Journal of Mathematical Systems, Estimation, and Control
Vol. 6, No. 1, 1996, pp. 79–103

The Multivariable Parabola Criterion for Robust Controller Synthesis: A Riccati Equation Approach*

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Abstract

In 1967 Bergen and Sapiro derived an absolute (frequency domain) stability criterion that unifies the classical circle and Popov criteria. A slightly weaker version of this combined criterion has a graphical interpretation in the Popov (rather than Nyquist) plane in terms of a parabola. Our goal in this paper is to reformulate and generalize the parabola criterion in terms of Riccati equations. Besides providing a multivariable extension, this formulation clarifies connections to state space bounded real and positive real theory and provides the basis for robust controller synthesis.

Key words: robust stability and performance, Popov criterion, circle criterion, parameter-dependent Lyapunov functions

1 Introduction

One of the most basic issues in system theory is the stability of feedback interconnections. Four of the most fundamental results concerning stability of feedback systems are the small gain, positivity, circle, and Popov theorems. In a recent paper [8], each result was specialized to the problem of robust stability involving linear uncertainty, and a Lyapunov function framework was established providing connections between these classical results and robust stability via state space methods. Furthermore, it was pointed out in [8] that both gain and phase properties can be simultaneously accounted for by means of the circle criterion which yields the small

*Received January 15, 1993; received in final form July 13, 1993.
†This research was supported in part by the Air Force Office of Scientific Research under Grant F49620-92-J-0127 and Contract F49620-91-C-0019 and the National Science Foundation under Research Initiation Grant ECS-9109558.

gain theorem and positivity theorem as special cases. It is important to note that since positivity theory and bounded real theory can be obtained from the circle criterion and vice versa, all three results can be viewed as equivalent from a mathematical point of view. However, the engineering ramifications of the ability to include phase information can be significant [3]. As shown in [8], the main difference between the small gain, positivity, and circle theorems versus the Popov theorem is that the former results guarantee robustness with respect to arbitrarily, time-varying uncertainty while the latter does not. This is not surprising since the Lyapunov function foundation of the small gain, positivity, and circle theorems is based upon conventional or fixed quadratic Lyapunov functions which guarantee stability with respect to arbitrarily time-varying perturbations. Since time-varying parameter variations can destabilize a system even when the parameter variations are confined to a region in which constant variations are nondestabilizing, a feedback controller designed for time-varying parameter variations may unnecessarily sacrifice performance when the uncertain real parameters are actually constant.

Whereas the small gain, positivity, and circle results are based upon fixed quadratic Lyapunov functions, the Popov result is based upon a quadratic Lyapunov function that is a *function* of the parametric uncertainty. Thus, in effect, the Popov result guarantees stability by means of a *family* of Lyapunov functions. For robust stability, this situation corresponds to the construction of a parameter-dependent quadratic Lyapunov function [9,10]. A key aspect of this approach (see [9,10]) is the fact that it does *not* apply to arbitrarily time-varying uncertainties, which renders it less conservative than fixed quadratic Lyapunov functions (such as the small gain, positivity, and circle results) in the presence of constant real parameter uncertainty. An immediate application of the parameter-dependent Lyapunov function framework of [9,10] is the reinterpretation and generalization of the classical Popov criterion as a parameter-dependent Lyapunov function for fully coupled constant linear parametric uncertainty.

The main contribution of this paper is the unification of the circle and Popov criteria via a parameter-dependent Lyapunov function framework that yields both results as special cases. The unification of the circle and Popov criteria per se is not new to this paper. Indeed, a parabola test which accomplishes this goal was originally developed in [2] and further studied in [19]. However, these results are confined to SISO systems and rely on graphical techniques. The present paper thus has four specific goals:

1. to provide a general framework for the parabola test in terms of parameter-dependent Lyapunov functions in the spirit of [9,10];
2. to obtain a state space characterization of the parabola test via Riccati equations;

3. to obtain a *multivariable* extension of the parabola test for parametric uncertainty; and
4. to use these results for robust controller synthesis.

To illustrate how the parabola test unifies the circle and Popov criteria, consider the plant G in a feedback configuration with uncertainty block Δ as shown in Figure 1. Introducing the multiplier $I + Ns$ into the loop yields the configuration in Figure 2. Applying positivity to the transfer function $(I+Ns)G(s)$ now yields the familiar Popov test. Next consider the equivalent formulation shown in Figure 3 which involves the introduction of an offset transfer function M_1 in parallel with Δ and in feedback about G [20,21]. The resulting configuration (Figure 4) now involves a shifted Δ (by M_1) and a bilinear transformation of G. Letting $M_1 = 0$ recovers the Popov formulation, while $N = 0$ yields the circle formulation. The simultaneous presence of both N and M_1 leads to a slightly stronger version of the parabola test [2] (see Remarks 4.7 and 4.8 for details).

Since these transformations in general do not commute with controller optimization techniques, they must be introduced at an early stage prior to the synthesis procedure. Specifically, if uncertainty in the control and measurement matrices is considered, then the resulting transformations will be functions of the controller gains.

Although from a mathematical point of view the use of shifts and bilinear transformations leads to equivalent results [19-21], the use of these transformations can yield less conservative results in practice [3]. In the present paper we give a *new* uncertainty characterization of the form $M_1 \leq \Delta \leq M_2$, where M_1 and M_2 are symmetric. The advantage of this new characterization is that it does not place limitations on the sign of the model uncertainty. Of particular interest is the special case $-M_1 = M_2$, where M_1 is symmetric.

Notation

$\mathcal{R}, \mathcal{R}^{r\times s}, \mathcal{R}^r$	real numbers, $r \times s$ real matrices, $\mathcal{R}^{r\times 1}$
$\mathcal{C}, \mathcal{C}^{r\times s}, \mathcal{C}^r$	complex numbers, $r \times s$ complex matrices, $\mathcal{C}^{r\times 1}$
$\mathcal{E}$,tr, $0_{r\times s}$	expectation, trace, $r \times s$ zero matrix
$\bar{\lambda}$	complex conjugate of $\lambda \in \mathcal{C}$
$I_r, (\ )^{\mathrm{T}}, (\ )^*$	$r \times r$ identity, transpose, complex conjugate transpose
$\rho(\ ), \sigma_{\max}(\ )$	spectral radius, largest singular value
$\mathcal{S}^r, \mathcal{N}^r, \mathcal{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \leq Z_2,\ Z_1 < Z_2$	$Z_2 - Z_1 \in \mathcal{N}^r, Z_2 - Z_1 \in \mathcal{P}^r, Z_1, Z_2 \in \mathcal{S}^r$
$\Vert Z\Vert_{\mathrm{F}}$	$[\mathrm{tr}\ ZZ^*]^{1/2}$ (Frobenius matrix norm)
$\Vert G(s)\Vert_2$	$[(1/2\pi)\int_{-\infty}^{\infty} \Vert G(\jmath\omega)\Vert_{\mathrm{F}}^2 \mathrm{d}\omega]^{1/2}$

2 Robust Stability and Performance Problems

Let $\mathcal{U} \subset \mathcal{R}^{n\times n}$ denote a set of perturbations ΔA of a given nominal dynamics matrix $A \in \mathcal{R}^{n\times n}$. We begin by considering the question of whether or not $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$.

Robust Stability Problem Determine whether the linear system

$$\dot{x}(t) = (A + \Delta A)x(t), \quad t \in [0, \infty), \tag{2.1}$$

is asymptotically stable for all $\Delta A \in \mathcal{U}$.

To consider the problem of robust performance, we introduce an external disturbance model involving white noise signals as in standard LQG (H_2) theory. The robust performance problem concerns the worst-case H_2 norm, that is, the worst-case (over $\mathcal{U}$) of the expected value of a quadratic form involving outputs $z(t) = Ex(t)$, where $E \in \mathcal{R}^{q\times n}$, when the system is subjected to a standard white noise disturbance $w(t) \in \mathcal{R}^d$ with weighting $D \in \mathcal{R}^{n\times d}$.

Robust Performance Problem For the disturbed linear system

$$\dot{x}(t) = (A + \Delta A)x(t) + Dw(t), \quad t \in [0, \infty), \tag{2.2}$$

$$z(t) = Ex(t), \tag{2.3}$$

where $w(\cdot)$ is a zero-mean $d-$dimensional white noise signal with intensity I_d, determine a performance bound β satisfying

$$J(\mathcal{U}) \triangleq \sup_{\Delta A\in\mathcal{U}} \limsup_{t\to\infty} \mathcal{E}\{\|z(t)\|_2^2\} \leq \beta. \tag{2.4}$$

As shown in Section 5, (2.2) and (2.3) will denote a control system in closed-loop configuration subjected to external white noise disturbances and for which $z(t)$ denotes the state and control regulation error.

Of course, since D and E may be rank deficient, there may be cases in which a finite performance bound β satisfying (2.4) exists while (2.1) is not asymptotically stable over $\mathcal{U}$. In practice, however, robust performance is mainly of interest when (2.1) is robustly stable. Next, we express the H_2 performance measure (2.4) in terms of the observability Gramian for the pair $(A + \Delta A, E)$. For convenience, define the $n \times n$ nonnegative-definite matrices $R \triangleq E^TE$ and $V \triangleq DD^T$.

Lemma 2.1 *Suppose $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then*

$$J(\mathcal{U}) = \sup_{\Delta A\in\mathcal{U}} \operatorname{tr} P_{\Delta A}V = \sup_{\Delta A\in\mathcal{U}} \|G_{\Delta A}(s)\|_2^2, \tag{2.5}$$

where $P_{\Delta A} \in \mathcal{R}^{n\times n}$ is the unique, nonnegative-definite solution to

$$0 = (A + \Delta A)^{\mathrm{T}}P_{\Delta A} + P_{\Delta A}(A + \Delta A) + R, \tag{2.6}$$

and

$$G_{\Delta A}(s) \triangleq E[sI - (A + \Delta A)]^{-1} D. \tag{2.7}$$

Proof: See [9,10]. □

In the present paper our approach is to obtain robust stability as a consequence of sufficient conditions for robust performance. Such conditions are developed in the following sections.

3 Robust Stability and Performance

ROBUST STABILITY AND PERFORMANCE VIA PARAMETER–DEPENDENT LYAPUNOV FUNCTIONS

The key step in obtaining robust stability and performance is to bound the uncertain terms $\Delta A^{\mathrm{T}} P_{\Delta A} + P_{\Delta A} \Delta A$ in the Lyapunov equation (2.6) by means of a parameter-dependent bounding function $\Omega(P, \Delta A)$ which guarantees robust stability by means of a family of Lyapunov functions. As shown in [9,10], this framework corresponds to the construction of a parameter-dependent Lyapunov function that guarantees robust stability. As discussed in [9,10], a key feature of this approach is the fact that it constrains the class of allowable time-varying uncertainties thus reducing conservatism in the presence of constant real parameter uncertainty. The following result is fundamental and forms the basis for all later developments.

Theorem 3.1 *Let* $\Omega_0\colon \mathcal{N}^n \to \mathcal{S}^n$ and $P_0\colon \mathcal{U} \to \mathcal{S}^n$ *be such that*

$$\Delta A^{\mathrm{T}} P + P \Delta A \le \Omega_0(P) - [(A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)], \quad \Delta A \in \mathcal{U}, P \in \mathcal{N}^n, \tag{3.1}$$

and suppose that there exists $P \in \mathcal{N}^n$ *satisfying*

$$0 = A^{\mathrm{T}} P + PA + \Omega_0(P) + R \tag{3.2}$$

and such that $P + P_0(\Delta A)$ *is nonnegative definite for all* $\Delta A \in \mathcal{U}$. *Then*

$$(A + \Delta A, E) \text{ is detectable}, \ \ \Delta A \in \mathcal{U}, \tag{3.3}$$

if and only if

$$A + \Delta A \text{ is asymptotically stable}, \ \ \Delta A \in \mathcal{U}. \tag{3.4}$$

In this case,

$$P_{\Delta A} \le P + P_0(\Delta A), \quad \Delta A \in \mathcal{U}, \tag{3.5}$$

where $P_{\Delta A}$ is given by (2.6). Therefore,

$$J(\mathcal{U}) \leq \text{tr } PV + \sup_{\Delta A \in \mathcal{U}} \text{ tr } P_0(\Delta A)V. \tag{3.6}$$

If, in addition, there exists $\bar{P}_0 \in \mathcal{S}^n$ such that

$$P_0(\Delta A) \leq \bar{P}_0, \quad \Delta A \in \mathcal{U}, \tag{3.7}$$

then

$$J(\mathcal{U}) \leq \beta, \tag{3.8}$$

where

$$\beta \triangleq \text{tr}[(P + \bar{P}_0)V]. \tag{3.9}$$

Proof: We stress that in (3.1) P denotes an arbitrary element of $\mathcal{N}^n$, whereas in (3.2) P denotes a specific solution of the modified Lyapunov equation. This minor abuse of notation considerably simplifies the presentation. To begin, note that for all $\Delta A \in \mathcal{R}^{n\times n}$, (3.2) is equivalent to

$$0 = (A + \Delta A)^{\mathrm{T}} P + P(A + \Delta A) + \Omega_0(P) - (\Delta A^{\mathrm{T}} P + P\Delta A) + R. \tag{3.10}$$

Adding and subtracting $(A+\Delta A)^{\mathrm{T}} P_0(\Delta A)+P_0(\Delta A)(A+\Delta A)$ to and from (3.10) yields

$$\begin{aligned} 0 = {} & (A + \Delta A)^{\mathrm{T}}(P + P_0(\Delta A)) + (P + P_0(\Delta A))(A + \Delta A) \\ & + \Omega_0(P) - [(A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)] \\ & - (\Delta A^{\mathrm{T}} P + P\Delta A) + R. \end{aligned} \tag{3.11}$$

Hence, by assumption, (3.11) has a solution $P \in \mathcal{N}^n$ for all $\Delta A \in \mathcal{R}^{n\times n}$. If ΔA is restricted to the set $\mathcal{U}$ then, by (3.1), the expression

$$\Omega_0(P) - [(A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)] - (\Delta A^{\mathrm{T}} P + P\Delta A)$$

is nonnegative definite. Thus, if the detectability condition (3.3) holds for all $\Delta A \in \mathcal{U}$, then it follows from Theorem 3.6 of [23] that $(A + \Delta A, [R + \Omega(P, \Delta A) - (\Delta A^{\mathrm{T}} P + P\Delta A)]^{1/2})$ is detectable for all $\Delta A \in \mathcal{U}$, where

$$\Omega(P, \Delta A) \triangleq \Omega_0(P) - [(A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)]. \tag{3.12}$$

It now follows from (3.11) and Lemma 12.2 of [23] that $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Conversely, if $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$, then (3.3) is immediate. Now, subtracting (2.6) from (3.11) yields

$$\begin{aligned} 0 = &(A+\Delta A)^{\mathrm{T}}(P+P_0(\Delta A)-P_{\Delta A})+(P+P_0(\Delta A)-P_{\Delta A})(A+\Delta A) \\ &+\Omega_0(P)-[(A+\Delta A)^{\mathrm{T}}P_0(\Delta A)+P_0(\Delta A)(A+\Delta A)] \\ &-(\Delta A^{\mathrm{T}}P+P\Delta A), \quad \Delta A \in \mathcal{U}, \end{aligned} \tag{3.13}$$

or, equivalently, since $A+\Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$,

$$\begin{aligned} P+P_0(\Delta A)-P_{\Delta A} = &\int_0^\infty e^{(A+\Delta A)^{\mathrm{T}}t}[\Omega(P,\Delta A)-(\Delta A^{\mathrm{T}}P+P\Delta A)] \\ &\cdot e^{(A+\Delta A)t}\mathrm{d}t \geq 0, \quad \Delta A \in \mathcal{U}, \end{aligned} \tag{3.14}$$

which implies (3.5). The performance bounds (3.6), (3.8) are now an immediate consequence of (2.5), (3.5), and (3.7). □

Remark 3.1 If R is positive definite then the detectability hypothesis of Theorem 3.1 is automatically satisfied.

Remark 3.2 Theorem 3.1 can be strengthened by noting that the detectability assumption is, in a sense, superfluous. To see this, first note that robust stability concerns only the undisturbed system (2.1) while R involves the H_2 performance weighting. Hence robust stability is guaranteed by the existence of a solution $P \in \mathcal{N}^n$ satisfying (3.2) with R replaced by αI_n for some $\alpha > 0$. For this replacement detectability is automatic (see previous remark). For robust performance, however, P in (3.5) must be obtained from (3.2).

Note that, with $\Omega(P, \Delta A)$ defined by (3.12), condition (3.1) can be written as

$$\Delta A^{\mathrm{T}}P+P\Delta A \leq \Omega(P,\Delta A), \quad \Delta A \in \mathcal{U}, \quad P \in \mathcal{N}^n, \tag{3.15}$$

where $\Omega(P, \Delta A)$ is a function of the uncertain parameters ΔA. For convenience we shall say that $\Omega(\cdot,\cdot)$ is a *parameter-dependent bounding function* or, to be consistent with [9,10], a *parameter-dependent Ω-bound.*

Finally, we note that the parameter-dependent Ω-bound framework establishing robust stability given by Theorem 3.1 is equivalent to the existence of a parameter-dependent Lyapunov function of the form $V(x) = x^T(P+P_0(\Delta A))x$ which also establishes robust stability. For further details see [8-10].

4 Construction and Connections

CONSTRUCTION OF PARAMETER–DEPENDENT LYAPUNOV FUNCTIONS AND CONNECTIONS TO THE MULTIVARIABLE PARABOLA CRITERION

In this section we assign explicit structure to the set $\mathcal{U}$ and the parameter-dependent bounding function $\Omega(\cdot,\cdot)$. Specifically, the uncertainty set $\mathcal{U}$ is defined by

$$\mathcal{U} \triangleq \{\Delta A \in \mathcal{R}^{n\times n}\colon\ \Delta A = B_0 F C_0,\ \text{where } F \in \mathcal{F}\}, \tag{4.1}$$

where $\mathcal{F}$ is a given subset of the set $\hat{\mathcal{F}}$, which is defined by

$$\hat{\mathcal{F}} \triangleq \{F \in \mathcal{R}^{m_0\times m_0}\colon (F-M_1)^{\mathrm{T}}[(M_2-M_1)^{-1}+(M_2-M_1)^{-\mathrm{T}}](F-M_1) \leq (F-M_1)+(F-M_1)^{\mathrm{T}}\}. \tag{4.2}$$

In (4.1) and (4.2), $B_0 \in \mathcal{R}^{n\times m_0}$ and $C_0 \in \mathcal{R}^{m_0\times n}$ are fixed matrices denoting the structure of the uncertainty, $F \in \mathcal{R}^{m_0\times m_0}$ is an uncertain matrix, and M_1, M_2 are given $m_0 \times m_0$ matrices such that $(M_2-M_1)^{-1}$ exists.

As an aside we simplify the set $\hat{\mathcal{F}}$ in the case in which F, M_1, and M_2 are symmetric and $M_2 - M_1$ is positive definite.

Lemma 4.1 *Let $F, M_1, M_2 \in \mathcal{S}^{m_0}$ and $M_2 - M_1 \in \mathcal{P}^{m_0}$. Then $(F - M_1)(M_2-M_1)^{-1}(F-M_1) \leq F - M_1$ if and only if $M_1 \leq F \leq M_2$.*

Proof: The proof follows as in the proof of Lemma 4.4 of [9,10]. □

Thus, suppose that M_1, M_2 are symmetric and $M_2 - M_1$ is positive definite. Then the set of symmetric matrices F in $\hat{\mathcal{F}}$ is given by

$$\hat{\mathcal{F}}_{\mathrm{s}} \triangleq \{F \in \mathcal{S}^{m_0}\colon\ M_1 \leq F \leq M_2\}. \tag{4.3}$$

If, furthermore, F in $\hat{\mathcal{F}}$ is constrained to have the diagonal structure $\mathrm{diag}[F_1, F_2, \ldots, F_{m_0}]$, then $M_{1i} \leq F_i \leq M_{2i}$, $i = 1, \ldots, m_0$, where $M_1 = \mathrm{diag}[M_{11}, M_{12}, \ldots, M_{1m_0}]$ and $M_2 = \mathrm{diag}\,[M_{21}, M_{22}, \ldots, M_{2m_0}]$. More generally, $\mathcal{F}$ may consist of those matrices having repeated elements and/or blocks on the diagonal of the form $\mathrm{diag}\,[F_1, F_1, F_1, F_2, \ldots, F_{m_0}]$. As shown in [7,8] the uncertainty set $\mathcal{U}$ involves a magnitude and phase constraint on the uncertainty unlike small gain or structured stability radii type uncertainty structures [12,13] which only involve a magnitude constraint on the uncertainty of the form $F^{\mathrm{T}}F \leq \gamma I$.

For the structure of $\mathcal{U}$ satisfying (4.1), the parameter-dependent bound $\Omega(\cdot,\cdot)$ satisfying (3.12) can now be given a concrete form. Since the elements ΔA in $\mathcal{U}$ are parameterized by the elements F in $\mathcal{F}$, for convenience in the following results we shall write $P_0(F)$ in place of $P_0(\Delta A)$. Furthermore, we introduce a key definition that will be used in subsequent developments.

Definition 4.1 *Let $M_1, M_2, N \in \mathcal{R}^{m_0 \times m_0}$. Then $\mathcal{F}$ and N are compatible if $(F - M_1)^{\mathrm{T}} N$ is symmetric for all $F \in \mathcal{F}$. Furthermore, $\mathcal{F}$ and N are strongly compatible if, in addition, $(F - M_1)^{\mathrm{T}} N$ is nonnegative-definite for all $F \in \mathcal{F}$.*

Proposition 4.1 *Let $M_1, M_2, N \in \mathcal{R}^{m_0 \times m_0}$ be such that $\mathcal{F}$ and N are compatible, $M_2 - M_1$ is positive definite, and*

$$(M_2 - M_1)^{-1} - NC_0B_0 + [(M_2 - M_1)^{-1} - NC_0B_0]^{\mathrm{T}} > 0. \tag{4.4}$$

Then the functions

$$\begin{aligned} \Omega_0(P) = &\left[C_0 + NC_0(A + B_0M_1C) + B_0^{\mathrm{T}}P\right]^{\mathrm{T}} \\ &\cdot \left[(M_2 - M_1)^{-1} - NC_0B_0 + [(M_2 - M_1)^{-1} - NC_0B_0]^{\mathrm{T}}\right]^{-1} \\ &\cdot [C_0 + NC_0(A + B_0M_1C) + B_0^{\mathrm{T}}P] + PB_0M_1C_0 \\ &+ C_0^{\mathrm{T}}M_1^{\mathrm{T}}B_0^{\mathrm{T}}P, \end{aligned} \tag{4.5}$$

$$P_0(F) = C_0^{\mathrm{T}}(F - M_1)^{\mathrm{T}}NC_0, \tag{4.6}$$

satisfy (3.1) with $\mathcal{U}$ given by (4.1).

Proof: Since by (4.4) $(M_2 - M_1)^{-1} - NC_0B_0 + [(M_2 - M_1)^{-1} - NC_0B_0]^{\mathrm{T}} > 0$ and by (4.2) $F - M_1 + (F - M_1)^{\mathrm{T}} - (F - M_1)^{\mathrm{T}}[2(M_2 - M_1)^{-1}](F - M_1) \geq 0$, it follows that

$$\begin{aligned} 0 \leq &\left[[C_0 + NC_0(A + B_0M_1C_0) + B_0^{\mathrm{T}}P]\right. \\ &\left. -[(M_2 - M_1)^{-1} - NC_0B_0 + (M_2 - M_1)^{-1} - NC_0B_0)^{\mathrm{T}}](F - M_1)C_0\right]^{\mathrm{T}} \\ &\cdot [(M_2 - M_1)^{-1} - NC_0B_0 + \left((M_2 - M_1)^{-1} - NC_0B_0\right)^{\mathrm{T}}]^{-1} \\ &\cdot \left[[C_0 + NC_0(A + B_0M_1C_0) + B_0^{\mathrm{T}}P] - [(M_2 - M_1)^{-1} - NC_0B_0\right. \\ &\left. + \left((M_2 - M_1)^{-1} - NC_0B_0\right)^{\mathrm{T}}](F - M_1)C_0\right] \\ &+ C_0^{\mathrm{T}}\left[(F - M_1) + (F - M_1)^{\mathrm{T}} - (F - M_1)^{\mathrm{T}}\right. \\ &\left. \cdot [2(M_2 - M_1)^{-1}](F - M_1)\right] C_0 \\ = &\Omega_0(P) - PB_0M_1C_0 - C_0^{\mathrm{T}}M_1^{\mathrm{T}}B_0^{\mathrm{T}}P \\ &- [C_0 + NC_0(A + B_0M_1C_0) + B_0^{\mathrm{T}}P]^{\mathrm{T}} \\ &\cdot (F - M_1)C_0 - C_0^{\mathrm{T}}(F - M_1)^{\mathrm{T}}[C_0 + NC_0(A + B_0M_1C_0) + B_0^{\mathrm{T}}P] \\ &+ C_0^{\mathrm{T}}(F - M_1)^{\mathrm{T}}[(M_2 - M_1)^{-1} - NC_0B_0 + \left((M_2 - M_1)^{-1} - NC_0B_0\right)^{\mathrm{T}}] \\ &\cdot (F - M_1)C_0 + C_0^{\mathrm{T}}[(F - M_1) + (F - M_1)^{\mathrm{T}} \\ &- (F - M_1)^{\mathrm{T}}[2(M_2 - M_1)^{-1}](F - M_1)]C_0 \end{aligned}$$

$$\begin{aligned}
&= \Omega_0(P) - A^{\mathrm{T}} C_0^{\mathrm{T}} N^{\mathrm{T}} (F - M_1) C_0 - C_0^{\mathrm{T}} M_1^{\mathrm{T}} B_0^{\mathrm{T}} C_0^{\mathrm{T}} N^{\mathrm{T}} (F - M_1) C_0 \\
&\quad - P B_0 F C_0 - C_0^{\mathrm{T}} (F - M_1)^{\mathrm{T}} N C_0 A - C_0^{\mathrm{T}} (F - M_1)^{\mathrm{T}} N C_0 B_0 M_1 C_0 \\
&\quad - C_0^{\mathrm{T}} F^{\mathrm{T}} B_0^{\mathrm{T}} P - C_0^{\mathrm{T}} (F - M_1)^{\mathrm{T}} N C_0 B_0 (F - M_1) C_0 \\
&\quad - C_0^{\mathrm{T}} (F - M_1)^{\mathrm{T}} B_0^{\mathrm{T}} C_0^{\mathrm{T}} N^{\mathrm{T}} (F - M_1) C_0 \\
&= \Omega_0(P) - [(A + \Delta A)^{\mathrm{T}} P_0(F) + P_0(F)(A + \Delta A)] - [\Delta A^{\mathrm{T}} P + P \Delta A],
\end{aligned}$$

which proves (3.1) with $\mathcal{U}$ given by (4.1). □

Remark 4.1 Note that by setting $M_1 = 0$, one recovers the parameter-dependent Ω-bound considered in [9,10] which corresponds to a generalized multivariable version of the Popov criterion for fully coupled linear uncertainty.

Remark 4.2 Note that, *unlike* the results of [9,10], $P_0(0) = -C_0^{\mathrm{T}} M_1^{\mathrm{T}} N C_0 \neq 0$ and $\Omega_0(P)$ is *not* necessarily nonnegative definite. See [4] for further discussion on indefinite parameter-dependent Ω-bounds resulting in indefinite Riccati/Lyapunov type equations.

If $M_2 - M_1$ is positive definite, then we have the following norm bound inequality on $F - M_1$ for all $F \in \hat{\mathcal{F}}$.

Lemma 4.2 *Let $F, M_1, M_2 \in \mathcal{R}^{m_0 \times m_0}$ and assume that $M_2 - M_1$ is positive definite. Then*

$$\sigma_{\max}(F - M_1) \leq \sigma_{\max}(M_2 - M_1). \tag{4.7}$$

Proof: The proof is a slight generalization of the proof of Lemma 4.1 of [10] and hence is omitted. □

If $\mathcal{F}$ and N are compatible and $M_2 - M_1$ is positive definite, then it follows from Lemma 4.2 that there exists a matrix $\mu \in \mathcal{N}^{m_0}$ such that $(F - M_1)^{\mathrm{T}} N \leq \mu$ for all $F \in \mathcal{F}$.

Next, using Theorem 3.1 and Proposition 4.1 we have the following immediate result.

Theorem 4.1 *Let $M_1, M_2, N \in \mathcal{R}^{m_0 \times m_0}$ be such that $\mathcal{F}$ and N are strongly compatible, $M_2 - M_1$ is positive definite, and (4.4) is satisfied. Furthermore, suppose there exists a nonnegative-definite matrix P satisfying*

$$\begin{aligned}
0 = {} & (A + B_0 M_1 C_0)^{\mathrm{T}} P + P(A + B_0 M_1 C_0) \\
& + [C_0 + N C_0 (A + B_0 M_1 C_0) + B_0^{\mathrm{T}} P]^{\mathrm{T}} [(M_2 - M_1)^{-1} \\
& - N C_0 B_0 + \left((M_2 - M_1)^{-1} - N C_0 B_0\right)^{\mathrm{T}}]^{-1} \\
& \cdot [C_0 + N C_0 (A + B_0 M_1 C_0) + B_0^{\mathrm{T}} P] + R.
\end{aligned} \tag{4.8}$$

Then

$$(A+\Delta A, E) \text{ is detectable}, \quad \Delta A \in \mathcal{U}, \tag{4.9}$$

if and only if

$$A+\Delta A \text{ is asymptotically stable}, \quad \Delta A \in \mathcal{U}. \tag{4.10}$$

In this case,

$$J(\mathcal{U}) \le \text{tr } PV + \sup_{F\in\mathcal{F}} tr\, C_0^{\mathrm{T}}(F-M_1)^{\mathrm{T}}NC_0 \le tr[(P+C_0^{\mathrm{T}}\mu C_0)V]. \tag{4.11}$$

Proof: The result is a direct specialization of Theorem 3.1 using Proposition 4.1. We only note that $P_0(\Delta A)$ now has the form $P_0(F) = C_0^{\mathrm{T}}(F - M_1)^{\mathrm{T}}NC_0$. Since by the strong compatibility assumption $(F-M_1)^{\mathrm{T}}N \ge 0$ for all $F \in \mathcal{F}$ it follows that $P+P_0(F)$ is nonnegative definite for all $F \in \mathcal{F}$ as required by Theorem 3.1. □

Note that asymptotic stability in Theorem 4.1 is guaranteed by the parameter-dependent Lyapunov function

$$V(x) = x^{\mathrm{T}}(P + C_0^{\mathrm{T}}[F-M_1]^{\mathrm{T}}NC_0)x.$$

Remark 4.3 The condition that $(F-M_1)^{\mathrm{T}}N = N^{\mathrm{T}}(F-M_1),\ F \in \mathcal{F}$, represents an intimate relationship between the matrix N and the structure of the matrices F in $\mathcal{F}$. In fact, this relationship is analogous to the commuting assumption between the D-scales and Δ blocks used in μ-analysis and synthesis and serves to explicitly enforce structure in the uncertainty F. It is easy to see that there *always exists* such a matrix N even if $F \in \mathcal{F}$ is neither diagonal nor symmetric. For example, if $F = F_0 I_{m_0}$, and $M_1 = M_0 I_{m_0}$, where F_0 and M_0 are scalars, then N can be an arbitrary nonnegative-definite matrix. Alternatively, if $N = N_0 I_{m_0}$, then $F - M_1$ may be nondiagonal. Of course, $F - M_1$ and N may have more intricate structure, for example, they may be block diagonal with commuting blocks situated on the diagonal.

As discussed later in this section, generalizations of the classical parabola criterion apply only to decoupled multivariable nonlinearities. Although limited to linear uncertainty, the set $\mathcal{U}$, however, allows a richer class of multivariable uncertainty inasmuch as F may represent a fully populated uncertain matrix. To see how such multivariable uncertainty may be useful in practice, consider the multiple degree of freedom vibrational system

$$M_0\ddot{x}(t) + C_0\dot{x}(t) + (K_0+\Delta K)x(t) = 0,$$

where M_0, C_0, and K_0 denote generalized mass, damping, and stiffness coefficients, respectively, and where ΔK denotes stiffness uncertainty. In state space form this system can be written as

$$\dot{z}(t) = \begin{bmatrix} 0 & I \\ -M_0^{-1}(K_0 + \Delta K) & -M_0^{-1}C_0 \end{bmatrix} z(t),$$

where $z(t) = [x^{\mathrm{T}}(t)\ \dot{x}^{\mathrm{T}}(t)]^{\mathrm{T}}$. In accordance with (4.1), a representation of the uncertain component of the system dynamics is thus given by

$$B_0 F C_0 = \begin{bmatrix} 0 \\ -M_0^{-1} \end{bmatrix} \Delta K [I \quad 0].$$

Assuming the stiffness uncertainty satisfies the bounds $M_1 \leq \Delta K \leq M_2$, which with $F = \Delta K$, is precisely the condition considered in Lemma 4.1.

Next, we establish connections between the parameter-dependent bounding function formed by (4.5) and (4.6) and the classical parabola test [2,19]. Furthermore, by exploiting results from positivity theory it is possible to guarantee the existence of a positive-definite solution to (4.8). First, however, we present additional notation and definitions and a key lemma concerning strongly positive real transfer functions. Let $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denote a state space realization of a transfer function $G(s)$, that is, $G(s) = C(sI - A)^{-1}B + D$. The notation "$\overset{\min}{\sim}$" denotes a minimal realization. Furthermore, an *asymptotically stable transfer function* is a transfer function each of whose poles is in the open left half plane.

A square transfer function $G(s)$ is called *positive real* [1, p. 216] if 1) all poles of $G(s)$ are in the closed left half plane and 2) $G(s) + G^*(s)$ is nonnegative definite for all $\mathrm{Re}[s] > 0$. A square transfer function $G(s)$ is called *strictly positive real* [16,22] if 1) $G(s)$ is asymptotically stable and 2) $G(j\omega) + G^*(j\omega)$ is positive definite for all real ω. Finally, a square transfer function $G(s)$ is *strongly positive real* if it is strictly positive real and $D + D^{\mathrm{T}} > 0$, where $D \triangleq G(\infty)$.

Lemma 4.3 *The transfer function* $G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is strongly positive real if and only if* $D + D^{\mathrm{T}} > 0$ *and there exist positive-definite matrices* P *and* R *such that*

$$0 = A^{\mathrm{T}}P + PA + (C - B^{\mathrm{T}}P)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(C - B^{\mathrm{T}}P) + R. \qquad (4.12)$$

Proof: See [7]. □

Remark 4.4 Lemma 4.3 is a special case of the Kalman-Yakubovich-Popov lemma (KYP) [1,18]. Specifically, it follows from the KYP lemma that $G(s)$ is positive real if and only if there exist matrices P, L, W, where P is positive definite, such that

$$\begin{aligned} 0 &= A^{\mathrm{T}}P + PA + L^{\mathrm{T}}L, \\ 0 &= B^{\mathrm{T}}P - C + W^{\mathrm{T}}L, \\ 0 &= D + D^{\mathrm{T}} - W^{\mathrm{T}}W. \end{aligned}$$

Now, it can be shown that in the case where $G(s)$ is strongly positive real the above three equations are equivalent to the single Riccati equation given by (4.12). For further details see [8,9].

Next, using Lemma 4.3 we obtain a sufficient condition for the existence of a solution to (4.8).

Theorem 4.2 *Let*

$$\mathcal{G}(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A + B_0M_1C_0 & -B_0 \\ \hline C_0 + NC_0(A + B_0M_1C_0) & (M_2 - M_1)^{-1} - NC_0B_0 \end{array}\right].$$

Then $\mathcal{G}(s)$ is strongly positive real, if and only if there exist positive definite matrices P and R satisfying (4.8).

Proof: The proof is an immediate consequence of Lemma 4.3. □

Remark 4.5 Note that the frequency domain condition in Theorem 4.2 is similar in principle to the frequency domain condition given by Hinrichsen and Pritchard [12,13] in terms of an H_∞ stability radius of a transfer function associated with A, B_0, and C_0. However, the Lyapunov function that establishes robust stability of the uncertain system in [12,13] is a fixed Lyapunov function in contrast to the parameter-dependent Lyapunov function that establishes robust stability in Theorem 4.2.

Next, we show that Theorem 4.1 is a generalization of the classical parabola test [2] for the case in which the loop sector-bounded nonlinearity is used to represent uncertainty. First, however, we provide a generalization of the parabola criterion for multivariable systems with diagonal nonlinearity structure. Specifically, we define the set Φ characterizing a class of sector-bounded memoryless *time-invariant* nonlinearities. Let M_1, M_2 and $M_2 - M_1$ be given positive-definite diagonal matrices and define

$$\begin{aligned} \Phi \triangleq \{\phi\colon \mathcal{R}^{m_0} \to \mathcal{R}^{m_0}\colon\ & (\phi - M_1y)^{\mathrm{T}}[(M_2 - M_1)^{-1}(\phi - M_1y) - y] \le 0, \\ & y \in \mathcal{R}^{m_0}, \text{ and } \phi(y) = [\phi_1(y_1), \phi_2(y_2), \ldots, \phi_{m_0}(y_{m_0})]^{\mathrm{T}}\}. \end{aligned}$$

In the scalar case $m_0 = 1$, the sector condition Φ is equivalent to the more familiar condition

$$\underline{m}y^2 \le \phi(y)y \le \overline{m}y^2,\ y \in \mathcal{R}.$$

For notational convenience in stating the multivariable generalization of the parabola criterion we define $M_1 \triangleq \text{diag}[\underline{m}_1, \underline{m}_2, \ldots, \underline{m}_{m_0}]$ and $N \triangleq \text{diag}[N_1, N_2, \ldots, N_{m_0}]$.

Theorem 4.3 (The Multivariable Parabola Criterion) *If there exists a nonnegative-definite diagonal matrix N such that $(M_2 - M_1)^{-1} + (I + Ns)(I+G(s)M_1)^{-1}G(s)$ is strongly positive real, where $G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$, then the negative feedback interconnection of $G(s)$ and $\phi(\cdot)$ is asymptotically stable for all $\phi(\cdot) \in \Phi$.*

Proof: First note that the negative feedback interconnection of $G(s)$ and $\phi(\cdot)$ has the state-space description

$$\dot{x}(t) = Ax(t) - B\phi(y(t)), \tag{4.13}$$

$$y(t) = Cx(t). \tag{4.14}$$

Now, noting that $[I + G(s)M_1]^{-1}G(s)$ corresponds to a plant $G(s)$ with negative feedback gain M_1, it follows from feedback interconnection manipulations that a minimal realization for $[I + G(s)M_1]^{-1}G(s)$ is given by

$$[I + G(s)M_1]^{-1}G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A - BM_1C & B \\ \hline C & 0 \end{array}\right].$$

Similarly, noting that $sG(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$, it follows that $(M_2-M_1)^{-1}+(I + Ns)(I + G(s)M_1)^{-1}G(s)$ has a minimal realization given by

$$\left[\begin{array}{c|c} A - BM_1C & B \\ \hline C + NC(A - BM_1C) & (M_2 - M_1)^{-1} + NCB \end{array}\right].$$

Now it follows from Lemma 4.3 that, since $(M_2 - M_1)^{-1} + (I + Ns)(I + G(s)M_1)^{-1}G(s)$ is strongly positive real, there exist positive-definite matrices P and R such that

$$\begin{aligned} 0 = {} & (A - BM_1C)^{\mathrm{T}}P + P(A - BM_1C) \\ & + [C + NC(A - BM_1C) - B^{\mathrm{T}}P]^{\mathrm{T}}[(M_2 - M_1)^{-1} \\ & + NCB + \left((M_2 - M_1)^{-1} + NCB\right)^{\mathrm{T}}]^{-1} \\ & \cdot [C + NC(A - BM_1C) - B^{\mathrm{T}}P] + R. \end{aligned} \tag{4.15}$$

Next, for $\phi \in \Phi$ define the Lyapunov function candidate

$$V(x) = x^{\mathrm{T}}Px + 2\sum_{i=1}^{m}\int_0^{y_i}[\phi_i(\sigma) - \underline{m}_i\sigma]N_i \mathrm{d}\sigma. \tag{4.16}$$

The corresponding Lyapunov derivative is given by

$$\dot{V}(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - \phi^{\mathrm{T}}B^{\mathrm{T}}Px - x^{\mathrm{T}}PB\phi + 2(\phi - M_1 y)^{\mathrm{T}}N\dot{y}. \quad (4.17)$$

Next, using (4.15), noting that $\dot{y} = CAx - CB\phi$, and adding and subtracting $2(\phi - M_1y)^{\mathrm{T}}(M_2 - M_1)^{-1}(\phi - M_1 y)$, $2(\phi - M_1 y)^{\mathrm{T}}y$, $2x^{\mathrm{T}}C^{\mathrm{T}}M_1NCB\phi$, $2x^{\mathrm{T}}A^{\mathrm{T}}C^{\mathrm{T}}NM_1Cx$, $2x^{\mathrm{T}}C^{\mathrm{T}}M_1B^{\mathrm{T}}C^{\mathrm{T}}N\phi$, and $2x^{\mathrm{T}}C^{\mathrm{T}}M_1B^{\mathrm{T}}C^{\mathrm{T}}NM_1Cx$ to and from (4.17), it follows (after some algebraic manipulation) that

$$\dot{V}(x) = -x^{\mathrm{T}}Rx - z^{\mathrm{T}}z + 2(\phi - M_1y)^{\mathrm{T}}[(M_2 - M_1)^{-1}(\phi - M_1y) - y],$$

where

$$\begin{aligned} z \triangleq\ & [(M_2 - M_1)^{-1} + NCB + \left((M_2 - M_1)^{-1} + NCB\right)^{\mathrm{T}}]^{-1/2} \\ & \cdot [C + NC(A - BM_1C) - B^{\mathrm{T}}P]x \\ & - [(M_2 - M_1)^{-1} + NCB + \left((M_2 - M_1)^{-1} + NCB\right)^{\mathrm{T}}]^{1/2}[\phi - M_1Cx]. \end{aligned}$$

Since R is positive definite and $(\phi - M_1y)^{\mathrm{T}}[(M_2 - M_1)^{-1}(\phi - M_1y) - y] \leq 0$, it follows that $\dot{V}(x)$ is negative definite. □

Remark 4.6 A similar proof for the case where $M_1 = 0$ is given in [18] using the three equation form of the KYP lemma given in Remark 4.4. This case corresponds to the multivariable Popov criterion [8,17,18].

Remark 4.7 Theorem 4.3 provides a multivariable generalization of the classical parabola test [2]. To see this, consider the SISO case, assume $M_1M_2 > 0$, and let $G = x + \jmath y$. Next, note that Theorem 4.3 requires $1/(M_2 - M_1) + \mathrm{Re}[(1 + Ns)G(s)/(1 + M_1G(s))] > 0, \mathrm{Re}[s] > 0$, or, equivalently,

$$\mathrm{Re}\left[\frac{1 + M_2G(s)}{1 + M_1G(s)}(1 + Ns)\right] > 0, \quad \mathrm{Re}[s] > 0. \quad (4.18)$$

Now, (4.18) implies that $M_1M_2x^2 + (M_1 + M_2)x + 1 > N(M_2 - M_1)\omega y - M_1M_2y^2$. Since $M_1M_2 > 0$, a slightly weaker version of the above inequality is $(M_1x + 1)(M_2x + 1) > N(M_2 - M_1)\omega y$ which provides an absolute frequency domain stability condition with a graphical interpretation in the Popov plane in terms of a parabola [2,19]. It is important to note that in the linear parametric uncertainty case, the Riccati equation formulation provides a stronger and hence less conservative result since we do not require that $M_1M_2 > 0$.

Remark 4.8 Note that for the symmetric sector case with $-M_1 = M_2$ which corresponds to the shifted Popov criterion, it follows from (4.18) that

$$M_2^2x^2 + M_2^2y^2 + 2NM_2\omega y + 1 < 0, \quad (4.19)$$

or, equivalently,

$$x^2 + (y + \frac{N\omega}{M_2})^2 < \frac{1}{M_2^2} + (\frac{N\omega}{M_2})^2. \tag{4.20}$$

Equation (4.20) involves a frequency domain stability criterion in the Nyquist plane (rather than the Popov plane) in terms of a family of frequency dependent off-axis circles [11,14,15]. The circle centers vary as a function of the phase of the Popov multiplier, but each has the same real axis intercepts at $\pm M_2^{-1}$. This criterion is reminiscent of the classical off-axis circle criterion of [5], where a single bounding circle is employed as opposed to a family of frequency dependent ones. This frequency dependent off-axis circle interpretation along with its connection to real parameter uncertainty if further discussed in [11, 14].

Remark 4.9 The authors in [11, 14] also discuss connections between the frequency domain stability condition for the symmetric shifted Popov criterion and the upper bounds for real-μ discussed in [6,24]. It is shown that the Popov multiplier corresponds to a particular parameterization of the frequency dependent scaling matrices in mixed-μ theory. By considering other classes of nonlinear models of the uncertainty, Haddad *et al.* [11] extend these frequency domain criteria to include more refined classes of nonlinear functions, and, as a result, more general forms of the stability multipliers, which consequently provide a general parameterization of the D, N scales for mixed-μ analysis and synthesis.

In order to specialize the result of Theorem 4.3 to robust stability with constant linear parameter uncertainty, consider the system

$$\dot{x}(t) = (A + \Delta A)x(t), \tag{4.21}$$

where $\Delta A \in \mathcal{U}$ and $\mathcal{U}$ is defined by

$$\mathcal{U} \triangleq \{\Delta A\colon \Delta A = -BFC, \quad F = \mathrm{diag}[F_1, F_2, \ldots, F_{m_0}], \\ \underline{m}_i \le F_i \le \overline{m}_i,\ i = 1, \ldots, m_0\}.$$

It now follows from Theorem 4.3 by setting $\phi(y) = Fy = FCx$ that $A+\Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$.

It has thus been shown that in the special case that F and N are diagonal nonnegative-definite matrices, Theorem 4.1 (with B_0 replaced by $-B_0$) specializes to the multivariable parabola criterion when applied to linear parameter uncertainty. This is not surprising since in this case the Lyapunov function (4.16) that establishes robust stability takes the form

$$V(x) = x^{\mathrm{T}}Px + 2\sum_{i=1}^{m_0} \int_0^{y_i} (F_i - \underline{m}_i)\sigma N_i \mathrm{d}\sigma, \quad y_i = (C_0x)_i, \tag{4.22}$$

or, equivalently,

$$V(x) = x^{\mathrm{T}} P x + x^{\mathrm{T}} C_0^{\mathrm{T}} (F - M_1) N C_0 x, \tag{4.23}$$

and thus is a special case of the parameter-dependent Lyapunov function discussed earlier. Note that the uncertain parameters are not allowed to be arbitrarily time-varying, which is consistent with the fact that the classical parabola criterion is restricted to *time-invariant* nonlinearities.

Finally, we note that in the case in which $M_1 = 0$, Theorem 4.3 specializes to the multivariable Popov criterion considered in [9,10]. Alternatively, retaining M_1 and setting $N = 0$ yields a strongly positive real requirement on $(M_2 - M_1)^{-1} + (I + G(s)M_1)^{-1}G(s)$ or, equivalently, on $(I + G(s)M_2)(I + G(s)M_1)^{-1}$ which corresponds to the multivariable circle criterion considered in [8] with the restrictions that M_1, M_2 be diagonal and positive definite.

5 Dynamic Output Feedback Controller Synthesis

In this section we introduce the Dynamic Robust Stability and Performance Problem. For simplicity we restrict our attention to controllers of order $n_c = n$, that is, controllers whose order is equal to the dimension of the plant. This problem involves the set $\mathcal{U} \subset \mathcal{R}^{n \times n}$ of uncertain perturbations ΔA of the nominal system matrix A.

Dynamic Robust Stability and Performance Problem Given the nth-order stabilizable and detectable plant with constant structured real-valued plant parameter variations

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + D_1 w(t), \quad t \geq 0, \tag{5.1}$$

$$y(t) = Cx(t) + D_2 w(t), \tag{5.2}$$

where $u(t) \in \mathcal{R}^m, w(t) \in \mathcal{R}^d$, and $y(t) \in \mathcal{R}^\ell$, determine an nth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{5.3}$$

$$u(t) = C_c x_c(t), \tag{5.4}$$

that satisfies the following design criteria:

i) the closed-loop system (5.1)-(5.4) is asymptotically stable for all $\Delta A \in \mathcal{U}$; and

ii) the performance functional

$$J(A_c, B_c, C_c) \triangleq \sup_{\Delta A \in \mathcal{U}} \limsup_{t \to \infty} \frac{1}{t}\, \mathcal{E} \left\{ \int_0^t \left[x^{\mathrm{T}}(s) R_1 x(s) + u^{\mathrm{T}}(s) R_2 u(s) \right] ds \right\} \tag{5.5}$$

is minimized.

For each uncertain variation $\Delta A \in \mathcal{U}$, the closed-loop system (5.1)-(5.4) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}w(t), \quad t \geq 0, \tag{5.6}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \; \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \quad \Delta\tilde{A} = \begin{bmatrix} \Delta A & 0_{n\times n_c} \\ 0_{n_c\times n} & 0_{n_c\times n_c} \end{bmatrix},$$

and where the closed-loop disturbance $\tilde{D}w(t)$ has intensity $\tilde{V} = \tilde{D}\tilde{D}^{\mathrm{T}}$, where $\tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}, \tilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_cV_2B_c^{\mathrm{T}} \end{bmatrix}, V_1 = D_1D_1^{\mathrm{T}}, V_2 = D_2D_2^{\mathrm{T}}$. The closed-loop system uncertainty $\Delta\tilde{A}$ has the form

$$\Delta\tilde{A} = \tilde{B}_0F\tilde{C}_0,$$

where

$$\tilde{B}_0 \triangleq \begin{bmatrix} B_0 \\ 0_{n_c\times m_0} \end{bmatrix}, \quad \tilde{C}_0 \triangleq [C_0 \quad 0_{m_0\times n_c}].$$

Finally, if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$ and for a given compensator (A_c, B_c, C_c), then the performance measure (5.5) is given by

$$J(A_c, B_c, C_c) = \sup_{\Delta A\in\mathcal{U}} \operatorname{tr} \tilde{P}_{\Delta\tilde{A}}\tilde{V}, \tag{5.7}$$

where $P_{\Delta\tilde{A}}$ satisfies the $2n \times 2n$ algebraic Lyapunov equation

$$0 = (\tilde{A} + \Delta\tilde{A})^{\mathrm{T}}\tilde{P}_{\Delta\tilde{A}} + \tilde{P}_{\Delta\tilde{A}}(\tilde{A} + \Delta\tilde{A}) + \tilde{R}, \tag{5.8}$$

where

$$\tilde{E} = [E_1 \quad E_2C_c], \quad \tilde{R} = \tilde{E}^{\mathrm{T}}\tilde{E} = \begin{bmatrix} R_1 & 0 \\ 0 & C_c^{\mathrm{T}}R_2C_c \end{bmatrix}.$$

To apply Theorem 4.1 to controller synthesis we consider the performance bound (3.9) in place of the actual worst-case H_2 performance as in Theorem 4.1 with A, R replaced by $\tilde{A}$ and $\tilde{R}$ to address the closed-loop control problem. This leads to the following optimization problem.

Auxiliary Minimization Problem Determine (A_c, B_c, C_c) that minimizes

$$\mathcal{J}(K) \triangleq \operatorname{tr}[(\tilde{P} + \tilde{C}_0^{\mathrm{T}}\mu\tilde{C}_0)\tilde{V}] \tag{5.9}$$

where $\tilde{P} \in \mathcal{N}^{2n}$ satisfies

$$\begin{aligned}0 =&(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0)^{\mathrm{T}} \tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0)\\ &+ [\tilde{C}_0 + N\tilde{C}_0(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0) + \tilde{B}_0^{\mathrm{T}} \tilde{P}]^{\mathrm{T}}\\ &\cdot [(M_2 - M_1)^{-1} - N\tilde{C}_0 \tilde{B}_0 + ((M_2 - M_1)^{-1} - N\tilde{C}_0\tilde{B}_0)^{\mathrm{T}}]^{-1}\\ &\cdot [\tilde{C}_0 + N\tilde{C}_0(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0) + \tilde{B}_0^{\mathrm{T}}\tilde{P}] + \tilde{R} \end{aligned} \tag{5.10}$$

and such that (A_c, B_c, C_c) is minimal and $\mathcal{F}$ and N are strongly compatible.

It follows from Theorem 4.1 that the satisfaction of (5.10) along with the detectability condition $(\tilde{A}+\Delta\tilde{A}, \tilde{E})$ leads to closed-loop robust stability along with robust H_2 performance.

Next, we present sufficient conditions for robust stability and performance for the dynamic output feedback problem. For arbitrary $P, Q \in \mathcal{R}^{n\times n}$ define the notation

$$R_0 \triangleq (M_2 - M_1)^{-1} - NC_0B_0 + \big((M_2 - M_1)^{-1} - NC_0B_0\big)^{\mathrm{T}},$$

$$R_{2a} \triangleq R_2 + B^{\mathrm{T}} C_0^{\mathrm{T}} N^{\mathrm{T}} R_0^{-1} N C_0 B,\ \tilde{C} \triangleq\ C_0 + NC_0(A + B_0 M_1 C_0),$$

$$P_a \triangleq B^{\mathrm{T}} P + B^{\mathrm{T}} C_0^{\mathrm{T}} N^{\mathrm{T}} R_0^{-1}(\tilde{C} + B_0^{\mathrm{T}} P),\ A_P \triangleq A + B_0 M_1 C_0 + B_0 R_0^{-1}\tilde{C},$$

$$\bar{\Sigma} \triangleq C^{\mathrm{T}} V_2^{-1} C,\ A_{\hat{P}} \triangleq A_P - Q\bar{\Sigma} + B_0 R_0^{-1} B_0^{\mathrm{T}} P.$$

Theorem 5.1 *Assume $R_0 > 0$ and assume that $\mathcal{F}$ and N are strongly compatible. Furthermore, suppose there exist $n \times n$ nonnegative-definite matrices $P, Q, \hat{P}$ satisfying*

$$0 = A_P^{\mathrm{T}} P + PA_P + R_1 + \tilde{C}^{\mathrm{T}} R_0^{-1}\tilde{C} + PB_0R_0^{-1}B_0^{\mathrm{T}}P - P_a^{\mathrm{T}} R_{2a}^{-1} P_a, \tag{5.11}$$

$$\begin{aligned}0 = &(A_P + B_0R_0^{-1}B_0^{\mathrm{T}}[P + \hat{P}])Q + Q(A_P + B_0R_0^{-1}B_0^{\mathrm{T}}[P + \hat{P}])^{\mathrm{T}}\\ &+ V_1 - Q\bar{\Sigma}Q,\end{aligned} \tag{5.12}$$

$$0 = A_{\hat{P}}^{\mathrm{T}}\hat{P} + \hat{P}A_{\hat{P}} + \hat{P}B_0R_0^{-1}B_0^{\mathrm{T}}\hat{P} + P_a^{\mathrm{T}}R_{2a}^{-1}P_a, \tag{5.13}$$

and let A_c, B_c, C_c be given by

$$A_c = A_P - Q\bar{\Sigma} + B_0R_0^{-1}B_0^{\mathrm{T}}P - (I + B_0R_0^{-1}NC_0)BR_{2a}^{-1}P_a, \tag{5.14}$$

$$B_c = QC^{\mathrm{T}}V_2^{-1}, \tag{5.15}$$

$$C_c = -R_{2a}^{-1}P_a. \tag{5.16}$$

Then $(\tilde{A} + \Delta\tilde{A},\ \tilde{E})$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case, the performance of the closed-loop system (5.5) satisfies the bound

$$J(A_c, B_c, C_c) \leq \mathrm{tr}[(P + \hat{P})V_1 + \hat{P}Q\bar{\Sigma}Q + C_0^{\mathrm{T}}\mu C_0 V_1]. \tag{5.17}$$

Proof: The proof follows as in the proof given in [9]. □

Remark 5.1 Note that if the uncertainty in the plant dynamics is deleted, that is, $B_0 = 0$, $C_0 = 0$, then Theorem 5.1 specializes to the standard LQG result.

Theorem 5.1 provides constructive sufficient conditions that yield dynamic output feedback controllers for robust stability and performance. These conditions comprise a system of three modified algebraic Riccati equations in variables P, Q, and $\hat{P}$, respectively. When solving (5.11)-(5.13) numerically, the matrices M_1, M_2 and N and the structure matrices B_0 and C_0 appearing in the design equations can be adjusted to examine tradeoffs between performance and robustness. To further reduce conservatism, one can view the multiplier matrix N as a free parameter and optimize the H_2 performance bound $\mathcal{J}$ with respect to N. In particular, $\partial\mathcal{J}/\partial N$ is given by

$$\frac{\partial \mathcal{J}}{\partial N} = \mu \tilde{C}_0 \tilde{V} \tilde{C}_0^{\mathrm{T}} + R_0^{-1}[\tilde{C}_0 + N\tilde{C}_0(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0) + \tilde{B}_0^{\mathrm{T}} \tilde{P}]\tilde{Q}$$
$$\cdot [(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0) + \tilde{B}_0 R_0^{-1}(\tilde{C}_0 + N\tilde{C}_0(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0) + \tilde{B}_0^{\mathrm{T}} \tilde{P})]^{\mathrm{T}} \tilde{C}_0^{\mathrm{T}},$$

where $\tilde{Q}$ satisfies

$$0 = [\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0 + \tilde{B}_0 R_0^{-1}(\tilde{\tilde{C}} + \tilde{B}_0^{\mathrm{T}} \tilde{P})]\tilde{Q}$$
$$+ \tilde{Q}[\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0 + \tilde{B}_0 R_0^{-1}(\tilde{\tilde{C}} + \tilde{B}_0^{\mathrm{T}} \tilde{P})]^{\mathrm{T}} + \tilde{V},$$

and

$$\tilde{\tilde{C}} \triangleq \tilde{C}_0 + N\tilde{C}_0(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0).$$

Now, the basic approach is to employ a numerical algorithm to design the optimal controller and the multiplier N simultaneously, thus avoiding the need to iterate between controller design and optimal multiplier evaluation. For details see [10,11].

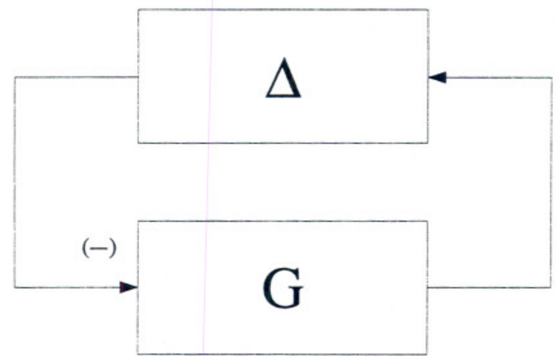


Figure 1: Uncertain Feedback System

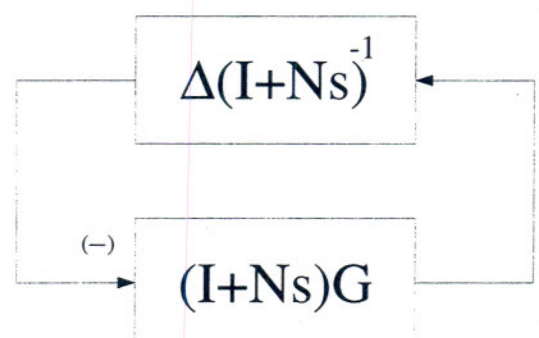


Figure 2: Uncertain Feedback System with Popov Multiplier

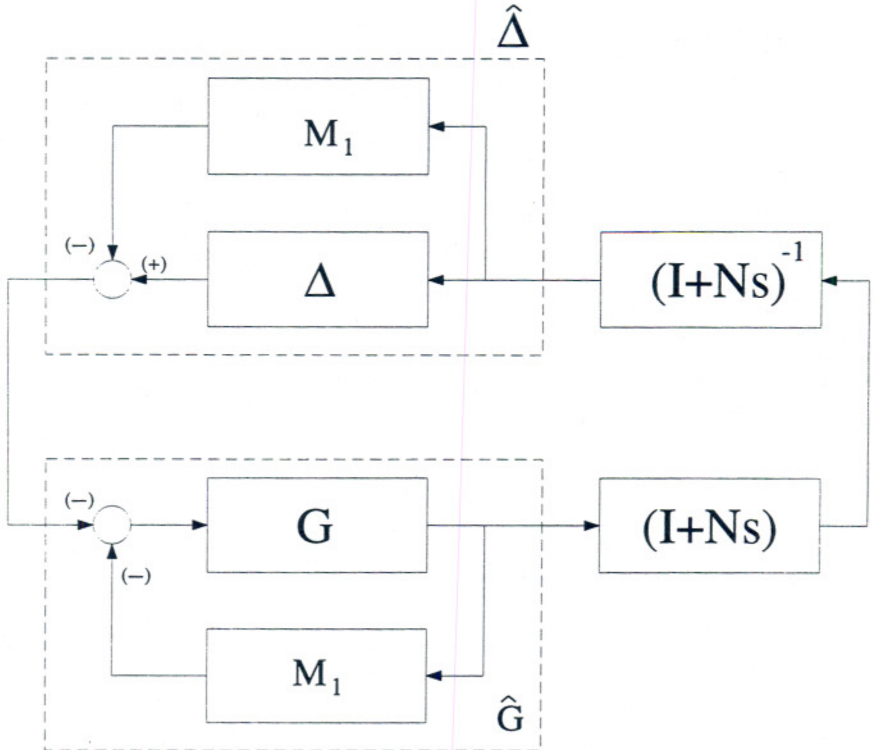
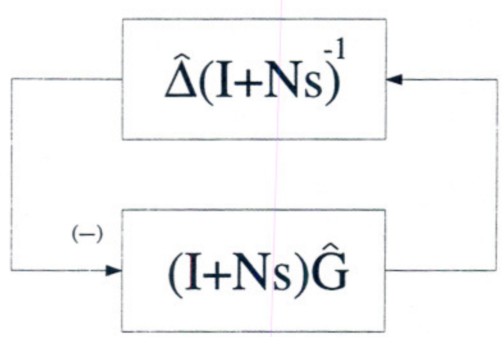


Figure 3: Uncertain Feedback System with Loop Transformation



$$\hat{G} = (I + GM_1)^{-1}G, \qquad \hat{\Delta} = \Delta - M_1.$$

Figure 4: Transformed Uncertain System

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Communicated by Clyde Martin