_k \left. \frac{\partial J_k}{\partial K} \right|_{K_k} = K_k - \eta_k B^T P \varepsilon_k(K_k) x_k^T \quad (5)$$

where $\eta_k \in \mathfrak{R}$ and $k \in Z^+$. Let $N \in \mathfrak{R}^{n \times n}$ satisfy $N^T N = P$. Then the following statements hold:

- (i) If $\varepsilon_k(K_k) = 0$, then $K_{k+1}(\eta_k) = K_k$ for all $\eta_k \in \mathfrak{R}$.
- (ii) If $\varepsilon_k(K_k) \neq 0$, then $\hat{\eta}_k$ given by

$$\hat{\eta}_k = \frac{\|N \varepsilon_k(K_k)\|_2^2}{\|B^T P \varepsilon_k(K_k) x_k^T\|_F^2} \quad (6)$$

is positive and minimizes $\hat{J}_k(K_{k+1}(\eta_k))$ with minimum value

$$\hat{J}_k(K_{k+1}(\hat{\eta}_k)) = \hat{J}_k(K_k) - \frac{\|N \varepsilon_k(K_k)\|_2^4}{\|B^T P \varepsilon_k(K_k) x_k^T\|_F^2} \quad (7)$$

(iii) Suppose $\varepsilon_k(K_k) \neq 0$. Then $\hat{J}_k(K_{k+1}(\eta_k)) < \hat{J}_k(K_k)$ if and only if $\eta_k \in (0, 2\hat{\eta}_k)$. Furthermore, $\hat{J}_k(K_{k+1}(\eta_k)) = \hat{J}_k(K_k)$ if and only if either $\eta_k = 0$ or $\eta_k = 2\hat{\eta}_k$.

Let $\{\eta_k\}_{k \in Z^+}$ be a sequence of positive real numbers, let $K_0 \in \mathfrak{R}^{n_u \times n_x}$, let $\{K_k\}_{k=1}^\infty$ be the sequence generated by (5), and let $S \triangleq \{k \in Z^+ : \varepsilon_k(K_k) \neq 0\}$. Then the following statements hold:

- (iv) If S is empty, then

$$\lim_{k \rightarrow \infty} x_k = 0 \quad (8)$$

- (v) If S is not empty and

$$\sup_{k \in S} \left| \frac{\eta_k}{\hat{\eta}_k} - 1 \right| < 1 \quad (9)$$

then

$$\lim_{k \rightarrow \infty} \frac{\|N\varepsilon_k(K_k)\|}{\|Nx_k\|} = 0 \quad (10)$$

Proof

To prove (i) let $\varepsilon_k(K_k) = 0$. Then (5) implies $K_{k+1}(\eta_k) = K_k$ for all η_k . To prove (ii), define

$$\hat{K}_k \triangleq K_k - K_s \quad (11)$$

and rewrite (5) as

$$\hat{K}_{k+1}(\eta_k) = \hat{K}_k - \eta_k B^T P \varepsilon_k(K_s) x_k^T \quad (12)$$

Now using (1), (2) we can write

$$x_{k+1}(K_k) = (A_s + B\hat{K}_k)x_k \quad (13)$$

which implies

$$\varepsilon_k(K_k) = B\hat{K}_k x_k \quad (14)$$

From (12) and (14) it follows that

$$\begin{aligned} \hat{J}_k(K_{k+1}(\eta_k)) &= \|\hat{K}_{k+1}(\eta_k)\|_F^2 \\ &= \|\hat{K}_k - \eta_k B^T P \varepsilon_k(K_k) x_k^T\|_F^2 \\ &= \hat{J}_k(K_k) + \|B^T P B \hat{K}_k x_k x_k^T\|_F^2 \eta_k^2 - 2\text{tr}(B^T P B \hat{K}_k x_k x_k^T \hat{K}_k^T) \eta_k \\ &= \hat{J}_k(K_k) + \|B^T P B \hat{K}_k x_k x_k^T\|_F^2 \eta_k^2 - 2\|NB\hat{K}_k x_k\|_2^2 \eta_k \\ &= \hat{J}_k(K_k) + \|B^T P \varepsilon_k(K_k) x_k^T\|_F^2 \eta_k^2 - 2\|N\varepsilon_k(K_k)\|_2^2 \eta_k \\ &= \hat{J}_k(K_k) + \eta_k(\eta_k - 2\hat{\eta}_k) \|B^T P \varepsilon_k(K_k) x_k^T\|_F^2 \\ &= \hat{J}_k(K_k) + ((\eta_k - \hat{\eta}_k)^2 - \hat{\eta}_k^2) \|B^T P \varepsilon_k(K_k) x_k^T\|_F^2 \end{aligned} \quad (15)$$

To minimize $J_k(K_{k+1}(\eta_k))$, we proceed as follows. By (14), $\varepsilon_k(K_k) \neq 0$ implies $\hat{K}_k x_k \neq 0$ and $x_k \neq 0$. Hence $\hat{K}_k x_k x_k^T \neq 0$. Since $B^T P B$ is non-singular, it follows that $\|B^T P \varepsilon_k(K_k) x_k^T\|_F^2 = \|B^T P B \hat{K}_k x_k x_k^T\|_F^2 \neq 0$. Therefore $\hat{\eta}_k$ can be defined by (6) and $\eta_k = \hat{\eta}_k$ minimizes $\hat{J}_k(K_{k+1}(\eta_k))$ with (7).

To prove (iii) assume $\hat{J}_k(K_{k+1}(\eta_k)) - \hat{J}_k(K_k) < 0$. Then by (15)

$$\eta_k(\eta_k - 2\hat{\eta}_k) \|B^T P \varepsilon_k(K_k) x_k^T\|_F^2 < 0 \quad (16)$$

which implies $0 < \eta_k < 2\hat{\eta}_k$. Conversely, $0 < \eta_k < 2\hat{\eta}_k$ implies (16), which implies $\hat{J}_k(K_{k+1}(\eta_k)) - \hat{J}_k(K_k) < 0$ by (15). Setting $\hat{J}_k(K_{k+1}(\eta_k)) = \hat{J}_k(K_k)$ in (15) yields $\eta_k = 0$ or $\eta_k = 2\hat{\eta}_k$.

To prove (iv) let $\varepsilon_k(K_k) = 0$ for all $k \in Z^+$. This implies $x_{k+1} = A_s x_k$ for all $k \in Z^+$. Since A_s is asymptotically stable, it follows that (8) holds.

To prove (v), define

$$\gamma \triangleq \sup_{k \in S} \left| \frac{\eta_k}{\hat{\eta}_k} - 1 \right| \quad (17)$$

By (9), $\gamma < 1$, hence $\eta_k \in [(1 - \gamma)\hat{\eta}_k, (1 + \gamma)\hat{\eta}_k] \subset (0, 2\hat{\eta}_k)$ for all $k \in S$. Hence $\eta_k \neq 0$ and $\eta_k \neq 2\hat{\eta}_k$. Furthermore, as in the proof of (ii), $\|B^T P \varepsilon_k(K_k) x_k^T\|_F^2 \neq 0$ and $\eta_k < 2\hat{\eta}_k$. Now let $k \in S$.

Using (6) and (15) we have

$$\begin{aligned} \|\hat{K}_{k+1}(\eta_k)\|_F^2 - \|\hat{K}_k\|_F^2 &= \eta_k(\eta_k \|B^T P \varepsilon_k(K_k) x_k^T\|_F^2 - 2\|N \varepsilon_k(K_k)\|_2^2) \\ &= \eta_k(\eta_k - 2\hat{\eta}_k) \|B^T P \varepsilon_k(K_k) x_k^T\|_F^2 \\ &< 0 \end{aligned} \quad (18)$$

Since S is not empty, there exists a positive integer $n > 0$ such that $\varepsilon_k(K_n) \neq 0$. Let $r_0 > n$ and, for all $r > r_0$, define the non-empty set $S_r \triangleq \{k : 0 \leq k \leq r \text{ and } \varepsilon_k(K_k) \neq 0\}$. For $r > r_0$, it follows from (18) that

$$\begin{aligned} \|\hat{K}_0\|_F^2 &\geq \|\hat{K}_0\|_F^2 - \|\hat{K}_{r+1}\|_F^2 \\ &= \sum_{k=0}^r (\|\hat{K}_k\|_F^2 - \|\hat{K}_{k+1}\|_F^2) \\ &= \sum_{k \in S_r} \eta_k(2\hat{\eta}_k - \eta_k) \|B^T P \varepsilon_k(K_k) x_k^T\|_F^2 \\ &> 0 \end{aligned} \quad (19)$$

Let $r > r_0$, let $k \in S_r$, and consider the function $g(\eta) = \eta(2\hat{\eta}_k - \eta)$ defined on the interval $L = [(1 - \gamma)\hat{\eta}_k, (1 + \gamma)\hat{\eta}_k]$. Since $g(\cdot)$ is quadratic, it follows that

$$(1 - \gamma^2)\hat{\eta}_k^2 = \min_{\eta \in L} g(\eta) = g((1 - \gamma)\hat{\eta}_k) = g((1 + \gamma)\hat{\eta}_k)$$

Hence,

$$\eta(2\hat{\eta}_k - \eta) \geq (1 - \gamma^2)\hat{\eta}_k^2 \text{ for all } \eta \in [(1 - \gamma)\hat{\eta}_k, (1 + \gamma)\hat{\eta}_k] \quad (20)$$

Using (6) and (20), we can rewrite (19) as

$$\begin{aligned} \|\hat{K}_0\|_F^2 &\geq (1 - \gamma^2) \sum_{k \in S_r} \frac{\|N \varepsilon_k(K_k)\|_2^4}{\|B^T P \varepsilon_k(K_k) x_k^T\|_F^2} \\ &= (1 - \gamma^2) \sum_{k \in S_r} \frac{\|N \varepsilon_k(K_k)\|_2^4}{\|B^T P \varepsilon_k(K_k) x_k^T N^T N^{-T}\|_F^2} \\ &\geq (1 - \gamma^2) \sum_{k \in S_r} \frac{\|N \varepsilon_k(K_k)\|_2^4}{\|NB\|_F^2 \|N \varepsilon_k(K_k)\|_2^2 \|N x_k\|_2^2 \|N^{-1}\|_F^2} \\ &\geq (1 - \gamma^2) \sum_{k \in S_r} \frac{\|N \varepsilon_k(K_k)\|_2^2}{\|NB\|_F^2 \|N x_k\|_2^2 \|N^{-1}\|_F^2} \end{aligned}$$

or

$$\sum_{k \in S_r} \frac{\|N\varepsilon_k(K_k)\|_2^2}{\|Nx_k\|_2^2} \leq \beta$$

where $\beta \triangleq \|\hat{K}_0\|_{\mathbb{F}}^2 \|NB\|_{\mathbb{F}}^2 \|N^{-1}\|_{\mathbb{F}}^2 / (1 - \gamma^2)$. Letting $r \rightarrow \infty$ yields

$$\sum_{k \in S} \frac{\|N\varepsilon_k(K_k)\|_2^2}{\|Nx_k\|_2^2} \leq \beta \quad (21)$$

Next, define the set $S' \triangleq Z^+ \setminus S$ and note that $\varepsilon_k(K_k) = 0$ for all $k \in S'$. If $k \in S'$ and $x_k = 0$ then $x_l = 0$ for all $l \geq k$. Hence assume that $x_k \neq 0$ for all $k \in S'$. For $k \in S'$, we have $\|N\varepsilon_k \times (K_k)\|_2^2 / \|Nx_k\|_2^2 = 0$. Therefore, it follows from (21) that (10) holds.

Theorem 1

Assume there exists $K_s \in \mathfrak{R}^{n_u \times n_x}$ such that $A_s \triangleq A + BK_s$ is asymptotically stable, let $R \in \mathfrak{R}^{n_x \times n_x}$ be positive definite, and let $P \in \mathfrak{R}^{n_x \times n_x}$ be the positive-definite solution to

$$P = A_s^T P A_s + R \quad (22)$$

Let the control be given by (2) with the gain update (5) and with $\{\eta_r\}_{k \in Z^+}$ satisfying (9). Then

$$\lim_{k \rightarrow \infty} x_k = 0 \quad (23)$$

Proof

If S is empty, the result follows from (iv) of Lemma 1. Hence assume S is not empty and consider the Lyapunov candidate $V(x_k) = x_k^T P x_k + \|\hat{K}_k\|_{\mathbb{F}}^2$. Then using (22) and (iii) of Lemma 1, we have

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= x_{k+1}^T P x_{k+1} - x_k^T P x_k + \|\hat{K}_{k+1}\|_{\mathbb{F}}^2 - \|\hat{K}_k\|_{\mathbb{F}}^2 \\ &\leq x_{k+1}^T P x_{k+1} - x_k^T P x_k \\ &= (A_s x_k + \varepsilon_k)^T P (A_s x_k + \varepsilon_k) - x_k^T P x_k \\ &= x_k^T (A_s^T P A_s - P) x_k + \varepsilon_k^T P \varepsilon_k + 2\varepsilon_k^T P A_s x_k \\ &= -x_k^T R x_k + \varepsilon_k^T P \varepsilon_k + 2\varepsilon_k^T P A_s x_k \\ &\leq -x_k^T R x_k + 2\|N\varepsilon_k\|_2 \|NA_s x_k\|_2 + \|N\varepsilon_k\|_2^2 \end{aligned}$$

where $N \in \mathfrak{R}^{n_x \times n_x}$ satisfies $N^T N = P$. Thus,

$$\begin{aligned} V(x_{k+1}) - V(x_k) &\leq -x_k^T R x_k + 2\|N\varepsilon_k\|_2 \|NA_s N^{-1} N x_k\|_2 + \|N\varepsilon_k\|_2^2 \\ &\leq -x_k^T R x_k + 2\|N\varepsilon_k\|_2 \|NA_s N^{-1}\|_{\mathbb{F}} \|N x_k\|_2 + \|N\varepsilon_k\|_2^2 \\ &\leq -x_k^T R x_k + 2\sigma_{\max}(NA_s N^{-1}) \|N\varepsilon_k\|_2 \|N x_k\|_2 + \|N\varepsilon_k\|_2^2 \end{aligned}$$

Now, since $N^T N = P$, (22) implies

$$I_{n_x} = \hat{A}_s^T \hat{A}_s + \hat{R} \quad (24)$$

where $\hat{A}_s \triangleq NA_s N^{-1}$ and $\hat{R} \triangleq N^{-T} R N^{-1}$ is positive definite. Thus, $\sigma_{\max}(\hat{A}_s) < 1$. Therefore,

$$V(x_{k+1}) - V(x_k) \leq -x_k^T R x_k + 2\|N\varepsilon_k\|_2 \|Nx_k\|_2 + \|N\varepsilon_k\|_2^2 \quad (25)$$

Let $\delta > 0$. By (v) of Lemma 1, there exists a positive integer k_δ such that $\|N\varepsilon_k\|_2 / \|Nx_k\|_2 < \delta$ for all $k > k_\delta$. Then for $k > k_\delta$ we can write

$$\begin{aligned} V(x_{k+1}) - V(x_k) &< -x_k^T R x_k + 2\delta \|Nx_k\|_2^2 + \delta^2 \|Nx_k\|_2^2 \\ &< -x_k^T [R - (2\delta + \delta^2)P] x_k \end{aligned} \quad (26)$$

Now choose δ sufficiently small such that $R - (2\delta + \delta^2)P$ is positive definite. Next, for $k > k_\delta$, define the translated system

$$\hat{x}_{\hat{k}+1} = (A + BK_k)\hat{x}_{\hat{k}}, \quad \hat{k} \geq 0 \quad (27)$$

where $\hat{k} = k - k_\delta$ and $\hat{x}_{\hat{k}} \triangleq x_{k_\delta + \hat{k}}$. Using (26), it follows from Theorem 6.3 in Reference [27] that, for the translated system (27) with initial condition $\hat{x}_0 = x_{k_\delta}$, $\hat{x}_{\hat{k}}^T [R - (2\delta + \delta^2)P] \hat{x}_{\hat{k}} \rightarrow 0$ as $\hat{k} \rightarrow \infty$. Hence $\lim_{\hat{k} \rightarrow \infty} \hat{x}_{\hat{k}} = 0$, and thus $\lim_{k \rightarrow \infty} x_k = 0$.

The following result provides an alternative step size that guarantees decrease of the cost function J_k . This result provides a one-step backward horizon interpretation for the gain update law (5).

Proposition 1

Let $\varepsilon_k(K_k) \neq 0$ and define

$$\eta_k^* = \frac{\|B^T P \varepsilon_k(K_k)\|_2^2}{\|NBB^T P \varepsilon_k(K_k) x_k^T\|_F^2} \quad (28)$$

Then the following statements hold:

(i) η_k^* is positive and minimizes $J_k(K_{k+1}(\eta_k))$ with minimum value

$$J_k(K_{k+1}(\eta_k^*)) = J_k(K_k) - \frac{x_k^T x_k \|B^T P \varepsilon_k(K_k)\|_2^4}{2\|NBB^T P \varepsilon_k(K_k) x_k^T\|_F^2} \quad (29)$$

(ii) $J_k(K_{k+1}(\eta_k)) < J_k(K_k)$ if and only if $\eta_k \in (0, 2\eta_k^*)$. Furthermore, $J_k(K_{k+1}(\eta_k)) = J_k(K_k)$ if and only if either $\eta_k = 0$ or $\eta_k = 2\eta_k^*$.

(iii) If $\varepsilon_k(K_k) \neq 0$, then $\eta_k^* \leq \hat{\eta}_k$.

(iv) If $\varepsilon_k(K_k) \neq 0$ and $n_u = 1$, then $\eta_k^* = \hat{\eta}_k$.

Proof

To prove (i), use (14) to write

$$J_k(K_k) = \frac{1}{2} x_k^T \hat{K}_k^T B^T P B \hat{K}_k x_k \quad (30)$$

Using (12) and (14) we obtain

$$\begin{aligned}
J_k(K_{k+1}(\eta_k)) &= \frac{1}{2}x_k^T \hat{K}_{k+1}(\eta_k)^T B^T P B \hat{K}_{k+1}(\eta_k) x_k \\
&= J_k(K_k) + \frac{1}{2}(x_k^T x_k)^2 [x_k^T \hat{K}_k^T (B^T P B)^3 \hat{K}_k x_k] \eta_k^2 - x_k^T x_k [x_k^T K_k^T (B^T P B)^2 \hat{K}_k x_k] \eta_k \\
&= J_k(K_k) + \frac{1}{2}x_k^T x_k \|N B B^T P B \hat{K}_k x_k x_k^T\|_F^2 \eta_k^2 - x_k^T x_k \|B^T P B \hat{K}_k x_k\|_2^2 \eta_k \\
&= J_k(K_k) + \frac{1}{2}x_k^T x_k (\eta_k - 2\eta_k^*) \|N B B^T P \varepsilon_k(K_k) x_k^T\|_F^2 \\
&= J_k(K_k) + \frac{1}{2}x_k^T x_k [(\eta_k - \eta_k^*)^2 - \eta_k^{*2}] \|N B B^T P \varepsilon_k(K_k) x_k^T\|_F^2
\end{aligned} \tag{31}$$

As in the proof of Lemma 1, part (ii), it follows that η_k^* globally minimizes (31) and satisfies (29)

The proof of (ii) is analogous to the proof of Lemma 1, part (iii).

To prove (iii), let $C = [v \quad B^T P B v]^T$, where $v \triangleq (B^T P B)^{1/2} \hat{K}_k x_k$. Since $\det(CC^T) \geq 0$ we have

$$v^T v [v^T (B^T P B)^2 v] - (v^T B^T P B v)^2 \geq 0 \tag{32}$$

Since $\varepsilon_k(K_k) \neq 0$, it follows that $\hat{K}_k x_k \neq 0$, $x_k \neq 0$ and $v \neq 0$. Therefore,

$$\eta_k^* = \frac{v^T B^T P B v}{x_k^T x_k v^T (B^T P B)^2 v} \leq \frac{v^T v}{x_k^T x_k v^T B^T P B v} = \hat{\eta}_k \tag{33}$$

To prove (iv), let $n_u = 1$. Then $B^T P B$ is a scalar and (33) holds with equality.

Remark 1

Note that K_{k+1} is computed using the knowledge of x_k and x_{k+1} at time $k+1$. The updated gain K_{k+1} is used to propagate the state from x_{k+1} to x_{k+2} .

To compute the updated gain matrix K_{k+1} we need the gradient direction of the cost function J_k as well as a step size η_k to move along this direction. To compute the step size η_k that minimizes the current cost $J_{k+1}(K_{k+1})$, it can be seen from the definition of J_k and ε_k that we require knowledge of the state x_{k+2} at time $k+1$. Since x_{k+2} is not available at time $k+1$, we instead minimize the *prior* cost $J_k(K_{k+1})$ with respect to the updated gain matrix K_{k+1} . However, the prior cost $J_k(K_k)$ has already been incurred by using K_k to move from x_k to x_{k+1} . Therefore minimizing $J_k(K_{k+1})$ is a *one-step backward horizon* cost optimization. Note that η_k^* may not satisfy (9), and thus, there is no guarantee of (23). Theorem 1 guarantees stability for an open interval around the larger step size $\hat{\eta}_k$ which minimizes the norm of the distance between K_{k+1} and K_s . The relation between the step sizes is shown in Figure 1.

3. IMPLEMENTATION

As an application of Theorem 1, consider the single-input system in companion form

$$A = \begin{bmatrix} 0_{(n_x-1) \times 1} & I_{n_x-1} \\ a & \end{bmatrix}, \quad B = \begin{bmatrix} 0_{(n_x-1) \times 1} \\ b \end{bmatrix} \tag{34}$$

system is the double companion form with decoupled inputs and coupled states

$$A = \begin{bmatrix} 0_{n \times 1} & I_n & 0_{n \times (n+1)} \\ & a_1 & \\ 0_{n \times 1} & I_n & 0_{n \times (n+1)} \\ & a_2 & \end{bmatrix} \quad B = |b| \begin{bmatrix} 0_{n-1 \times 2} \\ 1 & 0 \\ 0_{n-1 \times 2} \\ 0 & 1 \end{bmatrix} \quad (35)$$

where $a_1, a_2 \in \mathfrak{R}^{1 \times (2n+2)}$. This system can be stabilized without knowledge of the row vectors a_1, a_2 or the matrix K_s .

4. INTEGRAL CONTROL

Integral control for rejecting constant disturbances can be incorporated into the algorithm as follows. Consider the closed-loop system

$$x_{k+1} = A_s x_k + B u_k + d \quad (36)$$

$$u_k = K_k x_k + v_k \quad (37)$$

where $d \in \mathfrak{R}^n$ is an unknown constant disturbance. Assume that there exists K_s such that $A_s \triangleq A + B K_s$ is asymptotically stable, and also that there exists v_s such that $B v_s = -d$.

From (37) it follows that

$$u_k = \tilde{K}_k \tilde{x}_k \quad (38)$$

where $\tilde{K}_k \triangleq [K_k \quad v_k]$ and $x_k \triangleq [x_k^T \quad 1]^T$, and thus the closed-loop system can be written as

$$\tilde{x}_{k+1}(\tilde{K}_k) = (\tilde{A} + \tilde{B} \tilde{K}_k) \tilde{x}_k + \tilde{d} \quad (39)$$

where $\tilde{A} \triangleq \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$, $\tilde{B} \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix}$ and $\tilde{d} \triangleq \begin{bmatrix} d \\ 0 \end{bmatrix}$.
Next, define

$$e_k(\tilde{K}_k) \triangleq \tilde{x}_{k+1}(\tilde{K}_k) - (\tilde{A}_s x_k + \tilde{d}) \quad (40)$$

where $\tilde{A}_s \triangleq \tilde{A} + \tilde{B} \tilde{K}_s$ with $\tilde{K}_s \triangleq [K_s \quad v_s]$. Note that

$$e_k(\tilde{K}_k) = \begin{bmatrix} x_{k+1}(K) - A_s x_k \\ 0 \end{bmatrix} \quad (41)$$

which can be calculated using A_s and a measurement of x_k .

To derive the adaptive constant disturbance rejection law, we define the cost functions

$$J_k(\tilde{K}_k) \triangleq \frac{1}{2} e_k^T(\tilde{K}_k) \tilde{P} e_k(\tilde{K}_k) \quad (42)$$

$$\tilde{J}_k(\tilde{K}_k) \triangleq \|\tilde{K}_k - \tilde{K}_s\|_F^2 \quad (43)$$

where P is positive definite.

Lemma 2 Consider the gain update law

$$\tilde{\mathbf{K}}_{k+1}(\eta_k) = \tilde{\mathbf{K}}_k - \eta_k \left. \frac{\partial \mathcal{J}_k}{\partial \tilde{\mathbf{K}}} \right|_{\tilde{\mathbf{K}}_k} = \tilde{\mathbf{K}}_k - \eta_k \tilde{\mathbf{B}}^T \tilde{\mathbf{P}} \varepsilon_k(\tilde{\mathbf{K}}_k) \tilde{\mathbf{x}}_k^T \quad (44)$$

where $\eta_k \in \mathfrak{R}$ and $k \in \mathbb{Z}^+$. Let $\tilde{\mathbf{N}}$ satisfy $\tilde{\mathbf{N}}^T \tilde{\mathbf{N}} = \tilde{\mathbf{P}}$. Then the following statements hold:

- (i) If $\varepsilon_k(\tilde{\mathbf{K}}_k) = 0$, then $\tilde{\mathbf{K}}_{k+1}(\eta_k) = \tilde{\mathbf{K}}_k$ for all $\eta_k \in \mathfrak{R}$.
- (ii) If $\varepsilon_k(\tilde{\mathbf{K}}_k) \neq 0$, then $\hat{\eta}_k$ given by

$$\hat{\eta}_k = \frac{\|\tilde{\mathbf{N}} \varepsilon_k(\tilde{\mathbf{K}}_k)\|_2^2}{\|\tilde{\mathbf{B}}^T \tilde{\mathbf{P}} \varepsilon_k(\tilde{\mathbf{K}}_k) \tilde{\mathbf{x}}_k^T\|_F^2} \quad (45)$$

is positive and minimizes $\hat{\mathcal{J}}_k(\tilde{\mathbf{K}}_{k+1}(\hat{\eta}_k))$ with

$$\hat{\mathcal{J}}_k(\tilde{\mathbf{K}}_{k+1}(\hat{\eta}_k)) = \hat{\mathcal{J}}_k(\tilde{\mathbf{K}}_k) - \frac{\|\tilde{\mathbf{N}} \varepsilon_k(\tilde{\mathbf{K}}_k)\|_2^4}{\|\tilde{\mathbf{B}}^T \tilde{\mathbf{P}} \varepsilon_k(\tilde{\mathbf{K}}_k) \tilde{\mathbf{x}}_k^T\|_F^2} \quad (46)$$

- (iii) Suppose $\varepsilon_k(\tilde{\mathbf{K}}_k) \neq 0$. Then $\hat{\mathcal{J}}_k(\tilde{\mathbf{K}}_{k+1}(\eta_k)) < \hat{\mathcal{J}}_k(\tilde{\mathbf{K}}_k)$ if and only if $\eta_k \in (0, 2\hat{\eta}_k)$. Furthermore, $\hat{\mathcal{J}}_k(\tilde{\mathbf{K}}_{k+1}(\eta_k)) = \hat{\mathcal{J}}_k(\tilde{\mathbf{K}}_k)$ if and only if either $\eta_k = 0$ or $\eta_k = 2\hat{\eta}_k$.

Let $\{\eta_k\}_{k \in \mathbb{Z}^+}$ be a sequence of positive real numbers, let $\tilde{\mathbf{K}}_0 \in \mathfrak{R}^{n_u \times n_x}$, let $\{\tilde{\mathbf{K}}_k\}_{k=1}^\infty$ be the sequence generated by (5), and let $S \triangleq \{k \in \mathbb{Z}^+ : \varepsilon_k(\tilde{\mathbf{K}}_k) \neq 0\}$. Then the following statements hold:

- (iv) if S is non empty and

$$\sup_{k \in S} \left| \frac{\eta_k}{\hat{\eta}_k} - 1 \right| < 1 \quad (47)$$

then

$$\lim_{k \rightarrow \infty} \frac{\|\tilde{\mathbf{N}} \varepsilon_k(\tilde{\mathbf{K}}_k)\|_2}{\|\tilde{\mathbf{N}} \tilde{\mathbf{x}}_k\|_2} = 0 \quad (48)$$

Proof

The proof is identical to the proof of Lemma 1.

Theorem 2

Assume there exists $\mathbf{K}_s \in \mathfrak{R}^{n_u \times n_x}$ such that $A_s \triangleq A + \mathbf{B}\mathbf{K}_s$ is asymptotically stable, let $\mathfrak{R} \in \mathfrak{R}^{n_x \times n_x}$ be possible definite, and let $\mathbf{P} \in \mathfrak{R}^{n_x \times n_x}$ be the positive-definite solution to

$$\mathbf{P} = A_s^T \mathbf{P} A_s + \mathbf{R} \quad (49)$$

Define $\tilde{\mathbf{P}} \triangleq \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \lambda^2 \end{bmatrix}$ with $\lambda > 0$. Let the control be given by (38) with gain update (44) and with $\{\eta_k\}_{k \in \mathbb{Z}^+}$ satisfying (47). Then

$$\lim_{k \rightarrow \infty} x_k = 0 \quad (50)$$

Proof

From (48) in Lemma 2, it follows that for all $\delta > 0$, there exists a positive integer l_δ such that

$$\frac{\|\tilde{\mathbf{N}} \varepsilon_k(\tilde{\mathbf{K}}_k)\|_2}{\|\tilde{\mathbf{N}} \tilde{\mathbf{x}}_k\|_2} < \delta \quad (51)$$

for all $k > l_\delta$. Using (41), (51) and the definition of \tilde{N} , it follows that

$$\left\| \begin{bmatrix} N & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_{k+1} - A_s x_k \\ 0 \end{bmatrix} \right\|_2 < \delta \left\| \begin{bmatrix} N & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_k \\ 1 \end{bmatrix} \right\|_2 \quad (52)$$

or

$$\left\| \begin{bmatrix} Nx_{k+1} - NA_s N^{-1} Nx_k \\ 0 \end{bmatrix} \right\|_2 < \delta \left\| \begin{bmatrix} Nx_k \\ \lambda \end{bmatrix} \right\|_2 \quad (53)$$

Thus,

$$\left\| \begin{bmatrix} Nx_{k+1} - \hat{A}_s Nx_k \\ 0 \end{bmatrix} \right\|_2 < \delta \sqrt{\|Nx_k\|_2^2 + \lambda^2} < \delta(\|Nx_k\|_2 + \lambda) \quad (54)$$

where, $\hat{A}_s \triangleq NA_s N^{-1}$, or

$$\|Nx_{k+1} - \hat{A}_s Nx_k\|_2 < \delta(\|Nx_k\|_2 + \lambda) \quad (55)$$

Next, define

$$\mu \triangleq \delta + \sigma_{\max}(\hat{A}_s) \quad (56)$$

and note from (49) that $\sigma_{\max}(\hat{A}_s) < 1$. We choose δ such that $\mu < 1$, and thus, from (55), it follows that, for all $r \in Z^+$,

$$\|Nx_{k+r}\|_2 < \mu^r + \delta \lambda \sum_{j=0}^{r-1} \mu^j \quad (57)$$

Thus,

$$\limsup_{r \rightarrow \infty} \|Nx_{k+r}\|_2 \leq \frac{\delta \lambda}{1 - \mu} \quad (58)$$

for all $\delta > 0$ such that $\mu < 1$ and for all $\lambda > 0$. Hence,

$$\limsup_{r \rightarrow \infty} \|Nx_{k+r}\|_2 = 0 \quad (59)$$

which implies that

$$\lim_{r \rightarrow \infty} \|Nx_{k+r}\|_2 = 0 \quad (60)$$

and thus we obtain (50).

5. NUMERICAL EXAMPLES

In this section we illustrate the adaptive stabilization algorithm by means of a numerical example. We consider a two-input example. At time $k = 30$ the matrix A changes from A_1 , which is open-loop stable, to A_2 , which is open-loop unstable. The controller is unaware of this change.

The system is given by

$$A_1 = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ -0.1634 & -0.1443 & 0.0294 & -0.0140 \\ 0 & 0 & 0 & 1.0000 \\ -0.2261 & -0.0585 & -0.1165 & -0.1793 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (61)$$

$$A_2 = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0.2486 & -1.4897 & 0.3135 & -2.0251 \\ 0 & 0 & 0 & 1.0000 \\ 0.5290 & 0.3435 & 0.7582 & -0.6919 \end{bmatrix} \quad (62)$$

$$A_s = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0.2000 & 0.2000 & 0.1300 & 0.1600 \\ 0 & 0 & 0 & 1.0000 \\ 0.1000 & 0.1500 & 0.3000 & 0.3000 \end{bmatrix} \quad (63)$$

with $x_0 = [78.26 \ 52.42 \ -8.71 \ -96.30]^T$ and the system is perturbed to the state $x_{30} = [-12.32 \ 3.4 \ 8.1 \ -7.965]^T$. The simulation was run with $\eta_k = \hat{\eta}_k$. Figure 2 shows the open-loop and closed-loop performance using the adaptive disturbance rejection algorithm of Section 2. Figure 3 shows $\hat{\eta}_k$ for the same simulation.

6. EXPERIMENTAL RESULTS

In this section we implement the discrete-time adaptive algorithm with integral control for angular positioning of a Maxon brushless DC motor. The objective is to rotate a steel disc mounted on the shaft through a prescribed angle. The motor is driven in torque mode by a Copley Controls amplifier, which receives a voltage command from the controller. The disc inertia, motor inertia, motor torque-current ratio and current amplifier gain are all unknown. The controller is implemented on a dSPACE DS1103 system as a C-coded Simulink S-function.

Let the angular position of the disc be denoted by $\theta(t)$ and define the state vector $x(t) \triangleq \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$. The system dynamics are given by the continuous-time state space model

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{K_{\text{amp}} K_{\text{motor}}}{J} \end{bmatrix} (V(t) + \phi) \quad (64)$$

where $V(t)$ is the voltage input to the current amplifier, ϕ is an unknown constant input bias, K_{amp} is the amplifier gain, K_{motor} is the torque constant of the motor, and J is the total inertia of the disc and the motor armature.

Defining $\mathcal{K} \triangleq K_{\text{amp}} K_{\text{motor}}/J$ and discretizing (64) using a zero-order hold equivalent at a sampling rate of T , we obtain

$$x_{k+1} = Ax_k + BV_k + B\phi \quad (65)$$

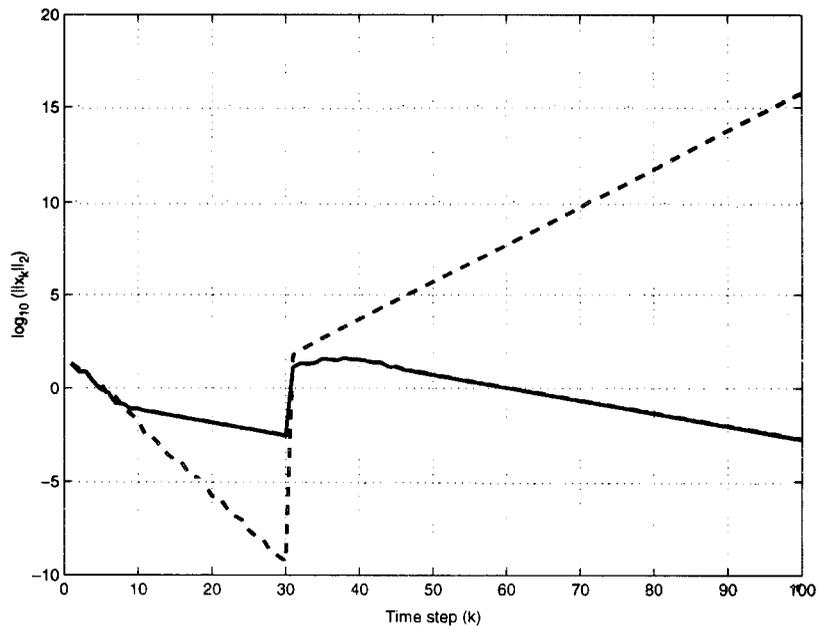


Figure 2. Open-loop and closed-loop performance from numerical simulation. Dashed line: open-loop, solid line: closed-loop.

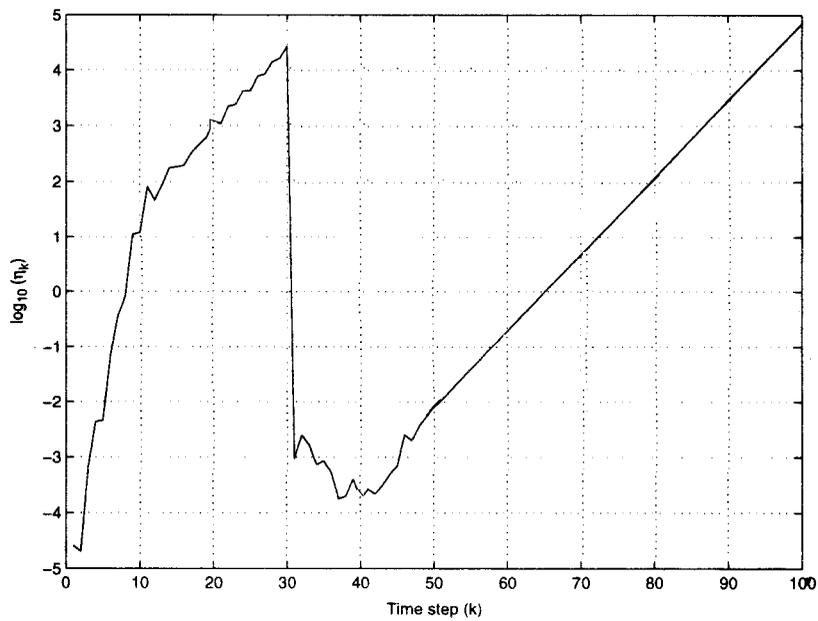


Figure 3. $\hat{\eta}_k$ from numerical simulation.

where

$$A \triangleq \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} \mathcal{H}T^2/2 \\ \mathcal{H}T \end{bmatrix} \quad (66)$$

We note that \mathcal{H} is unknown, and that the disc angular position and angular velocity are fed back to the controller to allow full-state-feedback control. The control objective is to asymptotically drive the system to a reference state $x_{\text{ref}} \triangleq \begin{bmatrix} \theta_{\text{ref}} \\ 0 \end{bmatrix}$, that is, to rotate the disc to an angular position θ_{ref} .

Next, we note that the reference state satisfies the constant state tracking condition

$$(A - I_{n_x})x_{\text{ref}} + Bu_{\text{ref}} = 0 \quad (67)$$

with $u_{\text{ref}} = 0$. Subtracting (67) from (65) we obtain

$$x_{k+1} - x_{\text{ref}} = A(x_k - x_{\text{ref}}) + BV_k + B\phi \quad (68)$$

or

$$e_{k+1} = Ae_k + BV_k + B\phi \quad (69)$$

where the *error state* $e_k \triangleq x_k - x_{\text{ref}}$. Defining $d \triangleq B\phi$, we obtain

$$e_{k+1} = Ae_k + Bu_k + d \quad (70)$$

We observe that (70) has the same form as (36), and hence we can use the adaptive integral control algorithm of Theorem 2, which implies $\lim_{k \rightarrow \infty} e_k = 0$.

The control algorithm is implemented as follows. First, we specify A_s to calculate P . Let $K_s = [K_{1s} \quad K_{2s}]$. Then, from (66) it follows that

$$A_s = A + BK_s = \begin{bmatrix} 1 + T\alpha_1 & T + \frac{T\alpha_2}{2} \\ \alpha_1 & 1 + \alpha_2 \end{bmatrix} \quad (71)$$

where $\alpha_1 \triangleq T\mathcal{H}K_{1s}$ and $\alpha_2 \triangleq T\mathcal{H}K_{2s}$. The coefficients of the second-order characteristic polynomial of A_s depend on α_1 and α_2 , and thus, we can specify α_1 and α_2 to ensure that A_s is asymptotically stable even though \mathcal{H} is unknown. The constants α_1 and α_2 are chosen such that the eigenvalues of A_s correspond to a damping ratio of 0.85 and a natural frequency of 10 rad/s with $T = 0.01$ s. A_s is used to determine P and to calculate ε_k . λ is chosen to be 0.001.

Next, from (66) we note that

$$B = b_0 \begin{bmatrix} \frac{T}{2} \\ 1 \end{bmatrix} \quad (72)$$

where $b_0 \triangleq T\mathcal{H}$. \mathcal{H} is assumed to be positive, and, although b_0 is unknown, the adaptive algorithm is robust to uncertainty in b_0 with Figures. 4–7 showing results of two tests with b_0 chosen to be 1 and 50.

7. CONCLUSIONS

In this paper we derived a discrete-time adaptive stabilization algorithm and proved closed-loop attractivity with respect to the plant states. Single and multiple input cases were simulated

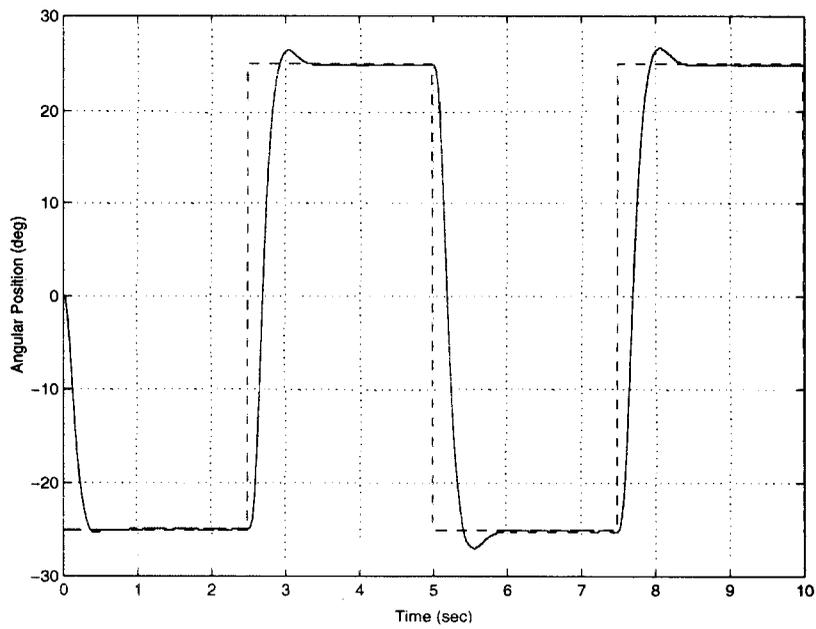


Figure 4. Angular position of disk for $b_0 = 1$. Solid line: θ_k , dashed line: θ_{ref} .

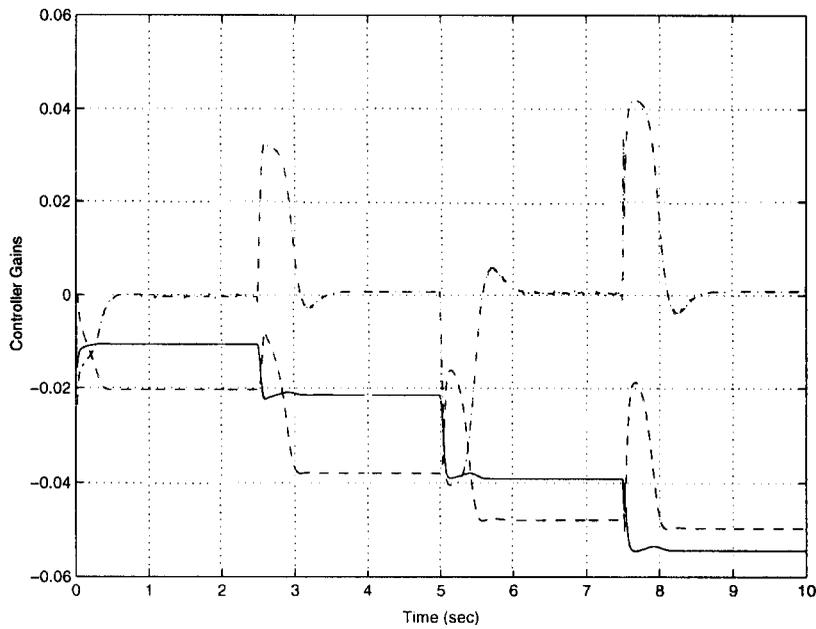


Figure 5. Controller gains for $b_0 = 1$. Solid line: K_{1k} , dashed line: K_{2k} , dash-dot line: v_k .

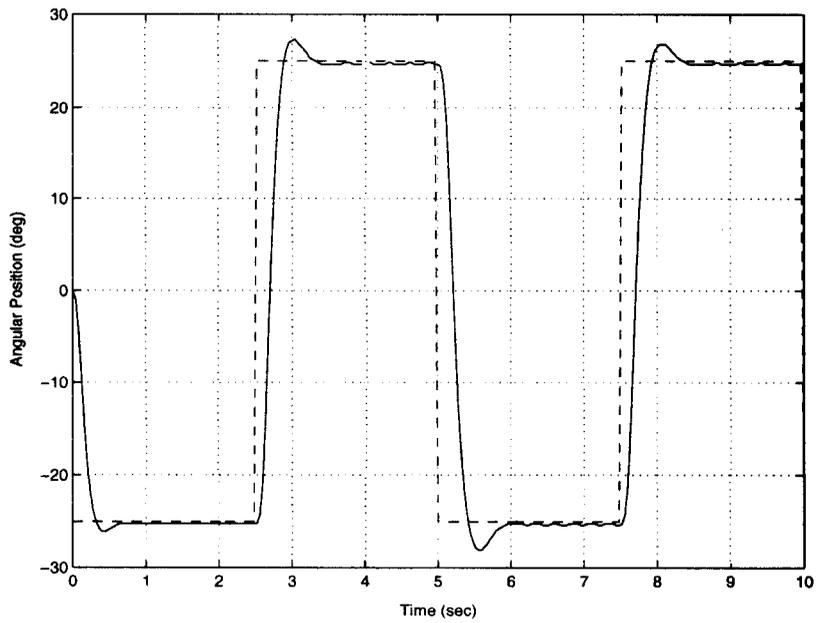


Figure 6. Angular position of disk for $b_0 = 50$. Solid line: θ_k , dashed line: θ_{ref} .

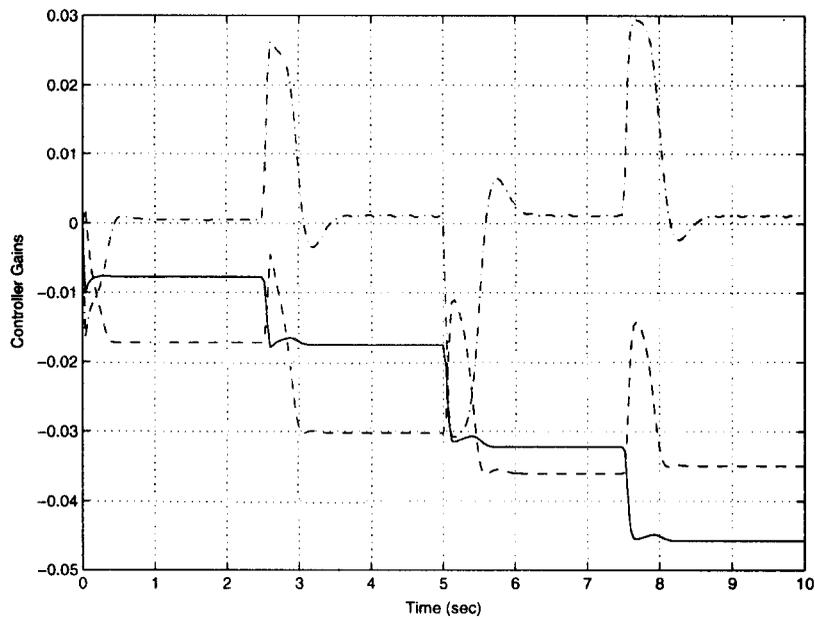


Figure 7. Controller gains for $b_0 = 50$. Solid line: K_{1k} , dashed line: K_{2k} , dash-dot line: v_k .

numerically for unstable and abruptly varying plants, and experimental results were obtained on a motor positioning system. Future work will involve extensions to output feedback.

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