where \([k_1, k_2], [l_1, l_2]\) are arbitrary polynomial matrices from \(R_{12}(d_1, d_2, d_3)\).

**CONCLUSIONS**

Sufficient conditions for the existence of a solution to the deadbeat servoproblem for multivariable \(n\)-linear systems are given. An algorithm based on elementary column and row operations for finding the matrices \(P, Q,\) and \(R\) of the linear \(n-D\) controller is presented and illustrated by a simple 3-D example.

**REFERENCES**


intensities $V_1 \geq 0$ and $V_2 > 0$, respectively, and cross intensity $V_{12} \in \mathbb{R}^{m \times l}$. It is further assumed that $v_i$, $w_i$, and $x(0)$ are mutually uncorrelated.

We require the technical assumption that, for each $i$, $B_i \neq 0$ implies $C_i = 0$, i.e., the control- and measurement-dependent noises are uncorrelated.

**Optimal Dynamic-Compensation Problem**

Given the controlled system

$$
\begin{align*}
\dot{x} &= (A + \sum_{i=1}^p v_i A_i) x + (B + \sum_{i=1}^p v_i B_i) u + w_i, \\
y &= (C + \sum_{i=1}^p v_i C_i) x + w_2,
\end{align*}
$$

(2.1)

design an $n$th-order dynamic compensator

$$
\begin{align*}
\dot{x}_c &= A_c x_c + B_c y, \\
u &= C_c x_c
\end{align*}
$$

(2.2)

which minimizes the performance criterion

$$
J(A_c, B_c, C_c) = J_x(A_c, B_c, C_c) + J_m(A_c, B_c, C_c) + J_u(A_c, B_c, C_c),
$$

(2.5)

where

$$
\begin{align*}
J_x(A_c, B_c, C_c) &= \lim_{t \to \infty} \mathbb{E}[x^T R_x x], \\
J_m(A_c, B_c, C_c) &= \lim_{t \to \infty} \mathbb{E}[2x^T R_m x], \\
J_u(A_c, B_c, C_c) &= \lim_{t \to \infty} \mathbb{E}[u^T R_u u].
\end{align*}
$$

To guarantee that $J$ is finite and independent of initial conditions, we restrict $(A_c, B_c, C_c)$ to the (open) set of second-moment-stabilizing triples

$$
\mathcal{S} = \{(A_c, B_c, C_c) \in \mathbb{R}^{n \times n} : A_c \oplus A_c + \sum_{i=1}^p A_i \otimes A_i \text{ is stable}\}
$$

where $\oplus$ and $\otimes$ denote Kronecker sum and product and

$$
\begin{align*}
\tilde{A}_i &= \begin{bmatrix} A_i & B_i C_i \\ B_i C_i^T & A_i \end{bmatrix}, \\
\tilde{A}_i &= \begin{bmatrix} A_i & B_i C_i \\ B_i C_i^T & A_i \end{bmatrix}, \\
A_c &\triangleq A + \frac{1}{2} \sum_{i=1}^p A_i + B + \frac{1}{2} \sum_{i=1}^p A_i B_i C_i + C + \frac{1}{2} \sum_{i=1}^p C_i A_i.
\end{align*}
$$

For convenience in stating the optimality conditions, define the following notation for $Q$, $P$, $\dot{Q}$, $\dot{P}$, $\hat{P}$ in $\mathbb{R}^{n \times n}$:

$$
\begin{align*}
R_{21} &= R_2 + \sum_{i=1}^p B_i^T (P + \hat{P}) B_i, \\
V_2 &= V_2 + \sum_{i=1}^p C_i (Q + \dot{Q}) C_i^T, \\
Q &= Q + \dot{Q}^T, \\
R &= R_1 + R_2 + 2 \sum_{i=1}^p B_i^T (P + \hat{P}) B_i.
\end{align*}
$$

For convenience in stating the optimality conditions, define the following notation for $Q$, $P$, $\dot{Q}$, $\dot{P}$ in $\mathbb{R}^{n \times n}$:

$$
\begin{align*}
R_{21} &= R_2 + \sum_{i=1}^p B_i^T (P + \hat{P}) B_i, \\
V_2 &= V_2 + \sum_{i=1}^p C_i (Q + \dot{Q}) C_i^T, \\
Q &= Q + \dot{Q}^T, \\
R &= R_1 + R_2 + 2 \sum_{i=1}^p B_i^T (P + \hat{P}) B_i.
\end{align*}
$$

For convenience in stating the optimality conditions, define the following notation for $Q$, $P$, $\dot{Q}$, $\dot{P}$ in $\mathbb{R}^{n \times n}$:

$$
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R_{21} &= R_2 + \sum_{i=1}^p B_i^T (P + \hat{P}) B_i, \\
V_2 &= V_2 + \sum_{i=1}^p C_i (Q + \dot{Q}) C_i^T, \\
Q &= Q + \dot{Q}^T, \\
R &= R_1 + R_2 + 2 \sum_{i=1}^p B_i^T (P + \hat{P}) B_i.
\end{align*}
$$

**Theorem 2.1:** Suppose $(A_c, B_c, C_c) \in \mathcal{S}$ solves the optimal dynamic-compensation problem. Then there exist $n \times n$ nonnegative-definite matrices $Q$, $P$, $\dot{Q}$, and $\hat{P}$ such that $A_c$, $B_c$, $C_c$ are given by

$$
\begin{align*}
A_c &= A \ominus B R_{21}^{-1} \sigma_1 \ominus Q_{12} V_2^{-1} C_c, \\
B_c &= Q_{12} V_2^{-1} C_c, \\
C_c &= -R_{21}^{-1} \sigma_1,
\end{align*}
$$

(2.6)

and such that the following conditions are satisfied:

$$
\begin{align*}
0 &= A_c Q + Q A_c^T + V_1 + \sum_{i=1}^p [A_i Q A_i^T \ominus (A_i \ominus B_i R_{21}^{-1} \sigma_1) \dot{Q} (A_i \ominus B_i R_{21}^{-1} \sigma_1)^T] - Q_{12} V_2^{-1} Q_{12}^T, \\
0 &= A_c^T P + P A_c + R_1 + \sum_{i=1}^p [A_i^T P A_i \ominus (A_i \ominus Q_{12} V_2^{-1} C_c) \dot{P} (A_i \ominus Q_{12} V_2^{-1} C_c)^T] - \sigma_1^T R_{21}^{-1} \sigma_1,
\end{align*}
$$

(2.7)

(2.8)

**Remark 2.1:** Letting $A_i = 0$, $B_i = 0$ and $C_i = 0$, $i = 1, \ldots, p$, it can readily be seen that (2.11) and (2.12) are superfluous and that (2.9) and (2.10) yield the standard separated LQG Riccati equations.

**Remark 2.2:** Since $R_2 \geq R_1$, so that $R_{21} \leq R_{21}^{-1}$, it is clear that the control-dependent noise effectively suppresses the regulator gain $C_c$. Similarly, since $V_2 \geq V_1$, the measurement-dependent noise suppresses the observer gain $B_c$. The effect of the terms $A_i Q A_i^T$ is discussed in [1] for modal systems.

**III. The Maximum Entropy Design Equations Applied to Doyle’s Example**

As shown in [2], LQG regulators for the example

$$
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad C = [1 \ 0],
$$

$$
V_1 = \sigma = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad V_{12} = 0, \quad V_2 = 1,
$$

$$
R_1 = \rho = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad R_{12} = 0, \quad R_2 = 1,
$$

have arbitrarily small stability margin with regard to variations $b + \Delta b$ when $\sigma$ and $\rho$ are sufficiently large and $b = 1$.

Setting $\sigma = \rho = 60$, it follows that the LQG regulator is only stable for $0.93 \leq b + \Delta b \leq 1.01$. Uncertainty in $b$ can be modeled by setting $p = 1$, $A_1 = 0$, $B_1 = [0 \ b_1]^T$, and $C_1 = 0$. Solving the optimality conditions (2.9)-(2.12) with $b_1 = 0.05$, 0.10, 0.15, and 0.20 yields a series of increasingly robust controller designs with respect to both positive and negative variations $\Delta b$ (see Table I and Figs. 1 and 2).

**CONCLUSION**

As demonstrated on the example of [2], the maximum entropy design equations provide a novel method for synthesizing robust feedback controllers. Since the design equations represent a fundamental generalization of standard LQG theory, the approach represents an alternative to LQG-modification techniques. Indeed, these equations are not intended as a device for recovering the gain and phase margins of LQ state-feedback regulators, but rather as a method for designing output-feedback dynamic compensators which are robust with respect to parametric deviations in...
### TABLE I

Dynamic Compensator Gains for LQG and Maximum Entropy Designs ($b = 1, \sigma = p = 60$)

<table>
<thead>
<tr>
<th>$b_1$</th>
<th>$a_k$</th>
<th>$b_k$</th>
<th>$c_k$</th>
<th>Stability Range of $b + ab$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (LQG)</td>
<td>$-9$</td>
<td>$-20$</td>
<td>$-9$</td>
<td>$10$</td>
</tr>
<tr>
<td>.05</td>
<td>$-9.253$</td>
<td>$1.0$</td>
<td>$10.25$</td>
<td>$12.31$</td>
</tr>
<tr>
<td>.1</td>
<td>$-9.639$</td>
<td>$1.0$</td>
<td>$10.64$</td>
<td>$15.95$</td>
</tr>
<tr>
<td>.15</td>
<td>$-10.10$</td>
<td>$1.0$</td>
<td>$11.10$</td>
<td>$20.53$</td>
</tr>
<tr>
<td>.2</td>
<td>$-10.69$</td>
<td>$1.0$</td>
<td>$11.40$</td>
<td>$26.67$</td>
</tr>
</tbody>
</table>

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The plant model. As discussed in [10], these are significantly different objectives.

### REFERENCES


### Stability of Multiloop LQ Regulators with Nonlinearities—Part I:

**Regions of Attraction**

S. M. JOSHI

Abstract—The closed-loop stability of linear, time-invariant systems controlled by linear quadratic (LQ) regulators is investigated when there are nonlinearities in the control channels which lie outside the $(0.5, \infty)$ stability sector in regions away from the origin (i.e., saturation-type nonlinearities). An estimate of the region of attraction is obtained which provides methods for selecting the performance function weights for more robust LQ designs.

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