Corollary 3.4: For any $\Theta \in \boldsymbol{BRG}(\boldsymbol{G})$

$$k^*[G] \geq \frac{1}{\sqrt{m}} \rho[\,|\Theta|\,]. \tag{3.5}$$

Proof: As in the proof of Corollary 3.3, we show that for any matrix $G = [g_{ij}] \in \boldsymbol{C}^{m \times m}$

$$\inf_{d \in \boldsymbol{D}_m} \bar{\sigma}[DGD^{-1}] \geq \frac{1}{\sqrt{m}} \rho[\,|G|\,]. \tag{3.6}$$

This can be accomplished by showing that

$$\bar{\sigma}[G] \geq \frac{1}{\sqrt{m}} \rho[\,|G|\,] \tag{3.7}$$

and noticing that $|DGD^{-1}| = D|G|D^{-1}$ for any $D \in \boldsymbol{D}_m$. Note first that $\bar{\sigma}[G] \geq (1/\sqrt{m})\|G\|_F = (1/\sqrt{m}\|\,|G|\,\|_F$, and let λ_i denote the eigenvalue of $|G|$. Then it follows from *Schur's decomposition* [6] that $\|\,|G|\,\|_F \geq \sqrt{\sum_{i=1}^{m} |\lambda_i|^2} \geq \rho[\,|G|\,]$. This proves (3.7) and thus completes the proof for the corollary. ∇∇∇

As a final remark, we note that before computing the lower bounds in (3.3) and (3.5), it is helpful to first inspect the diagonal elements of Θ. It follows from both corollaries that if these elements are large, then the minimal condition number will also be large.

IV. Conclusion

We have developed new relations between the block relative gain and condition number as well as the minimal condition number. Our results show that the condition number must be large if the corresponding block relative gain has a large maximum singular value, and the minimal condition number must be large if the block relative gain has a large structured singular value. These relations improve the previous result in [8] and are useful in clarifying the role of block relative gain as a measure of potential design difficulties associated with plants that have large block relative gains.

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References

[1] E. H. Bristol, "On a new measure of interaction for multivariable process control," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 133-134, Jan. 1966.

[2] J. Chen and J. S. Freudenberg, "Robust performance with respect to diagonal input uncertainty," in *Proc. 28th IEEE Conf. Decision Contr.*, Tampa, FL, Dec. 1989, pp. 441-446; also in *IEEE Trans. Automat. Contr.*, to be published.

[3] J. C. Doyle, "Analysis of feedback systems with structured uncertainties," *IEE Proc.*, vol. 129, Pt.D, no. 6, pp. 242-250, Nov. 1982.

[4] J. S. Freudenberg, "Analysis and design for ill-conditioned plants—Part 1: lower bounds on the structured singular value," *Int. J. Contr.*, vol. 49, no. 3, pp. 851-871, Mar. 1989.

[5] —, "Analysis and design for ill-conditioned plants, Part 2: Directionally uniform weightings and an example," *Int. J. Contr.*, vol. 49, no. 3, pp. 873-903, Mar. 1989.

[6] G. H. Golub and C. F. Van Loan, *Matrix Computations.* Baltimore, MD: Johns Hopkins Univ. Press, 1983.

[7] V. Manousiouthakis, R. Savage, and Y. Arkun, "Synthesis of decentralized control structures using the concept of block relative gain," *AIChE J.*, vol. 32, no. 6, pp. 991-1003, June 1986.

[8] C. N. Nett and V. Manousiouthakis, "Euclidean condition and block relative gain: Connections, conjectures, and clarifications," *IEEE Trans. Automat. Contr.*, vol. AC-32, no. 5, pp. 405-407, May 1987.

[9] S. Skogestad and M. Morari, "Implications of large RGA elements on control performance," *Ind. Eng. Chem. Res.*, vol. 26, pp. 2323-2330, 1987.

[10] S. Skogestad, M. Morari, and J. C. Doyle, "Robust control of ill-conditioned plants: High-purity distillation," *IEEE Trans. Automat. Contr.*, vol. AC-33, no. 12, pp. 1092-1105, Dec. 1988.

[11] G. Stein, *Lecture Notes of Workshop on Robust Multivariable Control.* Minneapolis, MN, June 1987.

Robust Controller Synthesis Using Kharitonov's Theorem

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***Abstract*—Kharitonov's theorem provides a necessary and sufficient *analysis* test for robust stability of polynomials whose coefficients lie within a hyperrectangle. In this note, we present a method based upon Kharitonov's theorem for *synthesizing* robustly stabilizing feedback controllers. Our approach is based upon a multiple plant model formulation with a quadratic cost functional. Sufficient conditions are obtained for characterizing robustly stabilizing static output feedback (proportional) controllers for MIMO plants with denominator polynomial uncertainty.**

I. Introduction

Kharitonov's theorem provides a necessary and sufficient *analysis* test for determining the robust stability of polynomials with perturbed coefficients [1]. Although Kharitonov's original result was limited to uncertain polynomials having independently varying coefficients, considerable progress has been achieved in generalizing this result to more general regions [2]-[8]. Although these results are useful for analyzing the stability robustness of a given feedback control system, there are relatively few results that exploit Kharitonov's theorem for *synthesizing* robust controllers. Notable exceptions are [9]-[12] which give necessary and sufficient conditions for robust *stabilizability* of uncertain plants.

The goal of the present note is to develop a technique that can be used for synthesizing such robustly stabilizing controllers. Our approach considers a class of MIMO systems in companion state-space form with denominator polynomial uncertainty. By limiting the controller to be proportional, i.e., static output feedback, the hyperrectangular structure of the parameter uncertainty is preserved. Hence it suffices to simultaneously stabilize the four "plants" corresponding to Kharitonov's theorem.

Although there exists extensive literature on simultaneous stabilization (see [13] and the references therein), we adopt here a fixed-structure optimization-based approach involving multiple models with a quadratic performance functional. This approach allows us to develop reasonably general conditions for robust static output feedback (proportional control) synthesis. Although extensions to dynamic compensation are more complex, we also show how SISO

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systems without zeros can be treated in a similar fashion by means of integral control.

II. Problem Formulation

We begin with a matrix formulation of Kharitonov's theorem. For $i = 0,\cdots, n-1$, Let $\underline{\beta}_i$ and $\bar{\beta}_i$ be given uncertainty bounds with $\underline{\beta}_i \le \bar{\beta}_i$.

Lemma 2.1: Consider the set of matrices

$$\mathscr{A} \triangleq \left\{ \begin{bmatrix} 0_{(n-1)\times 1} & & I_{n-1} \\ -\beta_0 & \cdots & -\beta_{n-1} \end{bmatrix} : \underline{\beta}_i \le \beta_i \le \bar{\beta}_i,\ i = 0,\cdots, n-1 \right\}.$$

Then every matrix in $\mathscr{A}$ is stable if and only if the four matrices

$$A_1 \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \cdots - \underline{\beta}_{n-4} - \underline{\beta}_{n-3} & -\bar{\beta}_{n-2} - \bar{\beta}_{n-1} \end{bmatrix},$$

$$A_2 \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \cdots - \bar{\beta}_{n-4} - \underline{\beta}_{n-3} & -\underline{\beta}_{n-2} - \bar{\beta}_{n-1} \end{bmatrix},$$

$$A_3 \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \cdots - \bar{\beta}_{n-4} - \bar{\beta}_{n-3} & -\underline{\beta}_{n-2} - \underline{\beta}_{n-1} \end{bmatrix},$$

$$A_4 \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \cdots - \underline{\beta}_{n-4} - \bar{\beta}_{n-3} & -\bar{\beta}_{n-2} - \underline{\beta}_{n-1} \end{bmatrix}$$

are stable.

Remark 2.1: As noted in [3], simplification is possible if $n = 2, 3, 4$. If $n = 2$, then it suffices to check A_3. If $n = 3$, then it suffices to check A_3 and $\beta_0 > 0$. Hence, for $n = 3$, either of the pairs (A_3, A_1) or $(A_3, \bar{A}_2)$ suffices. If $n = 4$, then it suffices to check A_2, A_3, and $\beta_0 > 0$. Hence either of the triples (A_1, A_2, A_3) or (A_2, A_3, A_4) suffices. Simplification to these cases of the results given in later sections is obvious and thus will not be noted explicitly.

For the statement of the robust controller synthesis problem, let A, B, C denote $n \times n$, $n \times m$, and $l \times n$ matrices, respectively, and let $x = x(t)$, $u = u(t)$, and $y = y(t)$ denote n-, m-, and l-dimensional vectors, respectively.

Robust Controller Syntehsis Problem: Consider the dynamical system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0, \tag{2.1}$$

$$y = Cx \tag{2.2}$$

where $A \in \mathscr{A}$. Then determine a static output feedback control law of the form

$$u = Ky \tag{2.3}$$

such that the closed-loop system

$$\dot{x} = (A + BKC)x \tag{2.4}$$

is stable for all $A \in \mathscr{A}$.

The key step in exploiting Lemma 2.1 is to assume that the $n \times m$ matrix B has the form

$$B = \begin{bmatrix} 0_{(n-1)\times m} \\ b \end{bmatrix}$$

where b has dimensions $1 \times m$. No assumptions are needed concerning the structure of C.

It can be seen that the assumed structure for (A, B, C) is sufficiently general to realize all single-output transfer functions with denominator polynomial uncertainty. Some, but not all, MIMO transfer functions can be realized with this structure. We now have the following corollary of Lemma 2.1.

Corollary 2.1: Let K be a given $m \times l$ matrix and consider the set of matrices

$$\tilde{\mathscr{A}} \triangleq \{ A + BKC : A \in \mathscr{A} \}.$$

Then every matrix in $\tilde{\mathscr{A}}$ is stable if and only if the four matrices $\tilde{A}_i \triangleq A_i + BKC$, $i = 1,\cdots,4$, are stable.

Proof: Every matrix in $\tilde{\mathscr{A}}$ is of the form

$$\begin{bmatrix} 0_{(n-1)\times 1} & & I_{n-1} \\ -\beta_0 + bKC_1 & -\beta_1 + bKC_2 & \cdots - \beta_{n-1} + bKC_n \end{bmatrix}$$

where C_i is the ith column of C. Defining, for $i = 0,\cdots, n-1$

$$\delta_i \triangleq \beta_i - bKC_{i+1}, \underline{\delta}_i \triangleq \underline{\beta}_i - bKC_{i+1}, \bar{\delta}_i \triangleq \bar{\beta}_i - bKC_{i+1}$$

it follows that $\tilde{\mathscr{A}}$ can be written as

$$\tilde{\mathscr{A}} = \left\{ \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ -\delta_0 & -\delta_1 \cdots - \delta_{n-1} \end{bmatrix} : \underline{\delta}_i \le \delta_i \le \bar{\delta}_i,\ i = 0,\cdots, n-1 \right\}.$$

Now note that the closed-loop matrices in $\tilde{\mathscr{A}}$ have the same structure as the open-loop matrices in $\mathscr{A}$. Furthermore, the matrices $\tilde{A}_1,\cdots, \tilde{A}_4$ now play the same role as $A_1,\cdots, A_4$ with uncertain parameters $\beta_0,\cdots, \beta_{n-1}$, lower bounds $\underline{\beta}_0,\cdots,\underline{\beta}_{n-1}$, and upper bounds $\bar{\beta}_0,\cdots,\bar{\beta}_{n-1}$ replaced by uncertain parameters $\delta_0,\cdots,\delta_{n-1}$, lower bounds $\underline{\delta}_0,\cdots,\underline{\delta}_{n-1}$, and upper bounds $\bar{\delta}_0,\cdots,\bar{\delta}_{n-1}$, respectively. □

Next, we consider an augmented system of dimension $4n$ that simultaneously includes the dynamics of $\tilde{A}_1,\cdots, \tilde{A}_4$. Specifically, consider

$$\dot{\tilde{x}} = \tilde{A}_a \tilde{x} \tag{2.5}$$

where

$$\tilde{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \tilde{A}_a = \begin{bmatrix} \tilde{A}_1 & 0 & 0 & 0 \\ 0 & \tilde{A}_2 & 0 & 0 \\ 0 & 0 & \tilde{A}_3 & 0 \\ 0 & 0 & 0 & \tilde{A}_4 \end{bmatrix}.$$

Note that

$$\tilde{A}_a = A_a + \sum_{i=1}^{4} B_{ia} K C_{ia} \tag{2.6}$$

where

$$A_a \triangleq \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix},$$

$$B_{1a} \triangleq \begin{bmatrix} B \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{2a} \triangleq \begin{bmatrix} 0 \\ B \\ 0 \\ 0 \end{bmatrix}, \quad B_{3a} \triangleq \begin{bmatrix} 0 \\ 0 \\ B \\ 0 \end{bmatrix}, \quad B_{4a} \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix},$$

$$C_{1a} \triangleq [C \ \ 0 \ \ 0 \ \ 0], \qquad C_{2a} \triangleq [0 \ \ C \ \ 0 \ \ 0],$$

$$C_{3a} = [0 \ \ 0 \ \ C \ \ 0], \qquad C_{4a} \triangleq [0 \ \ 0 \ \ 0 \ \ C].$$

(Note our notation scheme: $(\tilde{\ })$ denotes closed-loop, $(\ )_a$ denotes augmented system.)

We now turn to the problem of explicitly synthesizing a controller that stabilizes (2.5) and hence the original system (2.1), (2.2) for all $A \in \mathscr{A}$.

III. Controller Synthesis Via Quadratically Optimal Control

To synthesize a controller for the system (2.5), we consider a quadratic performance functional of the form

$$J(K) \triangleq \sum_{i=1}^{4} \int_0^\infty \left[x_i^{\mathrm{T}} R_1 x_i + u_i^{\mathrm{T}} R_2 u_i \right] dt \tag{4.1}$$

where R_1 and R_2 are $n \times n$ and $m \times m$ positive definite matrices and u_i is defined by

$$u_i \triangleq K C x_i.$$

It now follows that $J(K)$ is given by

$$J(K) = \int_0^\infty \tilde{x}^{\mathrm{T}} \tilde{R}_a \tilde{x}\, dt \tag{4.2}$$

where

$$\tilde{R}_a \triangleq \begin{bmatrix} R_1 + (KC)^{\mathrm{T}} R_2 KC & 0 & 0 & 0 \\ 0 & R_1 + (KC)^{\mathrm{T}} R_2 KC & 0 & 0 \\ 0 & 0 & R_1 + (KC)^{\mathrm{T}} R_2 KC & 0 \\ 0 & 0 & 0 & R_1 + (KC)^{\mathrm{T}} R_2 KC \end{bmatrix}.$$

Note that $\tilde{R}_a = R_a + \sum_{i=1}^4 C_{ia}^{\mathrm{T}} K^{\mathrm{T}} R_2 K C_{ia}$, where R_a is defined by

$$R_a \triangleq \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 \\ 0 & 0 & R_1 & 0 \\ 0 & 0 & 0 & R_1 \end{bmatrix}.$$

Writing

$$\tilde{x}(t) = e^{\tilde{A}_a t} \tilde{x}_0, \qquad \tilde{x}_0 \triangleq \begin{bmatrix} x_0 \\ x_0 \\ x_0 \\ x_0 \end{bmatrix} \tag{4.3}$$

leads to

$$J(K) = \int_0^\infty \tilde{x}_0^{\mathrm{T}} e^{\tilde{A}_a^{\mathrm{T}} t} \tilde{R}_a e^{\tilde{A}_a t} \tilde{x}_0\, dt \tag{4.4}$$

where we are now assuming that K is such that $\tilde{A}_a$ is stable. Now, as is common practice [14], we eliminate explicit dependence on the initial condition x_0 by assuming $x_0 x_0^{\mathrm{T}}$ has expected value I_n ($n \times n$ identity). Invoking this step leads to

$$\begin{aligned} J(K) &= \mathbb{E}\left[\operatorname{tr} \int_0^\infty \tilde{x}_0 \tilde{x}_0^{\mathrm{T}} e^{\tilde{A}_a^{\mathrm{T}} t} \tilde{R}_a e^{\tilde{A}_a t}\, dt \right] \\ &= \operatorname{tr} I_{4n} \int_0^\infty e^{\tilde{A}_a^{\mathrm{T}} t} \tilde{R}_a e^{\tilde{A}_a t}\, dt \\ &= \operatorname{tr} \tilde{P} \end{aligned}$$

where $\tilde{P}$ is the $4n \times 4n$ positive definite solution to

$$0 = \tilde{A}_a^{\mathrm{T}} \tilde{P} + \tilde{P} \tilde{A}_a + \tilde{R}_a. \tag{4.5}$$

To minimize $J(K)$ we form the Lagrangian

$$\mathscr{L}(K, \tilde{P}, \tilde{Q}) \triangleq \operatorname{tr}\left[\tilde{P} + \tilde{Q}\left(\tilde{A}_a^{\mathrm{T}} \tilde{P} + \tilde{P} \tilde{A}_a + \tilde{R}_a \right) \right]$$

where $\tilde{Q}$ is a $4n \times 4n$ Lagrange multiplier matrix. Note that

$$\mathscr{L}(K, \tilde{P}, \tilde{Q}) = \operatorname{tr}\left[\tilde{P} + 2 A_a \tilde{Q} \tilde{P} + 2 \sum_{i=1}^4 B_{ia} K C_{ia} \tilde{Q} \tilde{P} + \tilde{Q} R_a + \sum_{i=1}^4 \tilde{Q} C_{ia}^{\mathrm{T}} K^{\mathrm{T}} R_2 K C_{ia} \right].$$

Hence

$$\frac{\partial \mathscr{L}}{\partial K} = 2 \sum_{i=1}^4 \left[C_{ia} \tilde{Q} \tilde{P} B_{ia} + C_{ia} \tilde{Q} C_{ia}^{\mathrm{T}} K^{\mathrm{T}} R_2 \right]$$

so that $\partial \mathscr{L} / \partial K = 0$ yields

$$K = -R_2^{-1} \left[\sum_{i=1}^4 B_{ia}^{\mathrm{T}} \tilde{P} \tilde{Q} C_{ia}^{\mathrm{T}} \right] \left[\sum_{i=1}^4 C_{ia} \tilde{Q} C_{ia}^{\mathrm{T}} \right]^{-1}. \tag{4.6}$$

Similarly, evaluating $\partial K / \partial \tilde{P} = 0$ yields

$$0 = \tilde{A}_a \tilde{Q} + \tilde{Q} \tilde{A}_a^{\mathrm{T}} + I_{4n}. \tag{4.7}$$

We thus have the following result.

Theorem 3.1: Let $K \in \mathbb{R}^{m \times l}$ be a feedback gain that stabilizes (2.5) and minimizes (4.1). Then K is given by

$$K = -R_2^{-1} \left[\sum_{i=1}^4 B_{ia}^{\mathrm{T}} \tilde{P} \tilde{Q} C_{ia}^{\mathrm{T}} \right] \left[\sum_{i=1}^4 C_{ia} \tilde{Q} C_{ia}^{\mathrm{T}} \right]^{-1} \tag{4.8}$$

where $\tilde{Q}$ and $\tilde{P}$ satisfy

$$0 = \tilde{A}_a \tilde{Q} + \tilde{Q} \tilde{A}_a^{\mathrm{T}} + I_{4n}, \tag{4.9}$$

$$0 = \tilde{A}_a^{\mathrm{T}} \tilde{P} + \tilde{P} \tilde{A}_a + \tilde{R}_a. \tag{4.10}$$

Note that the matrices $\tilde{A}$ and $\tilde{R}_a$ depend upon K so that (4.8)–(4.10) must be solved numerically together. The expression (4.8) for K can be substituted into (4.9), (4.10) to eliminate this dependence. This optimal static output feedback solution is essentially a generalization of the standard theory [14].

IV. Sufficiency Result for Robust Stability and a Numerical Example

Since (4.9) and (4.10) are Lyapunov equations, they guarantee the stability of $\tilde{A}_a$ when they have solutions. Since, furthermore, $\tilde{A}$ is a block-diagonal matrix, each of its (four) diagonal blocks will be stable. Finally, by Corollary 2.1, the stability of these four matrices is sufficient to guarantee the stability of the closed-loop system (2.4) for *all* $A \in \mathscr{A}$, i.e., for all variations in the given uncertainty set.

Theorem 4.1: Suppose that a solution to (4.8)–(4.10) can be computed numerically. Then the resulting gain K solves the robust controller synthesis problem.

Since (4.8)–(4.10) arise from a parametric LQ problem, there exist a variety of numerical methods that can be used to solve them. Here we note the extensive survey [14] as well as the homotopy-based methods used in [16], [17]. For the following example we used the BFGS quasi-Newton algorithm [18], [19].

The example we consider was treated in [12] and is originally due to [20] as a model of an oblique wing aircraft. The model is given by

$$P(s, q, r) = \frac{q_1 s + q_0}{s^4 + r_3 s^3 + r_2 s^2 + r_1 s + r_0}$$

where $q_0 \in [90, 166]$, $q_1 \in [54, 74]$, $r_0 \in [-0.1, 0.1]$, $r_1 \in [30.1, 33.9]$, $r_2 \in [50.4, 80.8]$, and $r_3 \in [2.8, 4.6]$. Since our formulation does not include numerator uncertainty we set $q_0 = 128$ and $q_1 = 64$. For the remaining dominator uncertainty we introduced an expansion factor $\alpha \geq 1$ for $r_0, \cdots, r_3$ so that $r_3 \in [3.7 - 0.9\alpha, 3.7 + 0.9\alpha]$ and similarly for r_0, r_1, and r_2. Setting $R = 100 I_4$ and $R_2 = 1$ we obtained $K = -0.0687, -0.05996, -0.01486$ for $\alpha = 1, 2, 2.5$, respectively, for robustly stabilizing gains that minimize the quadratic cost. Note that the solution for $\alpha = 2.5$ represents a factor of $(2.5)^4 \approx 39$ increase in robustness "volume" compared to the $\alpha = 1$ solution. The solution given in [12] involved $\alpha = 1$ with uncertainty in q_0 and q_1 with a PI controller.

V. Integral Control

Static output feedback was considered in previous sections since it preserves the "Kharitonov" structure. Dynamic compensation is, of course, considerably more complex. We now consider, as in [10], the possibility of utilizing an integral controller. Our setting is more restrictive than [11], [12] with regard to the admissible plants.

Specifically, we consider the realization

$$A = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ -\beta_0 & -\beta_1 \cdots -\beta_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{(n-1)\times 1} \\ b \end{bmatrix}, \quad C = [1 \quad 0 \cdots 0]$$

corresponding to an SISO plant with no zeros. Letting the control u be given by

$$u = K_P y + K_I \int y$$

leads to closed-loop dynamics of the form

$$\dot{\tilde{x}} = \tilde{A}\tilde{x}$$

where

$$\tilde{x} = \begin{bmatrix} x_I \\ x \end{bmatrix} \in \mathbb{R}^{n+1}, \qquad x_I = \text{integrator state},$$

and

$$\tilde{A} = \begin{bmatrix} 0_{n\times 1} & I_n \\ bK_I & -\beta_0 + bK_P \cdots -\beta_{n-1} + bK_P \end{bmatrix}.$$

Note that $\tilde{A}$ preserves the companion structure needed to apply Lemma 2.1. Optimization can then proceed as in the static controller case.

VI. Conclusion

By formulating Kharitonov's result in terms of a MIMO state-space realization, a robust output feedback stabilization problem involving four plant models was formulated. A multimodel control problem involving a quadratic cost was then used to characterize robustly stabilizing controllers. Solving the optimality conditions by means of the BFGS quasi-Newton algorithm provided a constructive procedure for robust controller synthesis.

Although the results of the present paper were limited to plant denominator uncertainty, the multimodel simultaneous H_2 optimization approach is applicable to the 16 plant formulation developed in [12] for both denominator and numerator uncertainty with PI controllers. Extension to this case remains a topic for future research.

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References

[1] V. L. Kharitonov, "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations," *Differencial'nye Uravnenija*, vol. 14, pp. 2086–2088, 1978.
[2] S. Bialas and J. Garloff, "Stability of polynomials under coefficient perturbation," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 310–312, 1985.
[3] B. D. O. Anderson, E. I. Jury, and M. Mansour, "On robust Hurwitz polynomials," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 909–913, 1987.
[4] S. P. Bhattacharyya, *Robust Stabilization Against Structured Perturbations*. New York: Springer-Verlag, 1987.
[5] A. C. Bartlett, C. V. Hollot, and H. Lin, "Root locations of an entire polytope of polynomials: It suffices to check the edges," *Math. Contr. Signal Syst.*, vol. 1, pp. 61–71, 1988.
[6] B. R. Barmish, "New tools for robustness analysis," in *Proc. Conf. Decision Contr.*, Austin, TX, Dec. 1988, pp. 1–6.
[7] I. R. Petersen, "A class of stability regions for which a Kharitonov-like theorem holds," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 1111–1115, 1989.
[8] B. R. Barmish, "A generalization of Kharitonov's four-polynomial concept for robust stability problems with linearly dependent coefficient perturbations," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 157–165, 1989.
[9] B. K. Ghosh, "Some new results on the simultaneous stabilizability of a family of single input, single output systems," *Syst. Contr. Lett.*, vol. 6, pp. 39–45, 1985.
[10] H. Chapellat and S. P. Bhattacharyya, "A generalization of Kharitonov's theorem: Robust stability of interval plants," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 306–311, 1989.
[11] C. V. Hollot and F. Yang, "Robust stabilization of interval plants using lead or lag compensators," *Syst. Contr. Lett.*, vol. 14, pp. 9–12, 1990.
[12] C. V. Hollot, F. J. Kraus, R. Tempo, and B. R. Barmish, "Extreme point results for robust stabilization of interval plants with first-order compensators," in *Proc. Amer. Contr. Conf.*, San Diego, CA, May 1990, pp. 2533–2538.
[13] B. K. Ghosh, "An approach to simultaneous system design, Part II: Nonswitching gain and dynamic feedback compensation by algebraic geometric methods," *SIAM J. Contr. Optimiz.*, vol. 26, pp. 919–963, 1988.
[14] W. S. Levine and M. Athans, "On the determination of the optimal output feedback gains for linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 44–48, 1970.
[15] P. M. Makila and H. T. Toivonen, "Computational methods for parametric LQ problems—A survey," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 658–671, 1987.
[16] P. T. Kabamba, R. W. Longman, and S. Jian-Guo, "A homotopy approach to the feedback stabilization of linear systems," *J. Guidance Contr. Dynam.*, vol. 10, pp. 422–432, 1987.
[17] S. Richter and E. G. Collins, Jr., "A homotopy algorithm for reduced-order controller design using the optimal projection equations," in *Proc. Conf. Decision Contr.*, Tampa, FL, Dec. 1989, pp. 506–511.
[18] D. F. Shanno and K. H. Phua, "Numerical comparison of several variable-metric algorithms," *J. Optimiz. Theory Appl.*, vol. 25, pp. 507–518, 1978.
[19] ——, "Minimization of unconstrained multivariable functions," *ACM Trans. Math. Appl.*, vol. 6, pp. 618–622, 1980.
[20] R. Dorf, *Modern Control Systems*. Reading, MA: Addison-Wesley, 1989.

A New Controller Design for a Flexible One-Link Manipulator

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***Abstract*—A new controller design for controlling a flexible one-link manipulator based on variable structure theory is presented in this note.**

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