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THE INTERACTION OF PLANE AND CYLINDRICAL SOUND WAVES WITH A STATIONARY SHOCK WAVE

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LIST OF SYMBOLS

Symbol	Explanation
a	Amplitude of distortion
A	Amplitude of sound wave
B	Amplitude of vorticity wave
c_p, c_v	Specific heats
C	Undisturbed sound velocity, amplitude of the entropy wave
$\mathcal{C}$	Disturbed sound velocity
$\mathcal{d}, d$	Perturbation to density
D	Undisturbed density
$\mathcal{D}$	Disturbed density
$f(y,t)$	Shape of shock
$F(y)$	Reduced shape function
$G(y)$	Reduced vorticity function
$h(x)$	Unit step function
$H(\alpha,\beta)$	Integrand of entropy integral (supersonic)
$J(\alpha,\beta)$	Integrand of vorticity integral with poles removed (supersonic)
k	Amplitude of wave vector
K	Pole of sonic integral
$K(\alpha,\beta)$	Integrand of vorticity integral
ℓ	Phase constant
$L(\alpha,\beta)$	Integrand of sonic wave (supersonic)
m	Perturbation to Mach number
M	Undisturbed Mach number
$\mathcal{M}$	Disturbed Mach number
$\mathcal{p}, p$	Perturbation to pressure

LIST OF SYMBOLS (Continued)

Symbol	Explanation
P	Undisturbed pressure
$\mathcal{P}$	Disturbed pressure
r	Distance
s, S	Perturbation to entropy
S	Undisturbed entropy
$S(y)$	Reduced entropy function (supersonic)
$\vec{S}$	"Poynting" vector
$\mathcal{S}$	Disturbed entropy
t	Time
u, U	Perturbations to x-velocity
U	Undisturbed x-velocity
$\mathcal{U}$	Disturbed x-velocity
v, V	Perturbation to y-velocity
$\mathcal{V}$	Disturbed y-velocity
(x, y, z)	Coordinates
α	Cos θ
β	Sin θ
γ	Ratio of specific heats, c_p/c_v
Γ	Polynomial in α
δ	Dirac's delta function
$\vec{\nabla}$	Gradient operator
ϵ	Strength of incident disturbance
θ	Angle
λ	Ratio of sound speeds c_1/c_0
Λ	Polynomial in α'

LIST OF SYMBOLS (Concluded)

Symbol	Explanation
μ	$\frac{\gamma+1}{\gamma-1}$
ρ	Transformed distance
φ	Angle
$\phi(x,y)$	Reduced velocity potential
$\Phi(x,y,t)$	Velocity potential
$\chi(x,y,t)$	Vorticity potential
$X(y)$	Reduced vorticity potential
ψ	Angle
ω	Frequency

ABSTRACT

This investigation treats the problem of the interaction of plane and cylindrical sound waves with a stationary shock wave theoretically. The linearized Euler differential equations and the corresponding linearized shock conditions serve as the fundamental laws for this study.

Plane-wave solutions to the differential equations are found, and the analogues of Snell's laws of reflection and refraction are determined, together with the "Fresnel" formulae. Integral solutions to the differential equations corresponding to incident cylindrical waves are then found from the plane-wave solutions by a method devised by H. Weyl. These integrals are investigated to determine the reflected and refracted field of a line source.

Generalizations of the theory for moving shocks and point sources are set up but not investigated in detail.

OBJECTIVE

The purpose of this study is to determine how sound waves interact with a stationary shock wave in a compressible, inviscid gas. Two types of incident sound waves are considered, plane and cylindrical. For incident plane waves the analoques of the reflection and refraction laws of optics are desired. One may use these plane-wave laws in a manner devised by H. Weyl to solve the cylindrical wave problem.

I. INTRODUCTION

A. DESCRIPTION OF THE PROBLEM

This introductory chapter will be devoted to a purely qualitative discussion of the interaction of sound waves with a stationary shock wave.

The fluid medium which is to serve as a carrier for the shock wave is assumed to be a compressible inviscid gas. When no sound waves are present, we assume that the shock wave is a plane at rest normal to the flow of this gas. Such a plane normal shock wave constitutes a rigorous solution of the equations of hydrodynamics in which the physical properties of the gas assume constant values.[1] A situation of this type is illustrated in Fig. 1. It should be noticed that the flow enters the shock supersonically and leaves subsonically. The pressure, density, and entropy on the supersonic side are smaller than the corresponding quantities on the subsonic side.

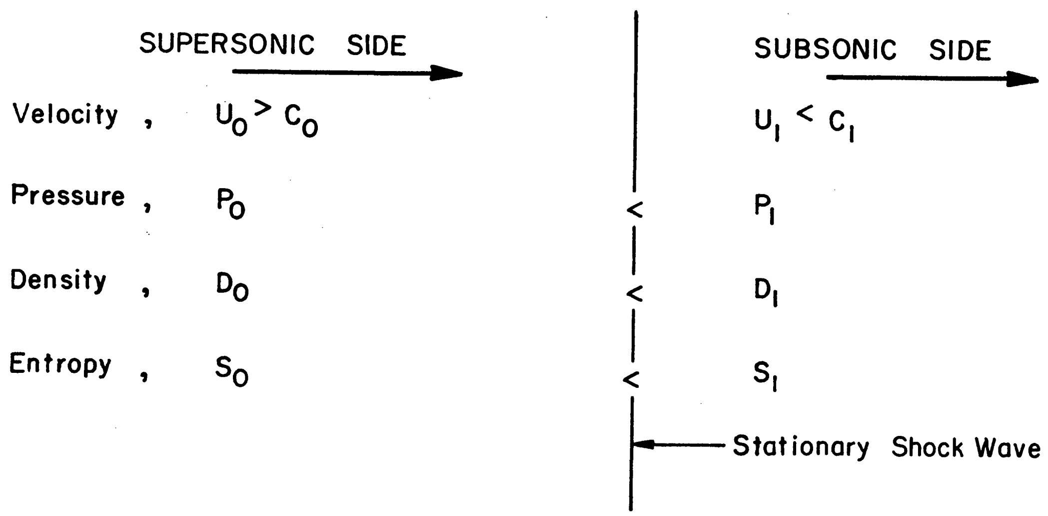

Fig. 1. Undisturbed state of the gas.

We now introduce a sound wave into this uniform but discontinuous flow. This sound wave disturbs the shock wave and thus the flow, presumably on both sides.

Let us suppose that the incident wave is a plane wave. Since the situation seems analogous to the reflection and refraction of plane light waves in a medium of discontinuous refractive index, one would expect to find reflected and refracted plane waves present in the flow. It is therefore natural to inquire after the analogues of Snell's law and of the Fresnel coefficients. In other words, we wish to find the relations between the angle of incidence and the angles of reflection or refraction as well as the amplitudes of the reflected and refracted waves.

However, the optical analogy is only superficial, as the two problems differ in certain important respects. Perhaps the most important way in which the problems differ is in the types of waves present.* In the optical case, there is only one type of wave present; this is referred to as a light wave. In the problem presented here, we may have two types of waves. One of these is an irrotational, isentropic wave, referred to as the sound wave; the other, a solenoidal, isobaric wave, referred to as the entropy-vorticity wave.

The sound waves here are not identical with the sound waves of classical acoustics since the medium carrying these waves is in motion. But they do behave like ordinary sound waves if they are observed from a frame of reference moving with the fluid. On the other hand, the entropy-vorticity waves are waves moving with the fluid; they would appear to be stationary to an observer moving with the flow. These waves transport entropy and

*In optics we have only a transverse wave to consider, while here we have both transverse and longitudinal waves. We should, therefore, compare our theory to the theory of propagation of waves in an elastic medium, but since this subject is apt to be less familiar to the general reader, we shall be content with the optical analogue.

vorticity changes into the flow, but do not transport pressure changes.

In the optical case, the light waves do not distort the reflecting surface, but it can readily be seen that in the present problem the shock itself changes shape, while at the same time an entropy wave is thrown off.

Another novel feature is illustrated in Fig. 2. It is seen that when the sound wave is incident from the supersonic side, only refracted waves appear, while waves incident from the subsonic side give rise to reflected waves only. In other words, the disturbance in the flow created at the shock front is apparent only on the subsonic side. The reason for this behavior in the case of the entropy-vorticity waves is obvious. These waves move with the fluid which in turn moves into the subsonic flow, away from the shock. Thus entropy-vorticity waves created at the shock will appear only on the subsonic side. The sound wave, on the other hand, moves relative to the flow with the sound speed. On the left, the flow speed is greater than the sound speed; thus, sound waves created at the shock wave will appear only on the subsonic side.

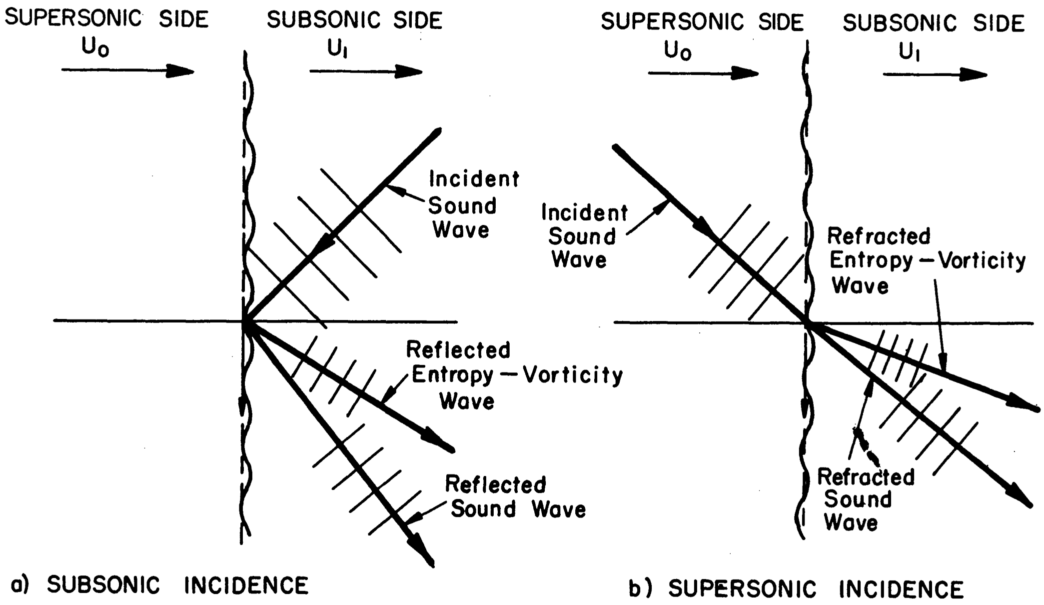

Fig. 2. Reflection and refraction of plane waves.

Let us consider the boundary conditions next. Small disturbances on one side of the shock wave are related to small disturbances on the other side by means of the linearized shock conditions. These conditions determine the disturbance in pressure, velocity, and entropy on the subsonic side as linear functions of the disturbances in pressure, velocity, and entropy on the supersonic side and the distortion of the shock wave. Since the disturbances are plane waves with amplitudes independent of the coordinates, it is necessary that the exponential factors match at the shock surface. This fact gives us the analogue of Snell's laws of reflection and refraction, as in optics.

The shock conditions may be written as four linear equations relating the amplitudes of the reflected or refracted waves to the amplitude of the incident wave. The four amplitudes appearing in the linear equations are those of the sound wave, entropy wave, vorticity wave, and the distortion of the shock. This latter amplitude has no optical analogue. These equations may be solved to give the amplitudes of the four waves, which are the required solutions to the plane-wave problem.

We now turn to the interaction of cylindrical sound waves with a shock wave. A three-dimensional electromagnetic analogue to this problem is the propagation of radio waves from a transmitter above a plane earth.

Let us first consider the properties of cylindrical sound waves in a moving medium with no shock wave present. If the medium were not in motion, one would be able to find the properties of such waves by solving the wave equation for an oscillating point source. Thus one is inclined to view this problem from a frame of reference moving with the fluid, with the result that the equation governing the velocity potential of the disturbance becomes the wave equation, but that the source is now in motion.

To transform the source back to rest and yet to preserve the form of the wave equation, an appropriate Lorentz transformation* on the moving-

*The Lorentz transformation referred to here is based on the speed of sound rather than the speed of light.

source equation is used. The resulting equation is still the wave equation but now with a stationary source. Thus from this peculiar Lorentz-Galileo frame of reference the waves should look like sound waves from a stationary source in a fluid at rest. One merely transforms the known solution back to the original frame of reference to find out what the waves actually look like.

This procedure gives a reasonable result when the fluid speed is subsonic. Further, upon using the same method with a "supersonic" Lorentz transformation, a perfectly acceptable solution is obtained. However, since it is essentially an elementary function of the transformed distance from the source [$\sqrt{x^2 - (M^2 - 1)y^2}$], it will be real not only in the downstream Mach "wedge" but also in the unphysical upstream Mach wedge. It is therefore clear that the desired solution cannot be written in closed form, but will have to be an integral or series which, when evaluated in various regions, will be represented by different analytical expressions.

The method adopted to overcome this difficulty is a technique commonly employed in wave mechanics.[2] The solution is expressed as a Fourier integral and the integrand is found. Various solutions to the problem may be obtained by choosing different paths of integration. Two of the solutions are of the form discarded above, but another is a physically reasonable result which is chosen to be the solution of our problem.

The solutions which are finally obtained are illustrated in Fig. 3. When the flow is subsonic,

$$\Phi = H_1^{(0)}\left(\frac{k}{1-M^2}\sqrt{x^2+(1-M^2)y^2}\right) e^{i\frac{kMx}{1-M^2} - i\omega t} .$$

The lines of constant phase are circles which are blown downstream with the flow (Fig. 3a). The supersonic case—

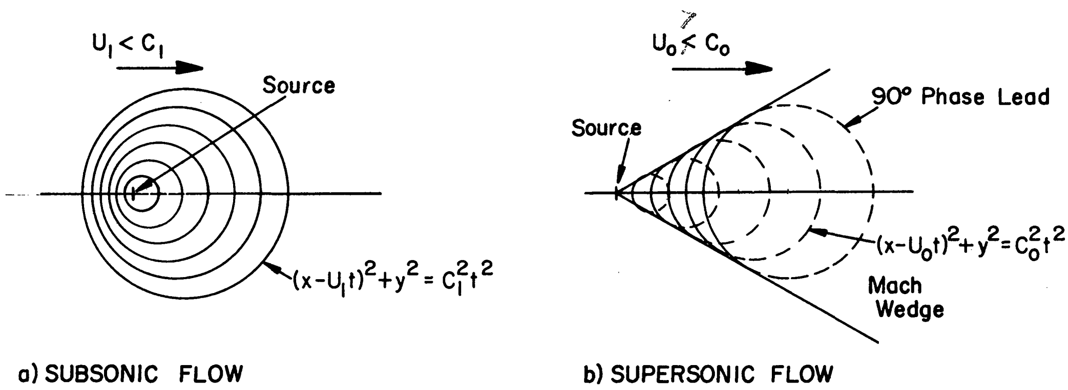

Fig. 3. Cylindrical waves in a moving gas.

$$\Phi = \begin{cases} J_0\left(\frac{k}{M^2-1}\sqrt{x^2-(M^2-1)y^2}\right)e^{-i\frac{kMx}{M^2-1}-i\omega t}, & x > \sqrt{M^2-1}\,|y| \\ 0 & \text{otherwise} \end{cases}$$

—is more interesting since the circles are confined to the interior of a wedge, the Mach wedge.

In Fig. 3b the part of the circle to the right of the intersection with the Mach "wedge" leads the remainder of the circle by 90°. This phase shift is analogous to the 90° phase shift, discovered by Debye,[3,4] of a wave passing through a focal line.

It should be noted that the fact that these circles cross one another brings about destructive interference along certain curves. The existence of the stationary zeros in the flow gives these waves the appearance of standing waves, although they are, in reality, progressing downstream, and preserving their nodal line.

The incidence of cylindrical waves from both the subsonic and supersonic side is treated in full, and rigorous closed form solutions for all flow variables of both sonic and entropy-vorticity waves are obtained. The results are qualitatively pictured in Fig. 4. In the case of subsonic incidence the reflected waves are obtained using the saddle-point method. The sonic waves are cylindrical waves whose centers move downstream with

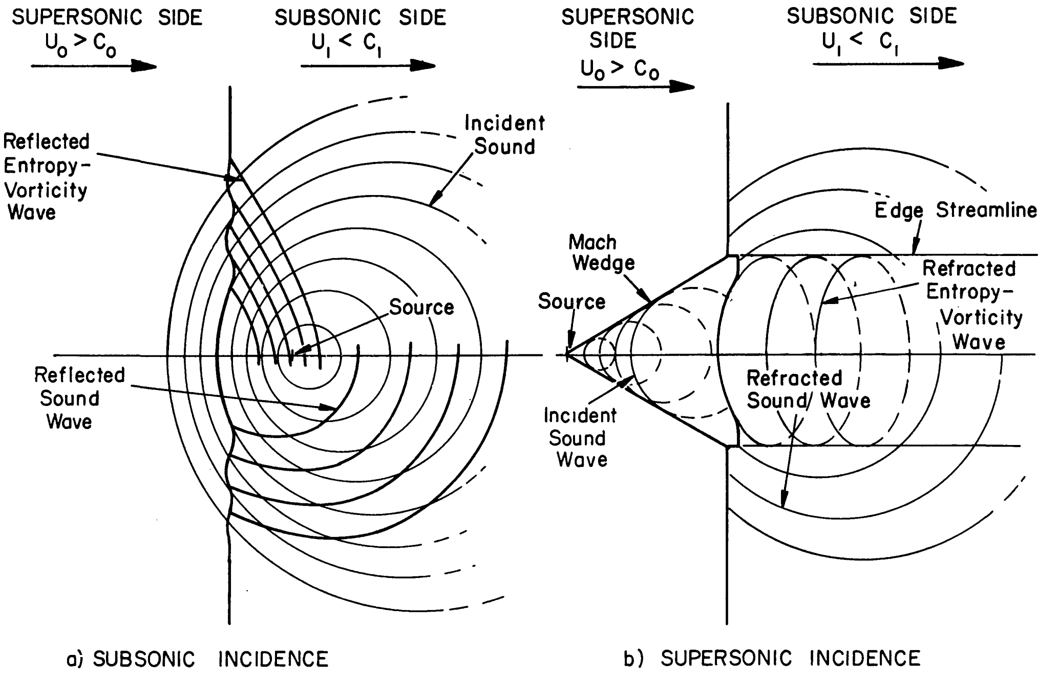

Fig. 4. Interaction of cylindrical waves.

the flow, while the entropy-vorticity waves are segments of hyperbolas. As is to be expected, the case of incidence from the supersonic side is more complex. The transmitted sonic waves are no longer circles, although the deviation from circularity is of the order x_0/ρ, x_0 being the distance of the source from the shock. The entropy-vorticity waves are confined to the strip $-\frac{x_0}{\sqrt{M_0^2 - 1}} < y < \frac{x_0}{\sqrt{M_0^2 - 1}}$, and are swept downstream within it, without distortion. They consist of ellipses.

The detailed presentation of the theory just outlined is to be found in Sections II to VI. Section II is devoted to a study of the linear differential equations and linear shock conditions governing the propagation of small disturbances. In Section III, plane-wave solutions to these differential equations are studied. Parts B and C of Section III are devoted to the interaction problem for the subsonic and supersonic cases, respectively. Each of these parts is again divided into two subsections, the first dealing with the analogue of Snell's law and the second with the analogue of the Fresnel coefficients.

In Section IV the problem of cylindrical waves in a medium without shocks is studied and the decomposition into plane waves is made. Section V is devoted to a detailed study of the integrals describing the interaction of the waves of Section IV with a plane normal shock wave. In Section VI we discuss the extension of this theory to the spherical waves generated by a point source, as well as the necessary modifications for the case of a moving shock wave interacting with a stationary source. This case could lend itself to observation in the shock tube.

Because of their relative complexity, certain computations pertaining to the shock conditions and Fresnel coefficients have been referred to Appendices A to E.

B. PREVIOUS INVESTIGATIONS

Several authors have interested themselves in problems involving the interaction of sound waves and shock waves.[5] The first of these was V. Bargmann,[6] who gave the solution to the problem of the diffraction of a shock wave by a thin wedge. In this problem a moving shock wave strikes a thin wedge normally. The tip of the wedge acts as a source of cylindrical sound waves. These sound waves interact with the shock wave and cause it to be slightly distorted. Bargmann's solution to the problem involved the assumption that the shock wave was so weak that the flow behind the shock was irrotational and isentropic.

Lighthill[7] examined this diffraction problem for strong shocks and thin wedges. He was able to eliminate consideration of the entropy and vorticity and find the pressure field and resulting distortion of the shock wave. Ludloff[8] was able to extend the results of Lighthill to the case of thin bodies of various shapes.

The diffraction problem, although it involves the interaction of the sound field and the shock wave in a manner different from that of our concern here, is of fundamental importance since it was the first in which the linear shock conditions were used as boundary conditions.

A problem which is closer to the one solved here was treated by Carrier.[9] This problem is that of interaction of a plane sound wave with a stationary shock wave inclined to the flow. Carrier introduces the entropy-vorticity wave as well as the sound wave. He assumes that a plane sound wave in the flow behind the inclined shock interacts with the shock, and he calculates the resulting distortion of the shock wave. No treatment of sound waves incident from the front of the shock is given.

Ribner[10] considers the problem of the convection of a plane vorticity wave of a given profile through a moving shock wave, and calculates the resulting sound, vorticity, and entropy waves.

F. K. Moore[11] calculates the interaction of sound waves of a given shape incident from either side of a moving shock. This problem is quite closely related to our plane-wave problem.

The problem of the cylindrical waves in a supersonic flow with no shock wave present may be compared with I. G. Tamm's[12] theory of Cerenkov radiation. In fact, we may formally compare our source with a body, whose charge oscillates in time, moving faster than the speed of light in a nondispersive medium.

The interaction of electromagnetic dipole radiation incident upon a plane earth has received a great deal of attention.[13] It is from this source that we are able to obtain the greatest amount of guidance for the present work.

Two related methods have been proposed for solving the radio problem. The first of these is due to Sommerfeld.[14] He writes the reflected and refracted waves as integrals which have integrands of a form suggested by the incident wave, except for an undetermined function, which is then determined by the boundary conditions. The solution so obtained is in integral form, and may be expanded asymptotically to give the far field.

Weyl[15] writes the incident wave as a superposition of plane waves. He then writes the reflected and refracted waves as superpositions of plane waves of the same form. He applies the boundary conditions to the integrals and finds the amplitudes of the reflected and refracted waves, which are (as one would expect) given by the Fresnel coefficients. Thus he also has the solution in the form of an integral which may be expanded asymptotically to give the far field.*

Our method of treatment of the cylindrical wave problem should be considered as the analogue of Weyl's method for the radio wave problem.

*It may be demonstrated that the integrals obtained by Sommerfeld are equivalent to those obtained by Weyl; however, the results arising from the discussion of the different formulations are in striking disagreement. This gave rise to a controversy which continued from 1919 until 1943, when it was resolved by Ott who used a method originally devised by Pauli for a different purpose. The difficulty occurs because poles and saddle-points may be near one another in the plane of integration. In Sommerfeld's treatment, this difficulty is not as obvious as in Weyl's method, and in fact was completely overlooked, giving rise to the disagreement. Van der Waerden has subjected the problem of evaluating integrals of this type to a rigorous mathematical treatment which has done much to clear up the problem. (See Refs. 16 and 17.)

We let the flow on either side of this plane normal shock be slightly disturbed. The dimensionless perturbations p, u, v, d, and s are introduced by means of:

$$
\begin{aligned}
&\text{Pressure:} && \mathcal{P} = P + p\,DCU \quad ;\\
&\text{x-velocity:} && \mathcal{U} = U + uU \quad ;\\
&\text{y-velocity:} && \mathcal{V} = \quad vU \quad ;\\
&\text{Density:} && \mathcal{D} = D + d\,D \quad ;\\
&\text{Entropy:} && \mathcal{S} = S + s\,c_p \quad .
\end{aligned}
\tag{2.2}
$$

Upon substituting Eqs. (2.2) and neglecting the squares of the small perturbing terms, Eqs. (2.1) become:

$$
\begin{aligned}
&\text{Continuity:} && \frac{\partial d}{\partial t} + U\frac{\partial d}{\partial x} + U\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad ;\\
&\text{Momentum x-component:} && \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + C\frac{\partial p}{\partial x} = 0 \quad ;\\
&\text{y-component:} && \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + C\frac{\partial p}{\partial y} = 0 \quad ;\\
&\text{Entropy:} && \frac{\partial s}{\partial t} + U\frac{\partial s}{\partial x} = 0 \quad ;\\
&\text{Equation of State:} && Up = C(d + s) \quad .
\end{aligned}
\tag{2.3}
$$

The entropy equation and equation of state in (2.3) may be used to eliminate d from the continuity equation. If this is done, we may rewrite the system as four linear differential equations involving p, u, v, and s, and one linear equation defining d in terms of p and s.

The resulting equations, which shall serve as the fundamental equations of this study, are:

$$
\begin{aligned}
&\frac{1}{C}\left(\frac{\partial p}{\partial t} + U\frac{\partial p}{\partial x}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad ; \\
&\frac{1}{C}\left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} = 0 \quad ; \\
&\frac{1}{C}\left(\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x}\right) + \frac{\partial p}{\partial y} = 0 \quad ; \\
&\frac{1}{C}\left(\frac{\partial s}{\partial t} + U\frac{\partial s}{\partial x}\right) = 0 \quad ; \\
&d = \frac{U}{C}p - s \quad .
\end{aligned}
\tag{2.4}
$$

In the following, only the first four of Eqs. (2.4) shall be considered. The fifth equation of (2.4) may be used to find the density after the first four have been solved. Subscripts 0 or 1 should be used on all the constants and dependent variables in Eqs. (2.4), 0 referring to the supersonic side of the shock, and 1 referring to the subsonic side.

In Appendix B the characteristic curves of Eqs. (2.4) are studied. The results found there lead us to the conclusion that the characteristic curves either move relative to the flow with the sound speed, or move with the flow.

In the former case there is a jump in pressure and a jump in velocity which is normal to the curve. There is no jump in entropy. These isentropic, longitudinal waves are called sound waves.

In the latter case there is no jump in pressure across the characteristics and the velocity jump is tangential to the curve. There is a jump in entropy which has an amplitude independent of the amplitude of the jump in velocity. These transverse, isobaric waves are called the entropy-vorticity waves.

We see here why the density was eliminated from Eqs. (2.3). If the density had been used, we would have had jumps in density across both families of characteristics, whereas pressure and entropy have jumps across one or the other of the characteristic families but not both.

Let us now derive the energy transport equation associated with Eqs. (2.4) found above.

The equation governing the transport of energy is found by multiplying the first three of (2.4) by p, u, and v, respectively, and adding the resulting equations. This gives the following expression:

$$\frac{1}{C}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(\frac{p^2+u^2+v^2}{2}\right)+p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}=0, \quad (2.5)$$

which may be rewritten in the form of a conservation equation as:

$$\frac{\partial}{\partial t}\left(\frac{p^2+u^2+v^2}{2}\right)+\frac{\partial}{\partial x}\left(Cpu+U\frac{(p^2+u^2+v^2)}{2}\right)+\frac{\partial}{\partial y}\left(Cpv\right)=0. \quad (2.6)$$

The term $\mathcal{E} = \frac{p^2+u^2+v^2}{2}$ in (2.6) is the "dimensionless" energy density of the wave, while the vector

$$\vec{S} = \left(Cpu + U\frac{(p^2+u^2+v^2)}{2},\ Cpv\right) ,$$

which has the dimensions of velocity, represents the energy flux, or "Poynting," vector. Equation (2.6) then expresses the conservation of energy as:*

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0 . \quad (2.7)$$

The vector $\vec{S}$ consists of two parts, $\vec{S}_1 = Cp(u,v)$ and $\vec{S}_2 = U(\mathcal{E},0)$ $\vec{S}_1$ is the "Poynting" vector in a frame of reference moving with the flow, while $\vec{S}_2$ is the contribution to the flux arising from the motion of the gas.

*This derivation of the energy-conservation law (as well as the related momentum-conservation law) is essentially due to O. Laporte.

B. THE SHOCK CONDITIONS

We have disturbed the flow on both sides of the shock wave, and since we have in no way constrained the shock wave, it will be disturbed also. Equations (2.4) describe the behavior of the disturbance on either side of the shock, and we shall now find the relations connecting these disturbances across the shock front.

The shock wave is assumed to be only slightly disturbed from a state of rest at $x = 0$. Let this disturbance be described by $x = f(y,t)$ The normal and tangent vectors to the shock wave are then given to first order by:

$$\begin{aligned} \vec{n} &= (1, -f_y) \\ \vec{t} &= (f_y, 1) \, . \end{aligned} \tag{2.8}$$

The shock velocity is given by:

$$\vec{U}_s = \left(f_y \frac{dy}{dt} + f_t \, , \, \frac{dy}{dt} \right) \, . \tag{2.9}$$

The relative velocities of the flow to the shock are given by:

$$\begin{aligned} \vec{U}_{R0} &= \vec{U}_0 - \vec{U}_s = \left(U_0 + u_0 U_0 - f_y \frac{dy}{dt} - f_t \, , \, v_0 U_0 - \frac{dy}{dt} \right) \\ \vec{U}_{R1} &= \vec{U}_1 - \vec{U}_s = \left(U_1 + u_1 U_1 - f_y \frac{dy}{dt} - f_t \, , \, v_1 U_1 - \frac{dy}{dt} \right) \, . \end{aligned} \tag{2.10}$$

The relative normal and tangential velocities of the flow are obtained by forming the scalar product of (2.8) and (2.10):

$$\begin{aligned} U_{0Rn} &= U_0 + u_0 U_0 - f_y \frac{dy}{dt} - f_t + f_y \frac{dy}{dt} = U_0 + (u_0 U_0 - f_t) \; ; \\ U_{1Rn} &= U_1 + u_1 U_1 - f_y \frac{dy}{dt} - f_t + f_y \frac{dy}{dt} = U_1 + (u_1 U_1 - f_t) \; ; \\ U_{0Rt} &= U_0 f_y + v_0 U_0 - \frac{dy}{dt} = -\frac{dy}{dt} + U_0 (v_0 + f_y) \; ; \\ U_{1Rt} &= U_1 f_y + v_1 U_1 - \frac{dy}{dt} = -\frac{dy}{dt} + U_1 (v_1 + f_y) \, . \end{aligned} \tag{2.11}$$

The situation with which we wish to deal is shown in Fig. 5.

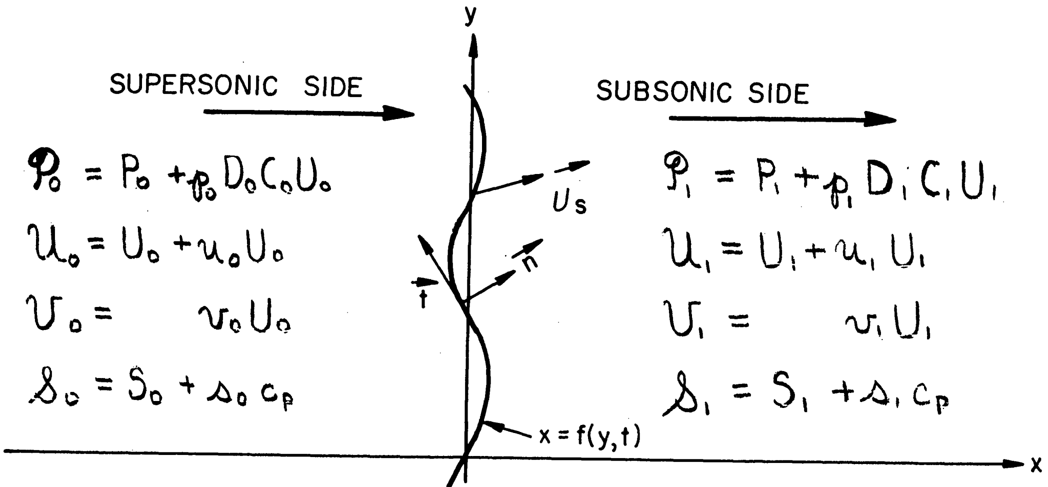

Fig. 5. The disturbed state of the gas.

The conditions governing a general curved, moving shock wave are:[20]

Continuity: $$\mathcal{D}_1 \mathcal{U}_{1Rn} = \mathcal{D}_0 \mathcal{U}_{0Rn}$$

Momentum
Normal component: $$\mathcal{P}_1 + \mathcal{D}_1 \mathcal{U}_{1Rn}^2 = \mathcal{P}_0 + \mathcal{D}_0 \mathcal{U}_{0Rn}^2$$

(2.12)

Tangential component: $$\mathcal{U}_{1Rt} = \mathcal{U}_{0Rt}$$

Energy: $$\frac{\gamma}{\gamma-1}\frac{\mathcal{P}_1}{\mathcal{D}_1} + \frac{1}{2}\mathcal{U}_{1Rn}^2 = \frac{\gamma}{\gamma-1}\frac{\mathcal{P}_0}{\mathcal{D}_0} + \frac{1}{2}\mathcal{U}_{0Rn}^2$$

In Appendix A we show that these may be written simply as:

$$\mathcal{P}_1 = \mathcal{P}_0\, G(\mathcal{M}_0) \quad ;$$

$$\mathcal{U}_{1Rn} = \mathcal{U}_{0Rn}\, F(\mathcal{M}_0) \quad ;$$

$$\mathcal{U}_{1Rt} = \mathcal{U}_{0Rt} \quad ; \qquad (2.13)$$

$$\frac{\mathcal{S}_1}{C_v} = \frac{\mathcal{S}_0}{C_v} + \ln G(\mathcal{M}_0) + \frac{\mu+1}{\mu-1}\ln F(\mathcal{M}_0),$$

where :

$$\mathcal{M}_0 = \frac{U_{0BN}}{C_0} \quad , \qquad F(\mathcal{M}_0) = \frac{\mathcal{M}_0^2 + (\mu - 1)}{\mu \mathcal{M}_0^2} \quad ,$$

$$G(\mathcal{M}_0) = \frac{(\mu+1)\mathcal{M}_0^2 - 1}{\mu} \quad , \qquad \mu = \frac{\gamma+1}{\gamma-1} \quad .$$

One may find the relations between the first order disturbances by differentiating the relations (2.13). The first order relations are:

$$\begin{aligned}
p_1 M_1 &= p_0 M_0 + \frac{\mu-1}{\mu+1}\frac{G'(M_0)}{G(M_0)} m_0 \quad ; \\
u_1 &= u_0 + \frac{F'(M_0)}{F(M_0)} m_0 + \frac{1}{U_0}\left[\frac{1-F(M_0)}{F(M_0)}\right] f_t \quad ; \\
v_1 &= \frac{v_0}{F(M_0)} + \left[\frac{1-F(M_0)}{F(M_0)}\right] f_y \quad ; \\
s_1 &= s_0 + \frac{\mu-1}{\mu+1}\left[\frac{G'(M_0)}{G(M_0)} + \frac{\mu+1}{\mu-1}\frac{F'(M_0)}{F(M_0)}\right] m_0 ,
\end{aligned} \tag{2.14}$$

where m_0 is defined by:

$$m_0 = \mathcal{M}_0 - M_0$$

These relations may be rewritten to give p_1, u_1 , v_1 , and s_1 as linear functions of p_0, u_0, v_0, s_0, f_t , and f_y . This rather lengthy manipulation is carried out in Appendix C. We summarize the linear shock conditions as follows:

$$\begin{aligned}
p_1 &= A_{11} p_0 + A_{12} u_0 + A_{14} s_0 + A_{15} f_t \\
u_1 &= A_{21} p_0 + A_{22} u_0 + A_{24} s_0 + A_{25} f_t \\
v_1 &= A_{33} v_0 + A_{36} f_y \\
s_1 &= A_{41} p_0 + A_{42} u_0 + A_{44} s_0 + A_{45} f_t
\end{aligned} \tag{2.15}$$

The coefficients A_{ij} are displayed explicitly in Appendix C.

Although this system of linear relations decomposes into four relations connecting the quantities p, u, Δ, f_t and one relation between v and f_y the actual boundary conditions between plane-wave amplitudes do not decompose in this manner.

In this section we have found the differential equations (2.4) which small disturbances satisfy, and the conditions (2.15) which connect the variables describing these small disturbances across a distorted shock wave In the next section we shall investigate the solutions to these equations in the case of plane-wave disturbances.

III. PLANE WAVES AND THEIR INTERACTION WITH A STATIONARY SHOCK WAVE

A. PLANE WAVES IN A MOVING FLUID

In this section we wish to consider solutions of the differential equations (2.4) which are harmonic in time and constant on lines perpendicular to a certain fixed unit vector $\vec{n} = (\alpha, \beta)$, α and β being the cosine and sine of the angle $\vec{n}$ makes with the positive x axis. From the discussion of characteristics in Section II, we expect the planes of constant phase to be of two types, those which move downstream with the fluid and those which move relative to the fluid with the sound speed. That this is so will be verified below.

Let us consider plane-wave solutions of (2.4) in the form:

$$\begin{aligned} \mathfrak{p} &= P\, e^{i\lambda(\alpha x+\beta y)-i\omega t}\,; \\ u &= U\, e^{i\lambda(\alpha x+\beta y)-i\omega t}\,; \\ v &= V\, e^{i\lambda(\alpha x+\beta y)-i\omega t}\,; \\ s &= S\, e^{i\lambda(\alpha x+\beta y)-i\omega t}\,. \end{aligned} \tag{3.1}$$

Equations (2.4) become:

$$\begin{aligned} (-k+\lambda M\alpha)P + \lambda\alpha U + \lambda\beta V &= 0\,; \\ \lambda\alpha P + (-k+\lambda M\alpha)U &= 0\,; \\ \lambda\beta P + (-k+\lambda M\alpha)V &= 0\,; \\ (-k+\lambda M\alpha)S &= 0\,, \end{aligned} \tag{3.2}$$

with $k = \frac{\omega}{c}$. These equations have a solution if the coefficient determinant vanishes. This gives the following equation for λ:

$$(-k+\lambda M\alpha)^2\left[(-k+\lambda M\alpha)^2-\lambda^2\right] = 0\,. \tag{3.3}$$

The roots of this equation are given by:

$$\lambda_1 = \frac{k}{M\alpha} \quad ; \tag{3.4}$$

$$\lambda_2 = \frac{k}{1+M\alpha} \quad ; \tag{3.5}$$

$$\lambda_3 = \frac{-k}{1-M\alpha} \quad . \tag{3.6}$$

The solutions to Eqs. (3.2) which correspond to the first eigenvalue $\lambda = \lambda_1$ are:

$$p = 0 \quad ;$$

$$u = -\beta B\, e^{ik\frac{(\alpha x+\beta y)}{M\alpha} - i\omega t} \quad ;$$

$$v = \alpha B\, e^{ik\frac{(\alpha x+\beta y)}{M\alpha} - i\omega t} \quad ; \tag{3.7}$$

$$\rho = C\, e^{ik\frac{(\alpha x+\beta y)}{M\alpha} - i\omega t} \quad ,$$

with **B** and **C** arbitrary constants. This wave corresponds to the entropy-vorticity wave of Section II. There are two important properties of this wave which we wish to mention here. These are:

1) The wave moves with the fluid.

Proof: The planes of constant phase are described by $\frac{k(\alpha x+\beta y)}{M\alpha} - \omega t =$ constant. The x-component of the phase velocity is given by $\frac{dx}{dt} = U =$ flow velocity (Fig. 6a).

2) The velocity may be derived from a vector potential.

Proof:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{from (3.7);}$$

hence:

$$u = \frac{\partial \chi}{\partial y} \quad , \quad v = -\frac{\partial \chi}{\partial x} \quad , \tag{3.8}$$

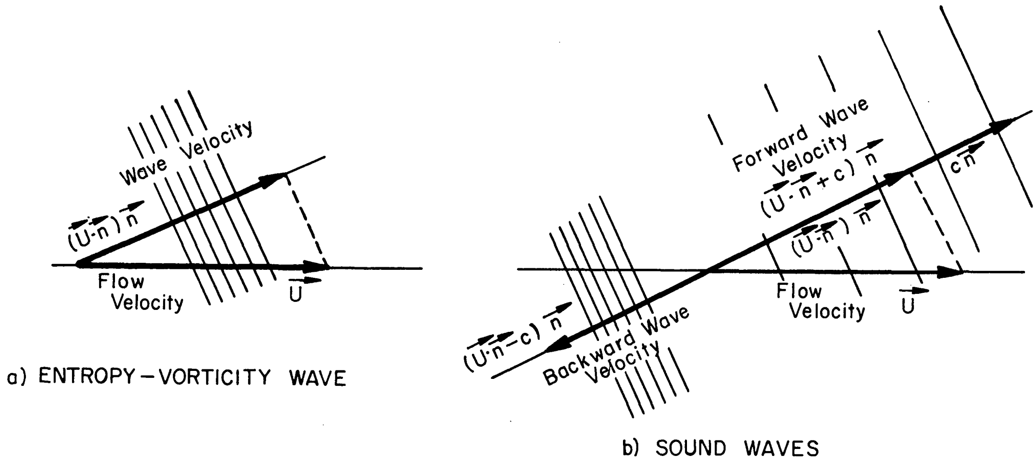

Fig. 6. Plane waves in a moving gas.

where

$$\chi = i\,\frac{M\alpha}{k}\,C\,e^{ik\frac{(\alpha x+\beta y)}{M\alpha}-i\omega t} \quad . \tag{3.9}$$

Now we turn to the second eigenvalue, $\lambda = \lambda_2$:

$$p = A\,e^{ik\frac{(\alpha x+\beta y)}{1+M\alpha}-i\omega t} \quad ;$$

$$u = \alpha A\,e^{ik\frac{(\alpha x+\beta y)}{1+M\alpha}-i\omega t} \quad ;$$

$$v = \beta A\,e^{ik\frac{(\alpha x+\beta y)}{1+M\alpha}-i\omega t} \quad ; \tag{3.10}$$

$$s = 0 \quad)$$

with A an arbitrary constant. This wave corresponds to the sound wave of Section II. There are two important properties of this wave analogous to, but differing from, those for the entropy-vorticity wave.

These are:

1) The wave moves with the sound speed relative to the fluid.

Proof: consider the surface of constant phase

$$k\frac{(\alpha x+\beta y)}{1+M\alpha} - \omega t = \text{constant.}$$

Hence, $\frac{d}{dt}(\alpha x+\beta y) = (1+M\alpha)\,c = (c+U\alpha)$. (Fig. 6b).

2) The velocity may be described by means of a scalar potential.

Proof:

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \qquad \text{from (3.10) ;}$$

hence:

$$u = \frac{\partial \Phi}{\partial x}\ , \quad v = \frac{\partial \Phi}{\partial y}\ , \tag{3.11}$$

where

$$\Phi = \frac{i(1+M\alpha)}{k} A e^{ik\frac{(\alpha x+\beta y)}{1+M\alpha} - i\omega t}, \tag{3.12}$$

Notice that p may be found also in terms of Φ as:

$$p = -\frac{1}{c}\left(\frac{\partial \Phi}{\partial t} + U\frac{\partial \Phi}{\partial x}\right) . \tag{3.13}$$

Finally for the third eigenvalue, $\lambda = \lambda_3$ we find from Eqs. (3.2):

$$\begin{aligned} p &= A e^{ik\frac{(-\alpha x-\beta y)}{1-M\alpha} - i\omega t} ; \\ u &= -\alpha A e^{ik\frac{(-\alpha x-\beta y)}{1-M\alpha} - i\omega t} ; \\ v &= -\beta A e^{ik\frac{(-\alpha x-\beta y)}{1-M\alpha} - i\omega t} ; \\ s &= 0 \ . \end{aligned} \tag{3.14}$$

This wave is identical with wave (3.10) if (α,β) is replaced by $(-\alpha, -\beta)$, i.e., if the normal vector is reversed. This is a sound wave also, but moving in the direction of $-\vec{n}$ rather than $\vec{n}$.

Each of the waves considered above transports energy into the gas. It is of interest to consider the direction of the energy flux associated with sound waves and with entropy-vorticity waves.

Energy is transported through the gas in the direction of $\vec{S}$ given in (2.6). The time average of $\vec{S}$ for the plane sound waves (3.10) is:

$$\tilde{\vec{S}} = \frac{A^2}{2}(c\vec{n} + U\vec{n}_1) = \tilde{\mathcal{E}}\,(c\vec{n} + U\vec{n}_1)\,, \tag{3.15}$$

where $\vec{n}_1 = (1, 0)$, and $\sim$ indicates the time average. Formula (3.15) is the generalization of the ordinary "Poynting" theorem for plane waves, $\tilde{\vec{S}} = c\tilde{\mathcal{E}}\vec{n}$. The direction of propagation of energy for the sound waves is found by the velocity addition law, as illustrated in Fig. 7a.

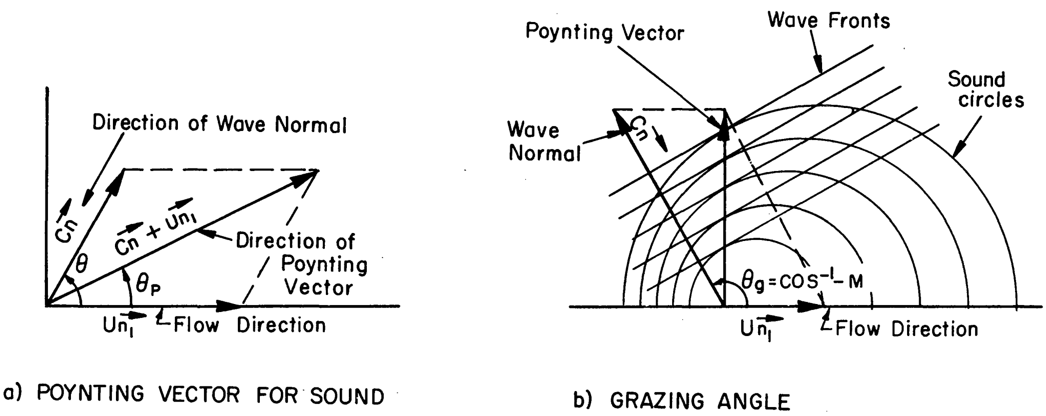

Fig. 7. Energy transport for sound waves.

If $U > c$ the vector $\tilde{\vec{S}}$ will always lie interior to the forward Mach "wedge" (and will vanish on this surface); thus the energy is always blown downstream. On the other hand, if $U < c$, the vector $\tilde{\vec{S}}$ will have a component in the direction of $-\vec{n}_1$, for angles in the range $\pi \leq \theta < \cos^{-1} - M_1$. For incidence upon a shock the angle $\theta_g = \cos^{-1} - M_1$ plays the same role as the 90° angle of grazing incidence in the "Fresnel" problem of electromagnetic theory (see Fig. 7b). In particular we must consider waves with angles in the range $0 \leq \theta < \theta_g$ as downstream waves, and waves with angles in the range $\theta_g < \theta \leq \pi$ as upstream waves.

The entropy-vorticity waves also have an associated "Poynting" vector, but since $p = 0$ for these waves, the term $\vec{S}_1$, which gives the energy flux relative to the flow, vanishes. Therefore the "Poynting" vector for the entropy-vorticity waves consists of the convective term only. From (3.7) we find the time average of $\vec{S}$ to be:

$$\tilde{\vec{S}} = \frac{B^2}{4} U \vec{n}_1 = U \tilde{\mathcal{E}} \vec{n}_1 \quad . \tag{3.16}$$

The energy of the entropy-vorticity wave is blown downstream with the flow.

B. INCIDENCE FROM THE SUBSONIC SIDE

Now that we have studied the behavior of plane waves in a moving fluid we are in a position to describe the interaction problem. First we shall describe the interaction of plane waves incident from the subsonic side of the shock wave (Fig. 2a, p.3). The incident plane wave is a sound wave moving in the subsonic flow toward the shock and is described by Eqs. (3.10). Let us choose the following notation:

$$\begin{aligned}
p_i &= \epsilon\, e^{i k_1 \frac{(\alpha_0 x + \beta_0 y)}{1 + M_1 \alpha_0} - i\omega t} ; \\
u_i &= \alpha_0 \epsilon\, e^{i k_1 \frac{(\alpha_0 x + \beta_0 y)}{1 + M_1 \alpha_0} - i\omega t} ; \\
v_i &= \beta_0 \epsilon\, e^{i k_1 \frac{(\alpha_0 x + \beta_0 y)}{1 + M_1 \alpha_0} - i\omega t} ; \\
s_i &= 0 \quad ,
\end{aligned} \tag{3.17}$$

where ϵ is the amplitude of the wave and is supposed much less than 1 to make the use of the linear equations reasonable.

We shall see that there is, in conjunction with this incident sound wave, a reflected sound wave as well as a reflected entropy-vorticity wave. These are described by:

Reflected sound wave:

$$
\begin{aligned}
p_s &= A\, e^{i k_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t}\ ; \\
u_s &= \alpha_1 A\, e^{i k_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t}\ ; \\
v_s &= \beta_1 A\, e^{i k_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t}\ ; \\
s_s &= 0\ .
\end{aligned}
\tag{3.18}
$$

Reflected entropy-vorticity wave:

$$
\begin{aligned}
p^* &= 0\ ; \\
u^* &= -\beta_2 B\, e^{i k_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t}\ ; \\
v^* &= \alpha_2 B\, e^{i k_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t}\ ; \\
s^* &= C\, e^{i k_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t}\ .
\end{aligned}
\tag{3.19}
$$

At x = 0, all the quantities p, u, v, and s are related linearly by the shock conditions (2.15). Thus we must require that the exponentials match at x = 0. This matching may be accomplished by requiring that the coefficients of y in the exponents all be the same. From this we conclude:

$$\frac{\beta_1}{1+M_1\alpha_1} = \frac{\beta_0}{1+M_1\alpha_0} , \tag{3.20}$$

and

$$\frac{\beta_2}{M_1\alpha_2} = \frac{\beta_0}{1+M_1\alpha_0} . \tag{3.21}$$

Equations (3.20) and (3.21) represent the analogues to Snell's law of reflection in optics.

1. Reflection Laws

In this section we wish to examine in detail the angle relations for the case of subsonic incidence.

a. Reflected Sound Waves.—The relation governing the angle of reflection for the sound wave was just found to be:

$$\frac{\beta_1}{1+M_1\alpha_1} = \frac{\beta_0}{1+M_1\alpha_0} . \tag{3.20}$$

Squaring, we find:

$$(1-\alpha_1^2)(1+M_1\alpha_0)^2 = \beta_0^2(1+2M_1\alpha_1+M_1^2\alpha_1^2) ,$$

or

$$(\beta_0^2M_1^2+(1+M_1\alpha_0)^2)\alpha_1^2+2M_1\beta_0^2\alpha_1+(\beta_0^2-(1+M_1\alpha_0)^2)=0.$$

This equation may be solved for α_1 ; we find:

$$\alpha_1 = \alpha_0 , \qquad \alpha_1 = -\frac{(1+M_1^2)\alpha_0+2M_1}{(1+M_1^2)+2M_1\alpha_0} .$$

The first of these corresponds to $\beta_1 = \beta_0$, whereas the second corresponds to:

$$\beta_1 = \frac{\beta_0}{1+M_1\alpha_0}(1+M_1\alpha_1) = \frac{(1-M_1^2)\beta_0}{(1+M_1^2)+2M_1\alpha_0} .$$

The solution $\alpha_1 = \alpha_0$, $\beta_1 = \beta_0$ is trivial and will not be considered further. The remaining nontrivial solution, which corresponds to the reflected sound wave, is:

$$\alpha_1 = -\frac{(1+M_1^2)\alpha_0 + 2M_1}{(1+M_1^2) + 2M_1\alpha_0} \quad ;$$

$$\beta_1 = \frac{(1-M_1^2)\beta_0}{(1+M_1^2) + 2M_1\alpha_0} \quad . \tag{3.22}$$

It should be noted that the angle of incidence θ_0 equals π for normal incidence, since the waves move toward the shock from the subsonic side, $x > 0$. As we have seen in Part A of this Section, only waves in the angle range $\pi \geqslant \theta_0 > \theta_g$, where θ_g is the grazing angle, transport energy upstream. We therefore limit our incident angles to this range. It can be seen easily from Eqs. (3.22) that as the incident wave normal goes from π to θ_g, the reflected wave normal goes from 0 to θ_g. The fact that $\theta_g \neq \frac{\pi}{2}$ is evidently due to aberration.

It is evident from Fig. 7a that θ_p, the angle of the "Poynting" vector, is related to θ, the angle of the associated wave normal, by the formula:

$$\tan\theta_p = \frac{\sin\theta}{M + \cos\theta} \quad . \tag{3.23}$$

Thus for the reflected sound waves treated above and illustrated in Fig. 8 we find:

$$\tan\theta_{p_1} = \frac{\beta_1}{M_1 + \alpha_1} = -\frac{\beta_0}{M_1 + \alpha_0} = -\tan\theta_{p_0} \quad . \tag{3.24}$$

This equation has as a solution:

$$\theta_{p_1} = -\theta_{p_0} \quad . \tag{3.25}$$

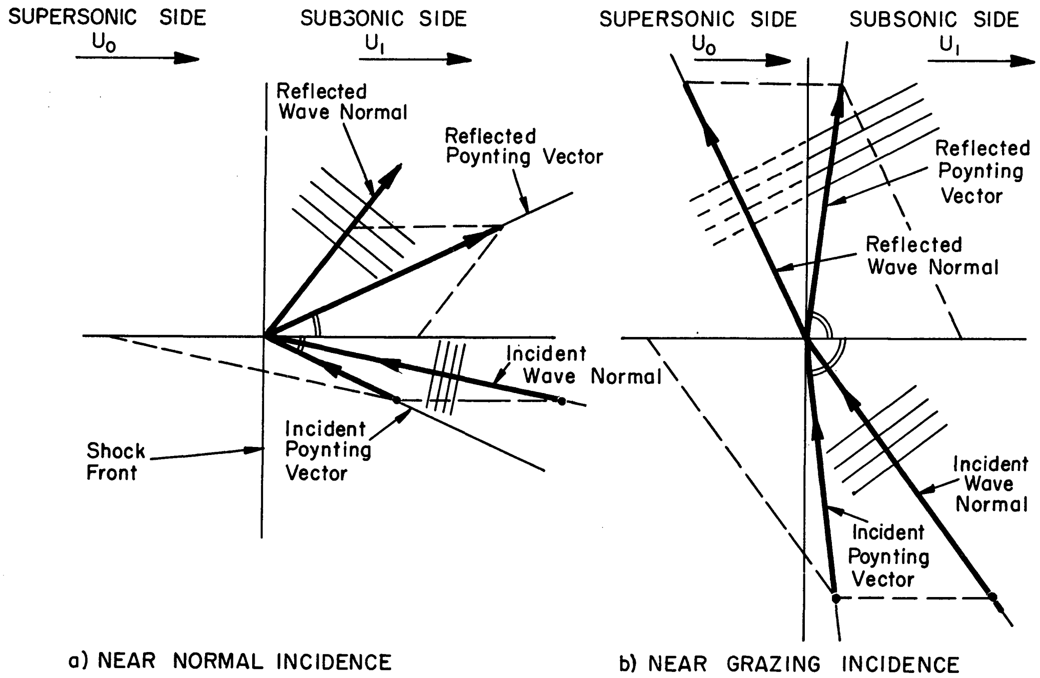

Fig. 8. Reflection of plane sound waves.

In other words, the ordinary form of Snell's law of reflection holds for the "Poynting" vectors.

b. <u>The Reflected Entropy-Vorticity Waves</u>.—The reflection governing the angle of reflection for the entropy-vorticity wave is:

$$\frac{\beta_2}{M_1\alpha_2} = \frac{\beta_0}{1 + M_1\alpha_0} \; . \tag{3.21}$$

Squaring this, we find:

$$M_1^2 \beta_0^2 \alpha_2^2 = (1 - \alpha_2^2)(1 + M_1 \alpha_0)^2 ,$$

or $\left((1+M_1^2) + 2M_1\alpha_0\right)\alpha_2^2 = (1 + M_1\alpha_0)^2$, which may be solved for α_2 to give:

$$\alpha_2 = \pm\frac{1 + M_1\alpha_0}{\sqrt{(1+M_1^2) + 2M_1\alpha_0}} \, , \qquad \beta_2 = \pm\frac{M_1\beta_0}{\sqrt{(1+M_1^2) + 2M_1\alpha_0}} \, .$$

If we choose the positive sign in both of these expressions, then $\theta_0 = \pi$ corresponds to $\theta_2 = 0$. This gives:

$$\alpha_2 = \frac{1 + M_1 \alpha_0}{\sqrt{(1+M_1^2) + 2M_1\alpha_0}} \; , \quad \beta_2 = \frac{M_1\beta_0}{\sqrt{(1+M_1^2) + 2M_1\alpha_0}} \tag{3.26}$$

The maximum angle of the entropy-vorticity wave corresponds to $\theta_0 = \theta_g = \cos^{-1} - M_1$. For this angle $\alpha_{2max.} = \sqrt{1 - M_1^2}$ or $\theta_{2max} = \cos^{-1}\sqrt{1-M_1^2}$. As θ_0 decreases from π to θ_g, θ_2 increases from 0 to θ_{2max} .

The relations expressed by Eqs. (3.22) and (3.26) are illustrated in Fig. 9.

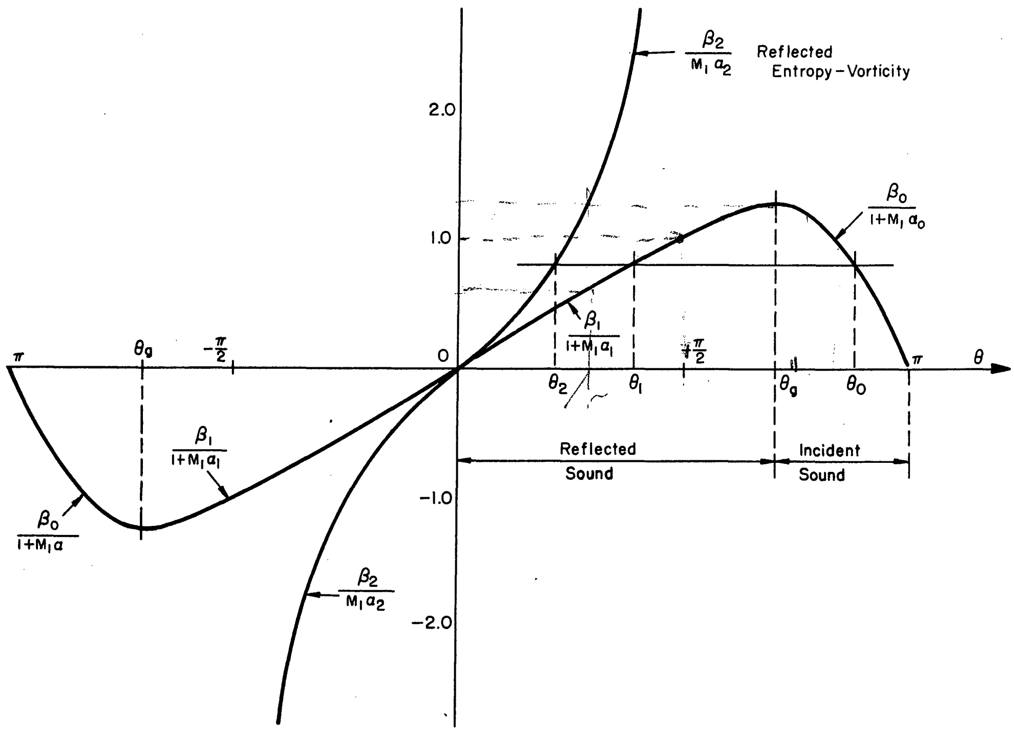

Fig. 9. Angle relations for reflection.

2. "Fresnel" Formulae for Reflection

The next step, after having explored the relations between the various angles involved, will be the calculation of the amplitudes A, B, and C of the reflected sound, velocity, and entropy waves in terms of the amplitude ϵ of the incident sound wave.

In the introduction we mentioned how this calculation was to be performed. We use the linear shock conditions (2.15) to relate the amplitudes of the waves on the subsonic side of the shock to the amplitude of the distortion of the shock. This gives four linear equations for the determination of the amplitudes A, B, C of the waves and the amplitude a of the distortion.

The pressure, velocity, and entropy on the subsonic side of the shock are given by (3.16), (3.17), and (3.18):

$$
\begin{aligned}
p_1 &= \left\{ \epsilon e^{ik_1\frac{(\alpha_0 x+\beta_0 y)}{1+M_1\alpha_0}} + A e^{ik_1\frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1}} \right\} e^{-i\omega t}; \\
u_1 &= \left\{ \alpha_0\epsilon e^{ik_1\frac{(\alpha_0 x+\beta_0 y)}{1+M_1\alpha_0}} + \alpha_1 A e^{ik_1\frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1}} - \beta_2 B e^{ik_1\frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2}} \right\} e^{-i\omega t}; \\
v_1 &= \left\{ \beta_0\epsilon e^{ik_1\frac{(\alpha_0 x+\beta_0 y)}{1+M_1\alpha_0}} + \beta_1 A e^{ik_1\frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1}} + \alpha_2 B e^{ik_1\frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2}} \right\} e^{-i\omega t}; \\
s_1 &= \left\{ C e^{ik_1\frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2}} \right\} e^{-i\omega t}.
\end{aligned}
\tag{3.27}
$$

We now carry out the transition to the other side using the linear shock conditions (2.15). Since all the quantities p_0, u_0, v_0, and s_0 are zero, these conditions reduce to:

$$
\begin{aligned}
p_1 &= \frac{M_1 M_0}{\mu M_0^2 F(M_0)} \left(-2(\mu-1)\frac{f_t}{c_0} \right); \\
u_1 &= \frac{1}{\mu M_0^2 F(M_0)} \left((\mu-1)\frac{M_0^2+1}{M_0}\frac{f_t}{c_0} \right); \\
v_1 &= \frac{1}{\mu M_0^2 F(M_0)} \left((\mu-1)(M_0^2-1) f_y \right); \\
s_1 &= \frac{M_1^2(M_0^2-1)^2}{\mu^2 M_0^4 F^2(M_0)} \left(-2\frac{(\mu-1)}{M_0}\frac{f_t}{c_0} \right),
\end{aligned}
\tag{3.28}
$$

where p_1, u_1, v_1, and Δ_1 are to be evaluated at x = 0.

We now make the assumption that f(y,t) depend upon its variables in the same way as p_1, u_1, v_1, and Δ_1:

$$f(y,t) = \qquad a\, e^{i k_1 \frac{\beta_0 y}{1+M_1\alpha_0} - i\omega t} \; ; \tag{3.29}$$

$$f_t(y,t) = -i\omega\, a\, e^{i k_1 \frac{\beta_0 y}{1+M_1\alpha_0} - i\omega t} \; ; \tag{3.30}$$

$$f_y(y,t) = i\frac{k_1\beta_0}{1+M_1\alpha_0}\, a\, e^{i k_1 \frac{\beta_0 y}{1+M_1\alpha_0} - i\omega t} \, . \tag{3.31}$$

Substituting (3.30) and (3.31) in Eqs. (3.28), dropping the common exponential factors, and noting that $M_0 C_0 F(M_0) = M_1 C_1$ and that $\frac{M_0^2-1}{M_0^2 F(M_0)} = \frac{1-M_1^2}{M_1^2}$, we find:

$$\begin{aligned} P_1 &= i\, k_1 a\, b_1 \; ; \\ U_1 &= i\, k_1 a\, b_2 \; ; \\ V_1 &= i\, k_1 a\, b_3 \; ; \\ S_1 &= i\, k_1 a\, b_4 \; ; \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} b_1 &= 2\frac{(\mu-1)}{\mu} \; ; \\ b_2 &= -\frac{(\mu-1)}{\mu}\,\frac{M_0^2+1}{M_0^2 M_1} \; ; \\ b_3 &= \frac{(\mu-1)}{\mu}\,\frac{(1-M_1^2)}{M_1^2}\,\frac{\beta_0}{1+M_1\alpha_0} \; ; \\ b_4 &= \frac{2(\mu-1)(M_0^2-1)(1-M_1^2)}{\mu^2 M_1 M_0^2} \, . \end{aligned} \tag{3.33}$$

To determine a, A, B, and C, consider the relations (3.27) at x = 0. Let us then remove the common exponential factors [by virtue of the reflection laws (3.20) and (3.21)] to find:

$$\begin{aligned} P_1 &= A + \epsilon &&= ik_1ab_1\,; \\ U_1 &= \alpha_1 A + \alpha_0\epsilon - \beta_2 B &&= ik_1ab_2\,; \\ V_1 &= \beta_1 A + \beta_0\epsilon + \alpha_2 B &&= ik_1ab_3\,; \\ S_1 &= C &&= ik_1ab_4\,. \end{aligned} \tag{3.34}$$

Since the unknown quantities to be determined in terms of ϵ are A, B, C, and a, these equations may be written in the form:

$$\begin{aligned} (ik_1b_1)a - A &= \epsilon\,; \\ (ik_1b_2)a - \alpha_1 A + \beta_2 B &= \alpha_0\epsilon\,; \\ (ik_1b_3)a - \beta_1 A - \alpha_2 B &= \beta_0\epsilon\,; \\ (ik_1b_4)a - C &= 0\,. \end{aligned} \tag{3.35}$$

The determinant of the system (3.35) is:

$$\Delta = k_1\left(\vec{n}_2\cdot\vec{b} - (\vec{n}_1\cdot\vec{n}_2)\,b_1\right), \tag{3.36}$$

where the vector $\vec{b} = (b_2, b_3)$ is parallel to the velocity. The solution to the system is:

$$\begin{aligned} a &= i\left[(\vec{n}_1\cdot\vec{n}_2) - (\vec{n}_0\cdot\vec{n}_2)\right]\epsilon/\Delta\,; \\ A &= k_1\left[\vec{b}\cdot\vec{n}_2 - b_1(\vec{n}_0\cdot\vec{n}_2)\right]\epsilon/\Delta\,; \\ B &= k_1\left[(\beta_1-\beta_0)b_2 - (\alpha_1-\alpha_0)b_3 + (\alpha_1\beta_0-\beta_1\alpha_0)b_1\right]\epsilon/\Delta\,; \\ C &= k_1\left[(\vec{n}_0\cdot\vec{n}_2) - (\vec{n}_1\cdot\vec{n}_2)\right]b_4\,\epsilon/\Delta\,. \end{aligned} \tag{3.37}$$

The last three expressions (3.37) are the analogues of the Fresnel coefficients in optics; the first has no optical analogue. It should be noted that a, which is the amplitude of the shock-wave distortion is imaginary, and thus 90° out of phase with all the waves.

In Appendix D these formulae are written entirely in terms of the incident wave-direction cosines α_0 and β_0. The resulting expressions will be used in Section V to calculate the amplitudes of the reflected cylindrical waves.

C. INCIDENCE FROM THE SUPERSONIC SIDE

In this section we shall describe the interaction of plane waves incident from the supersonic side of the shock wave (Fig. 2b, p. 3). Although many of these considerations are identical with those of the preceding section, the fact that complex angles occur here make it necessary to study this case in detail. The incident plane wave is a sound wave moving in the supersonic flow. In this case we must consider angles of incidence ranging from 0 to π since the energy, even of waves whose normals do not point toward the shock, is nevertheless swept downstream, the flow being supersonic.

The incident wave is described by:

$$
\begin{aligned}
p_i &= \epsilon\, e^{i k_0 \frac{(\alpha_0 x + \beta_0 y)}{1 + M_0 \alpha_0} - i\omega t}\,; \\
u_i &= \alpha_0 \epsilon\, e^{i k_0 \frac{(\alpha_0 x + \beta_0 y)}{1 + M_0 \alpha_0} - i\omega t}\,; \\
v_i &= \beta_0 \epsilon\, e^{i k_0 \frac{(\alpha_0 x + \beta_0 y)}{1 + M_0 \alpha_0} - i\omega t}\,; \\
s_i &= 0\,,
\end{aligned}
\qquad (3.38)
$$

where ϵ is the amplitude of the incident wave.

Together with this incident sound wave we consider a refracted sound wave as well as a refracted entropy-vorticity wave. These waves are described by:

Refracted Sound Wave:

$$\begin{aligned} p_s &= A\, e^{i k_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t} ; \\ u_s &= \alpha_1 A\, e^{i k_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t} ; \\ v_s &= \beta_1 A\, e^{i k_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t} ; \\ s_s &= 0 . \end{aligned} \tag{3.39}$$

Refracted Entropy-Vorticity Wave:

$$\begin{aligned} p^* &= 0 ; \\ u^* &= -\beta_2 B\, e^{i k_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t} ; \\ v^* &= \alpha_2 B\, e^{i k_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t} ; \\ s^* &= C\, e^{i k_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t} . \end{aligned} \tag{3.40}$$

Since at x = 0 the quantities p, u, v, and s are related linearly by means of the shock conditions (2.15), it is again necessary that the exponentials match at x =0 with the result that:

$$\frac{k_1 \beta_1}{1+M_1\alpha_1} = \frac{k_0 \beta_0}{1+M_0\alpha_0} , \tag{3.41}$$

and

$$\frac{k_1 \beta_2}{M_1\alpha_2} = \frac{k_0 \beta_0}{1+M_0\alpha_0} . \tag{3.42}$$

Equations (3.41) and (3.42) are the analogues of Snell's laws of refraction in optics.

1. Refraction Laws

In this subsection the angle relations for the case of supersonic incidence will be studied in detail.

a. Refracted Sound Waves.—The relation governing the angle of refraction for the sound wave is:

$$\frac{k_1 \beta_1}{1+M_1\alpha_1} = \frac{k_0 \beta_0}{1+M_0\alpha_0} . \tag{3.41}$$

If we let $\lambda = \frac{k_0}{k_1} = \frac{c_1}{c_0}$ and square both sides of (3.41), we find:

$$\lambda^2\beta_0^2(1+2M_1\alpha_1+M_1^2\alpha_1^2) = (1-\alpha_1^2)(1+M_0\alpha_0)^2$$

or $(M_1^2\lambda^2\beta_0^2+(1+M_0\alpha_0)^2)\alpha_1^2 + 2M_1\lambda^2\beta_0^2\alpha_1 + (\lambda^2\beta_0^2-(1+M_0\alpha_0)^2) = 0$

We may solve this equation to find:

$$\alpha_1 = \frac{-M_1\lambda^2\beta_0^2 \pm (1+M_0\alpha_0)\sqrt{(1+M_0\alpha_0)^2-(1-M_1^2)\lambda^2\beta_0^2}}{M_1^2\lambda^2\beta_0^2+(1+M_0\alpha_0)^2} .$$

$$\beta_1 = \frac{(1+M_0\alpha_0) \pm M_1\sqrt{(1+M_0\alpha_0)^2-(1-M_1^2)\lambda^2\beta_0^2}}{M_1^2\lambda^2\beta_0^2+(1+M_0\alpha_0)^2} \tag{3.43}$$

For the physically realized wave a definite choice of sign in Eqs. (3.43) has to be made. For this purpose let us consider first the angular range described by $(1+M_0\alpha_0)^2 \geq (1-M_1^2)\lambda^2\beta_0^2$. In this case the functions α_1 and β_1 are both real. Waves which are normally incident upon the shock wave must correspond to waves transmitted at an angle $\theta_1 = 0$, so both $\theta_0 = 0$ and $\theta_0 = \pi$ must correspond to $\theta_1 = 0$. Thus we are led to choose the positive sign in both Eqs. (3.43) for $(1+M_0\alpha_0) \geq \sqrt{1-M_1^2}\,\lambda\beta_0$, and to choose the negative sign in both equations for $(1+M_0\alpha_0) \leq -\sqrt{1-M_1^2}\,\lambda\beta_0$.

In the range $(1+M_0\alpha_0)^2 < (1-M_1^2)\lambda^2\beta_0^2$, the functions α_1 and β_1 are complex and thus do not correspond to the cosine and sine of the angle refraction. We find instead that:*

*The subscripts R and I will be used henceforth to denote the real and imaginary parts of a complex number.

$$\frac{i k_1}{1+M_1\alpha_1}(\alpha_1 x+\beta_1 y) = -\frac{k_1\alpha_{1I}\,x}{|1+M_1\alpha_1|^2} + i k_1\left\{\frac{\alpha_{1R}+M_1|\alpha_1|^2}{|1+M_1\alpha_1|^2}x + \frac{\lambda\beta_0}{1+M_0\alpha_0}y\right\}$$

$$= -\nu_1 x + i k_1'(\alpha_1' x + \beta_1' y) \;,$$

where

$$\nu_1 = \frac{k_1\alpha_{1I}}{|1+M_1\alpha_1|^2} \;,$$

and

$$k_1'\alpha_1' = \frac{\alpha_{1R}+M_1|\alpha_1|^2}{|1+M_1\alpha_1|^2} \;, \qquad k_1'\beta_1' = \frac{\lambda\beta_0}{1+M_0\alpha_0} \;.$$

The factor ν_1 is to be interpreted as a damping factor, while k_1' is the wave number of the damped wave. This wave moves in the direction $\vec{n}_1' = (\alpha_1', \beta_1')$. In particular we find:

$$\nu_1 = \pm\frac{\sqrt{\lambda^2\beta_0^2(1-M_1^2)-(1+M_0\alpha_0)^2}}{(1+M_0\alpha_0)(1-M_1^2)} \;, \qquad (3.44)$$

where the positive or negative sign arises from the ambiguity in (3.43). To achieve damping, we must require $\nu_1 > 0$. Thus we choose the positive sign in (3.43) for the range $\sqrt{1-M_1^2}\,\lambda\beta_0 > (1+M_0\alpha_0) > 0$, and we choose the negative sign for the range $-\sqrt{1-M_1^2}\,\lambda\beta_0 < (1+M_0\alpha_0) < 0$. Combining this condition with the condition determined for the case when the waves are undamped, we arrive at the following choice of signs:

choose the positive signs in (3.43) for $(1+M_0\alpha_0) > 0$;

choose the negative signs in (3.43) for $(1+M_0\alpha_0) < 0$.

Let us return to the case of damped waves and note that we have:

$$k_1'\alpha_1' = \frac{\alpha_{1R}+M_1|\alpha_1|^2}{|1+M_1\alpha_1|^2} = -\frac{M_1}{1-M_1^2} \;; \qquad (3.45)$$

$$k_1'\beta_1' = \frac{\lambda\beta_0}{1+M_0\alpha_0} \;.$$

Since $\alpha_1'^2 + \beta_1'^2 = 1$, we have:

$$k_1' = \pm \frac{\sqrt{M_1^2(1+M_0\alpha_0)^2 + (1-M_1^2)^2\lambda^2\beta_0^2}}{(1-M_1^2)(1+M_0\alpha_0)} \tag{3.46}$$

where the positive sign is to be chosen for $(1+M_0\alpha_0)>0$, and the negative sign for $(1+M_0\alpha_0)<0$.

Hence for $(1+M_0\alpha_0)>0$:

$$\alpha_1' = \frac{-M_1(1+M_0\alpha_0)}{\sqrt{M_1^2(1+M_0\alpha_0)^2+(1-M_1^2)^2\lambda^2\beta_0^2}} \; ; \; \beta_1' = \frac{\lambda M_1\beta_0(1-M_1^2)}{\sqrt{M_1^2(1+M_0\alpha_0)^2+(1-M_1^2)^2\lambda^2\beta_0^2}} , \tag{3.47}$$

and for $(1+M_0\alpha_0)<0$:

$$\alpha_1' = \frac{M_1(1+M_0\alpha_0)}{\sqrt{M_1^2(1+M_0\alpha_0)^2+(1-M_1^2)^2\lambda^2\beta_0^2}} \; ; \; \beta_1' = \frac{-\lambda M_1\beta_0(1-M_1^2)}{\sqrt{M_1^2(1+M_0\alpha_0)^2+(1-M_1^2)^2\lambda^2\beta_0^2}} , \tag{3.48}$$

The relations (3.47) and (3.48) are then the appropriate angle relations in the region where the wave is damped.

In the subsonic case a simple relation connected the incident and reflected "Poynting" vectors, but in the supersonic case no simple relation of this type (ordinary Snell's law for refraction) exists. One may easily show, though, that for the undamped waves region, $(1+M_0\alpha_0)^2>(1-M_1^2)\lambda^2\beta_0^2$, the refracted "Ponyting" vector is related to the incident wave angles by means of the formulae:

$$\alpha_p = \frac{\sqrt{(1+M_0\alpha_0)^2-(1-M_1^2)\lambda^2\beta_0^2}}{\sqrt{M_1^2\lambda^2\beta_0^2+(1+M_0\alpha_0)^2}} \; ;$$

$$\beta_p = \frac{\pm\lambda\beta_0}{\sqrt{M_1^2\lambda^2\beta_0^2+(1+M_0\alpha_0)^2}} , \tag{3.49}$$

where the positive sign is chosen for $(1+M_0\alpha_0)>0$ and the negative sign is chosen for $(1+M_0\alpha_0)<0$.

At the critical angles, $(1+M_0\alpha_0)^2=(1-M_1^2)\lambda^2\beta_0^2$, these formulae predict that the "Poynting" vector will be directed along the shock.

In the damped region $(1+M_0\alpha_0)^2 < (1-M_1^2)\lambda^2\beta_0^2$ the formula (3.15) relating $\tilde{\vec{S}}_1$ to $\vec{n}$ is no longer valid but must be replaced by:

$$\tilde{\vec{S}}_1 = \frac{1}{2} A^2 e^{-2\nu_1 x}\left(C_1\alpha_{1R} + U_1\frac{|\alpha_1|^2+|\beta_1|^2+1}{2}\,,\ C_1\beta_{1R}\right) \tag{3.50}$$

which is the obvious generalization of (3.15) for complex angles. Substituting the formulae (3.43) for α_1 and β_1 one finds:

$$\tilde{\vec{S}} = \frac{1}{2}|A|^2 e^{-2\nu_1 x}\left(0\ ,\ \frac{C_1\lambda\beta_0(1+M_0\alpha_0)}{M_1^2\lambda^2\beta_0^2+(1+M_0\alpha_0)^2}\right),$$

or

$$\tilde{\vec{S}} = C_1\left[\frac{1+M_0\alpha_0}{\lambda\beta_0}\right]\tilde{\mathcal{E}}\,\vec{n}_2\ , \tag{3.51}$$

where $\vec{n}_2=(0,1)$. This formula shows that the "Poynting" vector is directed along the shock in the entire damped region, directed upward for $(1+M_0\alpha_0)>0$ and downward for $(1+M_0\alpha_0)<0$, as is to be expected. It should also be noted that this vector field is damped in the direction normal to the shock. The relations discussed here are illustrated in Fig. 10.

b. Refracted Entropy-Vorticity Waves.—The relation governing the angle of refraction for the entropy-vorticity wave is:

$$\frac{k_1\beta_2}{M_1\alpha_2} = \frac{k_0\beta_0}{1+M_0\alpha_0}\ . \tag{3.42}$$

Setting $\lambda = \frac{k_0}{k_1} = \frac{C_1}{C_0}$ and squaring both sides, we find:

$$(1-\alpha_2^2)(1+M_0\alpha_0)^2 = \lambda^2 M_1^2\beta_0^2\alpha_2^2\ ,$$

or

$$\left[(1+M_0\alpha_0)^2+\lambda^2M_1^2\beta_0^2\right]\alpha_2^2 = (1+M_0\alpha_0)^2 .$$

Thus

$$\alpha_2 = \pm\frac{(1+M_0\alpha_0)}{\sqrt{\lambda^2M_1^2\beta_0^2+(1+M_0\alpha_0)^2}}\ , \tag{3.52}$$

and

$$\beta_2 = \pm\frac{M_1\lambda\beta_0}{\sqrt{\lambda^2M_1^2\beta_0^2+(1+M_0\alpha_0)^2}}\ .$$

Again we choose the positive sign for $(1+M_0\alpha_0)>0$, and choose the negative sign for $(1+M_0\alpha_0)<0$. This choice of signs corresponds to

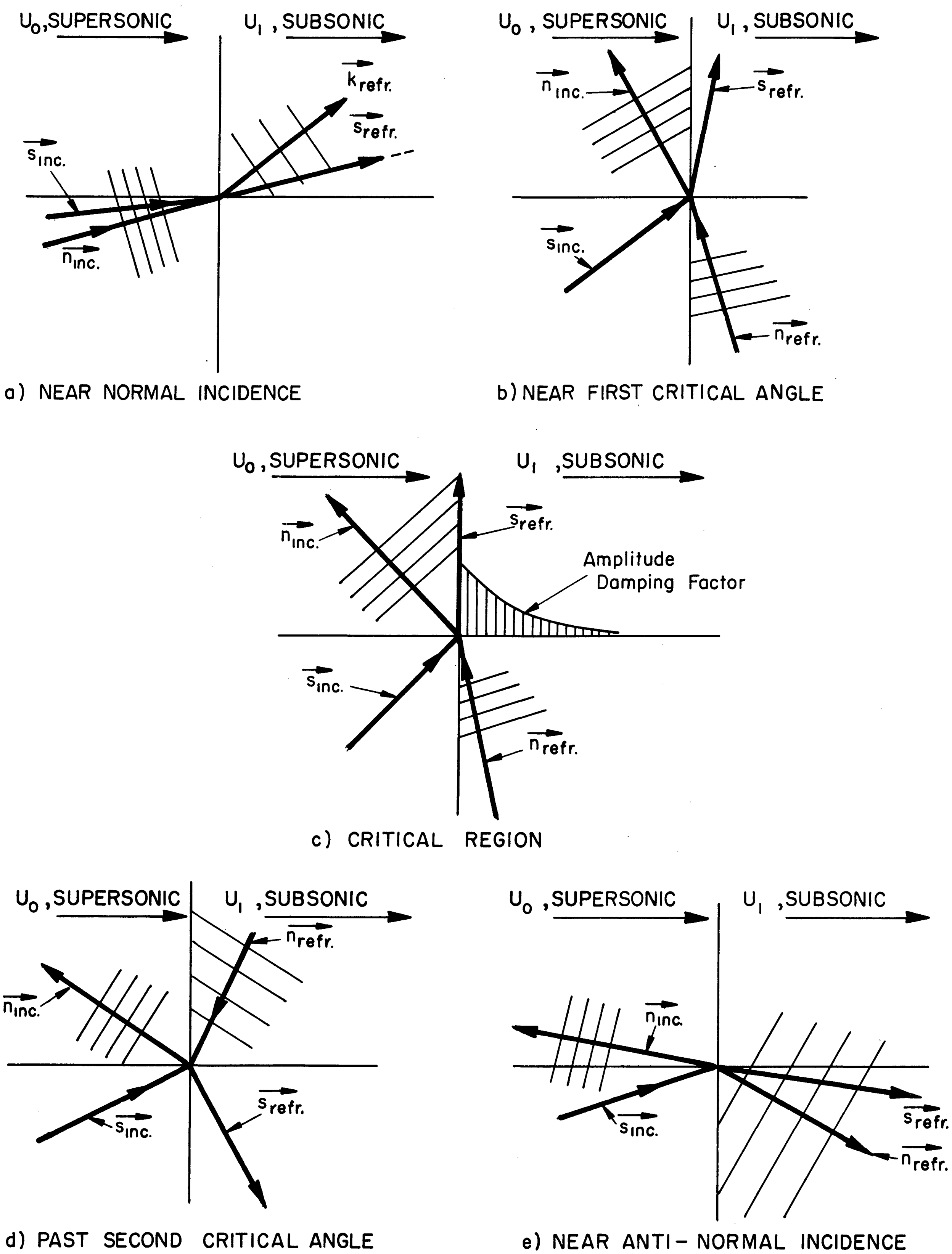

Fig. 10. Refraction of plane sound waves.

a choice of $\theta_2 = 0$ for $\theta_0 = 0$ or $\theta_0 = \pi$.

The relations expressed in Eqs. (3.43) and (3.52) are illustrated in Fig. 11.

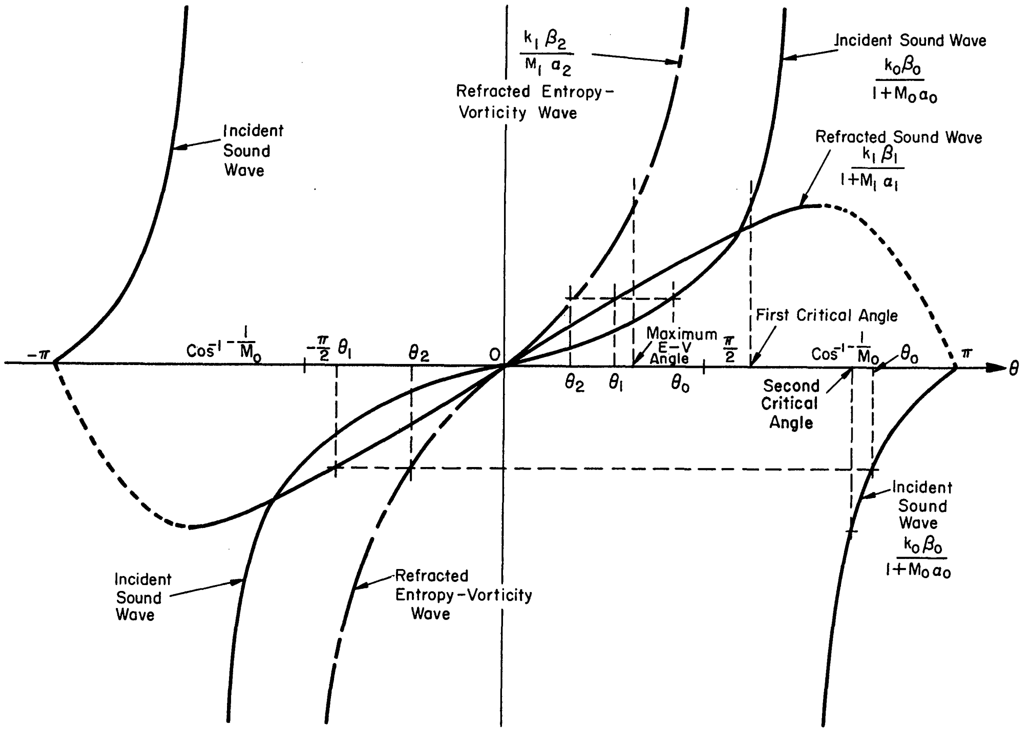

Fig. 11. Angle relations for refraction.

2. "Fresnel" Formulae for Refraction

Knowing the "Snell's" law of refraction we may now calculate the amplitudes A, B, and C of the refracted sound, vorticity, and entropy waves, in terms of ϵ , the amplitude of the incident wave.

On the supersonic side of the shock wave the pressure, velocity, and entropy are given by:

$$
\begin{aligned}
p_0 &= \epsilon\, e^{i k_0 \frac{(\alpha_0 x + \beta_0 y)}{1+M_0\alpha_0} - i\omega t} ; \\
u_0 &= \alpha_0\, \epsilon\, e^{i k_0 \frac{(\alpha_0 x + \beta_0 y)}{1+M_0\alpha_0} - i\omega t} ; \\
v_0 &= \beta_0\, \epsilon\, e^{i k_0 \frac{(\alpha_0 x + \beta_0 y)}{1+M_0\alpha_0} - i\omega t} ; \\
s_0 &= 0 ,
\end{aligned}
\tag{3.53}
$$

where ϵ is the amplitude of the incident wave.

On the subsonic side of the shock wave the pressure, velocity, and entropy are given by:

$$
\begin{aligned}
p_1 &= \left\{ A\, e^{ik_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1}} \right\} e^{-i\omega t}; \\
u_1 &= \left\{ \alpha_1 A\, e^{ik_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1}} - \beta_2 B\, e^{ik_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2}} \right\} e^{-i\omega t}; \\
v_1 &= \left\{ \beta_1 A\, e^{ik_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1}} + \alpha_2 B\, e^{ik_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2}} \right\} e^{-i\omega t}; \\
s_1 &= \left\{ C\, e^{ik_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2}} \right\} e^{-i\omega t}.
\end{aligned}
\tag{3.54}
$$

At x = 0 all the exponential factors are equal; thus we may drop this common factor from the Eqs. (3.53) and (3.54) and find:

$$
\begin{aligned}
P_0 &= \epsilon , \\
U_0 &= \alpha_0 \epsilon , \\
V_0 &= \beta_0 \epsilon , \\
S_0 &= 0 ,
\end{aligned}
\tag{3.55}
$$

and

$$
\begin{aligned}
P_1 &= A , \\
U_1 &= \alpha_1 A - \beta_2 B , \\
V_1 &= \beta_1 A + \beta_2 B , \\
S_1 &= C .
\end{aligned}
\tag{3.56}
$$

At x = 0 the quantities p_1, u_1, v_1, and s_1 on the subsonic side of the shock are related to the quantities p_0, u_0, v_0, and s_0 on the supersonic side of the shock by means of the linearized shock conditions (2.15).

Let us suppose again that $f(y,t)$ depends upon its variables in the same fashion as the remaining quantities in these equations:

$$f(y,t) = a\, e^{i k_0 \frac{\beta_0 y}{1+M_0\alpha_0} - i\omega t} ; \tag{3.57}$$

$$f_t(y,t) = -i\omega a\, e^{i k_0 \frac{\beta_0 y}{1+M_0\alpha_0} - i\omega t} ; \tag{3.58}$$

$$f_y(y,t) = i k_0 \frac{\beta_0}{1+M_0\alpha_0} a\, e^{i k_0 \frac{\beta_0 y}{1+M_0\alpha_0} - i\omega t} . \tag{3.59}$$

Using (3.58) and (3.59) for f_t and f_y and (3.55) for P_0, U_0, V_0, and S_0, the linearized shock conditions (2.15) reduce to:

$$\begin{aligned} P_1 &= a_1\epsilon + b_1(ik_0a) ; \\ U_1 &= a_2\epsilon + b_2(ik_0a) ; \\ V_1 &= a_3\epsilon + b_3(ik_0a) ; \\ S_1 &= a_4\epsilon + b_4(ik_0a) , \end{aligned} \tag{3.60}$$

where

$$\begin{aligned} a_1 &= \frac{M_1M_0}{\mu M_0^2F(M_0)}\left[2(\mu-1)M_0\alpha_0+(\mu-1)M_0^2-1\right] ; \quad b_1 = \frac{2(\mu-1)M_1M_0}{\mu M_0^2F(M_0)} ; \\ a_2 &= \frac{1}{\mu M_0^2F(M_0)}\left[(M_0^2-(\mu-1))\alpha_0+2M_0\right] ; \quad b_2 = \frac{-(\mu-1)(M_0^2+1)}{\mu M_0^3F(M_0)} ; \\ a_3 &= \frac{1}{\mu M_0^2F(M_0)}\left[\mu M_0^2\beta_0\right] ; \quad b_3 = \frac{(\mu-1)(M_0^2-1)\beta_0}{\mu M_0^2F(M_0)(1+M_0\alpha_0)} ; \\ a_4 &= \frac{2M_1^2(M_0^2-1)^2}{\mu^2M_0^4F^2(M_0)}\left[(\mu-1)\alpha_0-M_0\right] ; \quad b_4 = \frac{2M_1^2(M_0^2-1)^2(\mu-1)}{\mu^2M_0^5F^2(M_0)} . \end{aligned} \tag{3.61}$$

These shock conditions may be written as four linear equations for the four unknowns a, A, B, and C, by utilizing Eqs. (3.56):

$$\begin{aligned} -ik_0b_1a &+ A & &= a_1\epsilon ; \\ -ik_0b_2a &+ \alpha_1A - \beta_2B & &= a_2\epsilon ; \\ -ik_0b_3a &+ \beta_1A + \alpha_2B & &= a_3\epsilon ; \\ -ik_0b_4a & &+ C &= a_4\epsilon . \end{aligned} \tag{3.62}$$

The determinant of this system is:

$$\Delta = k_0(\vec{b}\cdot\vec{n}_2 - b_1(\vec{n}_1\cdot\vec{n}_2)), \tag{3.63}$$

where $\vec{b} = (b_2, b_3)$.

The solution to the system is:

$$\begin{aligned}
a &= i\left[\vec{n}_2\cdot\vec{a} - (\vec{n}_1\cdot\vec{n}_2)a_1\right]\epsilon/\Delta\,; \\
A &= k_0\left[a_1(\vec{n}_2\cdot\vec{b}) - b_1(\vec{n}_2\cdot\vec{a})\right]\epsilon/\Delta\,; \\
B &= k_0\left[\alpha_1(a_1b_3 - b_1a_3) + \beta_1(a_2b_1 - b_2a_1) + (a_3b_2 - b_3a_2)\right]\epsilon/\Delta\,; \\
C &= k_0\left[(\vec{n}_1\cdot\vec{n}_2)(a_1b_4 - b_1a_4) + a_4(\vec{n}_2\cdot\vec{b}) - b_4(\vec{n}_2\cdot\vec{a})\right]\epsilon/\Delta\,,
\end{aligned} \tag{3.64}$$

with $\vec{a} = (a_2, a_3)$.

These equations are the analogues of the "Fresnel" equations for the supersonic case. In Section V we shall use these equations in determining the interaction of a cylindrical wave with a shock.

IV. CYLINDRICAL SOUND WAVES IN A MOVING GAS

A. THE CONVECTIVE WAVE EQUATION

This entire section will be devoted to a study of the properties of sound waves in a moving gas. In treating this problem quantitatively, it is convenient to replace the system of first-order equations used in Section III by a single second-order equation.

Let us define a sound wave as an irrotational, isentropic disturbance in a compressible inviscid gas.* From Section II, Eqs. (2.4), we find that the differential equations governing such disturbances are:

$$
\begin{aligned}
&\frac{1}{c}\left(\frac{\partial p}{\partial t} + U\frac{\partial p}{\partial x}\right) + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad ; \\
&\frac{1}{c}\left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} = 0 \quad ; \\
&\frac{1}{c}\left(\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x}\right) + \frac{\partial p}{\partial y} = 0 \quad ; \\
&\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \, .
\end{aligned}
\tag{4.1}
$$

It should be noticed that these equations are not identical with the system (2.4), as the irrotationality condition in (4.1) replaces the entropy equation in (2.4).

The last three equations in (4.1) imply the existence of a scalar function $\Phi(x,y,t)$ such that:

*We are here using the properties described in Section III, Part A, for plane sound waves as defining properties for sound waves in general.

$$u = \frac{\partial \Phi}{\partial x} ;$$
$$v = \frac{\partial \Phi}{\partial y} ; \qquad (4.2)$$
$$p = -\frac{1}{c}\left(\frac{\partial \Phi}{\partial t} + U\frac{\partial \Phi}{\partial x}\right) .$$

This function $\Phi(x,y,t)$ is called the velocity potential. The first equation in (4.1) becomes by virtue of (4.2):

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \Phi = 0 . \qquad (4.3)$$

If we subject this equation to the transformation $x' = x - Vt$, $y' = y$, $t' = t$, we find:

$$\frac{\partial^2 \Phi}{\partial x'^2} + \frac{\partial^2 \Phi}{\partial y'^2} - \frac{1}{c^2}\left(\frac{\partial}{\partial t'} + (U-V)\frac{\partial}{\partial x'}\right)^2 \Phi = 0. \qquad (4.4)$$

In particular, if we move with the flow, by choosing $V = U$ the equation reduces to the ordinary wave equation.

The single second-order equation, (4.3), determines the behavior of the sound field when no sources are present in the flow. If there are sources, one must modify this equation as in electrostatics by adding the source term to the right side. In particular for an oscillating line source located at (x_0, y_0), we have:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \Phi = 4\pi\epsilon\,\delta(x-x_0)\,\delta(y-y_0)\,e^{-i\omega t} . \qquad (4.5)$$

This "convective" wave equation will serve as the fundamental equation for cylindrical waves.

B. LINE SOURCE IN A SUBSONIC GAS

Let us now find the solution to Eq. (4.5) when the speed of the flow is subsonic. We place the source at (0,0) for convenience, and find:

$$\nabla^2 \Phi - \frac{1}{C_1^2}\left(\frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x}\right)^2 \Phi = 4\pi \epsilon\, \delta(x)\,\delta(y)\, e^{-i\omega t} . \tag{4.6}$$

If we set

$$\Phi(x,y,t) = \varphi(x,y)\, e^{-i \frac{M_1 k_1 x}{1-M_1^2} - i\omega t} , \tag{4.7}$$

then Eq. (4.6) becomes:

$$(1-M_1^2)\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{k_1^2}{1-M_1^2}\varphi = 4\pi\epsilon\, \delta(x)\,\delta(y) . \tag{4.8}$$

Let us now simplify Eq. (4.8) by the following transformations:

$$x' = \frac{x}{\sqrt{1-M_1^2}} , \quad y' = y , \quad k' = \frac{k}{\sqrt{1-M_1^2}} , \quad \epsilon' = \frac{\epsilon}{\sqrt{1-M_1^2}} . \tag{4.9}$$

Equation (4.8) becomes the inhomogeneous Helmholtz equation:*

$$\frac{\partial^2 \varphi}{\partial x'^2} + \frac{\partial^2 \varphi}{\partial y'^2} + k'^2 \varphi = 4\pi\epsilon'\, \delta(x')\,\delta(y') . \tag{4.10}$$

Equation (4.10) arises frequently in optics and wave mechanics, where φ is interpreted as a Green's function. We are therefore led to use the common technique employed in these fields to solve the equation.[21]

Let us assume that the solution to (4.10) may be written as a Fourier Integral:

$$\varphi(x',y') = \iint_{-\infty}^{+\infty} \Psi(k_x,k_y)\, e^{i(k_x x' + k_y y')}\, dk_x\, dk_y . \tag{4.11}$$

We may write $4\pi\epsilon'\delta(x')\delta(y')$ as:

$$4\pi\epsilon'\delta(x')\,\delta(y') = \frac{\epsilon'}{\pi}\iint_{-\infty}^{+\infty} e^{i(k_x x' + k_y y')}\, dk_x\, dk_y . \tag{4.12}$$

Equation (4.10) for φ is thus equivalent to the following equation for Ψ:

*We have used the property of the δ-function: $a\,\delta(ax) = \delta(x)$.

$$\iint_{-\infty}^{+\infty} \left[(k'^2 - k_x^2 - k_y^2)\Psi - \frac{\epsilon'}{\pi} \right] e^{i(k_x x' + k_y y')} \, dk_x \, dk_y = 0, \tag{4.13}$$

which may be satisfied by choosing

$$\Psi(k_x, k_y) = \frac{\epsilon'}{\pi(k'^2 - k_x^2 - k_y^2)} \; . \tag{4.14}$$

Choosing this value for $\Psi(k_x, k_y)$, Eq. (4.11) becomes:

$$\varphi(x', y') = \frac{\epsilon'}{\pi} \iint_{-\infty}^{+\infty} \frac{e^{i(k_x x' + k_y y')}}{k'^2 - k_x^2 - k_y^2} \, dk_x \, dk_y \, . \tag{4.15}$$

This integral is formally the solution to our problem. It is possible to carry out the integrations in (4.13) in the following manner. Define $\chi(x')$ by:

$$\chi(x') = - \int_{-\infty}^{+\infty} \frac{e^{i k_x x'} \, dk_x}{(k_x - K)(k_x + K)} \; , \tag{4.16}$$

with

$$K = \sqrt{k'^2 - k_y^2} \; .$$

Then

$$\varphi(x', y') = \frac{\epsilon'}{\pi} \int_{-\infty}^{+\infty} e^{i k_y y'} \chi(x') \, dk_y \; . \tag{4.17}$$

The integrand of $\chi(x')$ has poles at $k_x = \pm K$. When K is imaginary, $k'^2 < k_y^2$, the integral is well defined, but when K is real, $k' > k_y^2$, the question arises as to how the Fourier Integral is to be interpreted in the vicinity of the poles. One may extend the path of integration into the complex plane near the poles, and by choosing various paths arrive at different solutions to Eq. (4.10). Sommerfeld[22] shows that, in the one-dimensional case, the choice of path shown in Fig. 12 is equivalent to his radiation condition, $\frac{\partial \varphi}{\partial x} - ik\varphi \rightarrow 0$, as $x \rightarrow \infty$. We shall use this path in our subsonic case since the resulting integral in the limit $M_1 = 0$ must satisfy the Sommerfeld condition.

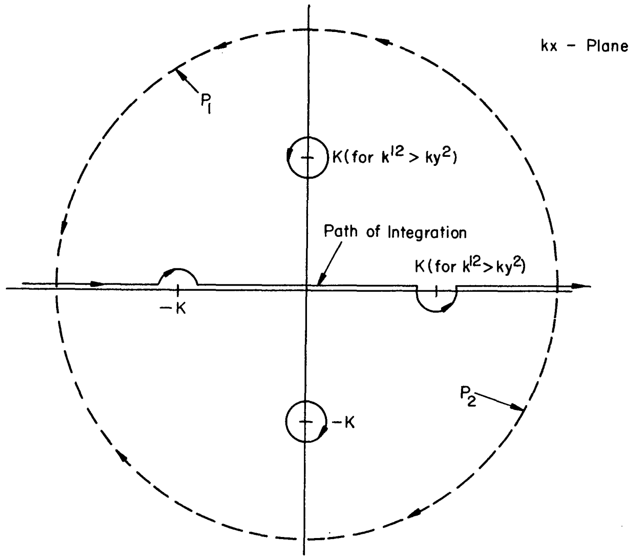

Fig 12. Path of integration for $\chi(x')$; subsonic case.

The integral (4.16) may then be converted into a contour integral by adding the semi-circular segment P_1 to the path of integration for $x' > 0$, and P_2 to the path when $x' < 0$. The contour integrals over paths P_1 and $P_2 \to 0$ as their radii $\to \infty$, and thus we find:

$$\chi(x') = \begin{cases} -2\pi i \text{ Res. at } K & \text{for } x' > 0 \\ 2\pi i \text{ Res. at} -K & \text{for } x' < 0, \end{cases}$$

or $\chi(x') = \frac{\pi}{i} e^{iK|x'|}$ for all x'. (4.18)

The integral (4.17) then reduces to:

$$\varphi(x', y') = -\frac{\epsilon'}{i} \int_{-\infty}^{+\infty} \frac{e^{i(\sqrt{k'^2 - k_y^2}|x'| + k_y y')}}{\sqrt{k'^2 - k_y^2}} dk_y. \quad (4.19)$$

The integral (4.19) may be reduced to the Sommerfeld integral for the Hankel function by the transformation:

$$k_y = k' \sin\theta \quad ; \quad dk_y = k' \cos\theta \, d\theta \, ;$$
$$\sqrt{k'^2 - k_y^2} = k' \cos\theta \quad ; \quad \frac{dk_y}{\sqrt{k'^2 - k_y^2}} = d\theta \, , \quad (4.20)$$

with the result:

$$\phi(x',y') = \frac{\epsilon'}{i}\int_P e^{ik'(\cos\theta|x'|+\sin\theta y')}\,d\theta \,, \tag{4.21}$$

where P is the path in the θ -plane shown in Fig. 13.

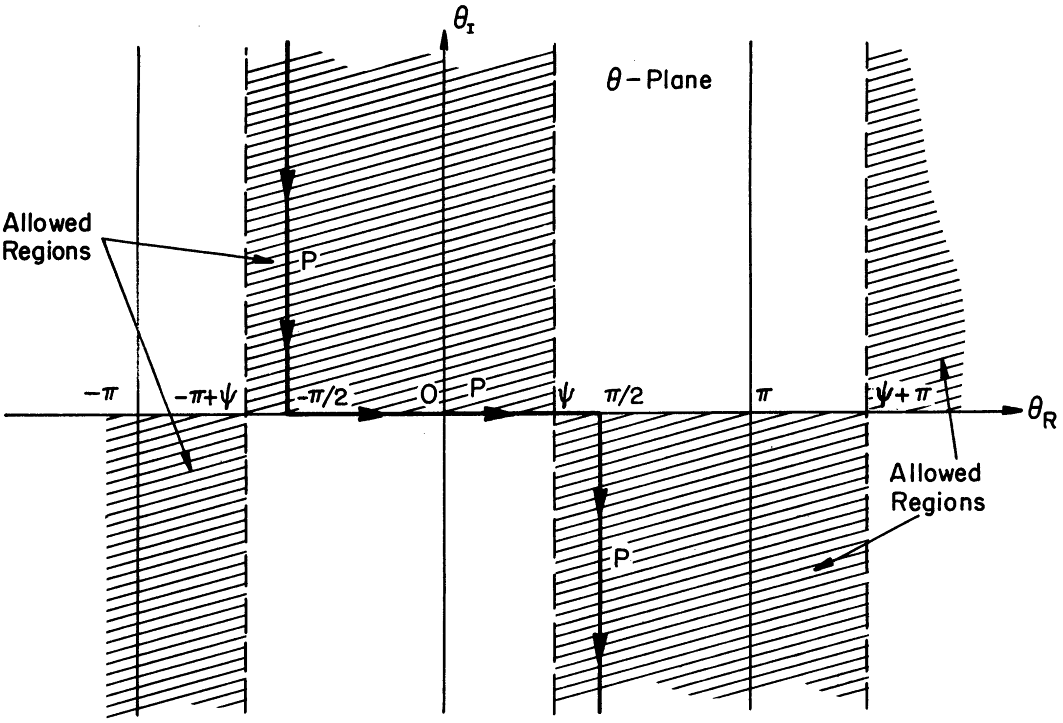

Fig. 13. Path of integration for $\phi(x',y')$; subsonic case.

Now let $|x'| = \rho'\cos\psi$ and $y' = \rho'\sin\psi$,

with $\rho' = \sqrt{x'^2+y'^2}$, and $-\pi/2 < \psi < \pi/2$.

Then integral (4.21) becomes:

$$\phi(x',y') = \frac{\epsilon'}{i}\int_P e^{ik'\rho'\cos(\theta-\psi)}\,d\theta. \tag{4.22}$$

We may translate the imaginary axis to the right by an amount ψ: $(\theta'=\theta-\psi)$ and find:

$$\phi(x',y') = \frac{\epsilon'}{i}\int_{P'} e^{ik'\rho'\cos\theta'}\,d\theta' \tag{4.23}$$

The integral (4.23) is Sommerfeld's integral for the Hankel function, and the path P' is the well-known path of integration.[23] Thus we find for ϕ :

$$\phi(x',y') = \frac{\epsilon'\pi}{i} H_0^{(1)}(k'\rho') \tag{4.24}$$

and the solution to our problem is given by:

$$\Phi(x,y,t) = \frac{\epsilon\pi}{i\sqrt{1-M_1^2}} H_0^{(1)}\left(\frac{k_1}{1-M_1^2}\sqrt{x^2+(1-M_1^2)y^2}\right) e^{-i\frac{M_1 k_1 x}{1-M_1^2}-i\omega t} \, . \qquad (4.25)$$

Notice that as $M_1 \to 0$ this solution gives:

$$\Phi(x,y,t) \longrightarrow \frac{\epsilon\pi}{i} H_0^{(1)}\left(k_1\sqrt{x^2+y^2}\right)e^{-i\omega t} \, , \qquad (4.26)$$

which is the well-known Green's function for an oscillating line source.[24] The solution (4.25) could have been found directly from the differential equation (4.6) by first applying a Galileo transformation to a frame of reference moving with the flow, and then a Lorentz transformation to put the source at rest. The solution to the resulting wave equation could be written down immediately, and is in fact identical with (4.25). We use the above method, though, for comparison with the supersonic case, which we cannot solve by the transformation method.

Let us examine the "far field" due to this source. We know the asymptotic behavior of $H_0^{(1)}(z)$, and thus of Φ :

$$\Phi(x,y,t) \longrightarrow \frac{\epsilon}{i}\sqrt{\frac{2\pi}{k_1\sqrt{x^2+(1-M_1^2)y^2}}}\; e^{i\frac{k_1\left(\sqrt{x^2+(1-M_1^2)y^2}-M_1x\right)}{1-M_1^2}-i\frac{\pi}{4}-i\omega t} \, , \qquad (4.27)$$

for

$$\sqrt{x^2+(1-M_1^2)y^2} \gg 1 \, .$$

The lines of constant phase are given by

$$\frac{k_1}{1-M_1^2}\left(\sqrt{x^2+(1-M_1^2)y^2} - M_1x\right) - \omega t - \pi/4 = \ell \, . \qquad (4.28)$$

Setting $t_1 = t + \frac{1}{\omega}(\ell + \frac{\pi}{4})$ this becomes

$$\sqrt{x^2+(1-M_1^2)y^2} = M_1 x + (1-M_1^2)c_1 t_1 \, , \qquad (4.29)$$

which is equivalent to:

$$(x - U_1 t_1)^2 + y^2 = c_1^2 t_1^2 \, . \qquad (4.30)$$

The lines of constant phase are therefore circles which are blown downstream with the flow. [See Fig. (3a), p. 6.] Thus the solution described by this method is entirely in accord with the results one would expect on physical grounds.

Although formula (4.25) represents a "closed form" expression for the cylindrical wave, it is not of a form appropriate, for reasons stated in the introduction, to the solution of the boundary value problem which interests us. A "Weyl-type" expression of (4.25) will therefore be developed.

The expression (4.21) represents the function φ, and hence Φ, as a superposition of plane waves of constant amplitude and wave number. It is for $x' < 0$ (the region of interest for us, since our subsonic source is to be located at $x_0 > 0$),

$$\varphi(x',y') = \frac{\epsilon'}{i}\int_P e^{ik'(-x'\cos\theta + y'\sin\theta)}\,d\theta, \tag{4.21}$$

which becomes in terms of x, y, t:

$$\Phi(x,y,t) = \frac{\epsilon}{i}\,\frac{e^{-i\omega t}}{\sqrt{1-M_1^2}}\int_P e^{ik_1\left(-\frac{M_1+\cos\theta}{1-M_1^2}x + \frac{\sin\theta}{\sqrt{1-M_1^2}}y\right)}\,d\theta\,. \tag{4.31}$$

However, these **plane** waves are different from those used in Section III. But the connection is established by means of the aberration relations:

$$-\frac{M_1+\cos\theta}{1-M_1^2} = \frac{\cos\theta_0}{1+M_1\cos\theta_0}\,; \qquad \frac{\sin\theta}{\sqrt{1-M_1^2}} = \frac{\sin\theta_0}{1+M_1\cos\theta_0}\,, \tag{4.32}$$

These equations may be solved to give:

$$\begin{aligned}\cos\theta &= -\frac{\cos\theta_0 + M_1}{1+M_1\cos\theta_0}\,; &\quad \sin\theta &= \frac{\sqrt{1-M_1^2}\,\sin\theta_0}{1+M_1\cos\theta_0}\,;\\ \cos\theta_0 &= -\frac{\cos\theta + M_1}{1+M_1\cos\theta}\,; &\quad \sin\theta_0 &= \frac{\sqrt{1-M_1^2}\,\sin\theta}{1+M_1\cos\theta}\,.\end{aligned} \tag{4.33}$$

The at first surpising occurrence of the relativistic aberration formulae is explained by the fact that the wave equation (4.4) is Lorentz invariant.

Differentiating the equation for sin θ in (4.33), we find:

$$d\theta = -\frac{\sqrt{1-M_1^2}\, d\theta_0}{1+M_1\cos\theta_0} \quad . \tag{4.34}$$

Expressions (4.31) and (4.34) may now be substituted into Eq. (4.31) to give the desired superposition:

$$\Phi(x,y,t) = -\frac{\epsilon}{i}\, e^{-i\omega t}\int_{P''} \frac{e^{ik_1(\alpha_0 x+\beta_0 y)}}{1+M_1\alpha_0}\, d\theta_0 \quad , \tag{4.35}$$

where $\alpha_0 = \cos\theta_0$, $\beta_0 = \sin\theta_0$ and the path P'' is the image of P under the transformations (4.33). The mapping of the path $P \rightarrow P''$ is illustrated in detail in Fig. 14. The appearance of two essential singularities on the lines $\theta_{0R} = n\pi$ may be noted.

We may find the pressure and velocity of the flow by differentiating (4.35) as in (4.2). Comparing the results of this differentiation with Eq. (3.10), we see that we may interpret the integral as a superposition of plane sound waves, each of amplitude

$$\epsilon^*(\theta_0) = -\frac{\epsilon k_1}{(1+M_1\alpha_0)^2} \quad . \tag{4.36}$$

The integral (4.35) will serve as the fundamental expression in the diffraction problem of Section V.

C. LINE SOURCE IN A SUPERSONIC GAS

Let us now consider the more complicated case of an oscillating line source in a supersonic flow. The fundamental equation (4.5) is for this case:

$$\nabla^2\Phi - \frac{1}{C_0^2}\left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)^2\Phi = 4\pi\epsilon\,\delta(x)\,\delta(y)\,e^{-i\omega t} \tag{4.37}$$

wherein we have placed the source at (0,0). If we make the transformation analogous to (4.7):

$$\Phi(x,y,t) = \varphi(x,y)\, e^{i\frac{M_0 k_0 x}{M_0^2-1} - i\omega t} \quad , \tag{4.38}$$

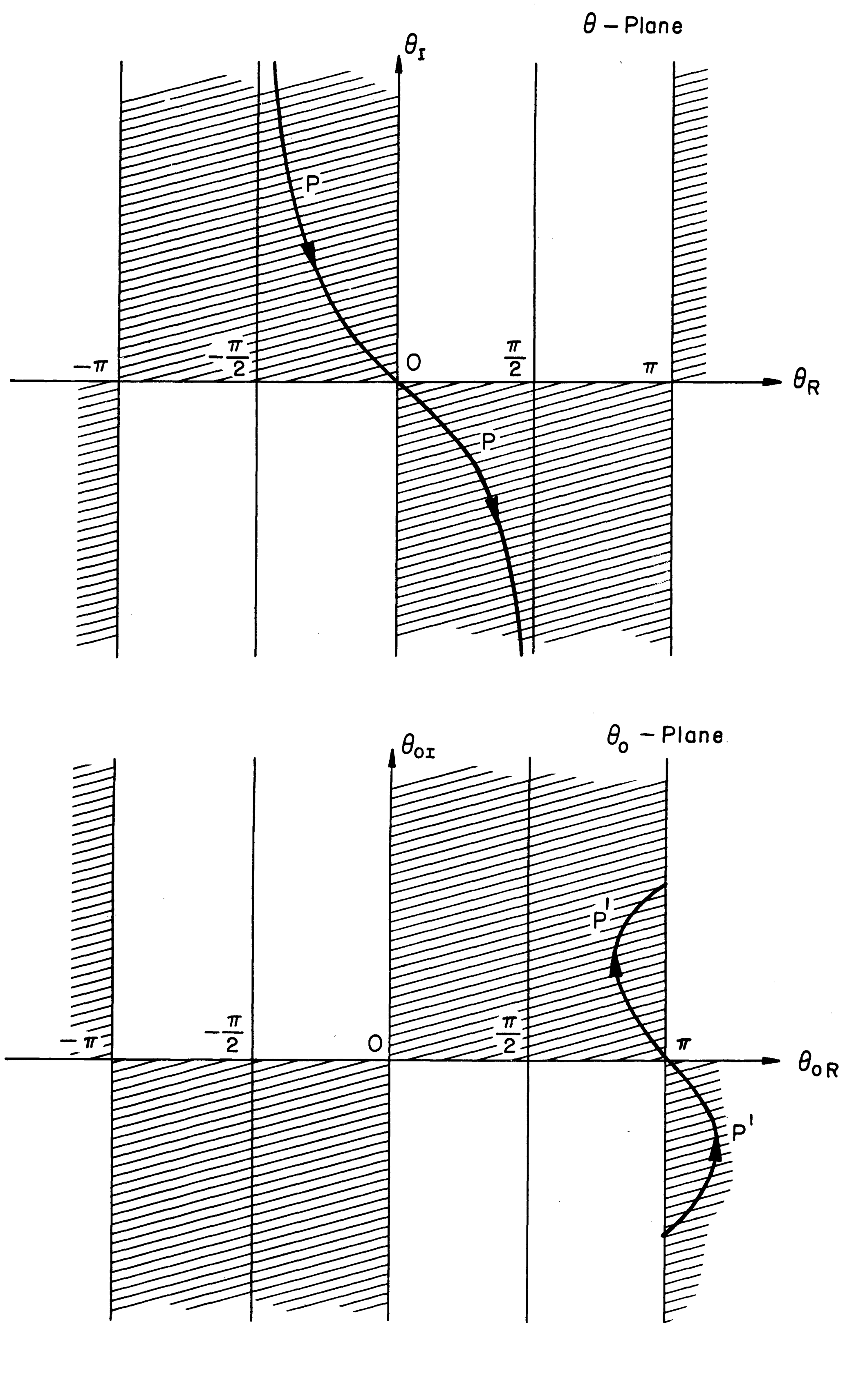

Fig. 14. Mapping of Θ to Θ_0; subsonic case.

then Eq. (4.37) becomes:

$$(M_0^2-1)\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial^2\varphi}{\partial y^2}+\frac{k_0^2}{M_0^2-1}\varphi=-4\pi\epsilon\,\delta(x)\delta(y) \quad . \tag{4.39}$$

We now simplify Eq. (4.39) by means of

$$x'=\frac{x}{\sqrt{M_0^2-1}}\ ,\ y'=y\ ,\ k'=\frac{k}{\sqrt{M_0^2-1}}\ ,\ \epsilon'=\frac{\epsilon}{\sqrt{M_0^2-1}}\ , \tag{4.40}$$

and find the analogue of the Helmholtz equation:

$$\frac{\partial^2\varphi}{\partial x'^2}-\frac{\partial^2\varphi}{\partial y'^2}+k'^2\varphi=-4\pi\epsilon'\,\delta(x')\,\delta(y') \ . \tag{4.41}$$

Let us seek solutions of (4.41) which may be expanded into a Fourier Integral as in the subsonic case:

$$\varphi(x',y')=\iint_{-\infty}^{+\infty}\Psi(k_x,k_y)\,e^{i(k_xx'+k_yy')}\,dk_x\,dk_y\,. \tag{4.42}$$

With the aid of (4.12) we may write Eq. (4.42) as:

$$\iint_{-\infty}^{+\infty}\left[(k_x^2-k_y^2-k'^2)\Psi-\frac{\epsilon'}{\pi}\right]e^{i(k_xx'+k_yy')}\,dk_x\,dk_y=0\ , \tag{4.43}$$

This equation may be satisfied by choosing:

$$\Psi(k_x,k_y)=\frac{\epsilon'}{\pi(k_x^2-k_y^2-k'^2)}\ , \tag{4.44}$$

whereupon the integral (4.41) becomes:

$$\varphi(x',y')=\frac{\epsilon'}{\pi}\iint_{-\infty}^{+\infty}\frac{e^{i(k_xx'+k_yy')}}{k_x^2-k_y^2-k'^2}\,dk_x\,dk_y\ . \tag{4.45}$$

This integral again is formally the solution to the problem. As in the subsonic case, it is possible to reduce (4.45) to a cylinder function.

Let us define: $$\chi(x')=\int_{-\infty}^{+\infty}\frac{e^{ik_xx'}\,dk_x}{(k_x-K)(k_x+K)}\ , \tag{4.46}$$

where $K=\sqrt{k'^2+k_y^2}$; then:

$$\varphi(x', y') = \int_{-\infty}^{+\infty} e^{ik_y y'} \chi(x')\, dk_y . \tag{4.47}$$

When choosing a path of integration, we cannot apply the reasoning of Part B, since $M_0 > 1$. We are thus forced to impose some other physical condition on the problem to replace the Sommerfeld radiation condition. We shall require instead that there be no radiation upstream of the source. This condition may be satisfied by requiring that $\chi(x') = 0$ for $x' < 0$. Referring to Fig. 15, we see that if this condition is to be satisfied, we must choose the path of integration as shown.

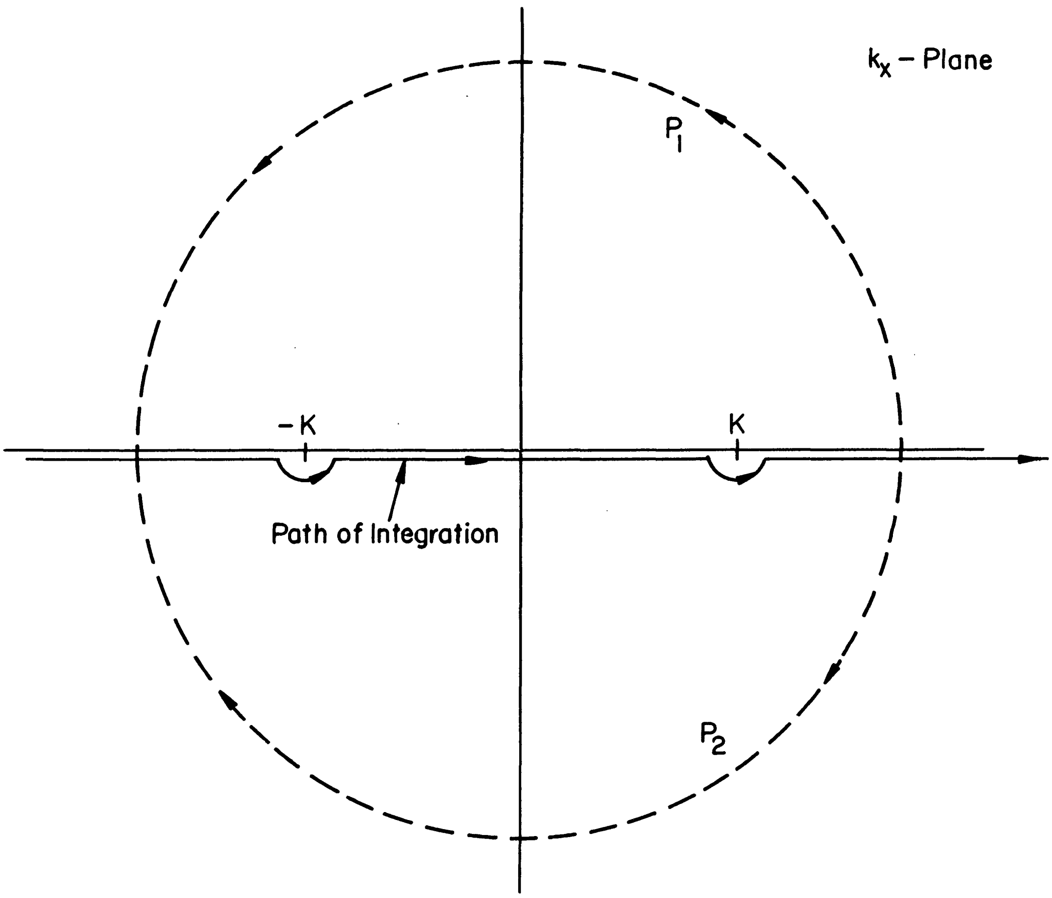

Fig. 15. Path of integration for $\chi(x')$; supersonic case.

The integral (4.46) may then be converted into a contour integral precisely as in the subsonic case. The path for $x' < 0$ encloses no poles, and hence our radiation condition is satisfied, as:

$$\chi(x') = \sum_{\text{Poles } K_i} \text{Res } K_i$$
$$= \begin{cases} 2\pi i\,(\text{Res } K + \text{Res} - K) & , \ x' > 0 \\ 0 & , \ x' < 0 \ , \end{cases}$$

or

$$\chi(x') = \begin{cases} \pi i \left[\dfrac{e^{iKx'}}{K} - \dfrac{e^{-iKx'}}{K} \right] & , \ x' > 0 \\ 0 & , \ x' < 0 \ . \end{cases} \tag{4.48}$$

Thus the integral for $\varphi(x', y')$, (4.47), becomes:

$$\varphi(x', y') = \epsilon' i \int_{-\infty}^{+\infty} \left\{ \frac{e^{i(\sqrt{k'^2 + k_y^2}\,x' + k_y y')}}{\sqrt{k'^2 + k_y^2}} - \frac{e^{i(-\sqrt{k'^2 + k_y^2}\,x' + k_y y')}}{\sqrt{k'^2 + k_y^2}} \right\} dk_y , \tag{4.49}$$

for $x' > 0$, and

$\varphi(x', y') = 0$ for $x' < 0$.

We may reduce this integral to Sommerfeld integrals for the two Hankel functions. To do this, let:

$$k_y = i k' \sin\theta \quad , \quad dk_y = i k' \cos\theta \, d\theta ,$$
$$\sqrt{k'^2 + k_y^2} = k' \cos\theta \quad , \quad \frac{dk_y}{\sqrt{k'^2 + k_y^2}} = i \, d\theta , \tag{4.50}$$

in the first term in (4.49), and in the second let:

$$k_y = i k' \sin\theta \quad , \quad dk_y = i k' \cos\theta \, d\theta ,$$
$$\sqrt{k'^2 + k_y^2} = - k' \cos\theta \quad , \quad \frac{dk_y}{\sqrt{k'^2 + k_y^2}} = - i \, d\theta . \tag{4.51}$$

The integral for $\varphi(x', y')$, (4.49) becomes:

$$\varphi(x', y') = -\epsilon' \int_{P_1 + P_2} e^{ik'(\cos\theta\, x' + \sin\theta\, y')} \, d\theta , \tag{4.52}$$

where P_1 and P_2 are the paths shown in Fig. 16. This integral will be seen to be represented by quite different mathematical expressions according as

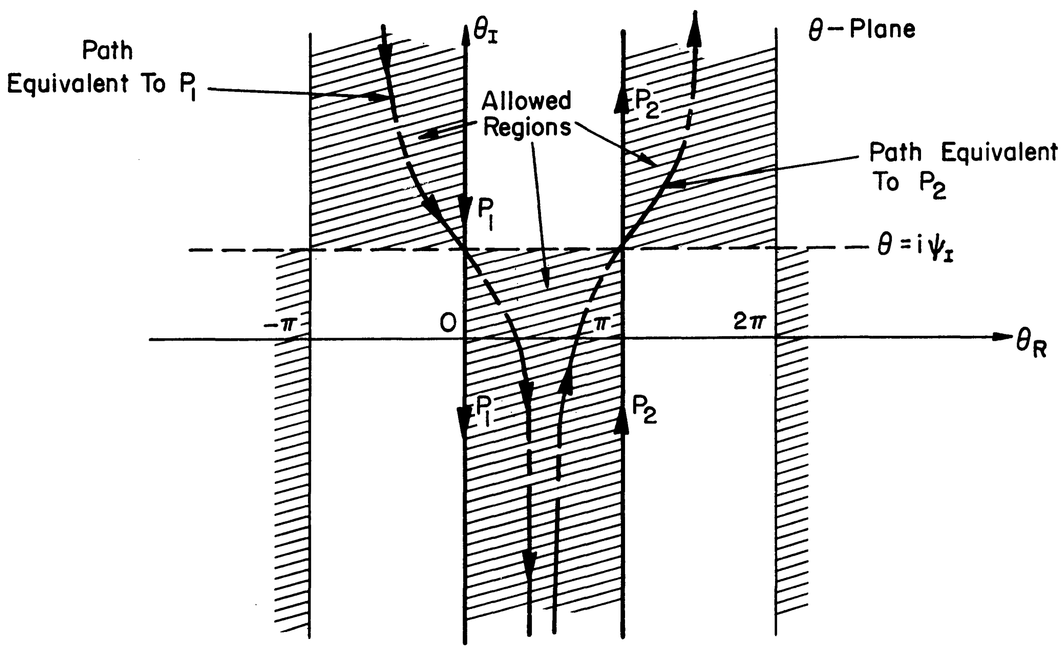

Fig. 16. Path of integration for $\phi(x',y')$; supersonic case, inside the Mach wedge.

$x' > |y'|$ inside the Mach wedge,

$x' < |y'|$ outside the Mach wedge.

Inside the Mach wedge we set $x' = \rho' \cos\psi$, $iy' = \rho' \sin\psi$, with $\rho' = \sqrt{x'^2 - y'^2}$ then we find $\psi_R = 0$ and $x' = \rho' \operatorname{ch}\psi_I$, $y' = \rho' \operatorname{sh}\psi_I$. This substitution gives $\psi_I = \operatorname{th}^{-1} y'/x'$, and thus we must have $|y'| < x'$. Hence we find for $x' > |y'|$:

$$\phi(x',y') = -\epsilon' \int_{P_1+P_2} e^{ik'\rho'\cos(\theta-\psi)}\, d\theta \quad , \tag{4.53}$$

If we let $\theta' = \theta - \psi$, the entire plane is translated along the imaginary axis, and we find:

$$\phi(x',y') = -\epsilon' \int_{P_1+P_2} e^{ik'\rho'\cos\theta'}\, d\theta' \quad . \tag{5.54}$$

The paths P_1 and P_2 may be distorted into the allowed region as is shown in

Fig. 16 to give the standard Sommerfeld integrals:

Inside the Mach wedge, $x' > |y'|$,

$$\varphi(x',y') = -\epsilon'\pi\left\{H_0^{(1)}(k'\rho') + H_0^{(2)}(k'\rho')\right\}$$

$$= -2\epsilon'\pi\, J_0(k'\rho') \quad . \tag{4.55}$$

On the other hand, outside the Mach wedge, $y' > x' > 0$,

we let $\quad -ix' = \rho'\cos\psi \;,\; y' = \rho'\sin\psi$,

where $\rho' = \sqrt{y'^2 - x'^2}$. Setting $\psi_R = \pi/2$ we find $x' = \rho'\,\mathrm{sh}\,\psi_I$, $y' = \rho'\,\mathrm{ch}\,\psi_I$. Thus $\psi_I = \mathrm{th}^{-1}\frac{x'}{y'}$, and:

$$\varphi(x',y') = -\epsilon'\int_{P_1+P_2} e^{i(ik'\rho')\cos(\theta-\psi)}\,d\theta \tag{4.56}$$

These paths are illustrated in Fig. 17.

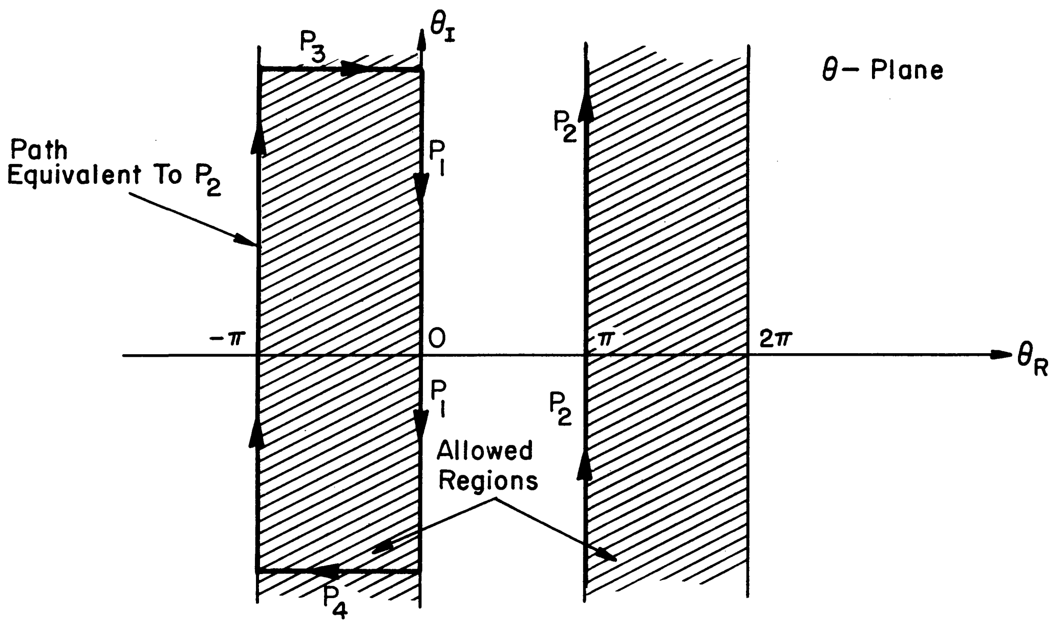

Fig. 17. Path of integration for $\phi(x',y')$; supersonic case, outside the Mach wedge.

We may replace the path P_2 by the equivalent path shown in Fig. 17, and close the paths with the segments P_3 and P_4. The integrals over P_3 and $P_4 \longrightarrow 0$ as the segments $\longrightarrow \pm i\infty$, and since the integrand has no poles in the allowed region, we conclude that:

For $y' > x' > 0$,

$$\varphi(x', y') = 0 \quad .$$

Similarly, for $y' < -x' < 0$,

$$\varphi(x', y') = 0 \quad .$$

Thus we find the solution to our problem:

$$\Phi(x,y,t) = \begin{cases} -\dfrac{2\pi\epsilon}{\sqrt{M_0^2-1}} J_0\left(\dfrac{k}{M_0^2-1}\sqrt{x^2-(M_0^2-1)y^2}\right) e^{i\frac{M_0 k_0 x}{M_0^2-1} - i\omega t} , & x > \sqrt{M_0^2-1}\,|y| \\ 0 \ , \quad \text{otherwise} \ . & \end{cases} \tag{4.57}$$

The somewhat surprising occurrence of the Bessel function was noted in the introduction. The calculation of the surfaces of constant plane follows. When the argument of the Bessel function is large, we may again carry out an asymptotic expansion, the first term being:

$$J_0(z) \longrightarrow \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right) ,$$

which gives the asymptotic formula for Φ :

$$\Phi(x,y,t) \longrightarrow -2\epsilon\sqrt{\frac{2\pi}{k_0\sqrt{x^2-(M_0^2-1)y^2}}} \cos\left(\frac{k_0}{M_0^2-1}\sqrt{x^2-(M_0^2-1)y^2} - \frac{\pi}{4}\right)$$

$$\times\, e^{i\frac{k_0 M_0 x}{M_0^2-1} - i\omega t}$$

Here we have a wave which is composed to two components. There are two lines of constant phase to examine. Let them be denoted by:

$$\ell_1 = \frac{k_0}{M_0^2-1}\left(\sqrt{x^2-(M_0^2-1)y^2} + M_0 x\right) - \frac{\pi}{4} - \omega t \quad ; \tag{4.58}$$

$$\ell_2 = -\frac{k_0}{M_0^2-1}\left(\sqrt{x^2-(M_0^2-1)y^2} - M_0 x\right) + \frac{\pi}{4} - \omega t \quad . \tag{4.59}$$

Now let:

$$t_1 = t + \frac{(\ell_1 + \frac{\pi}{4})}{\omega} \qquad \text{and} \qquad t_2 = t + \frac{(\ell_2 - \frac{\pi}{4})}{\omega} \;;$$

then

$$\sqrt{x^2 - (M_0^2 - 1)y^2} = \left[(M_0^2-1)c_0 t_1 - M_0 x\right], \tag{4.60}$$

and

$$\sqrt{x^2 - (M_0^2 - 1)y^2} = -\left[(M_0^2-1)c_0 t_2 - M_0 x\right]. \tag{4.61}$$

Squaring either we find:

$$(x - U_0 t_{1,2})^2 + y^2 = c_0^2 t_{1,2}^2 \tag{4.62}$$

Let us relate the phases ℓ_1 and ℓ_2 by $\ell_2 = \ell_1 + \frac{\pi}{2}$; then $t_1 = t_2$, and both waves correspond to the same circle. Thus each circle (4.62) is composed of two parts. The part corresponding to phase ℓ_2 leads the remaining part, ℓ_1 , by 90°. The expression $\sqrt{x^2 - (M_0^2-1)y^2} \geqslant 0$, the equality sign applying on the Mach wedge.

Thus for wave 1 $\quad (M_0^2-1)c_0 t_1 \geqslant M_0 x$, and

for wave 2 $\quad (M_0^2-1)c_0 t_1 \leq M_0 x$.

From this we may conclude that the part of the circle convex to the source is wave 1 and the part concave to the source is wave 2 (see Fig. 3b of page 6.

We may now inquire about the zeros of the cosine. These occur when

$$\ell_1 = \ell_2 + (2n-1)\pi \quad ,$$

or

$$\frac{k_0}{M_0^2-1}\sqrt{x^2 - (M_0^2-1)y^2} = \frac{4n-1}{4}\pi, \qquad n = 1, 2, 3, \cdots \tag{4.63}$$

The intersection of $\ell_2' = \ell_1 - \pi$ and $\ell_1' = \ell_1 - \frac{3\pi}{2}$, for example, corresponds to the zero $n=1$ of the cosine. This zero then moves with the flow as t increases, giving rise to an hyperbola (asymptotic to the Mach wedge) along which $\Phi = 0$. We have here an example of an interference phenomenon occurring for traveling waves.

Let us proceed now to the "Weyl-type" expansion of Φ in terms of the plane waves of Section III, Part A. Consider Eq. (4.52):

$$\varphi(x',y') = -\epsilon' \int_{P_1+P_2} e^{ik'(\cos\theta\, x' + \sin\theta\, i y')}\, d\theta \,, \quad x'>0, \tag{4.52}$$

from which it follows that:

$$\Phi(x,y,t) = -\frac{\epsilon\, e^{-i\omega t}}{\sqrt{M_0^2-1}} \int_{P_1+P_2} e^{ik\left(\frac{\cos\theta + M_0}{M_0^2-1} x + i \frac{\sin\theta}{\sqrt{M_0^2-1}} y\right)}\, d\theta. \tag{4.63}$$

Let

$$\frac{\cos\theta + M_0}{M_0^2-1} = \frac{\cos\theta_0}{1+M_0\cos\theta_0}, \qquad \frac{i\sin\theta}{\sqrt{M_0^2-1}} = \frac{\sin\theta_0}{1+M_0\cos\theta_0},$$

$$\cos\theta_0 = -\frac{\cos\theta + M_0}{1+M_0\cos\theta}, \qquad \sin\theta_0 = -i\,\frac{\sqrt{M_0^2-1}\,\sin\theta}{1+M_0\cos\theta};$$

$$\tag{4.64}$$

$$\cos\theta = -\frac{\cos\theta_0 + M_0}{1+M_0\cos\theta_0}, \qquad \sin\theta = i\,\frac{\sqrt{M_0^2-1}\,\sin\theta_0}{1+M_0\cos\theta_0};$$

These are the abberation formulae corresponding to a "supersonic" Lorentz transformation.

Differentiating the equation for $\sin\theta$ in (4.64), we find:

$$d\theta = \frac{-i\sqrt{M_0^2-1}}{1+M_0\cos\theta_0}\, d\theta_0 \tag{4.65}$$

We may now write our potential Φ as:

$$\Phi(x,y,t) = \frac{\epsilon}{i} \int_{P_1'+P_2'} \frac{e^{ik_0 \frac{(\alpha_0 x + \beta_0 y)}{1+M_0\alpha_0} - i\omega t}}{1+M_0\alpha_0}\, d\theta_0 \,, \tag{4.66}$$

where P_1' and P_2' are the images of P_1 and P_2 under the transformation (4.63), as shown in Fig. 18. By reference to (3.10), we see that we may interpret (4.66) as a superposition of plane waves of amplitude $\epsilon^*(\theta_0) = \dfrac{\epsilon k}{(1+M_0\alpha_0)^2}$.

The integral (4.66) will serve as the fundamental expression for the supersonic interaction problem of Section V.

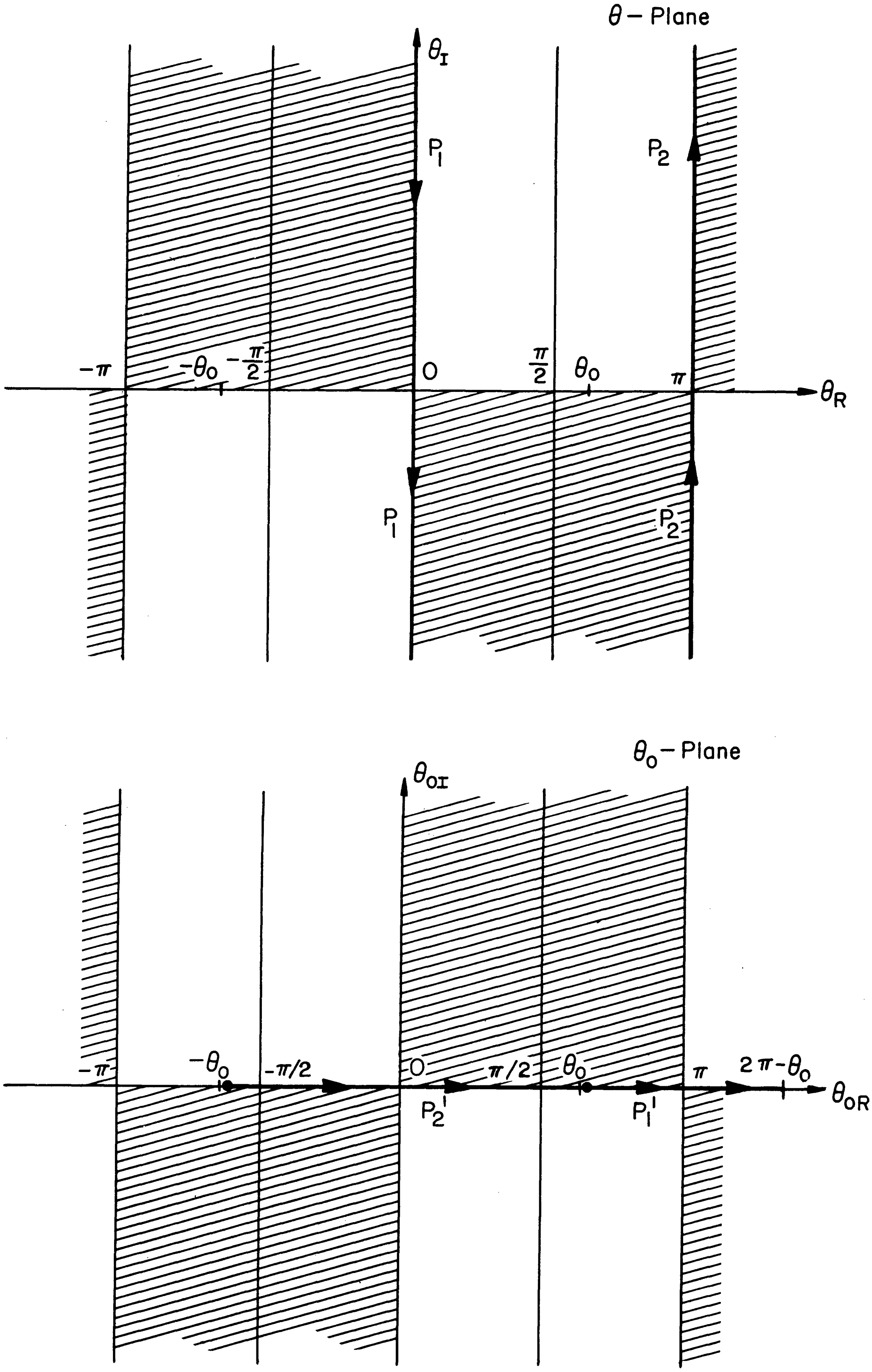

Fig. 18. Mapping of Θ to Θ_0 ; supersonic case.

V. THE INTERACTION OF CYLINDRICAL SOUND WAVES WITH A STATIONARY SHOCK WAVE

A. SUBSONIC INCIDENCE

1. The Interaction Integrals

Let us consider a cylindrical sound wave generated in the uniform subsonic flow to the right of our stationary shock (at $x_0 > 0$). In Part A of Section IV we found that such a sound wave can be decomposed into a weighted superposition of plane waves of the type considered in Section III. Since we have solved the interaction problem explicitly for each plane wave occurring in the superposition, we may thus obtain the solution of the interaction problem for incident cylindrical waves.

We shall write down the resulting integrals for the reflected waves below, and, in the remainder of Part A, evaluate these integrals approximately to find expressions for the far field of the reflected waves.

From Eq. (4.35) of Section IV we know that the incident wave potential from a source located at $x_0 > 0$ may be written as:

$$\Phi(x,y,t) = -\frac{\epsilon}{i}\int_{P''} \frac{e^{ik_1\frac{(\alpha_0(x-x_0)+\beta_0 y)}{1+M_1\alpha_0} - i\omega t}}{1+M_1\alpha_0}\, d\theta_0 \tag{5.1}$$

for $x_0 > x$, where P'' is the path shown in Fig. 14. Knowing Φ, we may calculate p_i, u_i, v_i, and ρ_i by (4.2). In particular:

$$p_i = \int_{P''} \epsilon^*(\theta_0)\, e^{ik_1\frac{(\alpha_0(x-x_0)+\beta_0 y)}{1+M_1\alpha_0} - i\omega t}\, d\theta_0 \tag{5.2}$$

where

$$\epsilon^*(\theta_0) = -\frac{\epsilon}{(1+M_1\alpha_0)^2} \quad .$$

The incident pressure wave is thus described as a superposition of plane-pressure waves, and likewise we may show that the incident velocities are described as superpositions of plane-velocity waves, the pressure and velocities all having the same amplitude (or weighting) factor, $\epsilon^*(\theta_0)$. An incident wave of the form:

$$
\begin{aligned}
p_i &= \epsilon^*(\theta_0)\, e^{ik_1 \frac{(\alpha_0(x-x_0)+\beta_0 y)}{1+M_1\alpha_0} - i\omega t} ; \\
u_i &= \alpha_0\, \epsilon^*(\theta_0)\, e^{ik_1 \frac{(\alpha_0(x-x_0)+\beta_0 y)}{1+M_1\alpha_0} - i\omega t} ; \\
v_i &= \beta_0\, \epsilon^*(\theta_0)\, e^{ik_1 \frac{(\alpha_0(x-x_0)+\beta_0 y)}{1+M_1\alpha_0} - i\omega t} ; \\
s_i &= 0 ,
\end{aligned}
\tag{5.3}
$$

gives rise to a reflected wave field:

$$
\begin{aligned}
p_r &= \left\{ A^*(\theta_0)\, e^{ik_1 \frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1}} \right\} e^{-i\frac{k_1\alpha_0 x_0}{1+M_1\alpha_0} - i\omega t} ; \\
u_r &= \left\{ \alpha_1 A^*(\theta_0)\, e^{ik_1 \frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1}} - \beta_2 B^*(\theta_0)\, e^{ik_1 \frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2}} \right\} e^{-i\frac{k_1\alpha_0 x_0}{1+M_1\alpha_0} - i\omega t} ; \\
v_r &= \left\{ \beta_1 A^*(\theta_0)\, e^{ik_1 \frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1}} + \alpha_2 B^*(\theta_0)\, e^{ik_1 \frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2}} \right\} e^{-i\frac{k_1\alpha_0 x_0}{1+M_1\alpha_0} - i\omega t} ; \\
s_r &= \left\{ C^*(\theta_0)\, e^{ik_1 \frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2}} \right\} e^{-i\frac{k_1\alpha_0 x_0}{1+M_1\alpha_0} - i\omega t} ,
\end{aligned}
\tag{5.4}
$$

and the corresponding distortion

$$
f(y,t) = a^*(\theta_0)\, e^{-ik_1 \frac{\alpha_0 x_0}{1+M_1\alpha_0} - i\omega t} . \tag{5.5}
$$

The functions $a^*(\theta_0)$, $A^*(\theta_0)$, $B^*(\theta_0)$, and $C^*(\theta_0)$ are the amplitudes $a, A, B,$ and C of Section III, Part B (and Appendix D) if ϵ is replaced by

$$
\epsilon^*(\theta_0) = -\frac{k_1 \epsilon}{(1+M_1\alpha_0)^2} .
$$

Thus:

$$a^*(\theta_0) = -\frac{k_1 a}{(1+M_1\alpha_0)^2} \quad ; \quad A^*(\theta_0) = -\frac{k_1 A}{(1+M_1\alpha_0)^2} \; ;$$

$$B^*(\theta_0) = -\frac{k_1 B}{(1+M_1\alpha_0)^2} \quad ; \quad C^*(\theta_0) = -\frac{k_1 C}{(1+M_1\alpha_0)^2} \; . \tag{5.6}$$

As in Section III, Part A, we introduce the scalar potential Φ_1 to describe the sound field, and the vector potential χ_1 to describe the vorticity wave. The sound and vorticity potentials are given by (3.12) and (3.9) of Section III, Part A:

$$\Phi_1 = A'(\theta_0)\, e^{ik_1\frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1} - i\frac{k_0\alpha_0 x_0}{1+M_1\alpha_0} - i\omega t} \; ; \tag{5.7}$$

$$\chi_1 = B'(\theta_0)\, e^{ik_1\frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2} - i\frac{k_0\alpha_0 x_0}{1+M_1\alpha_0} - i\omega t} \; ,$$

where

$$A' = -i\frac{(1+M_1\alpha_1)}{k_1} A^* = i\frac{(1+M_1\alpha_1)}{(1+M_1\alpha_0)^2} A \; , \tag{5.8}$$

$$B' = i\frac{M_1\alpha_2}{k_1} B^* = -i\frac{M_1\alpha_2}{(1+M_1\alpha_0)^2} B \; .$$

The entire reflected field may be described in terms of these two potentials and the entropy function. Thus the solution to the reflection problem is given by the four integrals:

$$\Phi_1 = i\int_{p11} \frac{(1+M_1\alpha_1)}{(1+M_1\alpha_0)^2} A\, e^{-i\frac{k_1\alpha_0 x_0}{1+M_1\alpha_0} + ik_1\frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1} - i\omega t}\, d\theta_0 \; ;$$

$$\chi_1 = -i\int_{p11} \frac{M_1\alpha_2}{(1+M_1\alpha_0)^2} B\, e^{-i\frac{k_1\alpha_0 x_0}{1+M_1\alpha_0} + ik_1\frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2} - i\omega t}\, d\theta_0 \; ;$$

$$s_1 = -k_1\int_{p11} \frac{C}{(1+M_1\alpha_0)^2} e^{-i\frac{k_1\alpha_0 x_0}{1+M_1\alpha_0} + ik_1\frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2} - i\omega t}\, d\theta_0 \; ; \tag{5.9}$$

$$f(y,t) = -k_1\int_{p11} \frac{a}{(1+M_1\alpha_0)^2} e^{ik_1\frac{(-\alpha_0 x_0+\beta_0 y_0)}{1+M_1\alpha_0} - i\omega t}\, d\theta_0 \; ,$$

where P'' is the path in the θ_0 plane described in Section IV, Part A (Fig. 14), and A, B, C, and a are the "Fresnel" coefficients for reflection calculated in Section III (and Appendix D).

These integrals give the exact behavior of the reflected sound wave. In the remainder of this section we shall study the approximate behavior of these integrals in detail.

2. The Shape Function: $f(y,t)$.

The shape of the shock wave was just shown to be:

$$f(y,t) = i \int_{P''} \frac{(i k_1 a)}{(1+M_1 \alpha_0)^2} e^{i k_1 \frac{(-\alpha_0 x_0 + \beta_0 y)}{1+M_1 \alpha_0} - i\omega t} d\theta_0 \tag{5.10}$$

where a is the amplitude of the shock distortion for plane waves, given in Section III, Part B.

To simplify this integral, and thus to facilitate the approximations, we transform back to the θ-plane of Section IV, by means of the formulae (4.33). From (4.34) we find:

$$\frac{d\theta_0}{1+M_1\alpha_0} = -\frac{d\theta}{\sqrt{1-M_1^2}} . \tag{5.11}$$

Thus in the θ-plane the integral for the shape is found to be:

$$f(y,t) = \frac{e^{i\frac{k_1 M_1 x_0}{1-M_1^2} - i\omega t}}{i(1-M_1^2)^{3/2}} \int_P (ika)(1+M_1\alpha) e^{ik'(\alpha x_0' + \beta y')} d\theta , \tag{5.12}$$

where P is the standard path for $H_0^{(1)}$, shown in Fig. 14 of Section IV, and

$$x_0' = \frac{x_0}{\sqrt{1-M_1^2}} , \quad y' = y , \quad k' = \frac{k}{\sqrt{1-M_1^2}} ,$$

as in Section III, Part B. In Appendix D we show that the expression for (ik_1a) is:

$$i k_1 a = \frac{2\mu M_1 M_0^2 \alpha}{(\mu - 1)\{M_0^2 \alpha^2 + 2 M_1 M_0^2 \alpha + 1\}} \epsilon , \qquad \text{(D.11)}$$

The formula for $f(y,t)$ in the Θ-plane reduces to:

$$f(y,t) = \frac{2\mu M_1 M_0^2 \, e^{i \frac{k_1 M_1 x_0}{1 - M_1^2} - i\omega t}}{i \, (\mu - 1) \, (1 - M_1^2)^{3/2}} \epsilon \, F(y) , \qquad (5.13)$$

where

$$F(y') = \int_P \frac{\alpha (1 + M_1 \alpha) \, e^{i k' (\alpha x_0' + \beta y')}}{M_0^2 \alpha^2 + 2 M_1 M_0^2 \alpha + 1} \, d\theta , \qquad (5.14)$$

and P is the path illustrated in Fig. 19.

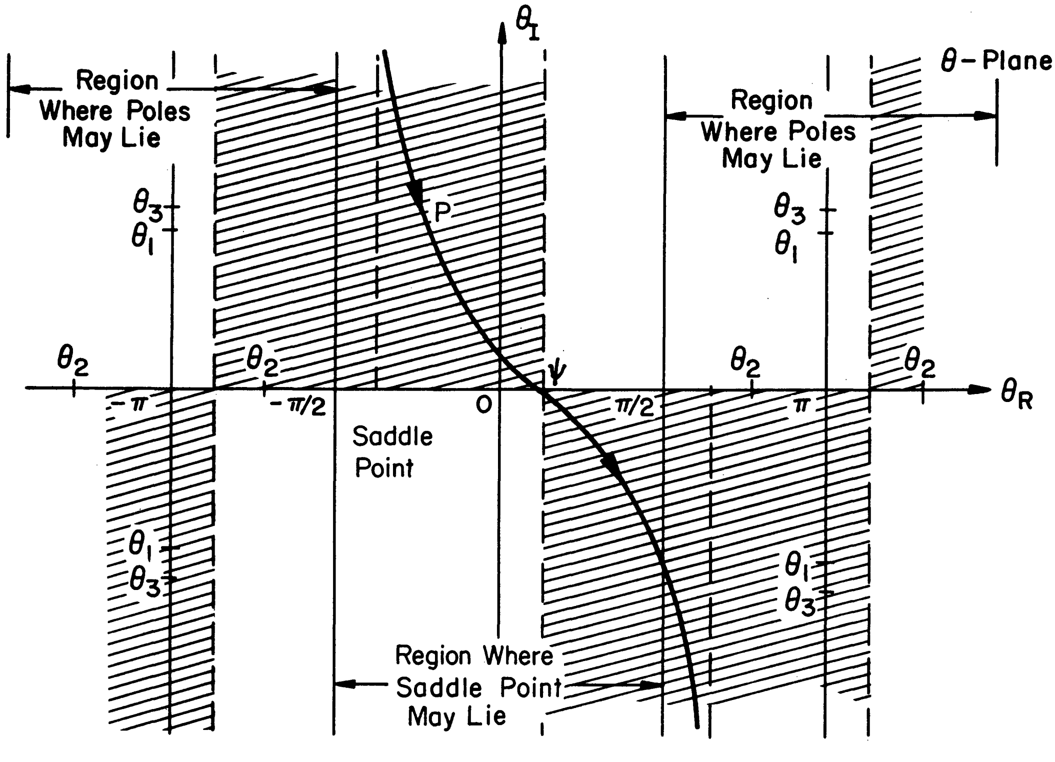

Fig. 19. Path of integration for F(y'); subsonic case.

We now obtain the first term in the asymptotic series for $F(y')$, in terms of $k' \sqrt{x_0'^2 + y'^2}$, by means of the saddle-point method. The saddle point of the integrand is located at:

$$\frac{d}{d\theta}(\alpha x_0' + \beta y') = -\beta x_0' + \alpha y' = 0 . \qquad (5.15)$$

Thus:

$$\alpha_{SP} = \cos\psi = \frac{x_0'}{\rho'} \quad , \quad \beta_{SP} = \sin\psi = \frac{y'}{\rho'} \quad , \tag{5.16}$$

with $\rho' = \sqrt{x_0'^2 + y'^2}$ and $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$. Near the saddle point the exponent becomes

$$ik'(\alpha x_0' + \beta y') = ik'\rho'\left(1 - \frac{(\theta-\psi)^2}{2} + ----\right) \quad , \tag{5.17}$$

which indicates that this is an ordinary saddle point.

Let us now examine the poles of the integrand. These poles occur at:

$$M_0^2 \alpha^2 + 2 M_1 M_0^2 \alpha + 1 = 0 \, ,$$

or

$$\alpha_{1,2} = - M_1 \pm \sqrt{M_1^2 - \frac{1}{M_0^2}} \; . \tag{5.18}$$

The values of $\alpha_{1,2}$ are always real, since $M_1^2 M_0^2 > 1$; and negative, since $-2 < -2M_1 < \alpha_{1,2} < 0$. Thus the poles lie outside the strip $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and are isolated from the saddle point.* We may therefore apply the ordinary saddle-point formula,

$$\int_P f(x)\, e^{i\lambda g(x)}\, dx \sim \sqrt{\frac{2\pi}{\lambda g''(x_0)}}\, f(x_0)\, e^{i(\lambda g(x_0) + \frac{\pi}{4})} \quad , \tag{5.19}$$

to find

$$F(y) \sim \sqrt{\frac{2\pi(1-M_1^2)}{k_1 \rho}} \; \frac{x_0\,(\rho + M_1 x_0)}{M_0^2 x_0^2 + 2 M_1 M_0^2 x_0 \rho + \rho^2}\, e^{i\frac{k_1\rho}{1-M_1^2} - i\frac{\pi}{4}} \quad , \tag{5.20}$$

where

$$\rho = \sqrt{x_0^2 + (1-M_1^2) y^2} \; .$$

---

*If this were not so, we would have the possibility of a pole being near the saddle point and thus the additional complications, discussed by Ott and Van der Waerden, referred to in the introduction.

The asymptotic formula for $f(y,t)$ is found by using the asymptotic expression for $F(y)$, (5.20), in the formula for $f(y,t)$, (5.13).

Let us now discuss briefly a few conclusions which may be drawn from the asymptotic formula (5.20). We see that the points of constant phase is given by:

$$(\rho + M_1 x_0) - (1-M_1^2) c_1 t = \ell \; . \tag{5.21}$$

Differentiating this expression we find that the velocity of these points is given by:

$$\frac{dy}{dt} = \frac{c_1}{2} \frac{\sqrt{x_0^2 + (1-M_1^2) y^2}}{y} \; . \tag{5.22}$$

On the axis $y = 0$, we have $\rho = x_0$, and the distortion formula reduces to:

$$f(0,t) = \sqrt{\frac{2\pi}{k_1 x_0}} \, \frac{2\mu M_1 M_0^2 \epsilon}{(\mu-1)(1-M_1)(M_0^2+2M_1M_0^2+1)} \, e^{i\frac{k_1 x_0}{1-M_1^2} - i\frac{3\pi}{4} - i\omega t} \, , \tag{5.23}$$

Near the axis, $\rho \sim x_0 \left(1 + \frac{1}{2}(1-M_1^2) y^2/x_0^2\right)$, the formula for $f(y,t)$ becomes:

$$f(y,t) = \left(1 + i\, \frac{k_1 x_0}{2} \frac{y^2}{x_0^2}\right) f(0,t) \; . \tag{5.24}$$

We may also obtain an expression for $f(y,t)$ when $y \gg x_0$:

$$f(y,t) = \frac{2\mu M_1 M_0^2 x_0 (2\pi)^{1/2} \epsilon}{(\mu-1)(1-M_1^2)^{1/4} y^{3/2} k_1^{1/2}} \, e^{i k_1 \frac{y}{\sqrt{1-M_1^2}} - i\frac{3\pi}{4} - i\omega t} \, . \tag{5.25}$$

This latter formula shows in particular that $|f(y,t)| \sim \frac{A}{y^{3/2}}$ as $y \to \infty$. The shape function is illustrated in Fig. 21.

3. The Entropy Function: $s_1(x,y,t)$.

The entropy function is given in formula (5.9) as:

$$\Delta_1(x,y,t) = -k_1 \int_{P''} \frac{C}{(1+M_1\alpha_0)^2} e^{-ik_1\frac{\alpha_0 x_0}{1+M_1\alpha_0} + ik_1\frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2} - i\omega t} d\theta_0 , \qquad (5.26)$$

where α_2 and β_2 are given by formula (3.26). We again transform from the θ_0-plane to the θ-plane of Section IV by means of the formulae (4.33), and find that α_2 and β_2 become:

$$\alpha_2 = \frac{\sqrt{1-M_1^2}}{1+M_1\alpha} , \qquad \beta_2 = \frac{M_1\beta}{1+M_1\alpha} . \qquad (5.27)$$

Thus in the θ-plane the integral for the entropy is:

$$\Delta_1(x,y,t) = \frac{k_1 e^{i\frac{k_1 M_1 x_0}{1-M_1^2} + i\frac{k_1}{M_1}(x-U_1 t)}}{(1-M_1^2)^{3/2}} \int_P C(\alpha)(1+M_1\alpha) e^{ik'(\alpha x_0' + \beta y')} d\theta . \qquad (5.28)$$

$C(\alpha)$ is shown in Appendix D to be:

$$C(\alpha) = \frac{4(1-M_1^2)(M_0^2-1)\alpha}{\mu\{M_0^2\alpha^2 + 2M_1 M_0^2\alpha + 1\}} \epsilon , \qquad (D.11)$$

and P is the path illustrated in Fig. 19. Thus we write the entropy integral as:

$$\Delta_1(x,y,t) = \frac{4k_1(M_0^2-1)\epsilon}{\mu(1-M_1^2)^{1/2}} e^{i\frac{k_1 M_1 x_0}{1-M_1^2} + i\frac{k_1}{M_1}(x-U_1 t)} F(y) , \qquad (5.29)$$

where $F(y)$ is precisely the same integral as occurred in (5.13) for the shape function. The asymptotic formulae for the entropy function is found by using the asymptotic expression for $F(y)$, (5.20), in Eq. (5.29).

Two facts are immediately apparent. First we see that the entropy function at (x, y, t) is proportional to the shape function at the same value of y but at time $t - \frac{x}{U_1}$. Thus the entropy field is given as the image of the shape blown downstream with the flow. Second, we see that the surfaces of constant phase are, choosing the constant to be $-\frac{\pi}{4}$:

$$\frac{k_1}{1-M_1^2}(\rho + M_1 x_0) + \frac{k_1}{M_1}(x - U_1 t) = 0, \qquad (5.30)$$

which may be written as:

$$\left(x + \frac{M_1^2}{1-M_1^2}x_0 - U_1 t\right)^2 - \frac{M_1^2}{1-M_1^2}y^2 = \frac{M_1^2 x_0^2}{(1-M_1^2)^2}. \qquad (5.31)$$

Thus the surfaces of constant entropy are hyperbolas, blown downstream with the flow. These surfaces are illustrated in Fig. 21.

4. The Vorticity Potential: $\chi_1(x,y,t)$

Let us now turn to the vorticity potential which is shown in (5.9) to be:

$$\chi_1(x,y,t) = -i\int_{P_{11}} \frac{M_1\alpha_2}{(1+M_1\alpha_0)^2} B\, e^{-ik_1\frac{\alpha_0 x_0}{1+M_1\alpha_0} + ik_1\frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t} d\theta_0. \qquad (5.32)$$

We again transform from the θ_0-plane to the θ-plane to simplify the integration. Using (5.27), we find:

$$\chi_1(x,y,t) = \frac{iM_1}{1-M_1^2}\left[e^{i\frac{k_1 M_1 x_0}{1-M_1^2} + i\frac{k_1}{M_1}(x-U_1 t)}\right]\int_P B(\alpha,\beta)\, e^{ik'(\alpha x_0' + \beta y')} d\theta, \qquad (5.33)$$

where

$$B(\alpha,\beta) = \frac{2\beta\left[\sqrt{1-M_1^2}(M_1^2+M_0^2)\alpha + M_1^2 M_0^2(1+M_1\alpha)\right]}{M_1\sqrt{1-M_1^2}(1+M_1\alpha)\left[M_0^2\alpha^2 + 2M_1 M_0^2\alpha + 1\right]}\epsilon, \qquad (D.11)$$

and P is the path illustrated in Fig. 19. Thus we write:

$$\chi_1(x,y,t) = \frac{2i\, e^{i\frac{k_1 M_1 x_0}{1-M_1^2} + i\frac{k_1}{M_1}(x-U_1 t)}}{(1-M_1^2)^{3/2}}\epsilon\, G(y), \qquad (5.34)$$

where

$$G(y') = \int_P \frac{\beta\left[\sqrt{1-M_1^2}(M_1^2+M_0^2) + M_1^2 M_0^2(1+M_1\alpha)\right]}{(1+M_1\alpha)\left[M_0^2\alpha^2 + 2M_1 M_0^2\alpha + 1\right]} e^{ik'(\alpha x_0' + \beta y')} d\theta. \qquad (5.35)$$

The saddle point of $G(y')$ is the same as that for $F(y')$; however, $G(y')$ has poles at $\alpha_3 = -\frac{1}{M_1}$ in addition to those of $F(y')$. These additional poles are located on the lines $\theta_R = (2n+1)\pi$ and thus are again isolated from the saddle point. We may therefore immediately write down the first term in the asymptotic series for $G(y')$ from the general saddle-point formula (5.19)

$$G(y) = \sqrt{\frac{2\pi}{k_1\rho}}\; \frac{(1-M_1^2)y\left[\sqrt{1-M_1^2}\,(M_1^2+M_0^2)x_0 + M_1^2M_0^2(\rho + M_1x_0)\right]}{\rho(\rho+M_1x_0)\left[M_0^2x_0^2 + 2M_1M_0^2x_0\rho + \rho^2\right]}\, e^{i\frac{k_1\rho}{1-M_1^2} - i\frac{\pi}{4}}, \qquad (5.36)$$

where

$$\rho = \sqrt{x_0^2 + (1-M_1^2)\,y^2} \quad .$$

We find the asymptotic formula for $\chi_1(x,y,t)$ by substituting (5.36) into (5.34). Formula (5.36) shows us that the vorticity vanishes on the axis $y = 0$, and that the lines of constant phase for the vorticity wave are the same as those for the entropy wave (see Fig. 21).

5. The Sound Potential: $\Phi_1(x,y,t)$

This is given in (5.9) as:

$$\Phi_1(x,y,t) = i\int_{P''} \frac{(1+M_1\alpha_1)}{(1+M_1\alpha_0)^2}\, A\, e^{-ik_1\frac{\alpha_0 x_0}{1+M_1\alpha_0} + ik_1\frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t}\, d\theta_0 \;, \qquad (5.37)$$

Let us again transform this integral to the θ-plane by means of Eqs. (4.33). Since α_1 and β_1 are given in terms of α_0 and β_0 by (3.22), we find in the θ-plane:

$$\alpha_1 = \frac{\alpha - M_1}{1 - M_1\alpha} \quad , \qquad \beta_1 = \frac{\sqrt{1-M_1^2}\,\beta}{1 - M_1\alpha} \;, \qquad (5.38)$$

Thus in the θ-plane the integral for the sound potential becomes:

$$\Phi_1(x,y,t) = \frac{e^{ik_1\frac{(x-x_0)M_1}{1-M_1^2}}}{i\sqrt{1-M_1^2}} \int_P \frac{(1+M_1\alpha)}{(1-M_1\alpha)}\, A(\alpha)\, e^{ik'(\alpha(x'+x_0') + \beta y')}\, d\theta, \qquad (5.39)$$

where:

$$A(\alpha) = -\frac{M_0^2\alpha^2 - 2M_1M_0^2\alpha + 1}{M_0^2\alpha^2 + 2M_1M_0^2\alpha + 1}\,\epsilon \tag{D.11}$$

and **P** is the path illustrated in Fig. 20.

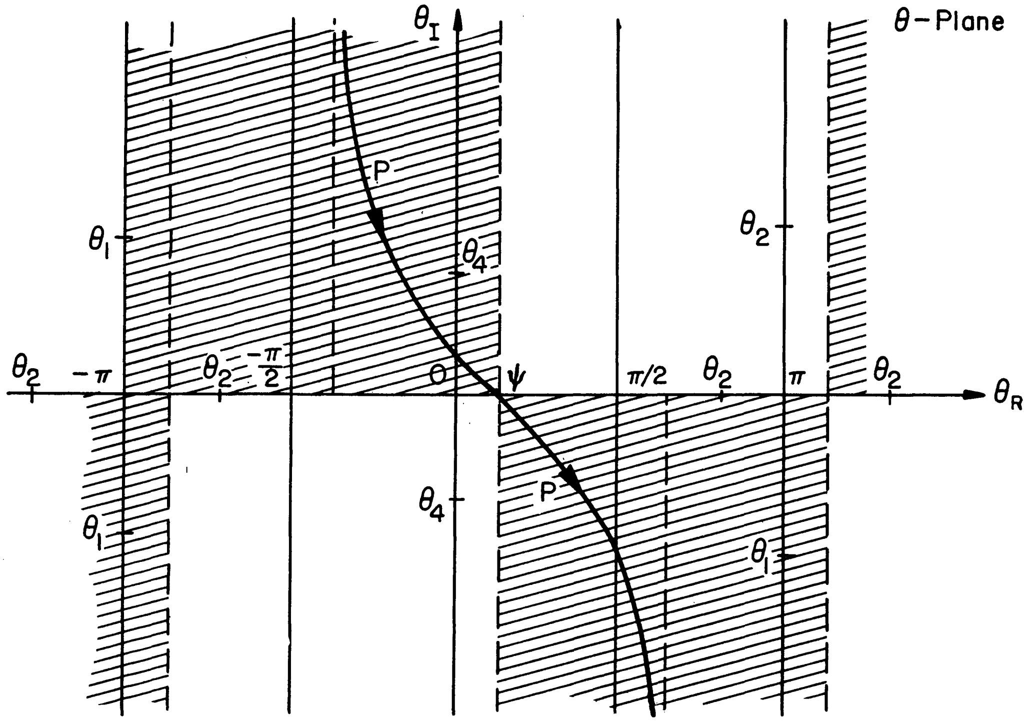

Fig. 20. Path of integration for $\phi_1(x', y')$; subsonic case.

The integral for the sound potential may thus be written:

$$\Phi_1(x,y,t) = -\frac{\epsilon}{i}\,\frac{e^{-ik_1\frac{(x-x_0)M_1}{1-M_1^2} - i\omega t}}{\sqrt{1-M_1^2}}\,\varphi_1(x,y) \quad , \tag{5.40}$$

where $\varphi_1(x,y)$ is given by:

$$\varphi_1(x',y') = \int_P \frac{(1+M_1\alpha)(M_0^2\alpha^2 - 2M_1M_0^2\alpha+1)}{(1-M_1\alpha)(M_0^2\alpha^2+2M_1M_0^2\alpha+1)}\, e^{ik'(\alpha(x'+x_0')+\beta y')}\, d\theta \,. \tag{5.41}$$

The saddle point of the integrand is at:

$$\alpha_{sp} = \cos\psi = \frac{x'+x_0'}{\rho_1'} \quad , \quad \beta_{sp} = \sin\psi = \frac{y'}{\rho_1'} \quad , \tag{5.42}$$

with $\rho_1' = \sqrt{(x'+x_0')^2 + y'^2}$ and $-\frac{\pi}{2} \le \psi \le \frac{\pi}{2}$.

The poles of the integrand are located at the points shown in Fig.

19, and in addition at $\alpha_4 = \frac{1}{M_1} > 1$. These latter poles occur on the imaginary axis, but even in this case are sufficiently distant from the saddle point.

Applying formula (5.19) to the integral (5.41) we find:

$$\varphi_1(x,y) = \sqrt{\frac{2\pi(1-M_1^2)}{k_1\,\rho_1}}\;\frac{[\rho_1 + M_1(x+x_0)][M_0^2(x+x_0)^2 - 2M_1M_0^2(x+x_0)\rho_1 + \rho_1^2]}{[\rho_1 - M_1(x+x_0)][M_0^2(x+x_0)^2 + 2M_1M_0^2(x+x_0)\rho_1 + \rho_1^2]}\;e^{i\frac{k_1\rho_1}{1-M_1^2} - i\frac{\pi}{4}}, \tag{5.43}$$

where

$$\rho_1 = \sqrt{(x+x_0)^2 + (1-M_1^2)y^2}\ .$$

The asymptotic formula for $\Phi_1(x,y,t)$ is found by substituting (5.43) into Eq. (5.40).

The lines of constant phase are given by (choosing the constant to be $-\pi/4$):

$$\rho_1 = M_1(x-x_0) + (1-M_1^2)\,c_1 t\ , \tag{5.44}$$

which reduces to:

$$\left(x + \frac{1+M_1^2}{1-M_1^2}x_0 - U_1 t\right)^2 + y^2 = \left(c_1 t - \frac{2M_1x_0}{1-M_1^2}\right)^2 . \tag{5.45}$$

These are circles moving downstream with the flow. The shape function, as well as the entropy, vorticity, and sound waves are illustrated in Fig. 21.

B. SUPERSONIC INCIDENCE

1. The Interaction Integrals

As in the preceding section, we shall develop the integrals which describe the refracted sound field. We have seen in Section IV, Part B, that the incident sound wave may be described as a superposition of plane sound waves of amplitude $\epsilon^*(\theta_0) = \frac{\epsilon k_0}{(1+M_0\alpha_0)^2}$.

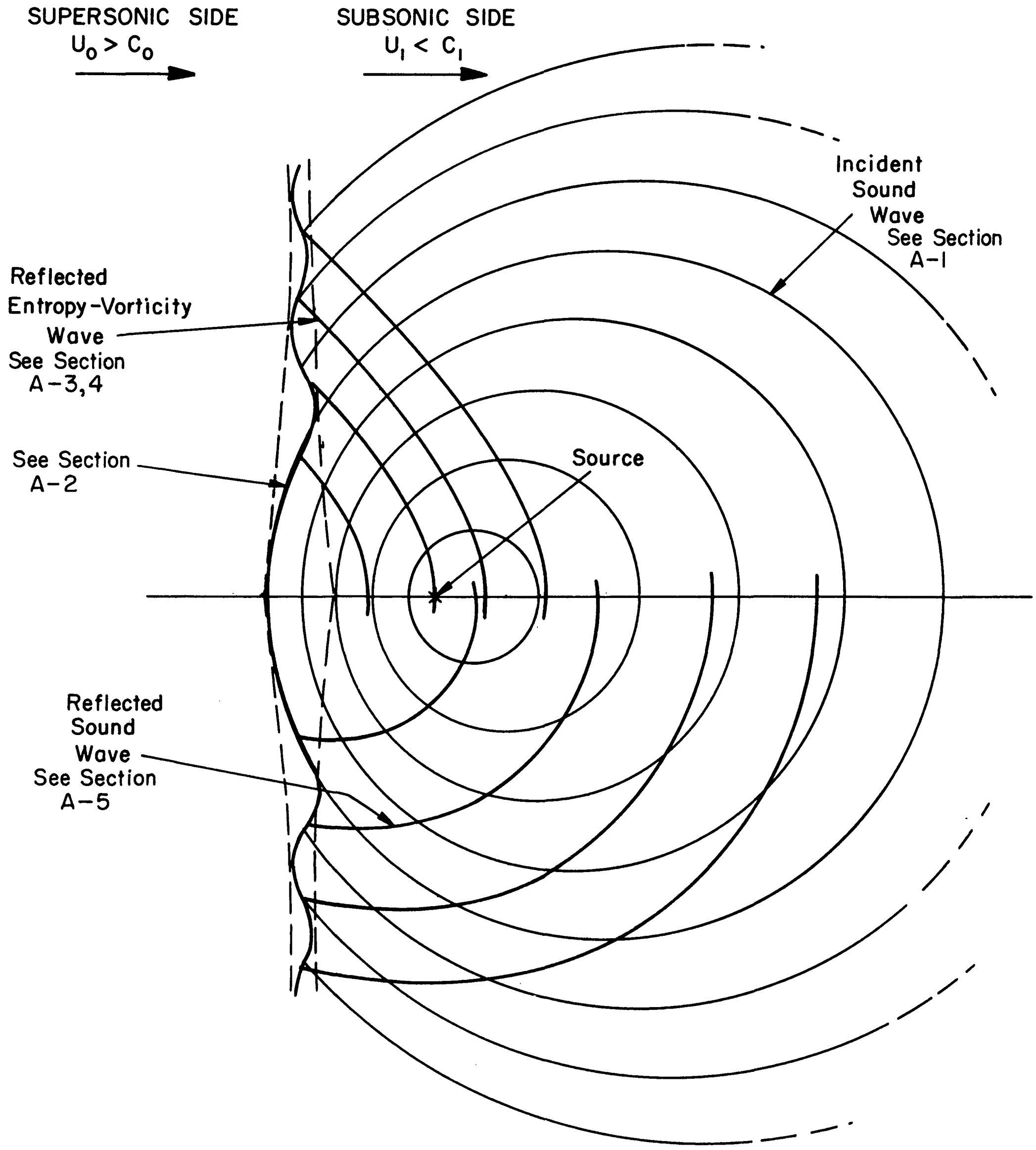

Fig. 21. Reflection of cylindrical waves.

Consider an incident plane sound wave:

$$p_i = \epsilon^* e^{i k_0 \frac{(\alpha_0 (x + x_0) + \beta_0 y)}{1 + M_0 \alpha_0} - i\omega t} \quad . \tag{5.46}$$

The resulting refracted wave will be described by:

$$\begin{aligned}
p_1 &= \left\{ A^* e^{i k_1 \frac{(\alpha_1 x + \beta_1 y)}{1 + M_1 \alpha_1}} \right\} e^{i k_0 \frac{\alpha_0 x_0}{1 + M_0 \alpha_0} - i\omega t} ; \\
u_1 &= \left\{ \alpha_1 A^* e^{i k_1 \frac{(\alpha_1 x + \beta_1 y)}{1 + M_1 \alpha_1}} - \beta_2 B^* e^{i k_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1 \alpha_2}} \right\} e^{i k_0 \frac{\alpha_0 x_0}{1 + M_0 \alpha_0} - i\omega t} ; \\
v_1 &= \left\{ \beta_1 A^* e^{i k_1 \frac{(\alpha_1 x + \beta_1 y)}{1 + M_1 \alpha_1}} + \alpha_2 B^* e^{i k_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1 \alpha_2}} \right\} e^{i k_0 \frac{\alpha_0 x_0}{1 + M_0 \alpha_0} - i\omega t} ; \\
s_1 &= \left\{ C^* e^{i k_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1 \alpha_2}} \right\} e^{i k_0 \frac{\alpha_0 x_0}{1 + M_0 \alpha_0} - i\omega t} ,
\end{aligned} \tag{5.47}$$

and the shape of the distortion will be:

$$f(y,t) = a^* e^{ik_0 \frac{\alpha_0 x_0}{1+M_0\alpha_0} + ik_0 \frac{\beta_0 y}{1+M_0\alpha_0} - i\omega t} , \tag{5.48}$$

where

$$A^* = \frac{k_0 A}{(1+M_0\alpha_0)^2} , \qquad B^* = \frac{k_0 B}{(1+M_0\alpha_0)^2} ,$$
$$a^* = \frac{k_0 a}{(1+M_0\alpha_0)^2} , \qquad C^* = \frac{k_0 C}{(1+M_0\alpha_0)^2} . \tag{5.49}$$

We shall again introduce the scalar and vector potentials. These potentials are given by:

$$\Phi_1 = A' e^{ik_0 \frac{\alpha_0 x_0}{1+M_0\alpha_0} + ik_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t} ,$$
$$\chi_1 = B' e^{ik_0 \frac{\alpha_0 x_0}{1+M_0\alpha_0} + ik_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t} , \tag{5.50}$$

where

$$A' = -i \frac{(1+M_1\alpha_1)}{k_1} A^* = -i\lambda \frac{(1+M_1\alpha_1)}{(1+M_0\alpha_0)^2} A ,$$
$$B' = i \frac{M_1\alpha_2}{k_1} B^* = i\lambda \frac{M_1\alpha_2}{(1+M_0\alpha_0)^2} B . \tag{5.51}$$

Thus the integrals describing the refracted sound field are:

$$\Phi_1 = -i\lambda \int_{P_1'+P_2'} \frac{(1+M_1\alpha_1)}{(1+M_0\alpha_0)^2} A e^{ik_0 \frac{\alpha_0 x_0}{1+M_0\alpha_0} + ik_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t} d\theta_0 ;$$

$$\chi_1 = i\lambda \int_{P_1'+P_2'} \frac{M_1\alpha_2}{(1+M_0\alpha_0)^2} B e^{ik_0 \frac{\alpha_0 x_0}{1+M_0\alpha_0} + ik_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t} d\theta_0 ;$$

$$\Delta_1 = k_0 \int_{P_1'+P_2'} \frac{C}{(1+M_0\alpha_0)^2} e^{ik_0 \frac{\alpha_0 x_0}{1+M_0\alpha_0} + ik_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t} d\theta_0 ; \tag{5.52}$$

$$f(y,t) = k_0 \int_{P_1'+P_2'} \frac{a}{(1+M_0\alpha_0)^2} e^{ik_0 \frac{(\alpha_0 x_0 + \beta_0 y)}{1+M_0\alpha_0} - i\omega t} d\theta_0 .$$

P_1' and P_2' are the paths illustrated in Fig. 18 in the θ_0 plane, and $A, B, C,$ and a are the "Fresnel" coefficients for refraction described in Section III, Part C, and Appendix E.

2. The Shape Function: $f(y,t)$

We have just seen in formula (5.52) that the shape of the distorted shock is given by:

$$f(y,t) = k_0 \int_{P_1'+P_2'} \frac{a}{(1+M_0\alpha_0)^2} e^{ik_0 \frac{(\alpha_0 x_0 + \beta_0 y)}{1+M_0\alpha_0} - i\omega t} \tag{5.53}$$

Let us again, for the convenience of calculation, transform this integral to the θ-plane of Section IV, Part B, by means of the supersonic aberration formulae (4.64).

The integral (5.53) becomes:

$$f(y,t) = \frac{e^{i\frac{k_0 M_0 x_0}{M_0^2-1} - i\omega t}}{(M_0^2-1)^{3/2}} \int_{P_1+P_2} (1+M_0\alpha_0)(ik_0 a) e^{i k_0 \frac{(\alpha x_0 + i\beta\sqrt{M_0^2-1}\, y)}{M_0^2-1}} d\theta, \tag{5.54}$$

where P_1 and P_2 are the paths in the θ-plane illustrated in Fig. 18. We may rewrite integral (5.54) as:

$$f(y,t) = \frac{e^{i\frac{k_0 M_0 x_0}{M_0^2-1} - i\omega t}}{(M_0^2-1)^{3/2}} F(y) , \tag{5.55}$$

where

$$F(y') = \int_{P_1+P_2} (1+M_0\alpha)(ik_0 a) e^{ik'(\alpha x_0' + i\beta y')} d\theta, \tag{5.56}$$

with $x_0' = \frac{x_0}{\sqrt{M_0^2-1}}$, $y' = y$, and $k' = \frac{k_0}{\sqrt{M_0^2-1}}$.

From Appendix E we find that:

$$(1+M_0\alpha)(ik_0 a) = \frac{\epsilon\left[\Gamma_3 - \Gamma_4\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}\right]}{(\mu-1)\left[\Gamma_1 + \Gamma_2\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}\right]} , \tag{E.7}$$

where the Γ's are polynomials in α of degree not higher than 2.

To make a saddle-point expansion of $F(y')$, we must first study the poles and branch points of the integrand. The branch points occur at:

$$(M_0^2 - 1) + (1 - M_1^2)\lambda^2\beta^2 = 0 \ .$$

Using $\quad (1 - M_1^2)\lambda^2 = \dfrac{[M_0^2 + (\mu - 1)]}{\mu M_0^2}[M_0^2 - 1] \ ,$

we find:

$$\beta^2 = -\frac{\mu M_0^2}{M_0^2 + (\mu - 1)} = -\frac{U_0}{U_1} < -1 \ ,$$

hence

$$\beta_\pm = \pm i\sqrt{\frac{\mu M_0^2}{M_0^2 + (\mu - 1)}} = \pm i\sqrt{\frac{U_0}{U_1}} \qquad (5.57)$$

The poles of the integrand are located at:

$$\Gamma_1^2 - \Gamma_2^2\left[(M_0^2 - 1) + (1 - M_1^2)\lambda^2\beta^2\right] = 0 \ ,$$

or

$$(M_0^2 + 1 + \lambda M_1 M_0 \beta^2)^2 = 4 M_1^2 M_0^4 \left(1 + \lambda \frac{M_1}{M_0^2}\beta^2\right) \ ,$$

which we may solve for β^2:

$$\beta_{1,2}^2 = \frac{2 M_1^2 M_0^2 - (M_0^2 + 1) \pm 2 M_1 M_0 \sqrt{M_1^2 M_0^2 - 1}}{\lambda M_1 M_0} \ . \qquad (5.58)$$

It is easy to verify that both of these poles belong to the negative branch of the square root.

$$\Gamma_1 - \Gamma_2\sqrt{(M_0^2 - 1) + (1 - M_1^2)\lambda^2\beta_{1,2}^2} = 0 \ .$$

a. Inside the Mach Wedge, $x_0' > |y'|$.—In Fig. 22 we illustrate the paths of integration together with the associated branch points and branch lines for $x_0' < y'$. The choice of branch lines depends on the location of the saddle point and is always made in such a way as to keep the path of integration on the positive branch of $\sqrt{(M_0^2 - 1) - (1 - M_1^2)\lambda^2\beta^2}$ Thus the poles of the integrand need never be considered.

For case i), $k_0'\rho' \gg 1$, we may carry out a saddle-point expansion. If we set $x_0' = \rho' \cos\psi_0$, $iy' = \rho' \sin\psi_0$, then $\rho' = \sqrt{x_0'^2 - y'^2}$, and ψ_0 is imaginary; $\psi_{0I} = \mathrm{th}^{-1}\, y'/x_0'$. The integral (5.56) may be written:

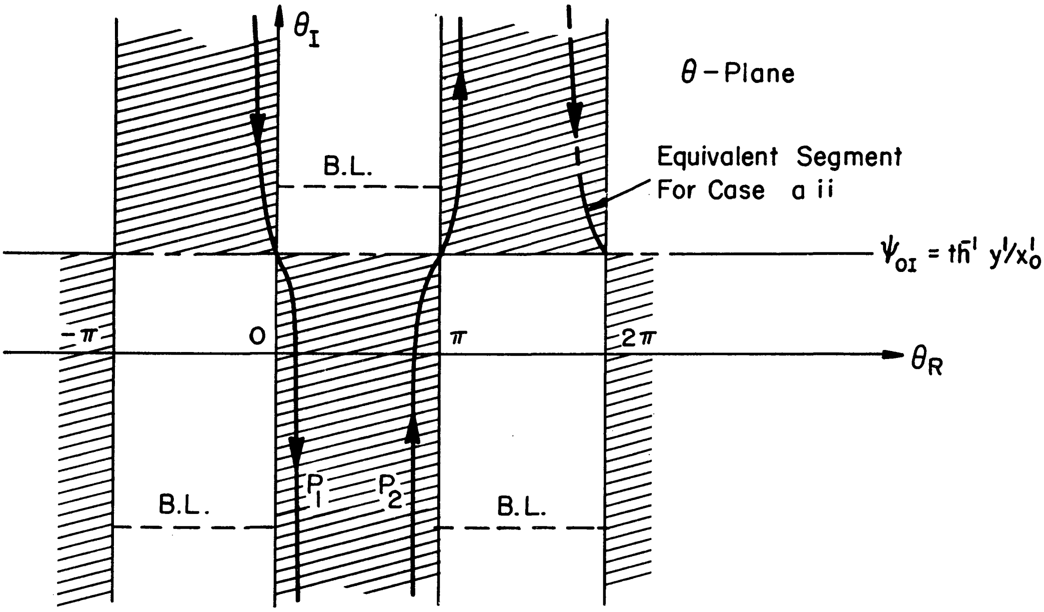

Fig. 22. Paths of integration for F(y'); supersonic incidence inside the Mach wedge.

$$F(y') = \frac{\epsilon}{\mu - 1}\int_{P_1+P_2} \frac{\Gamma_3 - \Gamma_4\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}}{\Gamma_1 + \Gamma_2\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}}\, e^{ik'\rho'\cos(\theta - i\psi_{0I})}\, d\theta. \quad (5.59)$$

On P_1 the saddle point is located at $\theta = i\psi_{0I}$. On P_2 the saddle point is located at $\theta = \pi + i\psi_{0I}$. If we set

$$G(\alpha,\beta) = \frac{\epsilon}{\mu - 1}\,\frac{\Gamma_3 - \Gamma_4\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}}{\Gamma_1 + \Gamma_2\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}}, \quad (5.60)$$

we find from formula (5.19)

$$F(y') = \sqrt{\frac{2\pi}{k'\rho'}}\Big[G(\cos\psi, \sin\psi)\, e^{i(k'\rho' - \frac{\pi}{4})} + G(-\cos\psi, -\sin\psi)\, e^{-i(k'\rho' - \frac{\pi}{4})}\Big] \quad (5.61)$$

The final form for the shape function is:

$$f(y,t) = \frac{1}{M_0^2 - 1}\sqrt{\frac{2\pi}{k_0\rho}}\Big[G\Big(\frac{x_0}{\rho}, \frac{i\sqrt{M_0^2-1}\,y}{\rho}\Big)\, e^{-i\frac{k_0}{M_0^2-1}(\rho + M_0x_0) - i\omega t - i\frac{\pi}{4}} + G\Big(-\frac{x_0}{\rho}, -\frac{i\sqrt{M_0^2-1}\,y}{\rho}\Big)\, e^{-i\frac{k_0}{M_0^2-1}(\rho - M_0x_0) - i\omega t + i\frac{\pi}{4}}\Big], \quad (5.62)$$

with

$$\rho = \sqrt{x_0^2 - (M_0^2-1)y^2} \quad .$$

Now the criterion $k'\rho' >> 1$ implies $x_0'^2 >> \frac{1}{k'^2} + y'^2$. This inequality may be satisfied by choosing first $x_0'^2 >> \frac{1}{k'^2}$ and then $y'^2 << x_0'^2 - \frac{1}{k'^2}$. Thus the approximation is valid in the vicinity of the X axis and then only for situations wherein the source is located sufficiently far from the shock wave.

For case ii), $k'\rho' << 1$, $y' \sim x_0'$, and we are dealing with the immediate vicinity of the intersection of the Mach wedge with the shock wave. Let us replace the part of the path P_1 above the saddle point in Fig. 22 by the equivalent path P_3. Then we let $\theta' = \theta - i\psi_{oI}$, and expand the exponential in powers of $k'\rho'$ to find:

$$F(y') = \int_{P_1'+P_2'} G(\cos(\theta+i\psi_{oI}), \sin(\theta+i\psi_{oI}))(1+ik'\rho'\cos\theta' - \tfrac{1}{2}k'^2\rho'^2\cos^2\theta' + --)\,d\theta . \quad (5.63)$$

G itself may be expanded in powers of $k'\rho'$ using:

$$\cos(\theta'+i\psi_{oI}) = \frac{x_0'}{\rho'}\cos\theta' - i\frac{y'}{\rho'}\sin\theta' \quad ;$$

$$\sin(\theta'+i\psi_{oI}) = \frac{iy'}{\rho'}\cos\theta' + \frac{x_0'}{\rho'}\sin\theta' \quad . \qquad (5.64)$$

Now we find that the segments of paths off the x axis cancel and we are left with:*

$$F(y') = \int_0^{2\pi} (G_0(\theta') + k'\rho' G_1(\theta') + --)(1 + ik'\rho'\cos\theta' + ---)\,d\theta'$$

$$= \int_0^{2\pi} G_0(\theta')\,d\theta' + k'\rho'\int_0^{2\pi}(G_1(\theta') + iG_0(\theta')\cos\theta')\,d\theta' + --- \quad . \qquad (5.65)$$

It is easily seen that $G_0(\theta')$ and $G_1(\theta')$ are constants and thus:

*We use $G(\theta')$ instead of $G(\cos(\theta'+i\psi_{oI}), \sin(\theta'+i\psi_{oI}))$, for the sake of brevity.

$$F(y') = 2\pi\left(G_0 + k'\rho' G_1 + O(k'^2\rho'^2)\right) \tag{5.66}$$

The first term gives for $f(y,t)$:

$$f(y,t) = \epsilon \frac{\mu M_0^2 + i M_0 \sqrt{\frac{1-M_1^2}{M_0^2-1}}\left((\mu-1)M_0^2 - (2\mu-1)\right)}{(\mu-1)\,(M_0^2-1)^{3/2}} e^{i\frac{k_0 M_0 x_0}{M_0^2-1} - i\omega t} \, . \tag{5.67}$$

b. Outside the Mach Wedge, $|y'| > x_0'$. If we assume $y' > x_0'$, then:

$$F(y') = \int_{P_1+P_2} G(\alpha,\beta)\, e^{ik'(x_0'\cos\theta + iy'\sin\theta)}\, d\theta \, . \tag{5.68}$$

Letting $-ix_0' = \rho'\cos\psi$, $y' = \rho'\sin\psi$, then $\psi_R = \frac{\pi}{2}$, $\psi_I = th^{-1}\frac{x_0'}{y'}$, $\rho' = \sqrt{y'^2 - x_0'^2}$, and the paths are as illustrated in Fig. 23.

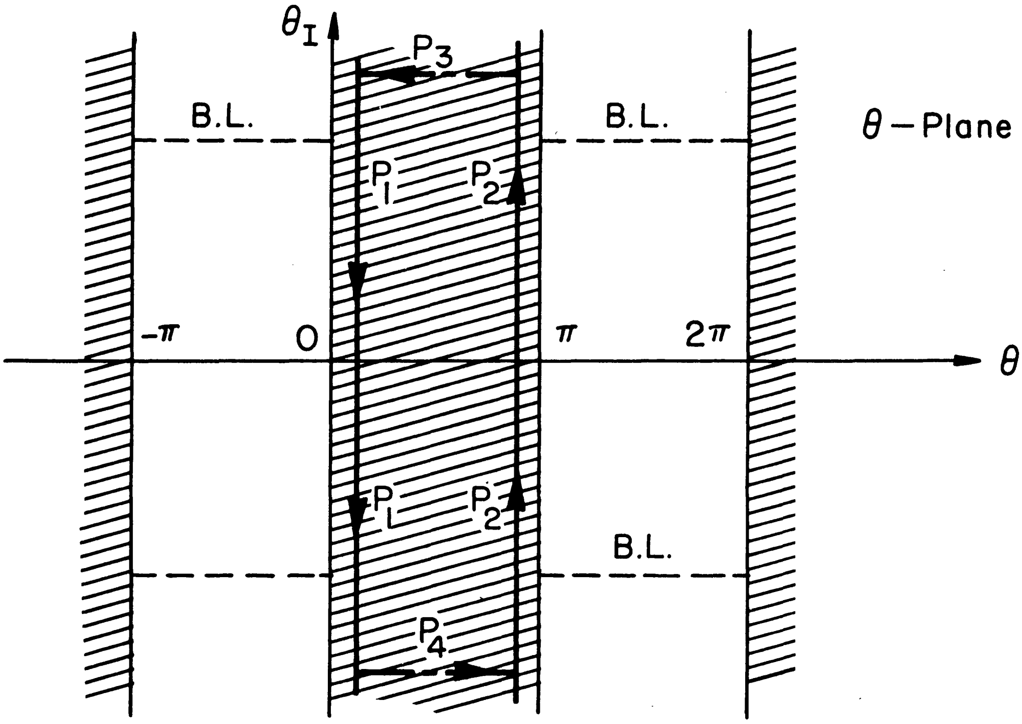

Fig. 23. Path of integration for F(y'); supersonic case outside the Mach wedge.

Since we have no poles in the allowed region, we may close the path and find $F(y') = 0$ for $y' > x_0'$. By similar considerations we find $F(y') = 0$ for $y' < -x_0'$. Therefore we conclude:

$$f(y,t) = 0, \quad \text{for} \quad |y'| > x_0' \; . \tag{5.69}$$

The results of these considerations are illustrated in Fig. 26.

3. The Entropy Wave: $\Delta_1(x,y,t)$

From formula (5.52) we see that the refracted entropy wave is given by:

$$\Delta_1(x,y,t) = k_0 \int_{P_1'+P_2'} \frac{C}{(1+M_0\alpha_0)^2} e^{ik_0 \frac{\alpha_0 x_0}{1+M_0\alpha_0} + ik_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1 \alpha_2} - i\omega t} \; . \tag{5.70}$$

This integral is transformed to the Θ-plane by means of the aberration formulae (4.64) to give:

$$\Delta_1(x,y,t) = \frac{ik_0 e^{i\frac{k_0 M_0 x_0}{M_0^2-1} + i\frac{k_1}{M_1}(x - U_1 t)}}{(M_0^2-1)^{3/2}} \int_{P_1+P_2} (1+M_0\alpha) C e^{ik_0 \frac{(\alpha x_0 + i\beta\sqrt{M_0^2-1}\,y)}{M_0^2-1}} d\theta . \tag{5.71}$$

Thus we write:

$$\Delta_1(x,y,t) = \frac{ik_0}{(M_0^2-1)^{3/2}} e^{i\frac{k_0 M_0 x_0}{M_0^2-1} + i\frac{k_1}{M_1}(x-U_1 t)} S(y) \tag{5.72}$$

where

$$S(y') = \int_{P_1+P_2} (1+M_0\alpha_0) C e^{ik'(\alpha x_0' + \beta y')} d\theta \, , \tag{5.73}$$

and

$$x_0' = \frac{x_0}{\sqrt{M_0^2-1}} \, , \quad y' = y \, , \quad k' = \frac{k_0}{\sqrt{M_0^2-1}} \; .$$

From Appendix E we find:

$$(1+M_0\alpha)C = \frac{2M_1^2(M_0^2-1)^2\left[\Gamma_7 - \Gamma_8\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}\right]}{\mu^2 M_0^4 F^2(M_0)\left[\Gamma_1 + \Gamma_2\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}\right]} \; . \tag{E.11}$$

The integrand of $S(y')$ has poles and branch points at precisely the same points as the shape integral $F(y')$. Therefore we may treat $S(y')$ by the method used for $F(y')$.

a. Inside the Edge Streamlines, $x_0' > |y'|$.—

For case i), $k'\rho' >> 1$, let $H(\alpha,\beta) = (1+M_0\alpha)\,C$; then by the saddle-point method we find:

$$S(y') = \sqrt{\frac{2\pi}{k'\rho'}}\left[H(\cos\psi,\sin\psi)\,e^{i(k'\rho'-\frac{\pi}{4})} + H(-\cos\psi,-\sin\psi)\,e^{-i(k'\rho'-\frac{\pi}{4})}\right], \quad (5.74)$$

and therefore the formula for the entropy wave becomes:

$$\begin{aligned}\Delta_1(x,y,t) = \frac{ik_0}{(M_0^2-1)}\sqrt{\frac{2\pi}{k_0\rho}}\Big[&H\Big(\frac{x_0}{\rho},\frac{iy\sqrt{M_0^2-1}}{\rho}\Big)e^{i\frac{k_0}{M_0^2-1}(\rho+M_0x_0)-i\frac{\pi}{4}} \\ &+H\Big(-\frac{x_0}{\rho},-\frac{iy\sqrt{M_0^2-1}}{\rho}\Big)e^{-i\frac{k_0}{M_0^2-1}(\rho-M_0x_0)+i\frac{\pi}{4}}\Big]e^{i\frac{k_1}{M_1}(x-U_1t)},\end{aligned} \quad (5.75)$$

with $\rho = \sqrt{x_0^2-(M_0^2-1)y^2}$.

An examination of the surfaces of constant phase conducted in the manner of Section IV, Part C, reveals that the waves have an elliptical shape. These ellipses are all tangent to the edge streamlines, and are blown downstream with no change in shape. There is a phase difference of 90° between the front and back surfaces again, just as in the case of the interaction-free sonic waves.

For case ii), $k'\rho' << 1$, we carry out a series expansion and find that the leading term in the series for H is $\frac{H_{-1}(\theta')}{k'\rho'}$. Therefore, since $H_{-1}(\theta')$ is a constant $S(y') \sim \frac{2\pi}{k'\rho'}H_{-1}$, and the entropy wave has a singularity on the streamlines through the intersection of the Mach wedge and the shock.

b. Outside the Edge Streamlines, $|y'| > x_0'$.— $S(y') = 0$ by the same reasoning as in Subsection 2b. Therefore the entropy wave vanishes outside the edge streamlines. The entropy wave is illustrated in Fig. 26.

4. The Vorticity Potential: $\chi_1(x,y,t)$

The refracted vorticity wave is shown in formula (5.52) to be:

$$\chi_1(x,y,t) = i\lambda \int_{P_1'+P_2'} \frac{M_1\alpha_2}{(1+M_0\alpha_0)^2} B\, e^{ik_0\frac{\alpha_0 x_0}{1+M_0\alpha_0} + ik_1\frac{(\alpha_2 x+\beta_2 y)}{M_1\alpha_2} - i\omega t}\, d\theta_0 . \quad (5.76)$$

Let us again transform this integral to the θ-plane of Section IV. Then:

$$\chi_1(x,y,t) = \frac{i\lambda\, e^{i\frac{k_0 M_0 x_0}{M_0^2-1} + i\frac{k_1}{M_1}(x-U_1 t)}}{(M_0^2-1)^{3/2}} \int_{P_1+P_2} M_1\alpha_2(1+M_0\alpha) B\, e^{ik_0\frac{(\alpha x_0 + i\beta\sqrt{M_0^2-1}\,y)}{}}\, d\theta . \quad (5.77)$$

Hence:

$$\chi_1(x,y,t) = \frac{i\lambda\, e^{i\frac{k_0 M_0 x_0}{M_0^2-1} + i\frac{k_1}{M_1}(x-U_1 t)}}{(M_0^2-1)^{3/2}}\, \mathfrak{X}(y) , \quad (5.78)$$

where

$$\mathfrak{X}(y') = \int_{P_1+P_2} M_1\alpha_2(1+M_0\alpha) B\, e^{ik'(\alpha x_0' + i\beta y')}\, d\theta , \quad (5.79)$$

with

$$x_0' = \frac{x_0}{\sqrt{M_0^2-1}} ,\quad y' = y' ,\quad k' = \frac{k_0}{\sqrt{M_0^2-1}}$$

Using formula (E.9) of Appendix E for B we find:

$$M_1\alpha_2(1+M_0\alpha)B = \frac{i\beta M_1 M_0\sqrt{M_0^2-1}\left[\Gamma_5 + \Gamma_6\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}\right]\epsilon}{\mu M_0^2 F(M_0)\left[(M_0^2-1) - M_1^2\lambda^2\beta^2\right]\left[\Gamma_1 + \Gamma_2\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}\right]} . \quad (5.80)$$

This expression has the same branch points as the previous amplitudes but in addition has poles at:

$$\left[(M_0^2-1) - M_1^2\lambda^2\beta^2\right] = 0. \quad (5.81)$$

Equation (5.81) has two solutions for β. Corresponding to each of these two roots, there are two possible values for α. Thus in each strip of width 2π in the θ-plane, we have 4 poles:

$$(\alpha_1,\beta_1)=\left(\frac{\sqrt{M_0^2-1}}{M_1\lambda},\ \frac{\sqrt{M_1^2\lambda^2-(M_0^2-1)}}{M_1\lambda}\right);\ (\alpha_2,\beta_2)=\left(\frac{\sqrt{M_0^2-1}}{M_1\lambda},\ \frac{-\sqrt{M_1^2\lambda^2-(M_0^2-1)}}{M_1\lambda}\right);$$

$$(\alpha_3,\beta_3)=\left(\frac{-\sqrt{M_0^2-1}}{M_1\lambda}\ \ \frac{\sqrt{M_1^2\lambda^2-(M_0^2-1)}}{M_1\lambda}\right);\ (\alpha_4,\beta_4)=\left(\frac{-\sqrt{M_0^2-1}}{M_1\lambda}\ \ \frac{-\sqrt{M_1^2\lambda^2-(M_0^2-1)}}{M_1\lambda}\right). \tag{5.82}$$

These poles are illustrated in Fig. 24.

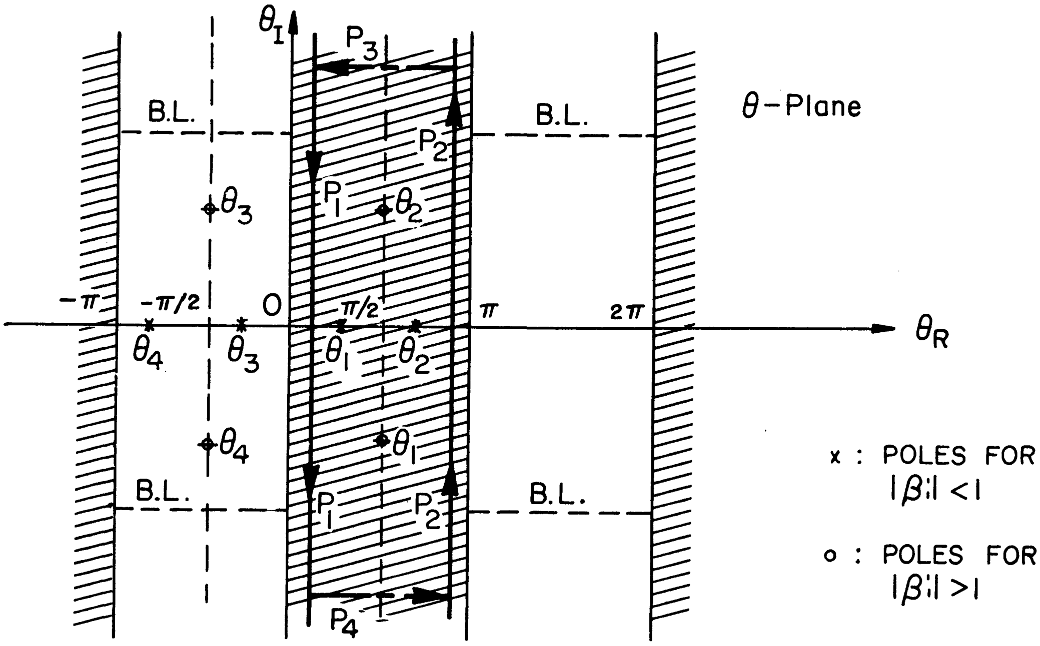

Fig. 24. Path of integration for X(y'); supersonic case outside the Mach wedge.

Again, we consider two cases, namely, inside and outside the streamlines emanating from the intersection of the Mach wedge with the shock. In the first case (inside the edge streamlines) the situation is identical with 2a and 3a since the poles are clearly isolated from the saddle point. In the other case (outside the edge streamlines), an interesting and different result arises. The shape function and entropy wave both vanished in the region $x_0' > |y'|$ but as we shall see presently, the vorticity potential does not vanish, due to the presence of the poles described above.

a. Inside the Edge Streamlines, $x_0' > |y'|$.—

For case i), $k'\rho' \gg 1$, we carry out a saddle-point expansion as before. Using the notation:

$$K(\alpha,\beta) = M_1 \alpha_2 (1 + M_0 \alpha) B , \tag{5.83}$$

we find:

$$\begin{aligned}\chi_1(x,y,t) = \frac{i\lambda}{M_0^2-1}\sqrt{\frac{2\pi}{k_0\rho}}\Bigg[&K\left(\frac{x_0}{\rho}, \frac{iy\sqrt{M_0^2-1}}{\rho}\right) e^{i\frac{k_0}{M_0^2-1}(\rho + M_0 x_0) - i\frac{\pi}{4}} \\ &+ K\left(-\frac{x_0}{\rho}, -\frac{iy\sqrt{M_0^2-1}}{\rho}\right) e^{-i\frac{k_0}{M_0^2-1}(\rho - M_0 x_0) + i\frac{\pi}{4}}\Bigg] e^{i\frac{k_1}{M_1}(x - U_1 t)} ,\end{aligned} \tag{5.84}$$

with

$$\rho = \sqrt{x_0^2 - (M_0^2-1)y^2} .$$

The surfaces of constant phase are identical to those of the entropy wave.

For case ii), $k'\rho' \ll 1$, we carry out a series expansion for K .

$$K(\theta') = K_0(\theta') + k'\rho' K_1(\theta') + \cdots \quad . \tag{5.85}$$

Therefore

$$\mathfrak{X}(y') \sim 2\pi K_0 \tag{5.86}$$

and $\chi_1(x,y,t)$ stays finite on the edge streamlines.

b. Outside the Edge Streamlines, $x_0' < |y'|$.—

For case i), suppose $y' > x_0'$; then the poles are either on the real or $\frac{2n+1}{2}\pi i$ axis, depending on whether or not $(M_0^2-1) < M_1^2\lambda^2$. Thus referring to Fig. 24, we find $\mathfrak{X}(y')$ by means of Cauchy's theorem as:

$$\begin{aligned}\mathfrak{X}(y') = 2\pi i \Big\{ &\mathrm{Res}\left[K(\alpha,\beta)\, e^{ik'(\alpha x_0' + i\beta y')}\right]_{\theta=\theta_1} \\ &+ \mathrm{Res}\left[K(\alpha,\beta)\, e^{ik'(\alpha x_0' + i\beta y')}\right]_{\theta=\theta_2} \Big\} .\end{aligned} \tag{5.87}$$

Let

$$J(\alpha,\beta) = -\frac{\left[(M_0^2-1) - M_1^2\lambda^2\beta^2\right]}{M_1^2\lambda^2} K(\alpha,\beta) \tag{5.88}$$

Therefore we have:

$$K(\alpha,\beta) = \frac{J(\alpha,\beta)}{(\beta-\beta_1)(\beta-\beta_2)} \,, \tag{5.89}$$

where β_1 and β_2 are defined by (5.82).

Both θ_1 and θ_2 lie in the allowed region. Therefore:

$$\begin{aligned} \mathrm{Res}\left(K(\alpha,\beta)\right)_{\alpha_1,\beta_1} &= \frac{2M_1\lambda}{\sqrt{M_0^2-1}} J(\alpha_1,\beta_1) \,; \\ \mathrm{Res}\left(K(\alpha,\beta)\right)_{\alpha_2,\beta_2} &= \frac{2M_1\lambda}{\sqrt{M_0^2-1}} J(\alpha_2,\beta_2) \,; \end{aligned} \tag{5.90}$$

and

$$\begin{aligned} \mathfrak{X}(y') = 2\pi i \frac{2M_1\lambda}{M_0^2-1} &\left[J\left(\sqrt{1-\frac{M_0^2-1}{\lambda^2 M_1^2}}, \frac{\sqrt{M_0^2-1}}{\lambda M_1}\right) e^{i\frac{k'}{M_1\lambda}\left(\sqrt{\lambda^2M_1^2-(M_0^2-1)}\,x_0' + i\sqrt{M_0^2-1}\,y'\right)} \right. \\ &\left. + J\left(-\sqrt{1-\frac{M_0^2-1}{\lambda^2 M_1^2}}, -\frac{\sqrt{M_0^2-1}}{\lambda M_1}\right) e^{i\frac{k'}{M_1\lambda}\left(-\sqrt{\lambda^2M_1^2-(M_0^2-1)}\,x_0' + i\sqrt{M_0^2-1}\,y'\right)} \right] . \end{aligned} \tag{5.91}$$

<u>For case ii)</u>, $y' < -x_0'$, since $J(\alpha,\beta)$ is a function of β^2, the effect of replacing $\beta_{1,2}$ by $\beta_{3,4}$ is to leave J unaltered. Thus the result of integration may be introduced by changing y' to $|y'|$ in the exponential of (5.91). Thus we find:

$$\begin{aligned} \chi_1(x,y,t) = -\frac{4\pi M_1\lambda^2}{(M_0^2-1)^{5/2}} &\left[J\left(\sqrt{1-\frac{M_0^2-1}{\lambda^2M_1^2}}, \frac{\sqrt{M_0^2-1}}{\lambda M_1}\right) e^{i\frac{k_0}{M_0^2-1}\sqrt{1-\frac{M_0^2-1}{\lambda^2M_1^2}}\,x_0} \right. \\ &\left. + J\left(\sqrt{1-\frac{M_0^2-1}{\lambda^2M_1^2}}, \frac{\sqrt{M_0^2-1}}{\lambda M_1}\right) e^{-i\frac{k_0}{M_0^2-1}\sqrt{1-\frac{M_0^2-1}{\lambda^2M_1^2}}\,x_0} \right] e^{i\frac{k_0M_1x_0}{M_0^2-1} + i\frac{k_1}{M_1}(x-U_1t) - \frac{k_0}{M_1\lambda}|y|} \,, \end{aligned} \tag{5.92}$$

which represents a damped vorticity wave outside the edge streamlines.

5. The Sound Potential: $\Phi_1(x,y,t)$.

From Eqs. (5.52) we see that the formula for the refracted sound potential is:

$$\Phi_1(x,y,t) = -i\lambda \int_{P_1'+P_2'} \frac{(1+M_1\alpha_1)}{(1+M_0\alpha_0)^2} A\, e^{ik_0\frac{\alpha_0 x_0}{1+M_0\alpha_0} + ik_1\frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t}\, d\theta_0 \,. \qquad (5.93)$$

Because of the complicated relationship between the functions α_1, β_1, and the aberrated angles functions α , β , transformation to the θ-plane is of no great help in simplifying this integral. To carry out a transformation which will simplify the integral we first transform θ_0 to θ as before, and then transform θ to θ' by means of the relations:

$$\alpha' = \frac{\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}}{\sqrt{M_0^2-1}} \,, \qquad \beta' = \frac{i\lambda\sqrt{1-M_1^2}}{\sqrt{M_0^2-1}}\,\beta \;;$$

$$\alpha = \pm\frac{\sqrt{\lambda^2(1-M_1^2)+(M_0^2-1)\beta'^2}}{\lambda\sqrt{1-M_1^2}} \,, \qquad \beta = -\frac{i\sqrt{M_0^2-1}}{\lambda\sqrt{1-M_1^2}}\,\beta' \,. \qquad (5.94)$$

With the aid of this transformation, the integral (5.93) becomes:

$$\Phi_1(x,y,t) = i\epsilon\frac{\sqrt{1-M_1^2}}{M_0^2-1}\, e^{i\frac{k_0 M_0 x_0}{M_0^2-1} - i\frac{k_1 M_1 x}{1-M_1^2} - i\omega t}\, \varphi_1(x,y) \,, \qquad (5.95)$$

where $\varphi_1(x,y)$ is defined by:

$$\varphi_1(x,y) = \int_{P_1''} L^+(\alpha',\beta')\, e^{i\frac{k_0 x_0}{M_0^2-1}\sqrt{1+\frac{(M_0^2-1)\beta'^2}{\lambda^2(1-M_1^2)}} + i\frac{k_1(\alpha' x + \beta'\sqrt{1-M_1^2}\,y)}{1-M_1^2}}\, d\theta'$$
$$-\int_{P_2''} L^-(\alpha',\beta')\, e^{-i\frac{k_0 x_0}{M_0^2-1}\sqrt{1+\frac{(M_0^2-1)\beta'^2}{\lambda^2(1-M_1^2)}} + i\frac{k_1(\alpha' x + \beta'\sqrt{1-M_1^2}\,y)}{1-M_1^2}}\, d\theta' \,. \qquad (5.96)$$

P_1'' and P_2'' are the paths illustrated in Fig. 25, and $L^\pm$ are given by:

$$L^\pm = \frac{\alpha'(1+M_0\alpha)\;A(\alpha',\beta')}{(1-M_1\alpha')\sqrt{1+\frac{(M_0^2-1)\beta'^2}{\lambda^2(1-M_1^2)}}} \,. \qquad (5.97)$$

Using the expression (E.14) of Appendix E for A, the integrands of (5.96) become:

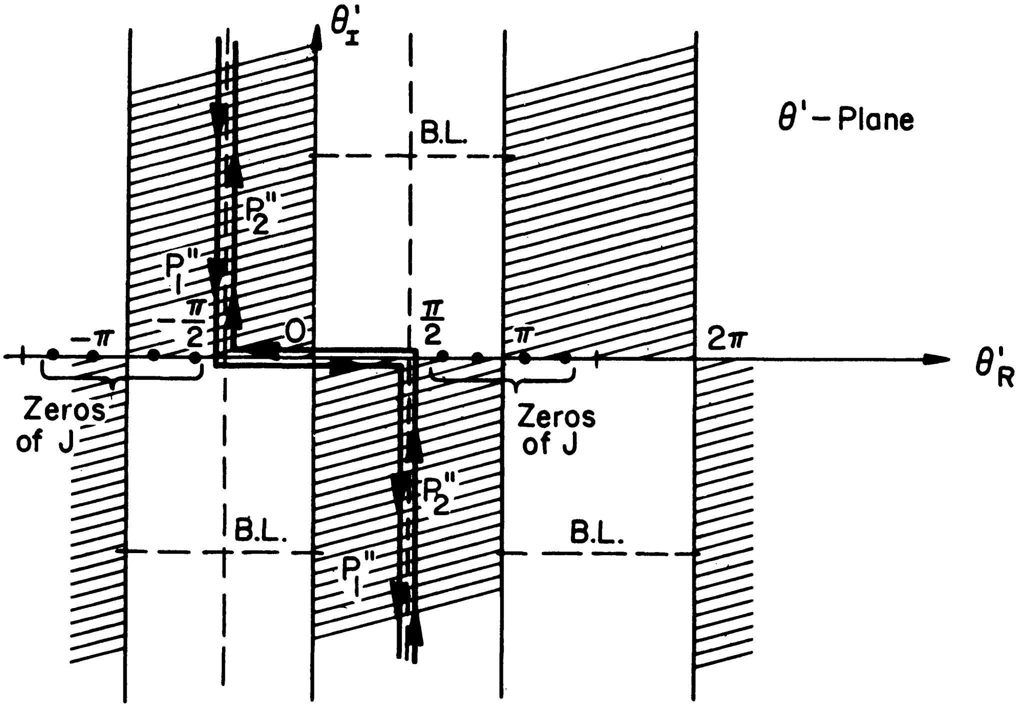

Fig. 25. Paths of integration for $\phi_1(x', y')$; supersonic case.

$$L^{\pm}(\alpha',\beta') = \frac{\alpha'}{1-M_1\alpha'} \; \frac{\Lambda_2 \pm \Lambda_3\sqrt{1+\frac{(M_0^2-1)\beta'^2}{\lambda^2(1-M_1^2)}}}{\Lambda_1\sqrt{1+\frac{(M_0^2-1)\beta'^2}{\lambda^2(1-M_1^2)}}} \tag{5.98}$$

Before we can proceed with an approximation of $\phi_1(x,y)$, we must examine the branch points and poles of the integrand. The branch points occur at:

$$\beta'^2 = -\frac{\lambda^2(1-M_1^2)}{M_0^2-1}$$

or (5.99)

$$\beta'_{\pm} = \pm i\sqrt{\frac{U_1}{U_0}} \quad , \; |\beta'_{\pm}| < 1$$

Thus these points are located on the lines $\theta'_R = n\pi$, as illustrated in Fig. 25. The poles are located at:

$$\alpha' = \frac{1}{M_1} > 1 \tag{5.100}$$

which occur on the lines $\theta'_R = 2\pi n$; and

$$\Lambda_1 = 0 \qquad \text{or} \qquad M_0^2\alpha'^2 + 2M_1M_0^2\alpha' + 1 = 0 \,.$$

Solving we find:

$$\alpha'_{1,2} = -M_1 \pm \sqrt{M_1^2 - \frac{1}{M_0^2}} \tag{5.101}$$

Thus $-2 < \alpha'_{1,2} < 0$. The expressions (5.100) and (5.101) show that the poles are again isolated from the saddle points of the integrands. Therefore, we proceed with the ordinary saddle-point expansion, using formula (5.19).

We may introduce $\frac{x}{\rho} = \cos\psi$, $\frac{\sqrt{1-M_1^2}\,y}{\rho} = \sin\psi$, $\rho = \sqrt{x^2+(1-M_1^2)y^2}$ and find:

$$\varphi_1 = \pm\int_{P_1'', P_2''} L^{\pm}(\alpha', \beta')\, e^{i\frac{k_1\rho}{1-M_1^2}\cos(\theta'-\psi) \pm \frac{k_0 M_0}{M_0^2-1}\sqrt{1+\frac{(M_0^2-1)\beta'^2}{\lambda^2(1-M_1^2)}}}\, d\theta' , \tag{5.102}$$

We may now approximate (5.102) as in subsection A5. The saddle-point expansion for $k_1\rho >> 1$ gives:

$$\varphi_1 = \sqrt{\frac{2\pi(1-M_1^2)}{k_1\rho}}\Big[L^+(\cos\psi, \sin\psi)\, e^{i\frac{k_1\rho}{1-M_1^2} - i\frac{\pi}{4} + i\frac{k_0 x_0}{M_0^2-1}\sqrt{1+\frac{(M_0^2-1)\sin^2\psi}{\lambda^2(1-M_1^2)}}}$$
$$+ L^-(\cos\psi, \sin\psi)\, e^{i\frac{k_1\rho}{1-M_1^2} - i\frac{\pi}{4} - i\frac{k_0 x_0}{M_0^2-1}\sqrt{1+\frac{(M_0^2-1)\sin^2\psi}{\lambda^2(1-M_1^2)}}}\Big]. \tag{5.103}$$

Thus:

$$\Phi_1(x,y,t) = i\epsilon\frac{(1-M_1^2)}{(M_0^2-1)}\sqrt{\frac{2\pi}{k_1\rho}}\Big[L^+\Big(\frac{x}{\rho}, \frac{\sqrt{1-M_1^2}\,y}{\rho}\Big) e^{i\frac{k_0 x_0}{M_0^2-1}\sqrt{1+\frac{(M_0^2-1)y^2}{\lambda^2\rho^2}}} +$$
$$L^-\Big(\frac{x}{\rho}, \frac{\sqrt{1-M_1^2}\,y}{\rho}\Big) e^{-i\frac{k_0 x_0}{M_0^2-1}\sqrt{1+\frac{(M_0^2-1)y^2}{\lambda^2\rho^2}}}\Big] e^{i\frac{k_1}{1-M_1^2}(\rho - M_1 x) + i\frac{k_0 M_0 x_0}{M_0^2-1} - i\frac{\pi}{4} - i\omega t} . \tag{5.104}$$

It is of interest to examine the formula (5.104) as $\frac{x_0}{\rho} \to 0$. In this limit the sound potential becomes:

$$\Phi_1 = ie\frac{(1-M_1^2)}{(M_0^2-1)}\sqrt{\frac{2\pi}{k_1\rho}}\left[L^+\left(\frac{x}{\rho},\frac{\sqrt{1-M_1^2}\,y}{\rho}\right)+L^-\left(\frac{x}{\rho},\frac{\sqrt{1-M_1^2}\,y}{\rho}\right)\right]e^{i\frac{k_1}{1-M_1^2}(\rho-M_1x)-i\frac{\pi}{4}-i\omega t} \quad (5.105)$$

These are cylindrical waves blown downstream with the flow.

To determine the deviation from cylindricality, we examine the surfaces of constant phase. Let $t' = t - \frac{M_0x_0}{c_0(M_0^2-1)}$, and we find the surfaces of constant phase from (5.104):

$$\rho\left(1 \pm \frac{k_0x_0}{k_1\rho}\left(\frac{1-M_1^2}{M_0^2-1}\right)\sqrt{1+\frac{(M_0^2-1)y^2}{\lambda^2\rho^2}}\right) - M_1x = (1-M_1^2)c_1t' \quad (5.106)$$

If we let $\frac{x_0}{\rho} \to 0$, we find:

$$\rho - M_1x = (1-M_1^2)c_1t' \quad , \quad (5.107)$$

or as before:

$$(x-U_1t')^2 + y^2 = c_1^2t'^2 \quad . \quad (5.108)$$

Thus at sufficiently large distances from the origin (in units of x_0), the sound waves become circular. The deviations from circularity as we get nearer to the shock may be calculated to first order.

Squaring (5.105) and keeping only first order terms in $\frac{x_0}{\rho}$, we find:

$$\pm\frac{2x_0}{(M_0^2-1)}\sqrt{\lambda^2\rho^2+(M_0^2-1)y^2} = \left[(x-U_1t')^2+y^2-c_1^2t'^2\right] \quad , \quad (5.109)$$

If we set

$$\begin{aligned} x &= U_1t + c_1t'\cos\theta + \delta\cos\theta \quad , \\ y &= \qquad\quad c_1t'\sin\theta + \delta\sin\theta \quad , \end{aligned} \quad (5.110)$$

then δ measures the deviation from circularity. In particular:

$$\delta c_1 t' = \mp \frac{x_0}{M_0^2 - 1} \sqrt{\lambda^2 (1 + M_1 \cos\theta)^2 + (M_0^2 - 1)\sin^2\theta}\,, \tag{5.111}$$

and hence the distorted circles are described by:

$$R(\theta) = c_1 t' \mp \frac{x_0}{M_0^2 - 1} \sqrt{\lambda^2 (1 + M_1 \cos\theta)^2 + (M_0^2 - 1)\sin^2\theta}\,. \tag{5.112}$$

Figure 26 illustrates this wave together with those described in the preceding subsections B1-B4.

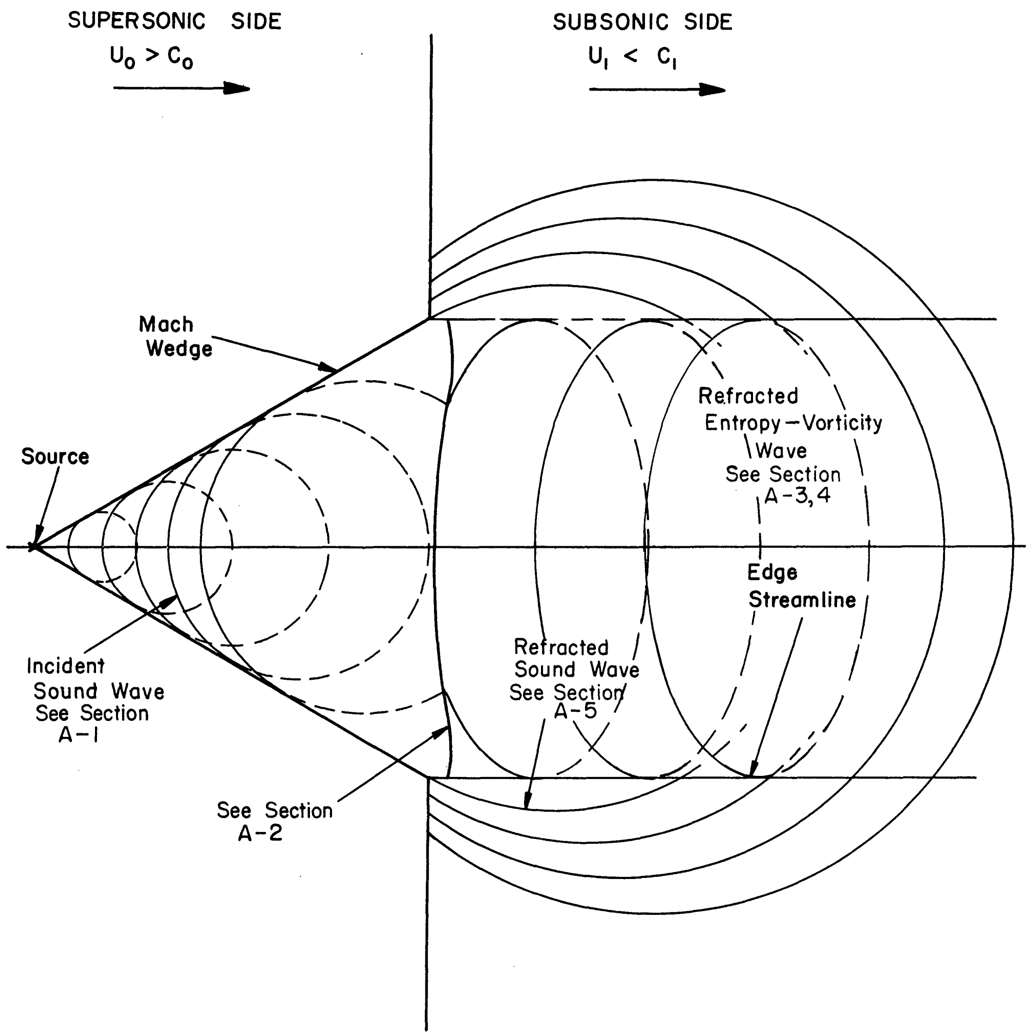

Fig. 26. Refraction of cylindrical waves.

VI. EXTENSIONS AND CONCLUSIONS

Although the theory presented in the preceding sections can in principle be checked, experiments dealing with the interaction of sound and shock waves can most easily be performed in the shock tube. However, for this purpose the theory will have to be generalized in several respects. On the one hand, tube shocks are not stationary but proceed into gas at rest. This causes the sound source which of course will be at rest also, to be moving supersonically with respect to the shock. Further, the sound source, which for reasons of simplicity, was assumed to be two dimensional will in practice be a point source. In Part A the point source interacting with a stationary shock is briefly discussed and in Part B the ground work is laid for a treatment of the problem of the moving shock.

A. POINT SOURCE

Let us first consider the generalization of the expressions for a line source as presented in Section IV to the case of a three dimensional point source.

The differential equation governing the propagation of spherical sound waves in a moving gas is found to be:

$$\nabla^2 \Phi - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)^2 \Phi = 4\pi\epsilon\, \delta(x)\,\delta(y)\,\delta(z)\, e^{-i\omega t}. \qquad (6.1)$$

The substitution

$$\Phi(x,y,z,t) = \varphi(x,y,z)\, e^{-i\frac{Mkx}{1-M^2} - i\omega t} \qquad (6.2)$$

with $M = \frac{U}{c}$ and $k = \frac{\omega}{c}$ reduces Eq. (6.1) to:

$$(1-M^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} + \frac{k^2}{1-M^2}\varphi = 4\pi\epsilon\,\delta(x)\,\delta(y)\,\delta(z)\,. \tag{6.3}$$

If $M = M_1 < 1$, we use the similarity transformation

$$x' = \frac{x}{\sqrt{1-M_1^2}}\,,\; y' = y\,,\; z' = z\,,\; k' = \frac{k}{\sqrt{1-M_1^2}}\,,\; \epsilon' = \frac{\epsilon}{\sqrt{1-M_1^2}} \tag{6.4}$$

to reduce (6.3) to:

$$\nabla^2\varphi + k'^2\varphi = 4\pi\,\epsilon'\,\delta(x')\,\delta(y')\,\delta(z')\,. \tag{6.5}$$

On the other hand, if $M = M_0 > 1$, we use the transformation

$$x' = \frac{x}{\sqrt{M_0^2-1}}\,,\; y' = y\,,\; z' = z\,,\; k' = \frac{k}{\sqrt{M_0^2-1}}\,,\; \epsilon' = \frac{\epsilon}{\sqrt{M_0^2-1}} \tag{6.6}$$

to reduce (6.3) to:

$$\frac{\partial^2\varphi}{\partial x'^2} - \frac{\partial^2\varphi}{\partial y'^2} - \frac{\partial^2\varphi}{\partial z'^2} + k'^2\varphi = -4\pi\,\epsilon'\,\delta(x')\,\delta(y')\,\delta(z')\,. \tag{6.7}$$

We may treat Eqs. (6.5) and (6.7), as in Section IV, by means of a Fourier Integral. Carrying out one integration, we find the solution to (6.5):

$$\varphi(x', y', z') = \frac{\epsilon'}{2\pi i}\iint_{-\infty}^{+\infty} \frac{e^{i(\sqrt{k'^2-k_y^2-k_z^2}\,|x'| + k_y y' + k_z z')}}{\sqrt{k'^2-k_y^2-k_z^2}}\, dk_y\, dk_z\,, \tag{6.8}$$

and the solution to (6.7):

$$\varphi = \begin{cases} \dfrac{\epsilon'}{2\pi i}\displaystyle\iint_{-\infty}^{+\infty}\left\{\frac{e^{i(\sqrt{k'^2+k_y^2+k_z^2}\,x' + k_y y' + k_z z')}}{\sqrt{k'^2+k_y^2+k_z^2}} - \frac{e^{i(-\sqrt{k'^2+k_y^2+k_z^2}\,x' + k_y y' + k_z z')}}{\sqrt{k'^2+k_y^2+k_z^2}}\right\} dk_y\, dk_z \\ \text{for } x' > 0\,; \\ 0\,, \quad \text{otherwise.} \end{cases} \tag{6.9}$$

After introducing the variables θ and φ into (6.8) by means of

$$k_y = k' \cos\varphi \sin\theta ;$$
$$k_z = k' \sin\varphi \sin\theta , \tag{6.10}$$

(6.8) becomes:

$$\varphi(x',y',z') = \frac{\epsilon' k'}{2\pi i}\int_0^{-\frac{\pi}{2}+i\infty}\int_0^{2\pi} e^{ik'(\cos\theta|x'|+\sin\theta\cos\varphi y'+\sin\theta\sin\varphi z')} \sin\theta\, d\theta\, d\varphi . \tag{6.11}$$

Since (6.5) is rotationally invariant, (6.11) reduces to

$$\varphi(x',y',z') = \frac{\epsilon' k'}{2\pi i}\int_P\int_0^{2\pi} e^{ik'r'\cos\theta} \sin\theta\, d\theta\, d\varphi , \tag{6.12}$$

where $r' = \sqrt{x'^2+y'^2+z'^2}$, and P' is illustrated in Fig. 27a. Equation (6.12) may be integrated immediately to give

$$\varphi(x',y',z') = -\epsilon' \frac{e^{ik'r'}}{r'} . \tag{6.13}$$

Thus for $M_1 < 1$ we find:

$$\Phi(x,y,z,t) = -\frac{\epsilon\, e^{i\frac{k_1}{1-M_1^2}\left(\sqrt{x^2+(1-M_1^2)(y^2+z^2)}-M_1x\right)-i\omega t}}{\sqrt{x^2+(1-M_1^2)(y^2+z^2)}} \tag{6.14}$$

We reduce (6.9) in a similar fashion. Let

$$k_y = ik' \cos\varphi \sin\theta ;$$
$$k_z = ik' \sin\varphi \sin\theta ; \tag{6.15}$$

then (6.9) reduces to:

$$\varphi(x',y',z') = \frac{\epsilon' k' i}{2\pi}\int_{P_1+P_2}\int_0^{2\pi} e^{ik'(\cos\theta x'+i\sin\theta\cos\varphi y'+i\sin\theta\sin\varphi z')} \sin\theta\, d\theta\, d\varphi , \tag{6.16}$$

$$x' > 0 ,$$

where P_1 and P_2 are the paths illustrated in Fig. 27b.

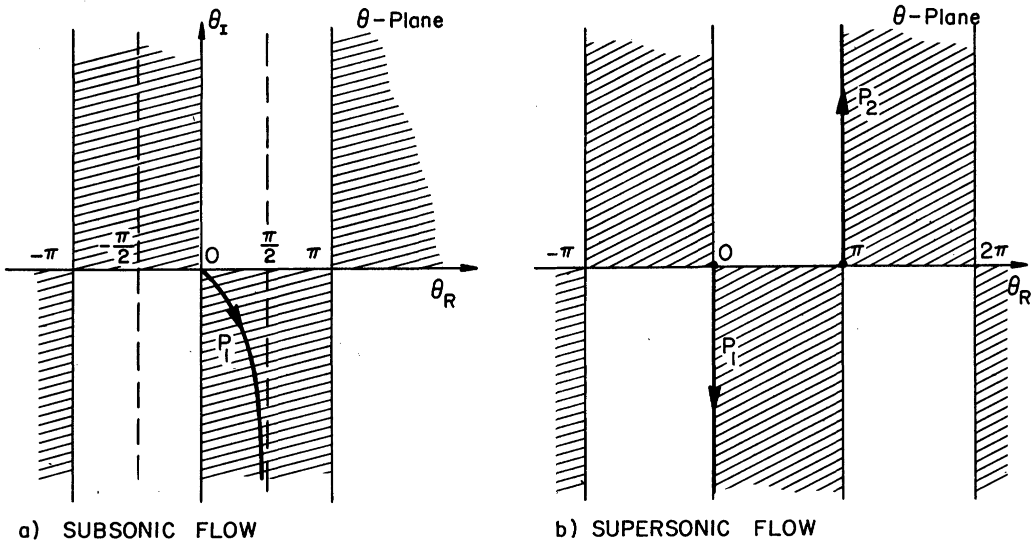

Fig. 27. Paths of integration for $\phi\ (x', y', z')$.

Since (6.7) is rotationally invariant in terms of the variables (x', iy', iz'), we find that (6.16) may be written:

$$\varphi(x',y',z') = \frac{\epsilon' k' i}{2\pi}\int_{P_1+P_2}\int_0^{2\pi} e^{ik'r'\cos\theta}\sin\theta\, d\theta\, d\varphi . \tag{6.17}$$

We integrate (6.17) to find:

$$\varphi(x',y',z') = \begin{cases} 2\dfrac{\epsilon'}{r'}\cos k'r' , & x' > \sqrt{y'^2+z'^2} \, ; \\ 0 , & \text{otherwise.} \end{cases} \tag{6.18}$$

Therefore for $M_0 > 1$ we conclude:

$$\Phi = \begin{cases} \dfrac{2\epsilon\cos\left(\frac{k_0}{M_0^2-1}\sqrt{x^2-(M_0^2-1)(y^2+z^2)}\right)}{\sqrt{x^2-(M_0^2-1)(y^2+z^2)}}\, e^{i\frac{k_0 M_0 x}{M_0^2-1}-i\omega t} \\ \text{for} \quad x > \sqrt{M_0^2-1}\sqrt{y^2+z^2} \\ 0 , \quad \text{otherwise.} \end{cases} \tag{6.19}$$

It should be noted that Eq. (6.19) reduces to the well-known result of I. G. Tamm[21] for Čerenkov radiation in the limit $\omega = 0$. Equations (6.14) and (6.19) give the exact behavior of sound waves generated by a point source in a moving medium when no shock wave is present.

When a shock wave is present in the flow, we use Eqs. (6.11) and (6.16) as the fundamental expression for our treatment. These equations are expressed in terms of aberrated angles as in Section IV, and the resulting integrals are interpreted as superpositions of plane waves. The corresponding reflected or refracted plane waves are found from Section III, and the resulting integrals give the appropriate behavior of the induced disturbance.

B. MOVING SHOCKS

The theory presented in Sections II, III, and V applies strictly to stationary shocks and thus must be modified if it is to be checked experimentally in the shock tube. In this part we shall consider the nature of the modifications to be made and the results which one would expect for the case of a moving shock.

Let us assume that the shock moves to the left into a gas at rest. The gas on the right then moves to the left as illustrated in Fig. 28.

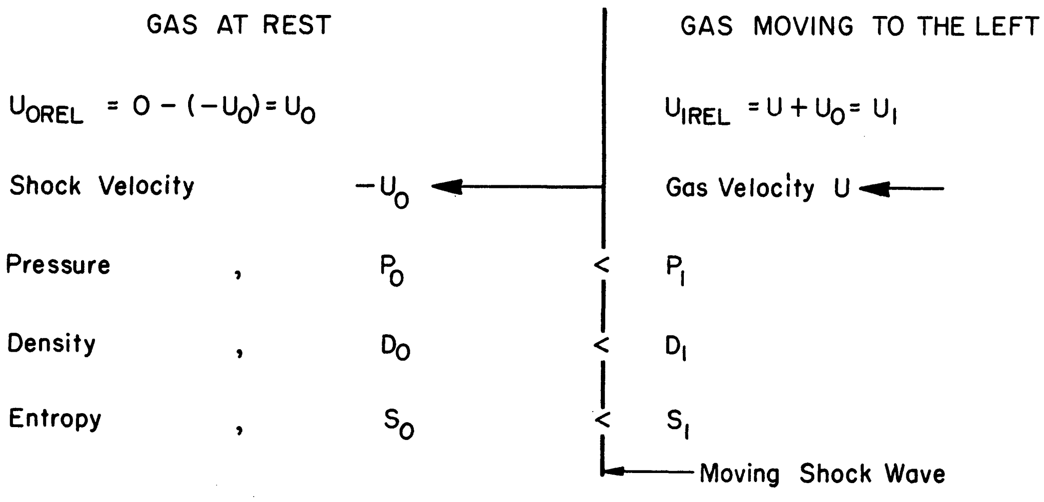

Fig. 28. Undisturbed state of a moving shock.

The situation illustrated in Fig. 28 may be arrived at by observing a stationary shock wave from a frame of reference moving to the right with the speed U_0. In this way we find that the speed of the gas on the supersonic side is $U_0 - U_0 = 0$, while the speed on the subsonic side is $U_1 - U_0 = U$, and the speed of the shock is $0 - U_0 = -U_0$.

The shock conditions, discussed in Appendix A, and used throughout the text, apply here without change since the relative velocities U_0 and U_1 are the same as before.

Let us choose our coordinate system so that at $t = 0$ the shock is located at $x = 0$. The undisturbed motion of the shock is described by $x = -U_0 t$; and a transformation of coordinates

$$x' = x + U_0 t \, , \quad y' = y \quad , t' = t \, , \tag{6.20}$$

gives us precisely the situation treated in the text.

The plane-wave problem may be solved immediately by reference to Section III. Let us indicate how this solution is obtained, first for incidence from the right $x > -U_0 t$.

1. Reflection of Plane Waves from a Moving Shock

Consider a plane-wave incident from the right of the shock:

$$\begin{aligned}
p_i &= \epsilon\, e^{i k_1 \frac{(\alpha_0 x + \beta_0 y)}{1 + M_0 \alpha_0} - i \omega t} \; ; \\
u_i &= \alpha_0 \epsilon\, e^{i k_1 \frac{(\alpha_0 x + \beta_0 y)}{1 + M_0 \alpha_0} - i \omega t} \; ; \\
v_i &= \beta_0 \epsilon\, e^{i k_1 \frac{(\alpha_0 x + \beta_0 y)}{1 + M_0 \alpha_0} - i \omega t} \; ; \\
s_i &= 0 \; ,
\end{aligned} \tag{6.21}$$

where

$$M_1 = \frac{U_1}{c_1} \quad \text{and} \quad k_1 = \frac{\omega}{c_1} \; .$$

If we apply the transformations (6.20) to Eqs. (6.21) we find the resulting incident waves of the form discussed in Section III, Part B. The frequency of these waves, however, will contain a Doppler factor due to the motion of the observer. The appropriate frequency in this moving frame of reference is given by:

$$\omega' = \frac{1 + M_1 \alpha_0}{1 + M \alpha_0}\,\omega \quad . \tag{6.22}$$

Now as mentioned above, the problem in the moving coordinate system is precisely the same as that treated in Section III. Thus we may apply the results of Section III to find the reflected sound field, and the reflected entropy-vorticity field, as well as the shape of the shock.

If we transform these reflected fields back to the original coordinates, the reflected sound wave becomes:

$$\begin{aligned} p_s &= A\, e^{i k_{11} \frac{(\alpha_1 x + \beta_1 y)}{1 + M' \alpha_1} - i \omega_1 t} \; ; \\ u_s &= \alpha_1 A\, e^{i k_{11} \frac{(\alpha_1 x + \beta_1 y)}{1 + M' \alpha_1} - i \omega_1 t} \; ; \\ v_s &= \beta_1 A\, e^{i k_{11} \frac{(\alpha_1 x + \beta_1 y)}{1 + M' \alpha_1} - i \omega_1 t} \; ; \\ s_s &= 0 \quad , \end{aligned} \tag{6.23}$$

where

$$\omega_1 = \frac{1 + M \alpha_1}{1 + M_1 \alpha_1}\, \frac{1 + M_1 \alpha_0}{1 + M \alpha_0}\, \omega \tag{6.24}$$

and

$$k_{11} = \frac{\omega_1}{c_1} \tag{6.25}$$

The entropy-vorticity wave reduces to:

$$
\begin{aligned}
p^* &= 0 \; ; \\
u^* &= -\beta_2 B e^{i k_{21} \frac{(\alpha_2 x + \beta_2 y)}{M_1 \alpha_2} - i \omega_2 t} \; ; \\
v^* &= \alpha_2 B e^{i k_{21} \frac{(\alpha_2 x + \beta_2 y)}{M_1 \alpha_2} - i \omega_2 t} \; ; \\
s^* &= C \; e^{i k_{21} \frac{(\alpha_2 x + \beta_2 y)}{M_1 \alpha_2} - i \omega_2 t} \; ,
\end{aligned}
\tag{6.26}
$$

where

$$
\omega_2 = \frac{M}{M_1} \frac{1 + M_1 \alpha_0}{1 + M \alpha_0} \omega \; , \tag{6.27}
$$

and

$$
k_{21} = \frac{\omega_2}{c_1} \; . \tag{6.28}
$$

The shape function is found to be

$$
x = -U_0 t + \frac{1 + M \alpha_0}{1 + M_1 \alpha_0} a \, e^{\, i \frac{1 + M_1 \alpha_0}{1 + M \alpha_0} \left(\frac{k_1 \beta_0}{1 + M_1 \alpha_0} y - \omega t \right)} \; . \tag{6.29}
$$

In Eqs. (6.23), (6.26), and (6.29), we use the values of A, B, C, a, and (α_1, β_1), (α_2, β_2) given in Section III, Part B, (3.37), (3.22), and (3.26). These relations completely solve the problem of reflection of a plane sound wave from the back of a moving shock wave.

2. Refraction of Plane Waves from a Moving Shock.

Let us now turn to the problem of a plane wave incident from the left of the shock:

$$
\begin{aligned}
p_i &= \epsilon\, e^{i k_0(\alpha_0 x+\beta_0 y)-i\omega t} \; ; \\
u_i &= \alpha_0 \epsilon\, e^{i k_0(\alpha_0 x+\beta_0 y)-i\omega t} \; ; \\
v_i &= \beta_0 \epsilon\, e^{i k_0(\alpha_0 x+\beta_0 y)-i\omega t} \; ; \\
s_i &= 0 \; ,
\end{aligned}
\tag{6.30}
$$

with $k_0 = \dfrac{\omega}{c_0}$.

We may apply the transformations (6.20) to the Eqs. (6.30) to find the incident sound field in a system of coordinates moving with the shock. We then find that the system (6.30) transforms to an incident wave of the type considered in Section III, Part C if we use the frequency

$$
\omega' = (1+M_0\alpha_0)\omega \quad . \tag{6.31}
$$

We find the refracted wave field corresponding to the incident field (6.16) by reference to Section III, Part C.

Transforming the refracted wave field to the original coordinate system, the refracted sound wave becomes:

$$
\begin{aligned}
p_s &= A\, e^{i k_{11}\frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1} - i\omega_1 t} \; ; \\
u_s &= \alpha_1 A\, e^{i k_{11}\frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1} - i\omega_1 t} \; ; \\
v_s &= \beta_1 A\, e^{i k_{11}\frac{(\alpha_1 x+\beta_1 y)}{1+M_1\alpha_1} - i\omega_1 t} \; ; \\
s_s &= 0 \; ,
\end{aligned}
\tag{6.32}
$$

where

$$\omega_1 = \frac{1 + M\alpha_1}{1 + M_1\alpha_1}(1 + M_0\alpha_0)\,\omega \tag{6.33}$$

and

$$k_{11} = \frac{\omega_1}{C_1} \tag{6.34}$$

The refracted entropy-vorticity wave is given by:

$$\begin{aligned} p^* &= 0 \;; \\ u^* &= -\beta_2 B\, e^{i k_{21} \frac{(\alpha_2 x + \beta_2 y)}{M_1 \alpha_2} - i\omega_2 t} \;; \\ v^* &= \alpha_2 B\, e^{i k_{21} \frac{(\alpha_2 x + \beta_2 y)}{M_1 \alpha_2} - i\omega_2 t} \;; \\ s^* &= C\, e^{i k_{21} \frac{(\alpha_2 x + \beta_2 y)}{M_1 \alpha_2} - i\omega_2 t} \;, \end{aligned} \tag{6.35}$$

with

$$\omega_2 = \frac{M}{M_1}(1 + M_0\alpha_0)\,\omega \tag{6.36}$$

and

$$k_{21} = \frac{\omega_2}{C_1} \tag{6.37}$$

The shape function is found to be:

$$X = -U_0 t + \frac{a}{1 + M_0\alpha_0}\, e^{i(1+M_0\alpha_0)\left(\frac{k_0\alpha_0}{1+M_0\alpha_0} y - \omega t\right)} \,. \tag{6.38}$$

Equations (6.32), (6.35), and (6.38) solve the problem of the refraction of a plane sound wave incident from the front of a moving shock. The coefficients A, B, C, and a and the angle functions (α_1, β_1), (α_2, β_2) used herein are given by (3.64), (3.43), and (3.52) of Section III, Part C.

3. Reflection and Refraction of Cylindrical Sound Waves by a Moving Shock

Since the plane wave interaction may be reduced to the problem treated in Section III, it might be expected that the cylindrical wave interaction would reduce to the problem treated in Section V. This is not the case though, since the shock wave is moving relative to the source here, and this motion causes the interaction integrals to be more complicated than those of Section V.

To make this fact apparent, let us discuss the appropriate interaction integrals. For incidence from the right we have according to Section IV: For $0 > M > -1$ (subsonic flow behind the moving shock),

$$\Phi(x,y,t) = \frac{\epsilon}{i}\int_{P''} \frac{e^{ik_1 \frac{(\alpha_0(x-x_0)+\beta_0 y)}{1+M\alpha_0} - i\omega t}}{(1+M\alpha_0)} \, d\theta_0 \quad , \tag{6.39}$$

P'' being the path shown in Fig. 14, p. 53. For $M < -1$ (supersonic flow behind the moving shock),

$$\Phi(x,y,t) = \frac{\epsilon}{i}\int_{P_1'+P_2'} e^{ik_1 \frac{(\alpha_0(x-x_0)+\beta_0 y)}{1+M\alpha_0} - i\omega t} \, d\theta_0 \quad . \tag{6.40}$$

P_1' and P_2' are the paths shown in Fig. 18.

Both of these integrals may be described as superpositions of plane waves of amplitudes

$$\epsilon^*(\theta_0) = \frac{\epsilon k_1}{(1+M\alpha_0)^2} \quad . \tag{6.41}$$

The corresponding integral for incidence from the left side is:

$$\Phi(x,y,t) = \frac{\epsilon}{i}\int_{P'} e^{ik_0(\alpha_0(x+x_0)+\beta_0 y) - i\omega t} \, d\theta_0 \quad , \tag{6.42}$$

P' being the Sommerfeld path of Fig. 14, p. 53. The corresponding plane wave amplitudes are

$$\epsilon^*(\theta_0) = \epsilon k_1 \quad . \tag{6.43}$$

The reflected waves with amplitudes given by (6.41), and the refracted waves corresponding to (6.43) may be written as integrals of the form considered in Section V. It should be remembered that now the time factor may not be removed from these integrals as in the cases considered in Section V, and thus the saddle points of the integrands are time-dependent. This gives rise to time-dependent wave amplitides, which is to be expected because of the motion of the shock. These interaction integrals will be considered in detail in a forthcoming paper.

An observer at rest in front of a moving shock will notice the following sequence of events if he is originally located between the source and the shock. First, he will see only the field of the sound source since there is no reflection into the gas at rest. Then he will notice the refracted field as the shock travels between him and the source. Finally, he will see a combination of the incident and reflected fields when the shock has passed the source.

If the observer were originally on the side of the source away from the approaching shock he would see first the incident field until the shock passed the source. After that time he would see no field as the shock moved between the source and him, and finally he would see a combination of the incident and reflected field as before.

C. CONCLUSIONS

To test this theory experimentally with the shock tube, one would use a point source and a moving shock wave. Therefore it is natural to carry out the extension to this case also. Such generalization will be made at a later date.

The theory of the interaction of plane and cylindrical waves with a stationary shock could be carried out in a wind tunnel. A schlieren photograph should reveal the distortion of the shock as well as the surfaces of constant phase of the sound wave. The density field could also be measured with the aid of a single-fringe interferometer. Photographs of the interaction taken with the aid of such an instrument give quantitative measurements of the density which could then be compared with the results of Section V.

In the limit $\omega = 0$, the Čerenkov limit, the line source becomes a stationary object in the gas. Such an object will give rise to a diffraction effect similar to that studied in the paper of Ludloff[8] mentioned in the introduction. One could easily set up such an experiment to check the theory in this limiting case.

APPENDICES

APPENDIX A

THE DIFFERENTIAL EQUATIONS AND SHOCK CONDITIONS

In this appendix we shall discuss the equations of compressible hydrodynamics and the conditions which apply across shock waves in a compressible flow.

The laws governing the motion of the fluid may be written in integral form as:

Conservation of Mass: $\int_V \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^\alpha} \rho U^\alpha \right) dV = 0$;

Conservation of Momentum: $\int_V \left(\frac{\partial}{\partial t} \rho U^i + \frac{\partial}{\partial x^\alpha} \left(\rho U^\alpha U^i + P\delta^{i\alpha} \right) \right) dV = 0$; (A.1)

Conservation of Energy: $\int_V \left(\frac{\partial}{\partial t} \rho \left(\mathcal{E} + \frac{U^\alpha U^\alpha}{2} \right) + \frac{\partial}{\partial x^\beta} \rho U^\beta \left(\mathcal{H} + \frac{U^\alpha U^\alpha}{2} \right) \right) dV = 0,$

where $\mathcal{E}$ = specific internal energy and $\mathcal{H}$ = specific enthalpy of the gas. All these equations are of the form:

$$\int_V \left(\frac{\partial y^i}{\partial t} + \frac{\partial S^{i\alpha}}{\partial x^\alpha} \right) dV = 0 \ , \tag{A.2}$$

where $y^i = \rho$, $S^{i\alpha} = \rho U^\alpha$,

or $y^i = \rho U^i$, $S^{i\alpha} = \rho U^i U^\alpha + P\delta^{i\alpha}$; (A.3)

or $y^i = \rho \left(\mathcal{E} + \frac{U^\beta U^\beta}{2} \right)$, $S^{i\alpha} = \rho U^\alpha \left(\mathcal{H} + \frac{U^\beta U^\beta}{2} \right)$.

We now have a general equation which governs the fluid flow. We would like to deduce from this equation the behavior of the fluid both in portions of the field where the properties behave continuously, and across surfaces of discontinuity (shock waves).

Let us assume that the fluid is continuous except for a single surface moving through the fluid, across which the properties may change discontinuously. We can describe this surface parametrically by means of the equations $x^i = f^i(\sigma_0, \sigma_1, t)$. The velocity of the surface is $U^i = f^i_t(\sigma_0, \sigma_1, t)$, and the tangent vectors to the surface are $t^i_0 = f^i_{\sigma_0}$, $t^i_1 = f^i_{\sigma_1}$. The normal vector is $\vec{n} = (\vec{t}_0 \times \vec{t}_1)/|\vec{t}_0 \times \vec{t}_1|$. The normal component of the velocity of the surface is $\vec{U} \cdot \vec{n}$, which is related to the Jacobian $J = \begin{pmatrix} f^1 & f^2 & f^3 \\ \sigma_0 & \sigma_1 & t \end{pmatrix}$ by means of the determinant:

$$J = \begin{vmatrix} f^1_{\sigma_0} & f^2_{\sigma_0} & f^3_{\sigma_0} \\ f^1_{\sigma_1} & f^2_{\sigma_1} & f^3_{\sigma_1} \\ f^1_t & f^2_t & f^3_t \end{vmatrix} = |\vec{t}_0 \times \vec{t}_1| \, \vec{U} \cdot \vec{n} . \tag{A.4}$$

We find the time at which this surface passes a point (x^1, x^2, x^3) by solving the equations $x^i = f^i(\sigma_0, \sigma_1, t)$ for t. These equations may be solved provided $J \neq 0$, i.e., $\vec{U} \cdot \vec{n} \neq 0$.

It will be necessary to calculate $\frac{\partial \tau}{\partial x^i}$, where $t = \tau(x^i)$ is the solution to the equations $x^i = f^i(\sigma_0, \sigma_1, t)$ for t. For this purpose consider:

$$\begin{aligned} \delta^{1i} &= f^1_{\sigma_0} \sigma_{0x^i} + f^1_{\sigma_1} \sigma_{1x^i} + f^1_t \tau_{x^i} \ ; \\ \delta^{2i} &= f^2_{\sigma_0} \sigma_{0x^i} + f^2_{\sigma_1} \sigma_{1x^i} + f^2_t \tau_{x^i} \ ; \\ \delta^{3i} &= f^3_{\sigma_0} \sigma_{0x^i} + f^3_{\sigma_1} \sigma_{1x^i} + f^3_t \tau_{x^i} \ ; \end{aligned} \tag{A.5}$$

The coefficient determinant of these equations is $|\vec{t}_0 \times \vec{t}_1| \, \vec{U} \cdot \vec{n}$, whereas the determinant:

$$\Delta \tau_{x^i} = \begin{vmatrix} f^1_{\sigma_0} & f^1_{\sigma_1} & \delta^{1i} \\ f^2_{\sigma_0} & f^2_{\sigma_1} & \delta^{2i} \\ f^3_{\sigma_0} & f^3_{\sigma_1} & \delta^{3i} \end{vmatrix} = |\vec{t}_0 \times \vec{t}_1| \, n^i \tag{A.6}$$

or

$$\tau_{x^i} = \frac{\Delta \tau_{x^i}}{J} = \frac{n^i}{\vec{U} \cdot \vec{n}} \ . \tag{A.7}$$

Now since the surface is to represent a discontinuity in the fluid, we let:

$$y^i = y_0^i(x^1,x^2,x^3,t)\,h(t-\tau) + y_1^i(x^1,x^2,x^3,t)\,h(\tau-t),$$
$$S^{i\alpha} = S_0^{i\alpha}(x^1,x^2,x^3,t)\,h(t-\tau) + S_1^{i\alpha}(x^1,x^2,x^3,t)\,h(\tau-t), \tag{A.8}$$

where h(x) is the unit step function defined by: $h(x) = 0$, $x < 0$; $h(x) = 1$, $x > 0$; then $dh/dx = \delta(x)$, the Dirac delta function, which is an even function. Hence for $t < \tau$, $y^i = y_1^i$, $S^{i\alpha} = S_1^{i\alpha}$ and for $t > \tau$, $y^i = y_0^i$, $S^{i\alpha} = S_0^{i\alpha}$. $t < \tau$ represents points ahead of the surface, whereas $t > \tau$ represents points behind the surface.

Now

$$\frac{\partial y^i}{\partial t} = \frac{\partial y_0^i}{\partial t}h(t-\tau) + \frac{\partial y_1^i}{\partial t}h(\tau-t) + (y_0^i - y_1^i)\,\delta(t-\tau),$$

and (A.9)

$$\frac{\partial S^{i\alpha}}{\partial x^\alpha} = \frac{\partial S_0^{i\alpha}}{\partial x^\alpha}h(t-\tau) + \frac{\partial S_1^{i\alpha}}{\partial x^\alpha}h(\tau-t) + (S_1^{i\alpha} - S_0^{i\alpha})\frac{\partial \tau}{\partial x^\alpha}\,\delta(t-\tau).$$

Thus our general hydrodynamic equation (A.10) becomes:

$$\int_{V_0}\left(\frac{\partial y_0^i}{\partial t} + \frac{\partial S_0^{i\alpha}}{\partial x^\alpha}\right)dV + \int_{V_1}\left(\frac{\partial y_1^i}{\partial t} + \frac{\partial S_1^{i\alpha}}{\partial x^\alpha}\right)dV$$
$$+ \int_V\left[(y_0^i - y_1^i) - (S_0^{i\alpha} - S_1^{i\alpha})\frac{n^\alpha}{\vec{U}\cdot\vec{n}}\right]\delta(t-\tau)\,dV = 0 \tag{A.10}$$

where V_0 is that portion of V behind the surface of discontinuity and V_1 in front of the surface. If we take V to lie wholly behind or in front of the surface, the fluid is continuous, and hence we may conclude that:

$$\frac{\partial y_0^i}{\partial t} + \frac{\partial S_0^{i\alpha}}{\partial x^\alpha} = 0, \quad \text{and} \quad \frac{\partial y_1^i}{\partial t} + \frac{\partial S_1^{i\alpha}}{\partial x^\alpha} = 0, \tag{A.11}$$

If the surface is included in V, then we have by the well-known property of $\delta(x)$,

$$\int f(x)\,\delta(x)\,dx = f(0) \;,$$

at $t = \tau$ $$(y_0^i - y_1^i)\vec{U}\cdot\vec{n} - (\vec{S}_0^i - \vec{S}_1^i)\cdot\vec{n} = 0 \qquad \text{(A.12)}$$

or $$(\vec{S}_0^i - y_0^i\vec{U})\cdot\vec{n} = (\vec{S}_1^i - y_1^i\vec{U})\cdot\vec{n} \;.$$

Thus we conclude that in the continuous portion of the fluid, the differential equations (A.11): $\frac{\partial y^i}{\partial t} + \frac{\partial s^{i\alpha}}{\partial x^\alpha} = 0$ must be satisfied, whereas the relations (A.12) $(\vec{S}_0^i - y^i\,\vec{U})\cdot\vec{n} = (\vec{S}_1^i - y_1^i\,\vec{U})\cdot\vec{n}$ must connect the variables on the two sides of a surface of discontinuity.

We wish to return from the general form of the differential equations and shock conditions given above to the equations in terms of the original variables. We would like also to put the energy equation into a simpler form by use of the laws of thermodynamics.

In terms of the original variables the differential equations are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial\, \rho U^\alpha}{\partial x^\alpha} = 0 \;;$$

$$\frac{\partial\, \rho U^i}{\partial t} + \frac{\partial}{\partial x^\alpha}\left(\rho U^i U^\alpha + P\,\delta^{i\alpha}\right) = 0 \;; \qquad \text{(A.13)}$$

$$\frac{\partial}{\partial t}\,\rho\left(\mathcal{E} + \frac{U^\beta U^\beta}{2}\right) + \frac{\partial}{\partial x^\alpha}\,\rho U^\alpha\left(\mathcal{H} + \frac{U^\beta U^\beta}{2}\right) = 0 \;.$$

If we now write the combined first and second laws of thermodynamics for systems in equilibrium in the form:

$$T\,dS = d\mathcal{E} - \frac{P}{\rho^2}\,d\rho \,, \qquad \text{(A.14)}$$

where S is the specific entropy, the third equation of (A.13) may be simplified to:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + U^\alpha \frac{\partial S}{\partial x^\alpha} = 0 \quad .$$

The second equation may be written:

$$D\left(\frac{\partial U^i}{\partial t} + U^\alpha \frac{\partial U^i}{\partial x^\alpha}\right) + \frac{\partial P}{\partial x^i} = 0 \quad .$$

Hence in the continuous portion of the fluid we have:

$$\frac{\partial D}{\partial t} + \frac{\partial}{\partial x^\alpha} D U^\alpha = 0 \; ;$$

$$D\left(\frac{\partial U^i}{\partial t} + U^\alpha \frac{\partial U^i}{\partial x^\alpha}\right) + \frac{\partial P}{\partial x^i} = 0 \; ; \qquad \text{(A.15)}$$

$$\frac{\partial S}{\partial t} + U^\alpha \frac{\partial S}{\partial x^\alpha} = 0 \, .$$

To these we must add the equation of state of the fluid:

$$P = A D^\gamma e^{S/c_v} \quad .$$

The shock conditions, too, may be written in terms of the original variables. The first becomes:

$$D_0(\vec{U}_0 - \vec{U})\cdot\vec{n} = D_1(\vec{U}_1 - \vec{U})\cdot\vec{n} \quad ,$$

letting $\vec{V} = \vec{U} - \vec{U}$;

then $$D_0(\vec{V}_0 \cdot \vec{n}) = D_1(\vec{V}_1 \cdot \vec{n}) \quad .$$

The second becomes:

$$D_0(\vec{V}_0 \cdot \vec{n})\vec{V}_0 + P_0 \vec{n} = D_1(\vec{V}_1 \cdot \vec{n})\vec{V}_1 + P_1 \vec{n} \quad .$$

Multiplying scalarly by $\vec{n}$ we find:

$$\mathcal{D}_0(\vec{\mathcal{V}}_0\cdot\vec{n})^2 + \mathcal{P}_0 = \mathcal{D}_1(\vec{\mathcal{V}}_1\cdot\vec{n})^2 + \mathcal{P}_1 \quad ,$$

and multiplying vectorially by $\vec{n}$:

$$\vec{\mathcal{V}}_0 \times \vec{n} = \vec{\mathcal{V}}_1 \times \vec{n}$$

The third may be written:

$$\mathcal{H}_0 + \frac{1}{2}(\vec{\mathcal{V}}_0\cdot\vec{n})^2 = \mathcal{H}_1 + \frac{1}{2}(\vec{\mathcal{V}}_1\cdot\vec{n})^2$$

Thus across the shock we have:

$$\begin{aligned} \mathcal{D}_0(\vec{\mathcal{V}}_0\cdot\vec{n}) &= \mathcal{D}_1(\vec{\mathcal{V}}_1\cdot\vec{n}) \quad ; \\ \mathcal{P}_0 + \mathcal{D}_0(\vec{\mathcal{V}}_0\cdot\vec{n})^2 &= \mathcal{P}_1 + \mathcal{D}_1(\vec{\mathcal{V}}_1\cdot\vec{n})^2 \; ; \\ \vec{\mathcal{V}}_0\times\vec{n} &= \vec{\mathcal{V}}_1\times\vec{n} \quad ; \\ \mathcal{H}_0 + \frac{1}{2}(\vec{\mathcal{V}}_0\cdot\vec{n})^2 &= \mathcal{H}_1 + \frac{1}{2}(\vec{\mathcal{V}}_1\cdot\vec{n})^2 \; . \end{aligned} \tag{A.16}$$

Now we have the standard differential equations (A.15) and the shock conditions (A.16) of compressible hydrodynamics. The differential equations apply to the flow in regions containing no shock waves, and the two sides of a shock. Notice that in the above treatment the shock waves are not necessarily planes.

We shall put the shock conditions into a form which will be useful to us for later calculations. In doing so we shall also discuss some of the important properties of the shock transition.

Let us consider a two-dimensional situation in which $\mathcal{V}_n$ denotes the normal component of velocity, and $\mathcal{V}_t$ denotes the tangential component. If we assume that the gas is an ideal gas with the equation of state $\mathcal{P} = A\,\mathcal{D}^{\gamma} e^{\mathcal{S}/c_v}$, then the enthalpy $\mathcal{H} = \frac{\gamma}{\gamma-1}\frac{\mathcal{P}}{\mathcal{D}}$, where $\gamma = c_p/c_v$.

Thus the shock conditions are:

$$\mathcal{D}_0\mathcal{V}_{0n} = \mathcal{D}_1\mathcal{V}_{1n} \quad ;$$

$$\mathcal{P}_0 + \mathcal{D}_0\mathcal{V}_{0n}^2 = \mathcal{P}_1 + \mathcal{D}_1\mathcal{V}_{1n}^2 \quad ;$$

$$\mathcal{V}_{0t} = \mathcal{V}_{1t} \quad ; \tag{A.17}$$

$$\frac{\gamma}{\gamma-1}\frac{\mathcal{P}_0}{\mathcal{D}_0} + \frac{1}{2}\mathcal{V}_{0n}^2 = \frac{\gamma}{\gamma-1}\frac{\mathcal{P}_1}{\mathcal{D}_1} + \frac{1}{2}\mathcal{V}_{1n}^2 .$$

We have here four equations in eight unknowns. We may solve these equations for four of the unknown terms of any other four. Let us decide to find the quantities $\mathcal{P}_1$, $\mathcal{D}_1$, $\mathcal{V}_{1n}$, and $\mathcal{V}_{1t}$ in terms of the quantities $\mathcal{P}_0$, $\mathcal{D}_0$, $\mathcal{V}_{0n}$, and $\mathcal{V}_{0t}$. To do this most easily, let us solve the first two of Eqs. (A.17) for $\mathcal{D}_1$ and $\mathcal{P}_1$ in terms of $\mathcal{V}_{1n}$. Then we may solve the last equation for $\mathcal{V}_{1n}$, using these values for $\mathcal{D}_1$ and $\mathcal{P}_1$. Thus:

$$\mathcal{D}_1 = \mathcal{D}_0\frac{\mathcal{V}_{0n}}{\mathcal{V}_{1n}} \quad , \qquad \mathcal{P}_1 = \mathcal{P}_0 + \mathcal{D}_0\mathcal{V}_{0n}(\mathcal{V}_{0n}-\mathcal{V}_{1n}) ,$$

and
$$\frac{\gamma}{\gamma-1}\frac{\mathcal{P}_0}{\mathcal{D}_0} + \frac{1}{2}\mathcal{V}_{0n}^2 = \frac{\gamma}{\gamma-1}\frac{\mathcal{P}_0 + \mathcal{D}_0\mathcal{V}_{0n}(\mathcal{V}_{0n}-\mathcal{V}_{1n})}{\mathcal{D}_0\mathcal{V}_{0n}}\mathcal{V}_{1n} + \frac{1}{2}\mathcal{V}_{1n}^2$$

$$= \frac{\gamma}{\gamma-1}\frac{\mathcal{P}_0}{\mathcal{D}_0}\frac{\mathcal{V}_{1n}}{\mathcal{V}_{0n}} + \frac{\gamma}{\gamma-1}(\mathcal{V}_{0n}-\mathcal{V}_{1n})\mathcal{V}_{1n} + \frac{1}{2}\mathcal{V}_{1n}^2 .$$

Now we let $\gamma\frac{\mathcal{P}}{\mathcal{D}} = C^2$, C_0^2 being the speed of sound in the gas on side "0."
Then we find:

$$\frac{C_0^2}{\gamma-1} + \frac{1}{2}\mathcal{V}_{0n}^2 = \frac{C_0^2}{\gamma-1}\frac{\mathcal{V}_{1n}}{\mathcal{V}_{0n}} + \frac{\gamma}{\gamma-1}(\mathcal{V}_{0n}\mathcal{V}_{1n}-\mathcal{V}_{1n})^2 + \frac{1}{2}\mathcal{V}_{1n}^2 .$$

For convenience let $\mu = \frac{\gamma+1}{\gamma-1}$; then we have:

$$(\mu-1)C_0^2 + \mathcal{V}_{0n}^2 = \left[(\mu-1)\frac{C_0^2}{\mathcal{V}_{0n}} + (\mu+1)\mathcal{V}_{0n}\right]\mathcal{V}_{1n} - \mu\mathcal{V}_{1n}^2 . \tag{A.18}$$

Now we define the normal Mach number $\mathfrak{M}_0$ by means of $\mathfrak{M}_0 = \frac{\mathcal{V}_{0n}}{C_0}$, and note that $\mathfrak{M}_0$ is dimensionless. In terms of $\mathfrak{M}_0$, our Eq. (A.18) becomes:

$$\mu v_{1n}^2 - c_0\left[(\mu-1)\frac{1}{\mathcal{M}_0} + (\mu+1)\mathcal{M}_0\right]v_{1n} + c_0^2\left[(\mu-1)+\mathcal{M}_0^2\right] = 0,$$

which may be solved to give:

$$v_{1n} = v_{0n}, \qquad \text{or} \qquad v_{1n} = \frac{c_0}{\mu}\left[\mathcal{M}_0 + \frac{(\mu-1)}{\mathcal{M}_0}\right] .$$

We rule out the first possibility since it does not represent a discontinuity in velocity, and we confine our attention to the second. The second may be written in the form:

$$v_{1n} = \frac{v_{0n}}{\mu}\left[1 + \frac{(\mu-1)}{\mathcal{M}_0^2}\right] \tag{A.19}$$

We are now in a position to calculate $\mathcal{P}_1$ and $\mathcal{D}_1$. Since

$$v_{0n} - v_{1n} = v_{0n} - \frac{v_{0n}}{\mu}\left[1 - \frac{\mu-1}{\mathcal{M}_0^2}\right]$$

we have

$$v_{0n} - v_{1n} = v_{0n}\,\frac{(\mu-1)}{\mu}\,\frac{(\mathcal{M}_0^2-1)}{\mathcal{M}_0^2} .$$

Thus

$$\mathcal{P}_1 = \mathcal{P}_0\left[1 + \frac{\mathcal{D}_0}{\mathcal{P}_0}v_{0n}(v_{0n}-v_{1n})\right]$$

becomes:

$$\mathcal{P}_1 = \frac{\mathcal{P}_0}{\mu}\left[(\mu+1)\mathcal{M}_0^2 - 1\right] . \tag{A.20}$$

We calculate $\mathcal{D}_1$ from (A.17) and (A.19):

$$\mathcal{D}_1 = \mathcal{D}_0\,\frac{\mu\,\mathcal{M}_0^2}{\mathcal{M}_0^2 + (\mu-1)} . \tag{A.21}$$

We have now found $\frac{\mathcal{P}_1}{\mathcal{P}_0}$ and $\frac{\mathcal{D}_1}{\mathcal{D}_0}$ in terms of $\mathfrak{M}_0$. Let us eliminate $\mathfrak{M}_0$ from these equations and find $\frac{\mathcal{D}_1}{\mathcal{D}_0}$ in terms of $\frac{\mathcal{P}_1}{\mathcal{P}_0}$. To do this we notice that $\mu\frac{\mathcal{P}_1}{\mathcal{P}_0} = (\mu+1)\mathfrak{M}_0^2 - 1$, hence $\mathfrak{M}_0^2 = \frac{\mu\,\mathcal{P}_1/\mathcal{P}_0 + 1}{\mu+1}$, so we have:

$$\frac{\mathcal{D}_1}{\mathcal{D}_0} = \frac{\mu\,\mathcal{P}_1/\mathcal{P}_0 + 1}{\mathcal{P}_1/\mathcal{P}_0 + \mu} \quad . \tag{A.22}$$

This is the well-known Rankine-Hugoniot equation.

The sound speed on side "1" satisfies the equation:

$$C_1^2 = \gamma\frac{\mathcal{P}_1}{\mathcal{D}_1} = C_0^2\,\frac{((\mu+1)\mathfrak{M}_0^2 - 1)(\mathfrak{M}_0^2 + (\mu-1))}{\mu^2\,\mathfrak{M}_0^2} \tag{A.23}$$

Hence:

$$\mathfrak{M}_1^2 = \frac{\mathcal{V}_{1n}^2}{C_1^2} = \frac{\mathfrak{M}_0^2 + (\mu-1)}{(\mu+1)\mathfrak{M}_0^2 - 1} \;,$$

and thus we have:

$$\mathfrak{M}_1^2 = \frac{\mathfrak{M}_0^2 + (\mu-1)}{(\mu+1)\mathfrak{M}_0^2 - 1} \;. \tag{A.24}$$

Solving for $\mathfrak{M}_0^2$ we have:

$$\mathfrak{M}_0^2 = \frac{\mathfrak{M}_1^2 + (\mu-1)}{(\mu+1)\mathfrak{M}_1^2 - 1} \tag{A.25}$$

The symmetry of these two relations is due to the symmetry of the original equations. Notice also that:

$$\mathfrak{M}_1^2 \lesseqqgtr 1 \qquad \text{implies} \qquad \mathfrak{M}_0^2 \gtreqqless 1 \;.$$

Now let us consider the entropy difference on the two sides of the shock wave.

$$\mathcal{S}_1 - \mathcal{S}_0 = C_v\,\ln\left[\frac{\mathcal{P}_1}{\mathcal{D}_1^{\gamma}}\cdot\frac{\mathcal{D}_0^{\gamma}}{\mathcal{P}_0}\right] \;.$$

Thus:

$$\frac{\mathcal{S}_1 - \mathcal{S}_0}{c_v} = \ln \frac{(\mu+1)\mathcal{M}_0^2 - 1}{\mu} - \left(\frac{\mu+1}{\mu-1}\right) \ln \frac{\mu \mathcal{M}_0^2}{\mathcal{M}_0^2 + (\mu-1)} . \qquad (A.26)$$

We give a summary of the equations which are important to us in the text. These are:

$$\begin{aligned} \mathcal{V}_{1n} &= \frac{\mathcal{V}_{0n}}{\mu}\left(1 + \frac{\mu-1}{\mathcal{M}_0^2}\right) \quad ; \\ \mathcal{P}_1 &= \mathcal{P}_0 \frac{(\mu+1)\mathcal{M}_0^2 - 1}{\mu} \quad ; \\ \mathcal{D}_1 &= \mathcal{D}_0 \frac{\mu \mathcal{M}_0^2}{\mathcal{M}_0^2 + (\mu-1)} \quad ; \\ \mathcal{V}_{1t} &= \mathcal{V}_{0t} \quad . \end{aligned} \qquad (A.27)$$

These equations may also be written:

$$\begin{aligned} \mathcal{V}_{1n} &= \mathcal{V}_{0n} F(\mathcal{M}_0) \quad ; \\ \mathcal{P}_1 &= \mathcal{P}_0 \, G(\mathcal{M}_0) \quad ; \\ \mathcal{D}_1 &= \mathcal{D}_0 \frac{1}{F(\mathcal{M}_0)} \quad ; \end{aligned} \qquad (A.28)$$

where

$$\begin{aligned} F(\mathcal{M}_0) &= \frac{\mathcal{M}_0^2 + (\mu-1)}{\mu \mathcal{M}_0^2} \quad , \\ G(\mathcal{M}_0) &= \frac{(\mu+1)\mathcal{M}_0^2 - 1}{\mu} \quad . \end{aligned}$$

Notice that:

$$\frac{\mathcal{S}_1 - \mathcal{S}_0}{c_v} = \ln G(\mathcal{M}_0) + \frac{\mu+1}{\mu-1} \ln F(\mathcal{M}_0) .$$

Let us now investigate a particular solution of the differential equations and the corresponding shock conditions. As was pointed out in the introduction, the problem with which we wish to deal concerns the interaction of a sound wave with a shock wave. If we wish to deal with a shock wave alone, we should first investigate the simplest solution of the differential equations, that in which all the dependent variables are held constant.

To do this, we set $\mathfrak{P} = P$, $\mathfrak{D} = D$, $\mathfrak{V}_n = U$, and $\mathfrak{V}_t = 0$. P, D, and S must satisfy the equation of state

$$P = A D^{\gamma} e^{S/c_v} \tag{A.29}$$

The shock conditions then imply that the variables on the two sides of the shock wave are related by:

$$\begin{aligned} U_1 &= U_0 \, F(M_0) \; ; \\ P_1 &= P_0 \, G(M_0) \; ; \\ D_1 &= D_0 \frac{1}{F(M_0)} \; . \end{aligned} \tag{A.30}$$

The entropy equation may be written as:

$$\frac{S_1 - S_0}{c_v} = \ln G(M_0) + \frac{\mu+1}{\mu-1} F(M_0) \; .$$

These equations thus determine the state "1" if the state "0" is known, and conversely. In this particular solution the flow is essentially one-dimensional, since we have set $\mathfrak{V}_{1t} = \mathfrak{V}_{0t} = 0$.

APPENDIX B

CHARACTERISTICS OF THE LINEAR FLOW EQUATIONS

The characteristics of the differential equations (2.4) are defined as curves across which p, u, v, and Ω may be discontinuous. In Appendix A the method of describing such curves is given in detail. We apply this method to Eqs. (2.4) and find that the following equations result:

$$
\begin{aligned}
(\vec{U}_c\cdot\vec{n} - Un_x)[p] - Cn_x[u] - Cn_y[v] &= 0; \\
-Cn_x[p] + (\vec{U}_c\cdot\vec{n} - Un_x)[u] &= 0; \\
-Cn_y[p] + (\vec{U}_c\cdot\vec{n} - Un_x)[v] &= 0; \\
(\vec{U}_c\cdot\vec{n} - Un_x)[\Omega] &= 0.
\end{aligned}
\tag{B.1}
$$

In these equations the [ ] symbol is used to represent the amplitude of the discontinuity. $\vec{U}_c$ represents the velocity of the characteristic curve in question, and $\vec{n}$ represents the normal vector to that curve.

Equations (B.1) have solutions for $[p]$, $[u]$, $[v]$, and $[\Omega]$, provided that:

$$
\begin{vmatrix}
(\vec{U}_c\cdot\vec{n} - Un_x) & -Cn_x & -Cn_y & 0 \\
-Cn_x & (\vec{U}_c\cdot\vec{n} - Un_x) & 0 & 0 \\
-Cn_y & 0 & (\vec{U}_c\cdot\vec{n} - Un_x) & 0 \\
0 & 0 & 0 & (\vec{U}_c\cdot\vec{n} - Un_x)
\end{vmatrix} = 0 ,
$$

or

$$(\vec{U}_c \cdot \vec{n} - U n_x)^2 \left((\vec{U}_c \cdot \vec{n} - U n_x)^2 - c^2 \right) = 0 \ . \tag{B.2}$$

There are two cases to consider:

$$(\vec{U}_c \cdot \vec{n} - U n_x)^2 = c^2 \ , \tag{B.3}$$

and

$$(\vec{U}_c \cdot \vec{n} - U n_x) = 0 \ . \tag{B.4}$$

In the former case, (B.3), the solutions to Eqs. (B.1) may be written:

$$\begin{aligned} [p] &= A \ ; \\ [u] &= \pm n_x A \ ; \\ [v] &= \pm n_y A \ ; \\ [s] &= 0 \ , \end{aligned} \tag{B.5}$$

where A is an arbitrary constant, and the choice of sign depends on the choice of sign in taking the square root in (B.3). In the latter case, (B.4), the solutions to Eqs. (B.1) may be written:

$$\begin{aligned} [p] &= 0 \ ; \\ [u] &= -n_y B \ ; \\ [v] &= n_x B \ ; \\ [s] &= C \ . \end{aligned} \tag{B.6}$$

where B and C are arbitrary constants.

If we describe the characteristic curves by means of a function

$\phi(x,y,t) = 0$, then $n_x = \frac{\phi_x}{|\vec{\nabla}\phi|}$, $n_Y = \frac{\phi_Y}{|\vec{\nabla}\phi|}$, and $\vec{U}_c \cdot \vec{n} = -\frac{\phi_t}{|\vec{\nabla}\phi|}$.

Equations (B.3) may be satisfied by choosing $\phi(x,y,t) = Ct - \psi(x-Ut,y)$, for which Eq. (B.3) becomes $|\vec{\nabla}\psi|^2 = 1$. The choice of the sign in the second and third of Eqs. (B.3) is the same as the choice of sign in $|\vec{\nabla}\psi| = \pm 1$.

Equations (B.4) may be satisfied by choosing $\phi(x,y,t) = \chi(x-Ut,y)$. This type of expression satisfies (B.4) identically.

APPENDIX C

SIMPLIFICATION OF THE LINEAR SHOCK CONDITIONS

To simplify Eqs. (2.14) of Section II, we must first calculate $\mathfrak{m}_0$ in terms of $\mathfrak{u}_0, \mathfrak{v}_0, \mathfrak{p}_0$, and $\mathfrak{d}_0$. Since $\mathcal{M}_0 = \frac{\mathcal{U}_{ORN}}{\mathcal{C}_0}$, we have:

$$\ln \mathcal{M}_0 = \ln \mathcal{U}_{ORN} - \tfrac{1}{2}\ln \mathcal{P}_0 + \tfrac{1}{2}\ln \mathcal{D}_0 + \tfrac{1}{2}\ln \gamma \ . \tag{C.1}$$

Differentiating (C.1), we find:

$$\frac{\mathfrak{m}_0}{M_0} = \frac{\mathfrak{u}_0 U_0 - f_t}{U_0} - \tfrac{1}{2}\mathfrak{p}_0 \frac{D_0 C_0 U_0}{P_0} + \tfrac{1}{2}\mathfrak{d}_0 \ ,$$

which becomes, by virtue of the fifth of Eqs. (2.4):

$$\mathfrak{m}_0 = M_0 \mathfrak{u}_0 - \frac{M_0^2}{\mu-1}\mathfrak{p}_0 - \frac{M_0}{2}\mathfrak{d}_0 - \frac{f_t}{C_0} \ . \tag{C.2}$$

We now use (C.2) to eliminate $\mathfrak{m}_0$ from (2.14). We find:

$$\begin{aligned}
\mathfrak{p}_1 M_1 &= \mathfrak{p}_0 M_0 + \frac{\mu-1}{\mu+1}\frac{G'(M_0)}{G(M_0)}\left(M_0\mathfrak{u}_0 - \frac{M_0^2}{\mu-1}\mathfrak{p}_0 - \frac{M_0}{2}\mathfrak{d}_0 - \frac{f_t}{C_0}\right) \\
&= \frac{\mu-1}{\mu+1}\frac{M_0 G'(M_0)}{G(M_0)}\mathfrak{u}_0 + M_0\left(1 - \frac{M_0 G'(M_0)}{(\mu+1)G(M_0)}\right)\mathfrak{p}_0 \\
&\quad - \frac{\mu-1}{2(\mu+1)}\frac{M_0 G'(M_0)}{G(M_0)}\mathfrak{d}_0 - \frac{(\mu-1)\,G'(M_0)}{(\mu+1)\,G(M_0)}\frac{f_t}{C_0} \qquad ; \qquad \text{(C.3)} \\
\mathfrak{u}_1 &= \mathfrak{u}_0 + \frac{F'(M_0)}{F(M_0)}\left(M_0\mathfrak{u}_0 - \frac{M_0^2}{\mu-1}\mathfrak{p}_0 - \frac{M_0}{2}\mathfrak{d}_0 - \frac{f_t}{C_0}\right) + \frac{1-F(M_0)}{M_0 F(M_0)}\frac{f_t}{C_0} \\
&= \frac{1}{F(M_0)}\Big[\big(F(M_0) + M_0 F'(M_0)\big)\mathfrak{u}_0 - M_0^2 F'(M_0)\mathfrak{p}_0 - \frac{M_0 F'(M_0)}{2}\mathfrak{d}_0 \\
&\quad + \frac{1}{M_0}\big(1 - F(M_0) - M_0 F'(M_0)\big)\frac{f_t}{C_0}\Big] \qquad ;
\end{aligned}$$

$$v_1 = \frac{1}{F(M_0)} v_0 + \left(\frac{1 - F(M_0)}{F(M_0)}\right) f_y$$

$$= \frac{1}{F(M_0)} \left(v_0 - (1 - F(M_0)) f_y\right) \text{ ; and} \qquad \text{(C.3)}$$

$$s_1 = s_0 + \left(\frac{\mu-1}{\mu+1}\frac{G'(M_0)}{G(M_0)} + \frac{F'(M_0)}{F(M_0)}\right)\left(M_0 u_0 - \frac{M_0^2}{\mu-1} p_0 - \frac{M_0}{2} s_0 - \frac{f_t}{C_0}\right).$$

Now $\frac{1}{G(M_0)} = \frac{M_1^2}{M_0^2 F(M_0)}$ and $G'(M_0) = 2\frac{\mu+1}{\mu} M_0$.

Substituting in the first of Eqs. (C.3), we find for p_1

$$p_1 M_1 = \frac{M_0 M_1^2}{\mu M_0^2 F(M_0)} \left\{ \frac{\mu-1}{\mu+1} \frac{2(\mu+1)}{\mu} M_0^2 u_0 + M_0 \left(\frac{(\mu+1)M_0^2 - 1}{\mu} - \frac{2(\mu+1)M_0^2}{(\mu+1)\mu}\right) p_0 \right.$$
$$\left. - \frac{\mu-1}{2(\mu+1)} \cdot \frac{2(\mu+1)}{\mu} M_0^2 s_0 - \frac{(\mu-1)}{\mu+1} \cdot \frac{2(\mu+1)}{\mu} M_0 \frac{f_t}{C_0} \right\} ,$$

or

$$p_1 = \frac{M_1 M_0}{\mu M_0^2 F(M_0)} \left\{ 2(\mu-1) M_0 u_0 + ((\mu-1)M_0^2 - 1) p_0 - (\mu-1) M_0 s_0 - 2\frac{(\mu-1)}{C_0} f_t \right\}. \qquad \text{(C.4)}$$

Using

$$F(M_0) = \frac{M_0^2 + (\mu-1)}{\mu M_0^2} , \qquad F'(M_0) = -\frac{2(\mu-1)}{\mu M_0^3}$$

in the second of Eqs. (C.3), we find for u_1:

$$u_1 = \frac{1}{F(M_0)} \left\{ \left(\frac{M_0^2 - (\mu-1)}{\mu M_0^2}\right) u_0 + \frac{2 M_0}{\mu M_0^2} p_0 + \frac{\mu-1}{\mu M_0^2} s_0 + \left(\frac{(\mu-1)(M_0^2+1)}{\mu M_0^3}\right) \frac{f_t}{C_0} \right\}$$

$$= \frac{1}{\mu M_0^2 F(M_0)} \left\{ (M_0^2 - (\mu-1)) u_0 + 2 M_0 p_0 + (\mu-1) s_0 + \frac{(\mu-1)(M_0^2+1)}{M_0} \frac{f_t}{C_0} \right\}. \qquad \text{(C.4)}$$

Using the value above for $F(M_0)$ in the third of Eqs. (C.3), we find for v_1:

$$v_1 = \frac{1}{\mu M_0^2 F(M_0)} \left\{ \mu M_0^2 v_0 + (\mu-1)(M_0^2 - 1) f_y \right\}. \qquad \text{(C.4)}$$

Noting that $\frac{F'(M_0)}{F(M_0)} = -\frac{2(\mu-1)}{M_0(M_0^2 + (\mu-1))}$ and $\frac{G'(M_0)}{G(M_0)} = \frac{2(\mu+1) M_0}{(\mu+1)M_0^2 - 1}$,

we find that $\left[\frac{\mu-1}{\mu+1}\frac{G'(M_0)}{G(M_0)} + \frac{F'(M_0)}{F(M_0)}\right] = \frac{M_1^2}{\mu^2 F^2(M_0) M_0^4}\left[\frac{2(\mu-1)}{M_0}(M_0^2 - 1)^2\right].$

Using this in the last of Eqs. (C.3), we find:

$$
\begin{aligned}
s_1 &= \frac{M_1^2}{\mu^2 M_0^4 F^2(M_0)} \Big\{ 2(\mu-1)(M_0^2-1)^2 u_0 - 2M_0(M_0^2-1)^2 p_0 \\
&\quad + \big(((\mu+1)M_0^2-1)(M_0^2+\mu-1)) - (\mu-1)(M_0^2-1)^2\big) s_0 - \frac{2(\mu-1)}{M_0}(M_0^2-1)\frac{f_t}{c_0} \Big\} \\
&= \frac{M_1^2}{\mu^2 M_0^4 F^2(M_0)} \Big\{ 2(\mu-1)(M_0^2-1)^2 u_0 - 2M_0(M_0^2-1)^2 p_0 \\
&\quad + \big(\mu^2 M_0^2 + 2(M_0^2+\mu-1)(M_0^2-1)\big) s_0 - 2\frac{(\mu-1)}{M_0}(M_0^2-1)^2 \frac{f_t}{c_0} \, .
\end{aligned}
\tag{C.4}
$$

APPENDIX D

TRANSFORMATION OF THE "FRESNEL" COEFFICIENTS FOR THE REFLECTED WAVES

Equations (3.37) for the "Fresnel" coefficients of the reflected waves may be rewritten in terms of α_0 and β_0 alone by using the reflection laws (3.22) and (3.26).

We note first that by virtue of the reflection laws:

$$\vec{n}_0 \cdot \vec{n}_2 = -\vec{n}_1 \cdot \vec{n}_2 = \frac{\alpha_0 + M_1}{\sqrt{1 + M_1^2 + 2M_1\alpha_0}} \quad , \tag{D.1}$$

and that:

$$\alpha_1\beta_0 - \alpha_0\beta_1 = -\frac{\beta_0}{\sqrt{1 + M_1^2 + 2M_1\alpha_0}} \quad ; \tag{D.2}$$

$$\alpha_1 - \alpha_0 = -\frac{2(M_1+\alpha_0)(1+M_1\alpha_0)}{1 + M_1^2 + 2M_1\alpha_0} \quad ; \tag{D.3}$$

$$\beta_1 - \beta_0 = -\frac{2(M_1+\alpha_0)M_1\beta_0}{1 + M_1^2 + 2M_1\alpha_0} \quad . \tag{D.4}$$

Using the expressions (D.1) to (D.4) Eqs. (3.37) for the "Fresnel" coefficients, we find:

$$\begin{aligned}
a &= 2i(\vec{n}_1 \cdot \vec{n}_2)\, \epsilon/\Delta \quad ; \\
A &= k_1(\vec{n}_2 \cdot \vec{b} + \vec{n}_1 \cdot \vec{n}_2\, b_1)\, \epsilon/\Delta \quad ; \\
B &= k_1\left(-\frac{2(M_1+\alpha_0)M_1\beta_0}{1+M_1^2+2M_1\alpha_0}\, b_2 + \frac{2(M_1+\alpha_0)(1+M_1\alpha_0)}{1+M_1^2+2M_1\alpha_0}\, b_3 - \frac{\beta_0 b_1}{\sqrt{1+M_1^2+2M_1\alpha_0}}\right)\frac{\epsilon}{\Delta} \quad ; \\
C &= -2k_1(\vec{n}_1 \cdot \vec{n}_2)\, b_4\, \epsilon/\Delta \quad ,
\end{aligned} \tag{D.5}$$

where

$$\Delta = k_1 (\vec{n}_2 \cdot \vec{b} - (\vec{n}_1 \cdot \vec{n}_2) b_1).$$

Using the values for b_1, b_2, b_3, and b_4 from (3.33), we find:

$$\vec{n}_2 \cdot \vec{b} - (\vec{n}_1 \cdot \vec{n}_2) b_1 = - \frac{(\mu-1)\{[\alpha_0(1-M_1^2) - M_1(1+M_1\alpha_0)](\alpha_0+M_1)M_0^2 + (1+M_1\alpha_0)^2\}}{\mu M_1 M_0^2 (1+M_1\alpha_0)\sqrt{1+M_1^2+2M_1\alpha_0}} ; \quad (D.6)$$

$$\vec{n}_2 \cdot \vec{b} + (\vec{n}_1 \cdot \vec{n}_2) b_1 = - \frac{(\mu-1)\{[(\alpha_0+M_1) + 2M_1(1+M_1\alpha_0)](\alpha_0+M_1)M_0^2 + (1+M_1\alpha_0)^2\}}{\mu M_1 M_0^2 (1+M_1\alpha_0)\sqrt{1+M_1^2+2M_1\alpha_0}} ; \quad (D.7)$$

$$\left\{ - \frac{2(M_1+\alpha_0)M_1\beta_0}{1+M_1^2+2M_1\alpha_0} b_2 + \frac{2(M_1+\alpha_0)(1+M_1\alpha_0)}{1+M_1^2+2M_1\alpha_0} b_3 - \frac{\beta_0 b_1}{\sqrt{1+M_1^2+2M_1\alpha_0}} \right\}$$

$$= \frac{2(1+M_1\alpha_0)\beta_0(\mu-1)\{(M_1^2+M_0^2)(M_1+\alpha_0) - M_1^2 M_0^2 \sqrt{1+M_1^2+2M_1\alpha_0}\}}{\mu M_1^2 M_0^2 (1+M_1\alpha_0)(1+M_1^2+2M_1\alpha_0)} . \quad (D.8)$$

The amplitudes a, A, B, and C become:

$$a = \frac{i}{k_1} \frac{2\mu M_1 M_0^2 (\alpha_0+M_1)(1+M_1\alpha_0)\,\epsilon}{(\mu-1)\{[\alpha_0(1-M_1^2) - M_1(1+M_1\alpha_0)](\alpha_0+M_1)M_0^2 + (1+M_1\alpha_0)^2\}} ;$$

$$A = - \frac{\{((\alpha_0+M_1) + 2M_1(1+M_1\alpha_0))(\alpha_0+M_1)M_0^2 + (1+M_1\alpha_0)^2\}\,\epsilon}{[\alpha_0(1-M_1^2) - M_1(1+M_1\alpha_0)](\alpha_0+M_1)M_0^2 + (1+M_1\alpha_0)^2} ;$$

$$B = - \frac{2(1+M_1\alpha_0)\beta_0\{(M_1^2+M_0^2)(M_1+\alpha_0) - M_1^2 M_0^2\sqrt{1+M_1^2+2M_1\alpha_0}\}\,\epsilon}{M_1\sqrt{1+M_1^2+2M_1\alpha_0}\{[\alpha_0(1-M_1^2) - M_1(1+M_1\alpha_0)](\alpha_0+M_1)M_0^2 + (1+M_1\alpha_0)^2\}} ; \quad (D.9)$$

$$C = - \frac{4(\alpha_0+M_1)(1+M_1\alpha_0)(M_0^2-1)(1-M_1^2)\,\epsilon}{\mu\{[\alpha_0(1-M_1^2) - M_1(1+M_1\alpha_0)](\alpha_0+M_1)M_0^2 + (1+M_1\alpha_0)^2\}} .$$

We shall transform the functions $a, A, B,$ and C into forms more useful for the calculations of Section V. In particular we use the relations (4.33) to find the appropriate formulae in the θ-plane of Section IV, Part A, Fig. 4. We use the relations:

$$\frac{\alpha_0 + M_1}{1 + M_1\alpha_0} = -\alpha \quad , \quad \frac{\beta_0}{1 + M_1\alpha_0} = \frac{\beta}{\sqrt{1 - M_1^2}} \quad , \quad \frac{(1 - M_1^2)\alpha_0}{1 + M_1\alpha_0} = -(M_1 + \alpha), \tag{D.10}$$

$$1 + M_1\alpha_0 = \frac{1 - M_1^2}{1 + M_1\alpha} \quad , \qquad \sqrt{1 + M_1^2 + 2M_1\alpha_0} = \sqrt{1 - M_1^2} \quad ,$$

to find:

$$a = -\frac{i}{k_1}\frac{2\mu M_1 M_0^2 \alpha}{(\mu - 1)\{M_0^2\alpha^2 + 2M_1M_0^2\alpha + 1\}}\,\epsilon \quad ;$$

$$A = -\frac{M_0^2\alpha^2 - 2M_1M_0^2\alpha + 1}{M_0^2\alpha^2 + 2M_1M_0^2\alpha + 1}\,\epsilon \quad ;$$

$$B = \frac{2\beta\{\sqrt{1 - M_1^2}\,(M_1^2 + M_0^2)\alpha + M_1^2M_0^2(1 + M_1\alpha)\}}{M_1\sqrt{1 - M_1^2}\,(1 + M_1\alpha)\{M_0^2\alpha^2 + 2M_1M_0^2\alpha + 1\}}\,\epsilon \quad ; \tag{D.11}$$

$$C = \frac{4(M_0^2 - 1)(1 - M_1^2)\alpha}{\mu\{M_0^2\alpha^2 + 2M_1M_0^2\alpha + 1\}}\,\epsilon \quad .$$

APPENDIX E

TRANSFORMATION OF THE "FRESNEL" COEFFICIENTS FOR THE REFRACTED WAVES

In Section III, Part C, (3.64), we have shown that the "Fresnel" coefficients for refraction may be written:

$$\mathcal{a} = \frac{i}{k_0}\,\frac{\vec{n}_2\cdot\vec{\mathcal{a}} - \vec{n}_1\cdot\vec{n}_2\,\mathcal{a}_1}{\vec{n}_2\cdot\vec{b} - \vec{n}_1\cdot\vec{n}_2\,b_1}\,\epsilon\;;$$

$$A = \frac{\mathcal{a}_1(\vec{n}_2\cdot\vec{b}) - b_1(\vec{n}_2\cdot\vec{\mathcal{a}})}{\vec{n}_2\cdot\vec{b} - \vec{n}_1\cdot\vec{n}_2\,b_1}\,\epsilon\;;$$

$$B = \frac{\alpha_1(\mathcal{a}_1 b_3 - b_1\mathcal{a}_3) + \beta_1(\mathcal{a}_2 b_1 - b_2\mathcal{a}_1) + (\mathcal{a}_3 b_2 - \mathcal{a}_2 b_3)}{\vec{n}_2\cdot\vec{b} - \vec{n}_1\cdot\vec{n}_2\,b_1}\,\epsilon\;; \tag{E.1}$$

$$C = \frac{(\vec{n}_1\cdot\vec{n}_2)(\mathcal{a}_1 b_4 - b_1\mathcal{a}_4) + \mathcal{a}_4(\vec{n}_2\cdot\vec{b}) - b_4(\vec{n}_2\cdot\vec{\mathcal{a}})}{\vec{n}_2\cdot\vec{b} - \vec{n}_1\cdot\vec{n}_2\,b_1}\,\epsilon\,.$$

It is convenient for the calculations of Section V, Part B, to transform $\mathcal{a}$, B, and C to the aberrated angles of Section IV by means of (4.64), and to transform A to a completely different angle variable.

Before discussing the transformations of A, let us first transform the other three quantities to the θ-plane. The transformation (4.64) of Section IV, Part B, when applied to (3.43) and (3.52) give:

$$\alpha_1 = \frac{M_1\lambda^2\beta^2 + \sqrt{M_0^2-1}\,\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}}{(M_0^2-1) - M_1^2\lambda^2\beta^2}\;;$$

$$\beta_1 = i\lambda\beta\,\frac{\sqrt{M_0^2-1} + M_1\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}}{(M_0^2-1) - M_1^2\lambda^2\beta^2}\;; \tag{E.2}$$

$$\alpha_2 = \frac{\sqrt{M_0^2 - 1}}{\sqrt{(M_0^2-1) - M_1^2\lambda^2\beta^2}} \quad ;$$

$$\beta_2 = \frac{i M_1 \lambda \beta}{\sqrt{(M_0^2-1) - M_1^2\lambda^2\beta^2}} \quad ; \tag{E.3}$$

$$n_1 \cdot n_2 = \sqrt{\frac{(M_0^2-1) + (1-M_1^2)\lambda^2\beta^2}{(M_0^2-1) - M_1^2\lambda^2\beta^2}} \, . \tag{E.4}$$

We also find from (3.61):

$$a_1\left[\mu M_0^2 F(M_0)\right] = \frac{M_1 M_0 \left[-2(\mu-1) M_0(\alpha+M_0) + ((\mu-1)M_0^2-1)(1+M_0\alpha)\right]}{1+M_0\alpha} \, ;$$

$$a_2\left[\mu M_0^2 F(M_0)\right] = \frac{-(M_0^2-(\mu-1))(\alpha+M_0) + 2M_0(1+M_0\alpha)}{1+M_0\alpha} \, ;$$

$$a_3\left[\mu M_0^2 F(M_0)\right] = \frac{-i\mu M_0^2 \beta \sqrt{M_0^2-1}}{1+M_0\alpha} \, ; \tag{E.5}$$

$$a_4\left[\mu M_0^2 F(M_0)\right] = -\frac{2M_1^2(M_0^2-1)^2 \left[(\mu-1)(\alpha+M_0) + M_0(1+\alpha M_0)\right]}{\mu M_0^2 F(M_0) \qquad (1+M_0\alpha)} \, ;$$

$$b_1\left[\mu M_0^2 F(M_0)\right] = 2 M_1 M_0 (\mu-1) \, ;$$

$$b_2\left[\mu M_0^2 F(M_0)\right] = -\left(\frac{M_0^2+1}{M_0}\right)(\mu-1) \, ; \tag{E.6}$$

$$b_3\left[\mu M_0^2 F(M_0)\right] = i\beta\sqrt{M_0^2-1}\,(\mu-1) \, ;$$

$$b_4\left[\mu M_0^2 F(M_0)\right] = \frac{2M_1^2(M_0^2-1)^2}{\mu M_0^3}(\mu-1) .$$

Hence:

$$i k_o a = \frac{\left(\Gamma_3 - \Gamma_4 \sqrt{(M_o^2-1)+(1-M_1^2)\lambda^2\beta^2}\right) \epsilon}{\left(\Gamma_1 + \Gamma_2 \sqrt{(M_o^2-1)+(1-M_1^2)\lambda^2\beta^2}\right)(\mu-1)(1+M_o\alpha)} , \tag{E.7}$$

where

$$\Gamma_1 = M_o^2+1 + \lambda M_1 M_o \beta^2 \quad ;$$

$$\Gamma_2 = 2 M_1 M_o^2 \quad ;$$

$$\Gamma_3 = M_o\left[-(M_o^2-(\mu-1))(\alpha+M_o) + 2M_o(1+M_o\alpha) + \lambda\mu M_1 M_o^2\beta^2\right]\sqrt{M_o^2-1} \quad ; \tag{E.8}$$

$$\Gamma_4 = M_1 M_o^2\left[-2(\mu-1)M_o(\alpha+M_o) + ((\mu-1)M_o^2-1)(1+M_o\alpha)\right] \quad ;$$

and

$$B = \frac{i\beta M_o\left[\Gamma_5 + \Gamma_6\sqrt{(M_o^2-1)+(1-M_1^2)\lambda^2\beta^2}\right]\epsilon}{\mu M_o^2 F(M_o)\left[\Gamma_1+\Gamma_2\sqrt{(M_o^2-1)+(1-M_1^2)\lambda^2\beta^2}\right](1+M_o\alpha)\sqrt{(M_o^2-1)-M_1^2\lambda^2\beta^2}} , \tag{E.9}$$

where

$$\Gamma_5 = -\sqrt{M_o^2-1}\, M_1 M_o\left[M_1\lambda^2\beta^2\right]\left[2\mu M_o^2 + ((\mu-1)M_o^2-1)(1+M_o\alpha) - 2(\mu-1)M_o(\alpha+M_o)\right]$$

$$-\lambda M_1\sqrt{M_o^2-1}\left[(-2(\mu-1)M_o(\alpha+M_o) + ((\mu-1)M_o^2-1)(1+M_o\alpha))(M_o^2+1) + 2M_o((\mu-1)-M_o^2)(\alpha+M_o)\right.$$

$$\left. + 2M_o(1+M_o\alpha)\right] + \sqrt{M_o^2-1}\left[(M_o^2-1)-M_1^2\lambda^2\beta^2\right]\left[-(M_o^2-(\mu-1))(\alpha+M_o)\right.$$

$$\left. + 2M_o(1+M_o\alpha)) - \mu M_o(M_o^2+1)\right] \quad ; \tag{E.10}$$

$$\Gamma_6 = -(M_o^2-1)M_1 M_o\left[2\mu M_o^2 + ((\mu-1)M_o^2-1)(1+M_o\alpha) - 2(\mu-1)M_o(\alpha+M_o)\right]$$

$$-\lambda M_1^2\left[(-2(\mu-1)M_o(\alpha+M_o) + ((\mu-1)M_o^2-1)(1+M_o\alpha))(M_o^2+1)\right.$$

$$\left. + 2M_o(-(M_o^2-(\mu-1))(\alpha+M_o) + 2M_o(1+M_o\alpha))\right] \quad ;$$

and

$$C = \frac{2M_1^2(M_0^2-1)^2\left[\Gamma_7 - \Gamma_8\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}\,\right]\epsilon}{\mu^2 M_0^4 F^2(M_0)\left[\Gamma_1+\Gamma_2\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}\,\right](1+M_0\alpha)}, \tag{E.11}$$

where

$$\Gamma_7 = \sqrt{M_0^2-1}\,\Big[-(M_0^2-(\mu-1))(\alpha+M_0) + 2M_0(1+M_0\alpha) + \lambda\mu M_1 M_0^2\beta^2$$

$$-\left((\mu-1)(\alpha+M_0) + M_0(1+\alpha M_0)(M_0^2+1+\lambda M_1 M_0\beta^2)\right)\Big] ; \tag{E.12}$$

$$\Gamma_8 = M_1M_0\Big[2M_0\left((\mu-1)(\alpha+M_0)+M_0(1+M_0\alpha)\right) - 2(\mu-1)M_0(\alpha+M_0)$$

$$+\left((\mu-1)M_0^2-1\right)(1+M_0\alpha)\Big] .$$

Thus we have the quantities a, B, C expressed as functions of θ. To express A in the appropriate form, we first transform θ_0 to θ by means of the aberration relations (4.64), and then transform θ to θ' by means of the relations:

$$\beta' = \frac{i\lambda\sqrt{1-M_1^2}}{\sqrt{M_0^2-1}}\,\beta \quad , \qquad \alpha' = \frac{\sqrt{(M_0^2-1)+(1-M_1^2)\lambda^2\beta^2}}{\sqrt{M_0^2-1}}$$

$$\beta = -\frac{i\sqrt{M_0^2-1}}{\lambda\sqrt{1-M_1^2}}\,\beta' \quad , \qquad \alpha = \pm\frac{\sqrt{\lambda^2(1-M_1^2)+(M_0^2-1)\beta'^2}}{\lambda\sqrt{1-M_1^2}} \tag{E.13}$$

where the + sign is chosen for path P_1' and the - sign is chosen for path P_2' of Fig. 18.

In terms of α' and β', the expression for A becomes:

$$A = \frac{\Lambda_2 \pm \Lambda_3\sqrt{1+\frac{(M_0^2-1)}{\lambda^2(1-M_1^2)}\beta'^2}}{\Lambda_1\,(1+M_0\alpha)}\,\epsilon , \tag{E.14}$$

where

$$\Lambda_1 = \mu\lambda\left[\lambda(1-M_1^2)(2M_1M_0^2\alpha' + (M_0^2+1)) - (M_0^2-1)M_1M_0\beta'^2\right] \;;$$

$$\Lambda_2 = 2M_0\left[-\lambda(1-M_1^2)M_0(M_0^2-(\mu-1)) - \mu M_0^2 M_1(M_0^2-1)\beta'^2\right]$$

$$+\left[(M_0^2+1)\lambda(1-M_1^2) - M_1M_0(M_0^2-1)\beta'^2\right]\left[(\mu-1)M_0^2+1\right] \;; \qquad \text{(E.15)}$$

$$\Lambda_3 = 2M_0\left[\lambda(1-M_1^2)(M_0^2+(\mu-1))\right]$$

$$-\left[(M_0^2+1)\lambda(1-M_1^2) - M_1M_0(M_0^2-1)\beta'^2\right]\left[M_0((\mu-1)M_0^2-(2\mu-1))\right] \;.$$

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