The Optimal Projection Equations for Fixed-Order Sampled-Data Dynamic Compensation with Computation Delay

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Abstract—**For an LQG-type sampled-data regulator problem which accounts for computational delay and utilizes an averaging A/D device, the equivalent discrete-time problem is shown to be of increased order due to the inclusion of delayed measurement states. The optimal projection equations for reduced-order, discrete-time compensation are applied to the augmented problem to characterize low-order controllers. The design results are illustrated on a tenth-order flexible beam example.**

I. INTRODUCTION

Classical sampled-data control theory has been extensively developed [1]–[7] and is widely used in practical applications. Sampled-data design based upon modern optimal control theory has also been developed, although to a considerably lesser extent [8]–[14]. The goals of the present note are twofold. First, for an LQG-type sampled-data regulator problem which explicitly accounts for computational delay, we obtain an equivalent discrete-time problem (Theorem 2.1 and Corollary 2.1). The timing diagram in Fig. 1 illustrates the unavoidable delay in the feedback loop (see Section II for notation).[1] A salient feature of this problem is that rather than replace the continuous-time white noise measurement model by a discrete-time version (which is often done in the literature since continuous-time white noise cannot be sampled), we employ an averaging-type A/D device as in [8, p. 82] [see (2.5)].

The second goal of the note is to present a novel design procedure which is applicable to the equivalent discrete-time problem, and which thus directly accounts for the delay effects. Since the discrete-time model is of augmented order $n + l$ (n = number of plant states, l = number of measurements), it seems natural to seek dynamic feedback of reduced order. To this end, we apply the optimal projection equations for discrete-time dynamic compensation to the equivalent discrete-time problem to characterize optimal controllers of order $n_c \leq n + l$. These equations, which were previously derived in [17] for the continuous-time case, are discussed in [15], [16]. Note that, in practice, the computational delay (and, hence, sample interval) in real-time controller implementation depends directly upon the controller order n_c. For example, by reducing n_c the sample rate can effectively be increased. Thus, the engineering tradeoffs of performance versus controller order and sample interval can be investigated using the results of this note.

This note also includes formulas for integrals of matrix exponentials arising in the sampled-data/discrete-time conversion, along with an algorithm for solving the optimal projection equations. The results are applied to a tenth-order flexible beam example.

II. SAMPLED-DATA PROBLEM AND EQUIVALENT DISCRETE-TIME FORMULATION

The following notation and definitions will be used throughout.

I_r, $0_{r\times s}$, 0_r	$r \times r$ identity matrix, $r \times s$ zero matrix, $0_{r\times r}$
$Z_{(i,j)}$	(i, j) element of matrix Z

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[1] For simplicity, the timing diagram Fig. 1 applies to the case in which the compensator is strictly proper. In the note the results are stated for the more general case in which a direct (static) feedthrough term $D_c\hat{y}(k)$ is included.

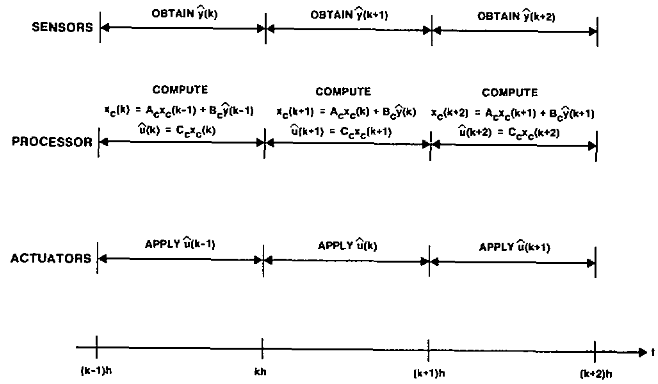

Fig. 1. Timing diagram for sampled-data controller.

Z^T, Z^{-T}	transpose of vector or matrix Z, $(Z^T)^{-1}$
E_i	matrix with unity in the (i, i) position and zeros elsewhere
$\mathbf{E}$, $\mathbf{R}$, $\mathbf{R}^{r\times s}$	expected value, real numbers, $r \times s$ real matrices
stable matrix	matrix with eigenvalues in open unit disk
nonnegative-semisimple matrix	diagonalizable matrix with nonnegative eigenvalues [18], [19]
n, m, l, n_c	positive integers, $1 \leq n_c \leq n + l$
x, u, y, x_c	n, m, l, n_c-dimensional vectors
A, B, C	$n \times n$, $n \times m$, $l \times n$ matrices
A_c, B_c, C_c, D_c	$n_c \times n_c$, $n_c \times l$, $m \times n_c$, $m \times l$ matrices
w_1, w_2	n, l-dimensional zero-mean continuous-time white noise processes
V_1	$n \times n$ nonnegative-definite intensity of w_1
V_2	$l \times l$ positive-definite intensity of w_2
V_{12}	$n \times l$ cross intensity of w_1, w_2
R_1	$n \times n$ nonnegative-definite state weighting matrix
R_2	$m \times m$ positive-definite control weighting matrix
R_{12}	$n \times m$ cross weighting matrix such that $R_1 - R_{12}R_2^{-1}R_{12}^T$ is nonnegative definite
k	discrete-time index, 1, 2, 3, $\cdots$.

In the statement of the sampled-data control problem the sample interval h and the controller order n_c are fixed and the optimization is performed over the controller parameters (A_c, B_c, C_c, D_c). For design tradeoff studies h and n_c can be varied and the problem can be solved for each pair of values of interest.

Fixed-Order, Sampled-Data Dynamic-Compensation Problem

Given the nth-order continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t) + w_1(t) \tag{2.1}$$

with continuous-time measurements

$$y(t) = Cx(t) + Du(t) + w_2(t) \tag{2.2}$$

design an n_cth-order discrete-time compensator

$$x_c(k+1) = A_cx_c(k) + B_c\hat{y}(k), \tag{2.3}$$

$$\hat{u}(k) = C_cx_c(k) + D_c\hat{y}(k), \tag{2.4}$$

which, with A/D averaged measurements

$$\hat{y}(k) \triangleq \frac{1}{h}\int_{(k-1)h}^{kh} y(t)\, dt \tag{2.5}$$

and D/A zero-order-hold controls

$$u(t)=\hat{u}(k), \qquad t \in [kh, (k+1)h), \tag{2.6}$$

minimizes the performance criterion

$$J(A_c, B_c, C_c, D_c) \triangleq \lim_{t\to\infty} E\frac{1}{t} \cdot \int_0^t [x(s)^T R_1 x(s) + 2x(s)^T R_{12} u(s) + u(s)^T R_2 u(s)]\, ds. \tag{2.7}$$

The main result of this section concerns propagation of the plant and digitized measurements over one time step. For notational convenience, define

$$H(s) \triangleq \int_0^s e^{Ar}\, dr.$$

Theorem 2.1: For the fixed-order, sampled-data dynamic-compensation problem, the plant dynamics (2.1), averaged measurements (2.5) and performance criterion (2.7) have the equivalent discrete-time representations

$$x'(k+1)=A'x'(k)+B'\hat{u}(k)+w_1'(k), \tag{2.8}$$

$$\hat{y}(k)=C'x'(k-1)+D'\hat{u}(k-1)+w_2'(k-1), \tag{2.9}$$

$$J(A_c, B_c, C_c, D_c)=\delta+\lim_{k\to\infty} E[x'(k)^T R_1' x'(k) + 2x'(k)^T R_{12}' \hat{u}(k) + \hat{u}(k)^T R_2' \hat{u}(k)] \tag{2.10}$$

where

$$x'(k) \triangleq x(kh),\ \delta \triangleq \frac{1}{h} \operatorname{tr} \int_0^h \int_0^s e^{Ar} V_1 e^{A^T r} R_1\, dr\, ds,$$

$$A' \triangleq e^{Ah},\ B' \triangleq H(h)B,\ C' \triangleq \frac{1}{h} CH(h),\ D' \triangleq \frac{1}{h} C \int_0^h H(s)\, ds B + D;$$

$w_1'(k)$ and $w_2'(k)$ are zero-mean, white noise processes with

$$E\left\{\begin{bmatrix} w_1'(k) \\ w_2'(k) \end{bmatrix}\begin{bmatrix} w_1'(k) \\ w_2'(k) \end{bmatrix}^T\right\} = \begin{bmatrix} V_1' & V_{12}' \\ V_{12}'^T & V_2' \end{bmatrix},$$

where

$$V_1' \triangleq \int_0^h e^{As} V_1 e^{A^T s}\, ds,\ V_{12}' \triangleq \frac{1}{h}\int_0^h e^{As} V_1 H^T(s)\, ds C^T + \frac{1}{h} H(h) V_{12},$$

$$V_2' \triangleq \frac{1}{h} V_2 + \frac{1}{h^2} C \int_0^h H(s) V_1 H^T(s)\, ds C^T + \frac{1}{h^2} C \int_0^h H(s)\, ds V_{12} + \frac{1}{h^2} V_{12}^T \int_0^h H^T(s)\, ds C^T;$$

and

$$R_1' \triangleq \frac{1}{h}\int_0^h e^{A^T s} R_1 e^{As}\, ds,\ R_{12}' \triangleq \frac{1}{h}\int_0^h e^{A^T s} R_1 H(s)\, ds B + \frac{1}{h} H^T(h) R_{12},$$

$$R_2' \triangleq R_2 + \frac{1}{h} B^T \int_0^h H^T(s) R_1 H(s)\, ds B + \frac{1}{h} B^T \int_0^h H^T(s)\, ds R_{12} + \frac{1}{h} R_{12}^T \int_0^h H(s)\, ds B.$$

The proof of this theorem is a straightforward calculation involving standard techniques, and hence is omitted. The result is more comprehensive than previous work, however, and includes several results as special cases. For example, the expressions for A' and B' are standard; C' is given by (10.9), [8, p. 83]; R_1', R_{12}', and R_2' are given in [10], [12]; and V_1', V_{12}', and V_2' can be found in [8, p. 85]. The expressions for δ and D' appear to be new.

Note that the averaged measurements depend upon delayed samples of the state. By augmenting the discretized state equation (2.8) to include these measurements, it is possible to state the original sampled-data problem as a discrete-time problem.

Corollary 2.1: With the notation

$$\hat{x}(k) \triangleq \begin{bmatrix} x'(k) \\ \hat{y}(k) \end{bmatrix},\ \hat{A} \triangleq \begin{bmatrix} A' & 0_{n\times l} \\ C' & 0_l \end{bmatrix},\ \hat{B} \triangleq \begin{bmatrix} B' \\ D' \end{bmatrix},$$

$$\hat{C} \triangleq [0_{l\times n}\ \ I_l],\ \hat{w}(k) \triangleq \begin{bmatrix} w_1'(k) \\ w_2'(k) \end{bmatrix},\ \hat{V} \triangleq \begin{bmatrix} V_1' & V_{12}' \\ V_{12}'^T & V_2' \end{bmatrix},$$

$$\hat{R}_1 \triangleq \begin{bmatrix} R_1' & 0_{n\times l} \\ 0_{l\times n} & 0_l \end{bmatrix},\ \hat{R}_{12} \triangleq \begin{bmatrix} R_{12}' \\ 0_{l\times m} \end{bmatrix},\ \hat{R}_2 \triangleq R_2'$$

the fixed-order, sampled-data dynamic-compensation problem is equivalent to the following discrete-time problem. Given the $(n + l)$th-order discrete-time system

$$\hat{x}(k+1)=\hat{A}\hat{x}(k)+\hat{B}\hat{u}(k)+\hat{w}(k) \tag{2.11}$$

with discrete-time measurements

$$\hat{y}(k)=\hat{C}\hat{x}(k) \tag{2.12}$$

design an n_cth-order discrete-time compensator of the form (2.3), (2.4), which minimizes

$$J(A_c, B_c, C_c, D_c)=\delta+\lim_{k\to\infty} E[\hat{x}(k)^T \hat{R}_1 \hat{x}(k) + 2\hat{x}(k)^T \hat{R}_{12} \hat{u}(k) + \hat{u}(k)^T \hat{R}_2 \hat{u}(k)]. \tag{2.13}$$

Remark 2.1: The equivalent cost (2.13) involves a constant offset δ which serves as a lower bound on the sampled-data performance, i.e., a "discretization floor." Note that

$$\delta = \frac{h}{2} \operatorname{tr} V_1 R_1 + O(h^2).$$

Remark 2.2: Although the measurements $\hat{y}(k)$ are noise free, the singularity is not so serious as singular measurement noise in the continuous-time case where the Kalman filter gains are expressed in terms of the inverse of the measurement noise intensity. In the discrete-time case, rather, it is required that $\bar{V} + \overline{CQC^T}$ be invertible, where $\bar{V}$ is the measurement noise covariance (see [11, p. 530], or [15], [16]).

Remark 2.3: The increase in plant order from n to $n + l$ is due to the computational delay and A/D process. Since discrete-time LQG theory yields a possibly unwieldy $(n + l)$th-order controller, we seek "reduced-order" controllers. Note that in this context an nth-order controller can be regarded as being of reduced order.

Remark 2.4: As pointed out in [10], particular choices of the sample interval h may result in a loss of controllability and observability for the equivalent discrete-time problem. Hence, these properties must be verified before applying control design methods.

III. Application of the Optimal Projection Equations to the Equivalent Discrete-Time Problem

We now apply the optimal projection equations for discrete-time dynamic compensation to the equivalent discrete-time problem. The following easily proved lemma will be needed.

Lemma 3.1: Let $\tau \in R^{(n+l)\times(n+l)}$. Then

$$\tau^2=\tau, \tag{3.1}$$

$$\rho(\tau)=n_c \tag{3.2}$$

if and only if there exist $G, \Gamma \in R^{n_c \times (n+l)}$ such that

$$G^T\Gamma = \tau, \tag{3.3}$$

$$\Gamma G^T = I_{n_c}. \tag{3.4}$$

Furthermore, G and Γ are unique to a change of basis in R^{n_c}.

Call G and Γ satisfying (3.3), (3.4) a *projective factorization* of τ. Furthermore, for $n \times n$ nonnegative-definite matrices $\mathcal{Q}$ and $\mathcal{P}$, define the set of *contragrediently diagonalizing* transformations

$$\mathcal{C}(\mathcal{Q}, \mathcal{P}) \triangleq \{\Psi \in R^{n \times n}: \Psi^{-1}\mathcal{Q}\Psi^{-T} \text{ and } \Psi^T\mathcal{P}\Psi \text{ are diagonal}\}.$$

It follows from [19, Theorem 6.2.5, p. 123] that $\mathcal{C}(\mathcal{Q}, \mathcal{P})$ is always nonempty. This set does not, however, have a unique element since basis rearrangements and sign transpositions may be incorporated into Ψ. Further nonuniqueness arises if $\mathcal{Q}\mathcal{P}$ has repeated eigenvalues.

To guarantee that J is finite and independent of initial conditions, we restrict our consideration to the (open) set

$$\mathcal{S} \triangleq \left\{(A_c, B_c, C_c, D_c): \begin{bmatrix} \hat{A}+\hat{B}D_c\hat{C} & \hat{B}C_c \\ B_c\hat{C} & A_c \end{bmatrix} \text{ is stable and } (A_c, B_c, C_c) \text{ is minimal}\right\}.$$

For the design problem it is required that $\mathcal{S}$ be nonempty, i.e., that the augmented system be stabilizable. We also require the notation

$$\tilde{R}_2 \triangleq \hat{R}_2 + \hat{B}^TP\hat{B}, \; \tilde{V}_2 \triangleq \hat{C}Q\hat{C}^T, \; \tau_\perp \triangleq I_{n+l} - \tau,$$

$$A_Q \triangleq \hat{A} - \hat{A}Q\hat{C}^T\tilde{V}_2^{-1}\hat{C}, \; A_P \triangleq \hat{A} - \hat{B}\tilde{R}_2^{-1}(\hat{B}^TP\hat{A} + \hat{R}_{12}^T),$$

$$\Sigma_Q \triangleq (\hat{A}Q\hat{C}^T + \hat{B}D_c\tilde{V}_2)\tilde{V}_2^{-1}(\hat{A}Q\hat{C}^T + \hat{B}D_c\tilde{V}_2)^T,$$

$$\Sigma_P \triangleq (\hat{B}^TP\hat{A} + \hat{R}_{12}^T + \tilde{R}_2D_c\hat{C})^T\tilde{R}_2^{-1}(\hat{B}^TP\hat{A} + \hat{R}_{12}^T + \tilde{R}_2D_c\hat{C}),$$

$$M \triangleq \begin{bmatrix} I_{n+l} \\ D_c\hat{C} \end{bmatrix}, \; \hat{M} \triangleq \begin{bmatrix} I_{n+l} \\ -\tilde{R}_2^{-1}(\hat{B}^TP\hat{A} + \hat{R}_{12}^T) \end{bmatrix},$$

$$\hat{R} \triangleq \begin{bmatrix} \hat{R}_1 & \hat{R}_{12} \\ \hat{R}_{12}^T & \hat{R}_2 \end{bmatrix}.$$

Theorem 3.1: Suppose $(A_c, B_c, C_c, D_c) \in \mathcal{S}$ solves the fixed-order, sampled-data dynamic-compensation problem. Then there exist $(n + l) \times (n + l)$ nonnegative-definite matrices Q, P, $\hat{Q}$, and $\hat{P}$ such that A_c, B_c, C_c, and D_c are given by

$$A_c = \Gamma[\hat{A} - \hat{A}Q\hat{C}^T\tilde{V}_2^{-1}\hat{C} - \hat{B}\tilde{R}_2^{-1}(\hat{B}^TP\hat{A} + R_{12}^T) - \hat{B}D_c\hat{C}]G^T, \tag{3.5}$$

$$B_c = \Gamma[\hat{A}Q\hat{C}^T\tilde{V}_2^{-1} + \hat{B}D_c], \tag{3.6}$$

$$C_c = -[\tilde{R}_2^{-1}(\hat{B}^TP\hat{A} + \hat{R}_{12}^T) + D_c\hat{C}]G^T, \tag{3.7}$$

$$D_c = -\tilde{R}_2^{-1}(\hat{B}^TP\hat{A}Q\hat{C}^T + \hat{R}_{12}^TQ\hat{C}^T)\tilde{V}_2^{-1}, \tag{3.8}$$

and such that Q, P, $\hat{Q}$, and $\hat{P}$ satisfy

$$Q = \hat{A}Q\hat{A}^T - \hat{A}Q\hat{C}^T\tilde{V}_2^{-1}\hat{C}Q\hat{A}^T + \hat{V} + \tau_\perp\hat{Q}\tau_\perp^T, \tag{3.9}$$

$$P = \hat{A}^TP\hat{A} - (\hat{B}^TP\hat{A} + \hat{R}_{12}^T)^T\tilde{R}_2^{-1}(\hat{B}^TP\hat{A} + \hat{R}_{12}^T) + \hat{R}_1 + \tau_\perp^T\hat{P}\tau_\perp, \tag{3.10}$$

$$\hat{Q} = \hat{A}_P\tau\hat{Q}\tau^T\hat{A}_P^T + \Sigma_Q, \tag{3.11}$$

$$\hat{P} = \hat{A}_Q^T\tau^T\hat{P}\tau\hat{A}_Q + \Sigma_P, \tag{3.12}$$

where

$$\tau = \sum_{i=1}^{n_c} \Psi E_i \Psi^{-1} \tag{3.13}$$

for some $\Psi \in \mathcal{C}(\hat{Q}, \hat{P})$ such that $(\Psi^{-1}\hat{Q}\hat{P}\Psi)_{(i,i)} \neq 0$, $i = 1, \cdots, n_c$, and some projective factorization G, Γ of τ. Furthermore, the minimal cost is given by

$$J(A_c, B_c, C_c, D_c) = \delta + \operatorname{tr}\,[(MQM^T + \hat{M}\tau\hat{Q}\tau^T\hat{M}^T)\tilde{R}]. \tag{3.14}$$

Remark 3.1: Theorem 3.1 can immediately be specialized to the more restrictive problem in which the compensator is strictly proper. This can be done in both the full- and reduced-order cases by ignoring (3.8) and setting $D_c = 0$ wherever it appears. See [15], [16].

IV. Numerical Evaluation of Integrals Involving Matrix Exponentials

To evaluate the exponential/integral expressions appearing in Theorem 2.1, we utilize the approach of [20]. The idea is to eliminate the need for integration by computing the matrix exponential of appropriate block matrices. Numerical matrix exponentiation is discussed in [21].

Proposition 4.1: Consider the following partitioned matrix exponentials of order $(3n + l) \times (3n + l)$, $(3n + m) \times (3n + m)$, $(2n + m) \times (2n + m)$, and $(3n) \times (3n)$, respectively:

$$\begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ 0_n & F_5 & F_6 & F_7 \\ 0_n & 0_n & F_8 & F_9 \\ 0_{l\times n} & 0_{l\times n} & 0_{l\times n} & I_l \end{bmatrix} \triangleq \exp \begin{bmatrix} -A & I_n & 0_n & 0_{n\times l} \\ 0_n & -A & V_1 & V_{12} \\ 0_n & 0_n & A^T & C^T \\ 0_{l\times n} & 0_{l\times n} & 0_{l\times n} & 0_l \end{bmatrix} h,$$

$$\begin{bmatrix} F_{10} & F_{11} & F_{12} & F_{13} \\ 0_n & F_{14} & F_{15} & F_{16} \\ 0_n & 0_n & F_{17} & F_{18} \\ 0_{m\times n} & 0_{m\times n} & 0_{m\times n} & I_m \end{bmatrix} \triangleq \exp \begin{bmatrix} -A^T & I_n & 0_n & 0_{n\times m} \\ 0_n & -A^T & R_1 & R_{12} \\ 0_n & 0_n & A & B \\ 0_{m\times n} & 0_{m\times n} & 0_{m\times n} & 0_m \end{bmatrix} h,$$

$$\begin{bmatrix} I_n & F_{19} & F_{20} \\ 0_n & F_{21} & F_{22} \\ 0_{m\times n} & 0_{m\times n} & I_m \end{bmatrix} \triangleq \exp \begin{bmatrix} 0_n & I_n & 0_{n\times m} \\ 0_n & A & B \\ 0_{m\times n} & 0_{m\times n} & 0_m \end{bmatrix} h,$$

$$\begin{bmatrix} F_{23} & F_{24} & F_{25} \\ 0_n & F_{26} & F_{27} \\ 0_n & 0_n & F_{28} \end{bmatrix} \triangleq \exp \begin{bmatrix} -A & I_n & 0_n \\ 0_n & -A & V_1 \\ 0_n & 0_n & A^T \end{bmatrix} h.$$

Then

$$A' = F_{17}, \; B' = F_{18}, \; C' = \frac{1}{h}F_9^T, \; D' = \frac{1}{h}CF_{20} + D, \; \delta = \frac{1}{h}\operatorname{tr} R_1F_{28}^TF_{25},$$

$$V_1' = F_8^TF_6, \; V_{12}' = \frac{1}{h}F_8^TF_7, \; V_2' = \frac{1}{h}\left(V_2 + \frac{1}{h}CF_8^TF_4 + \frac{1}{h}F_4^TF_8C^T\right),$$

$$R_1' = \frac{1}{h}F_{17}^TF_{15}, \; R_{12}' = \frac{1}{h}F_{17}^TF_{16}, \; R_2' = R_2 + \frac{1}{h}(B^TF_{17}^TF_{13} + F_{13}^TF_{17}B).$$

V. Numerical Solution of the Discrete-Time Optimal Projection Equations

The following algorithm is proposed for solving (3.9)–(3.12).

Algorithm 5.1:

Step 1: Initialize $k = 0$ and $\tau^{(0)} = I_{n+l}$.

Step 2: With $\tau \triangleq \tau^{(k)}$ solve (3.9)–(3.12) for $Q^{(k)} \triangleq Q$, $P^{(k)} \triangleq P$, $\hat{Q}^{(k)} \triangleq \hat{Q}$, and $\hat{P}^{(k)} \triangleq \hat{P}$.

Step 3: If $k \geq 1$ check for convergence: If $\|(Q^{(k)}, P^{(k)}, \hat{Q}^{(k)}, \hat{P}^{(k)}) - (Q^{(k-1)}, P^{(k-1)}, \hat{Q}^{(k-1)}, \hat{P}^{(k-1)})\| >$ tol then continue; else go to Step 6.

Step 4: Select $\Psi^{(k)} \in \mathcal{C}(\hat{Q}^{(k)}, \hat{P}^{(k)})$ and update $\tau^{(k+1)} = \Sigma_{i=1}^{n_c} \Psi^{(k)}E_i(\Psi^{(k)})^{-1}$.

Step 5: Increment k and go to Step 2.

Step 6: Evaluate (3.5)–(3.8) with $Q = Q^{(k)}$, $P = P^{(k)}$, $\hat{Q} = \hat{Q}^{(k)}$, $\hat{P} = \hat{P}^{(k)}$, $G^T\Gamma = \tau^{(k)}$, $\Gamma G^T = I_{n_c}$.

Remark 5.1: In solving the Riccati equation (3.9), the nonhomogeneous term is taken to be $\hat{V} + \tau_\perp\hat{Q}\tau_\perp^T$, which is nonnegative definite. Similar remarks apply to (3.10).

TABLE I
COST J FOR BEAM EXAMPLE

n_c \ h	0	.1	.5
12	---	1.3715	3.0134
10	1.1677	1.3723	3.0134
8	1.1883	1.3823	3.0162
6	1.2086	1.4421	3.0195
4	1.3330	1.5752	3.0812
2	1.4789	2.0425	3.3406
Open-Loop Cost (u=0) is 101.73			

Remark 5.2: The critical step of Algorithm 5.1 is the choice of $\Psi^{(k)}$ for constructing the projection $\tau^{(k+1)}$. Since $\Psi^{(k)}$ can include basis rearrangements, the choice of $\Psi^{(k)}$ essentially corresponds to a selection of n_c rank-1 eigenprojections out of $n + l$ possible eigenprojections. This selection is discussed at length in [22] where it is pointed out that the choice of eigenprojections determines which local extremal will be reached by the algorithm. Component-cost methods have thus been utilized as a promising selection criteria. Because of the eigenprojection structure of the necessary conditions, Algorithm 5.1 is fundamentally different from gradient search methods.

VI. Illustrative Example: Control of a Flexible Beam

Consider a simply supported beam of length two with two colocated sensor/actuator pairs placed at coordinates $a_1 = 55/172$ and $a_2 = 45/43$. Define

$$A = \underset{i=1,\cdots,5}{\text{block-diag}} \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix}, \quad \omega_i = i^2, \; i=1, \cdots, 5, \; \zeta = 0.005,$$

$$B_{(i,j)} = 0.5(-1)^{j+1}(1+(-1)^i) \sin(i\pi a_j/4),$$
$$i=1, \cdots, 10, \; j=1, 2, \; C=B^T,$$

$$V_1 = \underset{i=1,\cdots,5}{\text{block-diag}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_{12}=0, \; V_2 = 0.1 I_2,$$

$$R_1 = \underset{i=1,\cdots,5}{\text{block-diag}} \begin{bmatrix} 1 & 0 \\ 0 & \omega_i^{-2} \end{bmatrix}, \quad R_{12}=0, \; R_2 = 0.1 I_2.$$

For $n_c = 10, 8, 6, 4, 2$ continuous-time controllers were designed using the results of [17] and, for $n_c = 12, 10, 8, 6, 4, 2$ and $h = 0.1, 0.5$, strictly proper ($D_c = 0$) discrete-time controllers were obtained from Theorem 3.1. The results are summarized in Table I.

VII. Directions for Further Developments

The following extensions and related developments immediately suggest themselves: reduced-order, discrete-time modeling/estimation of continuous-time systems [22], [23]: robust sampled-data control of uncertain systems with multiplicative noise [24]-[27]; multirate sampling [28], [29]; alternative A/D and D/A devices and asynchronous sampling/control update; infinite-dimensional systems [30], [31].

References

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