$$\hat{\mathbf{D}}_{11}^{*}\hat{\mathbf{C}}_1 = (\hat{\mathbf{C}}_1^{*}\hat{\mathbf{D}}_{11})^T \tag{A.19}$$

$$\hat{\mathbf{D}}_{11}^{*}\hat{\mathbf{D}}_{12} = N^{\frac{1}{2}}W^{\dagger\frac{1}{2}}\mathbf{B}_1\mathbf{D}_{11}^{*}(I - \mathbf{D}_{11}\mathbf{D}_{11}^{*})^{-1}\mathbf{D}_{12} \tag{A.20}$$

$$\hat{\mathbf{D}}_{12}^{*}\hat{\mathbf{D}}_{11} = (\hat{\mathbf{D}}_{11}^{*}\hat{\mathbf{D}}_{12})^T \tag{A.21}$$

These equations lead to (A.15). The state-space computations of W_0, W, V_{cc}, V_{cd}, V_{dd}, M_1, M_2, N, and M based on three exponentials (A.1), (A.2) and (A.3) can be verified by the same technique as in [1]. □

Remark A.2:

1) We need three exponentiations of sizes $n + m_2$, $2n$, and $2(n + m_2)$, where n and m_2 are the dimensions of the state $x(t)$ and the control input $u(t)$, respectively.
2) If we consider a problem $J(K) < \gamma$ instead of $J(K) < 1$, only $\mathbf{C}_1$ and $\mathbf{D}_{12}$ should be replaced by $\mathbf{C}_1/\gamma$ and $\mathbf{D}_{12}/\gamma$ in the above formulas. Hence, recalculation is required only for Γ in the γ-iteration for the optimization, since Φ and Ψ are independent of γ.

References

[1] B. Bamieh and J. B. Pearson, "A general framework for linear periodic systems with applications to H_∞ sampled-data control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 418–435, 1992.

[2] R. W. Brockett, *Finite Dimensional Linear Systems.* New York: Wiley, 1970.

[3] T. C. Chen and B. A. Francis, "$\mathcal{H}_2$-optimal sampled-data control," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 387–397, 1991.

[4] T. C. Chen and B. A. Francis, "On the $\mathcal{L}_2$-induced norm of a sampled-data system," *Syst. Contr. Lett.*, vol. 15, pp. 211–219, 1990.

[5] T. C. Chen and B. A. Francis, "Sampled-data optimal design and robust stabilization," in *Proc. ACC*, 1991, pp. 2704–2709.

[6] G. Dullerud and K. Glover, "Robust stabilization of sampled-data systems to structured LTI perturbations," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1497–1508, 1993.

[7] B. A. Francis, *Lectures on $\mathcal{H}_\infty$ Control and Sampled-Data Systems.* CIME Lecture Notes, 1990.

[8] S. Hara and P. T. Kabamba, "Worst case analysis and design of sampled data control systems," in *Proc. 29th CDC*, 1990, pp. 202–203.

[9] S. Hara, M. Nakajima, and P. T. Kabamba, "Robust stabilization in digital control systems," in *Proc. MTNS-91*, Kobe, Japan, 1991.

[10] Y. Hayakawa, S. Amano, and S. Fujii, "On an equivalent discrete-time problem for the H_∞ problem of sampled-data systems," in *Proc. SICE DST*, 1991, pp. 99–104, (in Japanese).

[11] P. T. Kabamba and S. Hara, "Worst case analysis and design of sampled data control systems," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1337–1357, 1993.

[12] J. P. Keller and B. D. O. Anderson, "A new approach to the discretization of continuous-time controllers," in *Proc. ACC*, 1990, pp. 1127–1132.

[13] D. G. Luenburger, *Optimization by Vector Space Methods.* New York: Wiley, 1968.

[14] N. Sivashankar and P. P. Khargonekar, "Characterization of the $\mathcal{L}_2$-induced norm for linear systems with jumps with applications to sampled-data systems," Preprint, Dept. Elect. Eng., Univ. Michigan, 1991.

[15] ——, "Robust stability and performance analysis of sampled-data systems," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 58–69, 1993.

[16] W. Sun, K. M. Nagpal and P. P. Khargonekar, "$\mathcal{H}_\infty$ control and filtering for sampled-data systems," in *Proc. ACC*, 1991.

[17] H. T. Toivonen, "Sampled-data control of continuous-time systems with an $\mathcal{H}_\infty$ optimality criterion," *Automatica*, vol. 28, pp. 45–54, 1992.

[18] ——, "Sampled-data $\mathcal{H}_\infty$ optimal control of time-varying systems," *Automatica*, vol. 28, pp. 823–826, 1992.

[19] Y. Yamamoto, "New approach to sampled-data systems: a function space method," in *Proc. 29th CDC*, 1990, pp. 1882–1887.

Robust Strong Stabilization via Modified Popov Controller Synthesis

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Abstract—**In this paper we merge the parameter-dependent Lyapunov function framework used to construct robust Popov controllers with the *a priori* and *a posteriori* approaches given in [4], [5], [7], [8] for obtaining stable compensators. Specifically, we derive constructive sufficient conditions that yield robust Lyapunov and asymptotically stable stabilizing Popov dynamic compensators.**

I. Introduction

Many practical applications of robust feedback control involves constant real parameter uncertainty, whereas H_∞ theory guarantees robust stability against arbitrary time-varying uncertainty, thus entailing undue conservatism. In a recent series of papers [1]–[3] a parameter-dependent Lyapunov function framework was developed to address the problem of real parameter uncertainty. Since the uncertain parameters appear explicitly in the parameter-dependent Lyapunov functions, the ability of such a framework to guarantee robust stability with respect to arbitrary time-varying parameter variations is curtailed, thus reducing conservatism with respect to constant real parameter uncertainty. As an immediate application of the parameterized Lyapunov function framework, the authors in [1], [2] provide a generalization of the classical multivariable Popov criterion to the case of fully coupled linear uncertainty. These results were then used in conjunction with fixed-order optimization techniques to obtain Riccati characterizations of robust controllers.

In certain applications, the use of robust stable compensators greatly simplifies controller testing and implementation. The problem of synthesizing stable stabilizing controllers has been of interest for many years [10] and a variety of techniques have been proposed based on modification of existing synthesis methods to ensure stable compensation [4], [5], [7]–[9]. In particular, in [4], [5], [7]–[9], the authors guarantee suboptimal strong stabilization by modifying standard H_2 and H_2/H_∞ theory. Specifically, two approaches are proposed that guarantee strong stabilization, an *a priori* modification to H_2 theory (that is, prior to optimization) and an *a posteriori* modification to the standard H_2 design equations.

In this paper we merge the parameter-dependent Lyapunov function framework for designing robust Popov controllers with both the *a priori* and *a posteriori* approaches for strong stabilization to obtain robust strong stabilizing controllers. The results presented herein provide constructive sufficient conditions for robust stable compensators with robust H_2 performance bounds.

It is important to note that the conditions given in this paper for robust, strong stabilization are constructive and thus are sufficient but not necessary. The relative conservatism of the various proposed constructions is therefore problem dependent. Numerical techniques for

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implementing these conditions can be developed using the techniques applied in [4], [5] to the problem of (nominal) strong stabilization.

II. Robust Strong Stabilization

In this section we introduce the robust strong stabilization dynamic output feedback control problem. Specifically, we generalize the *a posteriori* approach for stable compensation [4], [5] to design robust stable full- and reduced-order Popov controllers. This problem involves a set $\mathcal{U} \subset \mathcal{R}^{n\times n}$ of uncertain perturbations ΔA of the nominal system matrix A.

Robust Strong Stabilization: Given the nth-order stabilizable and detectable plant

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + D_1 w(t), \qquad t \geq 0, \tag{1}$$
$$y(t) = Cx(t) + D_2 w(t) \tag{2}$$

where $u(t) \in \mathcal{R}^{n\times n}$, $w(t) \in \mathcal{R}^d$ is standard white noise, and $y(t) \in \mathcal{R}^l$, determine an n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{3}$$
$$u(t) = C_c x_c(t) \tag{4}$$

that satisfies the following design criteria:

i) the closed-loop system (1)–(4) is asymptotically stable for all $\Delta A \in \mathcal{U}$;
ii) the compensator dynamics matrix A_c is asymptotically stable; and
iii) the performance functional

$$J(A_c, B_c, C_c) = \sup_{\Delta A\in\mathcal{U}} \limsup_{t\to\infty} \frac{1}{t}\mathcal{E}\left\{\int_0^t [x^T(s)R_1 x(s) + u^T(s)R_2 u(s)]ds\right\} \tag{5}$$

is minimized.

Next, we assign explicit structure to the set $\mathcal{U}$. Specifically, the uncertainty set $\mathcal{U}$ is defined by

$$\mathcal{U} \triangleq \{\Delta A \in \mathcal{R}^{n\times n}: \Delta A = B_0 F C_0, \ F \in \mathcal{F}\} \tag{6}$$

where $\mathcal{F}$ satisfies

$$\mathcal{F} \subseteq \hat{\mathcal{F}} \triangleq \{F \in \mathcal{R}^{m_0\times m_0}: 0 \leq F \leq M\} \tag{7}$$

and $B_0 \in \mathcal{R}^{n\times m_0}$, $C_0 \in \mathcal{R}^{m_0\times n}$ are fixed matrices denoting the structure of the uncertainty, $F \in \mathcal{R}^{m_0\times m_0}$ is an uncertain symmetric matrix, and $M \in \mathcal{R}^{m_0\times m_0}$ is a given positive definite matrix. We restrict our attention to symmetric uncertainties F for convenience only. More general uncertainty sets as in [1], [2] can also be considered.

For each uncertain variation $\Delta A \in \mathcal{U}$, the closed-loop system (1)–(4) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}w(t), \qquad t \geq 0 \tag{8}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t)\end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_c C & A_c\end{bmatrix},$$
$$\Delta\tilde{A} \triangleq \begin{bmatrix} \Delta A & 0_{n\times n_c} \\ 0_{n_c\times n} & 0_{n_c\times n_c}\end{bmatrix}$$

and where the closed-loop disturbance $\tilde{D}w(t)$ has intensity $\tilde{V} \triangleq \tilde{D}\tilde{D}^T$, where

$$\tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2\end{bmatrix}, \quad \tilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T\end{bmatrix},$$
$$V_1 = D_1 D_1^T, \quad V_2 = D_2 D_2^T$$

and, for convenience, $V_{12} = D_1 D_2^T = 0$. The closed-loop system uncertainty $\Delta\tilde{A}$ has the form

$$\Delta\tilde{A} = \tilde{B}_0 F \tilde{C}_0 \tag{9}$$

where

$$\tilde{B}_0 \triangleq \begin{bmatrix} B_0 \\ 0_{n_c\times m_0}\end{bmatrix}, \quad \tilde{C}_0 \triangleq [C_0 \quad 0_{m_0\times n_c}].$$

Finally, if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$ for a given compensator (A_c, B_c, C_c), then it follows from Proposition 2.1 of [2] that the performance measure (5) is given by

$$J(A_c, B_c, C_c) = \sup_{\Delta A\in\mathcal{U}} \operatorname{tr} \tilde{P}_{\Delta\tilde{A}}\tilde{V} \tag{10}$$

where $\tilde{P}_{\Delta\tilde{A}}$ satisfies the $\tilde{n}\times\tilde{n}$ ($\tilde{n} \triangleq n + n_c$) Lyapunov equation

$$0 = (\tilde{A} + \Delta\tilde{A})^T\tilde{P}_{\Delta\tilde{A}} + \tilde{P}_{\Delta\tilde{A}}(\tilde{A} + \Delta\tilde{A}) + \tilde{R} \tag{11}$$

where

$$\tilde{E} = [E_1 \quad E_2 C_c], \quad \tilde{R} = \tilde{E}^T\tilde{E} = \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c\end{bmatrix}$$

and, for convenience, $R_{12} = E_1^T E_2 = 0$.

Next, define the set of compatible Popov multipliers $\mathcal{N}$ by

$$\mathcal{N} \triangleq \{N \in \mathcal{R}^{m_0\times m_0}: FN = N^T F \geq 0, \ F \in \mathcal{F}\}.$$

Now, as shown in [1] and [2], replacing the Lyapunov equation (11) with

$$0 = \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + (\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\tilde{P})^T[(M^{-1} - N\tilde{C}_0\tilde{B}_0) + (M^{-1} - N\tilde{C}_0\tilde{B}_0)^T]^{-1}(\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\tilde{P}) + \tilde{R} \tag{12}$$

and minimizing an upper bound for the H_2 cost given by

$$\mathcal{J}(A_c, B_c, C_c) = \operatorname{tr}(\tilde{P} + \tilde{C}_0^T\mu\tilde{C}_0)\tilde{V} \tag{13}$$

where μ satisfies $FN \leq \mu$, we can obtain sufficient conditions that characterize fixed-order dynamic output feedback controllers guaranteeing robust stability and performance. For convenience in stating this result, define

$$\overline{\Sigma} \triangleq C^T V_2^{-1} C, \ R_0 \triangleq (M^{-1} - NC_0B_0) + (M^{-1} - NC_0B_0)^T,$$
$$\hat{C} \triangleq C_0 + NC_0A, \ R_{2a} \triangleq R_2 + B^T C_0^T N^T R_0^{-1} NC_0 B,$$
$$P_a \triangleq B^T P + B^T C_0^T N^T R_0^{-1}(\hat{C} + B_0^T P),$$
$$A_P \triangleq A + B_0 R_0^{-1}\hat{C}, \ A_{\hat{P}} \triangleq A_P - Q\overline{\Sigma} + B_0 R_0^{-1} B_0^T P,$$
$$A_{\hat{Q}} \triangleq A_P + B_0 R_0^{-1} B_0^T P - (I + B_0 R_0^{-1} NC_0) B R_{2a}^{-1} P_a.$$

Theorem 2.1 [2]: Let $n_c \leq n$, assume $R_0 > 0$ and let $N \in \mathcal{N}$. Furthermore, suppose there exist $n\times n$ nonnegative-definite matrices $P, Q, \hat{P}, \hat{Q}$ satisfying

$$0 = A_P^T P + PA_P + R_1 + \hat{C}^T R_0^{-1}\hat{C} + PB_0R_0^{-1}B_0^T P - P_a^T R_{2a}^{-1} P_a + \tau_\perp^T P_a^T R_{2a}^{-1} P_a \tau_\perp, \tag{14}$$

$$0 = [A_P + B_0R_0^{-1}B_0^T(P + \hat{P})]Q + Q[A_P + B_0R_0^{-1}B_0^T(P + \hat{P})]^T + V_1 - Q\overline{\Sigma}Q + \tau_\perp Q\overline{\Sigma}Q\tau_\perp^T, \tag{15}$$

$$0 = A_{\hat{P}}^T\hat{P} + \hat{P}A_{\hat{P}} + \hat{P}B_0R_0^{-1}B_0^T\hat{P} + P_a^T R_{2a}^{-1}P_a - \tau_\perp^T P_a^T R_{2a}^{-1} P_a\tau_\perp, \tag{16}$$

$$0 = A_{\hat{Q}}\hat{Q} + \hat{Q}A_{\hat{Q}}^T + Q\overline{\Sigma}Q - \tau_\perp Q\overline{\Sigma}Q\tau_\perp^T, \tag{17}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c, \tag{18}$$

$$\hat{Q}\hat{P} = G^T M\Gamma, \ \Gamma G^T = I_{n_c}, \ M \in \mathcal{R}^{n_c\times n_c}, \tag{19}$$

$$\tau \triangleq G^T\Gamma, \quad \tau_\perp \triangleq I_n - \tau \tag{20}$$

and let

$$A_c = \Gamma(A_{\hat{Q}} - Q\overline{\Sigma})G^T, \tag{21}$$

$$B_c = \Gamma Q C^T V_2^{-1}, \tag{22}$$

$$C_c = -R_{2a}^{-1}P_a G^T. \tag{23}$$

Then $(\tilde{E}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case the performance of the closed-loop system satisfies the bound

$$J(A_c, B_c, C_c) \le \operatorname{tr}[(P + \hat{P})V_1 + \hat{P}Q\overline{\Sigma}Q + C_0^T\mu C_0 V_1]. \tag{24}$$

Next, we use the *a posteriori* approach developed in [4] to modify the synthesis equations given in Theorem 2.1 to construct stable Popov controllers.

Theorem 2.2: Let $n_c \le n$, assume $R_0 > 0$ and let $N \in \mathcal{N}$. Let $\Omega(P, \hat{P}) \ge 0$ satisfy

$$A_P^T P + PA_P + [(I + B_0R_0^{-1}NC_0)BR_{2a}^{-1}P_a]^T\hat{P} + \hat{P}(I + B_0R_0^{-1}NC_0)BR_{2a}^{-1}P_a + \Omega(P, \hat{P}) \ge 0 \tag{25}$$

for all nonnegative-definite P, $\hat{P}$. Furthermore, suppose there exist $n \times n$ nonnegative-definite matrices $P, Q, \hat{P}, \hat{Q}$ satisfying

$$0 = A_P^T P + PA_P + R_1 + \Omega(P, \hat{P}) + \hat{C}^T R_0^{-1}\hat{C} + PB_0R_0^{-1}B_0^T P - P_a^T R_{2a}^{-1}P_a + \tau_\perp^T P_a^T R_{2a}^{-1}P_a\tau_\perp, \tag{26}$$

$$0 = [A_P + B_0R_0^{-1}B_0^T(P + \hat{P})]Q + Q[A_P + B_0R_0^{-1}B_0^T(P + \hat{P})]^T + V_1 - Q\overline{\Sigma}Q + \tau_\perp Q\overline{\Sigma}Q\tau_\perp^T, \tag{27}$$

$$0 = A_{\hat{P}}^T\hat{P} + \hat{P}A_{\hat{P}} + \hat{P}B_0R_0^{-1}B_0^T\hat{P} + P_a^T R_{2a}^{-1}P_a - \tau_\perp^T P_a^T R_{2a}^{-1}P_a\tau_\perp, \tag{28}$$

$$0 = A_{\hat{Q}}\hat{Q} + \hat{Q}A_{\hat{Q}}^T + Q\overline{\Sigma}Q - \tau_\perp Q\overline{\Sigma}Q\tau_\perp^T, \tag{29}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c, \tag{30}$$

$$\hat{Q}\hat{P} = G^T M\Gamma, \quad \Gamma G^T = I_{n_c}, \quad M \in \mathcal{R}^{n_c\times n_c}, \tag{31}$$

$$\tau \triangleq G^T\Gamma, \quad \tau_\perp \triangleq I_n - \tau \tag{32}$$

and let

$$A_c = \Gamma(A_{\hat{Q}} - Q\overline{\Sigma})G^T, \tag{33}$$

$$B_c = \Gamma Q C^T V_2^{-1}, \tag{34}$$

$$C_c = -R_{2a}^{-1}P_a G^T. \tag{35}$$

Then the following results hold.

i) $(\tilde{E}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$.
ii) A_c is Lyapunov stable.
iii) If $R_1 > 0$, then A_c is asymptotically stable.
iv) The performance of the closed-loop system satisfies the bound

$$J(A_c, B_c, C_c) \le \operatorname{tr}[(P + \hat{P})V_1 + \hat{P}Q\overline{\Sigma}Q + C_0^T\mu C_0 V_1]. \tag{36}$$

Proof: Note that the H_2 performance bound and the closed-loop robust stability of $\tilde{A} + \Delta\tilde{A}$ are direct consequences of theorem 8.1 of [2]. To prove ii), we note that adding (26) to (28) yields

$$0 = A_{\hat{P}}^T\hat{P} + \hat{P}A_{\hat{P}} + \hat{P}B_0R_0^{-1}B_0^T\hat{P} + A_P^T P + PA_P + R_1 + \Omega(P, \hat{P}) + \hat{C}^T R_0^{-1}\hat{C} + PB_0R_0^{-1}B^T P.$$

Using the fact that

$$A_{\hat{P}} = A_{\hat{Q}} - Q\overline{\Sigma} + (I + B_0R_0^{-1}NC_0)BR_{2a}^{-1}P_a$$

and $P_2 = G\hat{P}G^T > 0$ [6], it follows that

$$A_c^T P_2 + P_2A_c = -G[A_P^T P + PA_P + R_1 + \hat{C}^T R_0^{-1}\hat{C} + PB_0R_0^{-1}B_0^T P + \hat{P}B_0R_0^{-1}B_0^T\hat{P} + [(I + B_0R_0^{-1}NC_0)BR_{2a}^{-1}P_a]^T\hat{P} + \hat{P}(I + B_0R_0^{-1}NC_0)BR_{2a}^{-1}P_a + \Omega(P, \hat{P})]G^T \le 0.$$

Thus, A_c is Lyapunov stable. If $R_1 > 0$, then $A_c^T P_2 + P_2A_c < 0$ which implies that A_c is asymptotically stable. □

Note that by letting $\Omega(P, \hat{P}) = 0$, we recover the standard Popov controller where A_c is not necessarily stable. Since the ordering induced by the cone of nonnegative-definite matrices is only a partial ordering, there does not exist a unique function $\Omega(\cdot, \cdot)$ satisfying (25). The next result gives four such functions. For convenience in stating this result let $\Phi \triangleq (I + B_0R_0^{-1}NC_0)BR_{2a}^{-1}P_a$.

Proposition 2.1: Given $\alpha, \beta > 0$, the following nonnegative definite functions $\Omega(P, \hat{P})$ satisfy (25).

i) $\Omega(P, \hat{P}) = \alpha^{-2}A_P^T A_P + \alpha^2P^2 + \beta^{-2}\Phi^T\Phi + \beta^2\hat{P}^2$.
ii) $\Omega(P, \hat{P}) = \alpha A_P^T PA_P + \alpha^{-1}P + \beta^{-2}\Phi^T\Phi + \beta^2\hat{P}^2$.
iii) $\Omega(P, \hat{P}) = \alpha^{-2}A_P^T A_P + \alpha^2P^2 + \beta\Phi^T\hat{P}\Phi + \beta^{-1}\hat{P}$.
iv) $\Omega(P, \hat{P}) = \alpha A_P^T PA_P + \alpha^{-1}P + \beta\Phi^T\hat{P}\Phi + \beta^{-1}\hat{P}$.

Proof: The proof follows from straightforward algebraic manipulations. □

Finally, note that in the full-order case $n_c = n$, it follows that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. Thus the last term in each of (26)–(29) can be deleted, and (33)–(35) become

$$A_c = A_{\hat{Q}} - Q\overline{\Sigma}, \tag{37}$$

$$B_c = QC^T V_2^{-1}, \tag{38}$$

$$C_c = -R_{2a}^{-1}P_a. \tag{39}$$

Remark 2.1: Note that by letting $\Omega(P, \hat{P}) = 0$ in the above formulation, we recover the Popov controller synthesis equations obtained in [1] and [2] for full-order dynamic compensation. It is interesting to note that in contrast to the full-order case given in [1] and [2], which involves three matrix equations for constructing Popov controllers, the full-order design equations for characterizing stable Popov controllers involve four matrix equations.

Remark 2.2: To consider uncertainties with upper and lower bounds of the form $M_1 \le F \le M_2$, where $F, M_1, M_2 \in \mathcal{R}^{m_0\times m_0}$ are symmetric matrices we use the shifting technique discussed in [2]. In this case, Theorem 2.2 holds with F, A, and M replaced by $F - M_1$, $A + B_0M_1C_0$, and $M_2 - M_1$, respectively. For further details, see [2].

III. An Alternative Approach to Robust Strong Stabilization Based Upon H_2 Cost Modification

In this section we present an alternative approach for designing stable Popov controllers. This approach involves an *a priori* modification to the Popov controller synthesis framework presented in [1] and [2] (that is, prior to optimization) in the vein of [4], [5], [7], and [8]. This approach addresses the minimization problem

$$\mathcal{J}(A_c, B_c, C_c) = \operatorname{tr}(\mathcal{P} + \tilde{C}_0^T\mu\tilde{C}_0)\tilde{V} \tag{40}$$

subject to

$$0 = \tilde{A}^T\mathcal{P} + \mathcal{P}\tilde{A} + (\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P})^T[(M^{-1} - N\tilde{C}_0\tilde{B}_0) + (M^{-1} - N\tilde{C}_0\tilde{B}_0)^T]^{-1} \cdot (\tilde{C}_0 + N\tilde{C}_0\tilde{A} + \tilde{B}_0^T\mathcal{P}) + \tilde{R} + \Omega(\mathcal{P}) \tag{41}$$

where $\Omega(\cdot)$ is a matrix function that satisfies $\Omega(\mathcal{P}) \ge 0$ for all $\mathcal{P} \ge 0$. As shown in [1] and [2], minimizing (40) subject to (41) with

$\Omega(\mathcal{P}) = 0$ guarantees robust stability and robust H_2 performance. This result follows from the fact that (40) is a bound for (5), that is, $J(A_c, B_c, C_c) \leq \mathcal{J}(A_c, B_c, C_c)$. To also guarantee stability of the compensator dynamics A_c, we set $\Omega(\mathcal{P})$ to

$$\Omega(\mathcal{P}) = \begin{bmatrix} 0 & 0 \\ 0 & P_{12}^T BR_{2a}^{-1} B^T P_{12} + P_a^T R_{2a}^{-1} P_a \end{bmatrix} \quad (42)$$

where $\mathcal{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}$. Now using the fixed-structure optimization approach outlined in [1] and [2] to minimize (40) subject to (41) with $\Omega(\mathcal{P})$ given by (42) yields the following theorem.

Theorem 3.1: Let $n_c = n$, assume $R_0 > 0$, and let $N \in \mathcal{N}$. Suppose there exists $n \times n$ positive definite matrix $\hat{P}$ and $n \times n$ nonnegative-definite matrices P, Q, and $\hat{Q}$ satisfying

$$\begin{aligned} 0 &= A_P^T P + PA_P + R_1 - \tilde{C}^T R_0^{-1} \tilde{C} - PB_0 R_0^{-1} B_0^T P \\ &\quad + \hat{P} BR_{2a}^{-1} B^T \hat{P}. \end{aligned} \quad (43)$$

$$\begin{aligned} 0 &= [A_P + B_0 R_0^{-1} B_0^T (P + \hat{P})]Q + Q[A_P + B_0 R_0^{-1} B_0^T \\ &\quad \cdot (P + \hat{P})]^T + V_1 - Q\overline{\Sigma} Q \\ &\quad + BR_{2a}^{-1} B^T \hat{P}\hat{Q} + \hat{Q}\hat{P} BR_{2a}^{-1} B^T. \end{aligned} \quad (44)$$

$$0 = A_{\hat{P}}^T \hat{P} + \hat{P} A_{\hat{P}} + \hat{P} B_0 R_0^{-1} B_0^T \hat{P} - \hat{P} BR_{2a}^{-1} B^T \hat{P}. \quad (45)$$

$$\begin{aligned} 0 &= (A_{\hat{P}} + Q\overline{\Sigma})\hat{Q} + \hat{Q}(A_{\hat{P}} + Q\overline{\Sigma})^T \\ &\quad + Q\overline{\Sigma} Q - BR_{2a}^{-1} B^T \hat{P}\hat{Q} - \hat{Q}\hat{P} BR_{2a}^{-1} B^T \end{aligned} \quad (46)$$

and let

$$A_c = A_{\hat{Q}} - Q\overline{\Sigma} - BR_{2a}^{-1} B^T \hat{P} - \hat{P}^{-1} P_a^T R_{2a}^{-1} P_a \quad (47)$$

and B_c, C_c be given by (38) and (39). Then the following results hold.

i) $(\tilde{E}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$.

ii) A_c is Lyapunov stable.

iii) The performance of the closed-loop system satisfies the modified cost (40).

Proof: Since $\Omega(\mathcal{P}) \geq 0$, it is shown in [1] and [2] that if there exist P and $\hat{P}$ satisfying (43) and (45), or, equivalently, (41) with $P_1 = P + \hat{P}$, $P_{12} = -\hat{P}$, and $P_2 = \hat{P}$, then $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Now applying the Lagrange multiplier method to minimize (40) subject to (41) yields

$$\begin{aligned} A_c^T \hat{P} + \hat{P} A_c = &-[(B^T \hat{P} - P_a)^T R_{2a}^{-1} (B^T \hat{P} - P_a) \\ &+ P_a^T R_{2a}^{-1} R_2 R_{2a}^{-1} P_a \\ &+ (B_0^T \hat{P} - NC_0 BR_{2a}^{-1} P_a)^T R_0^{-1} \\ &\cdot (B_0^T \hat{P} - NC_0 BR_{2a}^{-1} P_a)] \leq 0 \end{aligned}$$

which implies that A_c is stable in the sense of Lyapunov. Equations (43)–(46) follow from algebraic manipulations. □

Remark 3.1: Note that if the uncertainty in the plant dynamics is deleted, that is, $B_0 = 0$ and $C_0 = 0$, then Theorem 3.1 specializes to Theorem 3.2 of [5] in which $n_c = n$ and $\gamma \to \infty$.

IV. Conclusion

In this paper we extended the Popov controller synthesis technique proposed in [1], [2] to obtain H_2-suboptimal robust stable compensators. Two approaches were developed for obtaining robust strong compensators. The first approach involves modifying the Popov design equations given in [1], [2] to guarantee stability of the compensator, while the second approach is based upon the addition of a new term in the Popov Riccati equation given by (12).

References

[1] W. M. Haddad and D. S. Bernstein, "Parameter-dependent Lyapunov functions, constant real parameter uncertainty, and the Popov criterion in robust analysis and synthesis Part 1, Part 2," in *Proc. IEEE Conf. Dec. Contr.*, Brighton, U.K., Dec. 1991, pp. 2274–2279, 2632–2633.

[2] ——, "Parameter-dependent Lyapunov functions and the Popov criterion in robust analysis and synthesis," *IEEE Trans. Automat. Contr.*, to appear.

[3] W. M. Haddad, J. P. How, S. R. Hall, and D. S. Bernstein, "Extensions of mixed-μ bounds to monotonic and odd monotonic nonlinearities using absolute stability theory," in *Proc. IEEE Conf. Dec. Contr.*, Tucson, AZ, Dec. 1992, pp. 2813–2832.

[4] Y. W. Wang and D. S. Bernstein, "H_2-suboptimal stable stabilization," in *Proc. IEEE Conf. Dec. Contr.*, San Antonio, TX, 1993, pp. 1828–1829.

[5] Y. W. Wang, W. M. Haddad, and D. S. Bernstein, "Stable stabilization with H_2 and H_∞ performance constraints," *J. Math. Syst. Estimation, Contr.*, to appear.

[6] D. C. Hyland and D. S. Bernstein, "The optimal projection equations for fixed-order dynamic compensation," *IEEE Trans. Automat. Contr.*, pp. 1034–1037, 1984.

[7] M. J. Jacobus, "Stable, fixed-order dynamic compensation with applications to positive real and H^∞-constrained control design," Ph.D. dissertation, Univ. New Mexico, Albuquerque, 1990.

[8] M. J. Jacobus, M. Jamshidi, C. Abdallah, P. Dorato, and D. S. Bernstein, "Suboptimal strong stabilization using fixed-order dynamic compensation," in *Proc. Amer. Contr. Conf.*, San Diego, CA, 1990, pp. 2659.

[9] Y. Halevi, D. S. Bernstein, and W. M. Haddad, "On stable full-order and reduced-order LQG controllers," *Opt. Contr. Appl. Meth.*, vol. 12, pp. 163, 1991.

[10] D. C. Youla, J. J. Bongiorno, Jr., and C. N. Lu, "Single-loop feedback-stabilization of linear multivariable dynamical plants," *Automatica*, vol. 10, pp. 159, 1974.

Preconditioning of Transfer Matrices: Bounding the Frequency Dependent Structured Singular Value

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***Abstract*—The precondition of matrices by diagonal scaling is a useful tool for bounding the structured singular value. Although the constant matrix case has been well studied, comparatively little is known about the behavior of the scaling matrices as a function of frequency. In this paper this problem is addressed by considering the optimal Frobenius-norm scaling. It is shown that, under mild assumptions, there exist stable and minimum-phase diagonal transfer matrices which minimize the Frobenius-norm of a scaled transfer matrix.**

I. Introduction

During the past decade, $\mathcal{H}_\infty$ has emerged as a powerful synthesis method for linear time-invariant systems. It is not hard to see, though, that capturing all desirable specifications into a single norm objective is not possible in all but a small number of problems, without introducing potentially large degrees of conservatism. Consider, for instance, the problem illustrated in Fig. 1, where P is a generalized real rational plant, K is a controller, and Δ is an uncertainty transfer

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