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ROBUST STABILITY AND PERFORMANCE ANALYSIS FOR STATE-SPACE SYSTEMS VIA QUADRATIC LYAPUNOV BOUNDS*

DENNIS S. BERNSTEIN† AND WASSIM M. HADDAD‡

Abstract. For a given asymptotically stable linear dynamic system it is often of interest to determine whether stability is preserved as the system varies within a specified class of uncertainties. If, in addition, there also exist associated performance measures (such as the steady-state variances of selected state variables), it is desirable to assess the worst-case performance over a class of plant variations. These are problems of robust stability and performance analysis. In the present paper, quadratic Lyapunov bounds used to obtain a simultaneous treatment of both robust stability and performance are considered. The approach is based on the construction of modified Lyapunov equations, which provide sufficient conditions for robust stability along with robust performance bounds. In this paper, a wide variety of quadratic Lyapunov bounds are systematically developed and a unified treatment of several bounds developed previously for feedback control design is provided.

Key words. robust analysis, stability, performance, Lyapunov equations, structured uncertainty

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1. Introduction. Unavoidable discrepancies between mathematical models and real-world systems can result in degradation of control-system performance including instability [1], [2]. Ideally, feedback control systems should be designed to be *robust* with respect to uncertainties, or perturbations, in the plant characteristics. Such uncertainties may arise either due to limitations in performing system identification prior to control-system implementation or because of unpredictable plant changes that occur during operation. Thus robustness *analysis* must play a key role in control-system design. That is, given an existing or proposed control system, determine the performance degradation due to variations in the plant.

In performing robustness analysis there are two principal concerns, namely, stability robustness and performance robustness. Stability robustness addresses the qualitative question as to whether or not the system remains stable for all plant perturbations within a specified class of uncertainties. A related problem involves determining the largest class of plant perturbations under which stability is preserved. Once robust stability has been ascertained, it is of interest to investigate *quantitatively* the performance degradation within a given robust stability range. In practice it is often desirable to determine the *worst-case* performance as a measure of degradation.

The concern for both robust stability and performance can be traced back to the earliest developments in control theory. Design specifications such as gain and phase margin have traditionally been used to gauge system reliability in the face of uncertainty. In the modern control literature considerable effort has focused on rigorous robustness analysis and design techniques in a variety of settings. Analysis and synthesis results have been developed for both state-space and frequency-domain plant models to address structured parameter variations as well as normed-neighborhood uncertainty [3]–[7].

The present paper is concerned solely with the analysis of structured real-valued parameter uncertainty within the context of state-space models. One motivation for such

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† Harris Corporation, Government Aerospace Systems Division, MS 22/4842, Melbourne, Florida 32902.

‡ Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, Florida 32901.

problems is illustrated by the examples given in [1] and [2]. These examples show that standard linear-quadratic methods used to design either full-state feedback controllers or dynamic compensators may result in closed-loop systems that are arbitrarily sensitive to structured real-valued plant parameter variations. A particularly effective technique for analyzing robust stability is to construct a quadratic Lyapunov function $V(x) = x^T Px$, which guarantees stability of the system as the uncertain parameters vary over a specified range. This technique has been extensively developed for both analysis and synthesis (see, e.g., [8]–[37]).

Although both robust stability and performance are of interest in practice, most of the literature involving quadratic Lyapunov functions is confined to the problem of robust stability. A notable exception is the early work of Chang and Peng [9], which also provides bounds on worst-case quadratic performance within the context of full-state-feedback control design. In the present paper, we further extend the approach of [9] to obtain a series of results for analyzing both robust stability and performance. As will be seen, these results also provide substantial unification of more recent results pertaining to robust stability alone.

To illustrate the basis for our approach, consider the system

$$\dot{x}(t) = (A + \Delta A)x(t) + Dw(t), \qquad t \in [0, \infty), \quad x(0) = 0, \tag{1.1}$$

$$y(t) = Ex(t), \tag{1.2}$$

where $x(t)$ is an n-vector, A is an $n \times n$ matrix denoting the nominal dynamics matrix, ΔA denotes an uncertain perturbation of A belonging to a specified set $\mathscr{U}$, $Dw(t)$ is (for now) a white noise signal of intensity $V \triangleq DD^T$, and $y(t)$ is a q-vector of outputs. System (1.1), (1.2) may, for example, denote a control system in closed-loop configuration.

For the system (1.1) the performance measure involves the steady-state second moment of the outputs $y(t)$. In practice the diagonal elements of the second moment are measures of the ability of the external disturbances $Dw(t)$ to excite specified states. In the presence of uncertainties ΔA, it is of interest to determine the *worst-case* steady-state values of the second moments of selected states. Thus, we define the scalar performance criterion

$$J_S(\mathscr{U}) \triangleq \sup_{\Delta A \in \mathscr{U}} \limsup_{t \to \infty} \mathbb{E}\{y^T(t)y(t)\}, \tag{1.3}$$

where $\mathbb{E}$ denotes expectation and lim sup is a technicality to ensure that $J_S(\mathscr{U})$ is a well-defined quantity even when $A + \Delta A$ has eigenvalues in the closed right half plane. To evaluate (1.3) define the second-moment matrix

$$Q(t) \triangleq \mathbb{E}[x(t)x^T(t)],$$

which satisfies the Lyapunov differential equation

$$\dot{Q}_{\Delta A}(t) = (A + \Delta A)Q_{\Delta A}(t) + Q_{\Delta A}(t)(A + \Delta A)^T + V, \tag{1.4}$$

so that (1.3) becomes

$$J_S(\mathscr{U}) = \sup_{\Delta A \in \mathscr{U}} \limsup_{t \to \infty} \operatorname{tr} Q_{\Delta A}(t)R, \tag{1.5}$$

where $R \triangleq E^T E$. To guarantee both robust stability and performance we consider modified algebraic Lyapunov equations of the form

$$0 = AQ + QA^T + \Omega(Q) + V, \tag{1.6}$$

where $\Omega(\cdot)$ is a matrix operator satisfying

$$\Delta AQ + Q\Delta A^T \leqq \Omega(Q) \tag{1.7}$$

for all $\Delta A \in \mathcal{U}$ and all nonnegative-definite matrices Q. The ordering in (1.7) is defined with respect to the cone of nonnegative-definite matrices. Our results are based on the following robust stability and performance result (for convenience, assume that V is positive definite). If there exists a positive-definite solution Q to (1.6), where $\Omega(\cdot)$ satisfies (1.7), then $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$ and, furthermore,

$$J_S(\mathcal{U}) \leqq \operatorname{tr} QR. \tag{1.8}$$

The robust stability result is a direct consequence of Lyapunov theory, while the performance bound (1.8) follows from the fact that since $A + \Delta A$ is asymptotically stable, $Q_{\Delta A} \triangleq \lim_{t\to\infty} Q_{\Delta A}(t)$ exists, is independent of $Q_{\Delta A}(0)$, and satisfies

$$0 = (A + \Delta A)Q_{\Delta A} + Q_{\Delta A}(A + \Delta A)^T + V. \tag{1.9}$$

Now subtracting (1.9) from (1.6) yields

$$0 = (A + \Delta A)(Q - Q_{\Delta A}) + (Q - Q_{\Delta A})(A + \Delta A)^T + \Omega(Q) - (\Delta AQ + Q\Delta A^T) + V,$$

which, by (1.7) and the fact that $A + \Delta A$ is stable, implies

$$Q_{\Delta A} \leqq Q. \tag{1.10}$$

Now (1.5) and (1.10) yield the bound (1.8).

Since the ordering induced by the cone of nonnegative-definite matrices is only a partial ordering, it should not be expected that there exists an operator $\Omega(\cdot)$ satisfying (1.7), which is a least upper bound. Indeed, there are many alternative definitions for the bound $\Omega(\cdot)$. To illustrate some of these alternatives, assume for convenience that ΔA is of the form

$$\Delta A = \sigma_1 A_1, \qquad |\sigma_1| \leqq \delta_1, \tag{1.11}$$

where σ_1 is an uncertain real scalar parameter assumed only to satisfy the stated bounds, and A_1 is a known matrix denoting the structure of the parametric uncertainty. The bound $\Omega(\cdot)$ utilized in [9] and [12] for full-state-feedback design was chosen to be

$$\Omega(Q) = \delta_1 | A_1 Q + Q A_1^T |, \tag{1.12}$$

where $|\cdot|$ denotes the nonnegative-definite matrix obtained by replacing each eigenvalue by its absolute value. More recently, the quadratic (in Q) bound

$$\Omega(Q) = \delta_1 [A_L A_L^T + Q A_R^T A_R Q] \tag{1.13}$$

has been considered, where A_L, A_R are a factorization of A_1 of the form $A_1 = A_L A_R$. Bound (1.13) was studied in [29] for robustness analysis and in [17], [25], [28], [30], [33], and [36] for robust controller synthesis. A third bound that has also been considered is the linear (in Q) bound

$$\Omega(Q) = \delta_1 [\alpha Q + \alpha^{-1} A_1 Q A_1^T], \tag{1.14}$$

where α is an arbitrary positive scalar. As shown in [33], bound (1.14) arises from a multiplicative white noise model with exponential disturbance weighting. Control-design applications of bound (1.14) are given in [23], [27], [33]–[35]. The principal contribution of the present paper is thus a unified development of bounds (1.12)–(1.14) for both robust stability and performance analysis. In addition, we present a systematic approach that pays careful attention to the structure of the uncertainty set $\mathcal{U}$. For example, we show that bound (1.12) guarantees stability over a rectangular uncertainty set while (1.14) is most naturally associated with an ellipsoidal region. Furthermore, to provide a methodical development, we identify three classes of bounds (Types I, II, and III) that operate by exploiting, respectively, the symmetry of $\Delta AQ + Q\Delta A^T$, the structure of Q, and the structure of ΔA. This approach clarifies the relationships among different bounds and suggests several new bounds. The principal goal in this regard is to demonstrate the richness of quadratic Lyapunov bounds to stimulate future developments.

Finally, the present paper also considers an alternative cost functional for robust performance analysis. Specifically, in place of white noise disturbances, we reinterpret $w(t)$ in (1.1) as a deterministic L_2 signal as in H_∞ theory [6]. By imposing an L_∞ norm on the output $y(t)$ (rather than an L_2 norm as in H_∞ theory), the corresponding performance measure is given by (see [38])

$$J_D(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \limsup_{t \to \infty} \lambda_{\max}(Q_{\Delta A}(t)R), \tag{1.15}$$

in contrast to (1.5). Both performance measures $J_S(\mathcal{U})$ and $J_D(\mathcal{U})$ are considered in the paper.

The contents of the paper are as follows. After summarizing notation later in this section, the Robust Stability Problem, Stochastic Robust Performance Problem, and Deterministic Robust Performance Problem are introduced in § 2. In § 3 the basic result guaranteeing robust stability and performance (Theorem 3.1) is stated. This result is easily stated and forms the basis for all later developments. A dual version of Theorem 3.1 (Theorem 4.1) provides additional sufficient conditions and clarifies connections to traditional robust stability results. The bound $\Omega(\cdot)$ and its dual $\Lambda(\cdot)$ are given concrete forms in § 5. In § 6, the bounds of § 5 are merged with Theorem 3.1 to yield the main results guaranteeing robust stability and performance (Theorems 6.1–6.5) via modified Lyapunov equations. In § 7 we analyze the modified Lyapunov equations with regard to existence, uniqueness, and monotonicity of solutions. Additional bounds are derived in § 8 by utilizing a recursive substitution technique, while both upper and lower bounds are obtained in § 9. Finally, illustrative examples are considered in §§ 10 and 11.

Notation. Note: All matrices have real entries.

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$	real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expectation,
I_r	$r \times r$ identity matrix,
asymptotically stable matrix	matrix with eigenvalues in open left half plane,
$\mathbb{S}^r$	$r \times r$ symmetric matrices,
$\mathbb{N}^r$	$r \times r$ symmetric nonnegative-definite matrices,
$\mathbb{P}^r$	$r \times r$ symmetric positive-definite matrices,
$Z_1 \geqq Z_2$	$Z_1 - Z_2 \in \mathbb{N}^r$, $Z_1, Z_2 \in \mathbb{S}^r$,
$Z_1 > Z_2$	$Z_1 - Z_2 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$,
$\operatorname{tr} Z$, Z^T	trace of Z, transpose of Z,
$\lambda_i(Z)$	eigenvalue of matrix Z,
$\lambda_{\max}(Z)$	maximum eigenvalue of matrix Z having real spectrum,
$\lVert\cdot\rVert_2$	Euclidean vector norm,
$\lVert\cdot\rVert_s$	spectral matrix norm (largest singular value),
$\lVert\cdot\rVert_F$	Frobenius matrix norm.

2. Robust stability and performance problems. Let $\mathcal{U} \subset \mathbb{R}^{n\times n}$ denote a set of perturbations ΔA of a given nominal dynamics matrix $A \in \mathbb{R}^{n\times n}$. Throughout the paper it is assumed that A is asymptotically stable and that $0 \in \mathcal{U}$. We begin by considering the question of whether or not $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$.

ROBUST STABILITY PROBLEM. Determine whether the linear system

$$\dot{x}(t) = (A+\Delta A)x(t), \qquad t\in[0,\infty), \tag{2.1}$$

is asymptotically stable for all $\Delta A \in \mathcal{U}$.

To consider the problem of robust performance it is necessary to introduce external disturbances. In this paper we consider both stochastic and deterministic disturbance models. The stochastic disturbance model involves white noise signals as in standard LQG theory, whereas the deterministic disturbance model involves L_2 signals as in H_∞ theory [6]. By defining an appropriate performance measure for each disturbance class it turns out that we can provide a simultaneous treatment of both cases.

We first consider the case of stochastic disturbances. In this case the robust performance problem concerns the worst-case magnitude of the expected value of a quadratic form involving outputs $y(t) = Ex(t)$, where $E \in \mathbb{R}^{q\times n}$, when the system is subjected to a standard white noise disturbance $w(t) \in \mathbb{R}^d$ with weighting $D \in \mathbb{R}^{n\times d}$.

STOCHASTIC ROBUST PERFORMANCE PROBLEM. For the disturbed linear system

$$\dot{x}(t) = (A+\Delta A)x(t) + Dw(t), \qquad t\in[0,\infty), \quad x(0)=0, \tag{2.2}$$

$$y(t) = Ex(t), \tag{2.3}$$

where $w(\cdot)$ is a zero-mean d-dimensional white noise signal with intensity I_d, determine a performance bound β_S satisfying

$$J_S(\mathcal{U}) \triangleq \sup_{\Delta A\in\mathcal{U}} \limsup_{t\to\infty} \mathbb{E}\{\|y(t)\|_2^2\} \leqq \beta_S. \tag{2.4}$$

The system (2.2), (2.3) may denote, for example, a control system in closed-loop configuration subjected to external white noise disturbances for which $y(t)$ may be the state regulation error. Such specializations are not required for this development, however.

Of course, since D and E may be rank deficient, there may be cases in which a finite performance bound β_S satisfying (2.4) exists while (2.1) is not asymptotically stable over $\mathcal{U}$. In practice, however, robust performance is mainly of interest when (2.1) is robustly stable. In this case the performance $J_S(\mathcal{U})$ is given in terms of the steady-state second moment of the state. The following result from linear system theory will be useful. For convenience define the $n \times n$ nonnegative-definite matrices

$$R \triangleq E^TE, \qquad V \triangleq DD^T.$$

LEMMA 2.1. *Suppose $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then*

$$J_S(\mathcal{U}) = \sup_{\Delta A\in\mathcal{U}} \operatorname{tr} Q_{\Delta A}R, \tag{2.5}$$

where the $n\times n$ matrix $Q_{\Delta A} \triangleq \lim_{t\to\infty} \mathbb{E}[x(t)x^T(t)]$ is given by

$$Q_{\Delta A} = \int_0^\infty e^{(A+\Delta A)t}Ve^{(A+\Delta A)^Tt}\,dt, \tag{2.6}$$

which is the unique, nonnegative-definite solution to

$$0 = (A+\Delta A)Q_{\Delta A} + Q_{\Delta A}(A+\Delta A)^T + V. \tag{2.7}$$

To state the Deterministic Robust Performance Problem some additional notation is required. For a measurable function $z\colon [0,\infty) \to \mathbb{R}^r$ define

$$\|z(\cdot)\|_{2,2} \triangleq \left\{ \int_0^\infty \|z(t)\|_2^2 \, dt \right\}^{1/2}, \tag{2.8}$$

which is an L_2 function norm with a Euclidean spatial norm, and define

$$\|z(\cdot)\|_{\infty,2} \triangleq \text{ess.} \sup_{t\in[0,\infty)} \|z(t)\|_2,$$

which is an L_∞ function norm with a Euclidean spatial norm. We now reconsider (2.2) with $w(\cdot)$ interpreted as a square-integrable function. In this case the robust performance problem concerns the worst-case L_∞ norm of the output $y(t)$.

DETERMINISTIC ROBUST PERFORMANCE PROBLEM. For the disturbed linear system (2.2), (2.3), where $\|w(\cdot)\|_{2,2} \leqq 1$, determine a performance bound β_D satisfying

$$J_D(\mathscr{U}) \triangleq \sup_{\Delta A\in\mathscr{U}} \sup_{\|w(\cdot)\|_{2,2}\leqq 1} \|y(\cdot)\|_{\infty,2}^2 \leqq \beta_D. \tag{2.9}$$

The performance measure $J_D(\mathscr{U})$ in (2.9) is given by the following result.

LEMMA 2.2. *Suppose $A+\Delta A$ is asymptotically stable for all $\Delta A \in \mathscr{U}$. Then*

$$J_D(\mathscr{U}) = \sup_{\Delta A\in\mathscr{U}} \lambda_{\max}(Q_{\Delta A} R), \tag{2.10}$$

where $Q_{\Delta A}$ is the unique, nonnegative-definite solution to (2.7).

Proof. The result is an immediate consequence of Theorem 1(b) of [38]. □

Remark 2.1. Although $J_S(\mathscr{U})$ and $J_D(\mathscr{U})$ arise from different mathematical settings they are quite similar in form. Note that in general $J_D(\mathscr{U}) \leqq J_S(\mathscr{U})$, and $J_D(\mathscr{U}) = J_S(\mathscr{U})$ if rank $R = 1$.

Remark 2.2. In Lemma 2.2 $Q_{\Delta A}$ can be viewed as the controllability Gramian for the pair $(A+\Delta A, D)$ rather than the state covariance. Note that $Q_{\Delta A}$ is independent of $x(0)$ and $Q_{\Delta A}(0)$.

Remark 2.3. The stochastic performance measure $J_S(\mathscr{U})$ given by (2.5) can also be written as

$$J_S(\mathscr{U}) = \sup_{\Delta A\in\mathscr{U}} \int_0^\infty \|E e^{(A+\Delta A)t} D\|_F^2 \, dt, \tag{2.11}$$

which involves the L_2 norm of the impulse response of (2.2), (2.3). This stochastic performance measure can thus also be given a deterministic interpretation by letting $w(t)$ denote impulses at time $t=0$. For details of this formulation see [46, p. 331].

In the present paper our approach is to obtain robust stability as a consequence of sufficient conditions for robust performance. Such conditions are developed in the following sections.

3. Sufficient conditions for robust stability and performance. The key step in obtaining robust stability and performance is to bound the uncertain terms $\Delta A Q + Q \Delta A^T$ in the Lyapunov equation (2.7) by means of a function $\Omega(Q)$. The nonnegative-definite solution Q of this modified Lyapunov equation is then guaranteed to be an upper bound for $Q_{\Delta A}$. The following easily proved result is fundamental and forms the basis for all later developments. The result is based on Lyapunov function theory as applied to linear systems. For our purposes, a suitable statement of this result is given by Lemma 12.2 of [39]. Essentially this result states that if the matrix equation $0 = \Phi F + F\Phi^T + SS^T$ has a solution $F \geqq 0$ and (Φ, S) is stabilizable, then Φ is an asymptotically stable matrix. Of course, (Φ, S) is stabilizable (regardless of Φ) if S has full row rank, and we note (see [39, Thm. 3.6]) that if (Φ, S) is stabilizable then so is $(\Phi, [SS^T + H]^{1/2})$ for all nonnegative-definite matrices H.

THEOREM 3.1. *Let* $\Omega : \mathbb{N}^n \rightarrow \mathbb{N}^n$ *be such that*

$$\Delta A Q + Q \Delta A^T \leqq \Omega(Q), \qquad \Delta A \in \mathcal{U}, \quad Q \in \mathbb{N}^n, \tag{3.1}$$

and suppose there exists $Q \in \mathbb{N}^n$ *satisfying*

$$0 = AQ + QA^T + \Omega(Q) + V. \tag{3.2}$$

Then

$$(A + \Delta A, D) \text{ is stabilizable}, \qquad \Delta A \in \mathcal{U}, \tag{3.3}$$

if and only if

$$A + \Delta A \text{ is asymptotically stable}, \qquad \Delta A \in \mathcal{U}. \tag{3.4}$$

In this case,

$$Q_{\Delta A} \leqq Q, \qquad \Delta A \in \mathcal{U}, \tag{3.5}$$

where $Q_{\Delta A} \in \mathbb{N}^n$ *is given by* (2.7), *and*

$$J_S(\mathcal{U}) \leqq \operatorname{tr} QR, \tag{3.6}$$

$$J_D(\mathcal{U}) \leqq \lambda_{\max}(QR). \tag{3.7}$$

In addition, if there exists $\Delta A \in \mathcal{U}$ *such that* $(A + \Delta A, D)$ *is controllable, then* Q *is positive definite.*

Proof. We stress that in (3.1), Q denotes an arbitrary element of $\mathbb{N}^n$, whereas in (3.2) Q denotes a specific solution of the modified Lyapunov equation. This minor abuse of notation considerably simplifies the presentation. Now note that for all $\Delta A \in \mathbb{R}^{n \times n}$, (3.2) is equivalent to

$$0 = (A + \Delta A)Q + Q(A + \Delta A)^T + \Omega(Q) - (\Delta A Q + Q \Delta A^T) + V. \tag{3.8}$$

Hence, by assumption, (3.8) has a solution $Q \in \mathbb{N}^n$ for all $\Delta A \in \mathbb{R}^{n \times n}$. If ΔA is restricted to the set $\mathcal{U}$ then, by (3.1), $\Omega(Q) - (\Delta A Q + Q \Delta A^T)$ is nonnegative definite. Thus if the stabilizability condition (3.3) holds for all $\Delta A \in \mathcal{U}$, then it follows from Theorem 3.6 of [39] that $(A + \Delta A, [V + \Omega(Q) - (\Delta A Q + Q \Delta A^T)]^{1/2})$ is stabilizable for all $\Delta A \in \mathcal{U}$. It now follows from (3.8) and Lemma 12.2 of [39] that $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Conversely, if $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$, then (3.3) is immediate. Next, subtracting (2.7) from (3.8) yields

$$0 = (A + \Delta A)(Q - Q_{\Delta A}) + (Q - Q_{\Delta A})(A + \Delta A)^T + \Omega(Q) - (\Delta A Q + Q \Delta A^T), \qquad \Delta A \in \mathcal{U},$$

or, equivalently, since $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$

$$Q - Q_{\Delta A} = \int_0^\infty e^{(A + \Delta A)t}[\Omega(Q) - (\Delta A Q + Q \Delta A^T)]e^{(A + \Delta A)^T t}\, dt \geqq 0, \qquad \Delta A \in \mathcal{U}, \tag{3.9}$$

which implies (3.5). The performance bound (3.6) is now an immediate consequence of (2.5) and (3.5). To prove (3.7) we note that if $0 \leqq M_1 \leqq M_2$ then $\lambda_{\max}(M_1) \leqq \lambda_{\max}(M_2)$ (see, e.g., Corollary 7.7.4 of [40]). Thus

$$\begin{aligned} J_D(\mathcal{U}) &= \sup_{\Delta A \in \mathcal{U}} \lambda_{\max}(Q_{\Delta A} R) = \sup_{\Delta A \in \mathcal{U}} \lambda_{\max}(E Q_{\Delta A} E^T) \\ &\leqq \lambda_{\max}(E Q E^T) = \lambda_{\max}(QR). \end{aligned} \tag{3.10}$$

Finally, it follows from (3.8) that if $(A+\Delta A, D)$ is controllable for some $\Delta A \in \mathscr{U}$, then the controllability Gramian Q for the pair

$$(A+\Delta A,[V+\Omega(Q)-(\Delta A Q+Q\Delta A^T)]^{1/2})$$

is positive definite. □

For convenience we shall say that $\Omega(\cdot)$ *bounds* $\mathscr{U}$ if (3.1) is satisfied. To apply Theorem 3.1, we first specify a function $\Omega(\cdot)$ and an uncertainty set $\mathscr{U}$ such that $\Omega(\cdot)$ bounds $\mathscr{U}$. If the existence of a nonnegative-definite solution Q to (3.2) can be determined analytically or numerically and (3.3) is satisfied, then robust stability is guaranteed and the performance bounds (3.6), (3.7) can be computed. We can then enlarge $\mathscr{U}$, modify $\Omega(\cdot)$, and again attempt to solve (3.2). If, however, a nonnegative-definite solution to (3.2) cannot be determined, then $\mathscr{U}$ must be decreased in size until (3.2) is solvable. For example, $\Omega(\cdot)$ can be replaced by $\varepsilon\Omega(\cdot)$ to bound $\varepsilon\mathscr{U}$, where $\varepsilon>1$ enlarges $\mathscr{U}$ and $\varepsilon<1$ shrinks $\mathscr{U}$. Of course, the actual range of uncertainty that can be bounded depends on the nominal matrix A, the function $\Omega(\cdot)$, and the structure of $\mathscr{U}$. In § 5 the uncertainty set $\mathscr{U}$ and bound $\Omega(\cdot)$ satisfying (3.1) are given concrete forms. We complete this section with several observations.

Remark 3.1. If only robust stability is of interest, then the noise intensity V need not have physical significance. In this case we may set $D=I_n$ to satisfy (3.3).

Remark 3.2. Since A is asymptotically stable, Q satisfying (3.2) is given by

$$Q=\int_0^\infty e^{At}[\Omega(Q)+V]e^{A^Tt}\,dt, \tag{3.11}$$

or, equivalently,

$$Q=\int_0^\infty e^{At}\Omega(Q)e^{A^Tt}\,dt+Q_0, \tag{3.12}$$

where $Q_0\in\mathbb{N}^n$ is defined by

$$Q_0\triangleq\int_0^\infty e^{At}Ve^{A^Tt}\,dt \tag{3.13}$$

and satisfies

$$0=AQ_0+Q_0A^T+V. \tag{3.14}$$

Note that $Q_0\leqq Q$ and that the nominal performances $J_S(\{0\})$ and $J_D(\{0\})$ are given by $\operatorname{tr} Q_0R$ and $\lambda_{\max}(Q_0R)$, respectively.

Remark 3.3. Using (3.11) it is also useful to note that the bound for $J_S(\mathscr{U})$ given by (3.6) can be written as

$$\operatorname{tr} QR=\operatorname{tr}\int_0^\infty e^{At}[\Omega(Q)+V]e^{A^Tt}\,dtR=\operatorname{tr} P_0[\Omega(Q)+V], \tag{3.15}$$

where $P_0\in\mathbb{N}^n$ is defined by

$$P_0\triangleq\int_0^\infty e^{A^Tt}Re^{At}\,dt \tag{3.16}$$

and satisfies

$$0=A^TP_0+P_0A+R. \tag{3.17}$$

The bound tr $P_0[\Omega(Q) + V]$ can be viewed as a dual formulation of the bound tr QR since the roles of A and A^T are reversed. Dual bounds are developed in the following section. Note that tr Q_0R = tr P_0V.

Remark 3.4. If $\Omega(\cdot)$ bounds $\mathcal{U}$ then clearly $\Omega(\cdot)$ bounds the convex hull of $\mathcal{U}$. Hence, only convex uncertainty sets $\mathcal{U}$ need be considered. Next, we shall later use the obvious fact that if $\Omega'(\cdot)$ bounds $\mathcal{U}'$ and $\Omega''(\cdot)$ bounds $\mathcal{U}''$, then $\Omega'(\cdot) + \Omega''(\cdot)$ bounds $\mathcal{U}' + \mathcal{U}''$. Hence if $\mathcal{U}$ can be decomposed additively then it suffices to bound each component separately. Finally, if $\Omega(\cdot)$ bounds $\mathcal{U}$ and there exists $\Omega' : \mathbb{N}^n \rightarrow \mathbb{N}^n$ such that $\Omega(Q) \leqq \Omega'(Q)$ for all $Q \in \mathbb{N}^n$, then $\Omega'(\cdot)$ also bounds $\mathcal{U}$. That is, any *overbound* $\Omega'(\cdot)$ for $\Omega(\cdot)$ also bounds $\mathcal{U}$. Of course, as we shall see, it is quite possible that an overbound $\Omega'(\cdot)$ for $\Omega(\cdot)$ may actually bound a set $\mathcal{U}'$ that is larger than the "original" uncertainty set $\mathcal{U}$.

4. Dual sufficient conditions for robust stability and performance. As noted in Remark 3.3, the performance bound tr QR given by (3.6) can be expressed equivalently in terms of a dual variable P_0 for which the roles of A and A^T are reversed. Using a similar technique, additional conditions for robust stability and performance can be obtained by developing a dual version of Theorem 3.1. A prime motivation for developing such dual bounds is to draw connections with previous results in the literature relating to robust stability. Specifically, we shall show that traditional robust stability techniques based on the quadratic Lyapunov function $V(x) = x^TPx$ correspond to dual conditions. Robust performance bounds within the dual formulation, however, are difficult to motivate without first developing the primal performance bounds as was done in the previous section. In addition, the dual bounds may, for certain problems, yield larger stability regions and sharper performance bounds than the primal bounds.

LEMMA 4.1. *Suppose $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then*

$$J_S(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \operatorname{tr} P_{\Delta A} V, \tag{4.1}$$

where $P_{\Delta A} \in \mathbb{R}^{n \times n}$ is the unique, nonnegative-definite solution to

$$0 = (A + \Delta A)^T P_{\Delta A} + P_{\Delta A}(A + \Delta A) + R. \tag{4.2}$$

Proof. It need only be noted that

$$\operatorname{tr} Q_{\Delta A} R = \operatorname{tr} \int_0^\infty e^{(A + \Delta A)t} V e^{(A + \Delta A)^T t}\, dt R = \operatorname{tr} P_{\Delta A} V,$$

where

$$P_{\Delta A} \triangleq \int_0^\infty e^{(A + \Delta A)^T t} R e^{(A + \Delta A)t}\, dt$$

satisfies (4.2). □

The proof of Lemma 4.1 relies on the fact that tr $Q_{\Delta A}R$ = tr $P_{\Delta A}V$. However, it is not necessarily true that $\lambda_{\max}(Q_{\Delta A}R) = \lambda_{\max}(P_{\Delta A}V)$ even when $\Delta A = 0$. For example, if

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad R = I_2, \quad V = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

then

$$Q_0 R = \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \quad \text{and} \quad P_0 V = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

and thus $\lambda_{\max}(Q_0 R) = (15+\sqrt{145})/24$ and $\lambda_{\max}(P_0 V) = (5+\sqrt{17})/8$. Thus to obtain a suitable dual version of $J_D(\mathcal{U})$ we need to define a dual deterministic cost $\hat{J}_D(\mathcal{U})$, which is distinct from $J_D(\mathcal{U})$. This can be done if the disturbance signals are taken to be integrable rather than square integrable. Thus, for measurable $z:[0,\infty)\to\mathbb{R}^r$ define

$$\|z(\cdot)\|_{1,2} \triangleq \int_0^\infty \|z(t)\|_2\, dt, \tag{4.3}$$

which is an L_1 function norm with a Euclidean spatial norm. The dual deterministic cost $\hat{J}_D(\mathcal{U})$ is thus defined by

$$\hat{J}_D(\mathcal{U}) \triangleq \sup_{\Delta A\in\mathcal{U}} \sup_{\|w(\cdot)\|_{1,2}\leqq 1} \|y(\cdot)\|_{2,2}^2. \tag{4.4}$$

The following dual result follows from Theorem 1(a) of [38].

LEMMA 4.2. *Suppose $A+\Delta A$ is asymptotically stable for all $\Delta A\in\mathcal{U}$. Then*

$$\hat{J}_D(\mathcal{U}) = \lambda_{\max}(P_{\Delta A} V), \tag{4.5}$$

where $P_{\Delta A}\in\mathbb{R}^{n\times n}$ is the unique, nonnegative-definite solution to (4.2).

The dual version of Theorem 3.1 can now be stated.

THEOREM 4.1. *Let $\Lambda:\mathbb{N}^n\to\mathbb{N}^n$ be such that*

$$\Delta A^T P + P\Delta A \leqq \Lambda(P), \qquad \Delta A\in\mathcal{U}, \quad P\in\mathbb{N}^n, \tag{4.6}$$

and suppose there exists $P\in\mathbb{N}^n$ satisfying

$$0 = A^T P + PA + \Lambda(P) + R. \tag{4.7}$$

Then

$$(E, A+\Delta A) \text{ is detectable}, \qquad \Delta A\in\mathcal{U}, \tag{4.8}$$

if and only if

$$A+\Delta A \text{ is asymptotically stable}, \qquad \Delta A\in\mathcal{U}. \tag{4.9}$$

In this case,

$$P_{\Delta A}\leqq P, \qquad \Delta A\in\mathcal{U}, \tag{4.10}$$

where $P_{\Delta A}$ is given by (4.2), *and*

$$J_S(\mathcal{U}) \leqq \operatorname{tr} PV, \tag{4.11}$$

$$\hat{J}_D(\mathcal{U}) \leqq \lambda_{\max}(PV). \tag{4.12}$$

In addition, if there exists $\Delta A\in\mathcal{U}$ such that $(E, A+\Delta A)$ is observable, then P is positive definite.

Proof. The proof is completely analogous to the proof of Theorem 3.1. □

Remark 4.1. Note that $\hat{J}_D(\mathcal{U}) \leqq J_S(\mathcal{U})$ and that $\hat{J}_D(\mathcal{U}) = J_S(\mathcal{U})$ if rank $V=1$. Combining this fact with Remark 2.1, it follows that $J_D(\mathcal{U}) = \hat{J}_D(\mathcal{U})$ if both rank $R=1$ and rank $V=1$. In general, however, we should not expect that $J_D(\mathcal{U}) = \hat{J}_D(\mathcal{U})$.

It is quite possible that the *bounds* tr QR and tr PV for $J_S(\mathcal{U})$ given by (3.6) and (4.11) may be different in spite of the fact, as shown in the proof of Lemma 4.1, that tr $Q_{\Delta A}R$ = tr $P_{\Delta A}V$. That is, depending on $\Omega(\cdot)$ and $\Lambda(\cdot)$ either bound (3.6) or bound (4.11) may be better for a particular problem. In general, we have the following result.

PROPOSITION 4.1. *Let* $\Omega(\cdot)$, $\Lambda(\cdot)$, Q, *and* P *be as in Theorems* 3.1 *and* 4.1, *and let* Q_0 *and* P_0 *be given by* (3.13) *and* (3.16), *respectively. Then*

$$\text{tr } Q_0\Lambda(P) < \text{tr } P_0\Omega(Q) \Leftrightarrow \text{tr } QR > \text{tr } PV, \tag{4.13}$$

$$\text{tr } Q_0\Lambda(P) = \text{tr } P_0\Omega(Q) \Leftrightarrow \text{tr } QR = \text{tr } PV, \tag{4.14}$$

$$\text{tr } Q_0\Lambda(P) > \text{tr } P_0\Omega(Q) \Leftrightarrow \text{tr } QR < \text{tr } PV. \tag{4.15}$$

Proof. Note that

$$\text{tr } QR = \int_0^\infty e^{At}[\Omega(Q)+V]e^{A^Tt}\,dt\,R = \text{tr } P_0\Omega(Q) + \text{tr}\int_0^\infty e^{At}Ve^{A^Tt}\,dt\,R$$

and

$$\text{tr } PV = \text{tr}\int_0^\infty e^{A^Tt}[\Lambda(P)+R]e^{At}\,dt\,V = \text{tr } Q_0\Lambda(P) + \text{tr}\int_0^\infty e^{A^Tt}Re^{At}\,dt\,V$$

so that

$$\text{tr } QR - \text{tr } PV = \text{tr } P_0\Omega(Q) - \text{tr } Q_0\Lambda(P),$$

which yields (4.13)–(4.15). □

Remark 4.2. To draw connections with traditional Lyapunov theory, let R and V be positive definite and assume that there exists a positive-definite solution to (4.7). Then $V(x) \triangleq x^TPx$ satisfies $\dot{V}(x(t)) < 0$ for $x(\cdot)$ satisfying (2.1) and for all $\Delta A \in \mathcal{U}$. Thus $V(\cdot)$ is a Lyapunov function for (2.1) that guarantees robust asymptotic stability over $\mathcal{U}$.

5. Construction of the bounds $\Omega(\cdot)$ and $\Lambda(\cdot)$. As discussed in § 1, we consider three distinct classes of bounds $\Omega(\cdot)$ denoted by Type I, Type II, and Type III. Roughly speaking, these bounds exploit, respectively, the symmetry of the Lyapunov terms $\Delta AQ + Q\Delta A^T$, the structure of Q, and the structure of ΔA. The dual bounds $\Lambda(\cdot)$ can be constructed similarly by replacing Q and ΔA by P and ΔA^T. Hence these bounds will not be discussed separately. For convenience in discussing the set $\mathcal{U}$, we shall use the terms *rectangle* and *ellipse* to refer to closed regions bounded by such figures in multiple dimensions. As usual, a polytope is the convex hull of a finite number of points.

5.1. Type I bounds. We begin by constructing bounds $\Omega(\cdot)$ that exploit only the symmetry of the Lyapunov terms $\Delta AQ + Q\Delta A^T$. First we require the following well-known definition of a function of a symmetric matrix as an extension of a real-valued function (see, e.g., [40, p. 300]). Specifically, if $f: \mathbb{R} \to \mathbb{R}$, then (with a minor abuse of notation) $f: \mathbb{S}^n \to \mathbb{S}^n$ can be defined by setting

$$f(S) \triangleq Uf(D)U^T,$$

where $S = UDU^T$, U is orthogonal, D is real diagonal, and $f(D)$ is the diagonal matrix obtained by applying f to each diagonal element of D. Note that if f is the polynomial $f(x) = \sum_{i=0}^r a_ix^i$ then $f(S) = \sum_{i=0}^r a_iS^i$. Note also that if $f(x) = |x|$ then $f(S) = (S^2)^{1/2}$, where $(\cdot)^{1/2}$ denotes the (unique) nonnegative-definite square root. As in [41, p. 262], we use the notation $|S|$ to denote $(S^2)^{1/2}$. Finally, note that if $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are such that $f(x) \leqq g(x)$, $x \in \mathbb{R}$, then $f(S) \leqq g(S)$, $S \in \mathbb{S}^n$.

As a concretization of the uncertainty set $\mathcal{U}$, consider the set

$$\mathcal{U}_1 \triangleq \left\{\Delta A \in \mathbb{R}^{n\times n}: \Delta A = \sum_{i=1}^p \sigma_iA_i,\ |\sigma_i| \leqq \delta_i,\ i = 1, \cdots, p\right\}, \tag{5.1}$$

where, for $i = 1, \cdots, p$: $A_i \in \mathbb{R}^{n\times n}$ is a given matrix denoting the structure of the parametric uncertainty, σ_i is a real uncertain parameter, and δ_i denotes the range of parameter uncertainty. Clearly, the multidimensional set of uncertain parameters $(\sigma_1, \cdots, \sigma_p)$ is the rectangle $[-\delta_1, \delta_1] \times \cdots \times [-\delta_p, \delta_p]$ and $\mathcal{U}_1$ is a symmetric polytope of matrices in $\mathbb{R}^{n\times n}$. Note that the symmetry of the uncertainty interval $[-\delta_i, \delta_i]$ entails no loss of generality since the nominal value of A can be redefined if necessary. Furthermore, it is also possible, without loss of generality, to define $\delta_i = 1$ by replacing A_i by $\delta_i A_i$. For clarity, however, we choose not to employ this scaling. We begin by considering the bound utilized by Chang and Peng in [9].

PROPOSITION 5.1. *The function*

$$\Omega_1(Q) \triangleq \sum_{i=1}^{p} \delta_i |A_i Q + Q A_i^T| \tag{5.2}$$

bounds $\mathcal{U}_1$.

Proof. For $i = 1, \cdots, p$ and $|\sigma_i| \leqq \delta_i$,

$$\sigma_i(A_i Q + Q A_i^T) \leqq |\sigma_i(A_i Q + Q A_i^T)| = |\sigma_i|\,|A_i Q + Q A_i^T| \leqq \delta_i |A_i Q + Q A_i^T|.$$

Summing over i yields

$$\Delta A Q + Q \Delta A^T = \sum_{i=1}^{p} \sigma_i(A_i Q + Q A_i^T) \leqq \sum_{i=1}^{p} \delta_i |A_i Q + Q A_i^T|,$$

which implies (3.1) with $\Omega(\cdot) = \Omega_1(\cdot)$ and $\mathcal{U} = \mathcal{U}_1$. □

Remark 5.1. It is tempting to prove Proposition 5.1 by writing

$$\sum_{i=1}^{p} \sigma_i(A_i Q + Q A_i^T) \leqq \left| \sum_{i=1}^{p} \sigma_i(A_i Q + Q A_i^T) \right| \leqq \sum_{i=1}^{p} |\sigma_i(A_i Q + Q A_i^T)|.$$

However, counterexamples show that the inequality $|M_1 + M_2| \leqq |M_1| + |M_2|$ is not generally true for arbitrary symmetric matrices M_1, M_2.

Remark 5.2. Because of its simplicity it is tempting to conjecture that $\Omega_1(\cdot)$ is the best bound for $\Delta A Q + Q \Delta A^T$ over the set $\mathcal{U}_1$. To show that this is not the case, let $Q = \frac{1}{2} I_2$, $p = 1$, $A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, and $\delta_1 = 1$. Then $\sigma_1(A_1 Q + Q A_1^T) \leqq \delta_1 |A_1 Q + Q A_1^T| = I_2$, $|\sigma_1| \leqq 1$. However, it is also true that

$$\sigma_1(A_1 Q + Q A_1^T) \leqq \begin{bmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{bmatrix}, \qquad |\sigma_1| \leqq 1.$$

Neither bound, however, is an overbound for the other. This is a consequence of the fact that the nonnegative-definite matrix ordering is only a partial order.

As mentioned earlier, an overbound for $\Omega_1(\cdot)$ will also bound $\mathcal{U}_1$. The following result is immediate.

LEMMA 5.1. *For* $i = 1, \cdots, p$, *let* $f_i : \mathbb{R} \to \mathbb{R}$ *satisfy*

$$f_i(x) \geqq |x|, \qquad x \in \mathbb{R}. \tag{5.3}$$

Then the function

$$\Omega_2(Q) \triangleq \sum_{i=1}^{p} \delta_i f_i(A_i Q + Q A_i^T) \tag{5.4}$$

is an overbound for $\Omega_1(\cdot)$ *and hence also bounds* $\mathcal{U}_1$.

One particular choice of f_i satisfying (5.3) will be considered here, namely, the polynomial

$$f_i(x) = \tfrac{1}{4}\beta_i + \beta_i^{-1}x^2, \tag{5.5}$$

where β_i is an arbitrary positive constant. Thus $\Omega_2(\cdot)$ has the following specialization.

COROLLARY 5.1. *Let $\beta_1, \cdots, \beta_p$ be arbitrary positive constants. Then the function*

$$\Omega_3(Q) \triangleq \frac{1}{4}\sum_{i=1}^{p} \delta_i\beta_i I_n + \sum_{i=1}^{p}\left(\frac{\delta_i}{\beta_i}\right)(A_iQ + QA_i^T)^2 \tag{5.6}$$

is an overbound for $\Omega_1(\cdot)$ and hence also bounds $\mathscr{U}_1$.

Although overbounding $\Omega_1(\cdot)$ by $\Omega_3(\cdot)$ results in a looser bound for $\mathscr{U}_1$, it turns out that $\Omega_3(\cdot)$ actually bounds a set that is larger than $\mathscr{U}_1$. Specifically, in place of $\mathscr{U}_1$ consider

$$\mathscr{U}_2 \triangleq \left\{\Delta A \in \mathbb{R}^{n\times n}: \Delta A = \sum_{i=1}^{p} \sigma_i A_i,\ \sum_{i=1}^{p}\frac{\sigma_i^2}{\alpha_i^2} \leqq 1\right\}, \tag{5.7}$$

where $\alpha_1, \cdots, \alpha_p$ are given positive constants. Note that (5.7) replaces the rectangle of uncertain parameters $(\sigma_1, \cdots, \sigma_p)$ by an ellipse. Thus the set $\mathscr{U}_2$ of matrix perturbations is an ellipse of matrices in $\mathbb{R}^{n\times n}$ in contrast to the polytope $\mathscr{U}_1$. Of course, $\mathscr{U}_1 = \mathscr{U}_2$ if $p = 1$ and $\alpha_1 = \delta_1$. Again it is possible to take $\alpha_i = 1$ without loss of generality by replacing A_i by $\alpha_i A_i$. We again choose not to do this, however. The following result provides a convenient characterization of the relationship between the rectangle $\mathscr{U}_1$ and the ellipse $\mathscr{U}_2$.

PROPOSITION 5.2. *Suppose $\mathscr{U}_1$ is defined by the positive constants $\delta_1, \cdots, \delta_p$, and let $\mathscr{U}_2$ be characterized by*

$$\alpha_i = \left(\frac{\alpha\delta_i}{\beta_i}\right)^{1/2}, \qquad i = 1, \cdots, p, \tag{5.8}$$

where α is defined by

$$\alpha = \sum_{i=1}^{p} \delta_i\beta_i \tag{5.9}$$

and $\beta_1, \cdots, \beta_p$ are arbitrary positive constants. Then the ellipse

$$\left\{(\sigma_1, \cdots, \sigma_p): \sum_{i=1}^{p}\frac{\sigma_i^2}{\alpha_i^2} \leqq 1\right\}$$

circumscribes the rectangle $\{(\sigma_1, \cdots, \sigma_p): |\sigma_i| \leqq \delta_i, i = 1, \cdots, p\}$ and thus $\mathscr{U}_2$ contains $\mathscr{U}_1$. Furthermore, $\Omega_3(\cdot)$ actually bounds $\mathscr{U}_2$.

Proof. If $|\sigma_i| \leqq \delta_i$, $i = 1, \cdots, p$, then it follows from (5.8) and (5.9) that

$$\sum_{i=1}^{p}\frac{\sigma_i^2}{\alpha_i^2} = \alpha^{-1}\sum_{i=1}^{p}\frac{\beta_i\sigma_i^2}{\delta_i} \leqq \alpha^{-1}\sum_{i=1}^{p}\beta_i\delta_i = 1.$$

Thus the ellipse contains the rectangle. If, in addition, $(\sigma_1, \cdots, \sigma_p)$ is a vertex of the rectangle, i.e., $|\sigma_i| = \delta_i$, $i = 1, \cdots, p$, then $\sum_{i=1}^p \sigma_i^2/\alpha_i^2 = 1$, which corresponds to a point on the boundary of the ellipse. To show that $\Omega_3(\cdot)$ actually bounds $\mathcal{U}_2$ note that

$$0 \leqq \sum_{i=1}^p \left[\frac{1}{2}\left(\frac{\alpha^{1/2}\sigma_i}{\alpha_i}\right)I_n - \left(\frac{\alpha_i}{\alpha^{1/2}}\right)(A_iQ+QA_i^T)\right]^2$$

$$= \frac{\alpha}{4}\sum_{i=1}^p \left(\frac{\sigma_i^2}{\alpha_i^2}\right)I_n + \alpha^{-1}\sum_{i=1}^p \alpha_i^2(A_iQ+QA_i^T)^2 - (\Delta AQ + Q\Delta A^T).$$

Since $\sum_{i=1}^p \sigma_i^2/\alpha_i^2 \leqq 1$ in $\mathcal{U}_2$, it follows that

$$\Delta AQ + Q\Delta A^T \leqq \frac{\alpha}{4}I_n + \alpha^{-1}\sum_{i=1}^p \alpha_i^2(A_iQ+QA_i^T)^2.$$

Utilizing (5.8) and (5.9) to substitute for α and α_i yields (3.1) with $\Omega(\cdot) = \Omega_3(\cdot)$ and $\mathcal{U} = \mathcal{U}_2$. □

Proposition 5.2 shows that each choice of constants $\beta_1, \cdots, \beta_p > 0$ leads to a particular ellipse $\mathcal{U}_2$ that contains the polytope $\mathcal{U}_1$. Furthermore, $\Omega_3(\cdot)$, which by Corollary 5.1 bounds $\mathcal{U}_1$, actually bounds the larger set $\mathcal{U}_2$. For convenience, we now dispense with the constants $\beta_1, \cdots, \beta_p$ that relate the rectangle $\mathcal{U}_1$ to the ellipse $\mathcal{U}_2$ and we characterize $\Omega_3(\cdot)$ entirely in terms of $\alpha, \alpha_1, \cdots, \alpha_p$.

COROLLARY 5.2. *Let α be an arbitrary positive constant. Then the function*

$$\Omega_4(Q) \triangleq \frac{\alpha}{4}I_n + \alpha^{-1}\sum_{i=1}^p \alpha_i^2(A_iQ+QA_i^T)^2 \tag{5.10}$$

bounds $\mathcal{U}_2$.

Remark 5.3. Within the context of Corollary 5.2, the positive constant α plays no role in defining the set $\mathcal{U}_2$, although $\Omega_4(\cdot)$ is guaranteed to bound $\mathcal{U}_2$ for all choices of α. It can be expected, however, that certain choices of α provide better bounds than other choices. This will be seen by example in § 10.

The following variation of $\Omega_4(\cdot)$ was suggested by D. C. Hyland.

PROPOSITION 5.3. *Let α be an arbitrary positive constant. Then, for $Q > 0$,*

$$\Omega_{4'}(Q) \triangleq \frac{\alpha}{2}Q + \frac{\alpha^{-1}}{2}\sum_{i=1}^p \alpha_i^2[A_i^2Q + A_iQA_i^T + QA_i^TQ^{-1}A_iQ + QA_i^{2T}] \tag{5.10$'$}$$

bounds $\mathcal{U}_2$.

Proof. Note that

$$0 \leqq \sum_{i=1}^p \left[\frac{1}{2}\left(\frac{\alpha^{1/2}\sigma_i}{\alpha_i}\right)Q^{1/2} - \left(\frac{\alpha_i}{\alpha^{1/2}}\right)(A_iQ+QA_i^T)Q^{-1/2}\right]$$

$$\times\left[\frac{1}{2}\left(\frac{\alpha^{1/2}\sigma_i}{\alpha_i}\right)Q^{1/2} - \left(\frac{\alpha_i}{\alpha^{1/2}}\right)(A_iQ+QA_i^T)Q^{-1/2}\right]^T$$

$$= \frac{\alpha}{4}\sum_{i=1}^p \left(\frac{\sigma_i^2}{\alpha_i^2}\right)Q + \alpha^{-1}\sum_{i=1}^p \alpha_i^2(A_iQ+QA_i^T)Q^{-1}(A_iQ+QA_i^T) - (\Delta AQ + Q\Delta A^T),$$

which yields the desired result. □

Remark 5.4. The bound $\Omega_{4'}(Q)$ is of interest since it involves terms that arise from a multiplicative white noise model with a Stratonovich correction. Specifically, the term

$A_iQA_i^T$ arises from an Ito model [33], whereas the terms A_i^2Q and QA_i^{2T} can be viewed as the shift $A \to A + \frac{1}{2}\sum_{i=1}^p A_i^2$ due to the Stratonovich interpretation of stochastic integration [43]. These terms have interesting ramifications in designing controllers for flexible structures [23].

5.2. Type II bounds. We now consider additional bounds for $\mathcal{U}$ that exploit the structure of Q. For these bounds the natural uncertainty set is given by $\mathcal{U}_2$.

PROPOSITION 5.4. *Let α be an arbitrary positive number and, for each $Q \in \mathbb{N}^n$, let $Q_1 \in \mathbb{R}^{n\times m}$ and $Q_2 \in \mathbb{R}^{m\times n}$ satisfy*

$$Q = Q_1Q_2. \tag{5.11}$$

Then the function

$$\Omega_5(Q) \triangleq \alpha Q_2^TQ_2 + \alpha^{-1}\sum_{i=1}^p \alpha_i^2 A_iQ_1Q_1^TA_i^T \tag{5.12}$$

bounds $\mathcal{U}_2$.

Proof. Note that

$$0 \leqq \sum_{i=1}^p \left[\left(\frac{\alpha^{1/2}\sigma_i}{\alpha_i}\right)Q_2^T - \left(\frac{\alpha_i}{\alpha^{1/2}}\right)A_iQ_1\right]\left[\left(\frac{\alpha^{1/2}\sigma_i}{\alpha_i}\right)Q_2^T - \left(\frac{\alpha_i}{\alpha^{1/2}}\right)A_iQ_1\right]^T$$

$$= \alpha\sum_{i=1}^p\left(\frac{\sigma_i^2}{\alpha_i^2}\right)Q_2^TQ_2 + \alpha^{-1}\sum_{i=1}^p \alpha_i^2A_iQ_1Q_1^TA_i^T - \sum_{i=1}^p \sigma_i(A_iQ + QA_i^T),$$

which, since $\sum_{i=1}^p \sigma_i^2/\alpha_i^2 \leqq 1$, yields (3.1) with $\Omega(\cdot) = \Omega_5(\cdot)$ and $\mathcal{U} = \mathcal{U}_2$. □

We consider three specializations of $\Omega_5(\cdot)$. Specifically, we set $m = n$ and define

$$Q_1 = Q, \qquad Q_2 = I_n, \tag{5.13}$$

$$Q_1 = Q_2 = Q^{1/2}, \tag{5.14}$$

$$Q_1 = I_n, \qquad Q_2 = Q. \tag{5.15}$$

COROLLARY 5.3. *Let α be an arbitrary positive number. Then the functions*

$$\Omega_6(Q) \triangleq \alpha I_n + \alpha^{-1}\sum_{i=1}^p \alpha_i^2A_iQ^2A_i^T, \tag{5.16}$$

$$\Omega_7(Q) \triangleq \alpha Q + \alpha^{-1}\sum_{i=1}^p \alpha_i^2A_iQA_i^T, \tag{5.17}$$

$$\Omega_8(Q) \triangleq \alpha Q^2 + \alpha^{-1}\sum_{i=1}^p \alpha_i^2A_iA_i^T \tag{5.18}$$

bound $\mathcal{U}_2$.

Remark 5.5. Note that the term $A_iQ^2A_i^T$ appearing in $\Omega_6(\cdot)$ also appears in $\Omega_4(\cdot)$. Furthermore, both $\Omega_4(\cdot)$ and $\Omega_6(\cdot)$ involve a term proportional to I_n. Despite these similarities, neither bound $\Omega_4(\cdot)$ nor $\Omega_6(\cdot)$ is an overbound for the other. Furthermore, the term $A_iQA_i^T$ appears in both $\Omega_7(\cdot)$ and $\Omega_{4'}(\cdot)$. However, neither $\Omega_7(\cdot)$ nor $\Omega_{4'}(\cdot)$ is an overbound for the other.

Remark 5.6. The bound $\Omega_7(\cdot)$ given by (5.17) has the distinction that it is linear in Q. This bound was originally studied in [27] for systems with multiplicative white noise and was shown to yield robust stability and performance in [33] and [35]. A similar bound was studied in [34].

Remark 5.7. By using (5.11) additional bounds can be developed. For example, by setting

$$Q_1 = Q^{1/4}, \qquad Q_2 = Q^{3/4}, \tag{5.19}$$

$\Omega_5(\cdot)$ becomes

$$\Omega_9(Q) = \alpha Q^{3/2} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 A_i Q^{1/2} A_i^T. \tag{5.20}$$

Remark 5.8. When $p = 1$ and α is replaced by $\alpha\alpha_1$, $\Omega_7(\cdot)$ becomes

$$\Omega_7'(Q) = \alpha_1[\alpha Q + \alpha^{-1} A_1 Q A_1^T].$$

A sum of such terms with $\alpha_i = \delta_i$ can be used to bound the smaller rectangular set $\mathcal{U}_1$. Similar remarks apply to $\Omega_6(\cdot)$, $\Omega_8(\cdot)$, and $\Omega_9(\cdot)$.

5.3. Type III bounds. We now consider bounds that exploit the structure of ΔA itself. It turns out that these bounds permit consideration of an uncertainty set $\mathcal{U}$ that is larger than $\mathcal{U}_2$. Specifically, define

$$\mathcal{U}_3 \triangleq \{\Delta A \in \mathbb{R}^{n \times n} : \Delta A = A_L A_R, A_L A_L^T \leqq M, A_R^T A_R \leqq N\}, \tag{5.21}$$

where $A_L \in \mathbb{R}^{n \times r}$ and $A_R \in \mathbb{R}^{r \times n}$ are uncertain matrices, r is an arbitrary positive integer, and $M, N \in \mathbb{N}^n$ are given uncertainty bounds. The bound $\Omega_{10}(\cdot)$ for $\mathcal{U}_3$ is given by the following result.

PROPOSITION 5.5. *Let α be an arbitrary positive constant. Then the function*

$$\Omega_{10}(Q) \triangleq \alpha^{-1} M + \alpha Q N Q \tag{5.22}$$

bounds $\mathcal{U}_3$.

Proof. Note that

$$\begin{aligned} 0 &\leqq [\alpha^{-1/2} A_L - \alpha^{1/2} Q A_R^T][\alpha^{-1/2} A_L - \alpha^{1/2} Q A_R^T]^T \\ &= \alpha^{-1} A_L A_L^T + \alpha Q A_R^T A_R Q - [A_L A_R Q + Q(A_L A_R)^T] \\ &\leqq \alpha^{-1} M + \alpha Q N Q - (\Delta A Q + Q \Delta A^T), \end{aligned}$$

which yields (3.1) with $\Omega(\cdot) = \Omega_{10}(\cdot)$ and $\mathcal{U} = \mathcal{U}_3$. □

Remark 5.9. The bound $\Omega_{10}(\cdot)$ was developed in [29] for robust analysis and independently in [25] and [28] for robust full-state feedback. Applications to fixed-order dynamic compensation are given in [36].

Remark 5.10. Without loss of generality we can set $\alpha = 1$ in (5.22) by replacing M and N by $\alpha^{-1}M$ and αN, respectively. Again for clarity we choose not to employ this scaling.

Note that $\Omega_8(\cdot)$ is of the form $\Omega_{10}(\cdot)$ with $M = \sum_{i=1}^{p} \alpha_i^2 A_i A_i^T$ and $N = I_n$. Thus $\Omega_8(\cdot)$ also bounds $\mathcal{U}_3$ for this choice of M and N. It turns out in this case that $\mathcal{U}_3$ is actually larger than $\mathcal{U}_2$. To see this consider the more general case in which M and N satisfy

$$\sum_{i=1}^{p} \alpha_i^2 A_i A_i^T \leqq M, \qquad I_n \leqq N. \tag{5.23}$$

In this case $\Omega_{10}(\cdot)$ is an overbound for $\Omega_8(\cdot)$ and thus bounds $\mathcal{U}_2$. As in the case of $\Omega_3(\cdot)$ overbounding $\Omega_1(\cdot)$, we should not be surprised to find that $\Omega_{10}(\cdot)$ with (5.23) actually bounds a set that is larger than $\mathcal{U}_2$. Indeed, we now show that $\mathcal{U}_2$ is actually a very special subset of $\mathcal{U}_3$ when M and N defining $\mathcal{U}_2$ satisfy (5.23).

PROPOSITION 5.6. *If M and N satisfy (5.23) then $\mathcal{U}_2$ is a subset of $\mathcal{U}_3$. Hence $\Omega_{10}(\cdot)$ also bounds $\mathcal{U}_2$.*

Proof. If $\Delta A \in \mathcal{U}_2$ then $\Delta A = \sum_{i=1}^p \sigma_i A_i$, where $\sum_{i=1}^p \sigma_i^2/\alpha_i^2 \leqq 1$. Alternatively, we can write $\Delta A = A_L A_R$, where $r = pn$ and

$$(5.24) \qquad A_L = [\alpha_1 A_1 \cdots \alpha_p A_p], \qquad A_R = \begin{bmatrix} (\sigma_1/\alpha_1) I_n \\ \vdots \\ (\sigma_p/\alpha_p) I_n \end{bmatrix}.$$

Note that with M and N satisfying (5.23) and A_L and A_R defined by (5.24), it follows that $A_L A_L^T \leqq M$ and $A_R^T A_R \leqq N$. Thus $\Delta A \in \mathcal{U}_3$. □

The following result provides further conditions under which $\Omega_{10}(\cdot)$ bounds $\mathcal{U}_2$.

PROPOSITION 5.7. *Suppose $A_i = D_i E_i$, $i = 1, \cdots, p$, where $D_i \in \mathbb{R}^{n \times n_i}$ and $E_i \in \mathbb{R}^{n_i \times n}$, and suppose that*

$$(5.25) \qquad \sum_{i=1}^p \alpha_i^2 D_i D_i^T \leqq M, \qquad \sum_{i=1}^p E_i^T E_i \leqq N.$$

Then $\mathcal{U}_2$ is a subset of $\mathcal{U}_3$ and thus $\Omega_{10}(\cdot)$ also bounds $\mathcal{U}_2$.

Proof. The result follows as in the proof Proposition 5.6. □

Remark 5.11. When $p = 1$, $A_1 = D_1 E_1$, $M = \alpha_1^2 D_1 D_1^T$, and $N = E_1^T E_1$, it is convenient to replace α by $\alpha\alpha_1$ so that $\Omega_{10}(\cdot)$ becomes

$$(5.26) \qquad \Omega_{10'}(Q) = \alpha_1[\alpha^{-1} D_1 D_1^T + \alpha Q E_1^T E_1 Q].$$

In certain situations it is desirable to consider subsets of $\mathcal{U}_3$ of special structure. For example, define

$$\mathcal{U}_4 \triangleq \{\Delta A \in \mathbb{R}^{n \times n} : \Delta A = D_0 A_L A_R E_0, \|A_L\|_s \leqq 1, \|A_R\|_s \leqq 1\},$$

where $D_0 \in \mathbb{R}^{n \times n_1}$ and $E_0 \in \mathbb{R}^{n_2 \times n}$ are known matrices denoting the structure of the uncertainty, and $A_L \in \mathbb{R}^{n_1 \times r}$ and $A_R \in \mathbb{R}^{r \times n_2}$ are uncertain matrices [28]. Finer structure can be included within $\mathcal{U}_4$ by replacing $D_0 M N E_0$ by a sum of terms $D_i M_i N_i E_i$, where D_i, E_i are known and M_i, N_i are uncertain [36]. Note, however, that even though $\mathcal{U}_4$ is a proper subset of $\mathcal{U}_3$, the *form* of the bound $\Omega_{10}(\cdot)$ does not change. Thus such refinements render the bound $\Omega_{10}(\cdot)$ conservative with respect to $\mathcal{U}_4$ since the *larger* uncertainty set $\mathcal{U}_3$ is actually being bounded.

6. Robust stability and performance via modified Lyapunov equations. We now combine the principal results of §§ 3, 4, and 5 to obtain a series of conditions guaranteeing robust stability and performance. In particular, we focus on bounds Ω_1, Ω_4, Ω_6, Ω_7, and Ω_{10}. For simplicity we shall frequently assume that V is positive definite so that (3.3) is satisfied. In this case it follows that the solution Q of (3.2) is positive definite. Our first result is a corollary of Theorem 3.1 with $\Omega(\cdot) = \Omega_1(\cdot)$ and $\mathcal{U} = \mathcal{U}_1$.

THEOREM 6.1. *Let $V \in \mathbb{P}^n$, $\delta_1, \cdots, \delta_p > 0$, and suppose there exists $Q \in \mathbb{P}^n$ satisfying*

$$(\text{MLE1}) \qquad 0 = AQ + QA^T + \sum_{i=1}^p \delta_i |A_i Q + Q A_i^T| + V.$$

Then $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}_1$, and

$$(6.1) \qquad J_S(\mathcal{U}_1) \leqq \operatorname{tr} QR,$$

$$(6.2) \qquad J_D(\mathcal{U}_1) \leqq \lambda_{\max}(QR).$$

For the next result define

(6.3) $$A_\alpha \triangleq A + \frac{\alpha}{2} I_n$$

and

(6.4) $$\gamma_i \triangleq \frac{\alpha_i^2}{\alpha}, \qquad i = 1, \cdots, p.$$

Setting $\Omega(\cdot) = \Omega_4(\cdot)$, $\Omega_6(\cdot)$, $\Omega_7(\cdot)$ and $\mathcal{U} = \mathcal{U}_2$ yields the following corollary of Theorem 3.1.

THEOREM 6.2. *Let* $V \in \mathbb{P}^n$, $\alpha, \alpha_1, \cdots, \alpha_p > 0$, *and suppose there exists* $Q \in \mathbb{P}^n$ *satisfying either*

(MLE2) $$0 = AQ + QA^T + \sum_{i=1}^{p} \gamma_i (A_i Q + QA_i^T)^2 + \frac{\alpha}{4} I_n + V,$$

(MLE3) $$0 = AQ + QA^T + \sum_{i=1}^{p} \gamma_i A_i Q^2 A_i^T + \alpha I_n + V,$$

or

(MLE4) $$0 = A_\alpha Q + QA_\alpha^T + \sum_{i=1}^{p} \gamma_i A_i Q A_i^T + V.$$

Then $A + \Delta A$ *is asymptotically stable for all* $\Delta A \in \mathcal{U}_2$, *and*

(6.5) $$J_S(\mathcal{U}_2) \leqq \operatorname{tr} QR,$$

(6.6) $$J_D(\mathcal{U}_2) \leqq \lambda_{\max}(QR).$$

Next we set $\Omega(\cdot) = \Omega_{10}(\cdot)$ and $\mathcal{U} = \mathcal{U}_3$.

THEOREM 6.3. *Let* $V \in \mathbb{P}^n$, $\alpha > 0$, $M \in \mathbb{N}^n$, *and* $N \in \mathbb{N}^n$, *and suppose there exists* $Q \in \mathbb{P}^n$ *satisfying*

(MLE5) $$0 = AQ + QA^T + \alpha QNQ + \alpha^{-1} M + V.$$

Then $A + \Delta A$ *is asymptotically stable for all* $\Delta A \in \mathcal{U}_3$, *and*

(6.7) $$J_S(\mathcal{U}_3) \leqq \operatorname{tr} QR,$$

(6.8) $$J_D(\mathcal{U}_3) \leqq \lambda_{\max}(QR).$$

Remark 6.1. Note that (MLE5) is a Riccati equation. This is precisely the equation studied in [29].

Additional sufficient conditions can be obtained by considering "mixed" bounds. That is, we can construct modified Lyapunov equations by combining two or more different bounds. Although mixed bounds will not be considered further in this paper, we present one such result for illustrative purposes.

THEOREM 6.4. *Let* $V \in \mathbb{P}^n$, $\alpha, \delta_1, \cdots, \delta_p > 0$, $M \in \mathbb{N}^n$, *and* $N \in \mathbb{N}^n$, *and suppose there exists* $Q \in \mathbb{P}^n$ *satisfying*

(MLE1, 5) $$0 = AQ + QA^T + \sum_{i=1}^{p} \delta_i |A_i Q + QA_i^T| + \alpha QNQ + \alpha^{-1} M + V.$$

Then $A+\Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}_1+\mathcal{U}_3$, and

$$J_S(\mathcal{U}_1+\mathcal{U}_3) \leqq \operatorname{tr} QR, \tag{6.9}$$

$$J_D(\mathcal{U}_1+\mathcal{U}_3) \leqq \lambda_{\max}(QR). \tag{6.10}$$

As noted previously, the bound $\Lambda(\cdot)$ can readily be constructed by replacing ΔA by ΔA^T in the definitions of $\Omega_1(\cdot)$ through $\Omega_{10}(\cdot)$. Denote these bounds by $\Lambda_1(\cdot)$ through $\Lambda_{10}(\cdot)$, respectively. For illustration we state the dual of Theorem 6.1 involving $\Lambda_1(\cdot)$. The dual versions of (MLE1)–(MLE5) will be denoted by (MLED1)–(MLED5).

THEOREM 6.5. *Let $R \in \mathbb{P}^n$, $\delta_1, \cdots, \delta_p > 0$, and suppose there exists $P \in \mathbb{P}^n$ satisfying*

$$0 = A^T P + PA + \sum_{i=1}^{p} \delta_i |A_i^T P + PA_i| + R. \tag{MLED1}$$

Then $A+\Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}_1$, and

$$J_S(\mathcal{U}_1) \leqq \operatorname{tr} PV, \tag{6.11}$$

$$\hat{J}_D(\mathcal{U}_1) \leqq \lambda_{\max}(PV). \tag{6.12}$$

It is reasonable to expect that the sufficient conditions given by Theorems 3.1 and 4.1 are generally different. For example, the modified Lyapunov equations and their duals need not both possess a solution, while the bounds tr QR and tr PV need not be equal. An exception is the case in which $\Omega(\cdot)=\Omega_7(\cdot)$ and $\Lambda(\cdot)=\Lambda_7(\cdot)$. Note that the dual of (MLE4) is given by

$$0 = A_\alpha^T P + PA_\alpha + \sum_{i=1}^{p} \gamma_i A_i^T P A_i + V. \tag{MLED4}$$

PROPOSITION 6.1. *Let $\alpha, \alpha_1, \cdots, \alpha_p > 0$ and assume there exist $Q, P \in \mathbb{N}^n$ satisfying* (MLE4) *and* (MLED4). *Then*

$$\operatorname{tr} QR = \operatorname{tr} PV. \tag{6.13}$$

Proof. Note that

$$\begin{aligned}
\operatorname{tr} QR &= -\operatorname{tr} Q\left(A_\alpha^T P + PA_\alpha + \sum_{i=1}^{p} \gamma_i A_i^T P A_i\right) \\
&= -\operatorname{tr} P\left(A_\alpha Q + QA_\alpha^T + \sum_{i=1}^{p} \gamma_i A_i Q A_i^T\right) \\
&= \operatorname{tr} PV.
\end{aligned}$$

□

Remark 6.2. By setting $\Omega(\cdot)=\Omega_7(\cdot)$ and $\Lambda(\cdot)=\Lambda_7(\cdot)$ it follows from (4.14) that

$$\operatorname{tr} Q_0\left(\alpha P + \sum_{i=1}^{p} \gamma_i A_i^T P A_i\right) = \operatorname{tr} P_0\left(\alpha Q + \sum_{i=1}^{p} \gamma_i A_i Q A_i^T\right). \tag{6.14}$$

7. Existence, uniqueness, and monotonicity of solutions to the modified Lyapunov equations. It is important to stress that the sufficient conditions for robustness given by Theorems 6.1–6.5 assume only that there exist nonnegative-definite solutions Q, P satisfying the modified Lyapunov equations. Indeed, no *explicit* assumptions on the problem data A, V, R, and $\mathcal{U}$ were utilized for assuring robust stability and performance. In applying Theorems 6.1–6.5 to specific problems it thus suffices to show that a nonnegative-definite solution Q *exists* in order to obtain robust stability, while, for robust performance, the bounds (6.1), (6.2), (6.5)–(6.8) require explicit knowledge of Q. Thus, any computational method that yields a nonnegative-definite solution will suffice to guarantee both robust stability and performance.

Before considering the numerical solution of the modified Lyapunov equations, several relevant issues require discussion. For example, before seeking to compute solutions to (MLE1)–(MLE5) it would be desirable to determine a priori whether these equations actually possess nonnegative-definite solutions. For example, it may be useful to obtain sufficient and/or necessary conditions for the *existence* of nonnegative-definite solutions. Thus, if the sufficient conditions are satisfied then existence (and hence robustness) is assured, whereas if the necessary conditions are *not* satisfied then existence is ruled out. If, on the other hand, either the sufficient conditions are not satisfied or the necessary conditions *are* satisfied, then nothing can be surmised. Finally, such conditions need to be easily verifiable and reasonably nonconservative since otherwise it would be more prudent to attempt to numerically solve the modified Lyapunov equations themselves.

It is quite possible that at least some of the modified Lyapunov equations possess multiple nonnegative-definite solutions. In this case we may seek the minimal solution (i.e., the smallest with respect to the nonnegative-definite matrix ordering) to minimize the performance bounds. If multiple solutions exist, none of which is minimal, then the best bound would depend on the matrix R.

Since the matrix Q determines the performance bound, it is reasonable to expect Q to be *monotonic* in $\mathcal{U}$. That is, if $\mathcal{U}$ decreases in size, then the solution Q is more likely to exist while decreasing in the nonnegative-definite matrix ordering. For example, consider $\mathcal{U}'_1$ characterized by δ'_i, where $\delta'_i \leqq \delta_i$, $i = 1, \cdots, p$. Then we might expect $Q' \leqq Q$, where Q' is the solution to (MLE1) with δ_i replaced by δ'_i. Finally, monotonicity with respect to V should also be expected. Because of linearity, the analysis of bound $\Omega_7(\cdot)$ is simplest and it is possible to obtain necessary and sufficient conditions for the existence of solutions to (MLE4). The basic tool required is the Kronecker matrix algebra [42]. For convenience, define

$$\mathcal{A} \triangleq A_\alpha \oplus A_\alpha + \sum_{i=1}^{p} \gamma_i A_i \otimes A_i, \tag{7.1}$$

where $\otimes$ denotes the Kronecker product and $A_\alpha \oplus A_\alpha \triangleq A_\alpha \otimes I_n + I_n \otimes A_\alpha$ is the Kronecker sum.

PROPOSITION 7.1. *If* $V \in \mathbb{N}^n$ *and* $\mathcal{A}$ *is asymptotically stable, then there exists a unique* $Q \in \mathbb{R}^{n \times n}$ *satisfying* (MLE4), *and* $Q \geqq 0$. *Conversely, if for all* $V \in \mathbb{N}^n$ *there exists* $Q \geqq 0$ *satisfying* (MLE4), *then* $\mathcal{A}$ *is asymptotically stable.*

Proof. Since (MLE4) is equivalent to

$$Q = -\text{vec}^{-1}[\mathcal{A}^{-1} \text{vec } V], \tag{7.2}$$

existence and uniqueness hold. Here, vec and vec^{-1} denote the column-stacking operation [42] and its inverse. To prove that Q is nonnegative definite, we rewrite (7.2) as

$$Q = \int_0^\infty \text{vec}^{-1}[e^{\mathcal{A}t} \text{vec } V]\, dt \tag{7.3}$$

and show that the integrand is nonnegative-definite for all $t \in [0, \infty)$. (Note that the following argument for fixed $t \geqq 0$ does not require that $\mathscr{A}$ be stable.) Using the exponential product formula,[1] the exponential in (7.3) can be written as

$$(7.4)\qquad e^{\mathscr{A}t} = \lim_{k\to\infty}\left\{\exp\left[\frac{1}{k}(A_\alpha \oplus A_\alpha)t\right]\exp\left[\frac{1}{k}\sum_{i=1}^{p}\gamma_i(A_i\otimes A_i)t\right]\right\}^k.$$

For convenience, let S and N be $r \times r$ matrices with $N \geqq 0$. Since (see [42])

$$(7.5)\qquad \mathrm{vec}^{-1}\,[(S\otimes S)\,\mathrm{vec}\,N] = SNS^T \geqq 0$$

and

$$(7.6)\qquad (S\otimes S)^k = S^k \otimes S^k,$$

it follows that

$$(7.7)\qquad \mathrm{vec}^{-1}\,[e^{S\otimes S}\,\mathrm{vec}\,N] = \sum_{k=0}^{\infty}(k!)^{-1}S^kNS^{kT} \geqq 0.$$

Furthermore,

$$(7.8)\qquad \mathrm{vec}^{-1}\,[e^{S\oplus S}\,\mathrm{vec}\,N] = \mathrm{vec}^{-1}\,[(e^S\otimes e^S)\,\mathrm{vec}\,N] = e^SNe^{S^T} \geqq 0.$$

Applying (7.7) and (7.8) alternately with (7.4) and using induction on k, it follows that the integrand of (7.3) is nonnegative definite. To prove the converse, note that it follows from (MLE4) that Q satisfies

$$(7.9)\qquad Q = \mathrm{vec}^{-1}\,[e^{\mathscr{A}t}\,\mathrm{vec}\,Q] + \int_0^t \mathrm{vec}^{-1}\,[e^{\mathscr{A}s}\,\mathrm{vec}\,V]\,ds, \qquad t\in[0,\infty).$$

Since the integral term on the right-hand side of (7.9) is nonnegative definite, is bounded from above by Q, and $V \in \mathbb{N}^n$ is arbitrary, it follows that $\mathscr{A}$ is asymptotically stable. □

We now show that if $\mathscr{A}$ is asymptotically stable then actually A_α (and thus A) is asymptotically stable. This shows that the assumption that $\mathscr{A}$ is asymptotically stable is consistent with the original hypothesis that A is asymptotically stable.

PROPOSITION 7.2. *Assume $\mathscr{A}$ is asymptotically stable, let $\alpha_i' \in [0, \alpha_i]$, $i = 1, \cdots, p$, and define*

$$\mathscr{A}' \triangleq A_\alpha \oplus A_\alpha + \sum_{i=1}^{p}\left(\frac{\alpha_i'^2}{\alpha}\right)A_i\otimes A_i.$$

Then $\mathscr{A}'$ is also asymptotically stable. In particular, A_α and A are asymptotically stable.

Proof. Let $V \in \mathbb{N}^n$ be arbitrary and let Q be the unique, nonnegative-definite solution of (MLE4). Equivalently, Q satisfies

$$0 = A_\alpha Q + QA_\alpha^T + \sum_{i=1}^{p}\left(\frac{\alpha_i'^2}{\alpha}\right)A_iQA_i^T + V',$$

where

$$V' \triangleq \sum_{i=1}^{p}\alpha^{-1}(\alpha_i^2 - \alpha_i'^2)A_iQA_i^T + V.$$

[1] The exponential product formula is essential to the proof here since (1) $A_\alpha \oplus A_\alpha$ cannot be expressed as a Kronecker product $S\otimes S$, and (2) $A_\alpha \oplus A_\alpha$ and $\sum_{i=1}^{p}\gamma_iA_i\otimes A_i$ do not generally commute.

Since $V' \in \mathbb{N}^n$, the stability of $\mathscr{A}'$ now follows as in the proof of the converse of Proposition 7.1. Finally, if V is chosen to be positive definite then $\sum_{i=1}^{p} (\alpha_i'^2/\alpha) A_i Q A_i^T + V'$ is also positive definite and it follows from Lemma 12.2 of [39] that A_α, and hence A, is asymptotically stable. □

Hence it follows from Proposition 7.2 that a necessary condition for $\mathscr{A}$ to be asymptotically stable is that

$$\alpha < 2 \max_{i=1,\cdots,n} \operatorname{Re} \lambda_i(A). \tag{7.10}$$

We now have the following monotonicity result.

PROPOSITION 7.3. *Let $\mathscr{U}_2' \subset \mathscr{U}_2$, where $\mathscr{U}_2'$ is defined as in* (5.7) *with α_i replaced by $\alpha_i' \in [0, \alpha_i]$, $i = 1, \cdots, p$. Furthermore, let $V \in \mathbb{P}^n$, assume $\mathscr{A}$ is asymptotically stable, and let $Q \in \mathbb{P}^n$ satisfy* (MLE4). *Then there exists $Q' \in \mathbb{P}^n$ satisfying*

$$0 = A_\alpha Q' + Q' A_\alpha^T + \sum_{i=1}^{p} \left(\frac{\alpha_i'^2}{\alpha}\right) A_i Q' A_i^T + V, \tag{7.11}$$

and, furthermore,

$$Q' \leqq Q. \tag{7.12}$$

Consequently,

$$\operatorname{tr} Q'R \leqq \operatorname{tr} QR, \tag{7.13}$$

$$\lambda_{\max}(Q'R) \leqq \lambda_{\max}(QR). \tag{7.14}$$

Proof. Subtracting (7.11) from (MLE4) yields

$$0 = A_\alpha(Q - Q') + (Q - Q')A_\alpha^T + \sum_{i=1}^{p} \left(\frac{\alpha_i'^2}{\alpha}\right) A_i (Q - Q') A_i^T + V',$$

where V' is defined in the proof of Proposition 7.2. Since, by the converse portion of Proposition 7.1, $\mathscr{A}'$ is asymptotically stable, $Q - Q' \geqq 0$, which yields (7.12) and thus (7.13) and (7.14). □

Returning now to the existence question, Proposition 7.1 shows that a solution to (MLE4) exists so long as $\alpha_1, \cdots, \alpha_p$ are sufficiently small such that $\mathscr{A}$ remains asymptotically stable for some $\alpha > 0$. To this end we can treat this as a stability perturbation problem and apply results from [3]. Within our modified Lyapunov equation approach we have the following related result. For this and the following result let $\|\cdot\|$ denote an arbitrary vector norm on $\mathbb{R}^{n^2}$ and the corresponding induced matrix norm.

PROPOSITION 7.4. *If*

$$\left\| (A \oplus A)^{-1} \left(\alpha I_{n^2} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 A_i \otimes A_i \right) \right\| < 1, \tag{7.15}$$

then for all $V \in \mathbb{N}^n$ there exists $Q \in \mathbb{N}^n$ satisfying (MLE4) *and hence $\mathscr{A}$ is asymptotically stable.*

Proof. Define $\{Q_k\}_{k=0}^{\infty}$ where Q_0 satisfies (3.14) and Q_{k+1} satisfies

$$0 = AQ_{k+1} + Q_{k+1}A^T + \Omega_7(Q_k) + V.$$

Note that $Q_k \geqq 0$, $k = 1, 2, \cdots$. Hence it follows that

$$\text{vec } Q_{k+1} - \text{vec } Q_k = -(A \oplus A)^{-1}[\text{vec } \Omega_7(Q_k) - \text{vec } \Omega_7(Q_{k-1})]$$

and thus

$$\|\text{vec } Q_{k+1} - \text{vec } Q_k\| \leqq \left\| (A \oplus A)^{-1}\left(\alpha I_{n^2} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 A_i \otimes A_i\right) \right\| \|\text{vec } Q_k - \text{vec } Q_{k-1}\|.$$

Using (7.15) it follows that $Q \triangleq \lim_{k\to\infty} Q_k$ exists. Thus $Q \geqq 0$ and satisfies (MLE4). Finally, by the converse of Proposition 7.1, $\mathscr{A}$ is asymptotically stable. □

Since (MLE5) is nonlinear, a slightly different approach is required for existence. For the following result let κ, $\beta > 0$ satisfy

$$\|e^{At}\| \leqq \kappa e^{-\beta t}, \qquad t \geqq 0, \tag{7.16}$$

where $\|\cdot\|$ denotes an arbitrary submultiplicative matrix norm that is monotonic on $\mathbb{N}^n$, and define $\rho \triangleq 2\beta/\kappa^2$.

PROPOSITION 7.5. *Suppose* $V \in \mathbb{N}^n$ *and*

$$4\alpha\|N\| \|\alpha^{-1}M + V\| < \rho^2. \tag{7.17}$$

Then there exists $Q \in \mathbb{N}^n$ *satisfying* (MLE5).

Proof. Consider the sequence $\{Q_k\}_{k=0}^{\infty}$ where Q_0 satisfies (3.14) and Q_{k+1} is given by

$$0 = AQ_{k+1} + Q_{k+1}A^T + \alpha Q_k N Q_k + \alpha^{-1}M + V.$$

Clearly, $Q_k \geqq 0$, $k = 0, 1, \cdots$. Next we have

$$Q_{k+1} = \int_0^\infty e^{At}[\alpha Q_k N Q_k + \alpha^{-1}M + V]e^{A^T t}\, dt, \tag{7.18}$$

which yields

$$\|Q_{k+1}\| \leqq \alpha\rho^{-1}\|N\| \|Q_k\|^2 + \rho^{-1}\|\alpha^{-1}M + V\|. \tag{7.19}$$

Similarly, from (3.14) we obtain

$$\|Q_0\| \leqq \rho^{-1}\|V\| \leqq \rho^{-1}\|\alpha^{-1}M + V\|.$$

Now suppose that

$$\|Q_k\| \leqq 2\rho^{-1}\|\alpha^{-1}M + V\|.$$

Then (7.17) and (7.19) imply

$$\begin{aligned}\|Q_{k+1}\| &\leqq \alpha\rho^{-1}\|N\| [2\rho^{-1}\|\alpha^{-1}M + V\|]^2 + \rho^{-1}\|\alpha^{-1}M + V\| \\ &< 2\rho^{-1}\|\alpha^{-1}M + V\|.\end{aligned}$$

Thus $\|Q_k\| \leqq 2\rho^{-1}\|\alpha^{-1}M + V\|$, $k = 0, 1, \cdots$. Next, (7.18) yields

$$\begin{aligned}Q_{k+1} - Q_k &= \alpha \int_0^\infty e^{At}[Q_k N Q_k - Q_{k-1} N Q_{k-1}]e^{A^T t}\, dt \\ &= \alpha \int_0^\infty e^{At}[Q_k N(Q_k - Q_{k-1}) + (Q_k - Q_{k-1}) N Q_{k-1}]e^{A^T t}\, dt\end{aligned}$$

and thus

$$\|Q_{k+1}-Q_k\| \leqq \alpha\rho^{-1}\|N\|(\|Q_k\|+\|Q_{k-1}\|)\|Q_k-Q_{k-1}\|$$
$$\leqq 4\alpha\rho^{-2}\|N\|\|\alpha^{-1}M+V\|\|Q_k-Q_{k-1}\|$$
$$\leqq \varepsilon\|Q_k-Q_{k+1}\|,$$

where $\varepsilon \triangleq 4\alpha\rho^{-2}\|N\|\|\alpha^{-1}M+V\|$. Since by (7.17) $\varepsilon<1$, $\lim_{k\to\infty} Q_k$ exists, is nonnegative definite, and satisfies (MLE5). □

8. Additional upper bounds via recursive substitution. In this section we obtain additional upper bounds for $J_S(\mathcal{U})$ and $J_D(\mathcal{U})$ by utilizing a recursive substitution technique. The main idea involves rewriting (2.7) as

$$Q_{\Delta A}=-\operatorname{vec}^{-1}\{(A\oplus A)^{-1}(\Delta A\oplus\Delta A)\operatorname{vec} Q_{\Delta A}\}+Q_0 \tag{8.1}$$

and substituting this expression into the terms $\Delta A Q_{\Delta A}+Q_{\Delta A}\Delta A^T$ appearing in (2.7). This technique yields an equation that is, as expected, equivalent to (2.7) but that permits the development of additional bounds. As will be seen, the ability to develop new bounds exploits the fact that the substitution technique leads to terms that are quadratic in ΔA. We begin the development with the following technical result that does not require that A be asymptotically stable.

PROPOSITION 8.1. *Suppose $A\oplus A$ is invertible and let $\Delta A\in\mathbb{R}^{n\times n}$. If $Q_{\Delta A}$ satisfies (2.7), then $Q_{\Delta A}$ also satisfies*

$$\begin{aligned}0=AQ_{\Delta A}+Q_{\Delta A}A^T-\operatorname{vec}^{-1}[(\Delta A\oplus\Delta A)(A\oplus A)^{-1}(\Delta A\oplus\Delta A)\operatorname{vec} Q_{\Delta A}\\+(\Delta A\oplus\Delta A)(A\oplus A)^{-1}\operatorname{vec} V]+V.\end{aligned} \tag{8.2}$$

Conversely, if $Q_{\Delta A}$ satisfies (8.2) and $(A-\Delta A)\oplus(A-\Delta A)$ is invertible, then $Q_{\Delta A}$ also satisfies (2.7).

Proof. To obtain (8.2) substitute (8.1) into (2.7) as noted above. Conversely, adding the zero term $(\Delta A\oplus\Delta A)(A\oplus A)^{-1}(A\oplus A)\operatorname{vec} Q_{\Delta A}-(\Delta A\oplus\Delta A)\operatorname{vec} Q_{\Delta A}$ to (8.2), it follows that (8.2) can be written as

$$0=[(A-\Delta A)\oplus(A-\Delta A)](A\oplus A)^{-1}[(A+\Delta A)\oplus(A+\Delta A)\operatorname{vec} Q_{\Delta A}+\operatorname{vec} V],$$

which, under the invertibility assumption, implies that $Q_{\Delta A}$ satisfies (2.7). □

The following result is analogous to Theorem 3.1. We shall say that $\mathcal{U}$ is symmetric if $\Delta A\in\mathcal{U}$ implies $-\Delta A\in\mathcal{U}$.

THEOREM 8.1. *Suppose $\mathcal{U}$ is symmetric, let $\Omega_0\in\mathbb{N}^n$ satisfy*

$$\Delta A Q_0+Q_0\Delta A^T\leqq\Omega_0,\qquad \Delta A\in\mathcal{U}, \tag{8.3}$$

where Q_0 satisfies (3.14), let $\hat{\Omega}:\mathbb{N}^n\to\mathbb{N}^n$ satisfy

$$-\operatorname{vec}^{-1}[(\Delta A\oplus\Delta A)(A\oplus A)^{-1}(\Delta A\oplus\Delta A)\operatorname{vec} Q]\leqq\hat{\Omega}(Q),\qquad \Delta A\in\mathcal{U},\quad Q\in\mathbb{N}^n, \tag{8.4}$$

and suppose there exists $Q\in\mathbb{N}^n$ satisfying

$$0=AQ+QA^T+\hat{\Omega}(Q)+\Omega_0+V. \tag{8.5}$$

Then

$$(A+\Delta A,D)\ \textit{is stabilizable},\qquad \Delta A\in\mathcal{U}, \tag{8.6}$$

if and only if

$$(8.7) \qquad A+\Delta A \text{ is asymptotically stable}, \qquad \Delta A\in\mathcal{U}.$$

In this case,

$$(8.8) \qquad Q_{\Delta A}\leqq Q, \qquad \Delta A\in\mathcal{U},$$

where $Q_{\Delta A}$ *satisfies* (2.7), *and*

$$(8.9) \qquad J_S(\mathcal{U})\leqq \operatorname{tr} QR,$$

$$(8.10) \qquad J_D(\mathcal{U})\leqq \lambda_{\max}(QR).$$

Proof. The equivalence of (8.6) and (8.7) follows from (8.5) as in the proof of Theorem 3.1. Next (8.8) follows by comparing (8.5) and (8.2) while using (8.3) and (8.4). Since $\mathcal{U}$ is assumed to be symmetric, it follows from (8.7) that $A-\Delta A$ is asymptotically stable, $\Delta A\in\mathcal{U}$, and hence $(A-\Delta A)\oplus(A-\Delta A)$ is invertible, $\Delta A\in\mathcal{U}$. Thus, the converse portion of Proposition 8.1 implies that $Q_{\Delta A}$ satisfying (8.2) also satisfies (2.7). Thus, the bound (8.8) can be used to obtain (8.9) and (8.10). □

The principal difference between (8.4) and (3.1) is that ΔA appears linearly in (3.1), whereas it appears quadratically in (8.4). By exploiting this structure we can obtain new bounds for $Q_{\Delta A}$. To simplify matters, we now consider the bound in (8.4) in two special cases. In the first case we set $\mathcal{U}=\mathcal{U}_1$ and $p=1$ so that $\Delta A=\sigma_1 A_1$, $|\sigma_1|\leqq\delta_1$. In this case (8.4) becomes

$$(8.11) \qquad -\sigma_1^2 \operatorname{vec}^{-1}[(A_1\oplus A_1)(A\oplus A)^{-1}(A_1\oplus A_1)\operatorname{vec} Q]\leqq\hat{\Omega}(Q), \qquad |\sigma_1|\leqq\delta_1, \quad Q\in\mathbb{N}^n.$$

One choice of $\hat{\Omega}(\cdot)$ that immediately suggests itself can be obtained by defining the matrix function $|\cdot|_+$ on the set of symmetric matrices by

$$(8.12) \qquad |S|_+\triangleq\tfrac{1}{2}(S+|S|),$$

which effectively replaces the negative eigenvalues of S by zeros. We shall thus utilize the fact that

$$(8.13) \qquad \sigma_1^2 S\leqq\delta_1^2|S|_+, \qquad |\sigma_1|\leqq\delta_1,$$

for all symmetric S.

COROLLARY 8.1. *Let* $V\in\mathbb{P}^n$, $\mathcal{U}=\mathcal{U}_1$, $p=1$, *let* $\Omega_0\in\mathbb{N}^n$ *satisfy* (8.3), *and suppose there exists* $Q\in\mathbb{N}^n$ *satisfying*

$$(8.14) \qquad 0=AQ+QA^T+\delta_1^2|-\operatorname{vec}^{-1}[(A_1\oplus A_1)(A\oplus A)^{-1}(A_1\oplus A_1)\operatorname{vec} Q]|_++\Omega_0+V.$$

Then (8.7)–(8.10) *are satisfied*.

For the next specialization we shall assume that

$$(8.15) \qquad (\Delta A)A=A(\Delta A), \qquad \Delta A\in\mathcal{U},$$

which holds, for example, for modal systems with frequency uncertainty (see § 10). It thus follows that $(A\oplus A)^{-1}(\Delta A\oplus\Delta A)=(\Delta A\oplus\Delta A)(A\oplus A)^{-1}$ and thus (8.4) can be rewritten as

$$(8.16) \qquad \Delta A^2\hat{Q}+2\Delta A\hat{Q}\Delta A^T+\hat{Q}\Delta A^{2T}\leqq\hat{\Omega}(Q), \qquad \Delta A\in\mathcal{U}, \quad Q\in\mathbb{N}^n,$$

where $\hat{Q}\in\mathbb{N}^n$ satisfies

$$(8.17) \qquad 0=A\hat{Q}+\hat{Q}A^T+Q.$$

Assuming in addition to (8.15) that $\Delta A = \sigma_1 A_1$, $|\sigma_1| \leqq \delta_1$, (8.14) becomes

$$0 = AQ + QA^T + \delta_1^2 |A_1^2\hat{Q} + 2A_1\hat{Q}A_1^T + \hat{Q}A_1^{2T}|_+ + \Omega_0 + V. \tag{8.18}$$

Remark 8.1. It is interesting to note that the left-hand side of (8.16) is of the same form as $\Omega_{4'}(\cdot)$. Specifically, the term $\Delta A^2\hat{Q} + \hat{Q}\Delta A^{2T}$ is analogous to $A_i^2 Q + QA_i^{2T}$ whereas $2\Delta A\hat{Q}\Delta A^T$ is similar to $A_i Q A_i^T$.

9. An alternative approach yielding upper and lower bounds. In this section we develop a variation on the results of § 3 that has the additional benefit of yielding both upper and lower performance bounds. The basic approach was suggested by results obtained in [44]. To simplify the presentation we assume as in the preceding section that $\mathcal{U}$ is symmetric. This symmetry assumption of course holds for all of the uncertainty sets considered in previous sections. The underlying idea involves bounding the deviation of $Q_{\Delta A}$ from Q_0 rather than bounding $Q_{\Delta A}$ directly.

THEOREM 9.1. *Let* $\Omega_0 \in \mathbb{N}^n$ *satisfy*

$$\Delta A Q_0 + Q_0 \Delta A^T \leqq \Omega_0, \qquad \Delta A \in \mathcal{U}, \tag{9.1}$$

let $\Omega: \mathbb{N}^n \to \mathbb{N}^n$ *be such that* (3.1) *is satisfied, and suppose there exists* $\Delta\mathcal{Q} \in \mathbb{N}^n$ *satisfying*

$$0 = A\Delta\mathcal{Q} + \Delta\mathcal{Q}A^T + \Omega(\Delta\mathcal{Q}) + \Omega_0. \tag{9.2}$$

Then

$$(A + \Delta A, \Omega_0^{1/2}) \textit{ is stabilizable}, \qquad \Delta A \in \mathcal{U}, \tag{9.3}$$

if and only if

$$A + \Delta A \textit{ is asymptotically stable}, \qquad \Delta A \in \mathcal{U}. \tag{9.4}$$

In this case,

$$Q_0 - \Delta\mathcal{Q} \leqq Q_{\Delta A} \leqq Q_0 + \Delta\mathcal{Q}, \qquad \Delta A \in \mathcal{U}, \tag{9.5}$$

where $Q_{\Delta A}$ *is given by* (2.7), *and*

$$\operatorname{tr}(Q_0 + \Delta\mathcal{Q})R \leqq J_S(\mathcal{U}) \leqq \operatorname{tr}(Q_0 + \Delta\mathcal{Q})R, \tag{9.6}$$

$$\lambda_{\max}[(Q_0 - \Delta\mathcal{Q})R] \leqq J_D(\mathcal{U}) \leqq \lambda_{\max}[(Q_0 + \Delta\mathcal{Q})R]. \tag{9.7}$$

Proof. Define

$$\Delta Q \triangleq Q_{\Delta A} - Q_0 \tag{9.8}$$

and subtract (3.14) from (2.7) to obtain

$$0 = (A + \Delta A)\Delta Q + \Delta Q(A + \Delta A)^T + \Delta A Q_0 + Q_0 \Delta A^T. \tag{9.9}$$

Now rewrite (9.2) as

$$0 = (A + \Delta A)\Delta\mathcal{Q} + \Delta\mathcal{Q}(A + \Delta A)^T + \Omega(\Delta\mathcal{Q}) - (\Delta A\Delta\mathcal{Q} + \Delta\mathcal{Q}\Delta A^T) + \Omega_0. \tag{9.10}$$

Using (9.10), the equivalence of (9.3) and (9.4) is immediate as in the proof of Theorem 3.1. Next, subtracting (9.9) from (9.10) yields

$$\begin{aligned} 0 = {} & (A + \Delta A)(\Delta\mathcal{Q} - \Delta Q) + (\Delta\mathcal{Q} - \Delta Q)(A + \Delta A)^T + \Omega(\Delta\mathcal{Q}) \\ & - (\Delta A\Delta\mathcal{Q} + \Delta\mathcal{Q}\Delta A^T) + \Omega_0 - (\Delta A Q_0 + Q_0 \Delta A^T). \end{aligned} \tag{9.11}$$

Using (3.1) and (9.1) it follows from (9.11) that

$$\Delta\mathcal{Q} - \Delta Q \geqq 0,$$

or, equivalently,

$$Q_{\Delta A} \leqq Q_0 + \Delta \mathscr{Q}. \tag{9.12}$$

To obtain the lower bound rewrite (9.9) as

$$0 = (A+\Delta A)(-\Delta Q) + (-\Delta Q)(A+\Delta A)^T - (\Delta A Q_0 + Q_0 \Delta A^T). \tag{9.13}$$

Also, note that because of the assumed symmetry of $\mathscr{U}$, (9.1) holds with ΔA appearing in the inequality replaced by $-\Delta A$. Hence it can be shown similarly that

$$\Delta \mathscr{Q} + \Delta Q \geqq 0,$$

or, equivalently,

$$Q_0 - \Delta \mathscr{Q} \leqq Q_{\Delta A}. \tag{9.14}$$

Finally, (9.6) and (9.7) follow from (9.5). □

Remark 9.1. To compare the upper bound in (9.5) with (3.5), rewrite (9.2) as

$$0 = A(Q_0 + \Delta \mathscr{Q}) + (Q_0 + \Delta \mathscr{Q})A^T + \Omega(\Delta \mathscr{Q}) + \Omega_0 + V. \tag{9.15}$$

If $\Omega(\Delta \mathscr{Q}) + \Omega_0 = \Omega(Q_0 + \Delta \mathscr{Q})$ then (9.15) has the same form as (3.2) and thus the two upper bounds are identical. This will be the case, for example, if $\Omega(\cdot) = \Omega_7(\cdot)$ and Ω_0 is chosen to be $\Omega_7(Q_0)$ since $\Omega_7(\cdot)$ is linear. If, however, $\Omega(\Delta \mathscr{Q}) + \Omega_0 < \Omega(Q_0 + \Delta \mathscr{Q})$ then the upper bound in (9.5) will be sharper. In any case it is clear that the individual treatment of $\Delta \mathscr{Q}$ and Q_0 yields potentially new upper bounds.

Remark 9.2. Theorem 9.1 does not guarantee that the lower bound $Q_0 - \Delta \mathscr{Q}$ for $Q_{\Delta A}$ is nonnegative definite. However, $Q_{\Delta A}$ is always nonnegative definite and thus the lower bound in (9.5) may be of limited usefulness. Nevertheless, if $Q_0 - \Delta \mathscr{Q}$ is indefinite then, depending on R, the lower bounds in (9.6) and (9.7) may still be positive and thus be meaningful lower bounds.

10. Analytical examples. In this section we consider simple analytical examples that illustrate the principal results of the paper. These examples also provide insight into the individual characteristics of different bounds as a prelude to numerical examples considered in the following section.

To begin we consider the simplest possible example. Set $n = 1$, $A < 0$, $R > 0$, $V > 0$, $A_1 = 1$, and $\mathscr{U} = \{\Delta A : |\Delta A| \leqq \delta_1\}$. For $\delta_1 < -A$, $Q_{\Delta A} = V/2(|A| - \Delta A)$ and $J_S(\mathscr{U}) = J_D(\mathscr{U}) = RV/2(|A| - \delta_1)$, where this worst-case performance is achieved for $\Delta A = \delta_1$. Solving (MLE1) yields $Q = V/2(|A| - \delta_1)$, which is a nonconservative result for both robust stability and performance. The same result is obtained from (MLE4) by setting $\alpha = \alpha_1 = \delta_1$. To apply (MLE5), set $\delta_1 = \sqrt{MN}$. Choosing $\alpha = 2\delta_1(|A| - \delta_1)NV$ again yields the nonconservative result. Finally, the same result follows from Theorem 8.1.

For the second example we consider nondestabilizing uncertainty in the imaginary component of an uncertain eigenvalue, i.e., frequency uncertainty, in contrast to uncertainty in the real part considered in the previous example. Let $n = 2$,

$$A = \begin{bmatrix} -\nu & \omega \\ -\omega & -\nu \end{bmatrix}, \qquad \nu > 0, \quad \omega \geqq 0,$$

$V = R = I_2$, and $\mathscr{U} = \{\Delta A : \Delta A = \sigma_1 A_1, |\sigma_1| \leqq \delta_1\}$, where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Obviously, $A + \Delta A$ remains asymptotically stable for all values of σ_1 since ΔA affects only the imaginary part of the poles of ΔA. The question then is whether the robustness tests are able to guarantee this robustness. Note also that because of the choice of V, $Q_{\Delta A} = Q_0 = (2\nu)^{-1}I_2$ for all $\Delta A \in \mathcal{U}$. For this example we note that (MLE1) is satisfied by $Q = (2\nu)^{-1}I_2$, which is independent of δ_1. Thus (MLE1) possesses a nonnegative-definite solution for all $\delta_1 > 0$, which shows that (MLE1) is nonconservative with respect to robust stability and performance. Since $A(\Delta A) = (\Delta A)A$, it can also be seen that the same result holds for (8.18). The situation is considerably different for (MLE4) and (MLE5). To analyze (MLE4) note that $\mathscr{A}$ has an eigenvalue $-2\nu + \alpha + \delta_1$. (This can be shown by diagonalizing A and A_1 and thus $\mathscr{A}$.) Since, by Proposition 7.1, $\mathscr{A}$ must be asymptotically stable, we require $\delta_1 < 2\nu$. This is, of course, an extremely conservative result, especially when the damping ν is small. For (MLE5) we can factor $A_1 = D_1E_1$. Thus, let $D_1 = I_2$ and $E_1 = A_1$ and define $M = \delta_1^2 I_2$ and $N = I_2$. Assuming that Q is a multiple of I_2, it follows that Q is nonnegative definite only if $\delta_1 \leqq \nu$, which is again an extremely conservative result. The reason for this conservatism becomes clear by noting that M and N as given above will also serve as bounds for perturbations of the form $\sigma_1 I_2$ for which the range of nondestabilizing σ_1 is $|\sigma_1| < \delta_1$. This will also be the case for all factorizations D_1E_1 of A_1 since $D_1D_1^T$ and $E_1^TE_1$ must be positive definite and thus will also serve as bounds for destabilizing perturbations such as $\sigma_1 I_2$.

Finally, we consider a nondestabilizing uncertainty affecting the interaction of a pair of real poles. Let $n = 2$, $A = -I_2$, $V = R = I_2$, and $\mathcal{U} = \{\Delta A : \Delta A = \sigma_1 A_1,\ |\sigma_1| \leqq \delta_1\}$, where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Obviously, $A + \Delta A$ remains asymptotically stable for all values of σ_1 since ΔA does not affect the nominal poles. Note that

$$Q_{\Delta A} = \begin{bmatrix} \sigma_1^2/4 + \frac{1}{2} & \sigma_1/4 \\ \sigma_1/4 & \frac{1}{2} \end{bmatrix}$$

and $J_S(\mathcal{U}) = \frac{1}{4}\delta_1^2 + 1$, where this worst-case performance is achieved for $\sigma_1 = \delta_1$. In this case (MLE1) has the solution $Q = (2 - \delta_1)^{-1}I_2$, which is valid only for $\delta_1 < 2$, an extremely conservative robust stability result. Furthermore, the corresponding performance bound $\operatorname{tr} QR = 2(2 - \delta_1)^{-1}$ is conservative with respect to the actual worst-case performance $\frac{1}{4}\delta_1^2 + 1$. In contrast, (MLE4) has the solution

$$Q = \begin{bmatrix} (2-\alpha\delta_1)^{-1} + \alpha^{-1}\delta_1(2-\alpha\delta_1)^{-2} & 0 \\ 0 & (2-\alpha\delta_1)^{-1} \end{bmatrix},$$

which is nonnegative definite for all δ_1 so long as $\alpha < 2/\delta_1$. Hence (MLE4) is nonconservative with respect to robust stability. For robust performance,

$$\operatorname{tr} QR = 2(2-\alpha\delta_1)^{-1} + \alpha^{-1}\delta_1(2-\alpha\delta_1)^{-2},$$

which can be shown to be an upper bound for $\frac{1}{4}\delta_1^2 + 1$. Choosing, for example, $\alpha = \delta_1^{-1}$ yields $\operatorname{tr} QR = \delta_1^2 + 2$. The parameter α can also be chosen to minimize $\operatorname{tr} QR$, although this is somewhat tedious to carry out analytically. Finally, (MLE5) has the solution

$$Q = \begin{bmatrix} \frac{1}{2}(1+\alpha^{-1}\delta_1) & 0 \\ 0 & [1-(1-\alpha\delta_1)^{1/2}]/\alpha\delta_1 \end{bmatrix},$$

which exists so long as $\alpha \leqq 1/\delta_1$. Hence (MLE5) is also nonconservative with respect to robust stability. Choosing $\alpha = 1/\delta_1$ yields $\operatorname{tr} QR = \frac{1}{2}\delta_1^2 + \frac{3}{2}$, which lies above the nonconservative bound $\frac{1}{4}\delta_1^2 + 1$. Again, α can be chosen to minimize $\operatorname{tr} QR$.

11. Numerical examples. In this section we consider additional examples illustrating the results developed in earlier sections. In contrast to the analytical examples considered in § 10, however, we consider more complex examples by numerically solving the modified Lyapunov equations. Here we focus on (MLE4) and (MLE5), which are the easiest to solve numerically. Specifically, we solved (MLE4) by using the representation (7.2) (although this may not be practical when n is large), and we solved (MLE5) by means of a standard Riccati package. To simplify matters we consider only uncertainties ΔA of the form $\sigma_1 A_1$. Evaluation and presentation of robust stability and performance results for multiparameter uncertainty can be fairly complex and thus are deferred to a future numerical study.

Since both robustness tests (MLE4) and (MLE5) depend on an arbitrary positive constant α, it is desirable to determine the value of α that yields the tightest (i.e., lowest) performance bound for each robust stability range. To this end we performed a simple one-dimensional search to determine the best such α. Although analytical techniques may assist in determining optimal values of α more efficiently, the search technique proved to be adequate for the examples considered here.

As a first example we consider the control system given in [1] to demonstrate the lack of a guaranteed gain margin for LQG controllers. Hence consider

$$\dot{x}_0(t) = A_0 x_0(t) + B_0 u(t) + w_1(t), \tag{11.1}$$

$$y(t) = C_0 x_0(t) + w_2(t), \tag{11.2}$$

with controller

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{11.3}$$

$$u(t) = C_c x_c(t), \tag{11.4}$$

and performance

$$J = \lim_{t \to \infty} \mathbb{E}[x_0^T(t) R_1 x_0(t) + u^T(t) R_2 u(t)]. \tag{11.5}$$

The data are

$$A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_0 = [1 \quad 0],$$

$$V_1 = R_1 = \rho \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad V_2 = R_2 = 1,$$

where V_1 and V_2 are the intensities of $w_1(t)$ and $w_2(t)$, respectively. Uncertainty ΔB_0 in B_0 is thus represented by $\sigma_1 B_1$, where $B_1 = [0 \ 1]^T$. Thus the closed-loop system corresponds to

$$A = \begin{bmatrix} A_0 & B_0 C_c \\ B_c C_0 & A_c \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & B_1 C_c \\ 0 & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix},$$

where the zero in the (2, 2) block of R denotes the fact that we are considering the robust performance bound for the state regulation cost only. Choosing $\rho = 60$, it follows that the LQG gains are given by

$$A_c = \begin{bmatrix} -9 & 1 \\ -20 & -9 \end{bmatrix}, \quad B_c = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad C_c = [-10 \quad -10].$$

For this controller the actual stability region corresponds to $\sigma_1 \in (-.07, .01)$ so that the largest symmetric region about $\sigma_1 = 0$ is $|\sigma_1| < .01$. The worst-case performance over each stability region $|\sigma_1| < \delta_1$ is denoted by the solid line in Fig. 1, whereas the performance bounds obtained from (MLE4) and (MLE5) are shown for several values of δ_1. For (MLE5) we set $D_1 = [0\ 1\ 0\ 0]^T$ and $E_1 = [0\ 0\ C_c]$. Note that (MLE5) yields considerably tighter estimates of worst-case performance, particularly as δ_1 approaches .01. For (MLE4) optimal values of α were in the range .0012 to .0058, whereas for (MLE5) (with $\Omega_{10'}(\cdot)$, see (5.26)) α was in the range .0143 to .0020.

As a second example we consider a pair of nominally uncoupled oscillators with uncertain coupling. This example was considered in [45] using the majorant Lyapunov technique. Let

$$A = \begin{bmatrix} -\nu & \omega_1 & 0 & 0 \\ -\omega_1 & -\nu & 0 & 0 \\ 0 & 0 & -\nu & \omega_2 \\ 0 & 0 & -\omega_2 & -\nu \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\nu = .2, \quad \omega_1 = .2, \quad \omega_2 = 1.8, \quad R = V = I_4,$$

and, for (MLE5), define $D_1 = A_1$ and $E_1 = I_4$. We consider bounds on $J_S(\mathscr{U})$ only.

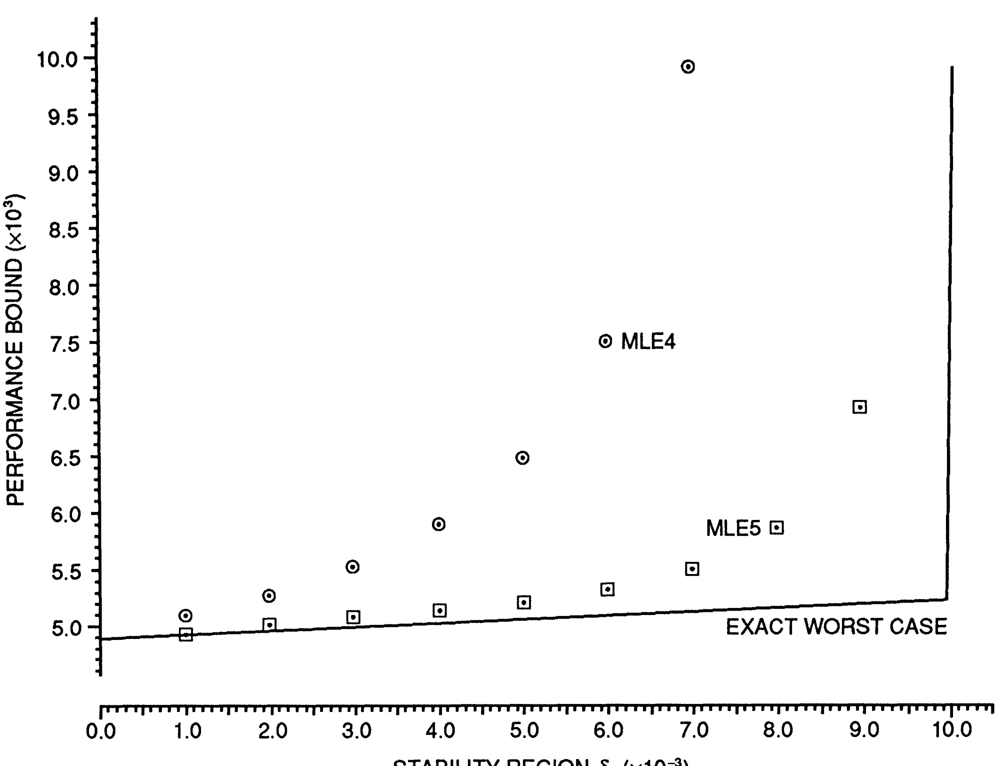

FIG. 1

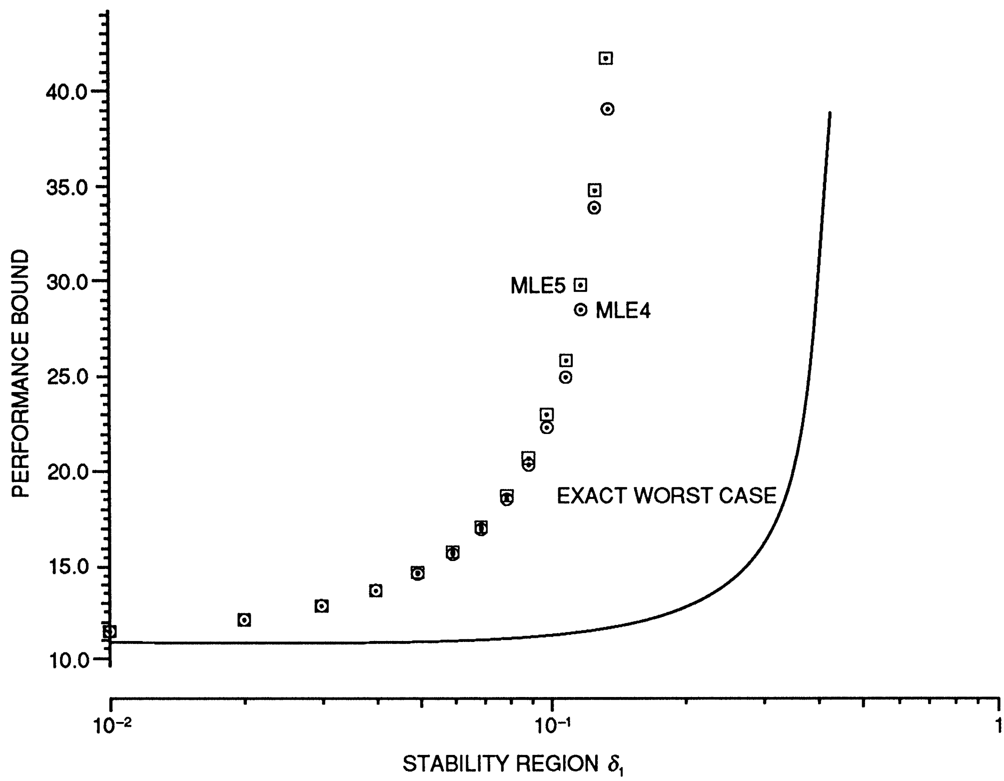

FIG. 2

Figure 2 illustrates the exact worst-case performance along with performance bounds obtained from (MLE4) and (MLE5). For (MLE4) optimal values of α ranged from .036 to .141, whereas for (MLE5) optimal α was between .361 and .096. Although (MLE4) was slightly less conservative than (MLE5), both bounds were able to guarantee robust stability only for $\delta_1 = .15$, whereas the largest stability region is actually $\delta_1 = .54$. It is interesting to contrast this result with [45] where the majorant Lyapunov technique yielded a robust stability range of $\delta_1 = .4$ for a richer class of off-diagonal blocks having maximum singular value less than δ_1.

12. Conclusion. A variety of quadratic Lyapunov bounds have been developed for both robust stability and performance. It seems clear, however, that no single quadratic Lyapunov bound is superior to the others. Although the conservatism of each bound is problem dependent, it is desirable to better understand the nature of the conservatism in order to utilize the bounds in an effective manner. In addition, the issue of *necessity* remains to be addressed. That is, if a system is robustly quadratically stable (i.e., robustly stable with a corresponding Lyapunov function), then is such a Lyapunov function necessarily given by one of the modified Lyapunov equations given in this paper? Furthermore, a better understanding is needed of the gap between robust stability and robust quadratic stability.

Acknowledgment. We thank A. W. Daubendiek for producing the numerical results in § 11.

Note added in proof. (1) The assumption $x(0) = 0$ in (2.2) is stronger than necessary for the treatment of (2.4). If $x(0) \neq 0$, then Lemma 2.1 remains unchanged since the effect of $x(0)$ vanishes as $t \to \infty$. If, however, $x(0) = 0$, then $Q_{\Delta A}(t)$ is increasing on $[0, \infty)$ and (2.4) is equivalent to

$$J_S(\mathscr{U}) = \sup_{\Delta A \in \mathscr{U}} \sup_{t \in [0,\infty)} \mathbb{E}\{\|y(t)\|_2^2\} \leqq \beta_S.$$

For $J_D(\mathscr{U})$, $x(0) = 0$ is essential since $\|y(\cdot)\|_{\infty,2}$ involves the supremum over $[0, \infty)$. If $x(0) \neq 0$, then the analysis can possibly be redone by considering the supremum over $[t, \infty)$ and letting $t \to \infty$ to eliminate the effect of the initial condition.

(2) A relationship between the linear bound $\Omega_7(\cdot)$ and the quadratic bound $\Omega_{10}(\cdot)$ can be seen as follows. If $\Delta A = \sigma_1 A_1$, $|\sigma_1| \leqq \delta_1$, then factor $\Delta A = A_L A_R$ as in $\mathscr{U}_3$ according to $A_L = \sigma_1 A_1 Q^{1/2}$ and $A_R = Q^{-1/2}$ with bounds $M = \delta_1^2 A_1 Q A_1^T$ and $N = Q^{-1}$. The unusual feature here is that the "splitting" of ΔA is Q-dependent. Then, by (5.22),

$$\Omega_{10}(Q) = \alpha^{-1}\delta_1^2 A_1 Q A_1^T + \alpha Q,$$

which has the form of $\Omega_5(Q)$.

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