Corollary 3.4: For any $\Theta \in \mathbb{BRG}(G)$

$$k^* \left[ G \right] \geq \frac{1}{\sqrt{m}} \rho \left[ \Theta \right].$$  \hspace{1cm} (3.5)

Proof: As in the proof of Corollary 3.3, we show that for any matrix $G = \{g_{ij}\} \in C^{m \times m}$

$$\sup_{D \in D_m} \sigma \left[ DGD^{-1} \right] \geq \frac{1}{\sqrt{m}} \rho \left[ G \right].$$  \hspace{1cm} (3.6)

This can be accomplished by showing that

$$\bar{\sigma} \left[ G \right] \geq \frac{1}{\sqrt{m}} \rho \left[ G \right]$$  \hspace{1cm} (3.7)

and noticing that $\left[ DGD^{-1} \right] = D \left[ G \right] D^{-1}$ for any $D \in D_m$. Note first that $\bar{\sigma}(G) \cong (1/\sqrt{m}) \rho \left( G \right)$, and let $\lambda_i$ denote the eigenvalue of $\left[ G \right]$. Then it follows from Schur's decomposition [6] that $\left\| G \right\|_{\infty} \geq \left( \sum_{i=1}^{m} |\lambda_i|^2 \right)^{1/2} \geq \rho \left[ G \right]$. This proves (3.7) and thus completes the proof for the corollary.

As a final remark, we note that before computing the lower bounds in (3.3) and (3.5), it is helpful to first inspect the diagonal elements of $\Theta$. It follows from both corollaries that if these elements are large, then the minimal condition number will also be large.

IV. CONCLUSION

We have developed new relations between the block relative gain and condition number as well as the minimal condition number. Our results show that the condition number must be large if the corresponding block relative gain has a large maximum singular value, and the minimal condition number must be large if the block relative gain has a large structured singular value. These relations improve the previous result in [8] and are useful in clarifying the role of block relative gain as a measure of potential design difficulties associated with plants that have large block relative gains.

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Robust Controller Synthesis Using
Khartinov's Theorem

Dennis S. Bernstein and Wassim M. Haddad

Abstract—Khartinov's theorem provides a necessary and sufficient analysis test for robust stability of polynomials whose coefficients lie within a hyperrectangle. In this note, we present a method based upon Khartinov's theorem for synthesizing robustly stabilizing feedback controllers. Our approach is based upon a multiple plant model formulation with a quadratic cost functional. Sufficient conditions are obtained for characterizing robustly stabilizing static output feedback (proportional) controllers for MIMO plants with denominator polynomial uncertainty.

I. INTRODUCTION

Khartinov's theorem provides a necessary and sufficient analysis test for determining the robust stability of polynomials with perturbed coefficients [1]. Although Khartinov's original result was limited to uncertain polynomials having independently varying coefficients, considerable progress has been achieved in generalizing this result to more general regions [2]-[8]. Although these results are useful for analyzing the stability robustness of a given feedback control system, there are relatively few results that exploit Khartinov's theorem for synthesizing robust controllers. Notable exceptions are [9]-[12] which give necessary and sufficient conditions for robust stabilizability of uncertain plants.

The goal of the present note is to develop a technique that can be used for synthesizing such robustly stabilizing controllers. Our approach considers a class of MIMO systems in companion state-space form with denominator polynomial uncertainty. By limiting the controller to be proportional, i.e., static output feedback, the hyperrectangular structure of the parameter uncertainty is preserved. Hence it suffices to simultaneously stabilize the four 'plants' corresponding to Khartinov's theorem.

Although there exists extensive literature on simultaneous stabilization (see [13] and the references therein), we adopt here a fixed-structure optimization-based approach involving multiple models with a quadratic performance functional. This approach allows us to develop reasonably general conditions for robust static output feedback (proportional control) synthesis. Although extensions to dynamic compensation are more complex, we also show how SISO controller synthesis can be performed.
systems without zeros can be treated in a similar fashion by means of integral control.

II. PROBLEM FORMULATION

We begin with a matrix formulation of Khartitonov’s theorem. For 
\( i = 0, \ldots, n - 1 \). Let \( \tilde{\beta}_i \) and \( \bar{\beta}_i \) be given uncertainty bounds with 
\( \tilde{\beta}_i \leq \bar{\beta}_i \).

**Lemma 2.1:** Consider the set of matrices
\[
\mathcal{A} = \left\{ \begin{bmatrix} I_{n-1} \\
0_{(n-1)\times 1} \\
-\bar{\beta}_0 \\
-\bar{\beta}_{n-1}
\end{bmatrix} : \tilde{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \quad i = 0, \ldots, n - 1 \right\}
\]

Then every matrix in \( \mathcal{A} \) is stable if and only if the four matrices
\[
A_1 \hat{=} \begin{bmatrix} I_{n-1} \\
0_{(n-1)\times 1} \\
-\bar{\beta}_0 \\
-\bar{\beta}_{n-1}
\end{bmatrix},
\]
\[
A_2 \hat{=} \begin{bmatrix} I_{n-1} \\
0_{(n-1)\times 1} \\
-\bar{\beta}_0 \\
-\bar{\beta}_{n-1}
\end{bmatrix},
\]
\[
A_3 \hat{=} \begin{bmatrix} I_{n-1} \\
0_{(n-1)\times 1} \\
-\bar{\beta}_0 \\
-\bar{\beta}_{n-1}
\end{bmatrix},
\]
\[
A_4 \hat{=} \begin{bmatrix} I_{n-1} \\
0_{(n-1)\times 1} \\
-\bar{\beta}_0 \\
-\bar{\beta}_{n-1}
\end{bmatrix}
\]

are stable.

**Remark 2.1:** As noted in [3], simplification is possible if \( n = 2, \) 
3, 4. If \( n = 2 \), then it suffices to check \( A_3 \). If \( n = 3 \), then it 
suffices to check \( A_4 \) and \( \bar{\beta}_0 > 0 \). Hence, for \( n = 3 \), either of the 
pairs \((A_3, A_4)\) or \((A_4, A_3)\) suffices. If \( n = 4 \), then it suffices to check 
\( A_4, A_3, \) and \( \bar{\beta}_0 > 0 \). Hence either of the triples \((A_1, A_2, \) 
\( A_3)\) or \((A_2, A_3, A_4)\) suffices. Simplification to these cases of the 
results given in later sections is obvious and thus will not be noted explicitly.

For the statement of the robust controller synthesis problem, let 
\( A, B, C \) denote \( n \times n, n \times m, \) and \( l \times n \) matrices, respectively, 
and let \( x = x(t), u = u(t), \) and \( y = y(t) \) denote \( n, m, \) and 
l-dimensional vectors, respectively.

**Robust Controller Synthesis Problem:** Consider the dynamical system
\[
\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad (2.1)
\]
\[
y = Cx \quad (2.2)
\]

where \( A \in \mathcal{A} \). Then determine a static output feedback control law of 
the form
\[
u = Ky \quad (2.3)
\]
such that the closed-loop system
\[
\dot{x} = (A + BKC) x \quad (2.4)
\]
is stable for all \( A \in \mathcal{A} \).

The key step in exploiting Lemma 2.1 is to assume that the 
\( n \times m \) matrix \( B \) has the form
\[
B = \begin{bmatrix} I_{n-1} \\
0 \\
\vdots \\
b
\end{bmatrix}
\]
where \( b \) has dimensions \( 1 \times m \). No assumptions are needed concerning 
the structure of \( C \).

It can be seen that the assumed structure for \( (A, B, C) \) is 
sufficiently general to realize all single-output transfer functions 
with denominator polynomial uncertainty. Some, but not all, MIMO 
transfer functions can be realized with this structure. We now have 
the following corollary of Lemma 2.1.

**Corollary 2.1:** Let \( K \) be a given \( m \times l \) matrix and consider the 
set of matrices
\[
\mathcal{A} = \left\{ A + BKC : A \in \mathcal{A}, B \right\}
\]

Then every matrix in \( \mathcal{A} \) is stable if and only if the four matrices 
\( A_i \hat{=} A_i + BKC, \quad i = 1, \ldots, 4 \), are stable.

**Proof:** Every matrix in \( \mathcal{A} \) is of the form
\[
\begin{bmatrix} I_{n-1} \\
0_{(n-1)\times 1} \\
-\bar{\beta}_0 + bKC_1 \\
-\bar{\beta}_1 + bKC_2 \\
-\bar{\beta}_{n-1} + bKC_{n-1}
\end{bmatrix}
\]

where \( C_i \) is the \( i \)-th column of \( C \). Defining, for \( i = 0, \ldots, n - 1 \)
\[
\tilde{\beta}_i \hat{=} \beta_i - bKC_{i+1}, \quad \bar{\beta}_i \hat{=} \beta_i - bKC_{i+1}
\]
it follows that \( \mathcal{A} \) can be written as
\[
\mathcal{A} = \left\{ \begin{bmatrix} I_{n-1} \\
0_{(n-1)\times 1} \\
-\bar{\beta}_0 \\
-\bar{\beta}_1 \\
-\bar{\beta}_{n-1}
\end{bmatrix} : \tilde{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \quad i = 0, \ldots, n - 1 \right\}
\]

Note that the closed-loop matrices in \( \mathcal{A} \) have the same 
structure as the open-loop matrices in \( \mathcal{A} \). Furthermore, the matrices 
\( A_1, \ldots, A_4 \) now play the same role as \( A_1, \ldots, A_4 \) with 
uncertain parameters \( \tilde{\beta}_0, \ldots, \tilde{\beta}_n, \) lower bounds \( \beta_0, \ldots, \beta_n, \) 
and upper bounds \( \tilde{\beta}_0, \ldots, \tilde{\beta}_n, \) replaced by uncertain parameters 
\( \bar{\beta}_0, \ldots, \bar{\beta}_n, \) lower bounds \( \beta_0, \ldots, \beta_n, \) and upper bounds 
\( \bar{\beta}_0, \ldots, \bar{\beta}_n, \), respectively.

Next, we consider an augmented system of dimension \( 4n \) that 
simultaneously includes the dynamics of \( A_1, \ldots, A_4 \). Specifically, 
consider
\[
\dot{x} = \hat{A} \hat{x} \quad (2.5)
\]

where
\[
\hat{x} = \begin{bmatrix} x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & A_3 & 0 \\
0 & 0 & 0 & A_4
\end{bmatrix}
\]

Note that
\[
\hat{A}_u = A_u + \sum_{i=1}^{4} B_i K C_{iu} \quad (2.6)
\]

where
\[
A_u = \begin{bmatrix} A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & A_3 & 0 \\
0 & 0 & 0 & A_4
\end{bmatrix},
\]
\[
B_{1u} = \begin{bmatrix} B \\
0 \\
0 \\
0
\end{bmatrix}, \quad B_{2u} = \begin{bmatrix} 0 \\
B \\
0 \\
0
\end{bmatrix}, \quad B_{3u} = \begin{bmatrix} 0 \\
0 \\
B \\
0
\end{bmatrix}, \quad B_{4u} = \begin{bmatrix} 0 \\
0 \\
0 \\
B
\end{bmatrix},
\]
\[
C_{1u} = \begin{bmatrix} C & 0 & 0 & 0
\end{bmatrix}, \quad C_{2u} = \begin{bmatrix} 0 & C & 0 & 0
\end{bmatrix}, \quad C_{3u} = \begin{bmatrix} 0 & 0 & C & 0
\end{bmatrix}, \quad C_{4u} = \begin{bmatrix} 0 & 0 & 0 & C
\end{bmatrix}
\]

(Note our notation scheme: (.) denotes closed-loop, (.) denotes 
augmented system.)
III. CONTROLLER SYNTHESIS VIA QUADRATICALLY OPTIMAL CONTROL

To synthesize a controller for the system (2.5), we consider a quadratic performance functional of the form

$$J(K) = \sum_{i=1}^{4} \int_{0}^{\infty} [x_i'R_i x_i + u_i'^T R_2 u_i] dt$$

(4.1)

where $R_1$ and $R_2$ are $n \times n$ and $m \times m$ positive definite matrices and $u_i$ is defined by

$$u_i = KC_{i,a}$$

It now follows that $J(K)$ is given by

$$J(K) = \int_{0}^{\infty} \tilde{x}^T \tilde{R}_a \tilde{x} dt$$

(4.2)

where

$$\tilde{R}_a = \begin{bmatrix} R_1 + (KC)^T R_2 KC & 0 \\ 0 & R_1 + (KC)^T R_2 KC \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Note that $\tilde{R}_a = R_a + \sum_{i=1}^{4} C_{i,a}^T K^T R_2 K C_{i,a}$, where $R_a$ is defined by

$$R_a = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 \\ 0 & 0 & R_1 & 0 \\ 0 & 0 & 0 & R_1 \end{bmatrix}$$

Writing

$$\tilde{x}(t) = e^{A_e t} x_0, \quad \tilde{x}_0 = \begin{bmatrix} x_0 \\ x_0 \\ x_0 \\ x_0 \end{bmatrix}$$

(4.3)

leads to

$$J(K) = \int_{0}^{\infty} \tilde{x}^T_0 e^{A_e t} \tilde{R}_a e^{A_e t} \tilde{x}_0 dt$$

(4.4)

where we are now assuming that $K$ is such that $\tilde{A}_a$ is stable. Now, as is common practice [14], we eliminate explicit dependence on the initial condition $x_0$ by assuming $x_0, x_0^T$ has expected value $I_n$ ($n \times n$ identity). Invoking this step leads to

$$J(K) = \mathbb{E} \left[ \text{tr} \int_{0}^{\infty} \tilde{x}^T_0 e^{A_e t} \tilde{R}_a e^{A_e t} \tilde{x}_0 dt \right]$$

$$= \text{tr} I_n \int_{0}^{\infty} e^{A_e t} \tilde{R}_a e^{A_e t} dt$$

$$= \text{tr} \tilde{R}$$

(4.5)

where $\tilde{R}$ is the $4n \times 4n$ positive definite solution to

$$0 = \tilde{A}_a^T \tilde{P} + \tilde{P} \tilde{A}_a + \tilde{R}_a.$$  

To minimize $J(K)$ we form the Lagrangian

$$\mathcal{L}(K, \tilde{P}, \tilde{Q}) = \text{tr} \left[ \tilde{P} + \tilde{Q} (\tilde{A}_a^T \tilde{P} + \tilde{P} \tilde{A}_a + \tilde{R}_a) \right]$$

where $\tilde{Q}$ is a $4n \times 4n$ Lagrange multiplier matrix. Note that

$$\mathcal{L}(K, \tilde{P}, \tilde{Q}) = \text{tr} \left[ \tilde{P} + 2 A_e \tilde{P} \tilde{Q} + 2 \sum_{i=1}^{4} B_{i,a} K C_{i,a} \tilde{Q} \tilde{P} \right]$$

$$+ \tilde{Q} R_a + \sum_{i=1}^{4} \tilde{Q} C_{i,a}^T K^T R_2 K C_{i,a}.$$  

Hence

$$\frac{\partial \mathcal{L}}{\partial K} = 2 \sum_{i=1}^{4} \left[ C_{i,a} \tilde{P} B_{i,a} + C_{i,a} \tilde{Q} C_{i,a}^T K^T R_2 \right]$$

so that $\partial \mathcal{L}/\partial K = 0$ yields

$$K = -R_2^{-1} \left[ \sum_{i=1}^{4} \tilde{Q} C_{i,a}^T \right] \left[ \sum_{i=1}^{4} C_{i,a} \tilde{Q} C_{i,a}^T \right]^{-1}.$$  

(4.6)

Similarly, evaluating $\partial K/\partial \tilde{P} = 0$ yields

$$0 = \tilde{A}_a \tilde{Q} + \tilde{Q} \tilde{A}_a^T + I_n.$$  

(4.7)

We thus have the following result.

**Theorem 3.1:** Let $K \in R^{m \times l}$ be a feedback gain that stabilizes

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & R_1 + (KC)^T R_2 KC & 0 \\ 0 & 0 & 0 & R_1 + (KC)^T R_2 KC \end{bmatrix}$$

(2.5) and minimizes (4.1). Then $K$ is given by

$$K = -R_2^{-1} \left[ \sum_{i=1}^{4} \tilde{Q} C_{i,a}^T \right] \left[ \sum_{i=1}^{4} C_{i,a} \tilde{Q} C_{i,a}^T \right]^{-1}$$

(4.8)

where $\tilde{Q}$ and $\tilde{P}$ satisfy

$$0 = \tilde{A}_a \tilde{Q} + \tilde{Q} \tilde{A}_a^T + I_n,$$  

(4.9)

$$0 = \tilde{A}_a^T \tilde{P} + \tilde{P} \tilde{A}_a + \tilde{R}_a.$$  

(4.10)

Note that the matrices $\tilde{A}$ and $\tilde{R}$ depend upon $K$ so that (4.8)-(4.10) must be solved numerically together. The expression (4.8) for $K$ can be substituted into (4.9), (4.10) to eliminate this dependence. This optimal static output feedback solution is essentially a generalization of the standard theory [14].

IV. SUFFICIENCY RESULT FOR ROBUST STABILITY AND A NUMERICAL EXAMPLE

Since (4.9) and (4.10) are Lyapunov equations, they guarantee the stability of $\tilde{A}_a$ when they have solutions. Since, furthermore, $\tilde{A}$ is a block-diagonal matrix, each of its (four) diagonal blocks will be stable. Finally, by Corollary 2.1, the stability of these four matrices is sufficient to guarantee the stability of the closed-loop system (2.4) for all $A \in \mathcal{A}$, i.e., for all variations in the given uncertainty set.

**Theorem 4.1:** Suppose that a solution to (4.8)-(4.10) can be computed numerically. Then the resulting gain $K$ solves the robust controller synthesis problem.

Since (4.8)-(4.10) arise from a parametric LQ problem, there exist a variety of numerical methods that can be used to solve them. Here we note the extensive survey [14] as well as the homotopy-based methods used in [16], [17]. For the following example we used the BFGS quasi-Newton algorithm [18], [19].

The example we consider was treated in [12] and is originally due to [20] as a model of an oblique wing aircraft. The model is given by

$$P(s, q, r) = \frac{q_s r + s_0}{s^4 + r s^3 + r_2 s^2 + r_1 s + r_0}$$
where \( q_0 \in [90, 166], q_1 \in [54, 74], r_0 \in [-0.1, 0.1], r_1 \in [30.1, 33.9], r_2 \in [50.4, 80.8], \) and \( r_3 \in [2.8, 4.6] \). Since our formulation does not include numerator uncertainty we set \( q_0 = 128 \) and \( q_1 = 46.4 \) for the remaining denominator uncertainty we introduced an expansion factor \( \alpha \geq 1 \) for \( r_0, \ldots, r_3 \) so that \( r_3 = (3.7 + 0.9\alpha) \) and similarly for \( r_0, r_1, r_2, r_3 \). Setting \( R = 100I_2 \) and \( R_1 = 1 \) we obtained \( K = -0.0687, -0.05996, -0.01486 \) for \( \alpha = 1, 0.25, 2.5 \), respectively, for robustly stabilizing gains that minimize the quadratic cost. Note that the solution for \( \alpha = 2.5 \) represents a factor of \((2.5)^3 \approx 39\) increase in robustness "volume" compared to the \( \alpha = 1 \) solution. The solution given in [12] involved \( \alpha = 1 \) with uncertainty in \( q_0 \) and \( q_1 \) with a PI controller.

V. INTEGRAL CONTROL

Static output feedback was considered in previous sections since it preserves the "Kharitonov" structure. Dynamic compensation is, of course, considerably more complex. We now consider, as in [10], the possibility of utilizing an integral controller. Our setting is more restrictive than [11], [12] with regard to the admissible plants.

Specifically, we consider the realization

\[
\begin{bmatrix}
A & I_{n-1} \\
-\beta_0 & -\beta_1 & \cdots & -\beta_{n-1}
\end{bmatrix}, \quad
B = \begin{bmatrix} 0_{n-1 \times 1} \\ b \end{bmatrix},
\]

\[
C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}
\]

corresponding to an SISO plant with no zeros. Letting the control \( u \) be given by

\[
u = K_p y + K_i \int y
\]

leads to closed-loop dynamics of the form

\[
\dot{x} = \tilde{A}x + \tilde{B}y
\]

where

\[
x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{n+1}, \quad x_1 = \text{integrator state},
\]

and

\[
\tilde{A} = \begin{bmatrix} 0_{n \times 1} & I_n \\ bK_1 & -\beta_0 + bK_p & \cdots & -\beta_{n-1} + bK_p \end{bmatrix}
\]

Note that \( \tilde{A} \) preserves the companion structure needed to apply Lemma 2.1. Optimization can then proceed as in the static controller case.

VI. CONCLUSION

By formulating Kharitonov's result in terms of a MIMO state-space realization, a robust output feedback stabilization problem involving four plant models was formulated. A multimodel control problem involving a quadratic cost was then used to characterize robustly stabilizing controllers. Solving the optimality conditions by means of the BFS quasi-Newton algorithm provided a constructive procedure for robust controller synthesis.

Although the results of the present paper were limited to plant denominator uncertainty, the multimodel simultaneous \( H_2 \) optimization approach is applicable to the 16 plant formulation developed in [12] for both denominator and numerator uncertainty with PI controllers. Extension to this case remains a topic for future research.

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A New Controller Design for a Flexible One-Link Manipulator

W. T. Qian and C. C. H. Ma

Abstract—A new controller design for controlling a flexible one-link manipulator based on variable structure theory is presented in this note.