
# Setpoint Tracking with Actuator Offset and Sensor Bias

## PROBING THE LIMITS OF INTEGRAL CONTROL

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In the classical control of single-input, single-output (SISO) systems, asymptotic tracking of commanded setpoints is achieved through integral control, a direct manifestation of the internal model principle [1]–[4]. In practice, however, the achievement of zero steady-state error must take into account the presence of constant signals that enter the loop in various ways. For example, the actuator input may include an unmodeled offset, while the sensor measurement may be corrupted by an unmodeled bias. These effects may be caused by imperfect calibration, which can be difficult to remove due to drift. We note that actuator offset can be viewed as a constant process disturbance, while sensor bias is effectively constant measurement noise.

In this article we consider the following question: Is it possible to achieve zero steady-state error in the presence of both unknown actuator offset and unknown sensor bias? To address this question, we consider a two-degree-of-freedom controller architecture that includes both forward- and backward-path controller transfer functions. As shown below, a feedforward gain is not helpful for setpoint tracking in the presence of actuator offset and sensor bias.

The answer to the question that we pose is summarized in Table 1. In particular, we show that the rejection of actuator offset requires an internal model controller, that is, an integral controller, in the loop transfer function, while sensor bias requires a zero at the origin in the backward-path transfer function. Since these controller properties are

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incompatible, it follows that, for the controller architecture we consider, zero steady-state error for a setpoint command is impossible in the presence of both unknown actuator offset and unknown sensor bias.

The development in this article is confined to the case of linear time-invariant SISO plants. In addition, we use only frequency-domain methods to make the development accessible to readers with a classical background. However, an alternative approach is to formulate the problem in state space and apply the results of [5] for the case of constant actuator offset. This approach has the advantage that it addresses multiple-input, multiple-output (MIMO) plants, which are difficult to address with frequency-domain approaches.

## PROBLEM FORMULATION

In the subsequent development we do not distinguish between a time-domain signal and its Laplace transform. Let $G(s)$ be a SISO plant with input $\hat{u}$ and output $y$. Let $w$ denote the actuator offset acting on $G(s)$ and let $v$ denote the sensor bias, so that the measurement $\hat{y}$ is given by

$$\hat{y} = y + v = G\hat{u} + v = G(u_{\text{fp}} + w) + v. \tag{1}$$

We assume that the actuator offset $w$ and sensor bias $v$ are constant but unknown. Define the output error $e$ by

**TABLE 1 Achievable asymptotic setpoint tracking. Asymptotic setpoint tracking is impossible in the presence of both unknown actuator offset and unknown sensor bias.**

| | Integral Control | Loop Zero at the Origin |
|---|---|---|
| No actuator offset or sensor bias | Possible | Possible |
| Actuator offset only | Possible | Never possible |
| Sensor bias only | Never possible | Possible |
| Actuator offset and sensor bias | Never possible | Never possible |

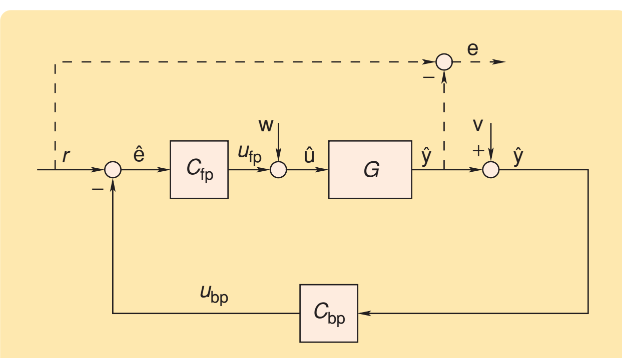


**FIGURE 1** Control architecture for setpoint tracking in the presence of actuator offset and sensor bias. The objective is to use the forward-path and backward-path controllers $C_{\text{fp}}$ and $C_{\text{bp}}$, respectively, to ensure that the plant output $y$ tracks the reference setpoint $r$.

$$e \triangleq r - y, \tag{2}$$

where $r$ is the prescribed constant setpoint. The control objective is to ensure that $\lim_{t\to\infty} e(t) = 0$, that is, to make the plant output $y$ asymptotically reach the specified setpoint $r$.

Since our interest is in asymptotic behavior, we assume for convenience that all initial conditions are zero. Nonzero initial conditions have no effect on the ability to achieve asymptotic setpoint tracking. Consider the control architecture shown in Figure 1, where $C_{\text{fp}}$ is the forward-path controller and $C_{\text{bp}}$ is the backward-path controller. Define the forward-path controller input $\hat{e}$ by

$$\hat{e} \triangleq r - u_{\text{bp}}, \tag{3}$$

where $u_{\text{bp}}$ is the backward-path controller output. Hence

$$\hat{e} = r - C_{\text{bp}}\hat{y} = r - C_{\text{bp}}\left(G(u_{\text{fp}} + w) + v\right). \tag{4}$$

Since the forward-path controller output $u_{\text{fp}}$ is given by

$$u_{\text{fp}} = C_{\text{fp}}\hat{e}, \tag{5}$$

substituting (5) into (4) yields

$$\hat{e} = \frac{1}{1+L}r - \frac{GC_{\text{bp}}}{1+L}w - \frac{C_{\text{bp}}}{1+L}v, \tag{6}$$

where the loop transfer function $L$ is defined by

$$L \triangleq C_{\text{fp}}GC_{\text{bp}}. \tag{7}$$

Substituting (6) into (5), and using $y = G(u_{\text{fp}} + w)$ in (2) yields

$$e = G_{er}r + G_{ew}w + G_{ev}v, \tag{8}$$

where

$$G_{er} \triangleq 1 - \frac{C_{\text{fp}}G}{1+L}, \quad G_{ew} \triangleq \frac{G}{1+L}, \quad G_{ev} \triangleq \frac{L}{1+L}. \tag{9}$$

To eliminate the possibility of closed-right-half-plane (CRHP) pole-zero cancellations in the closed-loop system, we introduce the notion of internal stability.

### Definition 1

The closed-loop system in Figure 1 is internally stable if, for all bounded inputs $r$, $w$, and $v$, the output $e$, the controller inputs $\hat{y}$ and $\hat{e}$, and the controller outputs $u_{\text{fp}}$ and $u_{\text{bp}}$ are bounded.

It follows from (1), (2), and (3) that

$$\hat{y} = r + v - e, \tag{10}$$

$$\hat{e} = r - u_{\text{bp}}, \tag{11}$$

while (5), (6), and (8) imply that

$$\begin{bmatrix} e \\ u_{\mathrm{fp}} \\ u_{\mathrm{bp}} \end{bmatrix} = \mathcal{G} \begin{bmatrix} r \\ w \\ v \end{bmatrix}, \tag{12}$$

where $\mathcal{G}$ is given by

$$\mathcal{G} \triangleq \begin{bmatrix} 1 - GC_{\mathrm{fp}}S & GS & T \\ C_{\mathrm{fp}}S & -T & -C_{\mathrm{fp}}C_{\mathrm{bp}}S \\ T & GC_{\mathrm{bp}}S & C_{\mathrm{bp}}S \end{bmatrix} \tag{13}$$

and

$$S \triangleq \frac{1}{1+L}, \quad T \triangleq \frac{L}{1+L}. \tag{14}$$

Next, we present a result that guarantees the boundedness of $e$, $u_{\mathrm{fp}}$, and $u_{\mathrm{bp}}$ when $r$, $w$, and $v$ are bounded. It follows from (10) and (11) that, if $r$, $v$, $e$, and $u_{\mathrm{bp}}$ are bounded, then $\hat{y}$ and $\hat{e}$ are also bounded.

### Lemma 1

The closed-loop system in Figure 1 is internally stable if and only if $\mathcal{G}$ is asymptotically stable.

Note that $\mathcal{G}$ has seven distinct entries. Furthermore, $S$ is asymptotically stable if and only if $T$ is asymptotically stable. The following result ensures that $\mathcal{G}$ is asymptotically stable.

### Lemma 2

$\mathcal{G}$ is asymptotically stable if and only if the following conditions hold:

1) $S$ is asymptotically stable
2) None of the cascades $C_{\mathrm{fp}}G$, $C_{\mathrm{fp}}C_{\mathrm{bp}}$, and $C_{\mathrm{bp}}G$ have CRHP pole-zero cancellations.

### *Feedforward Controller*

As a potential extension of the control architecture in Figure 1, we now include the feedforward controller $C_{\mathrm{ff}}$ shown in Figure 2 with the setpoint $r$ as the controller input and the controller output injected into the plant $G$. In this case, the error dynamics are given by (8), where

$$G_{er} = 1 - \frac{G(C_{\mathrm{fp}} + C_{\mathrm{ff}})}{1+L} \tag{15}$$

and $G_{ew}$ and $G_{ev}$ are given by (9). Since the transfer functions $G_{ew}$ and $G_{ev}$ are unchanged by the inclusion of $C_{\mathrm{ff}}$, the feedforward controller does not help in setpoint tracking in the presence of actuator offset or sensor bias. Hence, we do not consider a feedforward controller in the subsequent development.

## SETPOINT TRACKING IN THE ABSENCE OF ACTUATOR OFFSET AND SENSOR BIAS

To set the stage for later developments, we first consider the case in which there is no actuator offset or sensor bias, and determine controllers that achieve asymptotic tracking of a commanded setpoint. Assume that $w = 0$ and $v = 0$, and let $r = r_0/s$, where $r_0 \in \mathbb{R}$ is a constant. Note that $r$ represents a step command. Hence

$$e = G_{er}\frac{r_0}{s}. \tag{16}$$

### Lemma 3

Assume that $\mathcal{G}$ is asymptotically stable. Then, $\lim_{t\to\infty} e(t) = 0$ for all $r_0 \in \mathbb{R}$ if and only if

$$G_{er}(0) = 0. \tag{17}$$

### Proof

Since $G_{er}$ is asymptotically stable, it follows from (16) that

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sG_{er}(s)\frac{r_0}{s} = G_{er}(0)r_0. \qquad \square$$

### *Internal Model Control*

We now use an internal model controller to achieve setpoint tracking. This internal model includes an integrator in the forward-path controller to ensure that $C_{\mathrm{fp}}$ has infinite dc gain and satisfies (17). To simplify the presentation, we use the notation in Table 2.

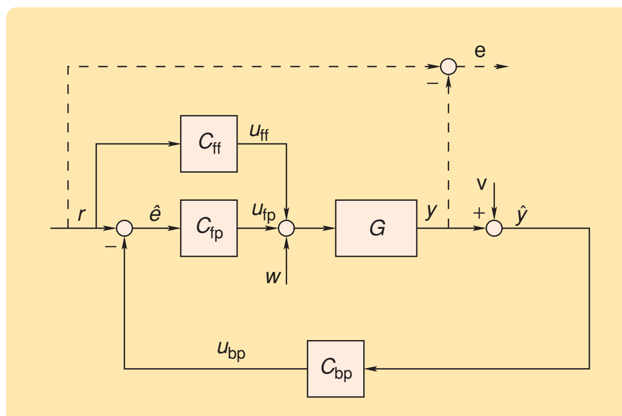


**FIGURE 2** Control architecture for setpoint tracking in the presence of actuator offset and sensor bias with a feedforward controller $C_{\mathrm{ff}}$. Although the feedforward controller can be used to ensure setpoint tracking in the absence of actuator offset and sensor bias, the feedforward controller does not help in rejecting actuator offset or sensor bias.

**TABLE 2 Verbal interpretation of mathematical conditions. These mathematical conditions are used throughout this article to streamline the presentation.**

| Notation | Definition |
|---|---|
| $G(0) = 0$ | $G$ has a zero at the origin |
| $G(0) \neq 0$ | $G$ has no zeros at the origin |
| $\lvert G(0)\rvert = \infty$ | $G$ has a pole at the origin |
| $\lvert G(0)\rvert < \infty$ | $G$ has no poles at the origin |

### Proposition 1
Assume that $G(0) \neq 0$, $|C_{\mathrm{fp}}(0)| = \infty$, $C_{\mathrm{bp}}(0) = 1$, and $\mathcal{G}$ is asymptotically stable. Then, $G_{er}(0) = 0$ and hence, $\lim_{t\to\infty} e(t) = 0$.

### Proof
Note that

$$\frac{GC_{\mathrm{fp}}}{1+L} = \frac{1}{\frac{1}{C_{\mathrm{fp}}G} + C_{\mathrm{bp}}}\,.$$

Since $|G(0)C_{\mathrm{fp}}(0)| = \infty$ and $C_{\mathrm{bp}}(0) = 1$, (9) implies that

$$\lim_{s\to 0} G_{er} = 1 - \lim_{s\to 0} \frac{1}{\frac{1}{C_{\mathrm{fp}}G} + C_{\mathrm{bp}}} = 1 - \frac{1}{C_{\mathrm{bp}}(0)} = 0\,.$$

Hence, it follows from Lemma 3 that $\lim_{t\to\infty} e(t) = 0$. □

The following result shows that when $G(0) = 0$ and a forward-path integral controller is not used, then asymptotic tracking of a setpoint command is not possible.

### Proposition 2
Assume that $G(0) = 0$, $|C_{\mathrm{fp}}(0)| < \infty$, and $\mathcal{G}$ is asymptotically stable. Then, $G_{er}(0) = 1$ and hence, $\lim_{t\to\infty} e(t) = r_0$.

### Proof
Since $G(0) = 0$ and $\mathcal{G}$ is asymptotically stable, it follows from Lemma 2 that $|C_{\mathrm{bp}}(0)| < \infty$. Hence, $L(0) = 0$ and

$$\lim_{s\to 0} G_{er} = 1 - \frac{C_{\mathrm{fp}}(0)G(0)}{1+L(0)} = 1\,.$$

Since $G_{er}(0) = 1$, using the final value theorem in (16) implies that $\lim_{t\to\infty} e(t) = r_0$. □

Note that Proposition 2 does not imply that an internal model controller can achieve asymptotic tracking of a setpoint command when $G(0) = 0$. In fact, if $G(0) = 0$ and $|C_{\mathrm{fp}}(0)| = \infty$, then it follows from Lemma 2 that, due to CRHP pole-zero cancellation between $C_{\mathrm{fp}}$ and $G$ at the origin, $\mathcal{G}$ is not asymptotically stable.

### *Loop Zero at the Origin*
We now show that an internal model controller is not necessary to achieve asymptotic setpoint tracking. In particular, setpoint tracking can be achieved by including a zero at the origin in the backward-path controller, thus satisfying (17). Unlike the internal model controller, which does not require specific information about the plant $G$, this controller requires knowledge of $G(0)$.

### Proposition 3
Assume that $|G(0)| < \infty$, $C_{\mathrm{bp}}(0) = 0$, and $\mathcal{G}$ is asymptotically stable. Then, $G_{er}(0) = 1 - C_{\mathrm{fp}}(0)G(0)$ and hence,

$$\lim_{t\to\infty} e(t) = (1 - C_{\mathrm{fp}}(0)G(0))r_0\,. \tag{18}$$

Therefore, if $C_{\mathrm{fp}}(0)G(0) = 1$, then $\lim_{t\to\infty} e(t) = 0$.

### Proof
Since $C_{\mathrm{bp}}(0) = 0$, it follows from Lemma 2 that $|C_{\mathrm{fp}}(0)| < \infty$. Furthermore, (7) implies that

$$\frac{GC_{\mathrm{fp}}}{1+L} = \frac{1}{\frac{1}{C_{\mathrm{fp}}G} + C_{\mathrm{bp}}}\,.$$

Since $G_{er}$ is asymptotically stable,

$$\lim_{s\to 0} G_{er} = 1 - \lim_{s\to 0} \frac{1}{\frac{1}{C_{\mathrm{fp}}G} + C_{\mathrm{bp}}} = 1 - C(0)G(0). \tag{19}$$

Hence, using the final value theorem in (16) yields (18). Substituting $C_{\mathrm{fp}}(0)G(0) = 1$ into (19) yields $G_{er}(0) = 0$ and Lemma 3 implies that $\lim_{t\to\infty} e(t) = 0$. □

### *Alternative Methods*
There exist controllers that do not fall into either of the above categories and still achieve asymptotic setpoint tracking. For example, let

$$G(s) = \frac{1}{s+2}, \quad C_{\mathrm{fp}}(s) = \frac{4}{4s+1}, \quad C_{\mathrm{bp}}(s) = \frac{1}{2}\,.$$

Then

$$G_{er}(s) = \frac{s(4s+5)}{4s^2+5s+4}\,,$$

which implies that $G_{er}$ is asymptotically stable. Since $G_{er}(0) = 0$ and there are no CRHP pole-zero cancellations between $C_{\mathrm{fp}}$, $G$, and $C_{\mathrm{bp}}$, it follows from Lemma 3 that $\lim_{t\to\infty} e(t) = 0$.

As noted in Table 1, asymptotic tracking of a setpoint command can thus be achieved in the absence of actuator offset and sensor bias by using either the integral control approach or the loop-zero-at-the-origin approach.

## SETPOINT TRACKING WITH ACTUATOR OFFSET ONLY
We now consider setpoint tracking in the presence of actuator offset. In particular, we present conditions that the forward- and backward-path controllers must satisfy to achieve tracking of a commanded setpoint in the presence of actuator offset. Assume that there is no sensor bias, that is, $v = 0$. Let $r = r_0/s$ and $w = w_0/s$, where $r_0, w_0 \in \mathbb{R}$ are constants and $w_0 \neq 0$. Note that $r$ represents a step command and $w$ represents a constant actuator offset. In this case, (8) implies that

$$e = G_{er}\frac{r_0}{s} + G_{ew}\frac{w_0}{s}. \tag{20}$$

### Lemma 4
Assume that $\mathcal{G}$ is asymptotically stable. Then, $\lim_{t\to\infty} e(t) = 0$ for all $r_0, w_0 \in \mathbb{R}$ if and only if

$$G_{er}(0) = 0 \tag{21}$$

and

$$G_{ew}(0) = 0. \tag{22}$$

### Proof

Since $G_{er}$ and $G_{ew}$ are asymptotically stable, we have

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sG_{er}(s)\frac{r_0}{s} + \lim_{s\to 0} sG_{ew}(s)\frac{w_0}{s} \\ &= G_{er}(0)r_0 + G_{ew}(0)w_0\,. \end{aligned} \quad \square$$

Again the internal model controller satisfies Lemma 4 and thus ensures setpoint tracking in the presence of actuator offset. Specifically, we have the following result.

### Proposition 4

Assume that $G(0) \neq 0$, $|C_{\text{fp}}(0)| = \infty$, $C_{\text{bp}}(0) = 1$, and $\mathcal{G}$ is asymptotically stable. Then, $G_{er}(0) = 0$ and $G_{ew}(0) = 0$ and hence, $\lim_{t\to\infty} e(t) = 0$.

### Proof

It follows from Proposition 1 that $G_{er}(0) = 0$. Note that

$$\frac{G}{1+L} = \frac{1}{\frac{1}{G} + C_{\text{fp}}C_{\text{bp}}}\,.$$

Since $G(0) \neq 0$ and $C_{\text{bp}}(0) = 1$, we have

$$\lim_{s\to 0} G_{ew} = \lim_{s\to 0}\frac{G}{1+L} = \lim_{s\to 0}\frac{1}{\frac{1}{G} + C_{\text{fp}}C_{\text{bp}}} = 0\,.$$

Hence, it follows from Lemma 4 that $\lim_{t\to\infty} e(t) = 0$. $\square$

The following result shows that, if $w \neq 0$, then setpoint tracking can be achieved only with an internal model controller.

### Proposition 5

Assume that $|C_{\text{fp}}(0)| < \infty$ and $\mathcal{G}$ is asymptotically stable. Then $G_{er}(0)$ and $G_{ew}(0)$ are not both zero. Hence, $\lim_{t\to\infty} e(t)$ is not zero for almost all values of $r_0, w_0 \in \mathbb{R}$.

### Proof

It follows from (9) that

$$G_{er} = 1 - C_{\text{fp}}G_{ew}\,.$$

Since $\mathcal{G}$ is asymptotically stable, $\lim_{s\to 0} G_{er}$ and $\lim_{s\to 0} G_{ew}$ exist. Furthermore, $|C_{\text{fp}}(0)| < \infty$ implies that

$$G_{er}(0) = 1 - C_{\text{fp}}(0)G_{ew}(0). \tag{23}$$

Therefore, $G_{er}(0)$ and $G_{ew}(0)$ are not both zero and using the final value theorem in (20) implies that $\lim_{t\to\infty} e(t) \neq 0$ for almost all $r_0, w_0 \in \mathbb{R}$. $\square$

Note that asymptotic tracking of a setpoint command in the presence of actuator offset cannot be achieved by any forward path controller $C_{\text{fp}}$ other than the integral controller with $|C_{\text{fp}}(0)| = \infty$. However, if $|C_{\text{fp}}(0)| = \infty$, then a backward path controller with $C_{\text{bp}}(0) = 0$ cannot be used because of the CRHP pole-zero cancellation at the origin, which renders $\mathcal{G}$ unstable. Hence, as shown in Table 1, asymptotic tracking of a setpoint command cannot be achieved by the loop-zero-at-the-origin approach.

## SETPOINT TRACKING WITH SENSOR BIAS ONLY

We now provide conditions under which setpoint tracking can be achieved in the presence of sensor bias $v \neq 0$ and with no actuator offset, that is, $w = 0$. Let $r = r_0/s$ and $v = v_0/s$, where $r_0, v_0 \in \mathbb{R}$ are constants and $v_0 \neq 0$, so that

$$e = G_{er}\frac{r_0}{s} + G_{ev}\frac{v_0}{s}. \tag{24}$$

Note that $r$ represents a step command and $v$ represents a constant sensor bias.

### Lemma 5

Assume that $\mathcal{G}$ is asymptotically stable. Then, $\lim_{t\to\infty} e(t) = 0$ for all $r_0, v_0 \in \mathbb{R}$ if and only if

$$G_{er}(0) = 0 \tag{25}$$

and

$$G_{ev}(0) = 0. \tag{26}$$

### Proof

Since $G_{er}$ and $G_{ew}$ are asymptotically stable, we have

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sG_{er}(s)\frac{r_0}{s} + \lim_{s\to 0} sG_{ev}(s)\frac{v_0}{s} \\ &= G_{er}(0)r_0 + G_{ev}(0)v_0\,. \end{aligned} \quad \square$$

Next, we use the backward-path controller introduced in Proposition 3, with a zero at the origin, to achieve setpoint tracking in the presence of a sensor bias.

### Proposition 6

Assume that $|G(0)| < \infty$, $C_{\text{bp}}(0) = 0$, and $\mathcal{G}$ is asymptotically stable. Then, $G_{er}(0) = 1 - C_{\text{fp}}(0)G(0)$ and $G_{ev}(0) = 0$ so that

$$\lim_{t\to\infty} e(t) = (1 - C_{\text{fp}}(0)G(0))r_0. \tag{27}$$

Therefore, if $C_{\text{fp}}(0)G(0) = 1$, then $\lim_{t\to\infty} e(t) = 0$.

### Proof

It follows from Proposition 3 that $G_{er}(0) = 1 - C_{\text{fp}}(0)G(0)$. Furthermore, it follows from Lemma 2 that $C_{\text{bp}}(0) = 0$ implies $|C_{\text{fp}}(0)| < \infty$. Therefore, $L(0) = 0$, and it follows

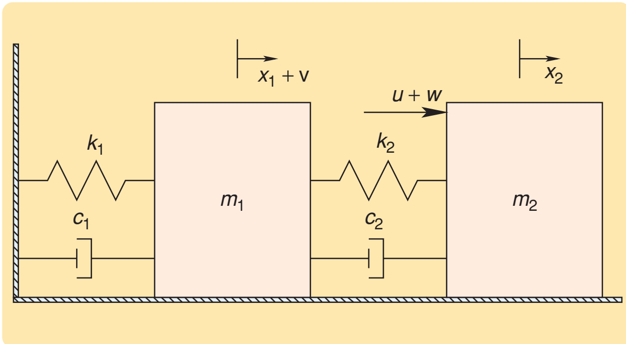


**FIGURE 3** Two-mass system. The positions of the masses $m_1$ and $m_2$ are denoted by $x_1$ and $x_2$, respectively. The stiffnesses of the springs are denoted by $k_1$ and $k_2$, while the damping coefficients of the dampers are denoted by $c_1$ and $c_2$. A force actuator on $m_2$ delivers the control input $u$. Unknown sensor bias $v$ and actuator offset $w$ are included to illustrate the effects of these disturbances.

from (9) that $G_{ev}(0) = 0$. Applying the final value theorem to (24) yields (27). Substituting $C_{\rm fp}(0)G(0) = 1$ into (27) yields $\lim_{t\to\infty} e(t) = 0$. □

It follows from Lemma 3 that if (25) is not satisfied then setpoint tracking in the presence of sensor bias is impossible. Consequently, a zero at the origin in the backward-path controller is necessary to achieve setpoint tracking in the presence of sensor bias.

### Proposition 7

Assume that $C_{\rm bp}(0) \neq 0$ and $\mathcal{G}$ is asymptotically stable. Then $G_{er}(0)$ and $G_{er}(0)$ are not both zero. Hence, $\lim_{t\to\infty} e(t)$ is not zero for almost all values of $r_0, v_0 \in \mathbb{R}$.

### Proof

It follows from (9) that

$$C_{\rm bp}(1 - G_{er}) = G_{ev}\,.$$

Since all the transfer functions in $\mathcal{G}$ are asymptotically stable, $\lim_{s\to 0} G_{er}$ and $\lim_{s\to 0} G_{er}$ exist. Hence

$$C_{\rm bp}(0)(1 - G_{er}(0)) = G_{ev}(0)\,.$$

Since $C_{\rm bp}(0) \neq 0$, it follows that $G_{er}(0)$ and $G_{ev}(0)$ are not both zero. Hence, using the final value theorem in (24) implies that $\lim_{t\to\infty} e(t) \neq 0$ for almost all values of $r_0, v_0 \in \mathbb{R}$. □

Since a backward-path controller with $C_{\rm bp}(0) = 0$ is necessary for asymptotic tracking of a setpoint command in the presence of sensor bias, a forward path integral controller with $|C_{\rm fp}(0)| = \infty$ cannot be used due to the CRHP pole-zero cancellation between $C_{\rm fp}$ and $C_{\rm bp}$ at the origin. Hence, as shown in Table 1, only the loop-zero-at-the-origin approach can be used to achieve asymptotic tracking of a setpoint command in the presence of sensor bias.

## SETPOINT TRACKING WITH ACTUATOR OFFSET AND SENSOR BIAS

Finally, we consider setpoint tracking in the presence of both actuator offset and sensor bias. In particular, we present a negative result that shows that setpoint tracking is impossible in the presence of both actuator offset and sensor bias for the control architecture in Figure 1. Let $r = r_0/s$, $w = w_0/s$, and $v = v_0/s$, where $r_0, w_0, v_0$ are constants and $w_0, v_0 \neq 0$. Note that $r$ represents a step command, $w$ represents a constant actuator offset, and $v$ represents a constant sensor bias. Then

$$e = G_{er}\frac{r_0}{s} + G_{ew}\frac{w_0}{s} + G_{ev}\frac{v_0}{s}. \tag{28}$$

### Proposition 8

Assume that $\mathcal{G}$ is asymptotically stable. Then $G_{er}(0)$, $G_{ew}(0)$, and $G_{ev}(0)$ are not all zero. Hence, $\lim_{t\to\infty} e(t)$ is not zero for almost all values of $r_0, w_0, v_0 \in \mathbb{R}$.

### Proof

Since $\mathcal{G}$ is asymptotically stable, $\lim_{s\to 0} G_{er}$, $\lim_{s\to 0} G_{ew}$, and $\lim_{s\to 0} G_{ev}$ exist. Next, we consider two cases, namely, $|C_{\rm fp}(0)| < \infty$ and $|C_{\rm fp}(0)| = \infty$. If $|C_{\rm fp}(0)| < \infty$, it follows from Proposition 5 that $G_{er}(0)$ and $G_{ew}(0)$ are not both zero.

Now suppose $|C_{\rm fp}(0)| = \infty$. Since $\mathcal{G}$ is asymptotically stable, Lemma 2 implies that $C_{\rm bp}(0) \neq 0$. Hence, it follows from Proposition 7 that $G_{er}(0)$ and $G_{ev}(0)$ are not both zero. Consequently, $G_{er}(0)$, $G_{ew}(0)$, and $G_{ev}(0)$ are not all zero. Using the final value theorem in (28) yields

$$\lim_{t\to\infty} e(t) = G_{er}(0)r_0 + G_{ew}(0)w_0 + G_{ev}(0)v_0\,.$$

Therefore, $\lim_{t\to\infty} e(t) \neq 0$ for almost all values of $r_0, w_0, v_0 \in \mathbb{R}$. □

## EXAMPLE: TWO-MASS SYSTEM

Consider the two-mass system shown in Figure 3 with force input $u$ and actuator offset $w$. The equations of motion are

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = 0, \tag{29}$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 - c_2\dot{x}_1 - k_2x_1 = u_{\rm fp} + w. \tag{30}$$

A state-space representation of (29) and (30) is

$$\dot{x} = Ax + B(u_{\rm fp} + w), \tag{31}$$

where $x \in \mathbb{R}^4$ is defined by

$$x \triangleq [\, x_1 \quad x_2 \quad \dot{x}_1 \quad \dot{x}_2 \,]^{\rm T} \tag{32}$$

and $A \in \mathbb{R}^{4\times 4}$ and $B \in \mathbb{R}^{4\times 1}$ are given by

$$A \triangleq \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix}. \tag{33}$$

Let the position measurement $\hat{y}$ of $x_1$ be given by

$$\hat{y} = Cx + v, \tag{34}$$

where

$$C \triangleq [1 \quad 0 \quad 0 \quad 0] \tag{35}$$

and $v \in \mathbb{R}$ is the unknown sensor bias. The SISO transfer function from $u$ to $x_1$ is given by

$$G(s) = C(sI_4 - A)^{-1}B. \tag{36}$$

Hence, it follows from (33) and (35) that we have (37). found at the bottom of the page.

The objective is to make $m_1$ track a constant position command $r = 0.1$, that is, to ensure that $x_1(t) \to 0.1$ as $t \to \infty$. We choose controllers to achieve asymptotic setpoint tracking of $x_1$ under the presence of actuator offset or sensor bias. The values of the masses $m_1$ and $m_2$, damping coefficients $c_1$ and $c_2$, and spring constants $k_1$ and $k_2$ are $m_1 = 1$ kg, $m_2 = 2$ kg, $c_1 = 1.5$ kg/s, $c_2 = 0.5$ kg/s, $k_1 = 0.5$ kg/s$^2$, and $k_2 = 0.5$ kg/s$^2$. The initial conditions in all of the simulations are assumed to be

$$x_1(0) = -0.1, \quad x_2(0) = 0.0, \\ \dot{x}_1(0) = -0.1, \quad \dot{x}_2(0) = 0.1. \tag{38}$$

First, we let $w(t) = 0.1$ and $v(t) = 0$ for all $t \geqslant 0$, which corresponds to a constant actuator offset and no sensor bias. We choose

$$C_{\rm fp}(s) = \frac{1}{10s(s+5)}, \quad C_{\rm bp}(s) = 1, \tag{39}$$

which ensures that $\mathcal{G}$ is asymptotically stable. The position of $m_1$ when the controller in (39) is used is shown in Figure 4. In accordance with Proposition 4, the position of $m_1$ asymptotically reaches the setpoint $r = 0.1$ in the presence of a constant actuator offset. Next, we let $w(t) = 0.1$ and $v(t) = 0.1$ for all $t \geqslant 0$ so that we have both actuator offset and sensor bias. The position $x_1$ of mass $m_1$ when the controller in (39) is used is shown in Figure 5. In accordance with Proposition 8, the position $x_1$ does not converge to the setpoint $r = 0.1$ when a sensor bias is present.

Next, we let $w(t) = 0$ and $v(t) = 0.1$ for all $t \geqslant 0$, which corresponds to no actuator offset and a constant sensor bias. We choose

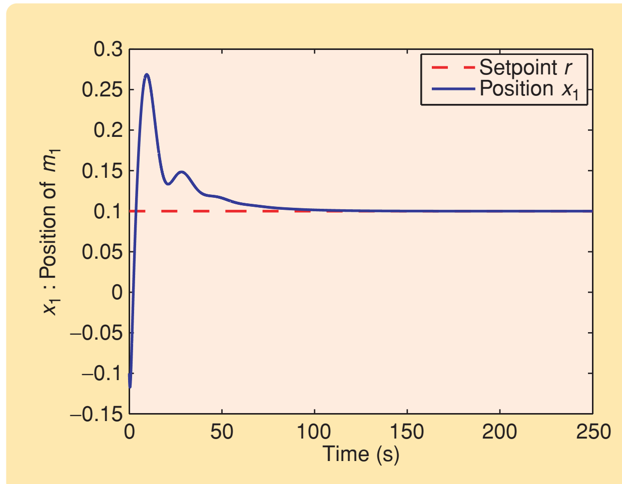


**FIGURE 4** Position $x_1$ of mass $m_1$ when $w(t) = 0.1$ and $v(t) = 0$ with the controller (39). Integral control achieves setpoint tracking in the presence of actuator offset.

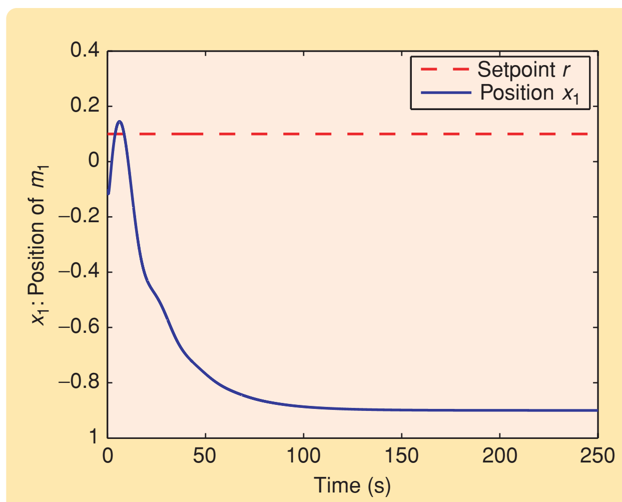


**FIGURE 5** Position $x_1$ of mass $m_1$ when $w(t) = 0.1$ and $v(t) = 0.1$ with the controller (39). Integral control cannot reject sensor bias and hence, in accordance with Proposition 8, setpoint tracking is not achieved.

$$G(s) = \frac{s\frac{c_2}{m_1 m_2} + \frac{k_2}{m_1 m_2}}{s^4 + s^3\left(\frac{c_2}{m_2} + \frac{1}{m_1}(c_1+c_2)\right) + s^2\left(\frac{k_2}{m_2} + \frac{1}{m_1}(k_1+k_2) + \frac{c_1 c_2}{m_1 m_2}\right) + s\left(\frac{c_1 k_2 + c_2 k_1}{m_1 m_2}\right) + \frac{k_1 k_2}{m_1 m_2}}. \tag{37}$$

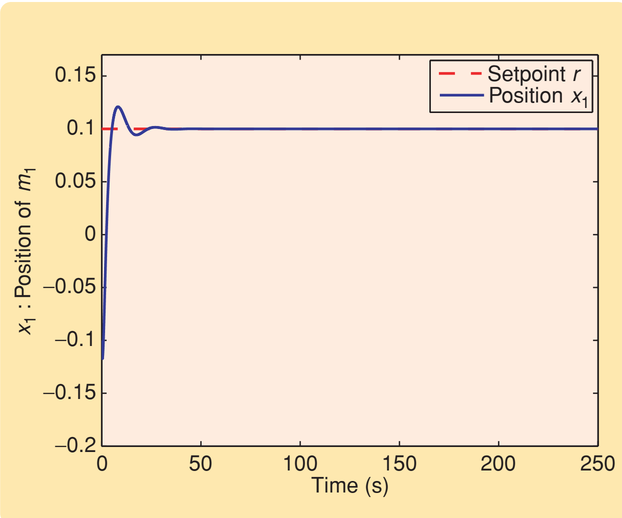


**FIGURE 6** Position $x_1$ of mass $m_1$ when $w(t) = 0.1$ and $v(t) = 0.1$ with the controller (40). Since the backward-path controller $C_{\rm bp}$ has a zero at the origin and $C_{\rm fp(0)}G(0) = 1$, setpoint tracking is achieved in the presence of sensor bias.

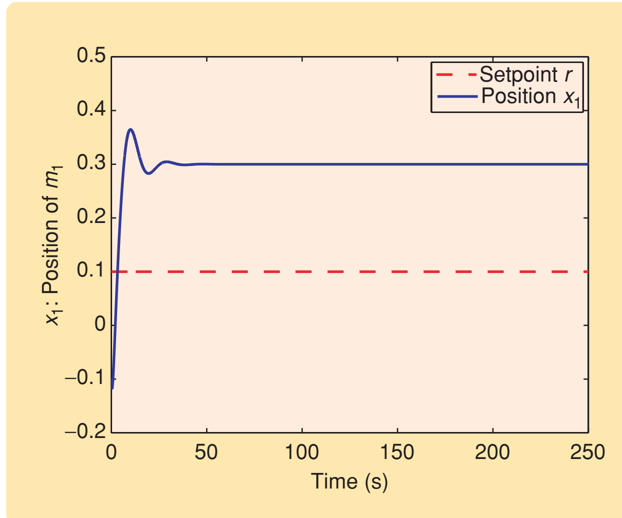


**FIGURE 7** Position $x_1$ of mass $m_1$ when $w(t) = 0.1$ and $v(t) = 0.1$ with the controller (40), (41). Due to the presence of an actuator offset, asymptotic setpoint tracking is not achieved by the controller (40), (41) in accordance with Proposition 8.

$$C_{\rm fp}(s) = \frac{1}{G(0)} = k_1 = 0.5, \tag{40}$$

$$C_{\rm bp}(s) = \frac{s}{s+5}, \tag{41}$$

which ensures that the assumptions of Proposition 6 are satisfied. The position $x_1$ when the controller in (40) is used is shown in Figure 6. In accordance with Proposition 6, $x_1(t) - r(t) \to 0$ as $t \to \infty$. However, when we let $w(t) = 0.1$ and use the same controller in (40), (41), the position $x_1$ does not reach the setpoint as shown in Figure 7 and in accordance with Proposition 8.

## CONCLUSIONS

In SISO systems, a servo-loop architecture with a forward- and backward-path controller cannot be used to achieve asymptotic setpoint tracking when both an actuator offset and sensor bias are present. We provide results for SISO systems that illustrate the limits of integral control in achieving setpoint tracking.

Although the results in this article are confined to the case of SISO transfer functions, extensions to the MIMO case based on state-space models are of interest. In particular, a MIMO treatment of the problem of command following in the presence of actuator offset is given in [5]. For the case of actuator and sensor disturbances that are both measured, setpoint tracking using feedforward control is considered in [6]. Consequently, our results on actuator offset are a special case of [5], whereas our results on sensor bias are not addressed by [6] due to the additional assumption in [6] of measured offsets.

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