

Some Open Problems in Matrix Theory Arising in Linear Systems and Control*

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ABSTRACT

Control theory has long provided a rich source of motivation for developments in matrix theory. Accordingly, we discuss some open problems in matrix theory arising from theoretical and practical issues in linear systems theory and feedback control. The problems discussed include robust stability, matrix exponentials, induced norms, stabilizability and pole assignability, and nonstandard matrix equations. A substantial number of references are included to acquaint matrix theorists with problems and trends in this application area.

1. INTRODUCTION

Feedback control theory has long provided a rich source of motivation for developments in matrix theory. The purpose of this paper is to discuss several open problems in matrix theory that arise from theoretical and practical issues in feedback control theory and the associated area of linear systems theory. Many of these problems are remarkably simple to state, are of intense interest in control theory and applications, and yet remain unsolved. Besides leading to the resolution of these problems, it is hoped that this paper will help to stimulate increased interaction between matrix and control theorists. Accord-

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ingly, the paper includes some brief tutorial discussions and provides motivation for these problems.

The problems we discuss are divided into five topics, namely, robust stability, matrix exponentials, induced norms, stabilizability and pole assignability, and nonstandard matrix equations. It is important to note that these problems are not my own, but have originated in a variety of control- and matrix-theory applications and are due to a multitude of researchers.

2. ROBUST STABILITY

A fundamental problem in the analysis of linear systems is the following: Given a collection of matrices $\mathcal{M} \subset \mathbb{R}^{n \times n}$, determine a subset $\mathcal{M}_0 \subset \mathcal{M}$ such that if every element of \mathcal{M}_0 is stable (that is, each of its eigenvalues has negative real part), then every element of \mathcal{M} is also stable. This problem arises when the modeling data are uncertain and guarantees of stability are desired. A related problem involves a set of polynomials \mathcal{P} rather than a set of matrices. Consider, for example, the set of polynomials

$$\mathcal{P} \triangleq \left\{ s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1s + \beta_0 : \underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, i = 0, \dots, n-1 \right\}, \quad (2.1)$$

where, for $i = 0, \dots, n-1$, the lower and upper coefficient bounds $\underline{\beta}_i$ and $\bar{\beta}_i$ are given. In this case the rather remarkable result of Kharitonov [48] states that every element of \mathcal{P} is stable if every element of \mathcal{P}_0 is stable, where \mathcal{P}_0 is the subset of \mathcal{P} consisting of the following *four* polynomials:

$$\begin{aligned} & s^n + \bar{\beta}_{n-1}s^{n-1} + \bar{\beta}_{n-2}s^{n-2} + \underline{\beta}_{n-3}s^{n-3} + \underline{\beta}_{n-4}s^{n-4} + \cdots, \\ & s^n + \bar{\beta}_{n-1}s^{n-1} + \underline{\beta}_{n-2}s^{n-2} + \underline{\beta}_{n-3}s^{n-3} + \bar{\beta}_{n-4}s^{n-4} + \cdots, \\ & s^n + \underline{\beta}_{n-1}s^{n-1} + \underline{\beta}_{n-2}s^{n-2} + \bar{\beta}_{n-3}s^{n-3} + \bar{\beta}_{n-4}s^{n-4} + \cdots, \\ & s^n + \underline{\beta}_{n-1}s^{n-1} + \bar{\beta}_{n-2}s^{n-2} + \bar{\beta}_{n-3}s^{n-3} + \underline{\beta}_{n-4}s^{n-4} + \cdots, \end{aligned} \quad (2.2)$$

where the 4-cyclic pattern of the coefficients is repeated for successively decreasing powers of s . Thus, to determine whether every polynomial in \mathcal{P} is stable, it suffices to check only these four polynomials. Kharitonov's result has generated considerable interest and has been generalized in numerous directions [6, 7, 9].

The corresponding problem for matrices is, however, much more difficult. Consider, for example, the case in which \mathcal{M} is a polytope of matrices, that is,

$$\mathcal{M} = \left\{ \sum_{i=1}^r \lambda_i M_i : \sum_{i=1}^r \lambda_i = 1, \lambda_i \geq 0, i = 1, \dots, r \right\}, \quad (2.3)$$

where M_1, \dots, M_r are given matrices. In contrast to the situation involving polynomials, it is shown by counterexample in [8] that it does *not* suffice to check the subset

$$\mathcal{M}_0 = \{ \lambda M_i + (1 - \lambda) M_j : 1 \leq i < j \leq r, 0 \leq \lambda \leq 1 \}, \quad (2.4)$$

which consists of all edges (2-dimensional faces) of the polytope. Furthermore, it is shown in [8] that checking \mathcal{M}_0 does not suffice even if each matrix M_i contains only one nonzero element, that is, the case of a hyperrectangle. A set \mathcal{M}_0 that does suffice is illustrated by a result given in [22]. There it is shown that it suffices to check every point in

$$\mathcal{M}_0 = \{ \text{faces of } \mathcal{M} \text{ of dimension } 2n - 4 \}. \quad (2.5)$$

To show that improvement is possible when the elements of \mathcal{M} have special structure, consider

$$\mathcal{M} = \left\{ \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ -\beta_0 & \dots & -\beta_{n-1} \end{bmatrix} : \underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, i = 0, \dots, n-1 \right\}, \quad (2.6)$$

where I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix. In this case it suffices to check the set

$$\mathcal{M}_0 = \left\{ \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ \dots & -\underline{\beta}_{n-4} & -\underline{\beta}_{n-3} & -\bar{\beta}_{n-2} & -\bar{\beta}_{n-1} \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ \dots & -\bar{\beta}_{n-4} & -\underline{\beta}_{n-3} & -\underline{\beta}_{n-2} & -\bar{\beta}_{n-1} \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ \dots & -\bar{\beta}_{n-4} & -\bar{\beta}_{n-3} & -\underline{\beta}_{n-2} & -\underline{\beta}_{n-1} \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ \dots & -\underline{\beta}_{n-4} & -\bar{\beta}_{n-3} & -\bar{\beta}_{n-2} & -\underline{\beta}_{n-1} \end{bmatrix} \right\}, \quad (2.7)$$

which consists of four matrices. That \mathcal{M}_0 given by (2.7) is sufficient is an immediate consequence of Kharitonov's result, since each matrix is in companion form. Note that \mathcal{M}_0 consists of four matrices *regardless* of n . Hence considerable simplification is possible when \mathcal{M} has special structure. A direct, matrix-based proof of this result is unknown. Such a proof could lead to improvements in treating more general matrix polytopes.

3. THE MATRIX EXPONENTIAL

The matrix exponential plays a central role in linear systems and control theory. Here we shall review the role of the matrix exponential, mention a few of its interesting properties, and point out some related unsolved problems.

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \quad (3.1)$$

where the state $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, the control $u(t) \in \mathbb{R}^m$, and $B \in \mathbb{R}^{n \times m}$. The state $x(t)$ is given explicitly by the well-known formula

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s) ds. \quad (3.2)$$

If $x_0 = 0$ and $u(\cdot)$ is allowed to be an arbitrary integrable function on the interval $[0, t]$, then the set of all states $x(t)$ reachable at time t is the subspace of \mathbb{R}^n given by the range of the nonnegative definite matrix $Q(t) \in \mathbb{R}^{n \times n}$ defined by

$$Q(t) \triangleq \int_0^t e^{As}BB^Te^{A^Ts} ds. \quad (3.3)$$

Furthermore, for $t > 0$ the range of $Q(t)$ is independent of t and is given by [18, 56, 74, 95]

$$\text{Im } Q(t) = \text{Im} [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B], \quad (3.4)$$

where Im denotes image or range. If A is asymptotically stable, then $Q \triangleq \lim_{t \rightarrow \infty} Q(t)$ exists and is given by the controllability Gramian

$$Q = \int_0^\infty e^{At}BB^Te^{A^Tt} dt, \quad (3.5)$$

which is the unique solution to the Lyapunov equation [53, 56, 74, 95]

$$0 = AQ + QA^T + BB^T. \tag{3.6}$$

Interesting problems arise immediately if the matrix A is perturbed by another matrix, say \hat{A} . For example, it may be of interest to understand the relationship between e^A and $e^{A+\hat{A}}$ (where $t = 1$ for convenience here). If A and \hat{A} commute, then clearly $e^A e^{\hat{A}} = e^{\hat{A}} e^A = e^{A+\hat{A}}$, whereas if A and \hat{A} do not commute, then $e^A e^{\hat{A}}$, $e^{\hat{A}} e^A$, and $e^{A+\hat{A}}$ are generally different [13]. Furthermore, as shown by examples in [85], $e^A e^{\hat{A}} = e^{\hat{A}} e^A$ does not imply $e^A e^{\hat{A}} = e^{A+\hat{A}}$, $e^A e^{\hat{A}} = e^{\hat{A}} e^A = e^{A+\hat{A}}$ does not imply $A\hat{A} = \hat{A}A$, and $e^A e^{\hat{A}} = e^{A+\hat{A}}$ does not imply $e^{\hat{A}} e^A = e^A e^{\hat{A}}$. If, however, A and \hat{A} have only algebraic entries, then $e^A e^{\hat{A}} = e^{\hat{A}} e^A$ implies that A and \hat{A} commute [85, 86]. If A and \hat{A} have algebraic entries and $e^A e^{\hat{A}} = e^{A+\hat{A}}$, then it is reasonable to conjecture that A and \hat{A} must commute, but this case is not discussed in [85] and remains open. Specializing to the case $\hat{A} = A^T$, a related open question is the following [14]: Does there exist a nonnormal matrix A satisfying either $e^A e^{A^T} = e^{A^T} e^A$ or $e^A e^{A^T} = e^{A+A^T}$? Some relevant results are given in [75].

In a somewhat different vein, the Campbell-Baker-Hausdorff formula from Lie group theory [10, 79, 83, 84, 87] states that if A and \hat{A} have sufficiently small norm, then there exists a matrix D in the Lie algebra generated by $\{A, \hat{A}\}$ that satisfies

$$e^A e^{\hat{A}} = e^D. \tag{3.7}$$

Specifically, D is given by the expansion

$$D = A + \hat{A} + \frac{1}{2} [A, \hat{A}] + \frac{1}{12} [A, [A, \hat{A}]] + \frac{1}{12} [[A, \hat{A}], \hat{A}] + \dots, \tag{3.8}$$

where $[A, \hat{A}] \triangleq A\hat{A} - \hat{A}A$. Of course, at least one such matrix D satisfying (3.7) must always exist, and it need not be unique [11, p. 202]. The expansion (3.8), however, is only locally convergent [83]. Thus (3.8) can only be used to determine the existence of D in the Lie algebra generated by $\{A, \hat{A}\}$ when the norm of $[A, \hat{A}]$ is sufficiently small.

A remarkable result of a related, but slightly different, nature is given in [80]. If A and \hat{A} have sufficiently small norm, then there exist invertible matrices S and T (depending upon A and \hat{A}) such that

$$e^A e^{\hat{A}} = e^{SAS^{-1} + T\hat{A}T^{-1}}. \tag{3.9}$$

Furthermore, it is known that S and T are of the form e^P and e^Q where P and Q are elements of the Lie algebra generated by A and \hat{A} .

An alternative, globally convergent expansion is given by [70]

$$e^{A+\hat{A}} = e^A e^{\hat{A}} + \sum_{k=2}^{\infty} C_k, \quad (3.10)$$

where, for $k = 0, 1, \dots$,

$$C_{k+1} = \frac{1}{k+1} \{ (A + \hat{A})C_k + [\hat{A}, D_k] \}, \quad C_0 = 0, \quad (3.11)$$

$$D_{k+1} = \frac{1}{k+1} \{ AD_k + D_k \hat{A} \}, \quad D_0 = I. \quad (3.12)$$

Another class of related results involves inequalities for spectral functions of products of exponentials. Such bounds may be useful for robust stability of sampled-data control systems [16]. For example, if A and \hat{A} are symmetric, then [24, 57]

$$\text{tr } e^{A+\hat{A}} \leq \text{tr } e^A e^{\hat{A}}, \quad (3.13)$$

while for arbitrary A we also have [12, 23]

$$\text{tr } e^A e^{A^T} \leq \text{tr } e^{A+A^T}. \quad (3.14)$$

A closely related result [76, p. 742] is

$$\lambda_{\max}(e^A e^{A^T}) \leq \lambda_{\max}(e^{A+A^T}). \quad (3.15)$$

An interesting open question that immediately arises is whether or not it is possible to derive (3.12)–(3.15) directly from any of the formulas (3.8), (3.9), or (3.10). In this regard (3.10) appears to be the most promising candidate.

Finally, note that for $t \in [0, \infty)$ (3.14) implies

$$\text{tr } e^{At} e^{A^T t} \leq \text{tr } e^{(A+A^T)t}. \quad (3.16)$$

Hence if A is stable, then the left-hand side of (3.16) will converge to zero, whereas the right-hand side may be unbounded, rendering the bound useless. A generalization of (3.14) in the spirit of (3.9) with $\hat{A} = A^T$ may be useful

here. To resolve the possible conservatism in (3.16) for $t \rightarrow \infty$, it is natural to conjecture generalizations of (3.14) to include terms of the form $e^{A^T P + PA}$, where the positive definite matrix P is chosen as in Lyapunov stability theory to render $A^T P + PA$ negative definite.

4. INDUCED NORMS

The performance of a control system is often measured by its ability to reject undesirable disturbances. One mathematical setting for this problem is to define a class \mathcal{D} of disturbances and a class \mathcal{E} of error signals and consider

$$\dot{x}(t) = Ax(t) + Dw(t), \tag{4.1}$$

$$z(t) = Ex(t), \tag{4.2}$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $w(t) \in \mathbb{R}^d$, $D \in \mathbb{R}^{n \times d}$, $z(t) \in \mathbb{R}^p$, $E \in \mathbb{R}^{p \times n}$, and where $w(\cdot) \in \mathcal{D}$ is the disturbance signal and $z(\cdot) \in \mathcal{E}$ is the error signal. In particular, we wish to determine the “size” of $z(\cdot)$ given that $w(\cdot) \in \mathcal{D}$.

Two settings for this problem are generally considered. In the stochastic case, $w(\cdot)$ denotes white noise, and performance is measured by the steady-state quadratic criterion [53]

$$J = \lim_{t \rightarrow \infty} \mathbb{E} [z^T(t)z(t)] = \text{tr} EQE^T, \tag{4.3}$$

where \mathbb{E} denotes expectation and the state covariance $Q \triangleq \lim_{t \rightarrow \infty} \mathbb{E}[x(t)x^T(t)]$ satisfies

$$0 = AQ + QA^T + DD^T, \tag{4.4}$$

or, equivalently,

$$Q = \int_0^\infty e^{At}DD^Te^{A^T t} dt. \tag{4.5}$$

Note the similarity between the steady-state covariance (4.5) and the controllability Gramian (3.5). It now follows from Parseval's theorem that

$$\begin{aligned} J &= \operatorname{tr} \int_0^\infty E e^{At} D D^T e^{A^T t} E^T dt \\ &= \frac{1}{2\pi} \operatorname{tr} \int_{-\infty}^\infty E(j\omega I_n - A)^{-1} D D^T (-j\omega I_n - A)^{-T} E^T d\omega, \end{aligned} \quad (4.6)$$

or, equivalently,

$$J = \int_0^\infty \|E e^{At} D\|_F^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty \|G(j\omega)\|_F^2 d\omega, \quad (4.7)$$

where $\|M\|_F \triangleq [\operatorname{tr} M M^*]^{1/2}$ denotes the Frobenius norm, and the transfer function $G(s)$ from w to z is given by

$$G(s) \triangleq E(sI_n - A)^{-1} D. \quad (4.8)$$

Note that (4.8) is just the Laplace-transform version of (4.1) and (4.2). Hence, by (4.7),

$$J = \|G(s)\|_2^2, \quad (4.9)$$

where $\|\cdot\|_2$ denotes the H_2 norm [31].

On the other hand, let \mathcal{D} and \mathcal{E} denote L_2 spaces on $[0, \infty)$, and define the induced norm

$$\hat{J} \triangleq \sup_{w(\cdot) \in \mathcal{D}} \frac{\|z\|_2}{\|w\|_2}. \quad (4.10)$$

Then

$$\hat{J} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}[G(j\omega)], \quad (4.11)$$

where σ_{\max} denotes the maximum singular value. That is,

$$\hat{J} = \|G(s)\|_\infty, \quad (4.12)$$

the H_∞ norm of $G(s)$ [31].

Immediately this raises the following question: Do there exist normed spaces \mathcal{D} and \mathcal{E} such that the H_2 norm is induced? From (4.7) the answer to this question would appear to be related to the following question: Is the Frobenius matrix norm induced? Since $\|I_n\|_F = \sqrt{n}$, the answer is no if the domain and ranges spaces of I_n are assigned the same norms. However, this does not rule out the possibility of assigning *different* norms to, say \mathbb{R}^m and \mathbb{R}^n for inducing the Frobenius norm of $M \in \mathbb{R}^{n \times m}$. This remains a fundamental open problem.

More generally, it is possible to define matrix norms that satisfy submultiplicativity ($\|AB\|_\alpha \leq \|A\|_\alpha \|B\|_\alpha$) or mixed submultiplicativity ($\|AB\|_\alpha \leq \|A\|_\beta \|B\|_\gamma$) conditions [37–39]. Is it possible to show that such norms are induced or *cannot* be induced? Note that mixed induced norms satisfy mixed submultiplicativity conditions.

The idea of mixed induced norms can be applied on the operator level as well. In [89], for example, it is shown that if \mathcal{D} is an L_2 space with a Euclidean spatial norm and \mathcal{E} is an L_∞ space also with a Euclidean spatial norm, then

$$\sup_{w(\cdot) \in \mathcal{D}} \frac{\|z\|_\infty}{\|w\|_2} = \lambda_{\max}(EQE^T). \tag{4.13}$$

Hence the H_2 norm (4.3) is an upper bound for the induced norm (4.13), which measures worst-case error amplitude due to bounded energy signals. Related issues involving the norm of the Hankel operator

$$z(t) = \int_0^\infty Ce^{A(t+s)}Bw(s) ds \tag{4.14}$$

are also of interest [33, 89, 90].

5. STABILIZABILITY AND POLE ASSIGNABILITY

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.1}$$

$$y(t) = Cx(t), \tag{5.2}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{l \times n}$, and where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$ denote the state, control, and output respectively. Furthermore,

consider the static output feedback control law

$$u(t) = Ky(t), \quad (5.3)$$

where $K \in \mathbb{R}^{m \times l}$. The word “static” here refers to the fact that the control signal $u(t)$ at time t is a linear function of the output $y(t)$ also at time t . Later we shall replace the static control law (5.3) with a dynamic control law.

The closed-loop system (5.1)–(5.3) is thus given by

$$\dot{x}(t) = (A + BKC)x(t). \quad (5.4)$$

One of the most basic unsolved problems in control theory is the problem of output feedback stabilizability: Determine necessary and sufficient conditions on (A, B, C) under which there exists K such that $A + BKC$ is asymptotically stable. If C is the identity matrix (the full-state feedback case), then the solution is remarkably simple and well known: The pair (A, B) must be stabilizable [96]. This condition can be expressed analytically as [46, p. 206]

$$\text{rank} \begin{bmatrix} \lambda I_n - A & B \end{bmatrix} = n, \quad \lambda \in \sigma(A) \cap \mathbb{C}^{\geq}, \quad (5.5)$$

where I_n is the $n \times n$ identity, $\sigma(A)$ denotes the spectrum of A , and \mathbb{C}^{\geq} denotes the closed right half plane. The condition (5.5) is weaker than controllability as characterized by the condition [42, 46]

$$\text{rank} \begin{bmatrix} \lambda I_n - A & B \end{bmatrix} = n, \quad \lambda \in \sigma(A), \quad (5.6)$$

which is equivalent to the Kalman controllability condition

$$\text{rank} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n. \quad (5.7)$$

If B is the identity matrix, then the dual condition for output feedback stabilizability is detectability, which can be written as

$$\text{rank} \begin{bmatrix} \lambda I_n - A^T & C^T \end{bmatrix} = n, \quad \lambda \in \sigma(A) \cap \mathbb{C}^{\geq}. \quad (5.8)$$

If neither B nor C is the identity, then the output feedback stabilizability problem becomes considerably more difficult. Partial results are given in [2, 4, 19], with [19] focusing on stabilizability for generic classes of systems. Conditions for exact stabilizability, however, are unknown.

For practical control-system implementation, it is often desirable to consider a decentralized static controller structure of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^r B_i u_i(t), \tag{5.9}$$

$$y_i(t) = C_i x(t), \quad i = 1, \dots, r, \tag{5.10}$$

with feedback law

$$u_i(t) = K_i y_i(t), \quad i = 1, \dots, r, \tag{5.11}$$

where, for $i = 1, \dots, r$, one has $u_i(t) \in \mathbb{R}^{m_i}$, $B_i \in \mathbb{R}^{n \times m_i}$, $y_i(t) \in \mathbb{R}^{l_i}$, $C_i \in \mathbb{R}^{l_i \times n}$, and $K_i \in \mathbb{R}^{m_i \times l_i}$. Note that in this problem the control $u_i(t)$ is constrained to depend solely upon the output $y_i(t)$. The number of feedback channels is denoted by r . For this problem the objective is to determine whether K_1, \dots, K_r exist such that the closed-loop dynamics matrix $A + \sum_{i=1}^r B_i K_i C_i$ is stable. Hence this problem can be viewed as a generalization of the centralized problem (5.1)–(5.3) to include a block-diagonal structure constraint on K . Although some results have been obtained for this problem [3], the theory is far from complete.

The decentralized static controller problem is of particular interest in that, as we shall now show, it encompasses the problem of stabilization via *dynamic* compensators of arbitrary fixed order (i.e., dimension). For example, consider

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.12}$$

$$y(t) = Cx(t), \tag{5.13}$$

with dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{5.14}$$

$$u(t) = C_c x_c(t), \tag{5.15}$$

where n_c is a given positive integer denoting the order of the dynamic compensator, $x_c(t) \in \mathbb{R}^{n_c}$, $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times l}$, and $C_c \in \mathbb{R}^{m \times n_c}$. Then the

dynamics matrix of the closed-loop system (5.12)–(5.15) can be written as

$$\begin{aligned} \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_{n_c} \end{bmatrix} A_c \begin{bmatrix} 0 & I_{n_c} \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ I_{n_c} \end{bmatrix} B_c \begin{bmatrix} C & 0 \end{bmatrix} + \begin{bmatrix} B & 0 \end{bmatrix} C_c \begin{bmatrix} 0 & I_{n_c} \end{bmatrix}, \end{aligned} \quad (5.16)$$

which is of the decentralized structure (5.9)–(5.11) with $r = 3$, $K_1 = A_c$, $K_2 = B_c$, and $K_3 = C_c$. If $n_c = n$, then necessary and sufficient conditions for stabilizability are well known, namely, (A, B) must be stabilizable and (C, A) must be detectable [53]. Stabilizability for the case $n_c < n$ is unsolved; for partial results see [59, 73]. Conversely, it is shown in [44] that (A, B) stabilizable and (C, A) detectable is actually a necessary condition for stabilization by means of arbitrary controllers.

A further extension of this problem is to require repetition of certain gain matrices in different feedback channels, that is, to impose constraints of the form $K_j = K_l$ for selected indices. It can be shown using the techniques of [63] that this generalization encompasses all affinely parameterized static and dynamic feedback structures.

Beyond the question of existence, it is of interest to be able to characterize (and perhaps parameterize) the set of stabilizing feedback gains. Such a parameterization would then be useful for selecting gains with desirable properties beyond stabilization.

A refinement of the stabilizability problem is the pole (i.e., eigenvalue) assignability problem. In this problem the goal is to determine feedback gains that place the closed-loop spectrum within a specified region or at specified locations in the left half plane. In the full-state-feedback case $C = I_n$, it is known that if $\lambda \in \sigma(A)$ and

$$\text{rank} \begin{bmatrix} \lambda I_n - A & B \end{bmatrix} = n, \quad (5.17)$$

then all repetitions of the open-loop eigenvalue λ can be arbitrarily reassigned in the closed-loop. Note, however, that if $\lambda \in \sigma(A)$ is a multiple eigenvalue and

$$\text{rank} \begin{bmatrix} \lambda I_n - A & B \end{bmatrix} < n, \quad (5.18)$$

then it may still be possible to place some repetitions of λ , although this is not discernible from (5.18). These observations follow by decomposing the state space into controllable and uncontrollable subspaces. A complete theory of

such questions for the full-state-feedback case in the absence of complete controllability is given in [98, 99]. Hence in the full-state-feedback case controllability is equivalent to pole assignability. In the output-feedback case, however, controllability and pole placement are not equivalent. Although this problem is essentially unsolved, some partial results are available [1, 17, 26, 30, 45, 49, 71, 78].

6. OPTIMAL FEEDBACK CONTROL AND NONSTANDARD MATRIX EQUATIONS

Consider the classical linear-quadratic regulator (LQR) problem: Given the plant

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t), \tag{6.1}$$

$$y(t) = x(t), \tag{6.2}$$

with full-state feedback controller

$$u(t) = Ky(t), \tag{6.3}$$

determine K to minimize

$$J(K) \triangleq \lim_{t \rightarrow \infty} \mathbb{E} [x^T(t)R_1x(t) + u^T(t)R_2u(t)], \tag{6.4}$$

where $R_1 \geq 0$, $R_2 > 0$, and $w(\cdot)$ is white noise. A mathematically equivalent formulation is to replace (6.1) by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{6.5}$$

and minimize

$$\hat{J}(K) \triangleq \int_0^\infty [x^T(t)R_1x(t) + u^T(t)R_2u(t)] dt. \tag{6.6}$$

We shall adopt the white-noise formulation (6.1), since it provides a consistent framework for treating problems with measurement noise considered below.

The solution to the LQR problem is well known: The minimizing feedback gain K is given by

$$K = -R_2^{-1}B^TP, \tag{6.7}$$

where $P \geq 0$ satisfies the matrix algebraic Riccati equation ($\Sigma \triangleq BR_2^{-1}B^T$)

$$0 = A^T P + PA - P \Sigma P + R_1 \quad (6.8)$$

with optimal performance ($V_1 \triangleq D_1 D_1^T$)

$$J(K) = \text{tr} P V_1. \quad (6.9)$$

Properties of solutions of the matrix algebraic Riccati equation such as existence, multiplicity, definiteness, monotonicity, and parameter dependence have been extensively studied [21, 25, 28, 29, 34–36, 50, 54, 55, 60, 62, 64, 66–68, 72, 91–94]. Here we note that the analysis of (6.8) is closely associated with the stabilizability of the pair (A, B) and the detectability of (A, R_1) . Furthermore, many properties of the solutions to the algebraic Riccati equation are obtained by considering the Hamiltonian matrix [66]

$$\mathcal{H} \triangleq \begin{bmatrix} -A & \Sigma \\ R_1 & A^T \end{bmatrix}, \quad (6.10)$$

while monotonicity results [68, 94] involve the symmetric comparison matrix

$$\mathcal{C} \triangleq \begin{bmatrix} R_1 & A^T \\ A & -\Sigma \end{bmatrix}. \quad (6.11)$$

If the full state $x(t)$ is not available for feedback as in (6.2), but rather only partial, noisy measurements of the form

$$y(t) = Cx(t) + D_2 w(t) \quad (6.12)$$

are available, then one may seek a dynamic controller of the form

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \quad (6.13)$$

$$u(t) = C_c x_c(t). \quad (6.14)$$

If $x_c(t) \in \mathbb{R}^n$, then this is the linear-quadratic Gaussian (LQG) problem. Now the solution is given by

$$A_c = A + BC_c - B_cC, \tag{6.15}$$

$$B_c = QC^T V_2^{-1}, \tag{6.16}$$

$$C_c = -R_2^{-1}B^T P, \tag{6.17}$$

where $Q \geq 0$ and $P \geq 0$ are given by the pair of matrix algebraic Riccati equations

$$0 = AQ + QA^T - Q\bar{\Sigma}Q + V_1, \tag{6.18}$$

$$0 = A^T P + PA - P\Sigma P + R_1, \tag{6.19}$$

where $\Sigma \triangleq BR_2^{-1}B^T$, $\bar{\Sigma} \triangleq C^T V_2^{-1}C$, and $V_2 \triangleq D_2 D_2^T$. Furthermore, the optimal performance is given by

$$J(A_c, B_c, C_c) = \text{tr}[QR_1 + PQ\bar{\Sigma}Q] = \text{tr}[PV_1 + QP\Sigma P]. \tag{6.20}$$

Here for simplicity we have assumed uncorrelated plant disturbance and measurement noise, that is, $D_1 D_2^T = 0$. The additional Riccati equation (6.18), which corresponds to the observer portion (6.13) of the dynamic compensator, is a dual version of (6.19). Note that (6.18) and (6.19) are not coupled and that the gain C_c given by (6.17) coincides with the full-state-feedback gain K given by (6.7). This is not a coincidence, but rather is the result of the separation principle of feedback control in the presence of partial, noisy measurements.

Now we consider some extensions of the LQR and LQG problems. Suppose that only partial measurements are available but that these are noise free, that is,

$$y(t) = Cx(t), \tag{6.21}$$

in place of (6.2) or (6.12). Then the optimal static (nondynamic) output-feedback controller of the form

$$u(t) = Ky(t) \tag{6.22}$$

is given by

$$K = -R_2^{-1}B^T P Q C^T (C Q C^T)^{-1}, \tag{6.23}$$

where now $P \geq 0$ and $Q \geq 0$ are given by [58]

$$0 = (A - \Sigma P \nu)Q + Q(A - \Sigma P \nu)^T + V_1, \quad (6.24)$$

$$0 = A^T P + PA - P \Sigma P + \nu_{\perp}^T P \Sigma P \nu_{\perp} + R_1, \quad (6.25)$$

with performance

$$J(K) = \text{tr} P V_1 = \text{tr} [Q(R_1 + \nu^T P \Sigma P \nu)]. \quad (6.26)$$

In (6.24)-(6.26), ν and ν_{\perp} are defined by

$$\nu \triangleq Q C^T (C Q C^T)^{-1} C, \quad \nu_{\perp} \triangleq I_n - \nu, \quad (6.27)$$

under the assumption that $C Q C^T$ is positive definite. A sufficient condition for $C Q C^T$ to be positive definite is $C V_1 C^T$ positive definite. Note that $\nu^2 = \nu$ and $\nu_{\perp}^2 = \nu_{\perp}$, that is, ν and ν_{\perp} are projections. If $C = I_n$, that is, the full-state-feedback case (6.2), then $\nu = I_n$, $\nu_{\perp} = 0$, (6.25) reduces to (6.8), and (6.24) plays no role. In the general case, however, (6.24) and (6.25) must be considered as a coupled system of matrix equations.

Although (6.24) and (6.25) provide a transparent and elegant generalization of (6.8), it is a remarkable fact that virtually nothing is known about their solution properties. It is quite reasonable to conjecture, however, that progress will depend upon the output-feedback stabilizability problem discussed in Section 5, that is, the existence of a matrix K such that $A + B K C$ is asymptotically stable. Hence, these two unsolved problems go hand in hand.

Next we consider a minor variation of the LQG dynamic compensation theory. Specifically, motivated by the practical need for controller simplicity, we constrain the state $x_c(t)$ of the dynamic compensator (6.13), (6.14) to have dimension $n_c \leq n$. With this constraint the optimal controller is now given by [43]

$$A_c = \Gamma A G^T + \Gamma B C_c - B_c C G^T, \quad (6.28)$$

$$B_c = \Gamma Q C^T V_2^{-1}, \quad (6.29)$$

$$C_c = -R_2^{-1} B^T P G^T, \quad (6.30)$$

where $Q \geq 0$, $P \geq 0$, $\hat{Q} \geq 0$, $\hat{P} \geq 0$, and $G, \Gamma \in \mathbb{R}^{n_c \times n}$ are given by

$$0 = AQ + QA^T + V_1 - Q\bar{\Sigma}Q + \tau_{\perp}Q\bar{\Sigma}Q\tau_{\perp}^T, \quad (6.31)$$

$$0 = A^T P + PA + R_1 - P\Sigma P + \tau_{\perp}^T P\Sigma P\tau_{\perp}, \quad (6.32)$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + Q\bar{\Sigma}Q - \tau_{\perp}Q\bar{\Sigma}Q\tau_{\perp}^T, \quad (6.33)$$

$$0 = (A - Q\bar{\Sigma})^T \hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_{\perp}^T P\Sigma P\tau_{\perp}, \quad (6.34)$$

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q}\hat{P} = n_c, \quad (6.35)$$

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^*, \quad (6.36)$$

$$\tau = G^T M \Gamma, \quad \Gamma G^T = I_{n_c}, \quad \tau_{\perp} \triangleq I_n - \tau, \quad M \in \mathbb{R}^{n_c \times n_c}, \quad (6.37)$$

with performance

$$\begin{aligned} J(A_c, B_c, C_c) &= \text{tr} \left[QR_1 + P(Q\bar{\Sigma}Q - \tau_{\perp}Q\bar{\Sigma}Q\tau_{\perp}^T) \right] \\ &= \text{tr} \left[PV_1 + Q(P\Sigma P - \tau_{\perp}^T P\Sigma P\tau_{\perp}) \right]. \end{aligned} \quad (6.38)$$

In (6.36), $(\)^*$ denotes the group generalized inverse [20], which is applicable to $\hat{Q}\hat{P}$, since the product of two nonnegative definite matrices has index one, that is, $\text{rank } \hat{Q}\hat{P} = \text{rank } (\hat{Q}\hat{P})^2$ [5, 69, 97]. Hence, it follows from properties of the group inverse that $\tau^2 = \tau$, that is, τ is idempotent. Note that if $n_c = n$, then \hat{Q} and \hat{P} are positive definite [this forces (A_c, B_c) to be controllable and (C_c, A_c) to be observable], $\tau = I_n$, and $\tau_{\perp} = 0$. In this case (6.31) and (6.32) specialize to (6.18) and (6.19), while (6.33) and (6.34) play no role except to guarantee that (A_c, B_c, C_c) is controllable and observable.

Again a seemingly minor extension of the standard theory has major consequences with respect to the algebraic matrix equations to which it gives rise. As with the static output-feedback problem, it can also be expected that the analysis of (6.13)–(6.34) is related to the existence of gains A_c, B_c, C_c such that the closed-loop dynamics matrix

$$\begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}$$

is asymptotically stable.

Finally, we discuss a further extension of LQR and LQG theory, namely, the enforcement of an H_∞ -norm constraint on the closed-loop transfer function $\tilde{G}(s)$ between disturbances $w(\cdot)$ and performance variables $z(t) = E_{1\infty}x(t) + E_{2\infty}u(t)$. It can be shown [18, p. 167; 28; 88] that the constraint

$$\|\tilde{G}(s)\|_\infty \leq \gamma \quad (6.39)$$

is equivalent to the existence of a solution to the matrix algebraic Riccati equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \gamma^{-2}\tilde{Q}\tilde{E}_\infty^T\tilde{E}_\infty\tilde{Q} + \tilde{V}, \quad (6.40)$$

where $(\tilde{\cdot})$ denotes matrices associated with the closed-loop system. Although the sign of the quadratic term in (6.40) is opposite to the sign appearing in the standard Riccati equation (6.8), it turns out that the existence, uniqueness, and monotonicity results of [54, 94] remain applicable.

In the full-state-feedback case, enforcement of the constraint $\|\tilde{G}(s)\|_\infty \leq \gamma$ leads to [65, 100]

$$0 = A^T P + PA - P\Sigma P + \gamma^{-2}PV_1P + R_{1\infty}, \quad (6.41)$$

with feedback gain $K = -R_2^{-1}B^T P$, where $R_{1\infty} \triangleq E_{1\infty}^T E_{1\infty}$, in place of (6.8). Now the quadratic term $P(\gamma^{-2}V_1 - \Sigma)P$ may be *indefinite*. With the exception of [64], virtually no results on the solutions to such indefinite Riccati equations are available. Similar extensions to dynamic feedback with and without a controller order constraint are given in [15, 27, 33, 41]. As can be expected, the complexity of the Riccati equations characterizing the optimal controllers grows significantly with the imposed constraints. Finally, problems involving *both* order reduction and H_∞ constraints lead to even more complex algebraic equations [15, 40, 41].

7. CONCLUSIONS

Linear systems and control theory have long been rich sources of problems in matrix theory. The objective of this paper is to demonstrate that this situation can be expected to continue strongly into the indefinite future. It goes without saying that such a relationship can only be mutually beneficial.

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