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OPTIMAL REDUCED-ORDER SUBSPACE-OBSERVER DESIGN WITH A FREQUENCY-DOMAIN ERROR BOUND

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I. INTRODUCTION

Constraints on implementation complexity often make it desirable in practice to design estimators of reduced order. Such low-order estimators are also of interest when estimates of only a few state variables are required. For example, although a large flexible space structure may involve numerous flexible modes, only estimates of the rigid body attitude may be desired. The literature on reduced-order estimator design is vast, and we note a representative collection of papers [1–11] as an indication of long-standing interest in this problem.

The starting point for this article is the Riccati equation approach developed in [1]. There it was shown that optimal reduced-order, steady-state estimators can be characterized by means of an algebraic system of equations consisting of one modified Riccati equation and two modified Lyapunov equations coupled by a projection matrix τ. As shown in [1] this projection arises directly from the fixed-order constraint on the estimator order. We note that the order projection τ derived in [1] is given by

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}, \tag{1}$$

where $(\ )^{\#}$ denotes group generalized inverse, and $\hat{Q}$ and $\hat{P}$ are rank-deficient nonnegative-definite matrices analogous to the controllability and observability Gramians of the estimator. As discussed in [1], the order projection τ arises as a direct consequence of optimality and is not the result of an *a priori* assumption on the structure of the reduced-order estimator. Indeed, no assumption was made in [1] concerning the internal structure of the estimator.

The solution given in [1], however, was confined to problems in which the plant is asymptotically stable though in practice it is often necessary to obtain estimators for plants with unstable modes. Intuitively, it is clear that finite, steady-state state-estimation error for unstable plants is only achievable when the estimator retains, or duplicates in some sense, the unstable modes. The solution given in [1] is inapplicable to the unstable plant problem for the simple reason that the range of the order projection τ may not fully encompass all of the unstable modes. Hence, in this article we derive a new and completely distinct reduced-order solution in which the observation subspace of the estimator is constrained *a priori* to include all of the unstable modes and selected stable modes. Specifically, for a plant with $\hat{n}_u$ unstable modes, we characterize the optimal estimator of order $n_u \geq \hat{n}_u$ which generates estimates of all of the $\hat{n}_u$ unstable states and $n_u - \hat{n}_u$ prespecified stable states. Hence this estimator effectively serves as an *observer* for a designated plant subspace.

The subspace observation constraint is embedded within the optimization process by fixing the internal structure of the reduced-order estimator. This structure gives rise to a projection μ defined by

$$\mu \triangleq \begin{bmatrix} I_{n_u} & P_u^{-1} P_{us} \\ 0_{n_s \times n_u} & 0_{n_s} \end{bmatrix}, \tag{2}$$

where $P_u \in \mathbb{R}^{n_u \times n_u}$ and $P_{us} \in \mathbb{R}^{n_u \times n_s}$ are subblocks of an $n \times n$ matrix P satisfying a modified algebraic Lyapunov equation, $n_u \geq \hat{n}_u$ is the dimension of the observation subspace of the estimator containing all of the $\hat{n}_u$ unstable modes and $n_u - \hat{n}_u$ selected stable modes, and $n_s \triangleq n - n_u$ is the dimension of the remaining subspace containing only stable modes. It turns out that the subspace projection μ, which is completely distinct from the order projection τ appearing in [1], plays a crucial role in characterizing the optimal estimator gains. Furthermore, in contrast to the lone observer Riccati equation of the standard full-order theory, in the constrained-subspace case the reduced-order solution consists of one modified Riccati equation and one modified Lyapunov equation coupled by the subspace projection μ.

In addition to the subspace-observation problem just discussed, this article includes the treatment of a worst-case frequency-domain design criterion for the state-estimation error. Specifically, we consider the least-squares state-estimation problem with a constraint on the frequency-domain (i.e., H_∞)

estimation error [12]. This generalization provides additional design flexibility by yielding a reduction of the frequency content of the estimation error in addition to its mean-square magnitude. The principal result in this case is a sufficient condition that yields subspace-constrained estimators satisfying an optimized L_2 bound as well as a prespecified H_∞ bound. The sufficient condition is a direct generalization of the subspace-observation problem developed previously for the least-squares estimation problem. Once again, the optimal reduced-order estimator is characterized by an algebraic system consisting of one modified Riccati equation and one modified Lyapunov equation coupled by the constrained-subspace projection μ with additional coupling arising due to the H_∞ constraint. This result is analogous to recent developments in H_∞ control theory [13–16].

An additional feature of this article is the inclusion of a static estimator gain in conjunction with the dynamic estimator. Thus, our results also represent a generalization of the standard steady-state Kalman filter result to the case of nonstrictly proper estimation. Specifically, noise-free measurements

$$\hat{y} = \hat{C}x(t) \tag{3}$$

multiplied by a static estimator gain lead to the static-gain projection

$$\nu \triangleq Q\hat{C}^{\mathrm{T}}(\hat{C}Q\hat{C}^{\mathrm{T}})^{-1}\hat{C}, \tag{4}$$

where Q is the steady-state estimation-error covariance. This projection has been discussed earlier, for example [17–19]. In the H_∞-constrained case, the static-gain projection ν becomes

$$\nu_\infty \triangleq (\mathcal{Q}\hat{C}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^{\mathrm{T}})(\hat{C}\mathcal{Q}\hat{C}^{\mathrm{T}} + \gamma^{-2}\hat{C}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^{\mathrm{T}})^{-1}\hat{C}, \tag{5}$$

where $\mathcal{Q}$ is a bound on the steady-state estimation-error covariance, $\mathcal{P}$ satisfies a modified Lyapunov equation, and γ is the prespecified frequency-domain error bound. If this bound is sufficiently relaxed (i.e., $\gamma \to \infty$), then $\nu_\infty \to \nu$ and the "pure" least-squares nonstrictly proper estimator is recovered. Of course, if nonnoisy measurements of the form (3) are not available for a particular application, then this design aspect can be ignored in both the least-squares and frequency-domain problems. Such specializations are pointed out in later sections.

It should be stressed that all three projections τ, μ, and ν are completely distinct and arise from different design objectives. Specifically, as discussed in [1,2], the order projection τ arises due to a constraint on the order of the estimator, the subspace projection μ arises from a constraint on the structure of the estimator, and the static-gain projection ν arises due to the presence of noise-free measurements. Designing a nonstrictly proper reduced-order estimator that includes all of the unstable modes and an optimal choice of some of the stable modes would involve all three projections and four matrix equations. This unified solution is considerably more complex and thus is deferred to a future paper.

After presenting notation in Section II, we give in Section III the statement of the optimal reduced-order subspace-observer problem. Theorem 1 shows that the reduced-order subspace-constrained estimator is characterized by one modified Riccati equation and one modified Lyapunov equation. The H_∞-constrained reduced-order subspace-observer problem is considered in Section IV. The principal result of this section (Lemma 1) shows that if the algebraic Lyapunov equation for the error covariance is replaced by a modified Riccati equation possessing a nonnegative-definite solution, then the H_∞ estimation constraint is satisfied, and the least-squares state-estimation error criterion is bounded above by an auxiliary cost function. The problem of determining reduced-order estimators that minimize this upper bound subject to the Riccati equation constraint is considered as the auxiliary minimization problem. Necessary conditions for the auxiliary minimization problem (Theorem 2) are again given in the form of a coupled system of algebraic Riccati and Lyapunov equations. To develop connections with the standard Kalman filter theory, the results of Theorem 2 are specialized to the full-order case (see Remark 11). In Section V the necessary conditions of Theorem 2 are combined with Lemma 1 to yield sufficient conditions for stability of the estimation-error dynamics, constrained H_∞ estimation error, and bounded least-squares state-estimation error.

II. NOTATION AND DEFINITIONS

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r, \mathbb{E}$	Real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value
$I_r, (\ )^{\mathrm{T}}, 0_{r\times s}, 0_r$	$r \times r$ identity matrix, transpose, $r \times s$ zero matrix, $0_{r\times r}$
tr	Trace
$\sigma_{\max}(Z)$	Largest singular value of matrix Z
$\lVert H(s)\rVert_\infty$	$\sup_{\omega\in\mathbb{R}} \sigma_{\max}[H(j\omega)]$
$\mathcal{N}(Z), \mathcal{R}(Z)$	Null space, range of matrix Z
$\mathbb{S}^r, \mathbb{N}^r, \mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \leq Z_2, Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r, Z_2 - Z_1 \in \mathbb{P}^r, Z_2, Z_1 \in \mathbb{S}^r$
$n, l, \hat{l}, n_e, n_u, n_s, q, p$	Positive integers
$x, y, \hat{y}, x_e, x_u, x_s, y_e$	$n, l, \hat{l}, n_e, n_u, n_s, q$-dimensional vectors
$A, C, \hat{C}$	$n \times n, l \times n, \hat{l} \times n$ matrices
A_u, A_{us}, A_s	$n_u \times n_u, n_u \times n_s, n_s \times n_s$ matrices
$C_u, C_s, \hat{C}_u, \hat{C}_s$	$l \times n_u, l \times n_s, \hat{l} \times n_u, \hat{l} \times n_s$ matrices

D_1, D_2, E, L	$n \times p, l \times p, r \times q, q \times n$ matrices
D_{1u}, D_{1s}, L_u, L_s	$n_u \times p, n_s \times p, q \times n_u, q \times n_s$ matrices
R	$E^T E$, estimation-error weighting in $\mathbb{P}^q$
A_e, B_e, C_e, D_e	$n_e \times n_e, n_e \times l, q \times n_e, q \times \hat{l}$ matrices
$w(\cdot)$	p-dimensional standard white noise process
V_1, V_2	Intensity of $D_1 w(\cdot)$, $D_2 w(\cdot)$; $V_1 = D_1 D_1^T \in \mathbb{N}^n$, $V_2 = D_2 D_2^T \in \mathbb{P}^l$
V_{12}	Cross intensity of $D_1 w(\cdot)$, $D_2 w(\cdot)$; $V_{12} = D_1 D_2^T \in \mathbb{R}^{n \times l}$
$\tilde{A}$	$A - \begin{bmatrix} I_{n_u} \\ 0_{n_s \times n_u} \end{bmatrix} B_e C$, $n_u < n$; $A - B_e C$, $n_u = n$
$\tilde{D}$	$D_1 - \begin{bmatrix} I_{n_u} \\ 0_{n_s \times n_u} \end{bmatrix} B_e D_2$, $n_u < n$; $D_1 - B_e D_2$, $n_u = n$
$\tilde{E}$	$E(L - D_e C)$
$\tilde{R}$	$\tilde{E}^T \tilde{E} = (L - D_e C)^T R (L - D_e C)$
$\tilde{V}$	$\tilde{D}\tilde{D}^T$

III. THE OPTIMAL REDUCED-ORDER SUBSPACE-OBSERVER PROBLEM

The problem is addressed as follows: Given the nth-order system

$$\dot{x}(t) = Ax(t) + D_1 w(t), \quad t \in [0, \infty), \tag{6}$$

with noisy and nonnoisy measurements

$$y(t) = Cx(t) + D_2 w(t), \tag{7}$$

$$\hat{y}(t) = \hat{C}x(t), \tag{8}$$

and with the partitioning

$$\begin{bmatrix} \dot{x}_u(t) \\ \dot{x}_s(t) \end{bmatrix} = \begin{bmatrix} A_u & A_{us} \\ 0_{n_s \times n_u} & A_s \end{bmatrix} \begin{bmatrix} x_u(t) \\ x_s(t) \end{bmatrix} + \begin{bmatrix} D_{1u} \\ D_{1s} \end{bmatrix} w(t), \tag{9}$$

$$y(t) = [C_u \quad C_s] \begin{bmatrix} x_u(t) \\ x_s(t) \end{bmatrix} + D_2 w(t), \tag{10}$$

$$\hat{y}(t) = [\hat{C}_u \quad \hat{C}_s] \begin{bmatrix} x_u(t) \\ x_s(t) \end{bmatrix}, \tag{11}$$

design an n_uth-order nonstrictly proper state estimator

$$\dot{x}_e(t) = A_e x_e(t) + B_e y(t), \tag{12}$$

$$y_e(t) = C_e x_e(t) + D_e \hat{y}(t), \tag{13}$$

such that the state-estimation error criterion

$$J(A_e, B_e, C_e, D_e) \triangleq \lim_{t \to \infty} \mathbb{E}[Lx(t) - y_e(t)]^T R[Lx(t) - y_e(t)] \tag{14}$$

is minimized and

$$\lim_{t \to \infty} [x_u(t) - x_e(t)] = 0, \tag{15}$$

for all $x(0)$ and $x_e(0)$ when $D_1 = 0$ and $D_2 = 0$.

Remark 1. Note that (13) allows the additional feature of a static feedthrough gain D_e when nonnoisy measurements (8) are available. This corresponds to a static least-squares estimator in conjunction with the dynamic (Wiener–Kalman) estimator. For the special case in which only noisy measurements are available, one needs only to set $D_e = 0$, which leads to a strictly proper state estimator.

Remark 2. Note that (14) is the usual least-squares state-estimation error criterion whereas (15) implies that perfect observation is achieved at steady state for the plant and observer dynamics under zero external disturbances and arbitrary initial conditions.

In this formulation the plant state $x(t)$ is partitioned into subsystems for $x_u(t)$ and $x_s(t)$ of dimension n_u and n_s, respectively. Furthermore, we assume that if λ is an eigenvalue of A such that $\text{Re}(\lambda) \geq 0$, then λ is also an eigenvalue of A_u with the same multiplicity. That is, the n_u-dimensional subspace for $x_u(t)$ contains all the unstable modes of the system (if there are any) and possibly selected stable modes. Thus, if the unstable subspace of A has dimension $\hat{n}_u$, then we have $n_u \geq \hat{n}_u$, and the n_s-dimensional subspace for $x_s(t)$ contains the remaining stable modes. Furthermore, the matrix L, which is partitioned as

$$L \triangleq [L_u \quad L_s], \tag{16}$$

where L_u and L_s are $q \times n_u$ and $q \times n_s$ matrices, identifies the states or linear combinations of states whose estimates are desired. The order n_e of the estimator state x_e is fixed to be equal to the order of the n_u-dimensional subspace for $x_u(t)$. Thus, the goal of the optimal reduced-order subspace-observer problem is to design an estimator of order n_u which yields quadratically optimal linear least-squares estimates of specified linear combinations of the states of the system. To satisfy the observation constraint (15), define the error state $z(t) \triangleq x_u(t) - x_e(t)$ satisfying

$$\begin{aligned} \dot{z}(t) &= \dot{x}_u(t) - \dot{x}_e(t) \\ &= (A_u - B_e C_u)x_u(t) - A_e x_e(t) + (A_{us} - B_e C_s)x_s(t) + D_{1u}w(t) - B_e D_2 w(t). \end{aligned} \tag{17}$$

Note that the explicit dependence of the error states $z(t)$ on the states $x_u(t)$ can be eliminated by constraining

$$A_e = A_u - B_e C_u. \tag{18}$$

Thus, (17) becomes

$$\dot{z}(t) = A_e z(t) + (A_{us} - B_e C_s)x_s(t) + D_{1u}w(t) - B_e D_2 w(t). \tag{19}$$

Furthermore, the explicit dependence of the estimation error (14) on the $x_u(t)$ subsystem can be eliminated by constraining

$$C_e = L_u - D_e \hat{C}_u. \tag{20}$$

Henceforth, we assume that A_e and C_e are given by (18) and (20). Now, from (9)–(13) it follows that

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \tag{21}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} z(t) \\ x_s(t) \end{bmatrix}, \qquad \tilde{A} \triangleq \begin{bmatrix} A_u - B_e C_u & A_{us} - B_e C_s \\ 0_{n_s \times n_u} & A_s \end{bmatrix}.$$

To guarantee that J is finite, consider the set of asymptotically stable reduced-order estimators,

$$\mathscr{S} \triangleq \{(A_e, B_e, C_e, D_e)\colon A_e = A_u - B_e C_u \quad \text{is asymptotically stable}\},$$

which is nonempty if (A_u, C_u) is detectable.

Before continuing, we note that if A_e is asymptotically stable, then since A_s is asymptotically stable, $\tilde{A}$ is also asymptotically stable. Hence the least-squares state-estimation error criterion (14) is given by

$$J(A_e, B_e, C_e, D_e) = \operatorname{tr} Q\tilde{R}, \tag{22}$$

where the $n \times n$ steady-state error covariance,

$$Q \triangleq \lim_{t \to \infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)] \geq 0, \tag{23}$$

exists and satisfies the algebraic Lyapunov equation

$$0 = \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + \tilde{V}. \tag{24}$$

Furthermore, for nondegeneracy we restrict our attention to the set of admissible estimators,

$$\mathscr{S}^+ \triangleq \{(A_e, B_e, C_e, D_e) \in \mathscr{S}\colon (A_e, C_e) \quad \text{is observable and} \quad \hat{C}Q\hat{C}^{\mathrm{T}} > 0\}.$$

The definiteness condition $\hat{C}Q\hat{C}^{\mathrm{T}} > 0$ holds if $\hat{C}$ has full row rank and Q is positive definite. Conversely, if $\hat{C}Q\hat{C}^{\mathrm{T}} > 0$, then $\hat{C}$ must have full row rank but Q need not necessarily be positive definite. As shown in the appendix, this condition implies the existence of the static-gain projection ν, which is defined in (35).

The following result gives necessary conditions that characterize solutions to the optimal reduced-order subspace-observer problem. For convenience in stating this result, define

$$Q_{\mathrm{a}} \triangleq QC^{\mathrm{T}} + V_{12}.$$

Theorem 1. If $(A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}, D_{\mathrm{e}}) \in \mathscr{S}^+$ solves the optimal reduced-order subspace-observer problem with constraints (18) and (19) and Q given by (24), then there exists $P \in \mathbb{N}^n$ such that

$$A_{\mathrm{e}} = \Phi(A - Q_{\mathrm{a}} V_2^{-1} C) F^{\mathrm{T}}, \tag{25}$$

$$B_{\mathrm{e}} = \Phi Q_{\mathrm{a}} V_2^{-1}, \tag{26}$$

$$C_{\mathrm{e}} = L \nu_{\perp} F^{\mathrm{T}}, \tag{27}$$

$$D_{\mathrm{e}} = L Q \hat{C}^{\mathrm{T}} (\hat{C} Q \hat{C}^{\mathrm{T}})^{-1}, \tag{28}$$

and such that Q and P satisfy

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q_{\mathrm{a}} V_2^{-1} Q_{\mathrm{a}}^{\mathrm{T}} + \mu_{\perp} Q_{\mathrm{a}} V_2^{-1} Q_{\mathrm{a}}^{\mathrm{T}} \mu_{\perp}^{\mathrm{T}}, \tag{29}$$

$$0 = (A - \mu Q_{\mathrm{a}} V_2^{-1} C)^{\mathrm{T}} P + P(A - \mu Q_{\mathrm{a}} V_2^{-1} C) + \nu_{\perp}^{\mathrm{T}} L^{\mathrm{T}} R L \nu_{\perp}, \tag{30}$$

where

$$P = \begin{bmatrix} P_{\mathrm{u}} & P_{\mathrm{us}} \\ P_{\mathrm{us}}^{\mathrm{T}} & P_{\mathrm{s}} \end{bmatrix} \in \mathbb{R}^{(n_{\mathrm{u}} + n_{\mathrm{s}}) \times (n_{\mathrm{u}} + n_{\mathrm{s}})}, \tag{31}$$

$$P_{\mathrm{u}} > 0, \tag{32}$$

$$F \triangleq [I_{n_{\mathrm{u}}} \quad 0_{n_{\mathrm{u}} \times n_{\mathrm{s}}}], \qquad \Phi \triangleq [I_{n_{\mathrm{u}}} \quad P_{\mathrm{u}}^{-1} P_{\mathrm{us}}], \tag{33}$$

$$\mu \triangleq F^{\mathrm{T}} \Phi = \begin{bmatrix} I_{n_{\mathrm{u}}} & P_{\mathrm{u}}^{-1} P_{\mathrm{us}} \\ 0_{n_{\mathrm{s}} \times n_{\mathrm{u}}} & 0_{n_{\mathrm{s}}} \end{bmatrix}, \qquad \mu_{\perp} \triangleq I_n - \mu, \tag{34}$$

$$\nu \triangleq Q \hat{C}^{\mathrm{T}} (\hat{C} Q \hat{C}^{\mathrm{T}})^{-1} \hat{C}, \qquad \nu_{\perp} \triangleq I_n - \nu. \tag{35}$$

Furthermore, the minimal cost is given by

$$J(A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}, D_{\mathrm{e}}) = \operatorname{tr} Q \nu_{\perp}^{\mathrm{T}} L^{\mathrm{T}} R L \nu_{\perp}. \tag{36}$$

Conversely, if there exist $Q, P \in \mathbb{N}^n$ satisfying (29) and (30), and such that $\hat{C} Q \hat{C}^{\mathrm{T}} > 0$, then Q satisfies (24) with $(A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}, D_{\mathrm{e}})$ given by (25)–(28). Furthermore, $(\tilde{A}, \tilde{D})$ is stabilizable if and only if A_{e} is asymptotically stable. In this case $(A_{\mathrm{e}}, C_{\mathrm{e}})$ is observable.

Proof. The result follows as a special case of Theorem 2. See Remark 9 for details. □

Remark 3. Equations (29) and (30) involve two distinct projections, namely, ν and μ. Note that ν and μ are idempotent since $\nu^2 = \nu$ and $\mu^2 = \mu$. As discussed earlier, the presence of noise-free measurements $\hat{y}(t) = \hat{C}x(t)$ gives rise to the static-gain projection ν whereas the observation constraint (15) gives rise to the subspace projection μ. It is easy to see that rank $\mu = n_u$; and with Sylvester's inequality, it follows that rank $\nu = \hat{l}$. Finally, it should be stressed that the subspace projection μ is completely distinct from the order projection τ appearing in [1].

Remark 4. Note that with B_e and D_e given by (26) and (28), the expressions (25) and (27) for A_e and C_e are equivalent to the constraints (28) and (29).

Remark 5. As a first step in analyzing these equations, consider the extreme case $\hat{l} = n$ and $\hat{C} = I_n$ so that perfect measurements of the entire state are available. It then follows from Theorem 1 with Q positive definite that $\nu = I_n$, $\nu_\perp = 0$, $C_e = 0$ (i.e., the dynamic filter is disabled), $D_e = L$, and by (36), $J = 0$. More generally, suppose that $\mathscr{R}(L) \subset \mathscr{R}(\hat{C})$, which implies that perfect measurements of Lx are available. In this case,

$$\operatorname{rank}\begin{bmatrix} \hat{C} \\ L \end{bmatrix} = \operatorname{rank} \hat{C},$$

and thus $L = \hat{L}\hat{C}$ for some $\hat{L} \in \mathbb{R}^{q \times \hat{l}}$ without loss of generality. Thus, it follows from Theorem 1 that $C_e = 0$, $D_e = L$, and $J = 0$ since $L^{\mathrm{T}}RL = \hat{C}^{\mathrm{T}}\hat{L}^{\mathrm{T}}R\hat{L}\hat{C}$ and $\hat{C}\nu_\perp = 0$. These are, of course, expected results because perfect estimation is achievable in both cases.

Remark 6. Note that for A_e, B_e, C_e, D_e given by (25)–(28), the estimator assumes the innovations form

$$\dot{x}_e(t) = \Phi A F^{\mathrm{T}} x_e(t) + \Phi Q_a V_2^{-1}[y(t) - CF^{\mathrm{T}} x_e(t)]. \tag{37}$$

By introducing the quasi full-state estimate $\hat{x}(t) \triangleq F^{\mathrm{T}} x_e(t) \in \mathbb{R}^n$, so that $\mu\hat{x}(t) = \hat{x}(t)$ and $x_e(t) = \Phi\hat{x}(t) \in \mathbb{R}^{n_u}$, we can write (37) as

$$\dot{\hat{x}}(t) = \mu A \mu \hat{x}(t) + \mu Q_a V_2^{-1}(y(t) - C\hat{x}(t)). \tag{38}$$

Note that although the implemented estimator (37) has the state $x_e(t) \in \mathbb{R}^{n_u}$ (38) can be viewed as a quasi full-order estimator whose geometric structure is entirely dictated by the projection μ. Specifically, error inputs $Q_a V_2^{-1}(y(t) - \hat{C}\hat{x}(t))$ are annihilated unless they are contained in $[\mathscr{N}(\mu)]^\perp = \mathscr{R}(\mu^{\mathrm{T}})$. Hence the observation subspace of the estimator is precisely $\mathscr{R}(\mu^{\mathrm{T}})$.

Remark 7. In the full-order case $n_u = n$, Theorem 1 yields a steady-state nonstrictly proper Kalman filter. To see this, formally set $\Phi = F = \mu = I_n$ and $\mu_\perp = 0$ so that (22) is superfluous and (21) becomes

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q_a V_2^{-1} Q_a^{\mathrm{T}}, \tag{39}$$

with gains

$$A_e = A - Q_a V_2^{-1} C, \tag{40}$$

$$B_e = Q_a V_2^{-1}, \tag{41}$$

$$C_e = L v_\perp, \tag{42}$$

$$D_e = L Q \hat{C}^T (\hat{C} Q \hat{C}^T)^{-1}. \tag{43}$$

Finally, to recover the standard steady-state Kalman filter, which involves only noisy measurements, set $\hat{C} = 0$, delete (43), and define $v = 0$ and $v_\perp = I_n$.

IV. THE OPTIMAL REDUCED-ORDER SUBSPACE-OBSERVER PROBLEM WITH AN H_∞ ERROR CONSTRAINT

We now introduce the reduced-order subspace-observer problem with an H_∞ constraint on the H_∞-norm of the state-estimation error. Specifically, we constrain the transfer function between disturbances and error states to have H_∞ norm less than γ. Given the nth-order observed system (6)–(11), determine an n_uth-order subspace observer, (12) and (13), that satisfies the following design criteria:

1. $A_e = A_u - B_e C_u$ is asymptotically stable.
2. The $r \times p$ transfer function

$$H(s) \triangleq \tilde{E}(sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D} \tag{44}$$

   from disturbances $w(t)$ to error states $E[Lx(t) - y_e(t)]$ satisfies the constraint

$$\|H(s)\|_\infty \leq \gamma, \tag{45}$$

   where $\gamma > 0$ is a given constant.
3. The least-squares state-estimation error criterion (14) is minimized, and the observation constraint (15) holds.

The key step in enforcing (45) is to replace the algebraic Lyapunov equation (24) by an algebraic Riccati equation. Justification for this technique is provided by the following result.

Lemma 1. Let (A_e, B_e, C_e, D_e) be given and assume there exists an $n \times n$ matrix $\mathcal{Q}$ satisfying

$$\mathcal{Q} \in \mathbb{N}^n \tag{46}$$

and

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V}. \tag{47}$$

Then,

$$(\tilde{A}, \tilde{D}) \text{ is stabilizable} \tag{48}$$

if and only if

$$A_e \text{ is asymptotically stable.} \tag{49}$$

Furthermore, in this case,

$$\|H(s)\|_\infty \leq \gamma, \tag{50}$$

$$Q \leq \mathcal{Q}, \tag{51}$$

and

$$J(A_e, B_e, C_e, D_e) \leq \mathcal{J}(A_e, B_e, C_e, D_e, \mathcal{Q}), \tag{52}$$

where

$$\mathcal{J}(A_e, B_e, C_e, D_e, \mathcal{Q}) \triangleq \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{53}$$

Proof. See [16]. □

Lemma 1 shows that the H_∞ constraint is automatically enforced when a nonnegative-definite solution to (47) is known to exist. Furthermore, the solution $\mathcal{Q}$ provides an upper bound for the actual closed-loop state covariance Q, which in turn yields an upper bound for the least-squares state-estimation error criterion. That is, given a fixed-order estimator (A_e, B_e, C_e, D_e) satisfying the H_∞ estimation constraint, the actual least-squares state-estimation error is guaranteed to be no worse than the bound given by $\mathcal{J}(A_e, B_e, C_e, D_e, \mathcal{Q})$ if (47) is solvable. Hence $\mathcal{J}(A_e, B_e, C_e, D_e, \mathcal{Q})$ can be interpreted as an auxiliary cost, which leads to the following optimization problem.

To solve the auxiliary minimization problem, determine the $(A_e, B_e, C_e, D_e, \mathcal{Q})$ that minimizes $\mathcal{J}(A_e, B_e, C_e, D_e, \mathcal{Q})$ subject to (46) and (47).

Rigorous derivation of the necessary conditions for the auxiliary minimization problem requires additional technical assumptions. Specifically, we restrict $(A_e, B_e, C_e, D_e, \mathcal{Q})$ to the open set

$$\begin{aligned}\mathcal{S}_\infty \triangleq \{(A_e, B_e, C_e, D_e, \mathcal{Q})\colon \tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R} \text{ is asymptotically stable,}\\ (A_e, B_e, C_e) \text{ is minimal, and } \hat{C}\mathcal{Q}\hat{C}^{\mathrm{T}} + \gamma^{-2}\hat{C}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^{\mathrm{T}} > 0\},\end{aligned} \tag{54}$$

where $\mathcal{P} \in \mathbb{N}^n$ satisfies

$$0 = (\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R})^{\mathrm{T}}\mathcal{P} + \mathcal{P}(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}) + \tilde{R}.$$

Remark 8. The set $\mathcal{S}_\infty$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the auxiliary minimization problem. Specifically, the requirement that $\mathcal{Q}$ be positive definite replaces (46) by an open-set constraint, the stability of $\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}$ serves as a normality condition, (A_e, B_e, C_e) minimal is a nondegeneracy condition, and the definiteness condition implies the existence of the static-gain projection ν_∞, defined in (62) for the H_∞-constrained problem. Finally, for arbitrary $\mathcal{Q} \in \mathbb{R}^{n \times n}$, define the notation

$$\mathcal{Q}_a \triangleq \mathcal{Q}C^T + V_{12}. \tag{55}$$

Theorem 2. If $(A_e, B_e, C_e, D_e, \mathcal{Q}) \in \mathcal{S}_\infty$ solves the auxiliary minimization problem with constraints (18) and (20) and $\mathcal{Q}$ given by (47), then there exists $\mathcal{P} \in \mathbb{N}^n$ such that

$$A_e = \Phi(A - \mathcal{Q}_a V_2^{-1} C)F^T, \tag{56}$$

$$B_e = \Phi \mathcal{Q}_a V_2^{-1}, \tag{57}$$

$$C_e = L\nu_{\infty\perp} F^T, \tag{58}$$

$$D_e = L(\mathcal{Q}\hat{C}^T + \gamma^{-2}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^T)(\hat{C}\mathcal{Q}\hat{C}^T + \gamma^{-2}\hat{C}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^T)^{-1}, \tag{59}$$

and such that $\mathcal{Q}$ and $\mathcal{P}$ satisfy

$$0 = A\mathcal{Q} + \mathcal{Q}A^T + V_1 + \gamma^{-2}\mathcal{Q}\nu_{\infty\perp}^T L^T R L \nu_{\infty\perp}\mathcal{Q} - \mathcal{Q}_a V_2^{-1}\mathcal{Q}_a^T$$
$$+ \mu_\perp \mathcal{Q}_a V_2^{-1} \mathcal{Q}_a^T \mu_\perp^T, \tag{60}$$

$$0 = (A - \mu\mathcal{Q}_a V_2^{-1} C + \gamma^{-2}\mathcal{Q}\nu_{\infty\perp}^T L^T R L \nu_{\infty\perp})^T \mathcal{P} + \mathcal{P}(A - \mu\mathcal{Q}_a V_2^{-1} C$$
$$+ \gamma^{-2}\mathcal{Q}\nu_{\infty\perp}^T L^T R L \nu_{\infty\perp}) + \nu_{\infty\perp}^T L^T R L \nu_{\infty\perp}, \tag{61}$$

where F, Φ, μ, and $\mu_\perp$ are defined by (33) and (34), $\mathcal{P}$ is partitioned as in (31), and ν_∞ and $\nu_{\infty\perp}$ are defined by

$$\nu_\infty \triangleq (\mathcal{Q}\hat{C}^T + \gamma^{-2}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^T)(\hat{C}\mathcal{Q}\hat{C}^T + \gamma^{-2}\hat{C}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^T)^{-1}\hat{C},$$
$$\nu_{\infty\perp} \triangleq I_n - \nu_\infty. \tag{62}$$

Furthermore, the auxiliary cost (53) is given by

$$\mathcal{J}(A_e, B_e, C_e, D_e, \mathcal{Q}) = \operatorname{tr} \mathcal{Q}\nu_{\infty\perp}^T L^T R L \nu_{\infty\perp}. \tag{63}$$

Conversely, if there exist $\mathcal{Q}, \mathcal{P} \in \mathbb{N}^n$ satisfying (60) and (61), and such that $\hat{C}\mathcal{Q}\hat{C}^T + \gamma^{-2}\hat{C}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^T > 0$, then $(A_e, B_e, C_e, D_e, \mathcal{Q})$ given by (56)–(60) satisfy (46) and (47) with the auxiliary cost (53) given by (63).

Proof. See Appendix. □

Remark 9. Theorem 2 presents necessary conditions for the auxiliary minimization problem that explicitly synthesize full- and reduced-order estimators (A_e, B_e, C_e, D_e). If the H_∞ estimation constraint is sufficiently relaxed (i.e., $\gamma \to \infty$), then $\nu_\infty = \nu$ and (60) and (61) reduce to (29) and (30), thus recovering the result of Theorem 1.

Remark 10. Since $\hat{C}\mathcal{Q}\hat{C}^T \leq \hat{C}\mathcal{Q}\hat{C}^T + \gamma^{-2}\hat{C}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^T$, it follows that if $\hat{C}\mathcal{Q}\hat{C}^T$

is positive definite, then so is $\hat{C}\mathcal{Q}\hat{C}^{\mathrm{T}} + \gamma^{-2}\hat{C}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^{\mathrm{T}}$. Also, note that since $Q \leq \mathcal{Q}$, it follows that if $\hat{C}Q\hat{C}^{\mathrm{T}}$ is positive definite, then so is $\hat{C}\mathcal{Q}\hat{C}^{\mathrm{T}}$. Hence, if v exists for an unconstrained problem, it follows that v_∞ will not fail to exist due to the singularity of $\hat{C}\mathcal{Q}\hat{C}^{\mathrm{T}} + \gamma^{-2}\hat{C}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^{\mathrm{T}}$ for the H_∞-constrained problem.

As discussed in Remark 7, in the full-order (Kalman-filter) case, set $n_{\mathrm{u}} = n$, $F = \Phi = \mu = I_n$, and $\mu_\perp = 0$. To develop further connections with standard steady-state Kalman filter theory assume that

$$V_{12} = 0. \tag{64}$$

In this case, the gain expressions (56)–(59) become

$$A_{\mathrm{e}} = A - \mathcal{Q}C^{\mathrm{T}}V_2^{-1}C, \tag{65}$$

$$B_{\mathrm{e}} = \mathcal{Q}C^{\mathrm{T}}V_2^{-1}, \tag{66}$$

$$C_{\mathrm{e}} = Lv_{\infty\perp}, \tag{67}$$

$$D_{\mathrm{e}} = L(\mathcal{Q}\hat{C}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^{\mathrm{T}})(\hat{C}\mathcal{Q}\hat{C}^{\mathrm{T}} + \gamma^{-2}\hat{C}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^{\mathrm{T}})^{-1}, \tag{68}$$

whereas (60) and (61) specialize to

$$0 = A\mathcal{Q} + \mathcal{Q}A^{\mathrm{T}} + V_1 + \gamma^{-2}\mathcal{Q}v_{\infty\perp}^{\mathrm{T}}L^{\mathrm{T}}RLv_{\infty\perp}\mathcal{Q} - \mathcal{Q}C^{\mathrm{T}}V_2^{-1}C\mathcal{Q}, \tag{69a}$$

$$0 = (A - \mathcal{Q}C^{\mathrm{T}}V_2^{-1}C + \gamma^{-2}\mathcal{Q}v_{\infty\perp}^{\mathrm{T}}L^{\mathrm{T}}RLv_{\infty\perp})^{\mathrm{T}}\mathcal{P} + \mathcal{P}(A - \mathcal{Q}C^{\mathrm{T}}V_2^{-1}C + \gamma^{-2}\mathcal{Q}v_{\infty\perp}^{\mathrm{T}}L^{\mathrm{T}}RLv_{\infty\perp}) + v_{\infty\perp}^{\mathrm{T}}L^{\mathrm{T}}RLv_{\infty\perp}. \tag{69b}$$

Remark 11. Note that the necessary conditions for the full-order non-strictly proper filter problem consist of one modified Riccati equation and one modified Lyapunov equation. To recover the case involving only noisy measurements, set $\hat{C} = 0$, delete (68), and define $v_\infty = 0$. In this case, (69) becomes

$$0 = A\mathcal{Q} + \mathcal{Q}A^{\mathrm{T}} + V_1 + \gamma^{-2}\mathcal{Q}L^{\mathrm{T}}RL\mathcal{Q} - \mathcal{Q}C^{\mathrm{T}}V_2^{-1}C\mathcal{Q}. \tag{70}$$

Finally, by relaxing the H_∞-constraint (i.e., $\gamma \to \infty$), (70) reduces to the standard observer Riccati equation.

V. SUFFICIENT CONDITIONS FOR COMBINED LEAST-SQUARES AND FREQUENCY-DOMAIN ERROR ESTIMATION

In this section we combine Lemma 1 with the converse of Theorem 2 to obtain our main result, guaranteeing H_∞-constrained estimation with an optimized least-squares bound on the state-estimation error criterion.

Theorem 3. Suppose there exist $\mathcal{Q}, \mathcal{P} \in \mathbb{N}^n$ satisfying (60) and (61), and let (A_e, B_e, C_e, D_e) be given by (56)–(59). Then (48) is satisfied if and only if A_e is asymptotically stable. In this case, the transfer function (44) satisfies the H_∞ estimation-error constraint

$$\|H(s)\|_\infty \leq \gamma, \tag{71}$$

and the least-squares state-estimation error criterion (14) satisfies the bound

$$J(A_e, B_e, C_e, D_e) \leq \operatorname{tr} \mathcal{Q} v_{\infty\perp}^{\mathrm{T}} L^{\mathrm{T}} R L v_{\infty\perp}. \tag{72}$$

Proof. The converse portion of Theorem 2 implies that $\mathcal{Q}$ given by (60) satisfies (46) and (47). It now follows from Lemma 1 that the stabilizability condition (48) is equivalent to the asymptotic stability of A_e, the H_∞ estimation-error constraint (50) holds, and the least-squares state-estimation error criterion satisfies the bound (53) which is equivalent to (72). □

APPENDIX. PROOF OF THEOREM 2

To optimize (53) over the open set $\mathcal{S}_\infty$ subject to the constraint (47), form the Lagrangian

$$\mathcal{L}(B_e, D_e, \mathcal{Q}, \mathcal{P}, \lambda) \triangleq \operatorname{tr}\{\lambda \mathcal{Q}\tilde{R} + [\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V}]\mathcal{P}\}, \tag{73}$$

where the Lagrange multipliers $\lambda \geq 0$ and $\mathcal{P} \in \mathbb{R}^{n\times n}$ are not both zero. We thus obtain

$$\frac{\partial \mathcal{L}}{\partial \mathcal{Q}} = (\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R})^{\mathrm{T}}\mathcal{P} + \mathcal{P}(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}) + \lambda\tilde{R}. \tag{74}$$

Setting $\partial\mathcal{L}/\partial Q = 0$ yields

$$0 = (\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R})^{\mathrm{T}}\mathcal{P} + \mathcal{P}(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}) + \lambda\tilde{R}. \tag{75}$$

Since $\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}$ is assumed to be stable, $\lambda = 0$ implies $\mathcal{P} = 0$. Hence it can be assumed without loss of generality that $\lambda = 1$. Furthermore, $\mathcal{P}$ is nonnegative definite.

Now partition $n \times n$ $\mathcal{Q}, \mathcal{P}$ into $n_u \times n_u$, $n_u \times n_s$, and $n_s \times n_s$ subblocks as

$$\mathcal{Q} = \begin{bmatrix} Q_u & Q_{us} \\ Q_{us}^{\mathrm{T}} & Q_s \end{bmatrix}, \qquad \mathcal{P} = \begin{bmatrix} P_u & P_{us} \\ P_{us}^{\mathrm{T}} & P_s \end{bmatrix}.$$

Thus, the stationarity conditions are given by

$$\frac{\partial \mathcal{L}}{\partial B_e} = P_u B_e V_2 - [P_u \quad P_{us}](\mathcal{Q}C^{\mathrm{T}} + V_{12}) = 0, \tag{76}$$

$$\frac{\partial \mathcal{L}}{\partial D_e} = D_e[\hat{C}\mathcal{Q}\hat{C}^{\mathrm{T}} + \gamma^{-2}\hat{C}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^{\mathrm{T}}] - L[\mathcal{Q}\hat{C}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\mathcal{P}\mathcal{Q}\hat{C}^{\mathrm{T}}] = 0. \tag{77}$$

Expanding the $n_u \times n_u$ subblock of (75) yields

$$0 = (A_e + \gamma^{-2}Q_u C_e^T R C_e)^T P_u + P_u(A_e + \gamma^{-2}Q_u C_e^T R C_e) + \gamma^{-2}Q_{us}P_{us}^T C_e^T R C_e P_{us} Q_{us}^T + C_e^T R C_e. \tag{78}$$

Since $(A_e, B_e, C_e, D_e) \in \mathscr{S}_\infty$, it follows from [20, Lemmas 2.1 and 12.2] that P_u is positive definite. Since P_u is thus invertible, define the $n_u \times n$ matrices

$$F \triangleq [I_{n_u} \quad 0_{n_u \times n_s}], \qquad \Phi \triangleq [I_{n_u} \quad P_u^{-1}P_{us}], \tag{79}$$

and the $n \times n$ matrix $\mu \triangleq F^T\Phi$. Note that since $\Phi F^T = I_n$, μ is idempotent, that is, $\mu^2 = \mu$.

Next note that (76), (77), and (79) imply (57) and (59). Similarly, (56) and (58) are equivalent to (18) and (20) with B_e and D_e given by (57) and (59), respectively. Now, using the expression for B_e, $\tilde{A}$ and $\tilde{V}$ become

$$\tilde{A} = A - \mu Q_a V_2^{-1} C, \tag{80}$$

$$\tilde{V} = V_1 - V_{12}V_2^{-1}Q_a^T\mu^T - \mu Q_a V_2^{-1}V_{12}^T + \mu Q_a V_2^{-1} Q_a^T \mu^T. \tag{81}$$

Now (60) and (61) follow from (47) and (75) by using (80) and (81).

Finally, to prove the converse, we use (56)–(61) to obtain (47) and (75)–(77). Let A_e, B_e, C_e, D_e, F, Φ, μ, $\mathscr{P}$ be as in the statement of Theorem 2. With $\Phi F^T = I_n$, it is easy to verify (76) and (77). Finally, substitute the definitions of F, Φ, and μ into (60) and (61), along with $\Phi F^T = I_n$, (33), and (34), to obtain (47) and (75). □

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REFERENCES

1. D. S. BERNSTEIN and D. C. HYLAND, "The Optimal Projection Equations for Reduced-Order State Estimation," *IEEE Trans. Autom. Control* **AC-30**, 583–585 (1985).
2. D. S. BERNSTEIN, L. D. DAVIS, and D. C. HYLAND. "The Optimal Projection Equations for Reduced-Order, Discrete-Time Modelling, Estimation and Control," *AIAA J. Guidance, Control Dyn.* **9**, 288–293 (1986).
3. D. A. WILSON and R. N. MISHRA, "Design of Low Order Estimators Using Reduced Models," *Int. J. Control* **23**, 447–456 (1979).
4. C. S. SIMS, "Reduced-order Modelling and Filtering," *Control Dyn. Syst.* **18**, 55–103 (1982).
5. R. B. ASHER, K. D. HERRING, and J. C. RYLES, "Bias Variance and Estimation Error in Reduced-Order Filters," *Automatica* **12**, 289–600 (1976).
6. F. W. FAIRMAN, "On Stochastic Observer Estimators for Continuous-Time Systems," *IEEE Trans. Autom. Control* **AC-22**, 874–876 (1977).
7. C. S. SIMS and R. B. ASHER, "Optimal and Suboptimal Results in Full and Reduced-Order Linear Filtering," *IEEE Trans. Autom. Control* **AC-23**, 469–472 (1978).

8. F. W. FAIRMAN and R. D. GUPTA, "Design of Multi-functional Reduced-Order Observers," *Int. J. Syst. Sci.* **11**, 1083–1094 (1980).
9. C. S. SIMS and L. G. STOTTS, "Linear Discrete Reduced-Order Filtering," *Proc. IEEE Conf. Decision Control*, 1172–1177 (1979).
10. T. E. FORTMAN and D. WILLIAMS, "Design of Low-Order Observers for Linear Feedback Control Laws," *IEEE Trans. Autom. Control* **AC-17**, 301–308 (1972).
11. C. S. SIMS, "An Algorithm for Estimating a Portion of a State Vector," *IEEE Trans. Autom. Control* **AC-19**, 391–393 (1974).
12. B. A. FRANCIS, "A Course in H_∞ Control Theory," Springer-Verlag, New York, 1987.
13. I. R. PETERSEN, "Disturbance Attenuation and H^∞ Optimization: A Design Method Based on the Algebraic Riccati Equation," *IEEE Trans. Autom. Control* **AC-32**, 427–429 (1987).
14. P. P. KHARGONEKAR, I. R. PETERSEN, and M. A. ROTEA, "H^∞ Optimal Control with State Feedback," *IEEE Trans. Autom. Control* **AC-33**, 786–788 (1988).
15. I. R. PETERSEN, "Complete Results for a Class of State Feedback Disturbance Attenuation Problems," *Proc. IEEE Conf. Decision Control,* 1349–1353 (1988).
16. D. S. BERNSTEIN and W. M. HADDAD, "LQG Control with an H_∞ Performance Bound: A Riccati Equation Approach," *Proc. Am. Control Conf., Atlanta, Ga.*, 796–802 (1988); *IEEE Trans. Autom. Control* **AC-34**, 293–305 (1989).
17. W. M. HADDAD and D. S. BERNSTEIN, "The Optimal Projection Equations for Reduced-Order State Estimation: The Singular Measurement Noise Case," *IEEE Trans. Autom. Control* **AC-32**, 1135–1139 (1987).
18. W. M. HADDAD and D. S. BERNSTEIN, "Robust, Reduced-Order Nonstrictly Proper State Estimation via the Optimal Projection Equations with Guaranteed Cost Bounds," *IEEE Trans. Autom. Control* **AC-33**, 591–595 (1988).
19. Y. HALEVI, "The Optimal Reduced-Order Estimator for Systems with Singular Measurement Noise," *IEEE Trans. Autom. Control* **34** (1989).
20. W. M. WONHAM, "Linear Multivariable Control: A Geometric Approach," Springer-Verlag, New York, 1979.