THE UNIVERSITY OF MICHIGAN

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THE INELASTIC BUCKLING GRADIENT

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INTRODUCTION

Various concepts of column behavior at and above the tangent modulus load are illustrated in Figure 1 for a geometrically perfect column at a particular slenderness ratio. The "geometrically perfect" column is initially straight, of constant cross-section, has frictionless hinged ends, and the load is applied at the centroidal axis. It is recognized that in the laboratory such perfection may be approached but even there cannot be realized; hence the modes of behavior to be discussed herein can be approached but not precisely duplicated in a column test. The concepts nevertheless permit a better understanding of column behavior.

For the concepts illustrated in Figure 1, the material is assumed to have a stress-strain curve similar to that of an aluminum alloy, as indicated. (Only the upper portions of the stress-strain curve and corresponding column load-deflection curves are shown in Figure 1.) The critical or buckling load in the inelastic range, (1) in Figure 1, is the tangent modulus load, P_t, as interpreted by Shanley,[1] and obtained by substituting the tangent modulus E_t in place of the elastic modulus E in the Euler formula:

$$P_t = \pi^2 \frac{E_t I}{L^2} \qquad (1)$$

For an infinitesimal amount of axial load above the buckling load, Shanley showed that bending would commence with no regression in compressive strain, but that for any finite added increment of load there would be strain regression, initiated on the convex side and at the mid-length of the buckled column.

The initial slope of the column load-deflection curve ((4) in Figure 1), is termed herein the "inelastic buckling gradient" and (because there is initially no strain regression) it is easily determined, as given for the symmetrical case by Duberg and Wilder,[2] who also determined the complete load deflection curve for an idealized H-section column (two concentrated areas) for which the material properties could be described by the Ramberg-Osgood[3] stress-strain curves. In considering an actual column area, part of the area, varying both along the column and with applied load, will experience strain regression at the elastic modulus rate. The remainder of the column area, also varying along the column and with column load, introduces the further complication that at any load finitely above the buckling load the variation in compressive strain in the cross-section requires a consideration of the changing magnitude of E_t over the column cross-section and along the column. Thus, for a realistic determination of a load-deflection curve, such as (6) of Figure 1, up to the maximum load (7) in Figure 1, a numerical analysis at successive increments of load is required, preferably with the aid of a digital computer. At each successive increment of load the deflected shape of the column is determined so that the internal resisting moment at a number of control points along the column is in equilibrium with the moment produced by the external load at the same locations. Also shown in Figure 1 is the "reduced modulus load" P_R, ((2) of Figure 1) which may be defined as <u>the load at which the inelastic buckling gradient becomes zero</u> if a column is prevented from buckling above the tangent modulus load.

Some authors, including von Karman, in his discussion of the Shanley paper,[1] have pointed out that if the tangent modulus remained constant above the buckling load, the column load-deflection curve would approach asymptotically the reduced modulus load, as shown by (3) and (5) on Figure 1. Such an assumption gives a very unrealistic picture of column behavior, since no true maximum load is reached, and the "false" reduced modulus load that is being approached is much greater than the true reduced modulus load which reflects the usual decrease in tangent modulus with increase in column stress.

It has often been pointed out, since the work of Shanley, that the reduced modulus load represents an upper bound to the maximum column load. This is true for the usual non-ferrous material for which the tangent modulus decreases monotonically with strain. As a corollary, it can be said for such a material that the inelastic buckling gradient together with the reduced modulus load form an upper bound to the column load deflection curve. Such an upper bound is shown shaded in Figure 1.

The inelastic buckling gradient will be considered herein both in reference to a continuously strain hardening material, typified by a structural aluminum alloy, and to the elasto-plastic behavior of structural steel in combination with an initial residual stress distribution. Further, for unsymmetrical sections, two different gradients are obtained, depending on the direction of buckling and suggesting the possibility of dualistic behavior.

THE INELASTIC BUCKLING GRADIENT FOR A CONTINUOUSLY STRAIN HARDENING MATERIAL

In Figure 2 is shown as a general case a monosymmetric cross section with buckling presumed to take place in the plane of symmetry. The uniform stress level at the tangent modulus load $\sigma_t = P_T/A$ is shown and the distribution of the first increment of stress for an infinitesimal increase in load is shown by the triangular area. $\Delta\phi_o$ represents the intensity of unit curvature at the mid-length of the column during the first increment of column buckling. At the buckling load the resistance to bending will be constant along the column and the initial buckled shape therefore will be a half-sine wave. Hence, at the column mid-length the initial increment of unit curvature will be

$$\Delta\phi_o = \Delta\delta_o \frac{\pi^2}{L^2} \tag{2}$$

For the stress distribution in Figure 2 the angle corresponding to the increment in compressive stress is shown opposite in sense to the angle representing the curvature intensity $\Delta\phi_o$ and the concave side of the buckled column would be at the left. Drawn this way, the "arrow" formed by the stress increment points in the buckling direction. The initial increment of load above the buckling load is:

$$\Delta P = E_T \, (\Delta\phi_o) \int_A x \, d \, A$$

or, introducing Equation (2)

$$\left| \frac{\Delta P}{\Delta\delta_o} \right|_{\delta \to o} = \left[\frac{\pi^2 E_T I}{L^2} \right] \left(\frac{1}{I} \right) \int_A x \, d \, A = \frac{P_T A x_o}{I} \tag{3}$$

or, alternatively

$$\left|\frac{\Delta P}{\Delta \delta_o}\right|_{\delta \to o} = \frac{P_T x_o}{r^2} \qquad (4)$$

where r is the radius of gyration of the cross section. For the doubly symmetric case

$$\left|\frac{\Delta P}{\Delta \delta_o}\right|_{\delta \to o} = \frac{P_T h}{2r^2} \qquad (5)$$

which is as introduced by Duberg and Wilder.(2)

A detailed study of the change in stress distribution and progressive inward movement of strain regression above the tangent modulus load presented in an earlier paper by the author.(4)

BUCKLING OF AN ALUMINUM ALLOY TEE SECTION

As a by-product of the calculation of the inelastic buckling gradient of an unsymmetrical section, consider now as a specific illustrative example the behavior of a structural aluminum alloy tee section, 8 inches x 6 inches, as shown in Figure 3 for which the properties* are as follows:

I_{xx} = 22.93 in.^{4} A = 9.56 in.^{2}

I_{yy} = 36.76 in.^{4} I_p = 59.69 in.^{4}

r_{xx} = 1.55 in.   J = 1.95 in.^{4}

The minimum elastic torsional buckling stress is

$$\sigma_c = \frac{JG}{I_p} = \frac{1.95 \times 3800}{59.69} = 124 \text{ ksi}$$

Tees are apt to be weak with regard to torsional buckling but since 124 ksi is far greater than the yield strength it may be presumed that buckling will not be initiated in the torsional mode.

Other dimensions are as indicated in Figure 3. By Equation (4), if the buckling deflection is to the right, the inelastic buckling gradient will be 1.981 P_T, but if the buckling is to the left the gradient would be 0.516 P_T. These gradients are shown in Figure 3 for the particular length L = 62 in., for which L/r_{xx} = 40. The following question arises: If the initial imperfections in the column are such that there is an equal tendency for buckling in either direction, would buckling be to the left, corresponding to the smaller buckling gradient? If the answer is yes, the most likely buckling would cause increasing compression in the web and regression of compression in the flange, as shown in Figure 3.

* from the "Alcoa Structural Handbook."

tangent modulus load of 328.1 kips, it is assumed that E_t remains at 5636 ksi, the corresponding "false reduced modulus load" for buckling to the left would be 366.4 kips and the actual tangent modulus of the material at the corresponding stress is only 1753 ksi. For buckling to the right, the false reduced modulus load would be 506.4 kips, which is more than 20 percent above the ultimate strength of the alloy 6061-T6 at which the tangent modulus would have reduced to zero. Thus it is to be noted that in some cases involving inelastic instability, the only realistic approach to a solution is an incremental one in which the change in E_t is considered as a function of varying increments of compressive strain both across and along a member.

Again referring to Figure 3, it is possible to suggest qualitatively possible modes of postbuckling behavior. If buckling were initiated to the right, corresponding the the larger buckling gradient, the flange of the tee section would be subjected to increasing compressive stress with a corresponding diminution of tangent modulus, E_T, whereas larger and larger portions of stem would be subject to strain regression at the elastic modulus rate. Thus the steam of the tee, being governed over an increasing area by the elastic modulus, would provide an increasing contribution to bending stiffness about the xx axis. Contrariwise, the flange of the tee section, under increasing compressive stress, will have a lower and lower tangent modulus as bending proceeds. Since the moment of inertia about the yy axis is less than twice that about the xx axis, it may be predicted that before reaching the maximum load for buckling in the plane of the web, the tee section will reach a new critical load for buckling normal to the plane of the web, or a still lower load involving interaction with torsional buckling. In either event, as soon as biaxial bending is initiated, the section will experience torsional moments which will add the consideration of twist to the problem. The only simplifying aspect of the problem would be the fact that in the tee section the internal torsional resistance could be presumed to be solely due to St. Venant torsion (together with the inelastic modifications thereof) and warping torsion resistance could be neglected. For the doubly symmetric wide flange shape, the direct problem of biaxial bending, including the effects of warping torsion in the inelastic range, has been solved by an incremental procedure, using the digital computer, by Birnstiel and Michalos.[6]

Now, suppose that buckling does commence in the most probable direction, to the left, for the smaller inelastic buckling gradient. Strain regression now will occur over an increasing amount of the flange area, whereas the stem will be under increasing compressive stress with corresponding decrease in tangent modulus. Thus the relative bending stiffnesses of the tee section for bending will be such as to increase the resistance against buckling out of the plane of the web and increase the tendency to continue bending in the plane of the web as originally initiated. Thus, if the tee section buckles in the most likely direction it will be increasingly stable within that plane and bending will proceed out to the maximum load capacity with no torsion and no tendency to buckle out of the plane of the web. (It is here assumed that the lowered E_T of the web is not sufficient to induce inelastic lateral-torsional buckling, which will be true of the tee sections similar to the one used herein for illustrative purposes.)

BUCKLING OF STEEL COLUMNS WITH RESIDUAL STRESS

Attention is now turned from the behavior of an aluminum alloy column to the case of the steel column in which there exists initial residual stresses that result either from welding or from the initial cooling of hot steel immediately after rolling. Although now well known, the essentials of this problem will be reviewed. The typical stress-strain diagram for steel is indicated in Figure 4, with near linearity between stress and strain up to the yield point, and in comparison, the dashed line shows the average stress-strain curve that will result if a short stub column is tested in compression. The stress distribution in the column is shown at three successive stages, first at zero applied load, second at a load for which the average applied stress plus the compressive residual stress exactly equals the yield point (which therefore corresponds to the proportional limit on the average stress-strain diagram) and finally, at an intermediate point for which two segments of the cross section have gone into the plastic range.

The research on this aspect of column behavior was commenced in 1949 through the Committee on Research of the Column Research Council, resulting in a general study of the residual stress effect by Osgood(7) and the initial Lehigh study(8) which was followed by much additional research that is still under way at this time as summarized by Beedle and Tall.(9) In Reference 8 it was shown that the initial buckling load could be determined by use of the Euler stress formula simply by substituting the moment of inertia of the portion of the cross section still elastic at any given stress level, I_e, in place of I. This substitution, of course, involves the same basic reasoning that underlies the Shanley interpretation of the tangent modulus load; i.e., if bending commences with a simultaneous increase in load it can do so without any strain reversal and thus the Shanley-type bifurcation is attained. However, should any finite increment of load be added, there would have to be some strain regression at the extreme fibers on the convex side of the buckled column, just as in the case of the continuously strain hardening material.

Referring to the average stress-strain curve in a stub column (Figure 4), above the proportional limit, σ_p, the apparent tangent modulus from the stub column test, E_t, is related to the elastic modulus E in the proportion A_e/A. Hence, the term τ which modifies E to give an apparent E_t is equal to A_e/A. This is readily shown as follows for $\sigma_P < \sigma_A < \sigma_Y$

$$dP = A_e E \, d\epsilon = A d\sigma_{avg}$$

$$E_t = \frac{d\sigma_{avg}}{d\epsilon} = E \left(\frac{A_e}{A}\right) = E \tau \qquad (6)$$

The buckling load for a steel column, for a bi-symmetric cross section containing a bi-symmetrical distribution of residual stress, is given(8) by

$$P_t = \frac{\pi^2 E I_e}{L^2} \qquad (7)$$

Alternatively, since

$$\frac{I_e}{I} = f \left(\frac{A_e}{A}\right) = f(\tau) \qquad (8)$$

we may express Equation (7) as follows

$$\sigma_t = \frac{P_t}{A} = \frac{\pi^2 E}{(\frac{L}{r})^2} f(\tau) \tag{9}$$

Equation (9) is convenient in obtaining directly a steel column strength curve in the inelastic range if the results of a stub column test are available. Only the pattern of residual stress need be known, its actual distribution can be parabolic, linear or any other distribution, provided it will be such that the plastic zones will develop as shown by the shaded areas in Figure 5.

Examples of the calculation of $\frac{I_e}{I} = f(\tau)$ for Equation (9) are shown below for different patterns of residual stress.

The first two of these were given in the paper by Yang, Beedle, and the author(8) and the third was developed by A. Nitta in 1960.(10,11) Reference 11 provides a very complete study of case 3, including formation of initial residual stress, triaxial effects, etc. The additional function of τ (Case 4) for the less usual situation where the residual stress is tension at the location of maximum bending strain is introduced here to illustrate the effect of such a distribution on the inelastic buckling gradient.

If stub column test results in the form of a plot of σ_{avg} vs. strain are available, any point on the inelastic portion of the column strength curve (plot of maximum column strength against slenderness ratio) may be calculated readily as follows:

(1) For any value of σ_{avg}, determine τ .

(2) For each value of σ_{avg}, calculate $\frac{L}{r} = \pi\sqrt{\frac{Ef(\tau)}{\sigma_{avg}}}$

If no stub column test is available, for an assumed residual stress distribution,

(1) Choose various values of $\tau = \frac{A_e}{A}$ between zero and one.

(2) Determine, for each τ

$$P = A_P \sigma_y + \int_{A_e} \sigma \, d A$$

hence

$$\sigma_{avg} = \frac{(A - A_e) \sigma_y}{A} + \frac{1}{A} \int_{A_e} \sigma \, d A \qquad (10)$$

and, as before

$$\frac{L}{r} = \pi \sqrt{\frac{Ef(\tau)}{\sigma_{avg}}}$$

Examples of four column strength curves, for assumed parabolic residual stress distribution, with maximum compressive residual stress equal to 0.3 σ_y, are plotted in terms of dimensionless parameters in Figure 6 for the four cases tabulated in Figure 5. Cases 2 and 3 yield the same curve. Note that it is not necessary to actually calculate the ratio I_e/I since it is given by the function $f(\tau)$ appropriate to the particular cross section and residual stress pattern that is involved. In the fourth case, where it is assumed that in some way tension residual stress can be produced in the outer fibers, the column strength curve reaches its closest approach to the upper limit of strength determined either by the yield point or the Euler buckling stress, whichever is the lesser. However, it is to be noted that in the case of steel columns, residual stress, no matter how distributed, has an adverse effect on the buckling strength.

BUCKLING GRADIENT FOR A STEEL COLUMN WITH RESIDUAL STRESS

As in the case of the continuously strain hardening material, a simple formula for the initial slope of the load deflection curve can be determined at the instant of buckling for the steel column containing residual stress. The discussion is limited to bi-symmetric sections and bi-symmetric residual stress distributions. Two cases will be considered:

Case (1) Compressive residual stress at extreme fibers in plane of bending (See Figure 7)

The shaded region represents the portion that is fully plastic and at the yield stress level, the unshaded region is still elastic, and the solid region represents the change in stress during an infinitesimal initial bending at the buckling load.

The initial increment of load for a differential initial mid-length curvature $(\Delta\phi_o)$ is

$$\Delta P = \int_{A_e} x(\Delta\phi_o)\ E\ d\ A = \frac{\pi^2 E(\Delta\delta_o)A_e d}{2L^2} \qquad (11)$$

From Equation (11) the buckling gradient may be expressed in very simple form in terms of the Euler load P_e that would be developed by the same column if it were elastic.

$$\frac{\Delta P}{\Delta\delta_o} = \frac{P_e \tau d}{2r^2} \qquad (12)$$

where τ is as defined by Equation (6).

Case (2) Tensile residual stress at extreme fibers in plane of bending

Since the external fibers are still elastic at the buckling load, and since the initial increment of load and initial increment of internal resisting moment will be a minimum, there will be strain regression during the initial infinitesimal increment of load, as indicated in Figure 8.

Equation (11) will not apply since the apex of the triangles determining the stress distribution will be at the juncture of the plastic and elastic regions. Let x_e now represent the distance from the extreme fiber in to the extreme range of the plastic region. The initial increment of load will be

$$\Delta P = \int_{A_e} (x - x_e)(\Delta\phi_o)E \, d\, A \tag{13}$$

$$\Delta P = \int_{A_e} x(\Delta\phi_o)E \, d\, A - \int_{A_e} x_e(\Delta\phi_o)E \, d\, A \tag{14}$$

$$\Delta P = \frac{\pi^2 E(\Delta\delta_o)A_e}{L^2}\left[\frac{d}{2} - x_e\right] \tag{15}$$

From which the buckling gradient is determined

$$\frac{\Delta P}{\Delta\delta_o} = \frac{P_{e\tau}}{r^2}\left[\frac{d}{2} - x_e\right] \tag{16}$$

If $x_e = \frac{d}{2}$ (elastic buckling limit), the buckling gradient is zero, as for Euler buckling. If $x_e = 0$, Equation (16) becomes the same as Equation (12). Such a limit takes on physical meaning for the initial residual stress pattern indicated by Figure 7b.

SUMMARY

(1) The "inelastic buckling gradient" is defined in relation to the behavior of a geometrically perfect column, considering both the continuously strain-hardening material and structural steel in the presence of residual stress.

(2) For a material that is continuously strain-hardening and for a column of monosymmetric cross section, buckling in the plane of symmetry may be initiated in either direction, but the inelastic buckling gradient will have different values for buckling in the alternative directions.

(3) In the case of the tee section buckling with the flange on the convex side (with the lesser buckling gradient) will become increasingly stable with respect to lateral buckling out of the plane of symmetry, provided the section is one for which inelastic lateral-torsional buckling does not develop. Contrarily, buckling with the flange on the concave side, will lead to inelastic lateral and torsional instability before reaching the maximum load that would be realized if lateral and torsional support were present.

(4) Formulas for the inelastic buckling gradient of structural steel columns with initial residual stresses are developed, and are shown to differ, depending on whether compressive or tensile residual stress is present initially at the extreme fiber.

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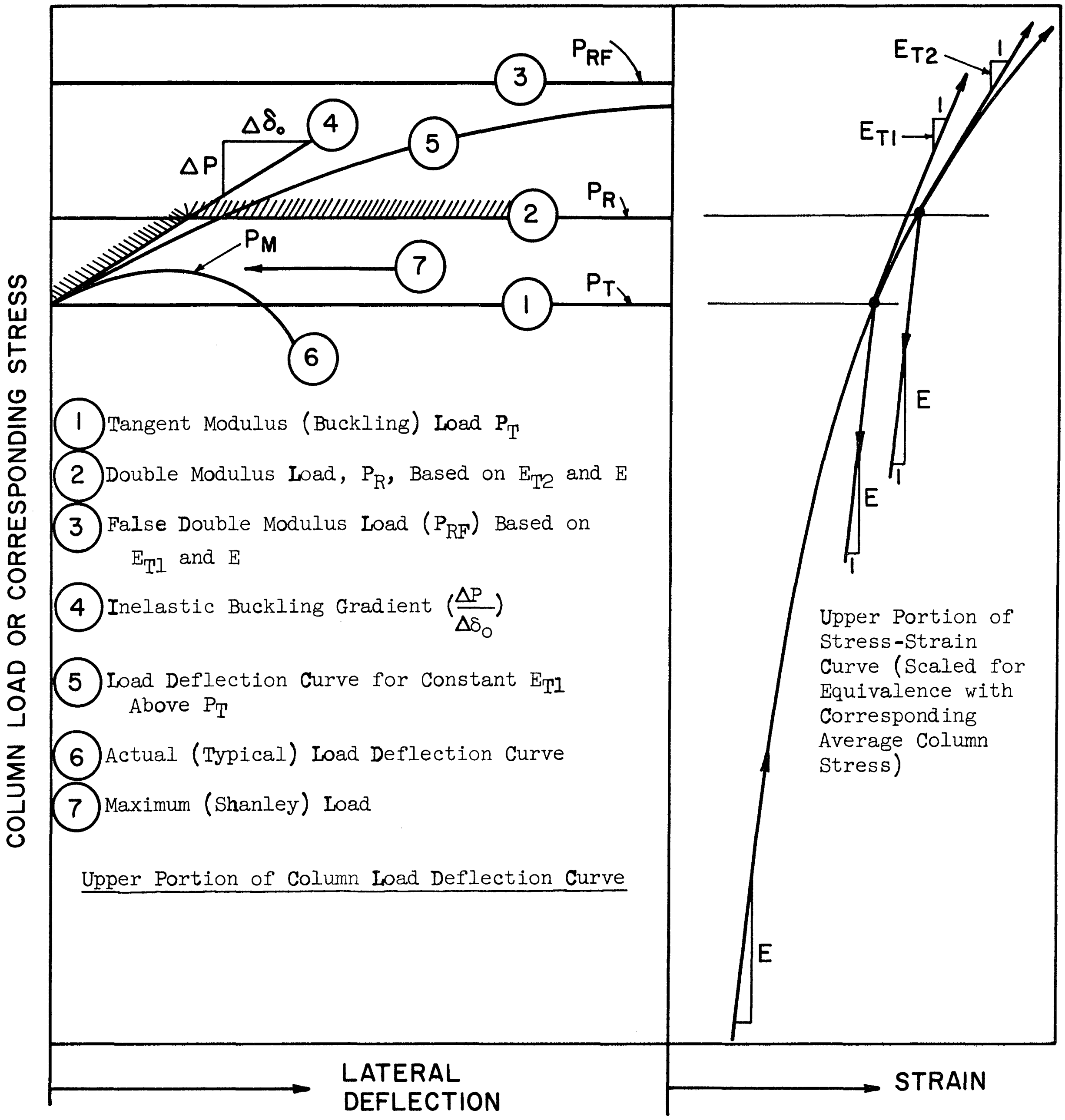


Figure 1. Concepts of Behavior At and Above the Tangent Modulus Load, Illustrated for a Geometrically Perfect Column At a Particular Slenderness Ratio.

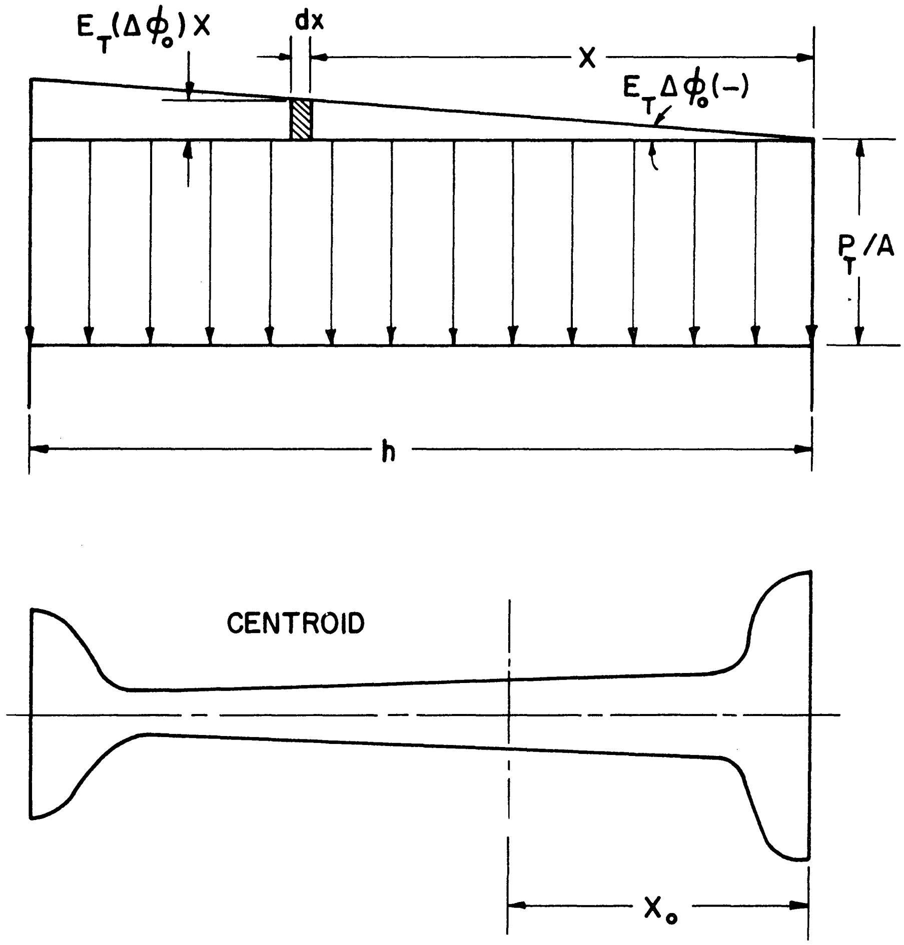


Figure 2

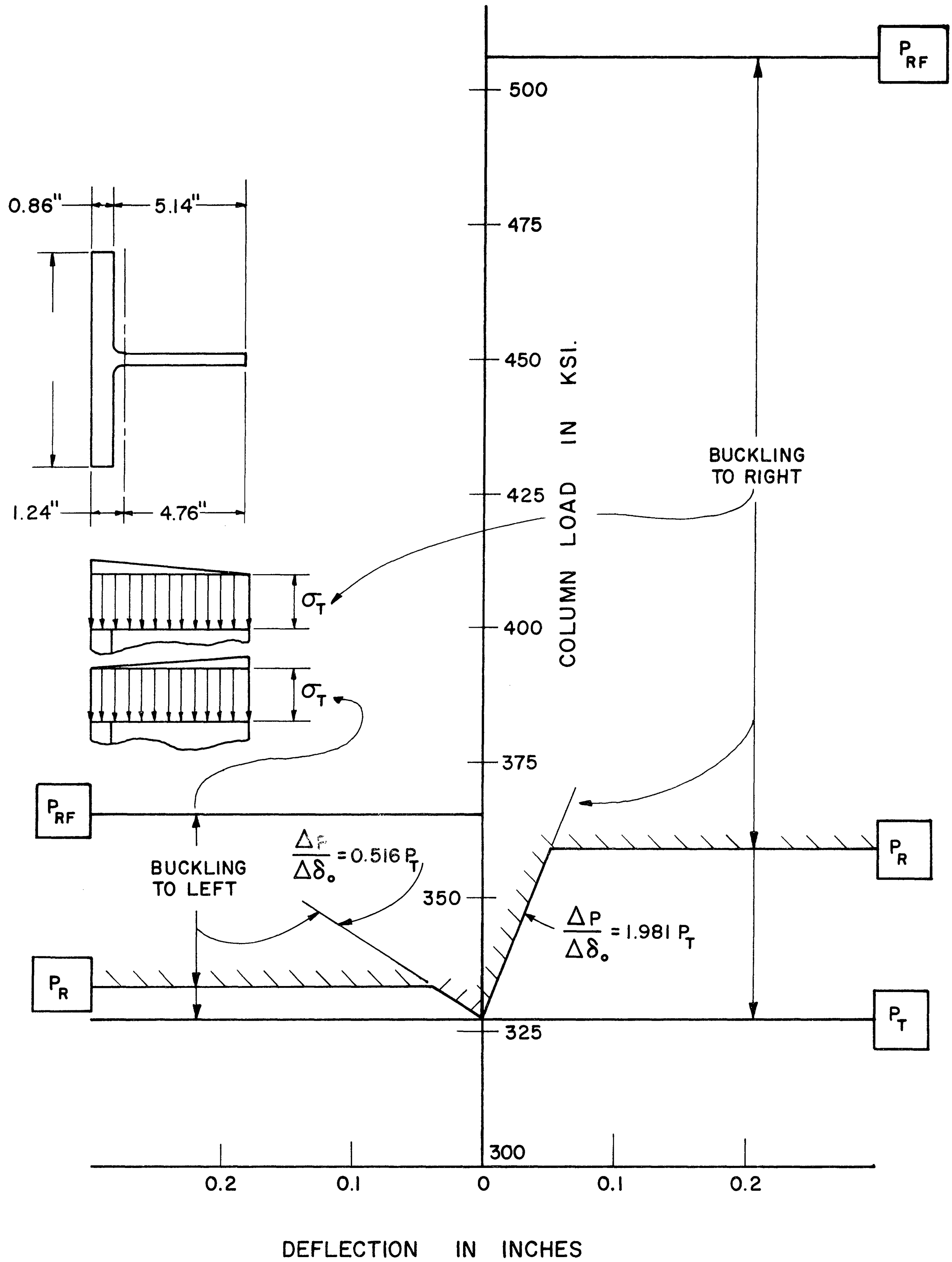


Figure 3. Tee Section Buckling - Illustrative Case.

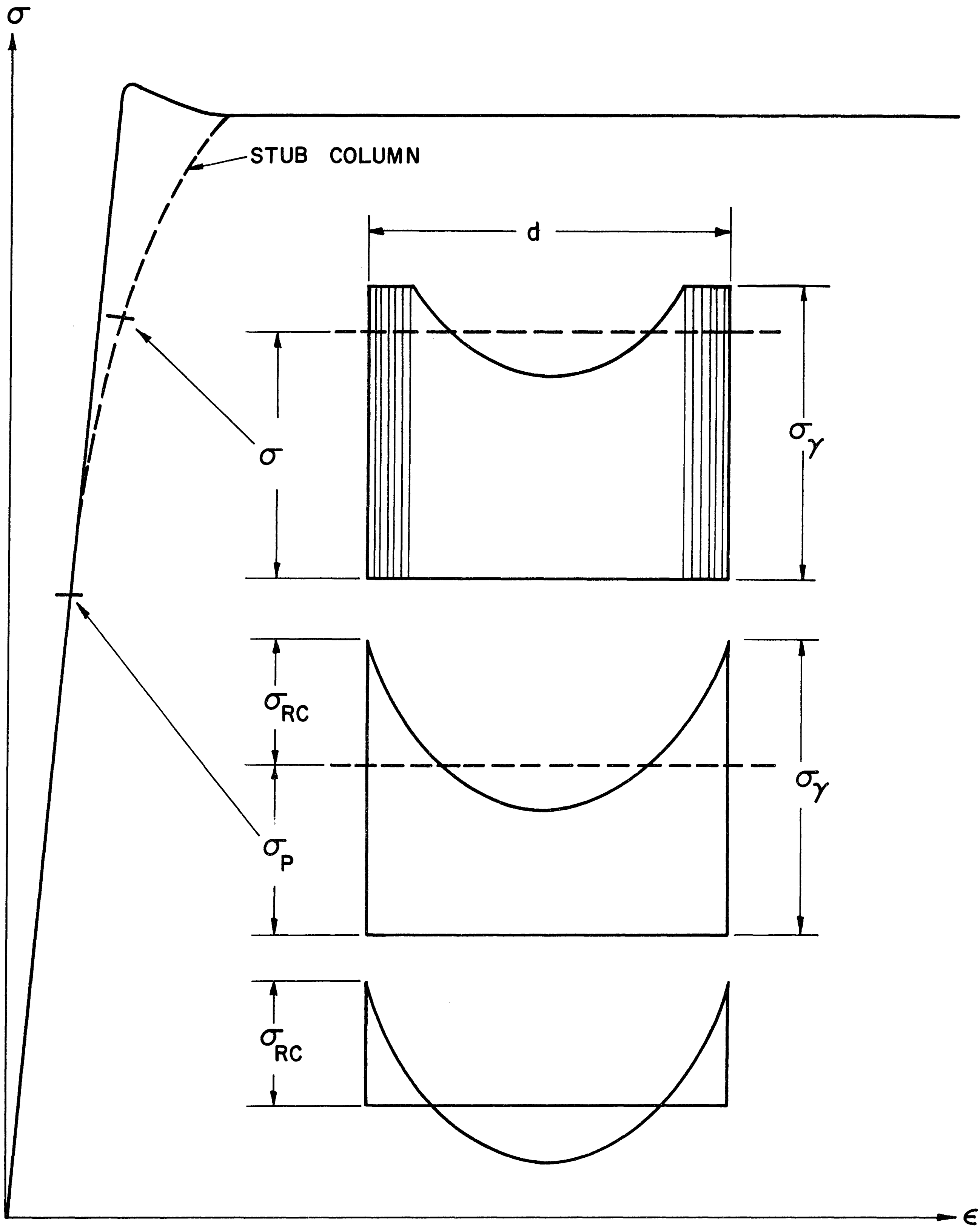
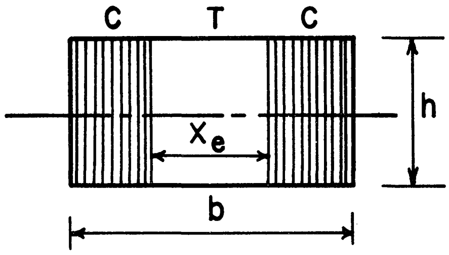


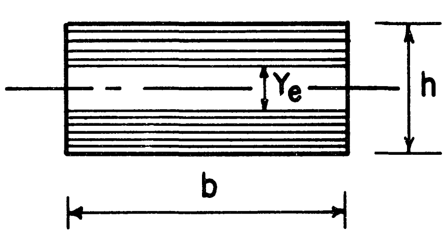
Figure 4

(1)



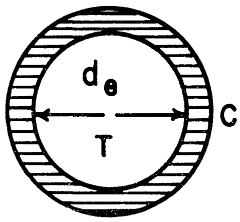
$$\frac{I_e}{I} = \frac{\frac{x_e h^3}{12}}{\frac{bh^3}{12}} = \frac{A_e}{A} = \tau$$

(2)



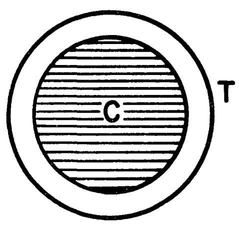
$$\frac{I_e}{I} = \frac{\frac{bY_e^3}{12}}{\frac{bh^3}{12}} = \frac{A_e^3}{A^3} = \tau^3$$

(3)



$$\frac{I_e}{I} = \frac{\frac{\pi d_e^4}{64}}{\frac{\pi d^4}{64}} = \frac{A_e^2}{A^2} = \tau^2$$

(4)



(Calculations omitted)

$$\frac{I_e}{I} = 2\tau - \tau^2$$

Figure 5

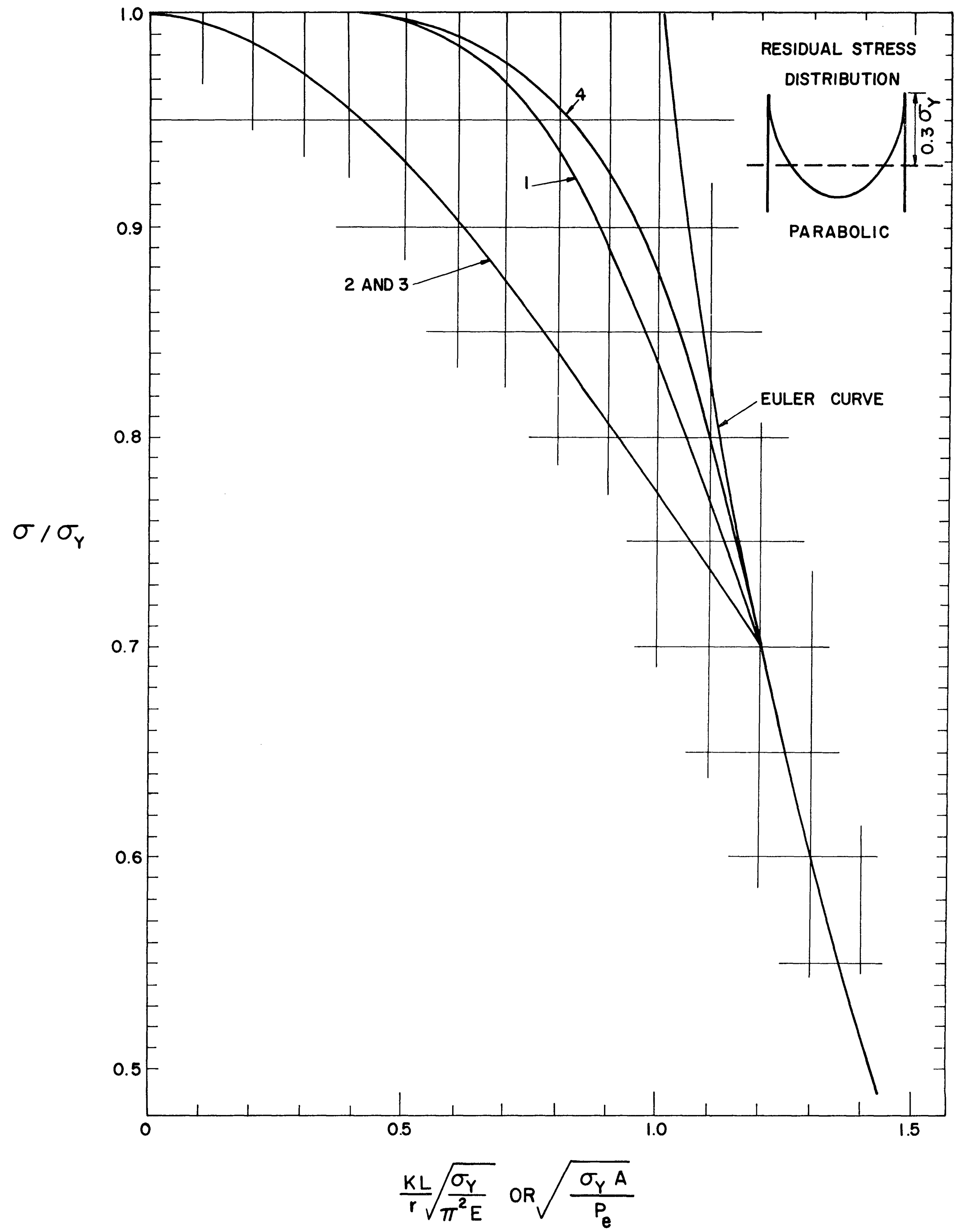


Figure 6. Column Strength Curves for Four Patterns of Residual Stress as Shown in Figure 4.

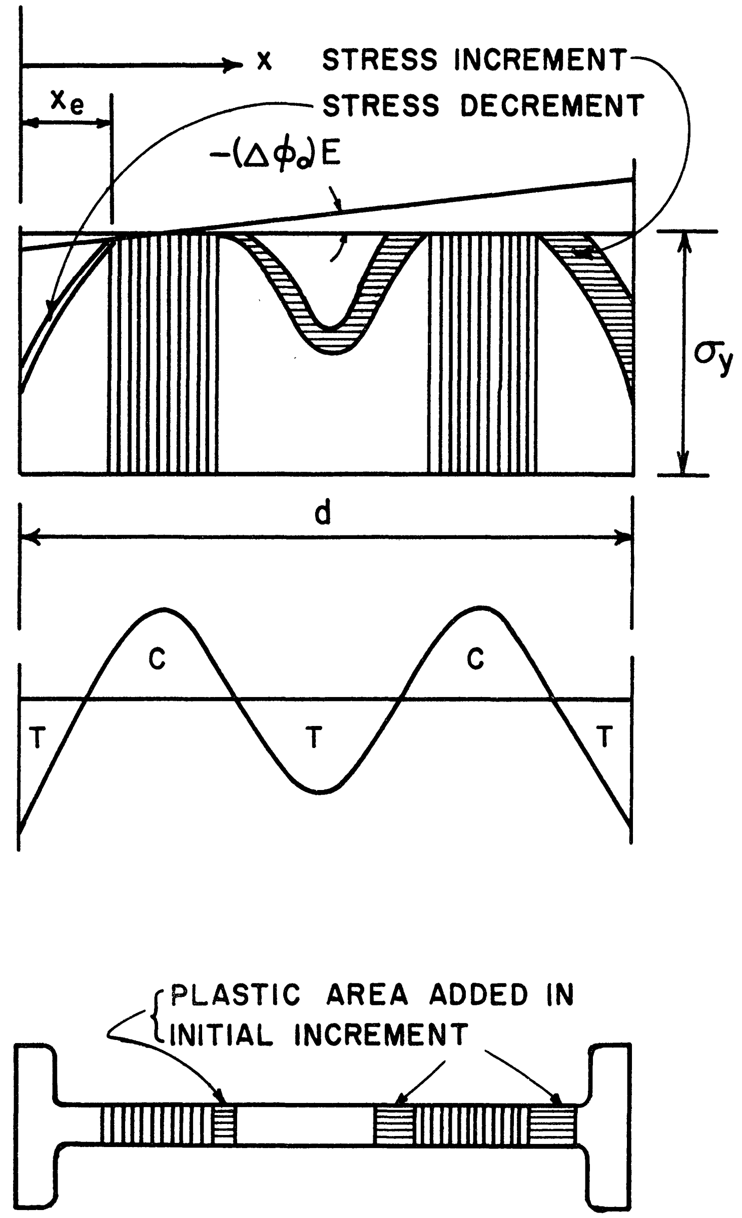

x
STRESS INCREMENT
STRESS DECREMENT
x_e
$-(\Delta\phi_o)E$
σ_y
d
C
C
T
T
T
PLASTIC AREA ADDED IN
INITIAL INCREMENT

Figure 8