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Steady-state Kalman filtering with an H_∞ error bound *

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Abstract: An estimator design problem is considered which involves both L_2 (least squares) and H_∞ (worst-case frequency-domain) aspects. Specifically, the goal of the problem is to minimize an L_2 state-estimation error criterion subject to a prespecified H_∞ constraint on the state-estimation error. The H_∞ estimation-error constraint is embedded within the optimization process by replacing the covariance Lyapunov equation by a Riccati equation whose solution leads to an upper bound on the L_2 state-estimation error. The principal result is a sufficient condition for characterizing fixed-order (i.e., full- and reduced-order) estimators with bounded L_2 and H_∞ estimation error. The sufficient condition involves a system of modified Riccati equations coupled by an oblique projection, i.e., idempotent matrix. When the H_∞ constraint is absent, the sufficient condition specializes to the L_2 state-estimation result given in [2].

Keywords: Kalman filter; H_∞ norm; reduced-order state estimation; optimal projection equations; Hankel norm.

1. Introduction

One of the fundamental problems in dynamic systems theory is the observation of state variables. Although an extensive theoretical foundation has been developed for the quadratic (least squares) error criterion, state estimation with a worst-case frequency-domain design objective has apparently not been considered. In the present paper we thus extend the least squares formulation to include a frequency-domain bound on the state-estimation error. The underlying idea involves the application of state-space techniques which have recently been developed for H_∞ control design in [1,4-6]. The results of the present paper are thus complementary to the results obtained in [1].

The principal result of the present paper is a sufficient condition which yields full- and reduced-order estimators satisfying an optimized L_2 error bound as well as a prespecified H_∞ error bound. In the full-order case, the H_∞-constrained estimator involves a modified Riccati equation which specializes to the standard steady-state Kalman filter when the H_∞ constraint is absent. In the reduced-order case the H_∞-constrained result leads to a direct generalization of the optimal projection approach developed in [2] for the unconstrained L_2 state-estimation problem. While the L_2-optimal reduced-order state estimator was characterized in [2] by means of a coupled system of one modified Riccati equation and two modified Lyapunov equations, the H_∞-constrained solution involves a coupled system consisting of three modified Riccati equations and one modified Lyapunov equation. As in [2], the coupling is due to the presence of an oblique projection (idempotent matrix) with additional coupling now arising from the H_∞ constraint. When the H_∞ constraint is sufficiently relaxed, these conditions again specialize directly to those given in [2].

We note that the development in the present paper is limited to the case in which the plant is asymptotically stable. These results can also be extended to the unstable plant case, although with additional complexity. This case will thus be treated in a future paper.

The contents of the paper are as follows. After collecting notation in Section 2, the statement of the H_∞-Constrained State-Estimation Problem is given in Section 3. The principal result of this section (Lemma 3.1) shows that if the algebraic

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Lyapunov equation for the covariance is replaced by a modified Riccati equation possessing a nonnegative-definite solution, then the H_∞ estimation-error constraint is enforced and the L_2 state-estimation error criterion is bounded above by an auxiliary cost function. The problem of determining a reduced-order estimator which minimizes this upper bound subject to the Riccati equation constraint is considered in Section 4 as the Auxiliary Minimization Problem. Necessary conditions for the Auxiliary Minimization Problem (Theorem 4.1) are given in the form of a coupled system of modified algebraic Riccati equations. To develop connections with standard Kalman filter theory the full-order estimator result is also given. In Section 5 the necessary conditions of Theorem 4.1 are combined with Lemma 3.1 to yield sufficient conditions for bounded H_∞ and L_2 estimation error. Although our result gives sufficient conditions for H_∞ estimation error, we also state hypotheses under which these conditions are also necessary (Proposition 5.1).

2. Notation and definitions

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$: real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value.

I_r, $()^{\mathrm{T}}$, $0_{r\times s}$, 0_r: $r\times r$ identity matrix, transpose, $r\times s$ zero matrix, $0_{r\times r}$.

tr: trace.

$\sigma_{\max}(Z)$: largest singular value of matrix Z.

$\lambda_{\max}(Z)$: largest eigenvalue of matrix Z with a real spectrum.

$\|Z\|_{\mathrm{F}}$: $[\operatorname{tr} ZZ^{\mathrm{T}}]^{1/2}$ (Frobenius matrix norm).

$\|H(s)\|_\infty$: $\sup_{\omega\in\mathbb{R}}\sigma_{\max}[H(\mathrm{j}\omega)]$.

$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$: $r\times r$ symmetric, nonnegative-definite, positive-definite matrices.

$Z_1\le Z_2$, $Z_1<Z_2$: $Z_2-Z_1\in\mathbb{N}^r$, $Z_2-Z_1\in\mathbb{P}^r$, $Z_1, Z_2\in\mathbb{S}^r$.

n, l, n_e, p, q, r; $\tilde{n}$ positive integers; $n+n_e$; $n_e\le n$.

x, y, y_e, x_e, $\tilde{x}$: n, l, q, n_e, $\tilde{n}$-dimensional vectors.

$$\tilde{x}\triangleq\begin{bmatrix} x\\ x_e\end{bmatrix}.$$

A, C: $n\times n$, $l\times n$ matrices.

D_1, D_2, E: $n\times p$, $l\times p$, $r\times q$ matrices.

L: $q\times n$ matrix.

A_e, B_e, C_e: $n_e\times n_e$, $n_e\times l$, $q\times n_e$ matrices.

$$\tilde{A}\triangleq\begin{bmatrix} A & 0_{n\times n_e}\\ B_eC & A_e\end{bmatrix},$$

$$\tilde{D}\triangleq\begin{bmatrix} D_1\\ B_eD_2\end{bmatrix},\qquad \tilde{E}\triangleq[EL\quad -EC_e].$$

R: $E^{\mathrm{T}}E$, estimation error weighting in $\mathbb{P}^q$.

$w(\cdot)$: p-dimensional standard white noise process.

V_1, V_2: intensity of $D_1w(\cdot)$, $D_2w(\cdot)$; $V_1\triangleq D_1D_1^{\mathrm{T}}\in\mathbb{N}^n$, $V_2\triangleq D_2D_2^{\mathrm{T}}\in\mathbb{P}^l$.

V_{12}: cross intensity of $D_1w(\cdot)$, $D_2w(\cdot)$; $V_{12}\triangleq D_1D_2^{\mathrm{T}}\in\mathbb{R}^{n\times l}$.

$$\tilde{R}\triangleq\begin{bmatrix} L^{\mathrm{T}}RL & -L^{\mathrm{T}}RC_e\\ -C_e^{\mathrm{T}}RL & C_e^{\mathrm{T}}RC_e\end{bmatrix},$$

$$\tilde{V}\triangleq\begin{bmatrix} V_1 & V_{12}B_e^{\mathrm{T}}\\ B_eV_{12}^{\mathrm{T}} & B_eV_2B_e^{\mathrm{T}}\end{bmatrix}.$$

γ: positive constant.

3. Statement of the problem

In this section we introduce the reduced-order state-estimation problem with a constraint on the H_∞ norm of the state-estimation error. Specifically, the transfer function between disturbances and error states is constrained to have H_∞ norm less than γ. In this paper it is assumed that the plant is asymptotically stable, i.e., the eigenvalues of A are in the open left half plane.

H_∞-Constrained State-Estimation Problem. Given the n-th-order observed system

$$\dot{x}(t)=Ax(t)+D_1w(t),\tag{3.1}$$

$$y(t)=Cx(t)+D_2w(t),\tag{3.2}$$

where $t\in[0,\infty)$, determine an n_e-th-order state estimator

$$\dot{x}_e(t)=A_ex_e(t)+B_ey(t),\tag{3.3}$$

$$y_e(t)=C_ex_e(t),\tag{3.4}$$

where $n_e\le n$, which satisfies the following design criteria:

(i) A_e is asymptotically stable;
(ii) the $r \times p$ transfer function

$$H(s) \triangleq \tilde{E}(sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D} \tag{3.5}$$

from disturbances $w(t)$ to error states $E[Lx(t) - y_e(t)] = \tilde{E}\tilde{x}(t)$ satisfies the constraint

$$\| H(s) \|_\infty \le \gamma, \tag{3.6}$$

where $\gamma > 0$ is a given constant; and
(iii) the L_2 state-estimation error criterion

$$J(A_e, B_e, C_e) \triangleq \lim_{t\to\infty} \mathbb{E}\left\{[Lx(t) - y_e(t)]^{\mathrm{T}} R [Lx(t) - y_e(t)]\right\} \tag{3.7}$$

is minimized.

It is useful to note that the augmented system (3.1)–(3.4) can be written as

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \quad t \in [0, \infty), \tag{3.8}$$

and that (3.7) is equivalent to

$$J(A_e, B_e, C_e) = \lim_{t\to\infty} \mathbb{E}\left\{[\tilde{E}\tilde{x}(t)]^{\mathrm{T}}[\tilde{E}\tilde{x}(t)]\right\} = \lim_{t\to\infty} \mathbb{E}[\tilde{x}^{\mathrm{T}}(t)\tilde{R}\tilde{x}(t)]. \tag{3.9}$$

Furthermore, if A_e is asymptotically stable for a given estimator (A_e, B_e, C_e) then the L_2 state-estimation error criterion is given by

$$J(A_e, B_e, C_e) = \operatorname{tr} \tilde{Q}\tilde{R}, \tag{3.10}$$

where the steady-state covariance defined by

$$\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)] \tag{3.11}$$

satisfies the $\tilde{n} \times \tilde{n}$ Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V}. \tag{3.12}$$

Using (3.10) and (3.12) we now show that the criterion (3.7) is an error measure involving the impulse response of (3.8) with respect to an L_2 norm.

Proposition 3.1. *If A_e is asymptotically stable then the L_2 state-estimation criterion* (3.7) *can be written as*

$$J(A_e, B_e, C_e) = \int_0^\infty \| \tilde{E} e^{\tilde{A}t}\tilde{D} \|_F^2 \, dt. \tag{3.13}$$

Proof. It need only be noted that (3.10) is equivalent to

$$\operatorname{tr} \int_0^\infty e^{\tilde{A}t}\tilde{V} e^{\tilde{A}^{\mathrm{T}}t} \, dt \tilde{R} = \operatorname{tr} \int_0^\infty (\tilde{E} e^{\tilde{A}t}\tilde{D})(\tilde{E} e^{\tilde{A}t}\tilde{D})^{\mathrm{T}} \, dt,$$

which is equivalent to (3.13). □

The key step in enforcing (3.6) is to replace the algebraic Lyapunov equation (3.12) by an algebraic Riccati equation. Justification for this technique is provided by the following result.

Lemma 3.1. *Let (A_e, B_e, C_e) be given and assume there exists $\mathcal{Q} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ satisfying*

$$\mathcal{Q} \in \mathbb{N}^{\tilde{n}} \tag{3.14}$$

and

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V}. \tag{3.15}$$

Then

$$(\tilde{A}, \tilde{D}) \text{ is stabilizable} \tag{3.16}$$

if and only if

$$A_e \text{ is asymptotically stable}. \tag{3.17}$$

Furthermore, in this case

$$\| H(s) \|_\infty \le \gamma, \tag{3.18}$$

$$\tilde{Q} \le \mathcal{Q}, \tag{3.19}$$

and

$$J(A_e, B_e, C_e) \le \mathcal{J}(A_e, B_e, C_e, \mathcal{Q}), \tag{3.20}$$

where

$$\mathcal{J}(A_e, B_e, C_e, \mathcal{Q}) \triangleq \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{3.21}$$

Proof. Theorem 3.6 of [9] and (3.16) imply that $(\tilde{A}, [\gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V}]^{1/2})$ is also stabilizable. Using Lemma 12.2 of [9] and the assumed existence of a nonnegative-definite solution to (3.15), it follows that $\tilde{A}$ is asymptotically stable. Since $\tilde{A}$ is lower block triangular, $\tilde{A}$ asymptotically stable implies A_e is asymptotically stable. Conversely, since A is assumed to be asymptotically stable, (3.17) implies $\tilde{A}$ is asymptotically stable and thus (3.16) holds. The proof of (3.18) follows from a standard manipulation of (3.15); for details see Lemma 1 of [8]. To prove (3.19) subtract (3.12) from (3.15) to obtain

$$0=\tilde{A}(\mathscr{Q}-\tilde{Q})+(\mathscr{Q}-\tilde{Q})\tilde{A}^{\mathrm{T}}+\gamma^{-2}\mathscr{Q}\tilde{R}\mathscr{Q}, \quad (3.22)$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\mathscr{Q}-\tilde{Q}=\int_0^{\infty}\mathrm{e}^{\tilde{A}t}\left[\gamma^{-2}\mathscr{Q}\tilde{R}\mathscr{Q}\right]\mathrm{e}^{\tilde{A}^{\mathrm{T}}t}\,\mathrm{d}t\geq 0. \quad (3.23)$$

Finally, (3.20) follows immediately from (3.19). □

Lemma 3.1 shows that the H_∞ constraint is automatically enforced when a nonnegative-definite solution to (3.15) can be shown to exist. Furthermore, the solution $\mathscr{Q}$ provides an upper bound for the steady-state covariance $\tilde{Q}$ along with a bound on the L_2 state-estimation error criterion. Next, we present a partial converse of Lemma 3.1 which guarantees the existence of a nonnegative-definite solution to (3.15) when (3.18) is satisfied.

Lemma 3.2. *Let (A_e, B_e, C_e) be given, suppose A_e is asymptotically stable, and assume the H_∞ state-estimation error constraint* (3.18) *is satisfied. Then there exists a unique nonnegative-definite solution $\mathscr{Q}$ satisfying* (3.15) *and such that $\tilde{A}+\gamma^{-2}\mathscr{Q}\tilde{R}$ is asymptotically stable. Furthermore, $\mathscr{Q}$ is the minimal solution to* (3.15).

Proof. The result is an immediate consequence of Theorems 3 and 2 of [3], pp. 150 and 167, along with the dual of Lemma 12.2 of [9]. □

Finally, we show that the quadratic term $\gamma^{-2}\mathscr{Q}\tilde{R}\mathscr{Q}$ in (3.15) also constrains the Hankel norm of the estimation error $E[Lx(t)-y_e(t)]$ when $\mathscr{Q}$ is positive definite. To show this let $\tilde{P}\in\mathbb{N}^{\tilde{n}}$ be the observability Gramian for the augmented system $(\tilde{A}, \tilde{D}, \tilde{E})$ which satisfies

$$0=\tilde{A}^{\mathrm{T}}\tilde{P}+\tilde{P}\tilde{A}+\tilde{R}. \quad (3.24)$$

Proposition 3.2. *Let (A_e, B_e, C_e) be given and assume there exists $\mathscr{Q}\in\mathbb{P}^{\tilde{n}}$ satisfying* (3.15) *and* (3.16) *or, equivalently,* (3.17). *Then*

$$\lambda_{\max}^{1/2}(\tilde{P}\tilde{Q})\leq\gamma. \quad (3.25)$$

Proof. Since $\mathscr{Q}$ is invertible, (3.15) implies

$$0=\gamma^2\tilde{A}^{\mathrm{T}}\mathscr{Q}^{-1}+\gamma^2\mathscr{Q}^{-1}\tilde{A}+\gamma^2\mathscr{Q}^{-1}\tilde{V}\mathscr{Q}^{-1}+\tilde{R}. \quad (3.26)$$

Next, subtract (3.24) from (3.26) to obtain

$$0=\tilde{A}^{\mathrm{T}}(\gamma^2\mathscr{Q}^{-1}-\tilde{P})+(\gamma^2\mathscr{Q}^{-1}-\tilde{P})\tilde{A} +\gamma^2\mathscr{Q}^{-1}\tilde{V}\mathscr{Q}^{-1}, \quad (3.27)$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\gamma^2\mathscr{Q}^{-1}-\tilde{P}=\int_0^{\infty}\mathrm{e}^{\tilde{A}^{\mathrm{T}}t}\left[\gamma^2\mathscr{Q}^{-1}\tilde{V}\mathscr{Q}^{-1}\right]\mathrm{e}^{\tilde{A}t}\,\mathrm{d}t\geq 0. \quad (3.28)$$

Thus (3.28) implies $\tilde{P}\leq\gamma^2\mathscr{Q}^{-1}$, or equivalently, $\mathscr{Q}^{1/2}\tilde{P}\mathscr{Q}^{1/2}\leq\gamma^2 I_{\tilde{n}}$. Hence,

$$\gamma^2\geq\lambda_{\max}(\mathscr{Q}^{1/2}\tilde{P}\mathscr{Q}^{1/2})=\lambda_{\max}(\tilde{P}^{1/2}\mathscr{Q}\tilde{P}^{1/2}) \geq\lambda_{\max}(\tilde{P}^{1/2}\tilde{Q}\tilde{P}^{1/2})=\lambda_{\max}(\tilde{P}\tilde{Q}). \quad □$$

4. The auxiliary minimization problem and necessary conditions for optimality

As discussed in the previous section, the replacement of (3.12) by (3.15) enforces the H_∞ state-estimation error constraint and results in an upper bound for the L_2 state-estimation error criterion. That is, given an estimator (A_e, B_e, C_e) satisfying the H_∞ estimation constraint, the *actual* L_2 state-estimation error criterion is guaranteed to be no worse than the bound $\mathscr{J}(A_e, B_e, C_e, \mathscr{Q})$ if (3.15) is solvable. Hence, $\mathscr{J}(A_e, B_e, C_e, \mathscr{Q})$ can be interpreted as an *auxiliary cost* which leads to the following optimization problem.

Auxiliary Minimization Problem. Determine $(A_e, B_e, C_e, \mathscr{Q})$ which minimizes $\mathscr{J}(A_e, B_e, C_e, \mathscr{Q})$ subject to (3.14) and (3.15).

It follows from Lemma 3.1 that the satisfaction of (3.14)–(3.16) leads to (1) A_e stable; (2) H_∞ estimation error bound γ; and (3) an upper bound (3.21) for the L_2 state-estimation error criterion. Hence it remains to determine (A_e, B_e, C_e) which minimizes $\mathscr{J}(A_e, B_e, C_e, \mathscr{Q})$ and thus provides an optimized bound for the *actual* L_2 criterion $J(A_e, B_e, C_e)$. Rigorous derivation of the necessary conditions for the Auxiliary Minimization Problem requires additional technical assumptions. Specifically, we restrict $(A_e, B_e, C_e, \mathscr{Q})$ to the open set

$$\begin{aligned}\mathscr{S} \triangleq \{(A_e, B_e, C_e, \mathscr{Q})\colon\ & \mathscr{Q} \in \mathbb{P}^{\tilde{n}},\\ & \tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R} \text{ is asymptotically stable,}\\ & \text{and } (A_e, B_e, C_e) \text{ is controllable}\\ & \text{and observable.}\}\end{aligned}$$

Remark 4.1. The set $\mathscr{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the Auxiliary Minimization Problem. Specifically, the requirement that $\mathscr{Q}$ be positive definite replaces (3.14) by an open set constraint, the stability of $\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}$ serves as a normality condition, and (A_e, B_e, C_e) minimal is a nondegeneracy condition.

The following lemma is needed for the statement of the main result.

Lemma 4.1. *Let $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and suppose* rank $\hat{Q}\hat{P} = n_e$. *Then there exist $n_e \times n$ G, Γ and $n_e \times n_e$ invertible M, unique except for a change of basis in $\mathbb{R}^{n_e}$, such that*

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma, \tag{4.1}$$

$$\Gamma G^{\mathrm{T}} = I_{n_e}. \tag{4.2}$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^{\mathrm{T}} \Gamma, \tag{4.3}$$

$$\tau_{\perp} \triangleq I_n - \tau, \tag{4.4}$$

are idempotent and have rank n_e and $n - n_e$, respectively. If, in addition,

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = n_e, \tag{4.5}$$

then

$$\hat{Q} = \tau \hat{Q}, \qquad \hat{P} = \hat{P}\tau. \tag{4.6), (4.7}$$

Finally, if $P \in \mathbb{N}^n$ then the inverse

$$S \triangleq (I_n + \gamma^{-2}\hat{Q}P)^{-1} \tag{4.8}$$

exists.

Proof. Conditions (4.1)–(4.7) are a direct consequence of Theorem 6.2.5 of [7]. To prove that the inverse in (4.8) exists, note that since the eigenvalues of $\hat{Q}P$ coincide with the eigenvalues of the nonnegative-definite matrix $P^{1/2}\hat{Q}P^{1/2}$, it follows that $\hat{Q}P$ has nonnegative eigenvalues. Thus, the eigenvalues of $I_n + \gamma^{-2}\hat{Q}P$ are all greater than one so that the above inverse exists. □

Finally, for arbitrary $Q \in \mathbb{R}^{n \times n}$ define

$$Q_a \triangleq QC^{\mathrm{T}} + V_{12}, \qquad \Sigma \triangleq L^{\mathrm{T}} R L. \tag{4.9}$$

Theorem 4.1. *If $(A_e, B_e, C_e, \mathscr{Q}) \in \mathscr{S}$ solves the Auxiliary Minimization Problem then there exist Q, P, $\hat{Q}$, $\hat{P} \in \mathbb{N}^n$ such that*

$$A_e = \Gamma(A - Q_a V_2^{-1} C - \gamma^{-4} Q\Sigma QPS) G^{\mathrm{T}}, \tag{4.10}$$

$$B_e = \Gamma Q_a V_2^{-1}, \tag{4.11}$$

$$C_e = L(I_n + \gamma^{-2} QPS) G^{\mathrm{T}}, \tag{4.12}$$

$$\mathscr{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix}, \tag{4.13}$$

and such that Q, P, $\hat{Q}$, $\hat{P}$ satisfy

$$\begin{aligned} 0 = {} & AQ + QA^{\mathrm{T}} + V_1 + \gamma^{-2} Q\Sigma Q \\ & - Q_a V_2^{-1} Q_a^{\mathrm{T}} + \tau_{\perp} Q_a V_2^{-1} Q_a^{\mathrm{T}} \tau_{\perp}^{T}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} 0 = {} & A^{\mathrm{T}} P + PA - \gamma^{-4} S^{\mathrm{T}} PQ\Sigma QPS \\ & + \tau_{\perp}^{\mathrm{T}} (I_n + \gamma^{-2} QPS)^{\mathrm{T}} \Sigma (I_n + \gamma^{-2} QPS) \tau_{\perp}, \end{aligned} \tag{4.15}$$

$$\begin{aligned} 0 = {} & (A - \gamma^{-4} Q\Sigma QPS)\hat{Q} + \hat{Q}(A - \gamma^{-4} Q\Sigma QPS)^{\mathrm{T}} \\ & + \gamma^{-6} \hat{Q} S^{\mathrm{T}} PQ\Sigma QPS\hat{Q} \\ & + Q_a V_2^{-1} Q_a^{\mathrm{T}} - \tau_{\perp} Q_a V_2^{-1} Q_a^{\mathrm{T}} \tau_{\perp}^{\mathrm{T}}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} 0 = {} & (A - Q_a V_2^{-1} C + \gamma^{-2} Q\Sigma)^{\mathrm{T}} \hat{P} \\ & + \hat{P}(A - Q_a V_2^{-1} C + \gamma^{-2} Q\Sigma) \\ & + (I_n + \gamma^{-2} QPS)^{\mathrm{T}} \Sigma (I_n + \gamma^{-2} QPS) \\ & - \tau_{\perp}^{\mathrm{T}} (I_n + \gamma^{-2} QPS)^{\mathrm{T}} \Sigma (I_n + \gamma^{-2} QPS) \tau_{\perp}, \end{aligned} \tag{4.17}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_e. \tag{4.18}$$

Furthermore, the auxiliary cost is given by

$$\begin{aligned} & \mathscr{J}(A_e, B_e, C_e, \mathscr{Q}) \\ & \quad = \operatorname{tr} L^{\mathrm{T}} R L (Q + \gamma^{-4} QPS\hat{Q}S^{\mathrm{T}} PQ). \end{aligned} \tag{4.19}$$

Conversely, if there exist Q, P, $\hat{Q}$, $\hat{P} \in \mathbb{N}^n$ satisfying (4.14)–(4.18), *then $(A_e, B_e, C_e, \mathscr{Q})$ given by*

(4.10)–(4.13) *satisfies* (3.14) *and* (3.15) *with the auxiliary cost* (3.21) *given by* (4.19).

Proof. See Appendix A. □

Remark 4.2. Theorem 4.1 presents necessary conditions for the Auxiliary Minimization Problem which explicitly synthesize extremal full- and reduced-order estimators (A_e, B_e, C_e). If the H_∞ estimation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then $S = I_n$. In this case equations (4.16) and (4.17) become decoupled from (4.15) and thus (4.15) becomes superfluous. Furthermore, (4.14), (4.16) and (4.17) specialize to the optimal projection equations obtained in [2].

As discussed in [2], in the full-order (Kalman filter) case $n_e = n$, $G = \Gamma^{-1}$ and thus $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$ without loss of generality. To develop further connections with the standard Kalman filter theory assume

$$V_{12} = 0. \tag{4.20}$$

In this case (4.15) implies that $P = 0$ so that the gain expressions (4.10)–(4.12) become

$$A_e = A - QC^{\mathrm{T}}V_2^{-1}C, \tag{4.21}$$

$$B_e = QC^{\mathrm{T}}V_2^{-1}, \tag{4.22}$$

$$C_e = L, \tag{4.23}$$

while equations (4.14)–(4.16) and auxiliary cost (4.19) specialize to

$$0 = AQ + QA^{\mathrm{T}} + V_1 + \gamma^{-2}QL^{\mathrm{T}}RLQ - QC^{\mathrm{T}}V_2^{-1}CQ, \tag{4.24}$$

$$\mathscr{J}(A_e, B_e, C_e, \mathscr{Q}) = \operatorname{tr} L^{\mathrm{T}}RLQ. \tag{4.25}$$

Remark 4.3. Note that the necessary conditions for the full-order problem involve one modified Riccati equation. This equation is similar to the observer Riccati equation with the additional quadratic term $\gamma^{-2}QL^{\mathrm{T}}RLQ$. Finally, note that when the H_∞ estimation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, (4.24) reduces to the standard observer Riccati equation of steady-state Kalman filter theory.

5. Sufficient conditions for combined L_2/H_∞ estimation

In this section we combine Lemma 3.1 with the converse of Theorem 4.1 to obtain our main result guaranteeing constrained H_∞ state-estimation error and an optimized L_2 state-estimation error bound.

Theorem 5.1. *Suppose there exist* $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ *satisfying* (4.14)–(4.18) *and let* $(A_e, B_e, C_e, \mathscr{Q})$ *be given by* (4.10)–(4.13). *Then* $(\tilde{A}, \tilde{D})$ *is stabilizable if and only if* A_e *is asymptotically stable. In this case, the transfer function* $H(s)$ *defined by* (3.5) *satisfies the* H_∞ *state-estimation error constraint*

$$\| H(s) \|_\infty \le \gamma, \tag{5.1}$$

and the L_2 *state-estimation error criterion* (3.7) *satisfies the bound*

$$J(A_e, B_e, C_e) \le \operatorname{tr} L^{\mathrm{T}}RL(Q + \gamma^{-4}QPS\hat{Q}S^{\mathrm{T}}PQ). \tag{5.2}$$

Proof. The converse portion of Theorem 4.1 implies that $\mathscr{Q}$ given by (4.13) satisfies (3.14) and (3.15). It now follows from Lemma 3.1 that the stabilizability condition (3.16) is equivalent to the asymptotic stability of A_e, the H_∞ state-estimation error constraint (3.18) holds, and the L_2 state-estimation error criterion (3.7) satisfies the bound (3.20) which, by (4.19), is equivalent to (5.2). □

In applying Theorem 5.1 the principal issue concerns conditions on the problem data under which the coupled Riccati equations (4.14)–(4.17) possess nonnegative-definite solutions. Clearly, for γ sufficiently large, (4.14)–(4.17) approximate the pure least squares problem considered in [2]. The important case of interest, however, involves small γ so that significant H_∞ estimation is enforced. Thus, if (5.1) can be satisfied for a given $\gamma > 0$, it is of interest to know whether one such fixed-order estimator can be obtained by solving (4.14)–(4.17). Lemma 3.2 guarantees that (3.15) possesses a solution for any fixed-order estimator satisfying (5.1). Thus our sufficient conditions will also be *necessary* so long as the Auxiliary Minimization Problem possesses at least one extremal over $\mathscr{S}$. When this is the case we have the following result.

Proposition 5.1. *Let* γ^* *denote the infimum of* $\| H(s) \|_\infty$ *over all asymptotically stable fixed-order estimators and suppose that the Auxiliary Minimization Problem has an extremal for all* $\gamma > \gamma^*$. *Then* *for all* $\gamma > \gamma^*$ *there exist* Q, P, $\hat{Q}$, $\hat{P} \in \mathbb{N}^n$ *satisfying* (4.14)–(4.17).

Appendix A: Proof of Theorem 4.1

To optimize (3.21) over the open set $\mathscr{S}$ subject to the constraint (3.15), form the Lagrangian

$$\mathscr{L}(A_e, B_e, C_e, \mathscr{Q}, \mathscr{P}, \lambda) \triangleq \operatorname{tr}\{\lambda \mathscr{Q}\tilde{R} + [\tilde{A}\mathscr{Q} + \mathscr{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathscr{Q}\tilde{R}\mathscr{Q} + \tilde{V}]\mathscr{P}\}, \tag{A.1}$$

where the Lagrange multipliers $\lambda \geq 0$ and $\mathscr{P} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are not both zero. We thus obtain

$$\frac{\partial\mathscr{L}}{\partial\mathscr{Q}} = (\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R})^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}) + \lambda\tilde{R}. \tag{A.2}$$

Setting $\partial\mathscr{L}/\partial\mathscr{Q} = 0$ yields

$$0 = (\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R})^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}) + \lambda\tilde{R}. \tag{A.3}$$

Since $\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}$ is assumed to be stable, $\lambda = 0$ implies $\mathscr{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, $\mathscr{P}$ is nonnegative definite.

Now partition $\tilde{n}\times\tilde{n}$, Q, P into $n\times n$, $n\times n_e$, and $n_e\times n_e$ subblocks as

$$\mathscr{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}, \qquad \mathscr{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix},$$

and for notational convenience define

$$\mathscr{P}\mathscr{Q} = \begin{bmatrix} Z_1 & Z_{12} \\ Z_{21} & Z_2 \end{bmatrix},$$

where

$$Z_1 \triangleq P_1Q_1 + P_{12}Q_{12}^{\mathrm{T}}, \qquad Z_{12} \triangleq P_1Q_{12} + P_{12}Q_2,$$

$$Z_{21} \triangleq P_{12}^{\mathrm{T}}Q_1 + P_2Q_{12}^{\mathrm{T}}, \qquad Z_2 \triangleq P_{12}^{\mathrm{T}}Q_{12} + P_2Q_2.$$

Thus, with $\lambda = 1$ the stationarity conditions are given by

$$\frac{\partial\mathscr{L}}{\partial\mathscr{Q}} = (\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R})^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}) + \tilde{R} = 0, \tag{A.4}$$

$$\frac{\partial\mathscr{L}}{\partial A_e} = Z_2 = 0, \tag{A.5}$$

$$\frac{\partial\mathscr{L}}{\partial B_e} = Z_{21}C^{\mathrm{T}} + P_{12}^{\mathrm{T}}V_{12} + P_2B_eV_2 = 0, \tag{A.6}$$

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial C_e} &= 2RC_eQ_2 + 2\gamma^{-2}RC_eZ_{12}^{\mathrm{T}}Q_{12} \\ &\quad - 2RLQ_{12} - \gamma^{-2}RLZ_1^{\mathrm{T}}Q_{12} \\ &\quad - \gamma^{-2}RLQ_1Z_{12} - \gamma^{-2}RLZ_{21}^{\mathrm{T}}Q_2 \\ &= 0.\end{aligned} \tag{A.7}$$

Expanding (3.15) and (A.4) yields

$$0 = AQ_1 + Q_1A^{\mathrm{T}} + V_1 + \gamma^{-2}(Q_1L^{\mathrm{T}} - Q_{12}C_e^{\mathrm{T}})R(Q_1L^{\mathrm{T}} - Q_{12}C_e^{\mathrm{T}})^{\mathrm{T}}, \tag{A.8}$$

$$\begin{aligned}0 &= AQ_{12} + Q_{12}A_e^{\mathrm{T}} + Q_1C^{\mathrm{T}}B_e^{\mathrm{T}} + V_{12}B_e^{\mathrm{T}} \\ &\quad + \gamma^{-2}Q_1L^{\mathrm{T}}RLQ_{12} - \gamma^{-2}Q_{12}C_e^{\mathrm{T}}RLQ_{12} \\ &\quad - \gamma^{-2}Q_1L^{\mathrm{T}}RC_eQ_2 + \gamma^{-2}Q_{12}C_e^{\mathrm{T}}RC_eQ_2,\end{aligned} \tag{A.9}$$

$$\begin{aligned}0 &= A_eQ_2 + Q_2A_e^{\mathrm{T}} + B_eCQ_{12} + Q_{12}^{\mathrm{T}}C^{\mathrm{T}}B_e^{\mathrm{T}} + B_eV_2B_e^{\mathrm{T}} \\ &\quad + \gamma^{-2}(Q_{12}^{\mathrm{T}}L^{\mathrm{T}} - Q_2C_e^{\mathrm{T}})R(Q_{12}^{\mathrm{T}}L^{\mathrm{T}} - Q_2C_e^{\mathrm{T}})^{\mathrm{T}},\end{aligned} \tag{A.10}$$

$$\begin{aligned}0 &= A^{\mathrm{T}}P_1 + P_1A + C^{\mathrm{T}}B_e^{\mathrm{T}}P_{12}^{\mathrm{T}} + P_{12}B_eC \\ &\quad + \gamma^{-2}L^{\mathrm{T}}RLZ_1^{\mathrm{T}} + \gamma^{-2}Z_1L^{\mathrm{T}}RL \\ &\quad - \gamma^{-2}L^{\mathrm{T}}RC_eZ_{12}^{\mathrm{T}} - \gamma^{-2}Z_{12}C_e^{\mathrm{T}}RL + L^{\mathrm{T}}RL,\end{aligned} \tag{A.11}$$

$$\begin{aligned}0 &= A^{\mathrm{T}}P_{12} + P_{12}A_e + C^{\mathrm{T}}B_e^{\mathrm{T}}P_2 \\ &\quad + \gamma^{-2}L^{\mathrm{T}}RLZ_{21}^{\mathrm{T}} - \gamma^{-2}Z_1L^{\mathrm{T}}RC_e \\ &\quad + \gamma^{-2}Z_{12}C_e^{\mathrm{T}}RC_e - L^{\mathrm{T}}RC_e,\end{aligned} \tag{A.12}$$

$$\begin{aligned}0 &= A_e^{\mathrm{T}}P_2 + P_2A_e - \gamma^{-2}C_e^{\mathrm{T}}RLZ_{21}^{\mathrm{T}} \\ &\quad - \gamma^{-2}Z_{21}L^{\mathrm{T}}RC_e + C_e^{\mathrm{T}}RC_e.\end{aligned} \tag{A.13}$$

Now define the $n\times n$ matrices

$$Q \triangleq Q_1 - Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \qquad P \triangleq P_1 - P_{12}P_2^{-1}P_{12}^{\mathrm{T}},$$

$$\hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \qquad \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^{\mathrm{T}},$$

$$\tau \triangleq -Q_{12}Q_2^{-1}P_2^{-1}P_{12}^{\mathrm{T}},$$

and the $n_e\times n$, $n_e\times n_e$, and $n_e\times n$ matrices

$$G \triangleq Q_2^{-1}Q_{12}^{\mathrm{T}}, \quad M \triangleq Q_2P_2, \quad \Gamma \triangleq -P_2^{-1}P_{12}^{\mathrm{T}}.$$

The existence of Q_2^{-1} and P_2^{-1} follows from the fact that (A_e, B_e, C_e) is minimal. See [1,2] for details. Note that $\tau = G^{\mathrm{T}}\Gamma$. Clearly, Q, P, $\hat{Q}$, and $\hat{P}$ are symmetric and nonnegative definite.

Next note that with the above definitions, (A.5) implies (4.2) and that (4.1) holds. Hence $\tau = G^{\mathrm{T}}\Gamma$ is idempotent, i.e., $\tau^2 = \tau$. Sylvester's inequality yields (4.18). Note also that (4.6) and (4.7) hold.

The components of $\mathcal{Q}$ and $\mathcal{P}$ can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G, and Γ as

$$Q_1 = Q + \hat{Q}, \qquad P_1 = P + \hat{P}, \tag{A.14}$$

$$Q_{12} = \hat{Q}\Gamma^{\mathrm{T}}, \qquad P_{12} = -\hat{P}G^{\mathrm{T}}, \tag{A.15}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^{\mathrm{T}}, \qquad P_2 = G\hat{P}G^{\mathrm{T}}. \tag{A.16}$$

Next note that by using (A.14)–(A.16), (A.7) becomes

$$C_e\hat{S} = L\left[I_n + \gamma^{-2}(Q + \hat{Q})P\right]G^{\mathrm{T}},$$

where

$$\hat{S} \triangleq I_{n_e} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}}.$$

To prove that $\hat{S}$ is invertible use (4.6) and (4.3) and note that

$$\begin{aligned} I_{n_e} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}} &= I_{n_e} + \gamma^{-2}\Gamma\hat{Q}\tau^{\mathrm{T}}PG^{\mathrm{T}} \\ &= I_{n_e} + \gamma^{-2}(\Gamma\hat{Q}\Gamma^{\mathrm{T}})(GPG^{\mathrm{T}}). \end{aligned}$$

Since $\Gamma\hat{Q}\Gamma^{\mathrm{T}}$ and GPG^{T} are nonnegative definite, their product has nonnegative eigenvalues. Thus each eigenvalue of $I_{n_e} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}}$ is real and is greater than unity. Hence $\hat{S}$ is invertible. Now note that by using (4.2) and (4.3) it can be shown that

$$G^{\mathrm{T}}\hat{S}^{-1} = SG^{\mathrm{T}}.$$

The expressions (4.11), (4.12) and (4.13) follow from (A.6), (A.7), (4.8) and the definition of $\mathcal{Q}$ by using the above identities. Next, computing either Γ(A.9) − (A.10) or G(A.12) + (A.13) yields (4.10). Substituting this expression for A_e into (A.8) − (A.13) it follows that (A.10) = Γ(A.9) and (A.13) = G(A.12). Thus, (A.10) and (A.13) are superfluous and can be omitted. Next, using

$$(\mathrm{A.8}) + G^{\mathrm{T}}\Gamma(\mathrm{A.9})G - (\mathrm{A.9})G - [(\mathrm{A.9})G]^{\mathrm{T}}$$

and

$$G^{\mathrm{T}}\Gamma(\mathrm{A.9})G - (\mathrm{A.9})G - [(\mathrm{A.9})G]^{\mathrm{T}}$$

yields (4.14) and (4.16). Using

$$(\mathrm{A.11}) + \Gamma^{\mathrm{T}}G(\mathrm{A.12})\Gamma - (\mathrm{A.12})\Gamma - [(\mathrm{A.12})\Gamma]^{\mathrm{T}}$$

and

$$\Gamma^{\mathrm{T}}G(\mathrm{A.12})\Gamma - (\mathrm{A.12})\Gamma - [(\mathrm{A.12})\Gamma]^{\mathrm{T}}$$

yields (4.15) and (4.17).

Finally, to prove the converse we use (4.10)–(4.18) to obtain (3.15) and (A.4)–(A.7). Let A_e, B_e, C_e, G, Γ, τ, Q, P, $\hat{Q}$, $\hat{P}$, $\mathcal{Q}$ be as in the statement of Theorem 4.1 and define Q_1, Q_{12}, Q_2, P_1, P_{12}, P_2 by (A.14)–(A.16). Using (4.4), (4.11) and (4.12) it is easy to verify (A.6) and (A.7). Finally, substitute the definitions of Q, P, $\hat{Q}$, $\hat{P}$, G, Γ and τ into (4.14)–(4.17) along with (4.2), (4.3), (4.6) and (4.7) to obtain (3.15) and (A.4). Finally, note that

$$\mathcal{Q} = \begin{bmatrix} Q & 0_{n\times n_e} \\ 0_{n_e\times n} & 0_{n_e} \end{bmatrix} + \begin{bmatrix} I_n \\ \Gamma \end{bmatrix}\hat{Q}\begin{bmatrix} I_n & \Gamma^{\mathrm{T}} \end{bmatrix},$$

which shows that $\mathcal{Q} \geq 0$. □

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